Free Convective Laminar Boundary Layer Flows about an Inclined Vertical Curved Surface

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Dedications

This work is dedicated to

My parents

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Declaration

None of the materials contained in this thesis will be submitted in support of any other degree or diploma at any other university or institution other than publications.

MAHKh
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Abstract

Possible similarity solutions for free convective laminar three dimensional boundary layer flows over an inclined vertical curvillinear surface, $h_1(\xi,\eta) > 0$, $h_2(\xi,\eta) > 0$, $h_3(\xi,\eta) = 1$, are discussed in different situations. The three dimensional boundary layer equations are considered in the curvillinear coordinate system and the relevant partial differential equations are transformed into ordinary differential equations by similarity transformations. The results thus obtained have a graphical illustration for different values of controlling parameters, the Prandtl number, Pr of the fluid, temperature power/exponent, m, scale factors power/exponent, n, a constant, c, and the angle δ (angle between the ξ-axis and the horizontal surface). Finally, the graphs and tables are displayed with discussion.

Contents

Introduction

There are essentially three factors which govern the natural convection process, the body force, the temperature variation in the flow field and the fluid density variation with temperature. Until recently, in studies of this mode of heat transfer, these factors were considered to be, respectively, the gravitational force, temperature differences and normal density temperature variations as encountered in such common fluids as oil, water and air. Such considerations correspond to rather restricted practical applications of the natural convection process.

However, an increase of one or more of these important physical factors should increase both the skin friction and heat transfer associated with the natural convection phenomenon. Currently, there are many practical situations in which these factors can be increased greatly beyond the previously considered limits. For example, in aircraft propulsion systems there are components (such as gas turbines and helicopter ramjets) which rotate at high speeds. Associated with these rotative speeds the large centrifugal forces similar to the gravitational force, are also proportional to the fluid density and hence can generate strong natural convection flows. Further, in nuclear power applications, very large temperature variations are encountered as are also unusual fluids whose density temperature variations may be more favourable for the natural convection process.

Laminar free convection from vertical surfaces (flat plates and cylinders) has been studied extensively when the temperature of the surface is uniform. The case of uniform heat flux rate at the surface, which is sometimes approximated in practical applications, was first discussed by Sparrow and Gregg[1955]. An exact solution has been obtained for Prandtl numbers in the range 0.1 to 100. He published many papers [1958 - 1961] on natural convections.

Ostrach [1953] analysed the new aspects of natural convection heat transfer. He studied the flow between two parallel infinite plates oriented to the direction of the generating body force . He [1954] later worked on combined natural and forced convection laminar flows and heat transfer of fluids.

Yang [1960] studied the unsteady laminar boundary layer equations for free convection on vertical plates and cylinders. He established some necessary and sufficient conditions for which the similarity solutions are possible. He dealt with steady and unsteady cases, but numerical works were found absent in his work.

Merkin [1969] started his work on free convection in early 60's of this century. He studied first the buoyancy effect on a semi infinite vertical flat plate in a uniform stream. Consequently,he[1985] showed the possible similarity solutions, then analysed the effects of Prandtl number. He continued his research related to prescribed surface heat flux with Mahmood [1990] and the conjugate free convection with Pop [1996]. Cohen and Reshotco [1955] worked on the similar solutions for the compressible boundary layer and Eckert and Jackson [1951] worked on the free convection on turbulent flow.

Recently Lee and Lin [1997] studied on transient conjugate heat transfer relating the heat conduction inside a solid body of arbitrary shape and the natural convection around the solid. In their computational work, they utilised Cartesian grid system.

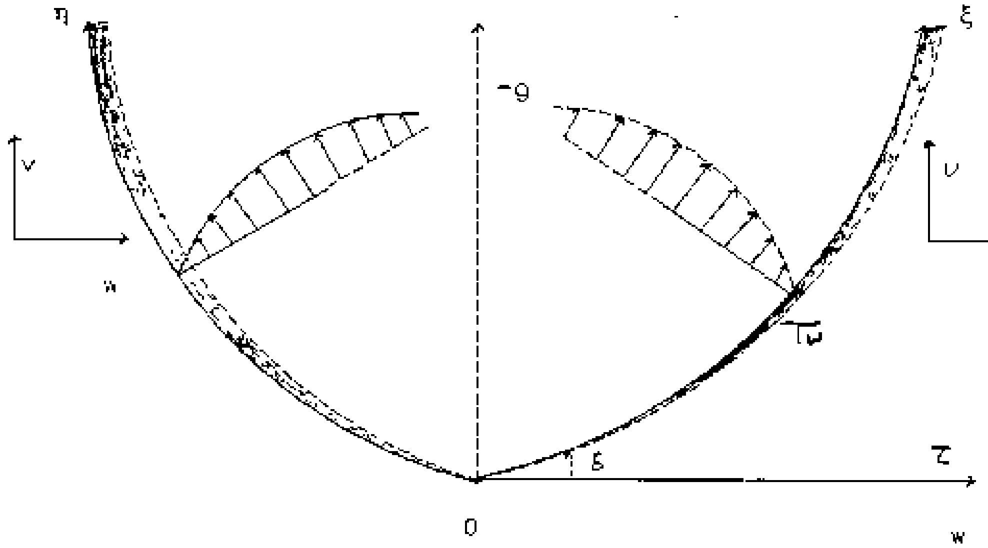
So far knowledge goes, no attempt has so far been made for the similarity solutions related to free convection on three dimensional surface with curvillinear coordinates. The similarity solutions for forced convection for three dimensional case was studied by Hansen [1958]. He presented similarity solutions of the three dimensional, laminar incompressible boundary layer equations on a general basis of analysis. Restrictions on potential flow velocity components and coordinate system which lead to similarity solutions were given in a table.

Similar to Hansen, Maleque [1996] studied the possible similarity solutions of Combined forced and free convective laminar boundary layer flows in curvilinear co-ordinates. Zakerullah etc. [1998] displayed the similarity requirements for orthogonal vertical curvilinear surfaces in tabular form

The similarity solutions of free convective three dimensional laminar boundary layer flows in curvilinear coordinates is more complicated in comparison with that of two dimensional boundary layer flow. In the present study, discussion is confined about the free convective three dimensional boundary layer flow over an inclined vertical curvilinear surfaces. The three dimensional boundary layer equations are developed for the curvilinear coordinate system and the relevant partial differential equations are transformed to ordinary differential equations by similarity transformations. The set of transformed equations are solved numerically to predict some essential flow parameters.

Chapter-1

Flow configuration of the problem



Free or natural convection flow arises under various reasons in nature. When the density variation of a particular fluid around heated or cooled object comes into play, a buoyancy effects, in general, is generated. Due to this effect heat is transferred from the surface of the object to the fluid layers in its neighborhood and then the body experiences friction due to velocity generated by the buoyancy effects.

Once the position of the edge of the surface, which is also to be the origin of the co-ordinate system, is decided, there are two combinations of the body force direction and the surface thermal condition that will yield flows that proceed away from the edge. If the edge of the semi infinite surface is taken at the bottom of the surface (i.e. the surface extends to $+\infty$ perpendicular to the ς-direction), the two combinations leading to flows in the proper direction are : the body force acting downward with a heated surface and the body force acting upward with a cooled surface.

The present problem is concerned with the three dimensional boundary layer free convective flows about an inclined vertical orthogonal curved surface. The investigation has led to the development of a technique somewhat different from that discussed in two dimensional case, which permits the systematic study of the conditions governing the existence of similarity solutions. In this problem, the coordinates ξ and η are considered to lie and be defined in the surface over which the boundary layer is flowing, while ζ extends into the boundary layer. Here we restrict ourselves $h_3(\xi,\eta) = 1$, so that ζ represents an actual distance measured along a straight normal from the surface. The surface is such vertically inclined with the horizontal surface so that it makes an angle δ with the horizontal surface. The body force is taken as the gravitational force $-\vec{g} = \left(-\vec{g}_{\xi}, -\vec{g}_{\eta}, 0\right)$. The surface thermal conditions are not uniform and the temperature variation along the surface $T = T_w(\xi,\eta)$, is greater than the ambient constant temperature T_∞. This temperature difference generates velocity as well as thermal boundary layer over the surface.

Governing equations :

The free convective flow about an inclined orthogonal vertical curvilinear surface are governed by the following equations :

continuity equation :

$$\frac{\partial}{\partial \xi}(h_2 u) + \frac{\partial}{\partial \eta}(h_1 v) + \frac{\partial}{\partial \zeta}(h_1 h_2 w) = 0, \qquad (1.1)$$

u- momentum equation along the ξ- direction :

$$\rho\left[\frac{u}{h_1}\frac{\partial u}{\partial \xi} + \frac{v}{h_2}\frac{\partial u}{\partial \eta} + w\frac{\partial u}{\partial \zeta} + \frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta} + \frac{uw}{h_1 h_2}\frac{\partial}{\partial \zeta}(h_1 h_2) - \left(\frac{v^2 + w^2}{h_1 h_2}\right)\frac{\partial h_2}{\partial \xi}\right]$$

$$= -\frac{1}{h_1}\frac{\partial P}{\partial \xi} + \rho g_\xi + \mu \nabla^2 u, \qquad (1.2)$$

v- momentum equation along the η- direction :

$$\rho\left[\frac{u}{h_1}\frac{\partial v}{\partial \xi} + \frac{v}{h_2}\frac{\partial v}{\partial \eta} + w\frac{\partial v}{\partial \zeta} + \frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi} + \frac{vw}{h_1 h_2}\frac{\partial}{\partial \zeta}(h_1 h_2) - \left(\frac{u^2 + w^2}{h_1 h_2}\right)\frac{\partial h_1}{\partial \eta}\right]$$

$$= -\frac{1}{h_2}\frac{\partial P}{\partial \eta} + \rho g_\eta + \mu \nabla^2 v, \qquad (1.3)$$

w- momentum equation along the ζ- direction :

$$\rho\left[\frac{u}{h_1}\frac{\partial w}{\partial \xi} + \frac{v}{h_2}\frac{\partial w}{\partial \eta} + w\frac{\partial w}{\partial \zeta} + \frac{uw}{h_1 h_2}\frac{\partial h_2}{\partial \xi} + \frac{vw}{h_1 h_2}\frac{\partial h_1}{\partial \eta} - \left(\frac{u^2 + v^2}{h_1 h_2}\right)\frac{\partial}{\partial \zeta}(h_1 h_2)\right]$$

$$= -\frac{\partial P}{\partial \zeta} + \mu \nabla^2 w, \qquad (1.4)$$

Energy equation :

$$\rho C_p\left[\frac{u}{h_1}\frac{\partial T}{\partial \xi}+\frac{v}{h_2}\frac{\partial T}{\partial \eta}+w\frac{\partial T}{\partial \zeta}\right]=\kappa\,\nabla^2 T+\mu\,\Phi\ , \tag{1.5}$$

$$\text{where } \nabla^2=\frac{1}{h_1h_2}\left[\frac{\partial}{\partial \xi}\left(\frac{h_2}{h_1}\frac{\partial}{\partial \xi}\right)+\frac{\partial}{\partial \eta}\left(\frac{h_1}{h_2}\frac{\partial}{\partial \eta}\right)+\frac{\partial}{\partial \zeta}\left((h_1h_2\frac{\partial}{\partial \zeta}\right)\right], \tag{1.6}$$

and the dissipation function

$$\Phi=2\left\{\left(\frac{\partial(h_2u)}{\partial\xi}\right)^2+\left(\frac{\partial(h_2v)}{\partial\eta}\right)^2+\left(\frac{\partial(h_1h_2w)}{\partial\zeta}\right)^2\right\}+\left(\frac{\partial(h_1v)}{\partial\xi}+\frac{\partial(h_2u)}{\partial\eta}\right)^2+\left(\frac{\partial(h_1h_2w)}{\partial\eta}+\frac{\partial(h_1v)}{\partial\zeta}\right)^2$$
$$+\left(\frac{\partial(h_2u)}{\partial\zeta}+\frac{\partial(h_1h_2w)}{\partial\xi}\right)^2-\frac{2}{3}\left(\frac{\partial(h_2u)}{\partial\xi}+\frac{\partial(h_1v)}{\partial\eta}+\frac{\partial(h_1h_2w)}{\partial\zeta}\right)^2$$

For simplicity, we set $h_3(\xi,\eta)=1$, so that ζ represent an actual distance measured along a straight normal from the surface. As a result of this simplifying assumptions only the choice of the two remaining surface coordinates ξ and η needs to be made. The ξ , η axes of the curvilinear coordinate system are along the vertically orthogonal curved surface and ζ axis normal to it. The thermal difference of the surface and ambient fluid generates the free convection flows. We assume that the viscosity and thermal conductivity coefficients are constants.

In our case, heating due to viscous dissipation is neglected and fluid is considered steady and incompressible.

Since the equation of state plays a vital rule for a fluid, we consider this in general form as

$$\rho = \rho(P.T.), \qquad (1.7)$$

For small change the above equation may be expressed as

$$\begin{aligned} d\rho &= \left(\frac{\partial\rho}{\partial P}\right)_T dP + \left(\frac{\partial\rho}{\partial T}\right)_P dT \\ &= \rho\left[\overline{K}\, dP - \beta_T\, dT\right] \end{aligned} \qquad (1.8)$$

where $\overline{K} = \frac{1}{\rho}\left[\left(\frac{\partial\rho}{\partial P}\right)\right]_T$ is the isothermal compressibility coefficient

and $\beta_T = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P$ is the volumetric expansion coefficient .

From the volumetric expansion, the relations may also be derived as follows

$$\beta_T = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P = -\frac{1}{\rho}\left(\frac{\rho-\rho_\infty}{T-T_\infty}\right), \qquad (1.8a)$$

$$\Rightarrow \rho-\rho_\infty = -\rho\beta_T\Delta T\theta = -\rho\beta_T\theta\Delta T, \qquad (1.8b)$$

$$T-T_\infty = \Delta T\theta, \qquad \Delta T = T_w - T_\infty \ , \qquad (1.8c)$$

for ideal gas , in fact $\beta_T = \frac{1}{T_\infty}$.

The boundary conditions to be imposed on the present problem may be determined as follows :

(a) The fluid must adhere to the surface (the no slip condition). That is , mathematically for the surface

$$u(\xi,\eta,0)=0=v(\xi,\eta,0)\ , \qquad (1.9)$$

(b) The temperature of the fluid at the surface must be function of ξ and η (non-isothermal surface):

$$T(\xi,\eta,0) = T_w(\xi,\eta), \qquad (1.10)$$

(c) The fluid at large distances from the surface must remain undisturbed :

$$u(\xi,\eta,\infty)=0=v(\xi,\eta,\infty)\,, \qquad (1.11)$$

(d) The temperature at large distances from the surface must be equal to the undisturbed fluid temperature

$$Lt_{\zeta\to\infty}T(\xi,\eta,\zeta)=T_{\infty}(=const.), \qquad (1.12)$$

The terms $\rho g_{\xi}, \rho g_{\eta}$ represent the body force components exerted on fluid particle. The pressure gradients in the ξ- & η- directions result from the change in elevation up the curved surface. Thus the hydrostatic conditions are,

$$-\frac{1}{h_1}\frac{\partial P}{\partial \xi}+\rho_{\infty}g_{\xi}=0.$$

$$\Rightarrow -\frac{1}{h_1}\frac{\partial P}{\partial \xi}=-\rho_{\infty}g_{\xi},$$

similarly,

$$-\frac{1}{h_2}\frac{\partial P}{\partial \eta}=-\rho_{\infty}g_{\eta},$$

Thus the elimination of pressure terms yields the equations (1.2), (1.3), (1.4) as,

$$\rho\left[\frac{u}{h_1}\frac{\partial u}{\partial \xi}+\frac{v}{h_2}\frac{\partial u}{\partial \eta}+w\frac{\partial u}{\partial \zeta}+\frac{uv}{h_1h_2}\frac{\partial h_1}{\partial \eta}+\frac{uw}{h_1h_2}\frac{\partial}{\partial \zeta}(h_1h_2)-\left(\frac{v^2+w^2}{h_1h_2}\right)\frac{\partial h_2}{\partial \xi}\right]$$

$$=(\rho-\rho_{\infty})g_{\xi}+\mu\nabla^2 u, \qquad (1.13)$$

Similarly,

$$\rho\left[\frac{u}{h_1}\frac{\partial v}{\partial \xi}+\frac{v}{h_2}\frac{\partial v}{\partial \eta}+w\frac{\partial v}{\partial \zeta}+\frac{uv}{h_1h_2}\frac{\partial h_2}{\partial \xi}+\frac{vw}{h_1h_2}\frac{\partial}{\partial \zeta}(h_1h_2)-\left(\frac{u^2+w^2}{h_1h_2}\right)\frac{\partial h_1}{\partial \eta}\right]$$

$$=(\rho-\rho_\infty)g_\eta+\mu\nabla^2 v, \qquad (1.14)$$

$$\rho\left[\frac{u}{h_1}\frac{\partial w}{\partial \xi}+\frac{v}{h_2}\frac{\partial w}{\partial \eta}+w\frac{\partial w}{\partial \zeta}+\frac{uw}{h_1h_2}\frac{\partial h_2}{\partial \xi}+\frac{vw}{h_1h_2}\frac{\partial h_1}{\partial \eta}-\left(\frac{u^2+v^2}{h_1h_2}\right)\frac{\partial}{\partial \zeta}(h_1h_2)\right]$$

$$=-\frac{\partial P}{\partial \zeta}+\mu\nabla^2 w, \qquad (1.15)$$

respectively.

Introducing the expression $\rho-\rho_\infty=-\rho\,\beta_T\,\theta\,\Delta T$ in equations (1.13-1.15),we get,

$$\rho\left[\frac{u}{h_1}\frac{\partial u}{\partial \xi}+\frac{v}{h_2}\frac{\partial u}{\partial \eta}+w\frac{\partial u}{\partial \zeta}+\frac{uv}{h_1h_2}\frac{\partial h_1}{\partial \eta}+\frac{uw}{h_1h_2}\frac{\partial}{\partial \zeta}(h_1h_2)-\left(\frac{v^2+w^2}{h_1h_2}\right)\frac{\partial h_2}{\partial \xi}\right]$$

$$=-\rho\,\beta_T\theta\,\Delta T g_\xi+\mu\nabla^2 u, \qquad (1.16)$$

$$\rho\left[\frac{u}{h_1}\frac{\partial v}{\partial \xi}+\frac{v}{h_2}\frac{\partial v}{\partial \eta}+w\frac{\partial v}{\partial \zeta}+\frac{uv}{h_1h_2}\frac{\partial h_2}{\partial \xi}+\frac{vw}{h_1h_2}\frac{\partial}{\partial \zeta}(h_1h_2)-\left(\frac{u^2+w^2}{h_1h_2}\right)\frac{\partial h_1}{\partial \eta}\right]$$

$$=-\rho\,\beta_T\theta\,\Delta T g_\eta+\mu\nabla^2 v, \qquad (1.17)$$

$$\rho\left[\frac{u}{h_1}\frac{\partial w}{\partial \xi}+\frac{v}{h_2}\frac{\partial w}{\partial \eta}+w\frac{\partial w}{\partial \zeta}+\frac{uw}{h_1h_2}\frac{\partial h_2}{\partial \xi}+\frac{vw}{h_1h_2}\frac{\partial h_1}{\partial \eta}-\left(\frac{u^2+v^2}{h_1h_2}\right)\frac{\partial}{\partial \zeta}(h_1h_2)\right]$$

$$=-\frac{\partial P}{\partial \zeta}+\mu\nabla^2 w, \qquad (1.18)$$

We now introduce the following non-dimensional variables,

$$\bar{\xi}=\frac{\xi}{L},\quad \bar{\eta}=\frac{\eta}{L},\quad \bar{\zeta}=\frac{\zeta}{L}$$

$$\bar{u}=\frac{u}{\bar{U}},\quad \bar{v}=\frac{v}{\bar{U}},\quad \bar{w}=\frac{w}{\bar{U}},$$

$$\bar{\rho}=\frac{\rho}{\rho_\infty},\ \bar{\mu}=\frac{\mu}{\mu_0},\quad \bar{k}=\frac{k}{k_0}, \tag{1.19a}$$

$$\bar{g}_\xi=\frac{g_\xi}{g},\bar{g}_\eta=\frac{g_\eta}{g},$$

$$\theta=\frac{T-T_\infty}{T_w-T_\infty}=\frac{T-T_\infty}{\Delta T}.$$

where L and $\bar{U}=\sqrt{g\,\beta_T\,\Delta T(\xi,\eta)L(\xi,\eta)}$ as some reference length and characteristic velocity generated by the buoyancy force, Introducing (1.19a) into the equations 1.1), (1.16), (1.17), (1.18) and (1.5) ,

we obtain the following non dimensional equations:

continuity equation:

$$\frac{\partial}{\partial\bar{\xi}}(h_2\bar{u})+\frac{\partial}{\partial\bar{\eta}}(h_1\bar{v})+\frac{\partial}{\partial\bar{\zeta}}(h_1h_2\bar{w})=0, \tag{1.19}$$

u-momentum equation:

$$\left[\frac{\bar{u}}{h_1}\frac{\partial\bar{u}}{\partial\bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial\bar{u}}{\partial\bar{\eta}}+\bar{w}\frac{\partial\bar{u}}{\partial\bar{\zeta}}+\frac{\bar{u}\bar{v}}{h_1h_2}\frac{\partial h_1}{\partial\bar{\eta}}+\frac{\bar{u}\bar{w}}{h_1h_2}\frac{\partial}{\partial\bar{\zeta}}(h_1h_2)-\left(\frac{\bar{v}^2+\bar{w}^2}{h_1h_2}\right)\frac{\partial h_2}{\partial\bar{\xi}}\right]$$

$$=(-g_{\bar{\xi}}\theta)+\frac{1}{R_e}\bar{v}\left[\frac{1}{h_1h_2}\left\{\frac{\partial}{\partial\bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial\bar{u}}{\partial\bar{\xi}}\right)+\frac{\partial}{\partial\bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial\bar{u}}{\partial\bar{\eta}}\right)+\frac{\partial}{\partial\bar{\zeta}}\left(h_1h_2\frac{\partial\bar{u}}{\partial\bar{\zeta}}\right)\right\}\right], \tag{1.20}$$

v-momentum equation:

$$\left[\frac{\bar{u}}{h_1}\frac{\partial\bar{v}}{\partial\bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial\bar{v}}{\partial\bar{\eta}}+\bar{w}\frac{\partial\bar{v}}{\partial\bar{\zeta}}+\frac{\overline{uv}}{h_1h_2}\frac{\partial h_2}{\partial\bar{\xi}}+\frac{\overline{vw}}{h_1h_2}\frac{\partial}{\partial\bar{\zeta}}(h_1h_2)-\left(\frac{\bar{u}^2+\bar{w}^2}{h_1h_2}\right)\frac{\partial h_1}{\partial\bar{\eta}}\right]$$

$$=(-\theta\bar{g}_{\bar{\eta}})+\frac{1}{R_F}\bar{\nu}\left[\frac{1}{h_1h_2}\left\{\frac{\partial}{\partial\bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial\bar{v}}{\partial\bar{\xi}}\right)+\frac{\partial}{\partial\bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial\bar{v}}{\partial\bar{\eta}}\right)+\frac{\partial}{\partial\bar{\zeta}}\left(h_1h_2\frac{\partial\bar{v}}{\partial\bar{\zeta}}\right)\right]\right], \tag{1.21}$$

w-momentum equation:

$$\left[\frac{\bar{u}}{h_1}\frac{\partial\bar{w}}{\partial\bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial\bar{w}}{\partial\bar{\eta}}+\bar{w}\frac{\partial\bar{w}}{\partial\bar{\zeta}}+\frac{\overline{uw}}{h_1h_2}\frac{\partial h_2}{\partial\bar{\xi}}+\frac{\overline{vw}}{h_1h_2}\frac{\partial h_1}{\partial\bar{\eta}}-\left(\frac{\bar{u}^2+\bar{v}^2}{h_1h_2}\right)\frac{\partial}{\partial\bar{\zeta}}(h_1h_2)\right]$$

$$=-\frac{\partial P}{\partial\zeta}+\frac{1}{R_F}\bar{\nu}\left[\frac{1}{h_1h_2}\left\{\frac{\partial}{\partial\bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial\bar{w}}{\partial\bar{\xi}}\right)+\frac{\partial}{\partial\bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial\bar{w}}{\partial\bar{\eta}}\right)+\frac{\partial}{\partial\bar{\zeta}}\left(h_1h_2\frac{\partial\bar{w}}{\partial\bar{\zeta}}\right)\right]\right], \tag{1.22}$$

Energy equation:

$$\Rightarrow\overline{\rho C}_P\left[\left[\frac{\bar{u}}{h_1}\frac{\partial\theta}{\partial\bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial\theta}{\partial\bar{\eta}}+\bar{w}\frac{\partial\theta}{\partial\bar{\zeta}}\right]+\theta\left[\frac{\bar{u}}{h_1}\frac{\partial(\ln\Delta T)}{\partial\bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial(\ln\Delta T)}{\partial\bar{\eta}}+\bar{w}\frac{\partial(\ln\Delta T)}{\partial\bar{\zeta}}\right]\right]$$

$$=\frac{\bar{k}}{P_rR_F}\frac{1}{h_1h_2}\left[\left[\frac{\partial}{\partial\bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial\theta}{\partial\bar{\xi}}\right)+\frac{\partial}{\partial\bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial\theta}{\partial\bar{\eta}}\right)+\frac{\partial}{\partial\bar{\zeta}}\left(h_1h_2\frac{\partial\theta}{\partial\bar{\zeta}}\right)\right]\right.$$

$$\left.+\theta\left[\frac{\partial}{\partial\bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial(\ln\Delta T)}{\partial\bar{\xi}}\right)+\frac{\partial}{\partial\bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial(\ln\Delta T)}{\partial\bar{\eta}}\right)+\frac{\partial}{\partial\bar{\zeta}}\left(h_1h_2\frac{\partial(\ln\Delta T)}{\partial\bar{\zeta}}\right)\right]\right]. \tag{1.23}$$

Here $R_f=\frac{\bar{U}L}{\nu}=\frac{\left(\sqrt{g\,\beta_T\,\Delta T\,L}\right)L}{\nu}$ is the Reynold's number based on fluid velocity generated by the buoyancy effects.

