SIMILARITY SOLUTIONS OF COMBINED FORCED AND FREE CONVECTIVE LAMINAR BOUNDARY LAYER FLOWS IN CURVILINEAR CO-ORDINATES

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ABSTRACT

Combined laminar forced and free convective three dimensional boundary layer flow over a vertical orthogonal curvilinear surface $z=0$, $x > 0$, $y > 0$, $h_1 > 0$, $h_2 > 0$, $h_3 = 1$ is discussed in different situations. The aim of this study is to look for the similarity solutions under some different conditions on the forcing velocity and temperature difference by which reductions of partial differential equations into ordinary differential equations are possible. The numerical solution of one set of the representative transformed equations for different values of controlling parameters $\frac{U_F^2}{U_e^2}$, $\frac{V_F^2}{V_e^2}$, m, n and ß, are obtained, where $h_1 = h_2 = 1$.

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CHAPTER I

Introduction

CHAPTER 1

INTRODUCTION

The natural flow originates from body force variations in fluids, whereas the forced convection is generally introduced by moving a body through a quiescent fluid or by forcing a fluid past astationary body. This flow regime is concerned with circumstances wherein both the natural and forced mechanisms of the flow must be considered simultaneously. The laminar boundary layer flow due to such combined forced and natural convection has received considerable attention for both steady and unsteady situations in evaluating flow parameters for technical purposes.

TANAEV (1956) published an approximate analytical analysis for laminar, compressible, mixed flow of air over an inclined flat plate. The external flow was assumed parallel to the plate for all plate inclination angles. His analysis is for low Mach numbers and for situations where the buoyancy effects could be considered small relative to the forced flow effects. ACRIVOS (1958) employed the Pohlhansen-Von Karman momentum integral method to consider incompressible laminar flow over a vertical isothermal plate with buoyancy effects. The form of the velocity and temperature profiles for the combined convection were assumed to be the sum of the purely forced and natural convection profiles. These assumptions are wrong due to the basic non-linearity of viscous flow problems. Numerical results were reported for the heating and cooling of upward flow past a vertical plate for Prandtl numbers of 0.73, 10, and 100. The influence of natural convection on separation flow showed that heating of the upward flow would stabilize the boundary layer, whereas cooling hastens the appearance of separsion.

SPARROW and GREGG (1959) attacked the isothermal vertical plate problem, considered by ACRIVOS, by a different approach. They perturbed the equations of the purely forced flow circumstance to include small effects of buoyancy. The study considered both the cases where the buoyancy effects are parallel and opposite to the external flow. The analysis was restricted to laminar boundary-layer flow and calculations were made for Prandtl number of 10, 1 and 0.01 respectively. For fluids with larger Prandtl numbers the effects of buoyancy were found to be of smaller importance in predicting the boundary layer characteristics.

SZEWCZYK (1963) also investigated combined flow over a vertical isothermal surface, considering two types of perturbations. First, purely forced flow was perturbed to include small buoyancy effects. This portion of the analysis is similar to that of SPARROW and GREGG (1959). However, SZEWCZYK considered the second-order term of perturbation series. The second part of the analysis considered a perturbation of the natural convection flow to include small forced flow effects. Results were tabulated for Prandtl numbers of 0.01, 0.72, 1.0, 5.0 and 10.0.

The case of laminar mixed flow over an iso-thermal horizontal flat plate was investigated by MORI (1961) and SPARROW and MINKOWYCZ (1962). MORI considered the case for a Prandtl number of 0.72 while SPARROW and MINKOWYCZ considered Prandtl numbers of 0.01, 0.7 and 10. In both the papers the technique of perturbing the purely forced flow equations to include small effects of buoyancy was applied. These investigations presented formulae for calculating the shear stress and heat transfer rate in the mixed flow regime.

GILL and DEL CASEL (1962) considered the influence of buoyancy for flow over a horizontal flat plate with non-uniform temperature at the surface. They found that the boundary-layer equations had a similarity solution for a surface temperature varying as one over the square root of distance from the leading edge. With this surface condition, however, there is no local heat transfer. The physical significance of this situation is not apparent. Prandtl numbers of 0.001, 0.72 and 10 were considered. For the plate temperature greater than that of the fluid external to the boundary-layer, the parallel velocity component was found to increase and temperature to decrease for the flow over the top of the plate, relative to the corresponding forced flow results. The opposite effects were encountered for the bottom of the plate.

MINKOWYCZ

These results are in agreement with those found MORI (1961) and SPARROW(1962). It was similarly observed that buoyancy has less influence as the Prandtl number is increased.

BRINDEY (1963) considered an approximate technique to solve the ordinary differential equations encountered by SPARROW, EICHHORN and GREGG (1959). His primary intent, however, was the comparison of his approximate method with respect to known numerical solutions.

GUNNESS and GEBHART (1965) considered mixed flow over isothermal wedges for a variety of included angles. The regime considered was incompressible, laminar, boundary-layer flow with Prandtl number of 0.73. Two types of wedge orientation were treated. The first case was discussed for horizontal wedge, i.e, the plane of symmetry normal to the body force and the second case was for vertical wedge.

MERKIN (1969) considered the boundary-layer flow over a semi-infinite vertical flat plate, heated to a constant temperature in a uniform free stream. He discussed two cases when the buoyancy forces aid and oppose the development of the boundary layer. In the former case, two series solutions were obtained, one of which was valid near the leading edge and other was valid asymptotically. In the latter case, a series, valid near the leading edge was obtained and it was extended by a numerical method to the point where the boundary layer was shown to separate.

COMBARNOUS and BIA (1971) considered the combined free and forced convection in a porous medium and had studied the effect of mean flow on the onset of stability in a porous medium bounded by two isothermal parallel plates. Numerical solutions were later obtained by HORNE and O'SULLIVAN (1974), CHENG and LAU (1977), and CHENG and TECKEHANDANI (1976) to study the effect of withdrawal of fluids in a hot-water geothermal reservoir. SCHROCK and LAIRD (1976) performed an experimental study on the simultaneous withdrawal and injection of fluids in a porous medium.

PING CHENG (1977) investigated the combined free and force convection boundary-layer flow along inclined surfaces embedded in porous media. It was found that similarity solutions exist when both the wall temperature distribution of the plate and the velocity parallel to the plate outside of the boundary-layer vary according to the same power function of distance, i.e x^{λ}. The value of $\frac{G_r}{R_e^2}$ was found to be the controlling parameter for the mixed convection from inclined plates in a porous medium. Numerical solutions were obtained for mixed convection from an iso-thermal vertical flat plate (i.e, λ=0) as well as an inclined plate with constant heat flux, having an angle of inclination 45^{o} (i.e, λ= 1/3).