The boundary conditions in dimensionless form are

$$\bar{u}(\bar{\xi},\bar{\eta},0)=\bar{v}(\bar{\xi},\bar{\eta},0)=0, \tag{1.24}$$

$$\bar{\theta}(\bar{\xi},\bar{\eta},0)=1, \tag{1.25}$$

$$\bar{u}(\bar{\xi},\bar{\eta},\infty)=\bar{v}(\bar{\xi},\bar{\eta},\infty)=0, \tag{1.26}$$

$$\theta(\bar{\xi},\bar{\eta},\infty)=0, \tag{1.27}$$

If δ be the boundary layer thickness, then the dimensionless boundary layer thickness is $\bar{\delta} = \frac{\delta}{L} << 1$, since $L >> 1$.

Now in order to determine the boundary layer equations, we have to estimate the order of the above equations (1.19 -- 1.23), so that very small terms can be neglected.

Since $\frac{\partial \bar{u}}{\partial \bar{\xi}}, \frac{\partial \bar{u}}{\partial \bar{\eta}}, \frac{\partial \bar{v}}{\partial \bar{\xi}}, \frac{\partial \bar{v}}{\partial \bar{\eta}}$ is of 0(1), so by equation of continuity (1.19),

$$\frac{\partial \bar{w}}{\partial \bar{\zeta}} \text{ is of } 0(1).$$

Since $\bar{\zeta}$ is of $0(\bar{\delta})$, so that $\bar{w}$ is of $0(\bar{\delta})$.

$$\text{and } \frac{\partial^2 \bar{u}}{\partial \bar{\zeta}^2} \sim 0\left(\frac{1}{\bar{\delta}^2}\right), \quad \frac{\partial^2 \bar{v}}{\partial \bar{\zeta}^2} \sim 0\left(\frac{1}{\bar{\delta}^2}\right), \quad \frac{\partial^2 \bar{w}}{\partial \bar{\zeta}^2} \sim 0\left(\frac{1}{\bar{\delta}}\right).$$

$$\frac{\partial \bar{w}}{\partial \bar{\zeta}} \sim 0(1), \quad \frac{\partial \bar{w}}{\partial \bar{\xi}} \sim 0(\bar{\delta}), \quad \frac{\partial \bar{w}}{\partial \bar{\eta}} \sim 0(\bar{\delta}).$$

$$R_F \sim 0\left(\frac{1}{\bar{\delta}_T^{\,2}}\right).$$

Let δ_T be the thermal boundary layer thickness, the conduction term becomes the same order of magnitude as the convectional term, only if the thickness of the thermal boundary layer is the order of $\left(\frac{\delta_T}{L}\right)^2 \sim \frac{1}{R_F \ P_r}$.

In view of the previously obtained estimation for the thickness of the velocity boundary layer $\bar{\delta} \sim \frac{1}{\sqrt{R_F}}$, it is found that

$$\frac{\delta_T}{L} \sim \frac{1}{\sqrt{R_F \ P_r}}, \Rightarrow \frac{\delta_T}{L} \sim \frac{\bar{\delta}}{\sqrt{P_r}} \Rightarrow \frac{\delta_T}{\delta} \sim \frac{1}{\sqrt{P_r}}.$$

Assuming that h_1, h_2 and all their first derivatives is of 0(1), setting the order of magnitude in each term of equations (1.19-1.23), we get

<u>equation of continuity</u>

$$\frac{\partial}{\partial \bar{\xi}}(h_2 \bar{u}) + \frac{\partial}{\partial \bar{\eta}}(h_1 \bar{v}) + \frac{\partial}{\partial \bar{\zeta}}(h_1 h_2 \bar{w}) = 0 ,$$

$$(0) \rightarrow \qquad (1) \qquad (1) \qquad (1)$$

<u>u - momentum equation</u>

$$\left[\frac{\bar{u}}{h_1}\frac{\partial \bar{u}}{\partial \bar{\xi}} + \frac{\bar{v}}{h_2}\frac{\partial \bar{u}}{\partial \bar{\eta}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{\zeta}} + \frac{\bar{u}\bar{v}}{h_1 h_2}\frac{\partial h_1}{\partial \bar{\eta}} + \frac{\bar{u}\bar{w}}{h_1 h_2}\frac{\partial}{\partial \bar{\zeta}}(h_1 h_2) - \left(\frac{\bar{v}^2 + \bar{w}^2}{h_1 h_2}\right)\frac{\partial h_1}{\partial \bar{\xi}}\right]$$

$$(0) \rightarrow \qquad (1) \qquad (1) \qquad (1) \qquad (1) \qquad (1) \qquad (1)(\bar{\delta}^2)$$

$$= (-\theta \bar{g}_{\xi}) + \frac{1}{R_F}\frac{\bar{\nu}}{h_1 h_2}\left\{\frac{\partial}{\partial \bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial \bar{u}}{\partial \bar{\xi}}\right) + \frac{\partial}{\partial \bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial \bar{u}}{\partial \bar{\eta}}\right) + \frac{\partial}{\partial \bar{\zeta}}\left(h_1 h_2 \frac{\partial \bar{u}}{\partial \bar{\zeta}}\right) + h_1 h_2 \frac{\partial^2 \bar{u}}{\partial \bar{\zeta}^2}\right\},$$

$$(1) \qquad \bar{\delta}^2 \qquad (1) \qquad (1) \qquad \frac{1}{\bar{\delta}} \qquad \frac{1}{\bar{\delta}^2}$$

<u>v - momentum equations</u>

$$\left[\frac{\bar{u}}{h_1}\frac{\partial \bar{v}}{\partial \bar{\xi}} + \frac{\bar{v}}{h_2}\frac{\partial \bar{v}}{\partial \bar{\eta}} + \bar{w}\frac{\partial \bar{v}}{\partial \bar{\zeta}} + \frac{\bar{u}\bar{v}}{h_1 h_2}\frac{\partial h_2}{\partial \bar{\xi}} + \frac{\bar{v}\bar{w}}{h_1 h_2}\frac{\partial}{\partial \bar{\zeta}}(h_1 h_2) - \left(\frac{\bar{u}^2 + \bar{w}^2}{h_1 h_2}\right)\frac{\partial h_1}{\partial \bar{\eta}}\right]$$

$$(0) \rightarrow \qquad (1) \qquad (1) \qquad (1) \qquad (1) \qquad (1) \qquad \bar{\delta}^2 \qquad (1)$$

$$= (-\theta \bar{g}_{\eta}) + \frac{1}{R_F}\bar{\nu}\left[\frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial \bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial \bar{v}}{\partial \bar{\xi}}\right) + \frac{\partial}{\partial \bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial \bar{v}}{\partial \bar{\eta}}\right) + \frac{\partial (h_1 h_2)}{\partial \bar{\zeta}}\left(\frac{\partial \bar{v}}{\partial \bar{\zeta}}\right) + h_1 h_2 \frac{\partial^2 \bar{v}}{\partial \bar{\zeta}^2}\right]\right.,$$

$$(1) \qquad (\bar{\delta}^2) \qquad (1) \qquad (1) \qquad (1) \quad \frac{1}{\bar{\delta}} \qquad \frac{1}{\bar{\delta}^2}$$

w - momentum equations

$$\left[\frac{\bar{u}}{h_1}\frac{\partial \bar{w}}{\partial \bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial \bar{w}}{\partial \bar{\eta}}+\bar{w}\frac{\partial \bar{w}}{\partial \bar{\zeta}}+\frac{\bar{u}\bar{w}}{h_1h_2}\frac{\partial h_2}{\partial \bar{\xi}}+\frac{\bar{v}\bar{w}}{h_1h_2}\frac{\partial h_1}{\partial \bar{\eta}}-\left(\frac{\bar{u}^2+\bar{v}^2}{h_1h_2}\right)\frac{\partial}{\partial \bar{\zeta}}(h_1h_2)\right]$$

$(0)\rightarrow$ $(\bar{\delta})$ $(\bar{\delta})$ $(\bar{\delta})$ $(\bar{\delta})$ $(\bar{\delta})$

$$=-\frac{\partial P}{\partial \zeta}+\frac{1}{R_F}\frac{\bar{v}}{h_1h_2}\left[\frac{\partial}{\partial \bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial \bar{w}}{\partial \bar{\xi}}\right)+\frac{\partial}{\partial \bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial \bar{w}}{\partial \bar{\eta}}\right)+\frac{\partial}{\partial \bar{\zeta}}\left(h_1h_2\frac{\partial \bar{w}}{\partial \bar{\zeta}}\right)+h_1h_2\frac{\partial \bar{w}}{\partial \bar{\zeta}^2}\right],$$

$(\bar{g}_\zeta=0)$ $(\bar{\delta}^2)$ $(\bar{\delta})$ $(\bar{\delta})$ (1) $\left(\frac{1}{\bar{\delta}}\right)$

$$\Rightarrow -\frac{\partial P}{\partial \zeta}\cong O(0)$$

Energy equation

$$\overline{\rho C_P}\left[\left\{\frac{\bar{u}}{h_1}\frac{\partial \theta}{\partial \bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial \theta}{\partial \bar{\eta}}+\bar{w}\frac{\partial \theta}{\partial \bar{\zeta}}\right\}+\theta\left\{\frac{\bar{u}}{h_1}\frac{\partial(\ln\Delta T)}{\partial \bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial(\ln\Delta T)}{\partial \bar{\eta}}+\bar{w}\frac{\partial(\ln\Delta T)}{\partial \bar{\zeta}}\right\}\right]$$

$(0)\rightarrow(1)$ (1) (1) (1) (1) (1) (1) (0)

$$=\frac{\bar{k}}{P_rR_F}\frac{1}{h_1h_2}\left[\left\{\frac{\partial}{\partial \bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial \theta}{\partial \bar{\xi}}\right)+\frac{\partial}{\partial \bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial \theta}{\partial \bar{\eta}}\right)+\frac{\partial}{\partial \bar{\zeta}}\left(h_1h_2\frac{\partial \theta}{\partial \bar{\zeta}}\right)\right\}\right.$$

(δ_T^2) (1) (1) (1)

$$\left.+\theta\left\{\frac{\partial}{\partial \bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial(\ln\Delta T)}{\partial \bar{\xi}}\right)+\frac{\partial}{\partial \bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial(\ln\Delta T)}{\partial \bar{\eta}}\right)+\frac{\partial}{\partial \bar{\zeta}}\left(h_1h_2\frac{\partial(\ln\Delta T)}{\partial \bar{\zeta}}\right)\right\}\right],$$

(1) (1) (0)

$$\Rightarrow \overline{\rho C_P}\left[\left\{\frac{\bar{u}}{h_1}\frac{\partial \theta}{\partial \bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial \theta}{\partial \bar{\eta}}+\bar{w}\frac{\partial \theta}{\partial \bar{\zeta}}\right\}+\theta\left\{\frac{\bar{u}}{h_1}\frac{\partial(\ln\Delta T)}{\partial \bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial(\ln\Delta T)}{\partial \bar{\eta}}\right\}\right]$$

$(0)\rightarrow$ (1) (1) $(1)'$ (1) (1) (1)

$$=\frac{\bar{k}}{P_rR_F}\frac{1}{h_1h_2}\left[\left\{\frac{\partial}{\partial \bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial \theta}{\partial \bar{\xi}}\right)+\frac{\partial}{\partial \bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial \theta}{\partial \bar{\eta}}\right)+\frac{\partial(h_1h_2)}{\partial \bar{\zeta}}\frac{\partial \theta}{\partial \bar{\zeta}}+\left(h_1h_2\frac{\partial^2 \theta}{\partial \bar{\zeta}^2}\right)\right\}\right]$$

$(\bar{\delta}_T^2)$ (1) (1) (1) $\left(\frac{1}{\delta_T^2}\right)$

$$+\theta\left\{\frac{\partial}{\partial\bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial(\ln\Delta T)}{\partial\bar{\xi}}\right)+\frac{\partial}{\partial\bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial(\ln\Delta T)}{\partial\bar{\eta}}\right)\right\}\Bigg],$$

(1) (1)

Neglecting the terms higher than order of $\bar{\delta}$ and $\bar{\delta}_T$ and omitting the dashes, we obtian,

$$\frac{\partial}{\partial\xi}(h_2u)+\frac{\partial}{\partial\eta}(h_1v)+\frac{\partial}{\partial\zeta}(h_1h_2w)=0, \tag{1.28}$$

$$\frac{u}{h_1}\frac{\partial u}{\partial\xi}+\frac{v}{h_2}\frac{\partial u}{\partial\eta}+w\frac{\partial u}{\partial\zeta}+\frac{uv}{h_1h_2}\frac{\partial h_1}{\partial\eta}+\frac{uw}{h_1h_2}\frac{\partial}{\partial\zeta}(h_1h_2)-\frac{v^2}{h_1h_2}\frac{\partial h_2}{\partial\xi}=-\theta\, g_\xi+\nu\frac{\partial^2u}{\partial\zeta^2}, \tag{1.29}$$

$$\frac{u}{h_1}\frac{\partial v}{\partial\xi}+\frac{v}{h_2}\frac{\partial v}{\partial\eta}+w\frac{\partial v}{\partial\zeta}+\frac{uv}{h_1h_2}\frac{\partial h_2}{\partial\xi}+\frac{vw}{h_1h_2}\frac{\partial}{\partial\zeta}(h_1h_2)-\frac{u^2}{h_1h_2}\frac{\partial h_1}{\partial\eta}=-\theta\, g_\eta+\nu\frac{\partial^2v}{\partial\zeta^2}, \tag{1.30}$$

and

$$\left(\frac{u}{h_1}\frac{\partial\theta}{\partial\xi}+\frac{v}{h_2}\frac{\partial\theta}{\partial\eta}+w\frac{\partial\theta}{\partial\zeta}\right)+\theta\left(\frac{u}{h_1}\frac{\partial(\ln\Delta T)}{\partial\xi}+\frac{v}{h_2}\frac{\partial(\ln\Delta T)}{\partial\eta}\right)=\frac{1}{P_r}\frac{\partial^2\theta}{\partial\zeta^2}, \tag{1.31}$$

where $P_r=\dfrac{\bar{\mu}\,\bar{C}_p}{\kappa}$ is the Prandtl number of the fluid,

The boundary conditions are,

$$u(\xi,\eta,0)=v(\xi,\eta,0)=0, \tag{1.32}$$

$$\theta(\xi,\eta,0)=1\quad, \tag{1.33}$$

$$u(\xi,\eta,\infty)=v(\xi,\eta,\infty)=0, \tag{1.34}$$

$$\theta(\xi,\eta,\infty)=0\ . \tag{1.35}$$

Chapter-2

Similarity transformations:

Equations (1.28-1.31) are non-linear simultaneous partial differential equations. Our aim is to reduce these equations to ordinary differential equations in order to predict some essential flow parameters.

Guided by the idea of the similarity analysis and following the method of Hansen [1958], the variables ξ, η, ζ be changed to a new set of variable X, Y and $\bar{\phi}$, where relations between two sets of variable are given by :

$$X = \xi, \quad Y = \eta \text{ and } \bar{\phi} = \frac{\zeta}{\gamma(X,Y)} \quad . \tag{2.1}$$

$\gamma(X,Y)$ is thought here to be proportional to the square root of the local boundary layer thickness. From equations (2.1), we have (by chain rule) the following expressions :

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial X} - \frac{\bar{\phi}}{\gamma}\gamma_x \frac{\partial}{\partial \bar{\phi}}, \tag{2.2}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial Y} - \frac{\bar{\phi}}{\gamma}\gamma_Y \frac{\partial}{\partial \bar{\phi}}, \tag{2.3}$$

$$\frac{\partial}{\partial \zeta} = \frac{1}{\gamma}\frac{\partial}{\partial \bar{\phi}}, \qquad \& . \qquad \frac{\partial^2}{\partial \zeta^2} = \frac{1}{\gamma^2}\frac{\partial^2}{\partial \bar{\phi}^2}. \tag{2.4}$$

Let two stream functions $\psi(\xi,\eta,\zeta)$ and $\phi(\xi,\eta,\zeta)$ be defined as the mass flow components within the boundary layer for the case of incompressible viscous flow. To satisfy the equation of continnity, we may introduce the components of the mass flow in the following way,

$$\psi_\zeta = h_2 u, \qquad \phi_\zeta = h_1 u \tag{2.5}$$

and $\quad -(\psi_\xi + \phi_\eta) = h_1 h_2 w.$

In order to seek the similarity functions, we introduce the following equations,

$$\int_0^{\bar{\phi}} \frac{u}{U(X,Y)} d\bar{\phi} = F(X,Y,\bar{\phi}), \tag{2.6}$$

where $\overline{U} = \sqrt{g\beta_T \Delta T L}$ represent the characteristic velocity (maximum) generated by the buoyancy effect & L denotes some characteristic length. Similarly we are allowed to write,

$$\int_0^{\bar{\phi}} \frac{v}{\overline{U}(X,Y)} d\bar{\phi} = S(X,Y,\bar{\phi}) \tag{2.7}$$

In attempting separation of variables of $F(X,Y,\bar{\phi}), S(X,Y,\bar{\phi})$ and $\theta(X,Y,\bar{\phi})$, it is assumed that

$$F(X,Y,\bar{\phi}) = \overline{L}(X,Y)\overline{F}(\bar{\phi}) \tag{2.8}$$

$$S(X,Y,\bar{\phi}) = \overline{M}(X,Y)\overline{S}(\bar{\phi}) \tag{2.9}$$

$$\theta(X,Y,\bar{\phi}) = \overline{N}(X,Y)\bar{\theta}(\bar{\phi}) \tag{2.10}$$

where $\overline{F}, \overline{S}$ and $\bar{\theta}$ are the functions of single variable $\bar{\phi}$. From (2.6) and (2.7), it is found that

$$\frac{u}{\overline{U}} = F_{\bar{\phi}} = \overline{L}\overline{F}_{\bar{\phi}}, \tag{2.11}$$

$$\Rightarrow u = \overline{U}\overline{L}\overline{F}_{\bar{\phi}}$$

$$\& \qquad v = \overline{U}\overline{M}\overline{S}_{\bar{\phi}}.$$

Therefore,

$$\psi_\zeta = h_2 u \Rightarrow \frac{\partial \psi}{\partial \zeta} = h_2 u$$

$$\Rightarrow \frac{u}{\overline{U}} = \frac{1}{h_2 \overline{U}} \frac{\partial \psi}{\partial \zeta} = \frac{1}{h_2} \frac{\partial}{\partial \zeta}\left(\frac{\psi}{\overline{U}}\right)$$

$$= \frac{1}{h_2 \gamma(X,Y)} \frac{\partial}{\partial \overline{\phi}}\left(\frac{\psi}{\overline{U}}\right)$$

$$= \frac{\partial}{\partial \overline{\phi}}\left(\frac{\psi}{h_2 \gamma \overline{U}}\right).$$

$$\therefore \int_0^{\overline{\phi}} \frac{u}{\overline{U}} d\overline{\phi} = \int_0^{\overline{\phi}} \frac{\partial}{\partial \overline{\phi}}\left(\frac{\psi}{h_2 \gamma \overline{U}}\right) d\overline{\phi} = \frac{1}{h_2 \gamma \overline{U}} [\psi(X,Y,\overline{\phi}) - \psi(X,Y,0)] \tag{2.12}$$

From equation (2.6), (2.8) and (2.12) , we get

$$F(X,Y,\overline{\phi}) = \frac{1}{h_2 \gamma \overline{U}} [\psi(X,Y,\overline{\phi}) - \psi(X,Y,0)].$$

$$\Rightarrow L(X,Y)\overline{F}\overline{\phi}) = \frac{1}{h_2 \gamma \overline{U}} [\psi(X,Y,\overline{\phi}) - \psi(X,Y,0)]$$

$$\Rightarrow \psi(X,Y,\overline{\phi}) = h_2 \gamma \overline{U}\overline{L}\overline{F}(\overline{\phi}) + \psi(X,Y,0) \tag{2.13}$$

Similarly,

$$\phi(X,Y,\overline{\phi}) = h_1 \gamma \overline{U}\overline{M}\overline{S}(\overline{\phi}) + \phi(X,Y,0) \tag{2.14}$$

and

$$h_1 h_2 w = -(\psi_\zeta + \phi_\eta)$$

$$= -[h_2 \gamma \overline{U}\overline{L}\overline{F}(\overline{\phi})]_\zeta - \psi_\zeta(X,Y,0) - [h_1 \gamma \overline{U}\overline{M}\overline{S(\overline{\phi})}]_\eta - \phi_\eta(X,Y\overline{U})$$

$$= -[h_2 \gamma \overline{U}\overline{L}\overline{F(\overline{\phi})}]_X - \psi_X(X,Y,0) + \frac{\overline{\phi}}{\gamma}\gamma_X [h_2 \gamma \overline{U}\overline{L}\overline{F(\overline{\phi})}]_{\overline{\phi}}$$

$$-[h_1 \gamma \overline{U}\overline{M}\overline{S(\overline{\phi})}]_Y - \phi_Y(X,Y,0) + \frac{\overline{\phi}}{\gamma}\gamma_Y [h_1 \gamma \overline{U}\overline{M}\overline{S(\overline{\phi})}]_{\overline{\phi}} \tag{2.15}$$

If $\bar{\phi} \to 0$, then

$$w_0(X,Y,0) = -\frac{1}{h_1 h_2}[\psi_X(X,Y,0) + \Phi_Y(X,Y,0)] \qquad (2.16)$$

If the surface be porous, w_0 represents the suction or injection velocity normal to the surface. Since $\bar{U}$ is independent of ζ, so $\bar{U}_{\bar{\phi}} = 0$.

Thus the equation (2.15) becomes,

$$h_1 h_2 w = -(h_2\gamma\bar{U}\bar{L}\bar{F})_X + \frac{\bar{\phi}}{\gamma}\gamma_x \ (h_2\gamma\bar{U}\bar{L}\bar{F})_{\bar{\phi}} - (h_1\gamma\bar{U}\bar{M}\bar{S})_Y + \frac{\bar{\phi}}{\gamma}\gamma_y \ (h_1\gamma\bar{U}\bar{M}\bar{S})_{\bar{\phi}} + h_1 h_2 w_0(X,Y,0)$$

$$= -(h_2\gamma\bar{U}\bar{L})_X\bar{F} + \bar{\phi}\gamma_x h_2\bar{U}\bar{L}\bar{F}_{\bar{\phi}} - (h_1\gamma\bar{U}\bar{M})_Y\bar{S} + \bar{\phi}\gamma_y h_1\bar{U}\bar{M}\bar{S}_{\bar{\phi}} + h_1 h_2 w_0(X,Y,0). \qquad (2.17)$$

The convective operator

$$\frac{d}{dt} = \frac{1}{h_1 h_2}\left[h_2 u\frac{\partial}{\partial\xi} + h_1 v\frac{\partial}{\partial\eta} + h_1 h_2 w\frac{\partial}{\partial\zeta}\right]$$

in terms of new set of variables X, Y and $\bar{\phi}$ may be derived. The convective operator in terms of new set of variables X, Y, $\bar{\phi}$ is

$$\frac{d}{dt} = \frac{1}{h_1 h_2}\left[h_2\bar{U}\bar{L}\bar{F}_{\bar{\phi}}\frac{\partial}{\partial X} + h_1\bar{U}\bar{M}\bar{S}_{\bar{\phi}}\frac{\partial}{\partial Y} - \frac{1}{\gamma}\{(h_2\gamma\bar{U}\bar{L})_X\bar{F} + (h_1\gamma\bar{U}\bar{M})_Y\bar{S} - h_1 h_2 w_0\}\frac{\partial}{\partial\bar{\phi}}\right]. \qquad (2.18)$$

In view of equation (2.18), equations (1.26), (1.27), (1.28) become,

$$\nu\bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{\gamma(\gamma h_2\bar{U}\bar{L})_X}{h_1 h_2}\bar{F}\bar{F}_{\bar{\phi}\bar{\phi}} + \frac{\gamma(\gamma h_1\bar{U}\bar{M})_Y}{h_1 h_2}\bar{S}\bar{F}_{\bar{\phi}\bar{\phi}} - \gamma w_0\bar{F}_{\bar{\phi}\bar{\phi}} - \frac{\gamma^2}{h_1}(\bar{U}\bar{L})_X\bar{F}^2_{\bar{\phi}}$$

$$-\gamma^2\frac{\bar{U}\bar{M}}{h_2}\left[\frac{(\bar{U}\bar{L})_Y}{\bar{U}\bar{L}} + \frac{h_{1Y}}{h_1}\right]\bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}} + \frac{\gamma^2}{h_1 h_2}\frac{\bar{U}^2\bar{M}^2}{\bar{U}\bar{L}}h_{2X}\bar{S}^2_{\bar{\phi}} - \frac{\gamma^2}{\bar{U}\bar{L}}g_{\xi}\beta_T\Delta T\theta = 0, \qquad (2.19)$$

$$\nu\bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{\gamma(\gamma h_1\overline{UM})_Y}{h_1 h_2}\overline{S}\overline{S}_{\bar{\phi}\bar{\phi}} + \frac{\gamma(\gamma h_2\overline{UL})_X}{h_1 h_2}\overline{F}\overline{S}_{\bar{\phi}\bar{\phi}} - \gamma w_0\bar{S}_{\bar{\phi}\bar{\phi}} - \frac{\gamma^2}{h_2}(\overline{UM})_Y\bar{S}^2_{\bar{\phi}}$$

$$-\frac{\gamma^2\overline{UL}}{h_1}\left[\frac{(\overline{UM})_X}{\overline{UM}} + \frac{h_{2X}}{h_2}\right]\bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}} + \frac{\gamma^2}{h_1 h_2}\frac{(\overline{UL})^2}{\overline{UM}}h_{1Y}\bar{F}^2_{\bar{\phi}} - \frac{\gamma^2}{\overline{UM}}g_n\beta_t\Delta T\theta = 0\,, \tag{2.20}$$

$$\frac{\nu}{P_r}\bar{\theta}_{\bar{\phi}\bar{\phi}} + \frac{\gamma(\gamma h_2\overline{UL})_X}{h_1 h_2}\overline{F}\theta_{\bar{\phi}} + \frac{\gamma(\gamma h_1\overline{UM})_Y}{h_1 h_2}\overline{S}\theta_{\bar{\phi}} - \gamma w_0\bar{\theta}_{\bar{\phi}} - \frac{\gamma^2\overline{UL}}{h_1}\left[(\ln\overline{N})_X + (\ln\Delta T)_X\right]\bar{F}_{\bar{\phi}}\bar{\theta}$$

$$-\frac{\gamma^2\overline{UM}}{h_2}\left[(\ln\overline{N})_Y + (\ln\Delta T)_Y\right]\bar{S}_{\bar{\phi}}\bar{\theta} = 0 \tag{2.21}$$

The associated boundary conditions are,

$$\overline{U}(X,Y,0) = 0 = \bar{F}_{\bar{\phi}}(0) = \bar{S}_{\bar{\phi}}(0)\,,$$

$$w(X,Y,0) = -w_0$$

where w_0 is considered to be the surface suction or injection velocity for the curved surface. For the impervious surface we may put w_0=0.Then from (1.10) & (1.33) we have,

$$T(X,Y,0) = T_w(X,Y).$$
$$\Rightarrow \theta(X,Y,0) = \overline{N}(X,Y)\bar{\theta}(0) = 1,$$
$$\overline{N}(X,Y) = 1, \quad \textit{and} \quad \bar{\theta}(0) = 1.$$

In order to satisfy the boundary conditions (1.32) & (1.34), without loss of generality, we put,

L=M=1.

The boundary conditions at large distance satisfy ,

$$\overline{U}L\overline{F}_{\overline{\varphi}}(\infty) = 0 \Rightarrow \quad \overline{F}_{\varphi}(\infty) = 0,$$
$$\overline{U}M\overline{S}_{\overline{\varphi}}(\infty) = 0 \Rightarrow \quad \overline{S}_{\overline{\varphi}}(\infty) = 0,$$
$$and \qquad \qquad \overline{\theta}(\infty) = 0,$$

Thus the two momentum and energy equations become:

$$\nu \overline{F}_{\overline{\phi}\overline{\phi}\overline{\phi}} + \frac{\gamma(\gamma h_2 \overline{U})_X}{h_1 h_2}\overline{F}\,\overline{F}_{\overline{\phi}\overline{\phi}} + \frac{\gamma(\gamma h_1 \overline{U})_Y}{h_1 h_2}\overline{S}\,\overline{F}_{\overline{\phi}\overline{\phi}} - \gamma w_0 \overline{F}_{\overline{\phi}\overline{\phi}} - \frac{\gamma^2}{h_1}\overline{U}_X \overline{F}^2_{\overline{\phi}}$$
$$-\gamma^2 \frac{\overline{U}}{h_2}\left[\frac{\overline{U}_Y}{\overline{U}} + \frac{h_{1Y}}{h_1}\right]\overline{F}_{\overline{\phi}}\overline{S}_{\overline{\phi}} + \frac{\gamma^2}{h_1 h_2}\overline{U}h_{2X}\overline{S}^2_{\overline{\phi}} - \frac{\gamma^2}{\overline{U}} g_{\xi}\beta_t \Delta T\theta = 0, \tag{2.22}$$

$$\nu \overline{S}_{\overline{\phi}\overline{\phi}\overline{\phi}} + \frac{\gamma(\gamma h_1 \overline{U})_Y}{h_1 h_2}\overline{S}\,\overline{S}_{\overline{\phi}\overline{\phi}} + \frac{\gamma(\gamma h_2 \overline{U})_X}{h_1 h_2}\overline{F}\,\overline{S}_{\overline{\phi}\overline{\phi}} - \gamma w_0 \overline{S}_{\overline{\phi}\overline{\phi}} - \frac{\gamma^2}{h_2}(\overline{U})_Y \overline{S}^2_{\overline{\phi}}$$
$$-\frac{\gamma^2 \overline{U}}{h_1}\left[\frac{\overline{U}_X}{\overline{U}} + \frac{h_{2X}}{h_2}\right]\overline{F}_{\overline{\phi}}\overline{S}_{\overline{\phi}} + \frac{\gamma^2}{h_1 h_2}\overline{U}h_{1Y}\overline{F}^2_{\overline{\phi}} - \frac{\gamma^2}{\overline{U}} g_{\eta}\beta_t \Delta T\theta = 0, \tag{2.23}$$

$$\frac{\nu}{P_r}\overline{\theta}_{\overline{\phi}\overline{\phi}} + \frac{\gamma(\gamma h_2 \overline{U})_X}{h_1 h_2}\overline{F}\,\overline{\theta}_{\overline{\phi}} + \frac{\gamma(\gamma h_1 \overline{U})_Y}{h_1 h_2}\overline{S}\,\overline{\theta}_{\overline{\phi}} - \gamma w_0 \overline{\theta}_{\overline{\phi}} - \frac{\gamma^2 \overline{U}}{h_1}(\ln \Delta T)_X \overline{F}_{\overline{\phi}}\overline{\theta}$$
$$-\frac{\gamma^2 \overline{U}}{h_2}(\ln \Delta T)_Y \overline{S}_{\overline{\phi}}\overline{\theta} = 0 \quad , \tag{2.24}$$

with the boundary conditions

$$\overline{F}_{\overline{\varphi}}(0) = \overline{S}_{\overline{\varphi}}(0) = 0 \quad ,$$
$$\overline{F}_{\varphi}(\infty) = \overline{S}_{\overline{\varphi}}(\infty) = 0 \quad ,$$
$$\overline{\theta}(0) = 1, \qquad \overline{\theta}(\infty) = 0,$$

The coefficients of $\overline{F}\overline{F}_{\bar\phi\bar\phi}$ & $\overline{S}\overline{S}_{\bar\phi\bar\phi}$ in (2.22) & (2.23) may be expressed as

$$\frac{\gamma(\gamma h_2\overline{U})_X}{h_1h_2} = \frac{1}{2}\left[\left(\frac{\gamma^2\overline{U}}{h_1}\right)_X + \frac{\gamma^2(h_2\overline{U})_X}{h_1h_2} - \gamma\,\overline{U}h_2\left(\frac{1}{h_1h_2}\right)_X\right]$$

$$and \quad \frac{\gamma(\gamma h_1\overline{U})_Y}{h_1h_2} = \frac{1}{2}\left[\left(\frac{\gamma^2\overline{U}}{h_2}\right)_Y + \frac{\gamma^2(h_1\overline{U})_Y}{h_1h_2} - \gamma\,\overline{U}h_1\left(\frac{1}{h_1h_2}\right)_Y\right],$$

Thus the momentum and energy equations become:

$$\nu\overline{F}_{\bar\phi\bar\phi\bar\phi} + \frac{1}{2}(a_0 + a_1 - a_2)\overline{F}\,\overline{F}_{\bar\phi\bar\phi} + \frac{1}{2}(a_3 + a_4 - a_5)\overline{S}\,\overline{F}_{\bar\phi\bar\phi} - a_6\overline{F}_{\bar\phi\bar\phi} - a_7\overline{F}_{\bar\phi}^2$$
$$-(a_8 + a_9)\overline{F}_{\bar\phi}\overline{S}_{\bar\phi} + a_{10}\overline{S}_{\bar\phi}^2 + a_{11}\theta = 0 \tag{2.25}$$

$$\nu\overline{S}_{\bar\phi\bar\phi\bar\phi} + \frac{1}{2}(a_3 + a_4 - a_5)\overline{S}\,\overline{S}_{\bar\phi\bar\phi} + \frac{1}{2}(a_0 + a_1 - a_2)\overline{F}\,\overline{S}_{\bar\phi\bar\phi} - a_6\overline{S}_{\bar\phi\bar\phi} - a_8\overline{S}_{\bar\phi}^2$$
$$-(a_7 + a_{10})\overline{F}_{\bar\phi}\overline{S}_{\bar\phi} + a_9\overline{F}_{\bar\phi}^2 + a_{12}\theta = 0 \tag{2.26}$$

$$\frac{\nu}{P_r}\overline{\theta}_{\bar\phi\bar\phi} + \frac{1}{2}(a_0 + a_1 - a_2)\overline{F}\overline{\theta}_{\bar\phi} + \frac{1}{2}(a_3 + a_4 - a_5)\overline{S}\overline{\theta}_{\bar\phi} - a_6\overline{\theta}_{\bar\phi} - (a_{13}\overline{F}_{\bar\phi} + a_{14}\overline{S}_{\bar\phi})\overline{\theta} = 0 \tag{2.27}$$

where the constants $a's$ with the differential equations involving the independent variables X and Y are given by the following relations :

$$a_0 = \left[\frac{\gamma^2 K\Delta T^{1/2}L^{1/2}}{h_1}\right]_X \tag{2.28.1}$$

$$a_1 = \left[\frac{\gamma^2(h_2K\Delta T^{1/2}L^{1/2})_X}{h_1h_2}\right] \tag{2.28.2}$$

$$a_2 = \gamma^2 h_2 K \Delta T^{1/2} L^{1/2} \left(\frac{1}{h_1 h_2} \right)_X \qquad (2.28.3)$$

$$a_3 = \left[\frac{\gamma^2 K \Delta T^{1/2} L^{1/2}}{h_2} \right]_Y \qquad (2.28.4)$$

$$a_4 = \left[\frac{\gamma^2 (h_1 K \Delta T^{1/2} L^{1/2})_Y}{h_1 h_2} \right] \qquad (2.28.5)$$

$$a_5 = \gamma^2 h_1 K \Delta T^{1/2} L^{1/2} \left(\frac{1}{h_1 h_2} \right)_Y \qquad (2.28.6)$$

$$a_6 = \gamma\, w_0 \qquad (2.28.7)$$

$$a_7 = \frac{\gamma^2 (K\, \Delta T^{1/2} L^{1/2})_X}{h_1} \qquad (2.28.8)$$

$$a_8 = \frac{\gamma^2 (K\, \Delta T^{1/2} L^{1/2})_Y}{h_2} \qquad (2.28.9)$$

$$a_9 = \frac{\gamma^2 K\, \Delta T^{1/2} L^{1/2}}{h_2} \frac{h_{1Y}}{h_1} \qquad (2.28.10)$$

$$a_{10} = \frac{\gamma^2 K\, \Delta T^{1/2} L^{1/2}}{h_1} \frac{h_{2X}}{h_2} \qquad (2.28.11)$$

$$a_{11} = -\frac{\gamma^2 g_\xi \beta_T\, \Delta T^{1/2}}{K L^{1/2}} \qquad (2.28.12)$$

$$a_{12} = -\frac{\gamma^2 g_\eta \beta_T\, \Delta T^{1/2}}{K L^{1/2}} \qquad (2.28.13)$$

$$a_{13} = \frac{\gamma^2 K\, \Delta T^{1/2} L^{1/2}}{h_1} (\ln \Delta T)_X \qquad (2.28.14)$$

$$a_{14} = \frac{\gamma^2 K\, \Delta T^{1/2} L^{1/2}}{h_2} (\ln \Delta T)_Y \qquad (2.28.15)$$

where $K = \sqrt{g \beta_T}$ and $\overline{U} = \sqrt{g\, \beta_T\, \Delta T\, L} = K \Delta T^{1/2} L^{1/2}$.

Similar solutions for(2.25)--(2.27) exist only when all the $a's$ are finite and independent of X and Y ;that is to say that all $a's$ must be constant. Thus the boundary layer momentum and energy equations will become non-linear ordinary differential equations if $\Delta T(X,Y),\ h_1(X,Y),\ h_2(X,Y)$ *and* $\gamma(X,Y)$ satisfy equations (2.28).

To find $\Delta T(X,Y),\ h_1(X,Y),\ h_2(X,Y)$ *and* $\gamma(X,Y)$ in different situations, we first ignore the suction or injection effects, i.e. $a_6 = 0$.

From the expressions for $a's$, we have,

$$a_1 + a_2 = \gamma^2 \left[\frac{K \Delta T^{1/2} L^{1/2}}{h_1} \right]_X \qquad (2.29.1)$$

$$a_3 + a_4 = \gamma^2 \left[\frac{K \Delta T^{1/2} L^{1/2}}{h_2} \right]_Y \qquad (2.29.2)$$

From (2.28.1),

$$a_0 = \left[\frac{\gamma^2 K \Delta T^{1/2} L^{1/2}}{h_1} \right]_X = \gamma^2 \left[\frac{K \Delta T^{1/2} L^{1/2}}{h_1} \right]_X + 2\gamma\gamma_X \frac{K \Delta T^{1/2} L^{1/2}}{h_1}$$

$$\Rightarrow \ 2\gamma\gamma_X = \frac{h_1}{K \Delta T^{1/2} L^{1/2}} (a_0 - a_1 - a_2) \qquad (2.29.3)$$

Similarly, from (2.28.4),

$$\Rightarrow \ 2\gamma\gamma_Y = \frac{h_2}{K \Delta T^{1/2} L^{1/2}} (a_3 - a_4 - a_5) \qquad (2.29.4)$$

By virtue of (2.28.1),

$$\frac{\gamma^2 K \Delta T^{1/2} L^{1/2}}{h_1} = a_0 X + A(Y) \ , \qquad (2.29.5)$$

where $A(Y)$is either constant or function of Y only.

Differentiating (2.29.5) with respect to Y, we get,

$$\frac{dA(Y)}{dY} = \left[\frac{\gamma^2 K \Delta T^{1/2} L^{1/2}}{h_1} \right]_Y = \frac{h_2}{h_1} \left[a_3 - a_4 - a_5 + a_8 - a_9 \right] \ . \qquad (2.29.6)$$

Similarly, in view of equation (2.28.4),we get

$$\frac{\gamma^2 K \Delta T^{1/2} L^{1/2}}{h_2} = a_3 Y + B(X) \tag{2.29.7}$$

where $B(X)$ is either constant or function of X only.

and $$\frac{dB(X)}{dX} = \frac{h_1}{h_2}[a_0 - a_1 - a_2 + a_7 - a_{10}] \tag{2.29.8}$$

Taking the product, we get

$$\frac{dA(Y)}{dY} \cdot \frac{dB(X)}{dX} = [a_3 - a_4 - a_5 + a_8 - a_9].[a_0 - a_1 - a_2 + a_7 - a_{10}] \tag{2.29.9}$$

The forms of similarity solution, the scale factors $\Delta T(X,Y),\ h_1(X,Y),\ h_2(X,Y)$ *and* $\gamma(X,Y)$ depend wholly on the equation (2.29.9). This situation leads to the following four possibilities :

Case (A): $\frac{dA(Y)}{dY} \neq 0 \quad (const.)$, $\frac{dB(X)}{dX} \neq 0 \quad (const.)$,

Case (B): $\frac{dA(Y)}{dY} \neq 0$ $\frac{dB(X)}{dX} = 0$,

Case (C): $\frac{dA(Y)}{dY} = 0$ $\frac{dB(X)}{dX} \neq 0$,

Case (D): $\frac{dA(Y)}{dY} = 0$ $\frac{dB(X)}{dX} = 0$.

Chapter-3

Study of different similarity cases:

3.1 *Case A:*

Let $\frac{dA(Y)}{dY} = \text{const.}$

$$\Rightarrow \frac{dA}{dY} = \frac{h_2}{h_1}(a_3 - a_4 - a_5 + a_8 - a_9)$$

$$= k_1 l_1 \tag{3.1.1}$$

where $\frac{h_2}{h_1} = k_1$ and $l_1 = a_3 - a_4 - a_5 + a_8 - a_9$.

$\frac{dB(X)}{dX} = \text{const.}$

$$\Rightarrow \frac{dB}{dX} = \frac{h_1}{h_2}(a_0 - a_1 - a_2 + a_7 - a_{10})$$

$$= \frac{l_2}{k_1} \tag{3.1.2}$$

where $l_2 = a_0 - a_1 - a_2 + a_7 - a_{10}$.

$\therefore \quad A(Y) = k_1 l_1 Y + A_0 \qquad$ and $\qquad B(X) = \frac{l_2}{k_1} X + B_0$.

Now from (2.28.1) and (2.28.4), we have

$$\frac{\gamma^2 K \Delta T^{1/2} L_X^{1/2}}{h_1} = a_0 X + A(Y) \tag{3.1.3}$$

where $L = L_X$, along $X - axis$, and $L = L_Y$, along $Y - axis$

$$\text{and} \quad \frac{\gamma^2 K \Delta T^{1/2} L_Y^{1/2}}{h_2} = a_3 Y + B(X) \tag{3.1.4}$$

Hence , $\frac{\gamma^2 K\Delta T^{1/2} L_X^{1/2}}{h_1} / \frac{\gamma^2 K\Delta T^{1/2} L_Y^{1/2}}{h_2} = \frac{a_0 X + A(Y)}{a_3 Y + B(X)}$

$$\Rightarrow \quad \frac{L_X^{1/2}}{L_Y^{1/2}} k_1 = \frac{a_0 X + k_1 l_1 Y + A_0}{a_3 Y + \frac{l_2}{k_2} X + B_0}$$

$$\Rightarrow \quad \frac{L_X^{1/2}}{L_Y^{1/2}} = \frac{a_0 X + k_1 l_1 Y + A_0}{l_2 X + a_3 k_1 Y + k_1 B_0}.$$

If we let. $l_2 = a_0$ and $l_1 = a_3$ & $A_0 = k_1 B_0$ then we get,

$$L_X = L_Y = L = a_0 X + k_1 a_3 Y + A(\text{constant}). \tag{3.1.5}$$

Threfore. (3.1.3) & (3.1.4) becomes,

$$\frac{\gamma^2 K\Delta T^{1/2} L^{1/2}}{h_1} = a_0 X + k_1 a_3 Y + A \text{ (constant)}. \tag{3.1.6}$$

and $a_4 + a_5 = a_8 - a_9$

$a_1 + a_2 = a_7 - a_{10}$.

Now from (2.28.10),

$$a_9 = \frac{\gamma^2 K\Delta T^{1/2} L^{1/2}}{h_2} \frac{h_{1Y}}{h_1}$$

$$\Rightarrow \quad \frac{h_{1Y}}{h_1} = a_9 k_1 \left(\frac{h_1}{\gamma^2 K\Delta T^{1/2} L^{1/2}} \right) = \frac{a_9 k_1}{a_0 X + k_1 a_3 y + A} = \frac{a_9}{a_3} \left[\frac{k_1 a_3}{a_0 X + k_1 a_3 Y + A} \right]$$

$$\Rightarrow \quad h_1(X, Y) = k_2(X) [a_0 X + k_1 a_3 Y + A]^{\frac{a_9}{a_3}}$$

Similarly, from (2.28.11)

$$h_2(X, Y) = k_3(Y) [a_0 X + k_1 a_3 Y + A]^{\frac{a_{10}}{a_0}}.$$

In order to the requirements $\frac{h_2(X,Y)}{h_1(X,Y)} = k_1$ (const.), we have to set $\frac{k_3(Y)}{k_2(X)} = k_1$,

$\frac{a_{10}}{a_0} = n = \frac{a_9}{a_3}$, Let $a_0 = a, \quad k_1 a_3 = b, \qquad A = aX_0 + bY_0$

$$x = X + X_0$$

$$y = Y + Y_0.$$

Then we get, $h_1(x,y) = (ax+by)^n$.

$$\text{and } h_2(x,y) = k_1(ax+by)^n. \tag{3.1.7}$$

Now from, (2.28.8) & (3.6), we get

$$\frac{(\gamma^2 K\Delta T^{1/2} L^{1/2})_x}{h_1} \Bigg/ \frac{\gamma^2 K\Delta T^{1/2} L^{1/2}}{h_1} = \frac{a_7}{a_0 X + k_1 a_3 Y + A}$$

$$\Rightarrow \frac{(K\Delta T^{1/2} L^{1/2})_x}{(K\Delta T^{1/2} L^{1/2})} = \frac{a_7}{a_0}\left[\frac{a_0}{a_0 X + k_1 a_3 Y + A}\right].$$

$$\Rightarrow (K\Delta T^{1/2} L^{1/2}) = (ax+by)^m \text{ where } m = \frac{a_7}{a_0}.$$

$$\Rightarrow \Delta T = \frac{1}{K^2}(ax+by)^{2m-1} \tag{3.1.8}$$

From (3.1.6), $\dfrac{\gamma^2 K\Delta T^{1/2} L^{1/2}}{h_1} = a_0 X + k_1 a_3 Y + A.$

$$\Rightarrow \gamma^2 = (ax+by)^{n+1-m} \tag{3.1.9}$$

Substituting, the values of $\gamma^2, \Delta T, L, h_1$ & h_2, we get the values of a's, i.e.

$a_0 = a,$ $a_1 = (m+n)a,$ $a_2 = -2na,$ $a_3 = b/k_1,$ $a_4 = \dfrac{(m+n)b}{k_1}.$ $a_5 = -\dfrac{2nb}{k_1}.$	$a_6 = 0,$ $a_7 = ma,$ $a_8 = \dfrac{mb}{k_1}.$ $a_9 = \dfrac{nb}{k_1}.$ $a_{10} = na$	$a_{11} = \dfrac{-g_\xi}{g} = \dfrac{a}{4}\cos\delta$ (say) $a_{12} = \dfrac{-g_\eta}{g} = \dfrac{a}{4}a\sin\delta$ (say) $a_{13} = (2m-1)a$ $a_{14} = (2m-1)\dfrac{b}{k_1}.$	(3.1.10)

where δ is the angle between the ξ- direction and the horizontal surface.

Hence the transform equations (2.25), (2.26) & (2.27). reduce to

$$\nu\overline{F}_{\bar\phi\bar\phi\bar\phi} + \frac{1}{2}(3n+m+1)a\overline{F}\overline{F}_{\bar\phi\bar\phi} + \frac{1}{2}(3n+m+1)\frac{b}{k_1}\overline{S}\overline{F}_{\bar\phi\bar\phi} - ma\overline{F}_{\bar\phi}^2$$

$$-(m-n)\frac{b}{k_1}\overline{F}_{\bar\phi}\overline{S}_{\bar\phi} + na\overline{S}_{\bar\phi}^2 + a\cos\delta.\theta = 0 \tag{3.1.11}$$

$$\nu\overline{S}_{\bar\phi\bar\phi\bar\phi} + \frac{1}{2}(3n+m+1)\frac{b}{k_1}\overline{S}\overline{S}_{\bar\phi\bar\phi} + \frac{1}{2}(3n+m+1)a\overline{F}\overline{S}_{\bar\phi\bar\phi} - \frac{mb}{k_1}\overline{S}_{\bar\phi}^2$$

$$-(m+n)a\overline{F}_{\bar\phi}\overline{S}_{\bar\phi} + \frac{nb}{k_1}\overline{F}_{\bar\phi}^2 + a\sin\delta.\theta = 0 \tag{3.1.12}$$

$$\frac{\nu}{P_r}\overline{\theta}_{\bar\phi\bar\phi} + \frac{1}{2}(3n+m+1)a\overline{F}\overline{\theta}_{\bar\phi} + \frac{1}{2}(3n+m+1)\frac{b}{k_1}\overline{S}\overline{\theta}_{\bar\phi}$$

$$-(2m-1)\left[a\overline{F}_{\bar\phi} + \frac{b}{k_1}\overline{S}_{\bar\phi}\right]\theta = 0 \tag{3.1.13}$$

In order to simplify the above type of equations, we substitute,

$\overline{F} = \alpha f$, $\overline{S} = \alpha s$, $\overline{\theta} = \theta$, $\overline{\phi} = \alpha\varphi$..