RAJU, LIU and LAW (1984) considered the boundary-layer flow over semi-infinite vertical and horizontal flat plate, heated to a constant temperature in a uniform free stream. They discussed when the buoyancy forces either aid or oppose the development of the boundary-layer. Different mixed-convection parameters were introduced in the formation of the respective problems involving horizontal and vertical surfaces such that smooth transition from one convective limit to the other was possible; in particular the governing equations for the purely forced and free convection cases were respectively recovered from the zero values of Grashoff and Reynolds numbers.

So far the author's knowledge goes no attempt has yet been made for combined convection for three dimensional body.

The similarity solutions exist only for very special types of main stream flow and for special types of co-ordinate systems. The similarity solutions of boundary layer for the three dimensional case was studied by Hansen (1958). He presented similarity solutions of the three-dimensional, laminar incompressible boundary layer equations along with a general method of analysis. Restrictions on main flow velocity components and co-ordinate system which lead to similarity solutions were tabulated. Finally a discussion was given of the practical application of similarity solution.

HOWARTH (1951) discussed the equations of boundary layer flow in the vicinity of a stagnation point on a general three dimensional surface. He showed that the equations were reducible to a pair of simultaneous ordinary third-order differential equations containing a single parameter C related to the exponent of the mainstream flow velocity. The variation of C would be effectively limited to the range from 0 (corresponding to two dimensional flow) to 1 (corresponding to the axial flow past a body of revolution), and solutions were obtained for the cases of C=0.25, 0.50, 0.75. DEVEY and SCHOFIELD (1967) studied the same case and presented numerical result in tabular form.

DAVEY (1961) studied the flow of a viscous incompressible fluid in (6)
the immediate neighborhood of saddle point of a attachment, near which the external flow was irrotational with components [ax, by, -(a+b)z], where $a>0$, $b<0$. It was shown that the flow was of a boundary layer character, and that part of the boundary layer flow was reversed when $b/a < -0.4294$. Numerical solutions were presented in the tables and diagrams.

SOWERBY (1965) derived series expansion for the three-dimensional (7)
boundary layer flow over a flat plate, arising from general mainstream flow over the flat plate. The series involved were calculated as far as terms of the order of ξ^2, where ξ is a non-dimensional parameter related to distance measured from the leading edge of the plate. These calculated results were applied to an example in which the main stream arises from the disturbance of a uniform stream by

uniform stream by a circular cylinder mounted downstream from the leading edge of the plate, the axis of the cylinder being normal to the plate. Calculations were made for shear-stress components on the plate, and for the deviation of direction of the limiting streamlines from those in the main stream.

DWYER (1968) discussed a method of calculating accurate solutions of the three-dimensional laminar boundary layer equations. This method was applied to a problem that exhibits interesting cross flow phenomena. The method of solution used was an implicit finite difference scheme, and the stability and convergence properties of this scheme were found to be good. Also the important question on initial conditions for three-dimensional boundary layer flow was discussed, and a method of obtaining initial conditions was derived.

WANG (1974) discussed general methods for calculating three-dimensional laminar boundary layers over inclined blunt bodies (not necessarily bodies of revolution). He used the traditional integral method and presented complete incompressible boundary layer results for a prolate spheroid at 30° incidence.[8]

The similarity solution of combined forced and free convective three-dimensional laminar boundary-layer flows in curvilinear co-ordinate is more complicated in comparison with that of two dimensional boundary-layer flow. In the present study, discussion is confined about the mixed three dimensional boundary-layer flow over a vertical orthogonal surface. The three dimensional boundary-layer equations are developed for the curvilinear co-ordinate system and relevent partial differential equations are transformed into ordinary differential equations by the technique of similarity. Only one set of transformed equations are solved numerically to predict some essential parameters.

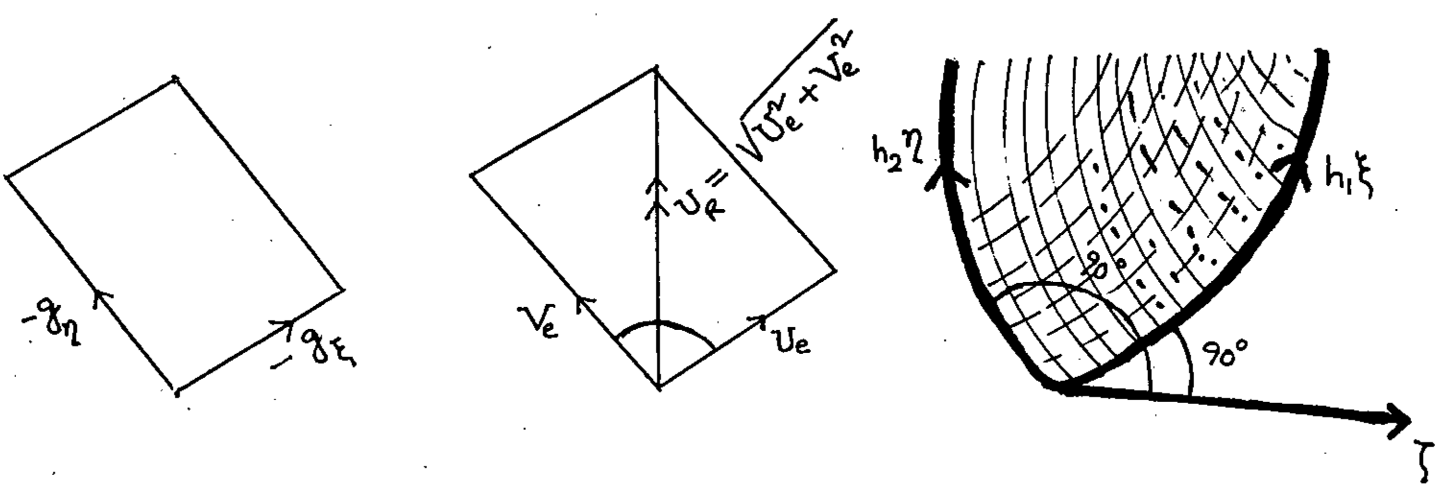
CHAPTER 2

Physical outline of the problem

CHAPTER 2

PHYSICAL OUTLINE OF THE PROBLEM

The situation discussed in the present problem is that of a resultant uniform free stream velocity $U_R(=\sqrt{U_e^2+V_e^2})$ flowing over a rectangular vertical curvilinear surface $\zeta = 0$, $h_1\,\xi \geq 0$, $h_2\eta \geq 0$ which is fixed with its leading edges perpendicular to ζ-axis. The body is heated to a temperature T_w above the ambient temperature T_e. Heat is transferred by convection from the plate, and this heating gives rise to buoyant body forces.