The constant α can be defined later so as to provide convenient simplifications in the above forms of equations. Thus the above equations are changed to

$$f_{\varphi\varphi\varphi} + \left(\frac{3n+m+1}{2}\right)\frac{a\alpha^2}{\nu}ff_{\varphi\varphi} + \left(\frac{3n+m+1}{2}\right)\frac{b}{k_1}\frac{\alpha^2}{\nu}sf_{\varphi\varphi} - \frac{ma\alpha^2}{\nu}f_\varphi^2$$

$$-(m+n)\frac{b}{k_1}\frac{\alpha^2}{\nu}f_\varphi s_\varphi + na\frac{\alpha^2}{\nu}s_\varphi^2 + a\cos\delta.\frac{\alpha^2}{\nu}\theta = 0 \tag{3.1.14}$$

$$s_{\varphi\varphi\varphi} + \left(\frac{3n+m+1}{2}\right)\frac{b}{k_1}\frac{\alpha^2}{\nu}ss_{\varphi\varphi} + \left(\frac{3n+m+1}{2}\right)a\frac{\alpha^2}{\nu}fs_{\varphi\varphi} - \frac{mb}{k_1}\frac{\alpha^2}{\nu}s_\varphi^2$$

$$-(m+n)\frac{a\alpha^2}{\nu}f_\varphi s_\varphi + \frac{nb}{k_1}\frac{\alpha^2}{\nu}f_\varphi^2 + a\sin\delta.\frac{\alpha^2}{\nu}\theta = 0 \tag{3.1.15}$$

$$P_r^{-1}\theta_{\varphi\varphi}+\left(\frac{3n+m+1}{2}\right)\frac{a\alpha^2}{\nu}f\theta_\varphi+\left(\frac{3n+m+1}{2}\right)\frac{b}{k_1}\frac{\alpha^2}{\nu}s\theta_\varphi$$

$$-(2m-1)\left[\frac{a\alpha^2}{\nu}f_\varphi+\frac{b}{k_1}\frac{\alpha^2}{\nu}s_\varphi\right]\theta=0 \tag{3.1.16}$$

Choosing $\frac{a\alpha^2}{\nu}=4$ and $\frac{b}{k_1 a}=c$, and $k_1=1$, then the final form of the similarity solutions stand as

$$f'''+2(3n+m+1)(f+cs)f''-4mf'^2-4(m+n)cf s' \tag{3.1.17}$$
$$+4ns'^2+(\cos\delta)\,\theta=0$$

$$S'''+2(3n+m+1)(f+cs)s''-4ms'^2-4(m+n)f s'$$
$$+4ncf'^2+(\sin\delta)\,\theta=0 \tag{3.1.18}$$

$$P_r^{-1}\theta''+2(3n+m+1)(f+cs)\theta'-4(2m-1)(f'+cs')\theta=0 \tag{3.1.19}$$

with the boundary conditions.

$$\begin{array}{ll} f(0)=f'(0)=0 & f'(\infty)=0, \\ s(0)=s'(0)=0 & s'(\infty)=0, \\ \theta(0)=1 & \theta(\infty)=0. \end{array} \tag{3.1.20}$$

Now

$$h_1(x,y)=a^n(x+cy)^n,$$

$$\Delta T=T_0(x+cy)^{2m-1}\quad where\quad T_0=\frac{a^{2m-1}}{g\beta_1}$$

$$\gamma^2=a^{n+1-m}(x+cy)^{n+1-m}$$

For n=0, m=0.5, c=0, δ=0 and f=s, the equations (3.17)—(3.19) with the boundary conditions coincide with the free convection flow of air subject to the gravitational

force about an isothermal, vertical flat plate, analysed and verified experimentally by Schmidt and Backmann [1930], which was also discussed by Ostrach [1953].
If we choose n=0 , then this case coincides with case (D). If we choose n=0, m=0.5, c=0, δ=0 ,then the problem coincide with the most noteworthy of the more general analysis given by Sparrow and Gregg [1958] for the power law case.

The transformed equations can be solved with the help of the controlling parameters P_r, c, m, n, and δ . The Prandtl number $\Pr = \frac{\mu\ C_p}{\kappa}$ depends on the properties of the media. For air at room temperature P_r =0.7, for water at temperature 62^0F, P_r=7.0, for Oil, P_r=1000.

The similarity variable φ is

$$\varphi = \frac{Z}{a\gamma} = \frac{Z}{\sqrt{\frac{4\upsilon}{a}}(ax+by)^{\frac{n+1-m}{2}}} = Gr_{xy}^{1/4}\ .\frac{z}{(x+cy)^{\frac{n}{2}+1}}$$

where, the modified Grashof number, $Gr_{xy} = \left[\frac{a^{4-\frac{n}{2}}}{4^2}.\frac{g\beta_T\Delta T(h_1^{2/3}(x+cy))^3}{\nu^2}\right]^{1/4}$

The velcity components

$$u = \overline{U}f'(\varphi)$$

$$v = \overline{U}s'(\varphi) \qquad \text{where } \overline{U} = \sqrt{(g\beta_T\Delta T(x+cy))}$$

and $w = \frac{1}{h_1h_2}\left[-\{h_2\gamma\overline{U}\}_X\overline{F} + \overline{\phi}\gamma_X h_2\overline{U}\overline{F}_{\overline{\phi}} - \{h_1\gamma\overline{U}\}_Y\overline{S} + \overline{\phi}\gamma_Y h_1\overline{U}\overline{S}_{\overline{\phi}} + h_1h_2w_0\right]$.

$$\Rightarrow\quad -w = \left(4\nu a^{m-n}\right)^{1/2}(x+cy)^{\frac{m-n-1}{2}}\left[\left(\frac{3n+m+1}{2}\right)(f+cs)+\left(\frac{n+1-m}{2}\right)\varphi\left(f'+cs'\right)\right].(3.1.21)$$

and the stream functions are

$$\psi = \left(4\nu a^{3n+m}\right)^{1/2}\left(x+cy\right)^{\frac{3n+m+1}{2}} f(\varphi)$$

$$\Phi = \left(4\nu a^{3n-m}\right)^{1/2}\left(x+cy\right)^{\frac{3n+m+1}{2}} s(\varphi) \qquad (3.1.22)$$

Skin frictions are

$$\tau_{w1} = \mu\left(x+cy\right)^{\frac{3m-n-1}{2}}\sqrt{\frac{a^{3m-n}}{4\nu}}\,f''(0)$$
$$\tau_{w2} = \mu\left(x+cy\right)^{\frac{3m-n-1}{2}}\sqrt{\frac{a^{3m-n}}{4\nu}}\,s''(0) \qquad (3.1.23)$$

and the heat transfer

$$q_w = -\kappa\Delta T\left(\frac{g\beta_T\Delta T}{L}\right)^{1/4}\frac{1}{\alpha}\left(\frac{\partial\theta}{\partial\omega}\right)_0$$
$$= -\left(\frac{\kappa}{g\beta_T}\right)\left(\frac{a^{5m-2}}{4\nu}\right)^{1/2}\left(x+cy\right)^{\frac{5m-3}{2}}\left(\frac{\partial\theta}{\partial\varphi}\right)_0 \qquad (3.1.24)$$

so the heat transfer coefficient is

$$-\theta'(0) = \frac{Nu_{xy}}{\left(Gr_{xy}\right)^{1/4}} \qquad (3.1.25)$$

where the modified Nusselt number , $Nu_{xy} = \dfrac{q_w(x+cy)}{\kappa\,\Delta T}$.

and primes denote derivatives with respect to the similarity variable φ.

3.2 ***Case B:***

Let $\frac{dA(Y)}{dY} = \text{const.}$ $\frac{dB(X)}{dX} \neq \text{const.}$

Let $\frac{h_2}{h_1} \neq$ const., then $a_3 - a_4 - a_5 + a_8 - a_9 = 0 = l_1$ (say)

and $a_0 - a_1 - a_2 + a_7 - a_{10} = l_2$ (say)

Let $h_1 = 1$, $a_3 = 0$ and $h_2 = h_2(x)$, then from (3.1.6), we get,

$$\gamma^2 K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}} = a_0 X + A \tag{3.2.1}$$

Now we have, $\frac{dB(X)}{dX} = \frac{h_1}{h_2}(a_0 - a_1 - a_2 + a_7 - a_{10}) = \frac{l_2}{h_2(x)}$

$$\Rightarrow B(X) = l_2 \int \frac{1}{h_2(X)} dX \tag{3.2.2}$$

Again, form $a_3 = \left(\frac{\gamma^2 K\Delta T^{1/2} L^{1/2}}{h_2} \right)$,

$\Rightarrow \frac{\gamma^2 K\Delta T^{1/2} L^{1/2}}{h_2(X)} = a_3 Y + B(X) = B(X)$, Since $a_3 = 0$.

$\Rightarrow \gamma^2 K\Delta T^{1/2} L^{1/2} = h_2(X).l_2 \int \frac{1}{h_2(X)} dX$ [By using (3.2.2)]

$$\Rightarrow a_0 X + A = l_2\, h_2(X) \int \frac{1}{h_2(X)} dX \text{ [By using (3.2.1)]} \tag{3.2.3}$$

(I) Choosing A = 0, we get

$a_0 X + A = l_2\, h_2(X) \int \frac{dX}{h_2(X)}$

$\Rightarrow h_2(X) = (a_0 X)^{1 - \frac{l_2}{a_0}}$.

$\Rightarrow \quad h_2(X) = (a_0 X)^n, \quad$ for $n = 1 - \frac{l_2}{a_0}$

where $a_0 \neq 0$ is an orbitrary constant.

Again, from $a_7 = \dfrac{\gamma^2\left(K\Delta T^{1/2} L^{1/2}\right)_X}{h_1}$

By using (3.2.1), we get,

$$\frac{\left(K\Delta T^{1/2} L^{1/2}\right)_X}{K\Delta T^{1/2} L^{1/2}} = \frac{a_7}{a_0 X}, \qquad \therefore \; A = 0.$$

$\Rightarrow \quad \Delta T = T_0 X^{2m-1} where, T_0 = \dfrac{a_0^{2m-1}}{K^2}$, By choosing $L = a_0 X$ and $m = \dfrac{a_7}{a_0}$.

Therefore, $\gamma^2 = a_0^{1-m} X^{1-m}$.

Therefore, the constants are,

$a_0 = a_0$,	$a_6 = 0$	$a_{11} = -\dfrac{g_\xi}{g} = a_0 \cos\delta$	
$a_1 = (m+n)a_0$,	$a_7 = ma_0$,		
$a_2 = -na_0$,	$a_8 = 0$	$a_{12} = -\dfrac{g_\eta}{g} = a_0 \sin\delta$	
$a_3 = 0$,	$a_9 = 0$	$a_{13} = (2m-1)a_0$	
$a_4 = 0$.	$a_{10} = na_0$	$a_{14} = 0$.	(3.2.4)
$a_5 = -0$			

where δ is the angle between the ξ- direction and the horizontal surface.

The corresponding equations are

$$\nu \overline{F}_{\overline{\phi\phi\phi}} + \left[\frac{2n+m+1}{2}\right] a_0 \overline{F}\overline{F}_{\overline{\phi\phi}} - ma_0 \overline{F}_{\overline{\phi}}^2 + na_0 \overline{S}_{\overline{\phi}}^2 + a_0 \cos\delta\, \overline{\theta} = 0 \qquad (3.2.5)$$

$$\nu \overline{S}_{\overline{\phi\phi\phi}} + \left[\frac{2n+m+1}{2}\right] a_0 \overline{F}\overline{S}_{\overline{\phi\phi}} - (m+n)a_0 \overline{F}_{\overline{\phi}} \overline{S}_{\overline{\phi}} + a_0 \sin\delta\, \overline{\theta} = 0 \qquad (3.2.6)$$

$$\frac{\nu}{P_r} \overline{\theta}_{\overline{\phi\phi}} + \left[\frac{2n+m+1}{2}\right] a_0 \overline{F}\overline{\theta}_{\overline{\phi}} - (2m-1)a_0 \overline{F}_{\overline{\phi}} \overline{\theta} = 0 \qquad (3.2.7)$$

In order to simplify the above type of equations, we substitute,

$\overline{F} = \alpha f$, $\overline{S} = \alpha s$, $\overline{\theta} = \theta$, $\overline{\phi} = \alpha\varphi$. and choosing $\frac{a_0 \alpha^2}{\nu} = 1$,

We get,

$$f''' + \left(n + \frac{m+1}{2}\right) ff'' - mf'^2 + ns'^2 + (\cos\delta)\theta = 0 \tag{3.2.8}$$

$$S''' + \left(n + \frac{m+1}{2}\right) fs'' - (m+n) f' s' + (\sin\delta)\,\theta = 0 \tag{3.2.9}$$

$$P_r^{-1}\theta'' + \left(n + \frac{m+1}{2}\right) f\theta' - (2m-1) f'\theta = 0 \tag{3.2.10}$$

With the boundary conditions,

$$\begin{aligned} &f(0) = f'(0) = 0 \qquad && f'(\infty) = 0, \\ &s(0) = s'(0) = 0 && s'(\infty) = 0. \\ &\theta(0) = 1 && \theta(\infty) = 0. \end{aligned} \tag{3.2.11}$$

If n=0, m=0.5, δ =0, then this equations is similar to the problem dealt with by Schmidt & Backmann [1930].

The transformed equations can be solved with the help of the controlling parameters P_r, m, n, and δ .

The similarity variable φ is,

$$\varphi = \frac{Z}{\alpha y} = \left(\frac{a_0}{\nu}\right)^{1/2} \frac{Z}{(a_0 x)^{\frac{1-m}{2}}} = \left(\frac{a_0^2 g \beta_T \Delta T}{\nu^2}\right)^{\frac{1}{4}} . \frac{Z}{(a_0 x)^{\frac{1}{4}}} = Gr_x^{1/4} . \frac{z}{x} \tag{3.2.12}$$

where, the modified Grashof number is, $Gr_x = \left(\frac{a_0^2 g \beta_T \Delta T x^3}{\nu^2}\right)$

The velocity components

$u = \overline{U} f'(\varphi)$

$v = \overline{U} s'(\varphi)$ where $\overline{U} = \sqrt{g\beta_T \Delta T x}$ (3.2.13)

and $w = \frac{1}{h_1 h_2}\left[-\{h_2 \gamma \overline{U}\}_X \overline{F} + \bar{\phi}\gamma_X h_2 \overline{U}\overline{F}_{\bar{\phi}} - \{h_1 \gamma \overline{U}\}_Y \overline{S} + \bar{\phi}\gamma_Y h_1 \overline{U}\overline{S}_{\bar{\phi}} + h_1 h_2 w_0\right]$.

$$\Rightarrow \quad -w = \left(a_0^m \nu\right)^{1/2} x^{\frac{m-1}{2}}\left[\left(\frac{2n+m+1}{2}\right) f - \left(\frac{1-m}{2}\right)\varphi f'\right]. \qquad (3.2.14)$$

The equation (3.2.10) & (3.2.14) is independent of s because of ΔT – variation in this case is free of y variations.

The stream functions are

$$\psi = \left(a_0^{2n+m} \nu\right)^{1/2} x^{\frac{2n+m+1}{2}} f(\varphi)$$

$$\phi = \left(a_0^m\right)^{1/2} x^{\frac{m-1}{2}} s(\varphi) \qquad (3.2.15)$$

Skin frictions are

$$\tau_{w1} = \mu x^{\frac{3m-1}{2}} \sqrt{\frac{a_0^{3m}}{\nu}} f''(0)$$

$$\tau_{w2} = \mu x^{\frac{3m-1}{2}} \sqrt{\frac{a_0^{3m}}{\nu}} s''(0) \qquad (3.2.16)$$

and the heat transfer

$$q_w = -\kappa \Delta T \left(\frac{g\beta_t \Delta T}{L}\right)^{1/4} \frac{1}{\alpha}\left(\frac{\partial\theta}{\partial\varphi}\right)_0$$

$$= -\left(\frac{\kappa}{g\beta_t}\right)\left(\frac{a_0^{5m-2}}{\nu}\right)^{1/2} x^{\frac{5m-3}{2}}\left(\frac{\partial\theta}{\partial\varphi}\right)_0$$

Hence the coefficient of heat transfer

$$-\theta'(0) = \frac{Nu_x}{(Gr_x)^{1/4}}, \qquad Nu_x = \frac{q_w x}{\kappa \Delta T}. \qquad (3.2.17)$$

where primes denotes derivatives with respect to the similarity variable φ.

(ii) If we choose $a_0 = 0$, A = arbitrary constant,

then $A = l_2\, h_2(X) \int \frac{1}{h_2(X)}\, dX$

$\Rightarrow \quad h_2(X) = e^{-\frac{l_2}{A}X}$.

$\Rightarrow \quad h_2(X) = e^{nX}$, for $n = -\frac{l_2}{A}$.

Now from (2.28.8) and (3.2.1) with $a_0 = 0$

$$\frac{\left(K\Delta T^{1/2} L^{1/2}\right)_X}{\left(K\Delta T^{1/2} L^{1/2}\right)} = \frac{a_7}{A}$$

$$\Rightarrow \quad K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}} = e^{mX},$$

$$\Rightarrow \quad \Delta T = T_0 e^{2mX} \quad where,\ T_0 = \frac{1}{g\beta_T},\ \text{where } L = 1 \text{ (let).}$$

$$\therefore \quad y^2 = Ae^{-mX}.$$

Therefore, the constants become,

$a_0 = 0$,	$a_6 = 0$	$a_{11} = -\frac{g_{\bar{\xi}}}{g} A = A\cos\delta$	
$a_1 = (m+n)A$,	$a_7 = mA$,		
$a_2 = -nA$,	$a_8 = 0$	$a_{12} = -\frac{g_{\bar{\eta}}}{g} A = A\sin\delta$	
$a_3 = 0$,	$a_9 = 0$		(3.2.18)
$a_4 = 0$.	$a_{10} = nA$	$a_{13} = 2mA$	
$a_5 = 0$		$a_{14} = 0$.	

The corresponding equations are

$$\nu\bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \left[\frac{m}{2} + n\right] A\bar{F}\bar{F}_{\bar{\phi}\bar{\phi}} - mA\bar{F}_{\bar{\phi}}^2 + nA\bar{S}_{\bar{\phi}}^2 + A\cos\delta\,\bar{\theta} = 0 \tag{3.2.19}$$

$$\nu\bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \left[\frac{m}{2} + n\right] A\bar{F}\bar{S}_{\bar{\phi}\bar{\phi}} - (m+n)A\bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}} + A\sin\delta\,\bar{\theta} = 0 \tag{3.2.20}$$

$$\frac{\nu}{P_r}\overline{\theta}_{\overline{\phi}\overline{\phi}} + \left[\frac{m}{2}+n\right]A\overline{F}\overline{\theta}_{\overline{\phi}} - 2mA\overline{F}_{\overline{\phi}}\overline{\theta} = 0 \qquad (3.2.21)$$

where δ is the angle between the ξ- direction and the horizontal surface

In order to simplify the above type of equations, we substitute,

$\overline{F} = \alpha f$, $\overline{S} = \alpha s$, $\overline{\theta} = \theta$, $\overline{\phi} = \alpha\varphi$. and then set $\frac{\alpha^2 A}{\nu} = 1$, we get the final form,

$$f''' + \left(\frac{m}{2}+n\right)ff'' - mf'^2 + ns'^2 + (\cos\delta)\theta = 0, \qquad (3.2.22)$$

$$S''' + \left(\frac{m}{2}+n\right)fs'' - (m+n)f's' + (\sin\delta)\theta = 0, \qquad (3.2.23)$$

$$P_r^{-1}\theta'' + \left(\frac{m}{2}+n\right)f\theta' - 2mf'\theta = 0, \qquad (3.2.24)$$

with the boundary conditions,

$$\begin{aligned} &f(0) = f'(0) = 0 && f'(\infty) = 0, \\ &s(0) = s'(0) = 0 && s'(\infty) = 0, \\ &\theta(0) = 1 && \theta(\infty) = 0. \end{aligned} \qquad (3.2.25)$$

If n=0, m=2 and δ =0 and f=s, then the problem may be comparable with the problem discussed by Ostrach [1964].

The transformed equations can be solved with the help of the controlling parameters P_r, m, n, and δ.

The similarity variable φ is,

$$\varphi = \frac{Z}{\alpha\gamma} = \left(\frac{g\beta_T\,\Delta T}{\nu^2}\right)^{\frac{1}{4}}.Z = Gr^{1/4}.z \qquad (3.2.26)$$

The velocity components

$$u = \overline{U} f'(\varphi)$$

$$v = \overline{U} s'(\varphi) \qquad \text{where, } \overline{U} = \sqrt{g\beta_T \Delta T.1} \qquad (3.2.27)$$

and $w = \frac{1}{h_1 h_2}\left[-\left\{h_2\gamma\overline{U}\right\}_X \overline{F} + \overline{\phi}\gamma_X h_2 \overline{UF}_{\overline{\varphi}} - \left\{h_1\gamma\overline{U}\right\}_Y \overline{S} + \overline{\phi}\gamma_Y h_1 \overline{US}_{\overline{\varphi}} + h_1 h_2 w_0\right].$

$$\Rightarrow \quad -w = (\nu)^{\frac{1}{2}} e^{\frac{m}{2}x}\left[\left(\frac{2n+m}{2}\right) f + \left(\frac{m}{2}\right)\varphi f'\right]. \qquad (3.2.28)$$

The equations (3.2.24) & (3.2.28) are independent of s because of ΔT – variation is independent of y.

The stream functions are

$$\psi = (\nu)^{\frac{1}{2}} e^{\left(\frac{m}{2}+n\right)x} f(\varphi)$$

$$\Phi = (\nu)^{\frac{1}{2}} e^{\frac{m}{2}x} s(\varphi) \qquad (3.2.29)$$

Skin frictions are

$$\tau_{w1} = \mu A e^{\frac{3mx}{2}} \sqrt{\frac{A}{\nu}} f''(0)$$

$$\tau_{w2} = \mu A e^{\frac{3mx}{2}} \sqrt{\frac{A}{\nu}} s''(0) \qquad (3.2.30)$$

and the heat transfer

$$q_w = -\kappa \Delta T \left(\frac{g\beta_T \Delta T}{L}\right)^{\frac{1}{4}} \frac{1}{\alpha}\left(\frac{\partial\theta}{\partial\varphi}\right)_0$$

$$= -\left(\frac{\kappa}{g\beta_T}\right)\left(\frac{A}{\nu}\right)^{\frac{1}{2}} e^{\frac{5mx}{2}}\left(\frac{\partial\theta}{\partial\varphi}\right)_0 \qquad (3.2.31)$$

Hence the coefficient of heat transfer

$$-\theta'(0) = \frac{Nu_x}{(Gr_x)^{\frac{1}{4}}} \qquad \textit{where, } \quad Nu_x = \frac{q_w . L}{\kappa \Delta T}. \qquad \textit{here} L = 1\text{m}$$

3.3 *Case C:*

Let $\frac{dA(Y)}{dY} \neq \text{const.} \quad \frac{dB(X)}{dX} = \text{const.}$

Let $\frac{h_1}{h_2} \neq$ const., then $a_3 - a_4 - a_5 + a_8 - a_n = l_1$ (say)

and $a_0 - a_1 - a_2 + a_7 - a_{10} = 0$

Let $h_1 = h_1(Y)$, $a_0 = 0$ and $h_2 = 1$ then from (3.1.4), we get,

$$\frac{\gamma^2 K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}}}{h_2} = a_3 Y + B \text{ (constant)} \tag{3.3.1}$$

Now we have, $\frac{dA(Y)}{dY} = \frac{h_2}{h_1} l_1 = \frac{l_1}{h_1(Y)}$

$$\Rightarrow A(Y) = l_1 \int \frac{1}{h_1(Y)} dY \tag{3.3.2}$$

Again, from $a_0 = \left(\frac{\gamma^2 K\Delta T^{1/2} L^{1/2}}{h_1} \right)_X$

$\Rightarrow \quad \frac{\gamma^2 K\Delta T^{1/2} L^{1/2}}{h_1} = a_0 X + A(Y) = A(Y)$, Since $a_0 = 0$.

$$\Rightarrow \quad a_3 Y + B = l_1\, h_1(Y) \int \frac{1}{h_1(Y)} dY \text{ [By using (3.3.1) \& (3.3.2)]} \tag{3.3.3}$$

(I) Choosing B = 0, we get

$a_3 Y = l_1\, h_1(Y) \int \frac{1}{h_1(Y)} dY$

$\Rightarrow \quad h_1(Y) = (a_3 Y)^n$. for $n = 1 - \frac{l_1}{a_3}$, where $a_3 \neq 0$ is an arbitrary constant.

Again. from $a_8 = \dfrac{\gamma^2\left(K\Delta T^{1/2}L^{1/2}\right)_Y}{h_2}$

By using (3.3.1), we get,

$$\frac{\left(K\Delta T^{1/2}L^{1/2}\right)_Y}{K\Delta T^{1/2}L^{1/2}} = \frac{a_8}{a_3 Y}, \quad \because \quad B = 0.$$

$\Rightarrow \quad \Delta T = T_0 Y^{2m-1}$, *where*, $T_0 = \dfrac{a_3^{2m-1}}{K^2}$, By choosing $L = a_3 Y$ and $m = \dfrac{a_8}{a_3}$.

Therefore, $\gamma^2 = a_3^{1-m} Y^{1-m}$.

Therefore, the constants are,

$$\begin{aligned}
&a_0 = 0,\\
&a_1 = 0,\\
&a_2 = 0.\\
&a_3 = a_3.\\
&a_4 = (m+n)a_3.\\
&a_5 = -na_3.\\
&a_6 = 0\\
&a_7 = 0\\
&a_8 = ma_3.\\
&a_9 = na_3.\\
&a_{10} = 0\\
&a_{11} = -\frac{g_\xi}{g} = a_3 \cos\delta \text{ (say)}\\
&a_{12} = -\frac{g_\eta}{g} = a_3 \sin\delta \text{ (say)}\\
&a_{13} = 0\\
&a_{14} = (2m-1)a_3
\end{aligned} \qquad (3.3.4)$$

Where δ is the angle between the ξ- direction and the horizontal surface

The corresponding equations are

$$\nu\overline{F}_{\bar\phi\bar\phi\bar\phi} + \frac{1}{2}(2n+m+1)a_3\overline{S}\overline{F}_{\bar\phi\bar\phi} - (m+n)a_3\overline{F}_{\bar\phi}\overline{S}_{\bar\phi} + a_3\cos\delta\,\theta = 0$$

$$\nu\overline{S}_{\bar\phi\bar\phi\bar\phi} + \frac{1}{2}(2n+m+1)a_3\overline{S}\overline{S}_{\bar\phi\bar\phi} - ma_3\overline{S}^2_{\bar\phi} + a_3\sin\delta\,\theta = 0$$

$$\frac{\nu}{P_r}\overline{\theta}_{\bar\phi\bar\phi} + \frac{1}{2}(2n+m+1)\overline{S}\overline{\theta}_{\bar\phi} - (2m-1)a_3\overline{S}_{\bar\phi}\overline{\theta} = 0\,.$$

In order to simplify the above type of equations, we substitute.

$\overline{F} = \alpha f$, $\overline{S} = \alpha s$, $\overline{\theta} = \theta$, $\bar\varphi = \alpha\varphi$. and choosing $\dfrac{a_3\alpha^2}{\nu} = 1$.

We get,

$$f''' + \left(n + \frac{m+1}{2}\right)sf'' - (m+n)f'\,s' + (\cos\delta)\,\theta = 0 \tag{3.3.5}$$

$$s''' + \left(n + \frac{m+1}{2}\right)ss'' - m\,s'^2 + (\sin\delta)\,\theta = 0 \tag{3.3.6}$$

$$P_r^{-1}\theta'' + \left(n + \frac{m+1}{2}\right)s\theta' - (2m-1)s'\,\theta = 0 \tag{3.3.7}$$

The boundary conditions are,

$$\begin{aligned} &f(0) = f'(0) = 0 && f'(\infty) = 0, \\ &s(0) = s'(0) = 0 && s'(\infty) = 0, \\ &\theta(0) = 1 && \theta(\infty) = 0. \end{aligned} \tag{3.3.8}$$

The transformed equations can be solved with the help of the controlling parameters P_r, m, n, and δ .

The similarity variable φ is,

$$\varphi = \frac{Z}{\alpha\gamma} = \left(\frac{a_3}{\nu}\right)^{\frac{1}{2}} \frac{Z}{(a_3 y)^{\frac{1-m}{2}}} = \left(\frac{a_3{}^2 g\beta_T\,\Delta T}{\nu^2}\right)^{\frac{1}{4}} . \frac{Z}{(a_3 y)^{\frac{1}{4}}} = Gr_y^{\frac{1}{4}} . \frac{z}{y}$$

where the modified Grashof number is $\quad Gr_y = \left(\dfrac{a_3{}^2 g\beta_T\,\Delta T\,y^3}{\nu^2}\right)^{\frac{1}{4}}$

The velocity components

$$u = \overline{U} f'(\varphi)$$

$$v = \overline{U} s'(\varphi) \qquad \text{where, } \overline{U} = \sqrt{g\beta_T \Delta T y}\,. \qquad (3.3.10)$$

$$\text{and } w = \frac{1}{h_1 h_2}\left[-\{h_2 \gamma \overline{U}\}_X \overline{F} + \overline{\phi}\gamma_X h_2 \overline{U}\overline{F}_{\overline{\phi}} - \{h_1 \gamma \overline{U}\}_Y \overline{S} + \overline{\phi}\gamma_Y h_1 \overline{U}\overline{S}_{\overline{\phi}} + h_1 h_2 w_0\right].$$

$$\Rightarrow \quad -w = \left(a_3^m v\right)^{1/2} y^{\frac{m-1}{2}}\left[\left(\frac{2n+m+1}{2}\right)s - \left(\frac{1-m}{2}\right)\varphi s'\right]. \qquad (3.3.11)$$

The equations (3.3.7) & (3.3.11) are independent of the stream function f due to the reason that ΔT – variation depends only on y.

The stream functions are

$$\psi = \left(a_3^{2n-m}\right)^{1/2} y^{\frac{2n+m+1}{2}} f(\varphi)$$

$$\phi = \left(a_3^{2n+m}\right)^{1/2} y^{\frac{m+1}{2}} s(\varphi) \qquad (3.3.12)$$

Skin frictions are

$$\tau_{w1} = \mu\, y^{\frac{3m-1}{2}} \sqrt{\frac{a_3^{3m}}{v}} f''(0)$$

$$\tau_{w2} = \mu\, y^{\frac{3m-1}{2}} \sqrt{\frac{a_3^{3m}}{v}} s''(0) \qquad (3.3.13)$$

and the heat transfer

$$q_w = -\kappa \Delta T \left(\frac{g\beta_T \Delta T}{L}\right)^{1/4} \frac{1}{\alpha}\left(\frac{\partial\theta}{\partial\varphi}\right)_0$$

$$= -\left(\frac{\kappa}{g\beta_T}\right)\left(\frac{a_3^{5m-2}}{4v}\right)^{1/2} y^{\frac{5m-3}{2}}\left(\frac{\partial\theta}{\partial\varphi}\right)_0$$

Hence the coefficient of heat transfer

$$-\theta'(0) = \frac{Nu_y}{\left(Gr_y\right)^{1/4}} \qquad where, \quad Nu_y = \frac{q_w y}{\kappa \Delta T}. \qquad (3.3.14)$$

where prime denote derivatives with respect to the similarity variable φ.

(ii) If we choose $a_3 = 0$, B = arbitrary constant.

then (5.3) implies.

$$B = l_1\, h_1(Y) \int \frac{1}{h_1(Y)} dY,$$

$$\Rightarrow \quad h_1(Y) = e^{-\frac{l_1}{B}Y},$$

$$\Rightarrow \quad h_1(Y) = e^{nY}, \text{ for } n = -\frac{l_1}{B}.$$

Now from (2.28.9) and (3.3.1) with $a_3 = 0$

$$\frac{\left(K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}}\right)_Y}{K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}}} = \frac{a_8}{B}$$

$$\Rightarrow \quad K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}} = e^{mY}, \text{ where } m = \frac{a_8}{B}.$$

If we choose $L = 1$,

$$\Delta T = T_0 e^{2mY}, \quad where, \quad T_0 = \frac{1}{K^2}, \tag{3.3.15}$$

and $\gamma^2 = Be^{-nY}$. (3.3.16)

Therefore, the constants become,

$a_0 = 0$,

$a_1 = 0$,

$a_2 = 0$,

$a_3 = 0$.

$a_4 = (m+n)B$.. (3.3.17)

$a_5 = -nB$.

$a_6 = 0$

$a_7 = 0$

$a_8 = mB$

$a_9 = nB$

$a_{10} = 0$

$$a_{11} = -\frac{Bg_{\xi}}{g} = B\cos\delta \text{ (say)}$$

$$a_{12} = -\frac{Bg_{\eta}}{g} = B\sin\delta \text{ .(say)}$$

$$a_{13} = 0$$

$$a_{14} = 2mB$$

The corresponding equations are

$$\nu\overline{F}_{\bar\varphi\bar\varphi\bar\varphi} + \frac{1}{2}(m+2n)B\overline{S}\overline{F}_{\bar\varphi\bar\varphi} - (m+n)B\overline{F}_{\bar\varphi}\overline{S}_{\bar\varphi} + B\cos\delta\,\overline{\theta} = 0$$

$$\nu\overline{S}_{\bar\varphi\bar\varphi\bar\varphi} + \frac{1}{2}(m+2n)B\overline{S}\overline{S}_{\bar\varphi\bar\varphi} - mB\overline{S}_{\bar\varphi}^{2} + nB\overline{F}_{\bar\varphi}^{2} + B\sin\delta\,\overline{\theta} = 0$$

$$\frac{\nu}{P_r}\overline{\theta}_{\bar\varphi\bar\varphi} + \frac{1}{2}(m+2n)B\overline{S}\overline{\theta}_{\bar\varphi} - 2mB\overline{S}_{\bar\varphi}\overline{\theta} = 0.$$

In order to simplify the above type of equations, we substitute,

$\overline{F} = \alpha f$, $\overline{S} = \alpha s$, $\overline{\theta} = \theta$. $\overline{\phi} = \alpha\varphi$. and then set $\frac{\alpha^2 B}{\nu} = 1$,

Then we get,

$$f''' + \left(\frac{m}{2}+n\right)sf'' - (m+n)f's' + \cos\delta\cdot\theta = 0$$

$$s''' + \left(\frac{m}{2}+n\right)ss'' - ms'^2 + nf'^2 + \sin\delta\cdot\theta = 0 \qquad (3.3.18)$$

$$P_r^{-1}\theta'' + \left(\frac{m}{2}+n\right)s\theta' - 2ms'\theta = 0.$$

Where prime denote derivatives with respect to φ.

With the boundary conditions,

$$\begin{aligned} &f(0) = f'(0) = 0 \qquad f'(\infty) = 0,\\ &s(0) = s'(0) = 0 \qquad s'(\infty) = 0,\\ &\theta(0) = 1 \qquad \theta(\infty) = 0. \end{aligned} \qquad (3.3.19)$$

The transformed equations can be solved with the help of the controlling parameters P_r, m, n, and δ .

The similarity variable φ is,

$$\varphi = \frac{Z}{\alpha\gamma} = \left(\frac{g\beta_T\ \Delta T}{\nu^2}\right)^{\frac{1}{4}}.Z = Gr^{1/4}.z \qquad (3.3.20)$$

The velcity components

$$u = \overline{U}f'(\varphi)$$

$$v = \overline{U}s'(\varphi) \qquad (3.3.21)$$

and $w = \frac{1}{h_1h_2}\left[-\left\{h_2\gamma\overline{U}\right\}_X\overline{F} + \overline{\phi}\gamma_X h_2\overline{UF}_{\overline{\phi}} - \left\{h_1\gamma\overline{U}\right\}_Y\overline{S} + \overline{\phi}\gamma_Y h_1\overline{U}\overline{S}_{\overline{\phi}} + h_1h_2w_0\right].$

$$\Rightarrow\ -w = (\nu)^{1/2}e^{\frac{m}{2}y}\left[\left(\frac{2n+m}{2}\right)s + \left(\frac{m}{2}\right)\varphi s'\right]. \qquad (3.3.22)$$

The equations (3.3.18) & (3.3.22) are independent of the stream function f due to the reason that ΔT – variation depends only on y.

The stream functions are

$$\psi = (\nu)^{1/2}e^{\frac{m}{2}y}f(\varphi)$$

$$\Psi = (\nu)^{1/2}e^{\left(n+\frac{m}{2}\right)y}s(\varphi) \qquad (3.3.23)$$

Skin frictions are

$$\tau_{w1} = \mu Be^{\frac{3my}{2}}\sqrt{\frac{B}{\nu}}f''(0)$$

$$\tau_{w2} = \mu Be^{\frac{3my}{2}}\sqrt{\frac{B}{\nu}}s''(0) \qquad (3.3.24)$$

and the heat transfer

$$q_w = -\kappa \Delta T \left(\frac{g\beta_T \Delta T}{L} \right)^{1/4} \frac{1}{\alpha} \left(\frac{\partial \theta}{\partial \varphi} \right)_0$$

$$= -\left(\frac{\kappa}{g\beta_T} \right) \left(\frac{B}{\nu} \right)^{1/2} e^{\frac{5m}{2}} \left(\frac{\partial \theta}{\partial \varphi} \right)_0$$

Hence the coefficient of heat transfer

$$-\theta'(0) = \frac{Nu}{\left(Gr\right)^{1/4}} \qquad \textit{where,} \qquad Nu = \frac{q_w L}{\kappa \Delta T}. \tag{3.3.25}$$

where primes denote derivatives with respect to the similarity variable φ.

3.4 *Case D:*

Let $\dfrac{dA(Y)}{dY}=0, \quad \dfrac{dB(X)}{dX}=0$

$\Rightarrow \quad A(Y)=A$ (const.), $\qquad B(X)=B$ (const.)

$$\text{and } \frac{h_2}{h_1}(a_3-a_4-a_5+a_8-a_9)=0 \tag{3.4.1}$$

$$\text{and } \frac{h_1}{h_2}(a_0-a_1-a_2+a_7-a_{10})=0 \tag{3.4.2}$$

Since $\dfrac{h_2}{h_1}\neq k_1(\neq 0)$ (say) So $a_3-a_4-a_5+a_8-a_9=0$

and $a_0-a_1-a_2+a_7-a_{10}=0.$

Therefore from (2.28.1), and (2.28.4), we get,

$$\left(\frac{\gamma^2 K\Delta T^{\frac{1}{2}}L_X^{\frac{1}{2}}}{h_1}\right)_X=a_0X+A \text{ and } \left(\frac{\gamma^2 K\Delta T^{\frac{1}{2}}L_Y^{\frac{1}{2}}}{h_2}\right)_Y=a_3$$

$$\frac{\gamma^2 K\Delta T^{\frac{1}{2}}L_X^{\frac{1}{2}}}{h_1}=a_0X+A \text{ and } \frac{\gamma^2 K\Delta T^{\frac{1}{2}}L_Y^{\frac{1}{2}}}{h_2}=a_3Y+B.$$

$$\therefore \quad \frac{\gamma^2 K\Delta T^{\frac{1}{2}}L_X^{\frac{1}{2}}}{h_1}\Big/\frac{\gamma^2 K\Delta T^{\frac{1}{2}}L_Y^{\frac{1}{2}}}{h_2}=\frac{a_0X+A}{a_3Y+B}.$$

$$\Rightarrow \quad \frac{L_X^{\frac{1}{2}}}{L_Y^{\frac{1}{2}}}\cdot\frac{h_2}{h_1}=\frac{a_0X+A}{a_3Y+B}. \tag{3.4.3}$$

$$\Rightarrow \quad L_X=a_0X+A, \text{ and } \quad L_Y=a_3Y+B.$$

and h_1 & h_2 must be constant, let $h_1=1$, $h_2=1$.

Then the constants (2.28), become,

$$
\left.
\begin{array}{l|l|l}
a_0 = \left(\gamma^2 K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}}\right)_X, & a_5 = 0 & a_{11} = -\dfrac{\gamma^2 g_{\bar{\xi}}\beta_T \Delta T^{\frac{1}{2}}}{KL^{\frac{1}{2}}} \\
a_1 = \gamma^2\left(K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}}\right)_X, & a_6 = 0 & \\
a_2 = 0, & a_7 = \gamma^2\left(K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}}\right)_X = a_1 & a_{12} = -\dfrac{\gamma^2 g_{\bar{\gamma}}\beta_T \Delta T^{\frac{1}{2}}}{KL^{\frac{1}{2}}} \\
a_3 = \left(\gamma^2 K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}}\right)_Y, & a_8 = \gamma^2\left(K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}}\right)_Y = a_4 & a_{13} = \gamma^2 K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}} (\ln \Delta T)_X \\
a_4 = \gamma^2\left(K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}}\right)_Y, & a_9 = 0 & \\
 & a_{10} = 0 & a_{14} = \gamma^2 K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}} (\ln \Delta T)_Y
\end{array}
\right| \quad (3.4.4)
$$

Now from (2.29.9), we have,

$$\frac{dA}{dY}\cdot\frac{dB}{dX} = (a_3 - a_4 - a_5 + a_8 - a_9)(a_0 - a_1 - a_2 + a_7 - a_{10}) = 0.$$

$$= a_3 . a_0 = 0 \qquad \text{[By using (3.4.4)]}$$

This implies either $a_0 = 0$ or $a_3 = 0$, not both.

Let $a_3 = 0$, $a_0 \neq 0$ orbitrary constant, then $L = a_0 X + A$ (from(3.3.3).

We have, $\gamma^2 K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}} = a_0 X + A$ and $a_1 = \gamma^2\left(K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}}\right)_X$,

$$\therefore \quad \frac{\gamma^2\left(K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}}\right)_X}{\gamma^2\left(K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}}\right)} = \frac{a_1}{a_0 X + A}.$$

$$\Rightarrow \quad K\Delta T^{\frac{1}{2}} L^{\frac{1}{2}} = \left(a_0 X + A\right)^m \quad \text{where} \quad m = \frac{a_1}{a_0}$$

$$\Rightarrow \quad \Delta T = \frac{1}{K^2}\left(a_0 X + A\right)^{2m-1} = T_0 x^{2m-1}, \qquad where, \quad x = X + X_0, \quad A = a_0 X_0 \;\; \& \; T_{0*} = \frac{a_0}{K^2} \qquad (3.4.5)$$

and $\gamma^2 = \left(a_0 X + A\right)^{1-m} = a_0^{1-m} x^{1-m}$.

Threfore the constant are,

$a_0 = a_0$ (say)	$a_6 = 0$	$a_{11} = -\frac{g_X}{g} = \frac{a_0}{4}\cos\delta$ (say)	
$a_1 = m a_0$,	$a_7 = m a_0$,		
$a_2 = 0$,	$a_8 = 0$	$a_{12} = -\frac{g_Y}{g} = \frac{a_0}{4}\sin\delta$ (say)	
$a_3 = 0$,	$a_9 = 0$		(3.4.6)
$a_4 = 0$,	$a_{10} = 0$	$a_{13} = (2m-1)a_0$	
$a_5 = -0$		$a_{14} = 0$.	

The corresponding equations are

$$\nu \overline{F}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \left[\frac{m+1}{2}\right] a_0 \overline{F}\,\overline{F}_{\bar{\phi}\bar{\phi}} - m a_0 \overline{F}_{\bar{\phi}}^{2} + a_0 \cos\delta\ \overline{\theta} = 0 \tag{3.4.7}$$

$$\nu \overline{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \left[\frac{m+1}{2}\right] a_0 \overline{F}\,\overline{S}_{\bar{\phi}\bar{\phi}} - m a_0 \overline{F}_{\bar{\phi}} \overline{S}_{\bar{\phi}} + a_0 \sin\delta\ \overline{\theta} = 0 \tag{3.4.8}$$

$$\frac{\nu}{P_r} \overline{\theta}_{\bar{\phi}\bar{\phi}} + \left[\frac{m+1}{2}\right] a_0 \overline{F}\,\overline{\theta}_{\bar{\phi}} - (2m-1) a_0 \overline{F}_{\bar{\phi}} \overline{\theta} = 0 \tag{3.4.9}$$

Let $\overline{F} = \alpha f$, $\overline{S} = \alpha\, s$, $\overline{\theta} = \theta$, $\overline{\phi} = \alpha\varphi$. and choosing $\frac{a_0 \alpha^2}{\nu} = 4$, we get,

$$f''' + 2(m+1) f f'' - 4m f'^2 + (\cos\delta)\ \theta = 0 \tag{3.4.10}$$

$$S''' + 2(m+1) f s'' - 4m f' s' + (\sin\delta)\ \theta = 0 \tag{3.4.11}$$

$$P_r^{-1} \theta'' + 2(m+1) f \theta' - 4(2m-1) f' \theta = 0 \tag{3.4.12}$$

with the boundary conditions,

$$\begin{aligned} &f(0) = f'(0) = 0 && f'(\infty) = 0, \\ &s(0) = s'(0) = 0 && s'(\infty) = 0, \\ &\theta(0) = 1 && \theta(\infty) = 0. \end{aligned} \tag{3.4.13}$$

For m=0.5, δ =0 and f=s, the present problem turns to a case discussed by Ostrach [1953], with the omission of the equation (3.4.11).

The transformed equations can be solved with the help of the controlling parameters P_r, m and δ.

The similarity variable φ is,

$$\varphi = \frac{Z}{\alpha\gamma} = \frac{Z}{\sqrt{\frac{4\upsilon}{a_0}}(a_0x + A)^{\frac{1-m}{2}}} = \left(\frac{a_0{}^2 g\beta_T \Delta T}{4^2 \nu^2}\right)^{\frac{1}{4}} \cdot \frac{Z}{(a_0x + A)^{\frac{1}{4}}} = Gr_x^{1/4} \cdot \frac{z}{x} \quad (3.4.14)$$

where, the modified Grashof number, $Gr_x = \left[\frac{a^{4-\frac{n}{2}}}{4^2} \cdot \frac{g\beta_T \Delta T x^3}{\nu^2}\right]^{1/4}$

The velcity components

$$u = \overline{U}f'(\varphi)$$

$$v = \overline{U}s'(\varphi) \quad (3.4.15)$$

and $w = \frac{1}{h_1h_2}\left[-\left\{h_2\gamma\overline{U}\right\}_x \overline{F} + \bar{\phi}\gamma_x h_2 \overline{UF}_{\bar{\phi}} - \left\{h_1\gamma\overline{U}\right\}_y \overline{S} + \bar{\varphi}\gamma_y h_1 \overline{US}_{\bar{\phi}} + h_1h_2w_0\right].$

$$\Rightarrow \quad -w = \left(4\nu a_0^m\right)^{1/2} x^{\frac{m-1}{2}}\left[\left(\frac{m+1}{2}\right)f + \left(\frac{1-m}{2}\right)\varphi f'\right]. \quad (3.4.16)$$

The equations (3.4.12) & (3.4.16) are independent of s because of ΔT – variation in this case is free of y variations.

The stream functions are

$$\psi = \left(4\nu a_0^m\right)^{1/2} x^{\frac{m+1}{2}} f(\varphi)$$

$$\phi = \left(4\nu a_c^m\right)^{1/2} x^{\frac{m+1}{2}} s(\varphi) \quad (3.4.17)$$

Skin frictions are

$$\tau_{w1} = \mu x^{\frac{3m-1}{2}} \sqrt{\frac{a_0^{3m}}{4\nu}} f''(0)$$
$$\tau_{w2} = \mu x^{\frac{3m-1}{2}} \sqrt{\frac{a_0^{3m}}{4\nu}} s''(0) \tag{3.4.18}$$

and the heat transfer

$$q_w = -\kappa \Delta T \left(\frac{g\beta_T \Delta T}{L}\right)^{1/4} \frac{1}{\alpha}\left(\frac{\partial\theta}{\partial\varphi}\right)_0$$
$$= -\left(\frac{\kappa}{g\beta_T}\right)\left(\frac{a_0^{5m-2}}{4\nu}\right)^{1/2} x^{\frac{5m-3}{2}} \left(\frac{\partial\theta}{\partial\varphi}\right)_0$$

so the heat transfer coefficient is

$$-\theta'(0) = \frac{Nu_x}{(Gr_x)^{1/4}} \tag{3.1.19}$$

where. the modified Nusselt number , $Nu_x = \frac{q_w x}{\kappa \Delta T}$.

Here primes denote derivatives with respect to the similarity variable φ.

Similarly, if we set $a_0 = 0$ and $a_3 \neq 0$, arbitrary constant, then we get

$$s''' + 2(n+1)ss'' - 4ns'^2 + \cos\delta'.\theta = 0 \tag{3.4.20}$$

$$f''' + 2(n+1)sf'' - 4ns'f' + \sin\delta'\theta = 0 \tag{3.4.21}$$

$$P_r^{-1}\theta'' + 2(n+1)s\theta' - 4(2n-1)s'\theta = 0 \tag{3.4.22}$$

where $n = \frac{a_4}{a_3}$, with the same bonndary conditions and δ' is the angle between the η-direction and the horizontal surface

$$\begin{aligned} &f(0) = f'(0) = 0 \quad f'(\infty) = 0, \\ &s(0) = s'(0) = 0 \quad s'(\infty) = 0, \\ &\theta(0) = 1 \quad \theta(\infty) = 0. \end{aligned} \tag{3.4.23}$$

The transformed equations can be solved with the help of the controlling parameters P_r, n, and δ.