GOVERNING EQUATIONS

Heating due to viscous dissipation can be neglected and the fluid may be considered incompressible. The flow of the fluid may be considered steady state laminar . The co-efficient of viscosity μ and the thermometric conductivity k can be taken as constant. The Navier-Stoke's equation including the body force term and energy equation are in vector form

$$\nabla.\vec{q} = 0$$
$$\varrho(\vec{q}.\nabla)\vec{q} = -\nabla P + \vec{F} + \mu\nabla^2\vec{q} \qquad 1(a,b,c)$$
$$\varrho c_p(\vec{q}.\nabla)T = \kappa\nabla^2 T$$

where $\vec{q} = \vec{q}(u, v, w)$

body force $\vec{F} = \varrho\vec{g} = \varrho(g_x, g_y, g_z)$

ϱg_x ——> body force in the x-direction

ϱg_y ——> body force in the y-direction

ϱg_z ——> body force in the z-direction.

To transform the governing equations into curvilinear co-ordinate:

let $$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{q^2}{2}\right) - 2(vW_3 - wW_2)$$

$$a_y = \frac{\partial}{\partial y}\left(\frac{q^2}{2}\right) - 2(wW_1 - uW_3) \qquad 2(a,b,c)$$

and $\quad a_z = \frac{\partial}{\partial z}\left(\frac{q^2}{2}\right) - 2\,(uW_2 - vW_1).$

Hence

$$\vec{a} = (a_x,\ a_y,\ a_z) = \nabla\left(\frac{q^2}{2}\right) - 2\,(\vec{q}\times\vec{W}), \tag{3}$$

where $\nabla\left(\frac{q^2}{2}\right) = \sum i\frac{\partial}{\partial x}\left(\frac{u^2+v^2+w^2}{2}\right)$,

and vorticity vector $\vec{W} = (W_1,\ W_2,\ W_3)$,

$$W_1 = \frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),$$

$$W_2 = \frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right),$$

$$W_3 = \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right),$$

and $\quad 2\vec{q}\times\vec{W} = \sum 2\,\vec{i}\,(vW_3 - wW_2).$

Equation (3) becomes,

$$\frac{d\vec{q}}{dt} = \nabla\left(\frac{q^2}{2}\right)-2\,(\vec{q}\times\vec{W})$$

$$\therefore\quad (\vec{q}.\nabla)\,\vec{q} = \nabla\left(\frac{q^2}{2}\right)-2\,(\vec{q}\times\vec{W}), \tag{4}$$

where steady convective operator $\quad \frac{d}{dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$, and $\frac{d\vec{q}}{dt} = \vec{a}.$

Equation 1(b) becomes,

$$\nabla\left(\frac{q^2}{2}\right) - 2(\vec{q} \times \vec{W}) = -\frac{1}{\varrho}\nabla p + \vec{g} + \nu \nabla^2 \vec{q}. \tag{5}$$

Convective operator of u, v & w - components in curvilinear co-ordinates are,

$$\frac{du}{dt} = \frac{u}{h_1}\frac{\partial u}{\partial \xi} + \frac{v}{h_2}\frac{\partial u}{\partial \eta} + w\frac{\partial u}{\partial \zeta} + \frac{uw}{h_1 h_2}\frac{\partial}{\partial \zeta}(h_1 h_2)$$

$$+ \frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta} - \frac{1}{h_1 h_2}(v^2 + w^2)\frac{\partial h_2}{\partial \xi}, \tag{6a}$$

$$\frac{dv}{dt} = \frac{u}{h_1}\frac{\partial v}{\partial \xi} + \frac{v}{h_2}\frac{\partial v}{\partial \eta} + w\frac{\partial v}{\partial \zeta} + \frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi}$$

$$+ \frac{vw}{h_1 h_2}\frac{\partial}{\partial \zeta}(h_1 h_2) - \left(\frac{u^2 + w^2}{h_1 h_2}\right)\frac{\partial h_1}{\partial \eta}, \tag{6b}$$

$$\frac{dw}{dt} = \frac{u}{h_1}\frac{\partial w}{\partial \xi} + \frac{v}{h_2}\frac{\partial w}{\partial \eta} + w\frac{\partial w}{\partial \zeta} + \frac{uw}{h_1 h_2}\frac{\partial h_2}{\partial \xi}$$

$$+ \frac{vw}{h_1 h_2}\frac{\partial h_1}{\partial \eta} - \left(\frac{u^2 + v^2}{h_1 h_2}\right)\frac{\partial}{\partial \zeta}(h_1 h_2). \tag{6c}$$

Here $h_3(\xi,\eta) = 1$ has been set such that ζ represents an actual distance measured normal to the surface.

The equations governing the flow field in general curvilinear co-ordinates become,

$$\frac{\partial}{\partial \xi}(h_2 u) + \frac{\partial}{\partial \eta}(h_1 v) + \frac{\partial}{\partial \zeta}(h_1 h_2 w) = 0 \quad , \tag{7}$$

$$\frac{u}{h_1}\frac{\partial u}{\partial \xi} + \frac{v}{h_2}\frac{\partial u}{\partial \eta} + w\frac{\partial u}{\partial \zeta} + \frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta} + \frac{uw}{h_1 h_2}\frac{\partial}{\partial \zeta}(h_1 h_2)$$

$$- \left(\frac{v^2 + w^2}{h_1 h_2}\right)\frac{\partial h_2}{\partial \xi} = -\frac{1}{\varrho h_1}\frac{\partial p}{\partial \xi} + g_\xi + \nu\nabla^2 u \quad , \tag{8a}$$

$$\frac{u}{h_1}\frac{\partial v}{\partial \xi} + \frac{v}{h_2}\frac{\partial v}{\partial \eta} + w\frac{\partial v}{\partial \zeta} + \frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi} + \frac{vw}{h_1 h_2}\frac{\partial}{\partial \zeta}(h_1 h_2)$$

$$- \left(\frac{u^2 + w^2}{h_1 h_2}\right)\frac{\partial h_1}{\partial \eta} = -\frac{1}{\varrho h_2}\frac{\partial p}{\partial \eta} + g_\eta + \nu\nabla^2 v \quad , \tag{8b}$$

$$\frac{u}{h_1}\frac{\partial w}{\partial \xi} + \frac{v}{h_2}\frac{\partial w}{\partial \eta} + w\frac{\partial w}{\partial \zeta} + \frac{uw}{h_1 h_2}\frac{\partial h_2}{\partial \xi} + \frac{vw}{h_1 h_2}\frac{\partial h_1}{\partial \eta}$$

$$- \left(\frac{u^2 + v^2}{h_1 h_2}\right)\frac{\partial}{\partial \zeta}(h_1 h_2) = -\frac{1}{\varrho}\frac{\partial p}{\partial \zeta} + g_\zeta + \nu\nabla^2 w \quad , \tag{8c}$$

and

$$\frac{u}{h_1}\frac{\partial T}{\partial \xi} + \frac{v}{h_2}\frac{\partial T}{\partial \eta} + w\frac{\partial T}{\partial \zeta} = \frac{\kappa}{\varrho c_p}\nabla^2 T \tag{9}$$

Where

$$\nabla^2 u = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \xi}\left(\frac{h_2}{h_1}\frac{\partial u}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{h_1}{h_2}\frac{\partial u}{\partial \eta}\right) + \frac{\partial}{\partial \zeta}\left(h_1 h_2 \frac{\partial u}{\partial \zeta}\right)\right]$$

Similarly, $\nabla^2 v$, $\nabla^2 w$ and $\nabla^2 T$.