The similarity variable φ is,

$$\varphi = \frac{Z}{\alpha\gamma} = \frac{Z}{\sqrt{\frac{4\upsilon}{a_0}}(a_3 y + B)^{\frac{1-m}{2}}} = \left(\frac{a_3^{\,2} g\beta_T \,\Delta T}{4^2 \nu^2}\right)^{\frac{1}{4}} \cdot \frac{Z}{(a_3 y + B)^{\frac{1}{4}}} = Gr_y^{1/4} \cdot \frac{z}{y} \qquad (3.4.24)$$

where, the modified Grashof number, $Gr_y = \left[\frac{a^{4-\frac{n}{2}}}{4^2} \cdot \frac{g\beta_T \Delta T y^3}{\nu^2}\right]^{1/4}$

The velcity components

$u = \overline{U} f'(\varphi)$

$v = \overline{U} s'(\varphi)$ where, $\overline{U} = \sqrt{g\beta_T \Delta T y}$ (3.4.25)

and $w = \frac{1}{h_1 h_2}\left[-\left\{h_2 \gamma \overline{U}\right\}_X \overline{F} + \overline{\phi}\gamma_X h_2 \overline{U} \overline{F}_{\overline{\varphi}} - \left\{h_1 \gamma \overline{U}\right\}_Y \overline{S} + \overline{\phi}\gamma_Y h_1 \overline{U} \overline{S}_{\overline{\varphi}} + h_1 h_2 w_0\right]$.

$$\Rightarrow \quad -w = \left(4\nu a_3^m\right)^{1/2} y^{\frac{m-1}{2}}\left[\left(\frac{m+1}{2}\right)s + \left(\frac{1-m}{2}\right)\varphi s'\right]. \qquad (3.4.26)$$

The equation (3.4.22) & (3.4.26) is independent of f because of ΔT – variation in this case is free of x-variations.

The stream functions are

$\psi = \left(4\nu a_3^m\right)^{1/2} y^{\frac{m+1}{2}} f(\varphi)$

$\Psi = \left(4\nu a_3^m\right)^{1/2} y^{\frac{m+1}{2}} s(\varphi)$ (3.4.27)

Skin frictions are

$$\tau_{w1} = \mu y^{\frac{3m-1}{2}} \sqrt{\frac{a_3^{3m}}{4\nu}} f''(0)$$
$$\tau_{w2} = \mu y^{\frac{3m-1}{2}} \sqrt{\frac{a_3^{3m}}{4\nu}} s''(0) \qquad (3.4.28)$$

and the heat transfer

$$q_w = -\kappa \Delta T \left(\frac{g\beta_T \Delta T}{L} \right)^{1/4} \frac{1}{a} \left(\frac{\partial \theta}{\partial \varphi} \right)_0$$
$$= -\left(\frac{\kappa}{g\beta_T} \right) \left(\frac{a_3^{5m-2}}{4\nu} \right)^{1/2} y^{\frac{5m-3}{2}} \left(\frac{\partial \theta}{\partial \varphi} \right)_0$$

so the heat transfer coefficient is

$$-\theta'(0) = \frac{Nu_y}{\left(Gr_y\right)^{1/4}} \qquad (3.4.29)$$

where the modified Nusselt number , $Nu_y = \frac{q_w y}{\kappa \, \Delta T}$.

where primes denote derivatives with respect to the similarity variable φ.

Graphs:

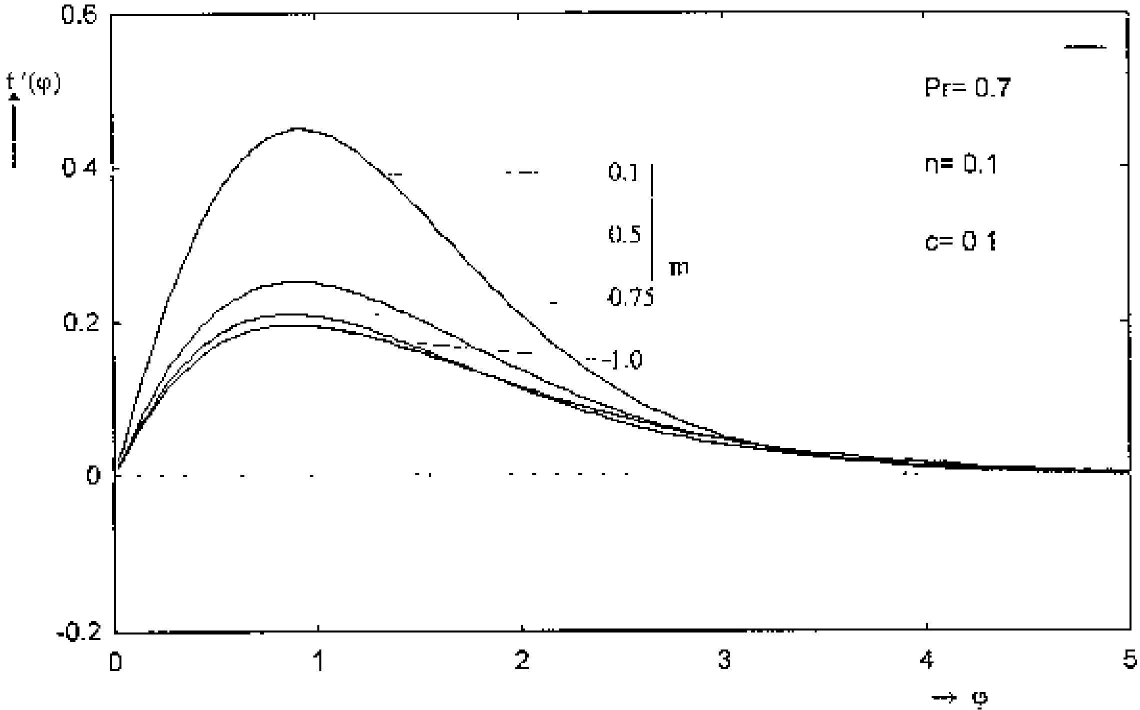


Fig (1): Dimensionless velocity distributions along u-direction for several values of $m(= 0.1, 0.5, 0.75, 1.0)$ $\left(\Delta T = T_0(x + cy)^{2m-1}\right)$

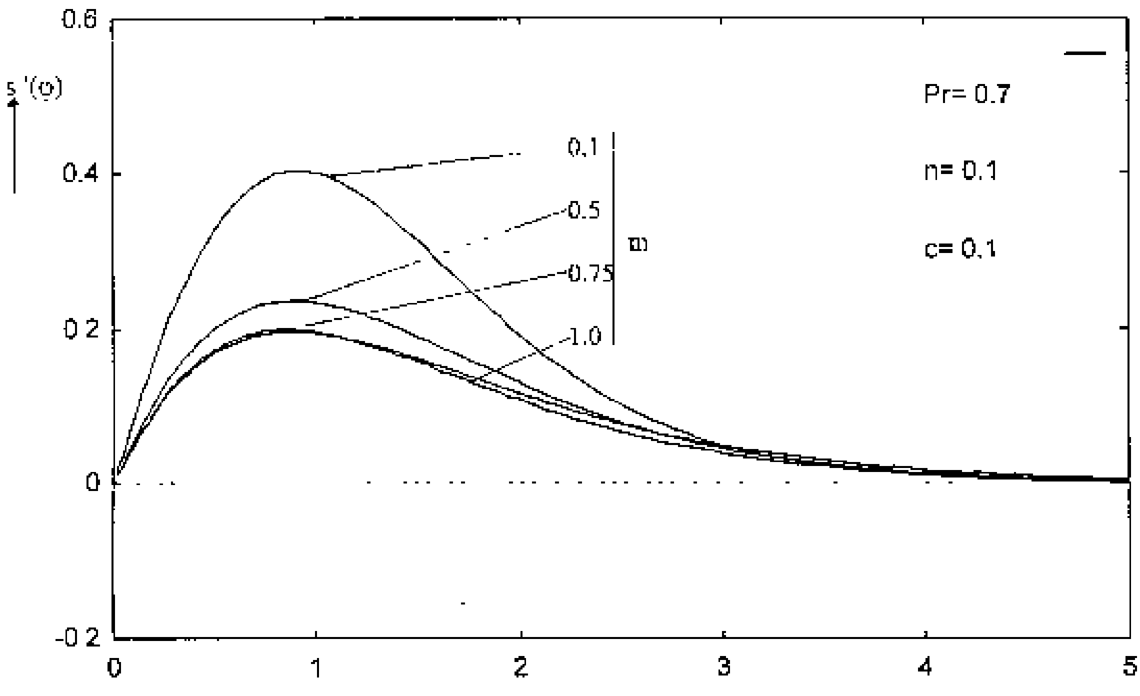


Fig (2): Dimensionless velocity distributions along v-direction for several values of $m(= 0.1, 0.5, 0.75, 1.0)$ $\left(\Delta T = T_0(x + cy)^{2m-1}\right)$

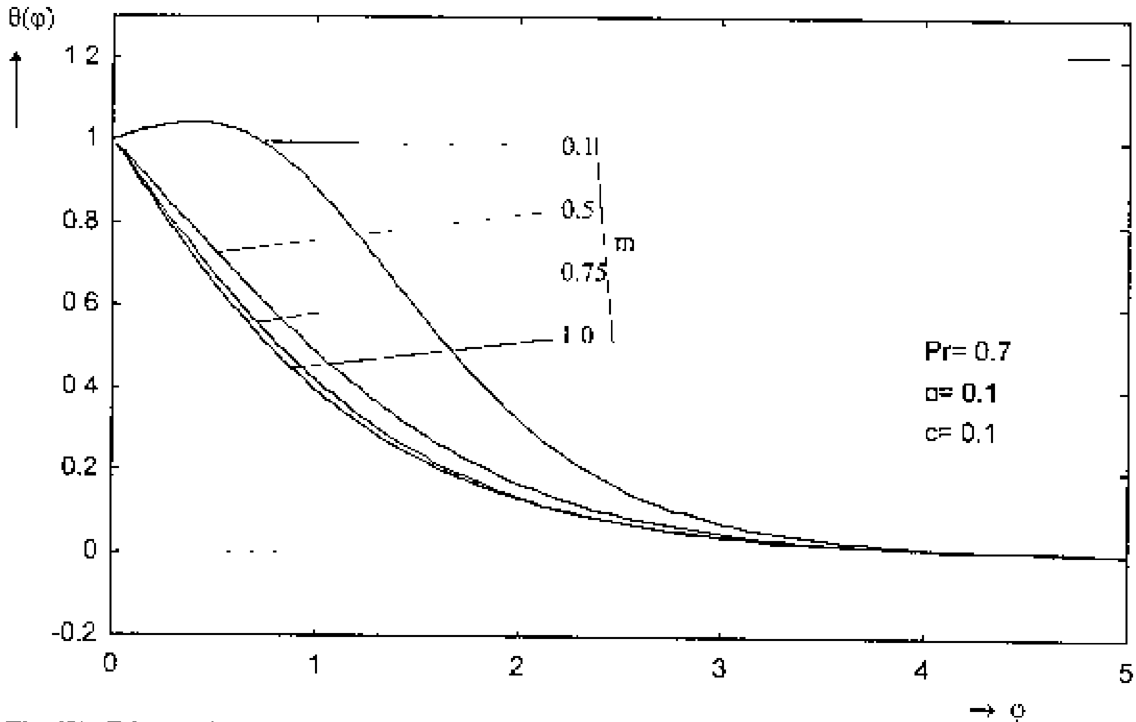
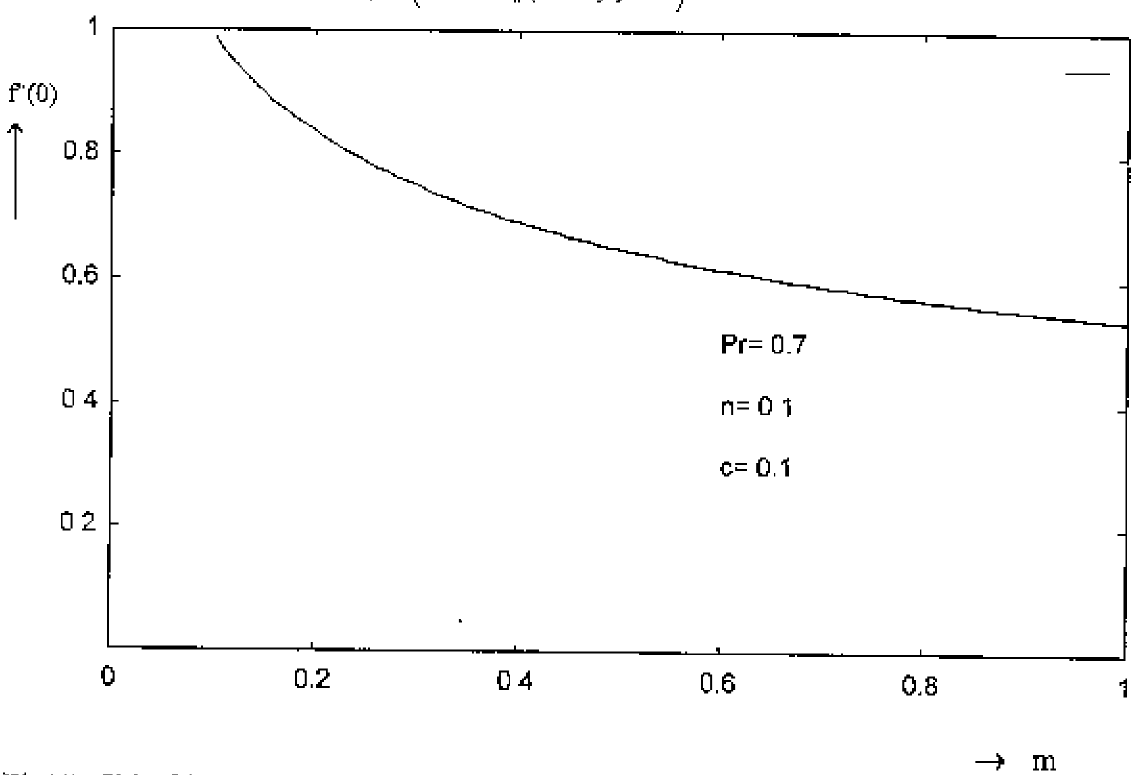
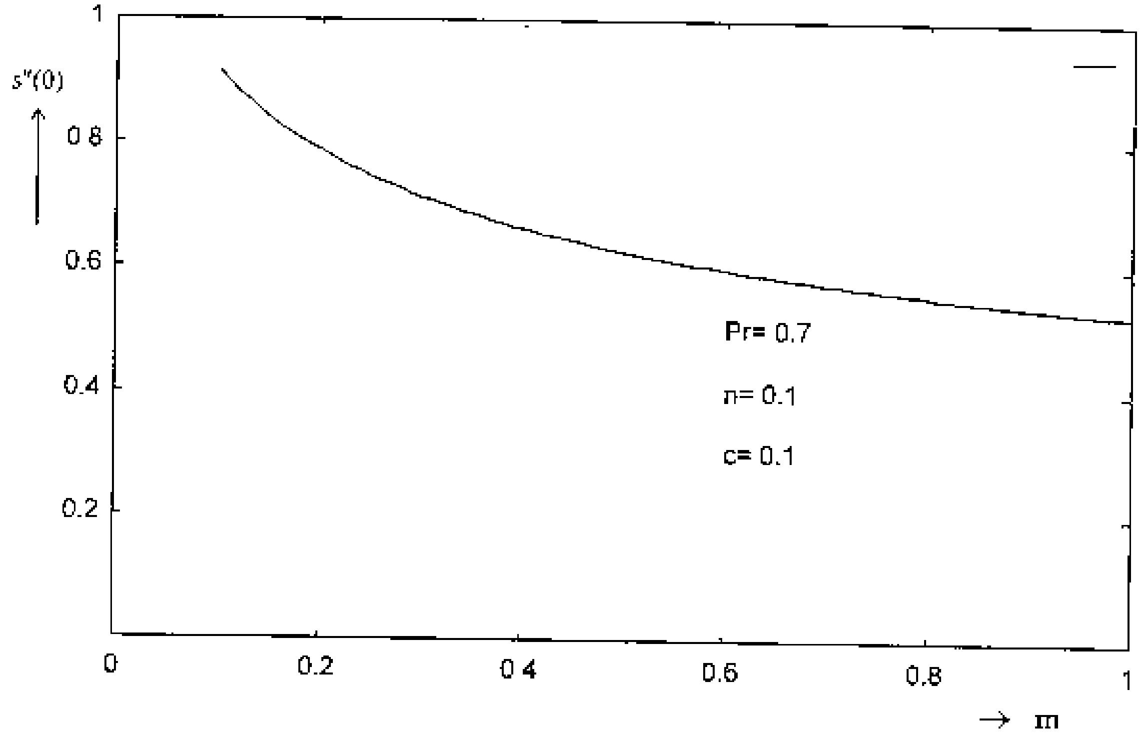


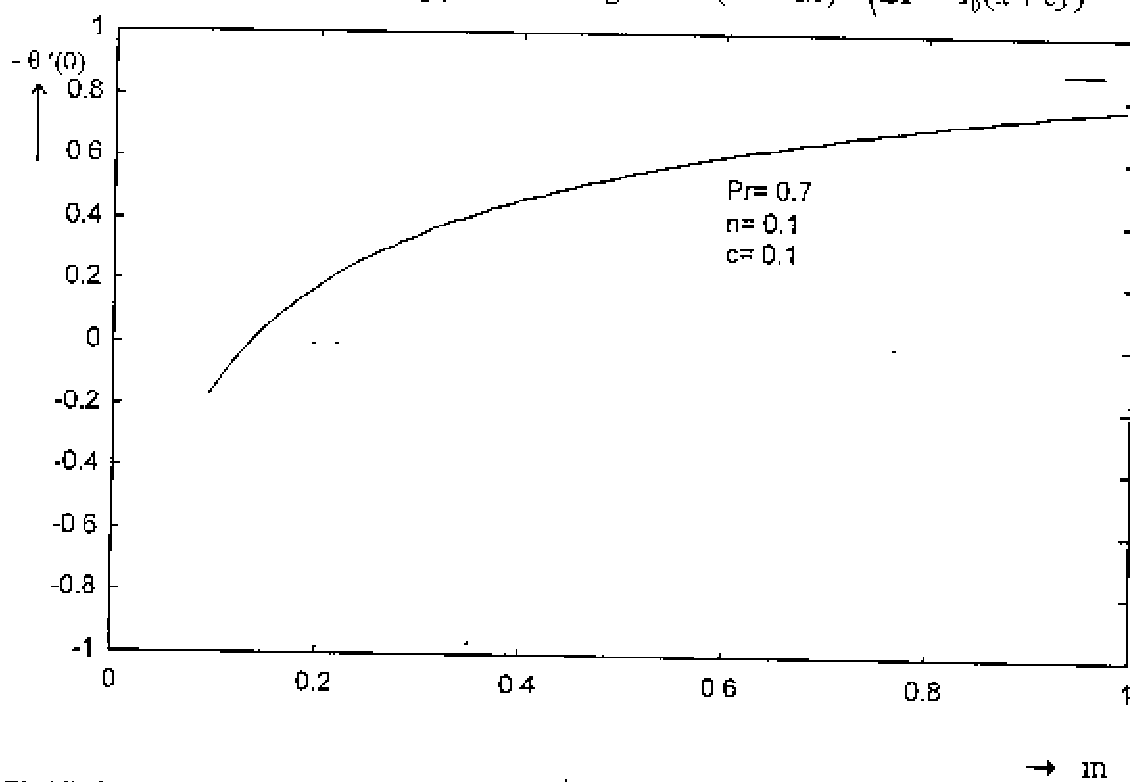
Fig (3): Dimensionless temperature distributions for several values of $m(= 0.1, 0.5, 0.75, 1.0)$ $\left(\Delta T = T_0(x+cy)^{2m-1}\right)$.



Fig(4): Skin friction factor along x-direction against $m(0.1-1.0)$ $\left(\Delta T = T_0(x+cy)^{2m-1}\right)$.



Fig(5): Skin friction factor along y-direction against $m(0.1-1.0)$ $\left(\Delta T = T_0(x+cy)^{2m-1}\right)$.



Fig(6): Heat transfer against $m(0.1-1.0)$ $\left(\Delta T = T_0(x+cy)^{2m-1}\right)$.

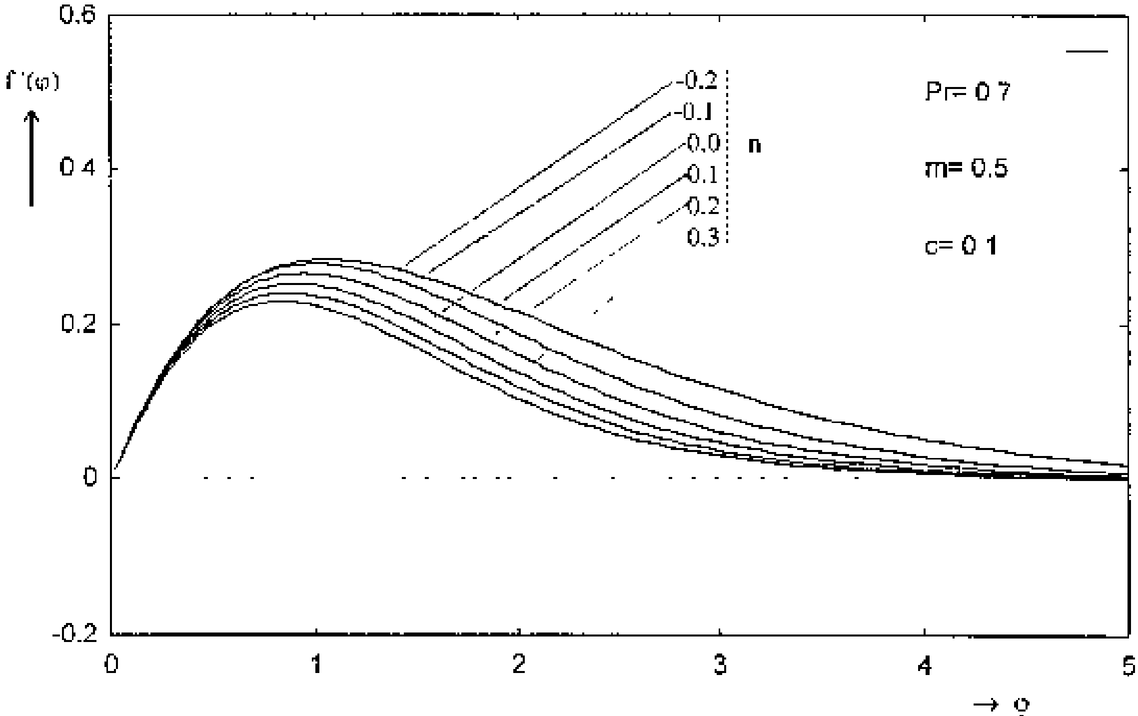


Fig (7): Dimensionless velocity distributions along u-direction for several values of $n(= -0.2, -0.1, 0.0, 0.1, 0.2, 0.3)$ $\left(h_1 = (x + cy)^n, \quad h_2 = k_1 h_1\right)$

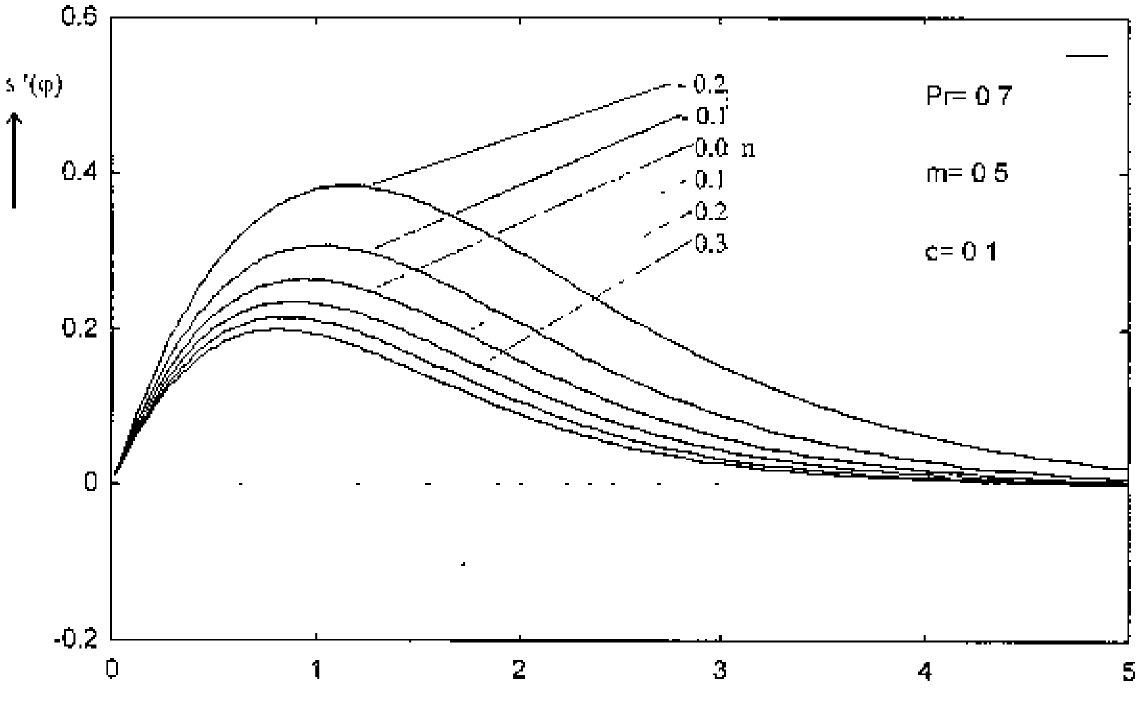


Fig (8): Dimensionless velocity distributions along v-direction for several values of $n(= -0.2, -0.1, 0.0, 0.1, 0.2, 0.3)$ $\left(h_1 = (x + cy)^n, \quad h_2 = k_1 h_1\right)$