For convenience dimensionless quantities have been introducted into the equations (7), (8) and (9) in the following way:

All lengths have been referred to a representative length 'L', the velocity has been made dimensionless with reference to the resultant free stream velocity U_R $(=\sqrt{U_e^2 + V_e^2})$, the density with respect to ϱ_e and the pressure with reference to $\varrho_e U_R^2$. The temperature in the energy equation (Eq.9) has been made dimensionless with reference to the temperature difference $\Delta T\,(=T_w - T_e)$ between the wall and the fluid at the large distance from the body. Thus, the non-dimensional temperature function is,

$$\theta = \frac{T-Te}{\Delta T}.$$

Thus the dimensionless quantities primes are:

$$\xi = L\xi', \quad \eta = L\eta', \quad \zeta = L\zeta', \quad u = u'U_R, \quad v = v'U_R,$$

$$w = w'U_R, \quad \varrho = \varrho_e\varrho', \quad p = p'\varrho_e U_R^2, \quad T-Te = \theta\Delta T, \quad \mu = \mu_o\mu',$$

$$k = k_o k', \quad g_\xi = g g_\xi{}', \quad g_\eta = g g_\eta', \quad \text{and} \quad g_\zeta = g g_\zeta'.$$

Hence the equations (7), (8) and (9) in dimensionless form are,

$$\frac{\partial}{\partial \xi'}\,(u'h_2) + \frac{\partial}{\partial \eta'}\,(h_1 v') + \frac{\partial}{\partial \zeta'}\,(h_1 h_2 w') = 0 \qquad (10)$$

$$\varrho'[\frac{u'}{h_1}\frac{\partial u'}{\partial \xi'} + \frac{v'}{h_2}\frac{\partial u'}{\partial \eta'} + w'\frac{\partial u'}{\partial \zeta'} + \frac{u'v'}{h_1h_2}\frac{\partial h_1}{\partial \eta'} + \frac{u'w'}{h_1h_2}\frac{\partial}{\partial \zeta'}(h_1h_2)$$

$$-(\frac{v'^2 + w'^2}{h_1h_2})\frac{\partial h_2}{\partial \xi'}] = -\frac{\partial p'}{\partial \xi'} + \frac{g'_\xi \varrho'}{F_r}$$

$$+ \frac{\mu'}{h_1h_2R_e}[\frac{\partial}{\partial \xi'}(\frac{h_2}{h_1}\frac{\partial u'}{\partial \xi'}) + \frac{\partial}{\partial \eta'}(\frac{h_1}{h_2}\frac{\partial u'}{\partial \eta'}) + \frac{\partial}{\partial \zeta'}(h_1h_2\frac{\partial u'}{\partial \zeta'})], \qquad (11a)$$

$$\varrho'[\frac{u'}{h_1}\frac{\partial v'}{\partial \xi'} + \frac{v'}{h_2}\frac{\partial v'}{\partial \eta'} + w'\frac{\partial v'}{\partial \zeta'} + \frac{u'v'}{h_1h_2}\frac{\partial h_2}{\partial \xi'} + \frac{v'w'}{h_1h_2}\frac{\partial (h_1h_2)}{\partial \zeta'}$$

$$-(\frac{v'^2+w'^2}{h_1h_2})\frac{\partial h_1}{\partial \eta'}] = -\frac{\partial p'}{\partial \eta'} + \frac{g'_\eta \varrho'}{F_r}$$

$$+ \frac{\mu'}{h_1h_2R_e}[\frac{\partial}{\partial \xi'}(\frac{h_2}{h_1}\frac{\partial v'}{\partial \xi'}) + \frac{\partial}{\partial \eta'}(\frac{h_1}{h_2}\frac{\partial v'}{\partial \eta'}) + \frac{\partial}{\partial \zeta'}(h_1h_2\frac{\partial v'}{\partial \zeta'})],$$

(11b)

$$\varrho'[\frac{u'}{h_1}\frac{\partial w'}{\partial \xi'} + \frac{v'}{h_2}\frac{\partial w'}{\partial \eta'} + w'\frac{\partial w'}{\partial \zeta'} + \frac{u'w'}{h_1h_2}\frac{\partial h_2}{\partial \xi'}$$

$$-(\frac{u'^2+v'^2}{h_1h_2})\frac{\partial}{\partial \zeta'}(h_1h_2) + \frac{v'w'}{h_1h_2}\frac{\partial h_1}{\partial \eta'}] = -\frac{\partial p'}{\partial \zeta'} + \frac{g'_\zeta \varrho}{F_r}$$

$$+ \frac{\mu'}{h_1h_2R_e}[\frac{\partial}{\partial \xi'}(\frac{h_2}{h_1}\frac{\partial w'}{\partial \xi'}) + \frac{\partial}{\partial \eta'}(\frac{h_1}{h_2}\frac{\partial w'}{\partial \eta'}) + \frac{\partial}{\partial \zeta'}(h_1h_2\frac{\partial w'}{\partial \zeta'})], \qquad (11c)$$

and

$$\varrho' c'_p [(\frac{u'}{h_1} \frac{\partial \theta}{\partial \xi'} + \frac{v'}{h_2} \frac{\partial \theta}{\partial \eta'} + w' \frac{\partial \theta}{\partial \zeta}) + \theta \{ \frac{u'}{h_1} (\ln \Delta T)_{\xi'} + \frac{v'}{h_2} (\ln \Delta T)_{\eta'} \}]$$

$$= \frac{k'}{P_r R_e} \frac{1}{h_1 h_2} [\{ \frac{\partial}{\partial \xi'} (\frac{h_2}{h_1} \frac{\partial \theta}{\partial \xi'}) + \frac{\partial}{\partial \eta'} (\frac{h_1}{h_2} \frac{\partial \theta}{\partial \eta'}) + \frac{\partial}{\partial \zeta'} (h_1 h_2 \frac{\partial \theta}{\partial \zeta'}) \}$$

$$+ \frac{\theta}{\Delta T} \{ \frac{\partial}{\partial \xi'} (\frac{h_2}{h_1} \frac{\partial}{\partial \xi'} (\Delta T)) + \frac{\partial}{\partial \eta'} (\frac{h_1}{h_2} \frac{\partial \Delta T}{\partial \eta'}) \}], \qquad (12)$$

where,

Prandtl number $P_r = \frac{\mu_o C_{p_o}}{k_o}$,

co-efficient of kinemetic viscosity $\nu_o = \frac{\mu_o}{\varrho_o}$,

Reynold number $R_e = \frac{U_R L}{\nu_o} = \frac{\varrho_o U_R L}{\mu_o}$

and Froude number $F_r = \frac{U_R^2}{gL}$.