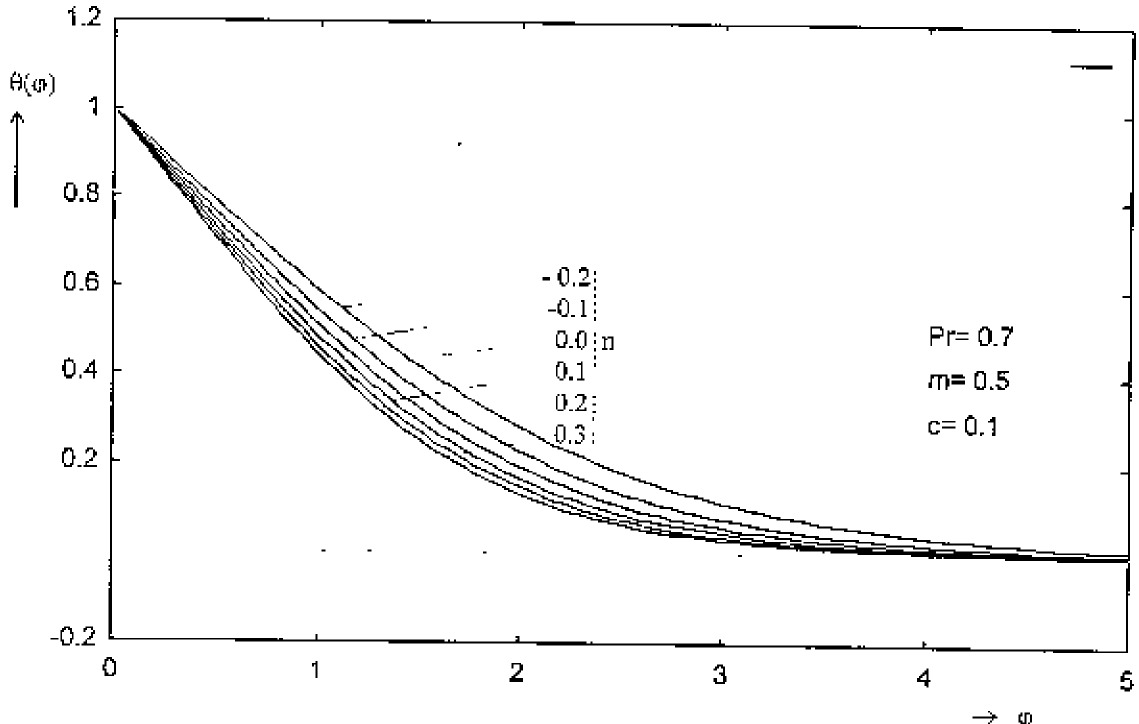


Fig (9): Dimensionless temperature distributions for several values of $n(= -0.2, -0.1, 0.0, 0.1, 0.2, 0.3)$ $\left(h_1 = (x + cy)^n, \quad h_2 = k_1 h_1\right)$

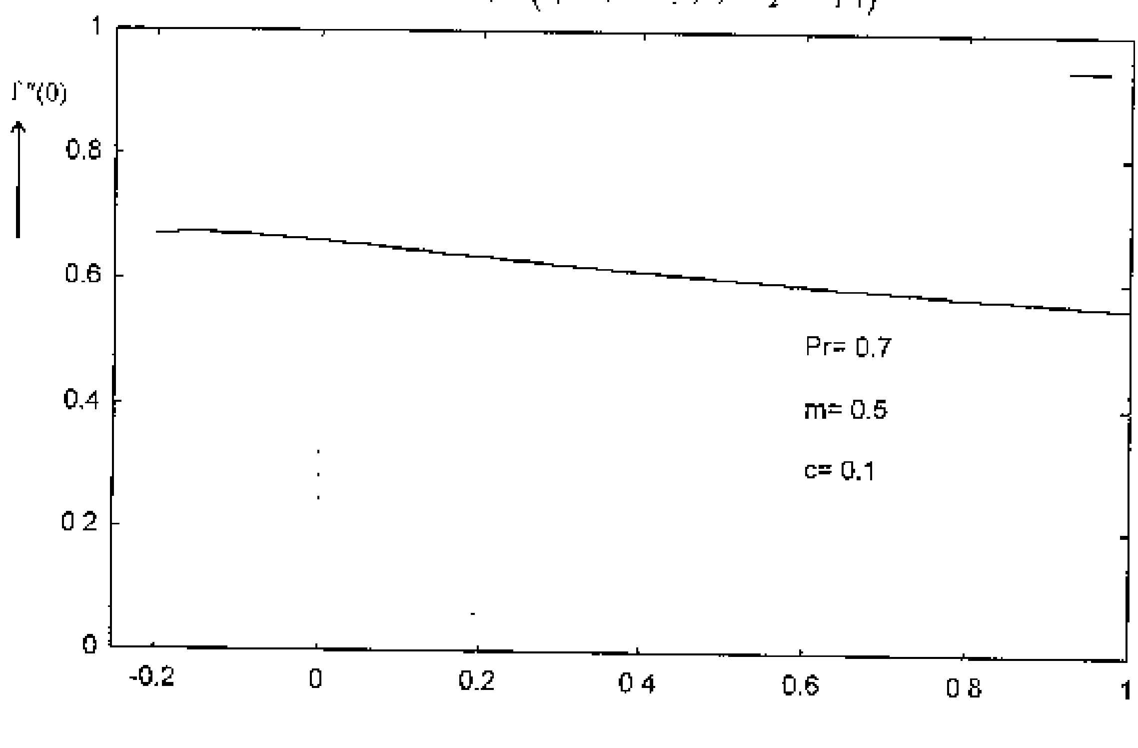


Fig (10): Skin friction factor against $n(= (-0.2) - 1.0)$ $\left(h_1 = (x + cy)^n, \quad h_2 = k_1 h_1\right)$

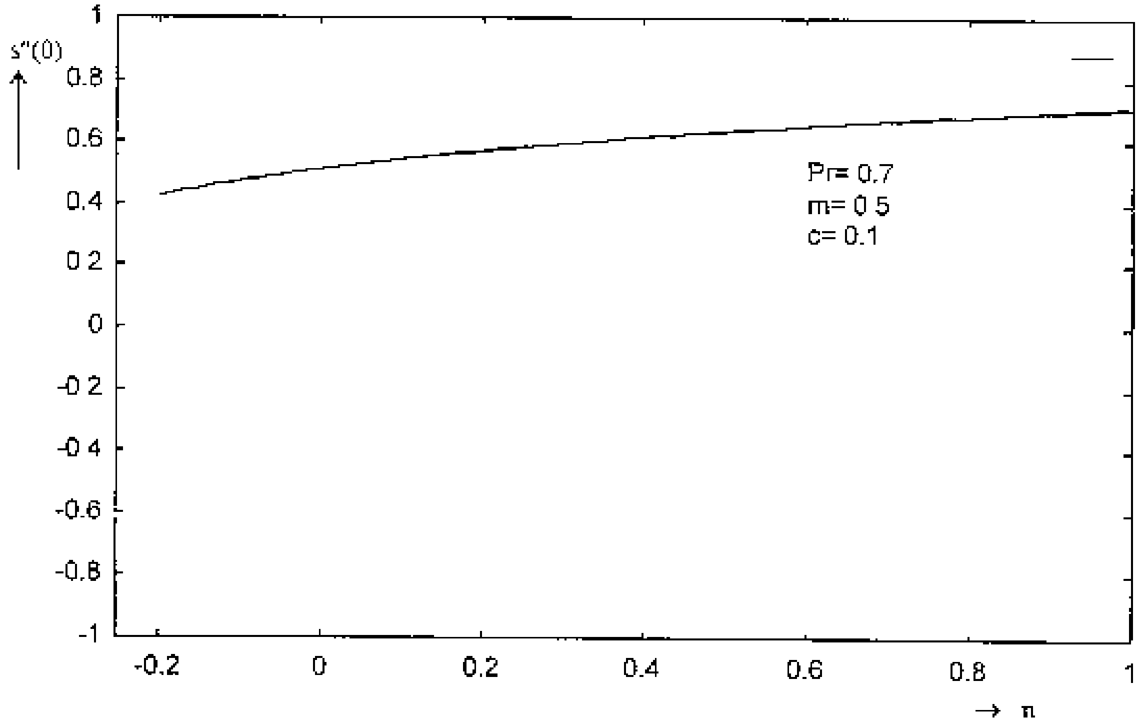


Fig (11)· Skin friction factor along y-direction against $n((-0.2) - 1.0)$ $\left(h_1 = (x + cy)^n,\quad h_2 = k_1 h_1\right)$.

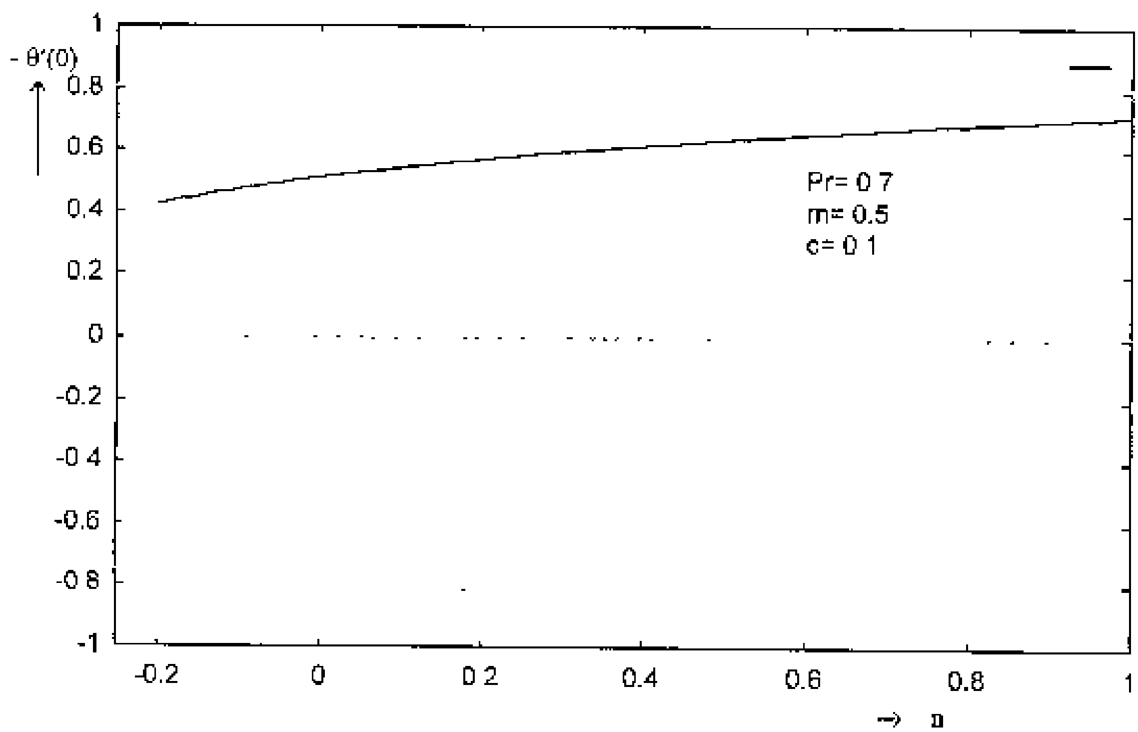


Fig (12): Heat transfer factor against $n(= (-0.2) - 1.0)$ $\left(h_1 = (x + cy)^n,\quad h_2 = k_1 h_1\right)$.

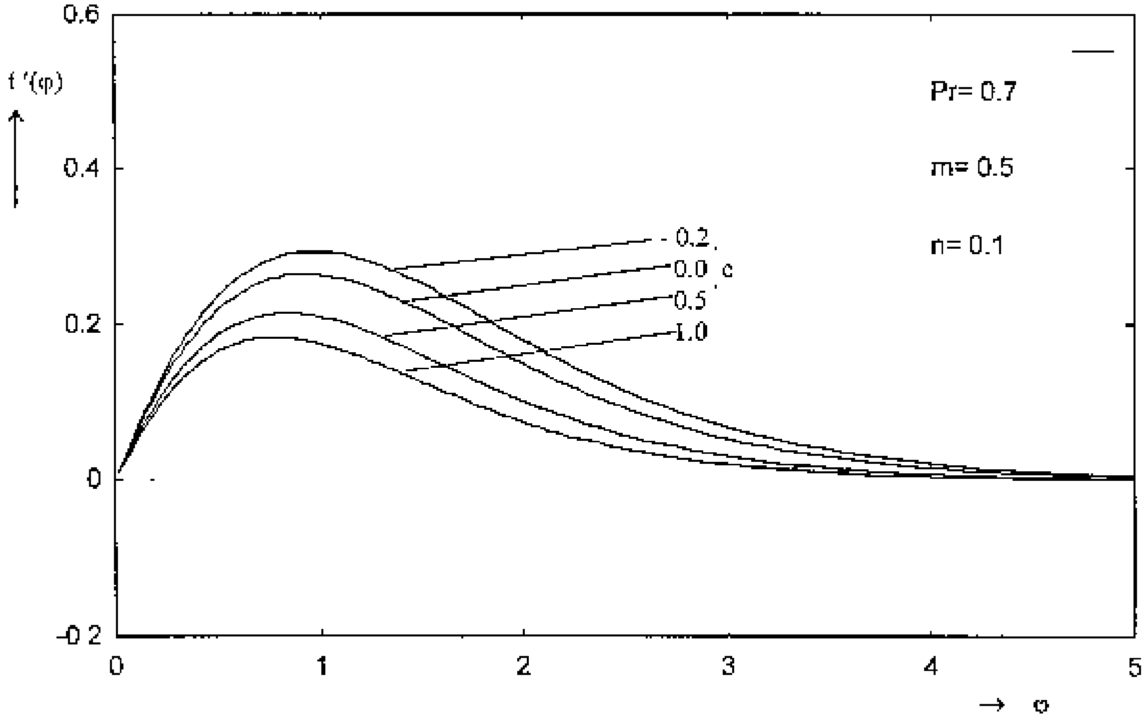


Fig (13): Dimensionless velocity distributions along u-direction for several values of $c(= -0.2, 0.0, 0.5, 1.0)$

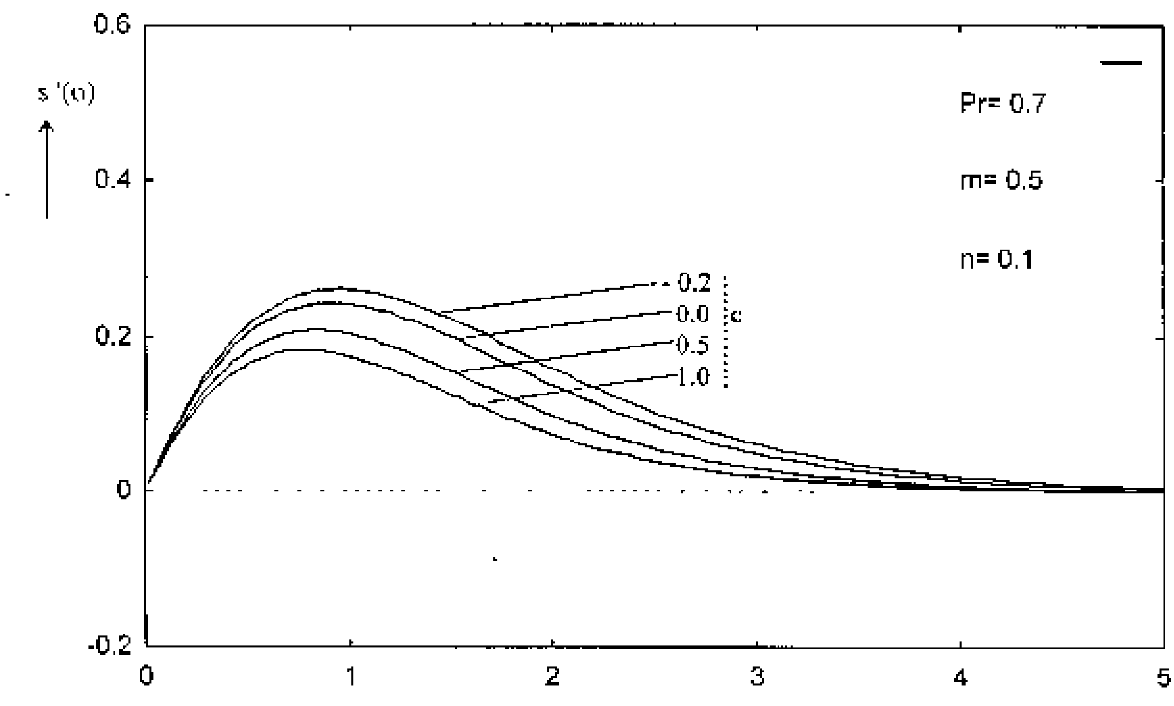


Fig (14): Dimensionless velocity distributions along v-direction for several values of $c(= -0.2, 0.0, 0.5, 1.0)$

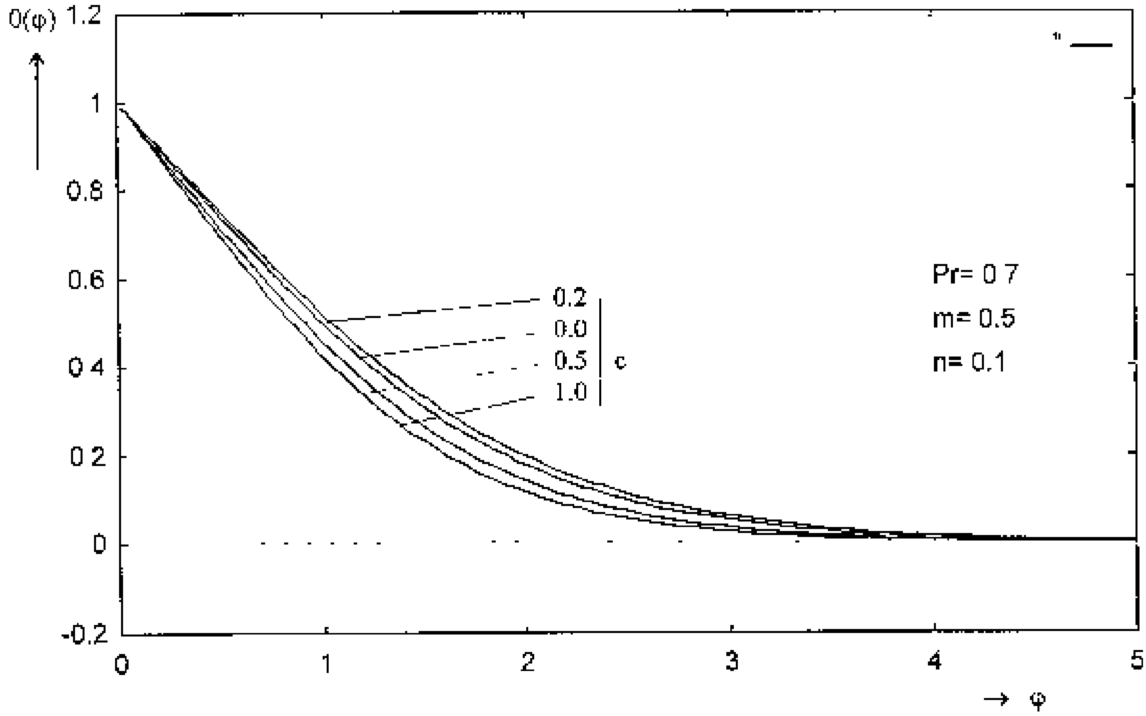


Fig (15): Dimensionless temperature distributions for several values of
$c(=-0.2, 0.0, 0.5, 1.0)$.

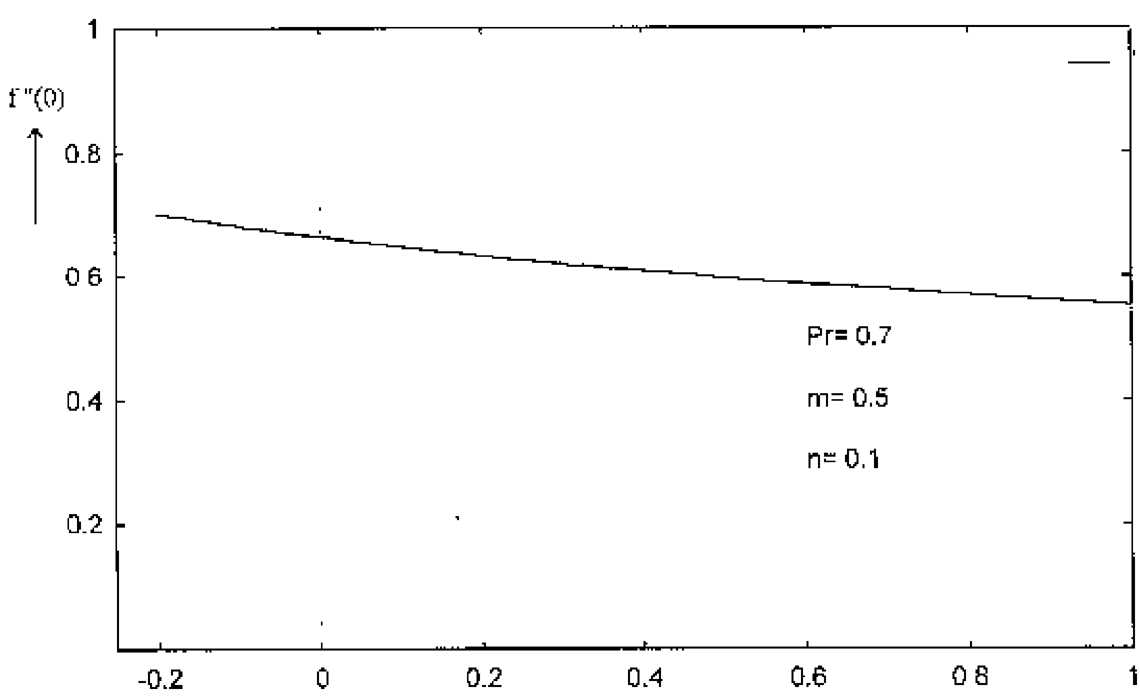


Fig (16): Skin friction factor against $c(=(-0.2) - 1.0)$.

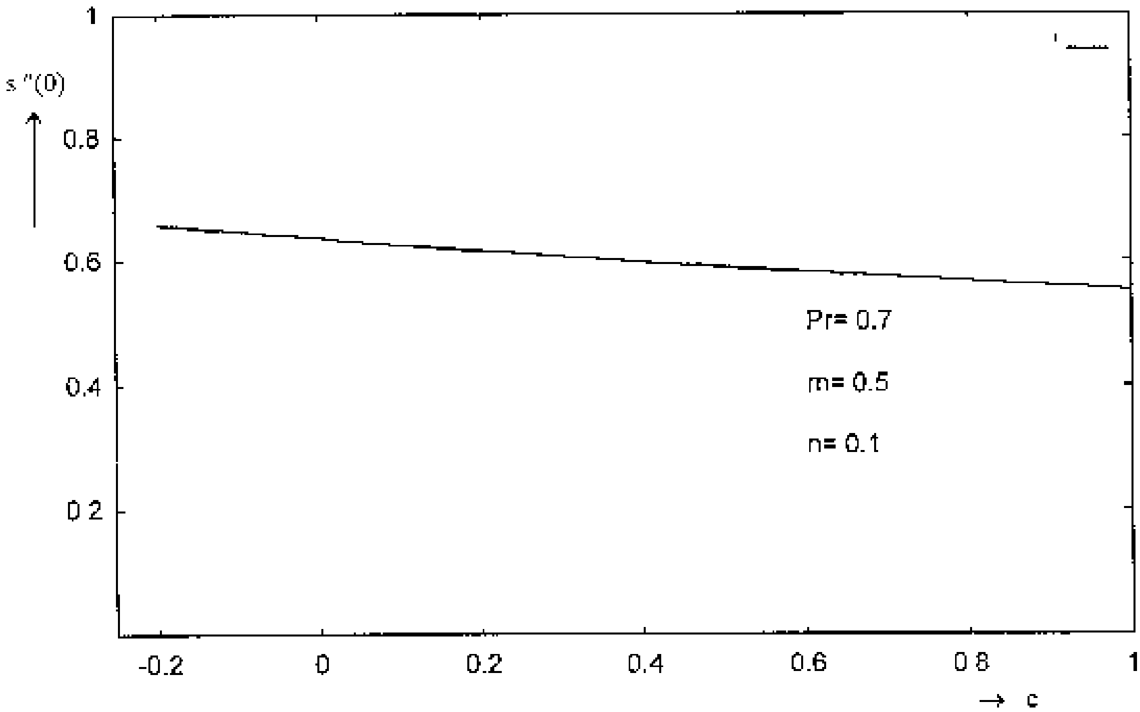


Fig (17): Skin friction factor along y-direction against $c((-0.2) - 1.0)$.