If δ be the boundary layer thickness, then the dimensionless boundary layer thickness is $\delta' = \frac{\delta}{L} << 1$ Since L > > 1.

Order of magnitude of each of the terms in equations, (10) , (11), and (12) are estimated, so that very small terms can be neglected.

Since $\frac{\partial u'}{\partial \xi'}$ is of O(1), $\frac{\partial u'}{\partial \eta'}$ is of O(1), $\frac{\partial v'}{\partial \xi'} \sim 0(1)$, $\frac{\partial v'}{\partial \eta'} \sim 0(1)$ then also $\frac{\partial w'}{\partial \zeta'} \sim O(1)$, since ζ' is of order δ', Similarly $w' \sim 0(\delta')$.

Therefore $\frac{\partial^2 u'}{\partial \xi'^2} \sim 0(1)$, $\frac{\partial^2 v'}{\partial \xi'^2} \sim 0(1)$, $\frac{\partial^2 u'}{\partial \eta'^2} \sim 0(1)$, $\frac{\partial^2 v'}{\partial \eta'^2} \sim 0(1)$.

Since the maximum values of each u' and v' is 1.

$\frac{\partial^2 u'}{\partial \zeta'^2} \sim 0(1/\delta'^2)$, $\frac{\partial^2 v'}{\partial \zeta'^2} \sim 0(1/\delta'^2)$, $\frac{\partial^2 w'}{\partial \zeta'^2} \sim 0(1/\delta')$,

$\frac{\partial w'}{\partial \zeta'} \sim 0(1)$, $\frac{\partial w'}{\partial \xi'} \sim 0(\delta')$, $\frac{\partial w'}{\partial \eta'} \sim 0(\delta')$, $R_e \sim 0(1/\delta'^2)$ and $F_r \sim 0(1)$.

Now, since $\delta' << 1$.

$\frac{\partial^2 w'}{\partial \xi'^2}$, $\frac{\partial^2 w'}{\partial \eta'^2}$, $\frac{\partial w'}{\partial \xi'}$ and $\frac{\partial w'}{\partial \eta'}$ can be neglected.

Outside the boundary layer Bernoulli's equation

$\frac{p}{\varrho} + \frac{1}{2}(U_e^2 + V_e^2)$ = constant,

gives $-\frac{1}{\varrho}\frac{\partial p}{\partial \xi} = U_e \frac{\partial U_e}{\partial \xi} + V_e \frac{\partial V_e}{\partial \xi}$,

and $-\frac{1}{\varrho}\frac{\partial p}{\partial \eta} = U_e \frac{\partial U_e}{\partial \eta} + V_e \frac{\partial V_e}{\partial \eta}$.

The dimensionless form of these equations are,

$$-\frac{1}{\varrho'}\frac{\partial p'}{\partial \xi'} = U'_e\frac{\partial U'_e}{\partial \xi'}+V'_e\frac{\partial V'_e}{\partial \xi'}$$

and $$-\frac{1}{\varrho'}\frac{\partial p'}{\partial \eta'} = U'_e\frac{\partial U'_e}{\partial \eta'}+V'_e\frac{\partial V'_e}{\partial \eta'}.$$

Thus $-\frac{1}{\varrho'}\frac{\partial p'}{\partial \xi'}$, and $-\frac{1}{\varrho'}\frac{\partial p'}{\partial \eta'}$ are of order $U'_e\frac{\partial U'_e}{\partial \xi'}+V'_e\frac{\partial V'_e}{\partial \xi'}$ and $U'_e\frac{\partial U'_e}{\partial \eta'}+V'_e\frac{\partial V'_e}{\partial \eta'}$ respectively.

Hence $-\frac{1}{\varrho'}\frac{\partial p'}{\partial \xi'}$, and $-\frac{1}{\varrho'}\frac{\partial p'}{\partial \eta'}$ are of $O(1)$.

Let δ_T be the thermal boundary layer thickness, the conduction term becomes of the same order of magnitude as the convectional term, only if the thickness of the thermal boundary layer is of the order of $(\frac{\delta_T}{L})^2 \sim \frac{1}{R_e P_r}$.

In view of the previously obtained estimation for the thickness of the velocity boundary layer $\delta \sim \frac{1}{\sqrt{R_e}}$, it is found that $\frac{\delta_T}{\delta} \sim \frac{1}{\sqrt{P_r}}$.

Assuming that h_1, h_2 and all their first derivative of 0(1) and setting the order of magnitude in each terms of equations (10), (11) and (12), one obtains

$$\frac{\partial}{\partial \xi'}(h_2 u') + \frac{\partial}{\partial \eta'}(h_1 v') + \frac{\partial}{\partial \zeta'}(h_1 h_2 w') = 0 \ , \tag{13}$$

0 ⟶ 1 1 1

$$\varrho'[\frac{u'}{h_1}\frac{\partial u'}{\partial \xi'} + \frac{v'}{h_2}\frac{\partial u'}{\partial \eta'} + w'\frac{\partial u'}{\partial \zeta'} + \frac{u'v'}{h_1 h_2}\frac{\partial h_1}{\partial \eta'} + \frac{u'w'}{h_1 h_2}\frac{\partial}{\partial \zeta'}(h_1 h_2)$$

0 ⟶ 1 1 1 1 1 δ'

$$- (\frac{v'^2 + w'^2}{h_1 h_2})\frac{\partial h_2}{\partial \xi'}] = - \frac{\partial p'}{\partial \xi'} + \frac{g'_\xi \rho'}{F_r} + \frac{\mu'}{h_1 h_2 R_e}[\frac{\partial}{\partial \xi'}(\frac{h_2}{h_1}\frac{\partial u'}{\partial \xi'})$$

1 δ'^2 1 1 1 δ'^2 1

$$+ \frac{\partial}{\partial \eta'}(\frac{h_1}{h_2}\frac{\partial u'}{\partial \eta'}) + \frac{\partial}{\partial \zeta'}(h_1 h_2)\frac{\partial u'}{\partial \zeta'}) + (h_1 h_2)\frac{\partial^2 u'}{\partial \zeta'}] \ , \tag{14a}$$

1 $1/\delta'$ $1/\delta'^2$

$$\rho'[\frac{u'}{h_1}\frac{\partial v'}{\partial \xi'} + \frac{v'}{h_2}\frac{\partial v'}{\partial \eta'} + w'\frac{\partial v'}{\partial \zeta'} + \frac{v'w'}{h_1 h_2}\frac{\partial}{\partial \zeta'}(h_1 h_2) + \frac{u'v'}{h_1 h_2}\frac{\partial h_2}{\partial \xi'}$$

$0 \longrightarrow 1 \quad 1 \quad 1 \quad 1 \quad \delta' \quad 1$

$$-(\frac{u'^2+w'^2}{h_1 h_2})\frac{\partial h_1}{\partial \eta'}] = -\frac{\partial p'}{\partial \eta'} + \frac{g'_\eta \rho'}{F_r} + \frac{\mu'}{h_1 h_2 R_e}[\frac{\partial}{\partial \xi'}(\frac{h_2}{h_1}\frac{\partial v'}{\partial \xi'})$$

$1 \quad \delta'^2 \quad 1 \qquad 1 \quad 1 \qquad \delta'^2 \quad 1$

$$+\frac{\partial}{\partial \eta'}(\frac{h_1}{h_2}\frac{\partial v'}{\partial \eta'}) + \frac{\partial}{\partial \zeta'}(h_1 h_2)\frac{\partial v'}{\partial \zeta'}) + h_1 h_2 \frac{\partial^2 v'}{\partial \zeta'^2}], \tag{14b}$$

$1 \qquad 1/\delta' \qquad 1/\delta'^2$

$$\rho'[\frac{u'}{h_1}\frac{\partial w'}{\partial \xi'} + \frac{v'}{h_2}\frac{\partial w'}{\partial \eta'} + w'\frac{\partial w'}{\partial \zeta'} + \frac{u'w'}{h_1 h_2}\frac{\partial h_2}{\partial \xi'} + \frac{v'w'}{h_1 h_2}\frac{\partial h_1}{\partial \eta'}$$

$0 \longrightarrow 1 \quad \delta' \quad \delta' \quad \delta' \quad \delta' \quad \delta'$

$$-(\frac{u'^2+v'^2}{h_1 h_2})\frac{\partial}{\partial \zeta'}(h_1 h_2)] = -\frac{\partial p'}{\partial \zeta'} + \frac{g'_\zeta \rho'}{F_r} + \frac{\mu'}{h_1 h_2 R_e}[\frac{\partial}{\partial \xi'}(\frac{h_2}{h_1}\frac{\partial w'}{\partial \xi'})$$

$1 \quad 1 \quad 1 \qquad 1/\delta' \quad 1 \qquad \delta'^2 \quad \delta'$

$$+\frac{\partial}{\partial \eta'}(\frac{h_1}{h_2}\frac{\partial w'}{\partial \eta'}) + \frac{\partial}{\partial \zeta'}(h_1 h_2)\frac{\partial w'}{\partial \zeta'} + h_1 h_2 \frac{\partial^2 w'}{\partial \zeta^2}], \tag{14c}$$

$\delta' \qquad 1 \qquad 1/\delta'$

$$\varrho' C_p' [(\frac{u'}{h_1} \frac{\partial \theta}{\partial \xi'} + \frac{v'}{h_2} \frac{\partial \theta}{\partial \eta'} + w' \frac{\partial \theta}{\partial \zeta'}) + \theta \{ \frac{u'}{h_1} (\ln \Delta T)_{\xi'} + \frac{v'}{h_2} (\ln \Delta T)_{\eta'} \}]$$

0 ⟶ 1   1   1   1   1   1

$$= \frac{\kappa'}{P_r R_e} \frac{1}{h_1 h_2} [\{ \frac{\partial}{\partial \xi'} (\frac{h_2}{h_1} \frac{\partial \theta}{\partial \xi'}) + \frac{\partial}{\partial \eta'} (\frac{h_1}{h_2} \frac{\partial \theta}{\partial \eta'}) + h_1 h_2 \frac{\partial}{\partial \zeta'} (\frac{\partial \theta}{\partial \zeta'})$$

δ'^2_T 1   1 $1/\delta'^2_T$

$$+ \frac{\partial}{\partial \zeta'} (h_1 h_2) \frac{\partial \theta}{\partial \zeta'} \} + \frac{\theta}{\Delta T} \{ \frac{\partial}{\partial \xi'} (\frac{h_2}{h_1} \frac{\partial \Delta T}{\partial \xi'}) + \frac{\partial}{\partial \eta'} (\frac{h_1}{h_2} \frac{\partial \Delta T}{\partial \eta'}) \}] . \qquad (15)$$

1   $1/\delta'_T$ 1   1   1

Order of each term of equation (14c) when multiplied by δ', then order of all the terms in equation (14c) become above δ' except $\frac{\partial p'}{\partial \zeta'}$. Hence equation (14c) can be neglected compared to the equation (13), 14(a,b) and (15). Furthermore, the term $\frac{1}{R_e} \frac{\partial^2 u'}{\partial \xi'^2}$; $\frac{1}{R_e} \frac{\partial^2 u'}{\partial \eta'^2}$; $\frac{1}{R_e} \frac{\partial^2 v'}{\partial \xi'^2}$ and $\frac{1}{R_e} \frac{\partial v'}{\partial \eta'^2}$ are of order δ'^2 also, $\frac{1}{R_e} \frac{\partial}{\partial \zeta'} (h_1 h_2) \frac{\partial u'}{\partial \zeta'}$ and $\frac{1}{R_e} \frac{\partial}{\partial \zeta'} (h_1 h_2) \frac{\partial v'}{\partial \zeta'}$ are of order δ'. These terms can be neglected compared to the other terms in equations 14(a,b).

Again, $\frac{\kappa'}{P_r R_e} \frac{1}{h_1 h_2} \frac{\partial}{\partial \xi'} (\frac{h_2}{h_1} \frac{\partial \theta}{\partial \xi'})$ and $\frac{\kappa'}{P_r R_e} \frac{1}{h_1 h_2} \frac{\partial}{\partial \eta'} (\frac{h_1}{h_2} \frac{\partial \theta}{\partial \eta'})$ are of order δ'^2_T also $\frac{\kappa'}{R_e P_r} \frac{1}{h_1 h_2} \frac{\partial}{\partial \zeta'} (h_1 h_2) \frac{\partial \theta'}{\partial \zeta'}$ is of order δ'_T. These can be neglected compared to the other terms of equations (15). Thus the simplified governing equations of the flow field in general orthogonal curvilinear co-ordinates become:

$$\frac{\partial}{\partial \xi}(h_2 u) + \frac{\partial}{\partial \eta}(h_1 v) + \frac{\partial}{\partial \zeta}(h_1 h_2 w) = 0, \tag{16}$$

$$\varrho(\frac{u}{h_1} \frac{\partial u}{\partial \xi} + \frac{v}{h_2} \frac{\partial u}{\partial \eta} + w\frac{\partial u}{\partial \zeta} + \frac{uv}{h_1 h_2} \frac{\partial h_1}{\partial \eta} - \frac{v^2}{h_1 h_2} \frac{\partial h_2}{\partial \xi})$$