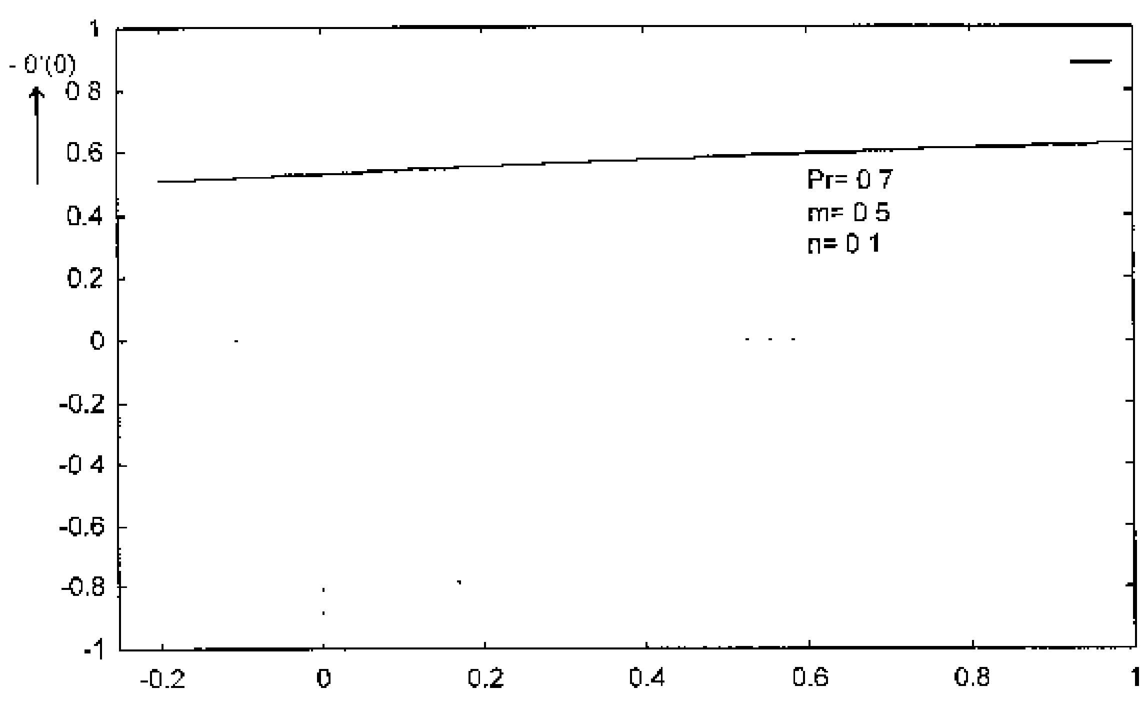


Fig (18): Heat transfer factor against $c(= (-0.2) - 1.0)$.

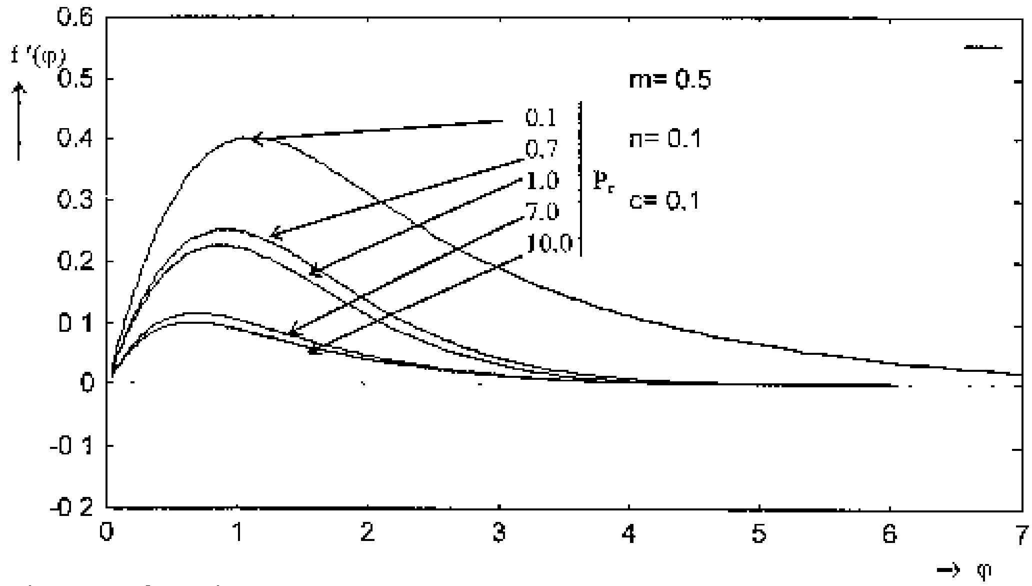


Fig (19): Dimensionless velocity distributions along u-direction for several Prandtl numbers P_r

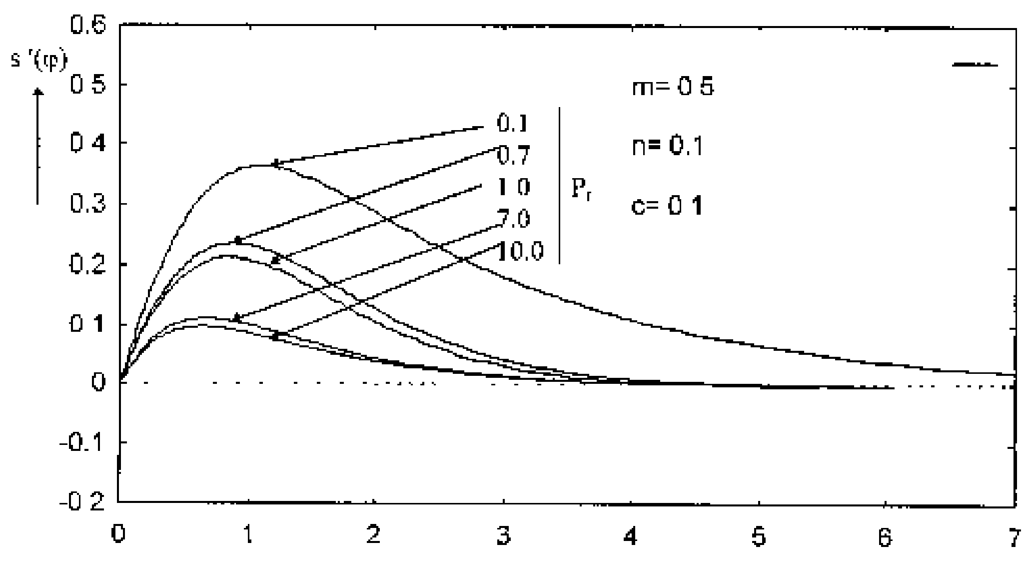


Fig (20): Dimensionless velocity distributions along v-direction for several Prandtl numbers P_r

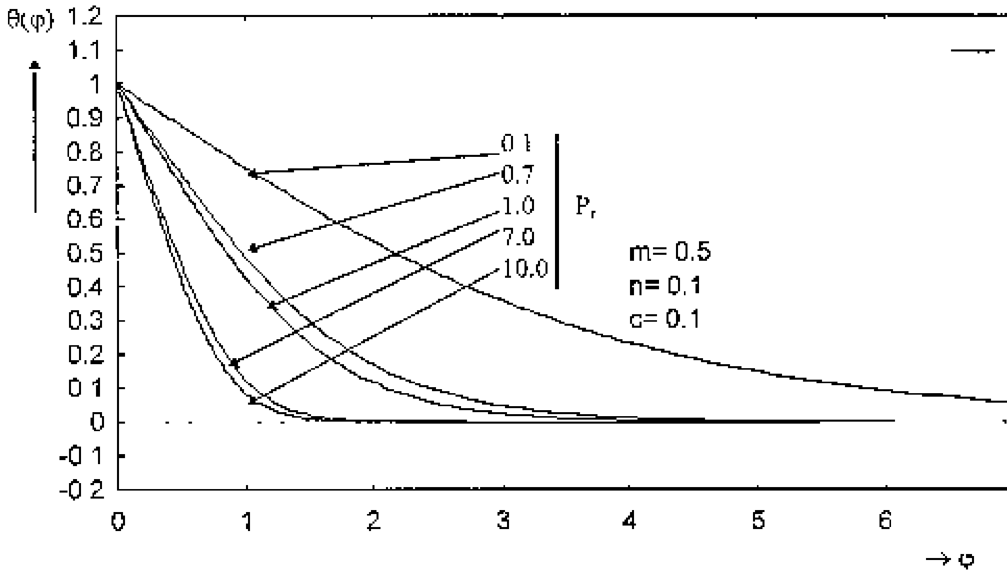


Fig (21): Dimensionless temperature distributions for several Prandtl numbers P_r.

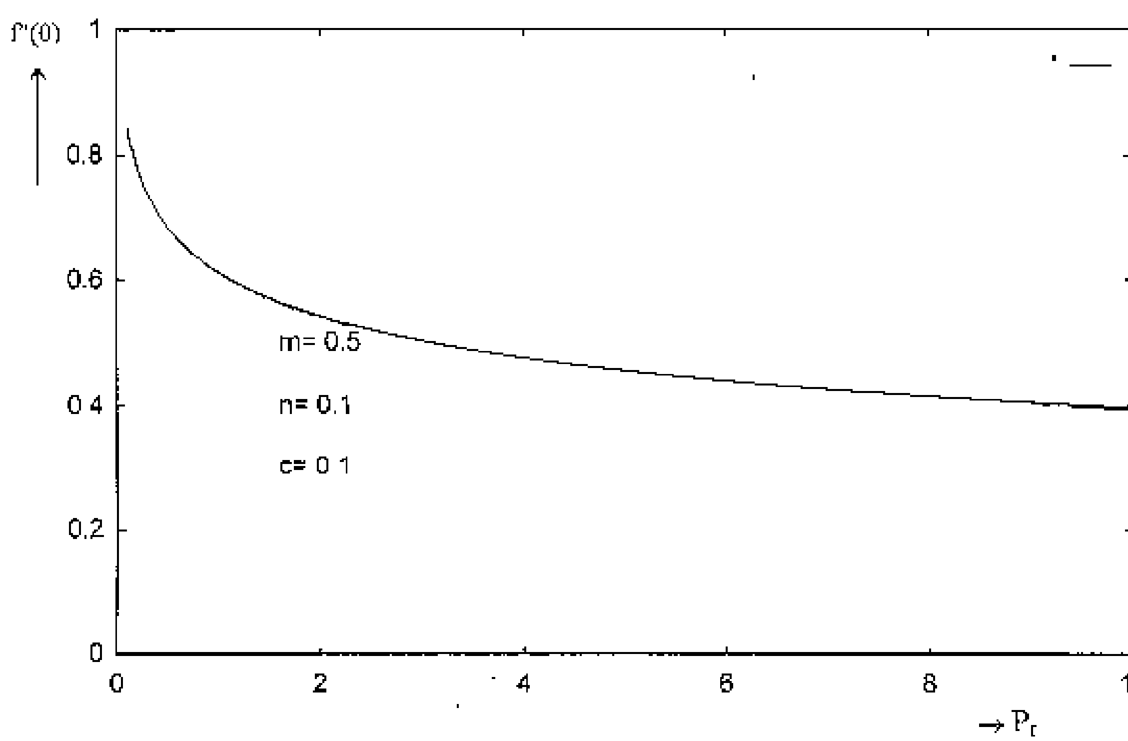


Fig (22): Skin friction factor against P_r= (0.1--10.0).

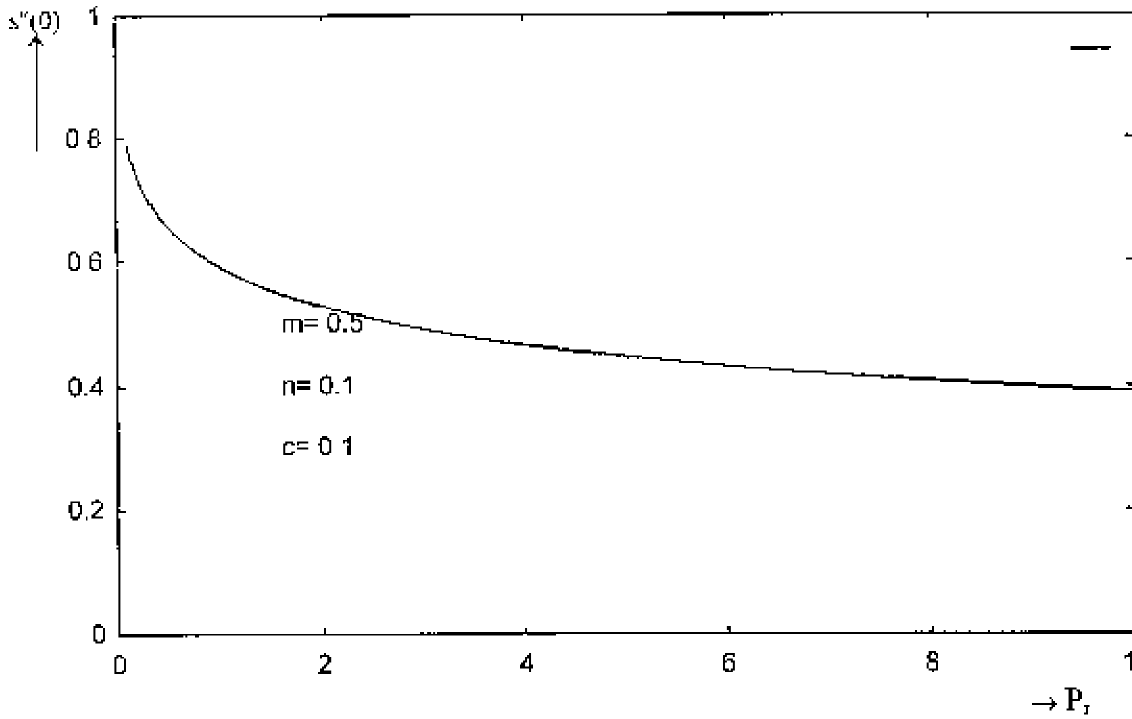


Fig (23): Skin friction factor along y-direction against P_r= (0.1--10.0).

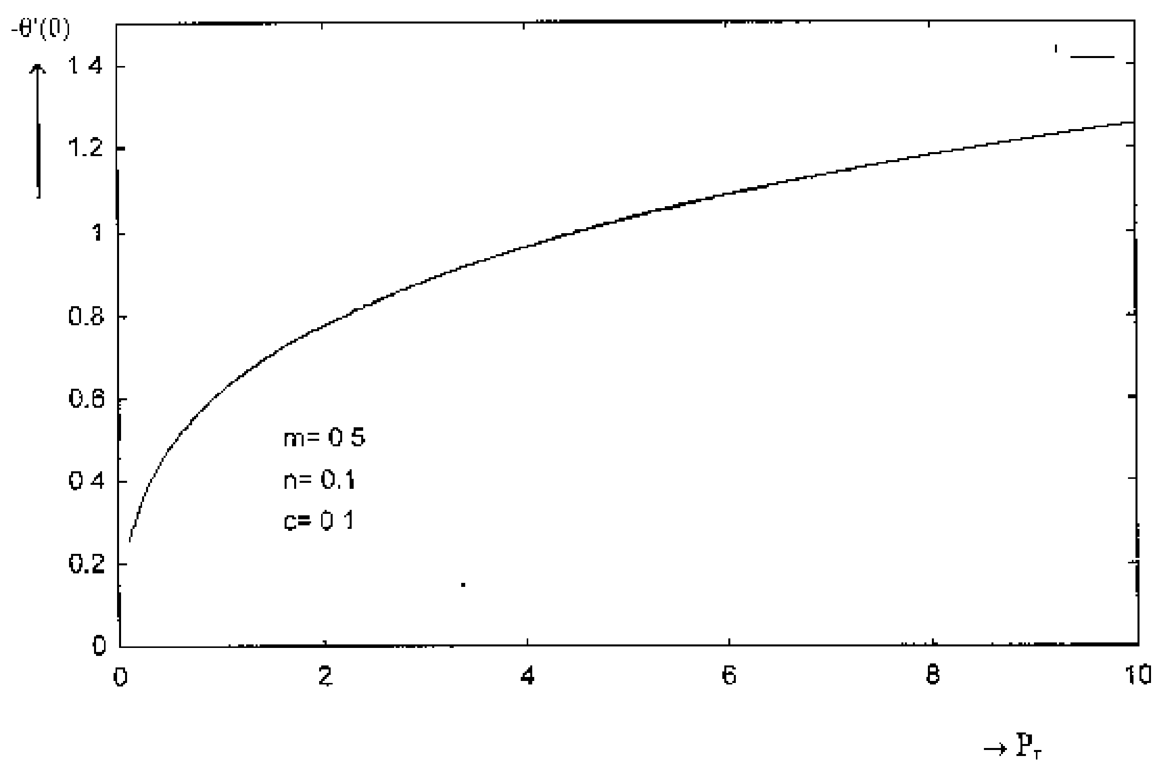


Fig (24): Heat transfer factor against P_r= (0.1--10.0).

Table - 1

m	f''(0)	s''(0)	-θ'(0)
0.10000	0.98723	0.91591	-0.16782
0.15000	0 89867	0.84046	0.03292
0.20000	0 83724	0.78819	0.16691
0.25000	0.78857	0.74641	0.26761
0.30000	0.75008	0.71320	0 34546
0 35000	0.71864	0.68595	0.40816
0 40000	0.69232	0.66302	0.46025
0.45000	0.66986	0.64337	0.50459
0 50000	0.65038	0.62625	0.54309
0.60000	0 61807	0.59766	0.60737
0.80000	0.57118	0.55543	0.70406
0 90000	0.55305	0.53904	0.74246
1.00000	0.53739	0.52480	0.77639

For Pr=0.7, n=0.1, c=0.1

Table - 2

n	f''(0)	s'' ((0)	-θ'(0)
-0.20000	0.67099	0.79445	0.42174
-0 15000	0 67450	0.74714	0.44978
-0.10000	0.67247	0.71302	0.47292
-0.05000	0 66815	0.68546	0.49322
0.00000	0.66270	0.66252	0.51139
0.05000	0.65670	0.64296	0.52788
0.10000	0.65048	0.62598	0.54302
0.20000	0.63802	0.59767	0.57011
0.30000	0.62603	0.57477	0.59394
0.40000	0.61475	0.55566	0.61528
0.50000	0.60420	0.53935	0.63467
0.70000	0.58520	0 51267	0.66894
0.90000	0.56862	0.49149	0.69872
1.00000	0.56111	0.48238	0.71230

For Pr=0.7.m=0.5, c=0.1,

Table - 3

c	f ''(0)	s'' ((0)	-θ'(0)
-0 20000	0.70397	0.65873	0.50739
-0.10000	0.68456	0.64755	0.51973
-0.05000	0.67540	0.64200	0.52573
0 00000	0.66668	0.63656	0.53162
0 10000	0.65047	0.62604	0.54303
0.15000	0.64292	0.62097	0 54856
0 20000	0.63572	0.61603	0.55397
0.30000	0.62226	0.60656	0.56445
0.40000	0.60994	0.59761	0.57449
0.50000	0.59862	0.58916	0.58413
0.60000	0.58817	0.58118	0.59338
0 75000	0.57393	0.57001	0.60661
0.90000	0.56114	0.55973	0 61911
0 95000	0.55716	0.55648	0.62313
1 00000	0.55331	0.55331	0 62708

For P_r=0.7, m=0.1, n=0.1

Table-4

P_r	f ''(0)	s''(0)	-θ'(0)
0.10000	0.84221	0.78884	025208
0.50000	0.68576	0.65767	0 48029
0.72000	0.64753	062381	0.54851
1.00000	0.61318	0.59294	0 61558
1.50000	0.57140	055491	0.70598
2.00000	0.54242	0.52823	0.77544
2.50000	0.52041	0.50781	0.83253
3.00000	0.50275	0.49135	0.88134
3.50000	0.48807	0.47761	0.92419
4.00000	0.47555	0.46584	0.96253
4.50000	0 46465	0.45557	0.99730
5.00000	0.45503	0.44648	1.02920
5.50000	0.44642	0 43834	1.05871
6.00000	0.43865	0.43097	1.08621
6.50000	0.43158	0.42425	111199
7.00000	0 42510	0.41809	1.13628
8.00000	0 41357	0.40710	1.18107
10.0000	0.39476	0 38911	1.25902

For m=0.5, n=0.1, c=0.1

Table -5

Similarity Cases in tabular form:

Case	$h_1(x,y) \propto$	$h_2(x,y) \propto$	$\Delta T(x,y) \propto$	Similarity variable φ
A	$(x+cy)^n$	h_1	$(x+cy)^{2m-1}$	$Gr_{x_1}^{1/4} \cdot \dfrac{z}{(x+cy)^{\frac{n}{2}+1}}$
B(i)	1	x^n	x^{2m-1}	$Gr_x^{1/4} \cdot \dfrac{z}{x}$
(ii)	1	e^{nx}	e^{2mx}	$Gr^{1/4} \cdot z$
C(i)	y^n	1	y^{2m-1}	$Gr_y^{1/4} \cdot \dfrac{z}{y}$
(ii)	e^{ny}	1	e^{2my}	$Gr^{1/4} \cdot z$
D(i)	1	1	x^{2m-1}	$Gr_x^{1/4} \cdot \dfrac{z}{x}$
(ii)	1	1	y^{2m-1}	$Gr_y^{1/4} \cdot \dfrac{z}{y}$

Result and Discussion:

The ordinary differential equations (3.1.17)-(3.1.19) are solved numerically by Sweggert iteration technique for $\delta = 45^0$. Dimensionless velocity and temperature profiles for the power law surface temperature case are presented in figures (1)-(3) respectively, for P_r =0.7, n = 0.1 and c = 0.1 with several values of m. The velocity profiles vary as usual with the parameter m. However, the temperature profiles for negative power (m=0 .1) differ notably in shape from the uniform wall temperature case (m=0.5). An unusual observation for m=0.1, we may infer that the surface receives heat from the fluid. Similar behaviour was noticed in 2-D situation also by Sparrow and Gregg [1955,1956,1958] for free convection over a vertical plate and by Schuh [1948] for forced convection over a plate with a power law surface temperature variation. For positive power, the temperature distributions are similar in shape to that of uniform wall temperature case.

Velocity profiles displayed in figures (4)-(5) & in table (1) show that the skin friction decreases as the power of the temperature increases. While the heat transfer factors are as usual as in Sparrow and Gregg[1958].

Representative velocity and temperature profiles for the power law curvature affect (different values of n) are shown in figures (7)-(9), for fixed values of P_r =0.7, m=0.5 and c=0.1. These figures shows the limitation of curvature affect.

Within the limit $-0.2 \le n \le 0.3$, the velocity and temperature distributions are regular. For negative values of n, the velocity distribution along y-direction is higher than the x-direction,so that , we find the variation of the skin friction at the edges in figures (10) & (11).

In our equation (3.1.17)-(3.1.19), if c=0.0, n=0.0, then the equation coincide with (6.9)-(6.11). In addition if we set m=0.5, then these equations are similar to the case defined by Ede,A.J.[1967].

The velocity and temperature profiles for different values of c are shown in fig.(13)-(15) and the associated skin friction and heat transfer factor are in figures (16)-(18) as well as in table(3).

Dimensionless velocity distributions along u and v direction for several values of Prandtl number, Pr are shown in fig (19) & (20). In this situation small Prandtl number, ($Pr \rightarrow 0$) generates large temperature distributions on the surface, shown in fig. (21). The variations of skin frictions ($f''(0), s''(0)$) are displayed in figures (22) & (23), heat transfer coefficients ($-\theta'(0)$) is shown in figure (24) for the variation of the fluid properties P_r (the Prandtl number). A numerical Table (4) displays the effects of skin friction factors and heat transfer coefficient with the variation of P_r.

Finally, the restricted variation in (x,y) of ΔT, h_1, h_2, under which the partial differential equation governing the natural convection flow in three dimentional curvilinear coordinates are reducible to ordinary differential equation, are displayed in table 5. This table also exhibits the nature of similarity variable in terms of modified Grashof number embeded with ΔT-variation.

Nomenclature

a,b,c	constants
C_p	specific heat at constant pressure
F,S	dimensionless scaled stream functions
f,s	dimensionless stream functions
g	acceleration due to gravity
h_1,h_2,h_3	scale factors for curvilinear surface
Gr_{xy}	modified Grashof number
K	constant
k	the coefficient of thermal diffusivity
L	characteristic length
m	temperature power/exponent parameter
n	power/exponent of h_1 & h_2
$Nu_{x,y}$	modified Nusselt number
P	Pressure
P_r	Prandtl number
q_w	heat flux
R_F	modified Reynolds number
T	temperature of fluid
T_∞	temperature of ambient fluid
T_w	surface temperature
u,v,w	velocity components in the boundary layer
$\overline{U}$	characteristic velocity generated by buoyancy effects
x,y	coordinates along the edges of surface
z	coordinate normal to surface

Greek letters

α	constant
β_T	the coefficient of volumetric expansion
δ	boundary layer thickness
δ_1	thermal boundary layer thickness
θ	dimensionless temperature function
Ψ, Φ	mass flow componants (stream functions)
Φ	dissipation function
φ	similarity variable
ν	the kinematic coefficient of viscosity
ρ	the density of ambient fluid
μ	coefficient of viscosity
κ	the coefficient of thermal diffusivity
τ_w	nondimensional skin friction
ξ, η, ζ	scaled coordinate defined in equations
γ	the square root of the local boundary layer thickness

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