$$= - \frac{1}{h_1} \frac{\partial p}{\partial \xi} + \varrho g_\xi + \mu \frac{\partial^2 u}{\partial \zeta^2}, \tag{17a}$$

$$\varrho(\frac{u}{h_1} \frac{\partial v}{\partial \xi} + \frac{v}{h_2} \frac{\partial v}{\partial \eta} + w\frac{\partial v}{\partial \zeta} + \frac{uv}{h_1 h_2} \frac{\partial h_2}{\partial \xi} - \frac{u^2}{h_1 h_2} \frac{\partial h_1}{\partial \eta})$$

$$= - \frac{1}{h_2} \frac{\partial p}{\partial \eta} + \rho g_\eta + \mu \frac{\partial^2 v}{\partial \zeta^2}, \tag{17b}$$

$$\varrho(\frac{u}{h_1} \frac{\partial T}{\partial \xi} + \frac{v}{h_2} \frac{\partial T}{\partial \eta} + w\frac{\partial T}{\partial \zeta}) = \frac{\mu}{P_r} \frac{\partial^2 T}{\partial \zeta^2}, \tag{18}$$

and $\frac{\partial p}{\partial \zeta} = 0(1).$ (19)

The potential flow in the main stream outside the boundary layer is the function of (ξ,η). Hence external velocity components U_e and V_e and external temperature T_e (= constant) be independent of ζ.

To elliminate pressure terms in equations 17(a,b), the conditions outside the boundary layer are imposed. Imposing the boundary conditions $u \longrightarrow U_e$, $v \longrightarrow V_e$, $\varrho \longrightarrow \varrho_e$, $T \longrightarrow T_e$, $\frac{\partial}{\partial\zeta} \longrightarrow 0$ the following equations are left for onward study:

$$\varrho_e\left[\frac{U_e}{h_1}\frac{\partial U_e}{\partial\xi} + \frac{V_e}{h_2}\frac{\partial U_e}{\partial\eta} + \frac{U_e V_e}{h_1 h_2}\frac{\partial h_1}{\partial\eta} - \frac{V_e^2}{h_1 h_2}\frac{\partial h_2}{\partial\xi}\right] = -\frac{1}{h_1}\frac{\partial p}{\partial\xi} + g_\xi \varrho_e, \qquad 20(a)$$

$$\varrho_e\left[\frac{U_e}{h_1}\frac{\partial V_e}{\partial\xi} + \frac{V_e}{h_2}\frac{\partial V_e}{\partial\eta} + \frac{U_e V_e}{h_1 h_2}\frac{\partial h_2}{\partial\xi} - \frac{U_e^2}{h_1 h_2}\frac{\partial h_1}{\partial\eta}\right] = -\frac{1}{h_2}\frac{\partial p}{\partial\eta} + g_\eta \varrho_e \qquad 20(b)$$

and
$$\varrho_e\left(\frac{U_e}{h_1}\frac{\partial T_e}{\partial\xi} + \frac{V_e}{h_2}\frac{\partial T_e}{\partial\eta}\right) = 0. \qquad (21)$$

But the outer temperature T_e is constant,

so $\frac{\partial T_e}{\partial\xi} = \frac{\partial T_e}{\partial\eta} = 0.$

In view of 20(a, b), (21) and $\frac{T-T_e}{T_w-T_e}=\theta$, $T_w-T_e=\Delta T$, the boundary layer equations with the elimination of pressure terms (by 20(a) and 20(b)) become

$$\frac{\partial}{\partial\xi}(h_2 u) + \frac{\partial}{\partial\eta}(h_1 v) + \frac{\partial}{\partial\zeta}(h_1 h_2 w) = 0 \ , \tag{22}$$

$$\varrho\left[\frac{u}{h_1}\frac{\partial u}{\partial\xi} + \frac{v}{h_2}\frac{\partial u}{\partial\eta} + w\frac{\partial u}{\partial\zeta} + \frac{uv}{h_1h_2}\frac{\partial h_1}{\partial\eta} - \frac{v^2}{h_1h_2}\frac{\partial h_2}{\partial\xi}\right]$$

$$= g_\xi(\rho-\rho_e) + \rho_e\left[\frac{U_e}{h_1}\frac{\partial U_e}{\partial\xi} + \frac{V_e}{h_2}\frac{\partial U_e}{\partial\eta} + \frac{U_eV_e}{h_1h_2}\frac{\partial h_1}{\partial\eta}\right.$$

$$\left. - \frac{V_e^2}{h_1h_2}\frac{\partial h_2}{\partial\xi}\right] + \mu\frac{\partial^2 u}{\partial\zeta^2} \ , \tag{23a}$$

$$\varrho\left[\frac{u}{h_1}\frac{\partial v}{\partial\xi} + \frac{v}{h_2}\frac{\partial v}{\partial\eta} + w\frac{\partial v}{\partial\zeta} + \frac{uv}{h_1h_2}\frac{\partial h_2}{\partial\xi} - \frac{u^2}{h_1h_2}\frac{\partial h_1}{\partial\eta}\right]$$

$$= (\varrho-\varrho_e)\, g_\eta + \varrho_e\left[\frac{U_e}{h_1}\frac{\partial V_e}{\partial\xi} + \frac{V_e}{h_2}\frac{\partial V_e}{\partial\eta} + \frac{U_eV_e}{h_1h_2}\frac{\partial h_2}{\partial\xi}\right.$$

$$\left. - \frac{U_e^2}{h_1h_2}\frac{\partial h_1}{\partial\eta}\right] + \mu\frac{\partial^2 v}{\partial\zeta^2} \ , \tag{23b}$$

and

$$\left(\frac{u}{h_1}\frac{\partial\theta}{\partial\xi} + \frac{v}{h_2}\frac{\partial\theta}{\partial\eta} + w\frac{\partial\theta}{\partial\zeta}\right) + \theta\left[\frac{u}{h_1}(\ln\Delta T)_\xi + \frac{v}{h_2}(\ln\Delta T)_\eta\right] = \frac{\nu}{P_r}\frac{\partial^2\theta}{\partial\zeta^2} \ . \tag{24}$$

where ν is the kinematic viscosity of the fluid.

Let $\varrho = \varrho(P, T)$ be the equation of state

i.e, $d\varrho = \frac{\partial\varrho}{\partial p}]_T\, dP + \frac{\partial\varrho}{\partial T}]_P\, dT$.

Then, $d\varrho = \varrho k dp - \varrho\beta_T\, dT$, (25)

where, $\beta_T = -\frac{1}{\varrho}(\frac{\partial\varrho}{\partial T})_{P=const}$, $k = \frac{1}{\varrho}(\frac{\partial\varrho}{\partial P})_{T=const}$. (26a,b)

For ideal gas $\beta_T = \frac{1}{T}$.

In non-dimensional form (25) may be written

as $\frac{d\varrho'}{\varrho'} = -\beta_T \Delta T d\theta + k P_e \frac{\varrho_e U_R^2}{P_e} dP'$. (27a)

$[\because\ \varrho = \varrho_e \varrho', \quad P = \varrho_e U_R^2 P', \quad T - T_e = \Delta T\theta]$

In the case of slow motion for a gas $k\ P_e \sim 0(1)$, $\frac{\varrho_e U_R^2}{P_e} \sim 0(M^2) << 1$,

$C = \sqrt{\nu \frac{P_e}{\varrho_e}}$ = the speed of sound, and $M = \frac{U_R}{C}$.

In case of liquid $\frac{\varrho_e U_R^2}{P_e}$ is more significant than for a gas.

However for a liquid $k\ P_0 << 1$,

Hence the equation (27a) becomes,

$$\frac{d\varrho'}{\varrho'} = -\beta_T \Delta T \theta \quad \text{or} \quad \rho = \rho(T) \tag{28}$$

$$= -\beta_T dT.$$

i.e, Dropping the primes $ln(\frac{\varrho}{\varrho_r}) = -\beta_T(T-T_r)$

$$\text{or} \quad \frac{\varrho}{\varrho_r} = e^{-\beta_T(T-T_r)}$$

$$= 1-\beta_T(T-T_r)$$

$$\therefore \varrho = \varrho_r\{1-\beta_T(T-T_r)\} . \tag{29a}$$

Similarely, $\varrho_e = \varrho_r\{1 - \beta_T(T_e - T_r)\}$. (29b)

Then it may be written that

$$(\varrho - \varrho_e) = -\varrho_r \beta_T (T - T_e) \quad [\because T_r = T_e, \; \varrho \simeq \varrho_r]$$

$$= -\varrho \beta_T \Delta T \theta \tag{29c}$$

considered to be the Boussinesq approximation.

Hence the simplified boundary layer equations become

$$\frac{\partial}{\partial\xi}(h_2 u) + \frac{\partial}{\partial\eta}(h_1 v) + \frac{\partial}{\partial\zeta}(h_1 h_2 w) = 0, \tag{30a}$$

$$\frac{u}{h_1}\frac{\partial u}{\partial\xi} + \frac{v}{h_2}\frac{\partial u}{\partial\eta} + w\frac{\partial u}{\partial\zeta} + \frac{uv}{h_1h_2}\frac{\partial h_1}{\partial\eta} - \frac{v^2}{h_1h_2}\frac{\partial h_2}{\partial\xi} = -g_\xi \beta_T \Delta T \theta$$

$$+ \frac{U_e}{h_1}\frac{\partial U_e}{\partial\xi} + \frac{V_e}{h_2}\frac{\partial U_e}{\partial\eta} + \frac{U_eV_e}{h_1h_2}\frac{\partial h_1}{\partial\eta} - \frac{V_e^2}{h_1h_2}\frac{\partial h_2}{\partial\xi} + \nu\frac{\partial^2 u}{\partial\xi^2}, \tag{30b}$$

$$\frac{u}{h_1}\frac{\partial v}{\partial \xi} + \frac{v}{h_2}\frac{\partial v}{\partial \eta} + w\frac{\partial v}{\partial \zeta} + \frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi} - \frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta} = - g_\eta \beta_T \Delta T \theta$$

$$+ \frac{U_e}{h_1}\frac{\partial V_e}{\partial \xi} + \frac{V_e}{h_2}\frac{\partial V_e}{\partial \eta} + \frac{U_e V_e}{h_1 h_2}\frac{\partial h_2}{\partial \xi} - \frac{U_e^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta} + \nu\frac{\partial^2 v}{\partial \zeta^2} , \qquad (30c)$$

and

$$\frac{u}{h_1}\frac{\partial \theta}{\partial \xi} + \frac{v}{h_2}\frac{\partial \theta}{\partial \eta} + w\frac{\partial \theta}{\partial \zeta} + \theta\left[\frac{u}{h_1}(\ln \Delta T)_\xi + \frac{v}{h_2}(\ln \Delta T)_\eta\right] = \frac{\nu}{P_r}\frac{\partial^2 \theta}{\partial \zeta^2} \qquad (30d)$$

CHAPTER 3

Similarity transformations

CHAPTER 3

SIMILARITY TRANSFORMATIONS

Equations (30) are non-linear, simultaneous partial differential equations and the solutions of these equations are extremely difficult to obtain. Hence our aim is to reduce equations 30(a,b,c,) to ordinary differential equations with the help of 30(a) which permits possible variations in ΔT, U_e, V_e, h_1, and h_2 with respect to ξ and η.

Let the variables ξ, η and ζ be changed to a new set of variables X, Y and $\overline{\phi}$, where relations between two sets of variables are given by,

$$X = \xi, \quad Y = \eta, \quad \overline{\phi} = \frac{\zeta}{\gamma(X, Y)} \tag{31}$$

$\gamma(X, Y)$ is considered here to be prportional to the square root of the local boundary layer thickness.

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial X} \cdot \frac{\partial X}{\partial \xi} + \frac{\partial}{\partial Y} \cdot \frac{\partial Y}{\partial \xi} + \frac{\partial}{\partial \overline{\phi}} \cdot \frac{\partial \overline{\phi}}{\partial \xi}$$

$$\therefore \frac{\partial}{\partial \xi} = \frac{\partial}{\partial X} - \frac{\overline{\phi}}{\gamma} \gamma_X \frac{\partial}{\partial \overline{\phi}} \tag{32a}$$

where $\frac{\partial X}{\partial \xi} = 1, \qquad \frac{\partial \overline{\phi}}{\partial \xi} = -\frac{\overline{\phi}}{\gamma}\gamma_X$

Similarly,

$$\frac{\partial}{\partial\eta} = \frac{\partial}{\partial Y} - \frac{\overline{\Phi}}{\gamma}\gamma_Y \frac{\partial}{\partial\overline{\Phi}} \tag{32b}$$

and $$\frac{\partial^2}{\partial\zeta^2} = \frac{1}{\gamma^2}\frac{\partial^2}{\partial\overline{\Phi}^2} \tag{32c}$$

Let two stream functions $\psi(\xi,\eta,\zeta)$ and $\varphi(\xi,\eta,\zeta)$ be defined as the mass flow components within the boundary layer for the case of incompressible flow.

The following equations can be written

$$\psi_\zeta = h_2 u, \qquad \varphi_\zeta = h_1 v \;,$$
$$-\quad (\psi_\xi + \varphi_\eta) = h_1 h_2 w \tag{33a,b,c}$$

to satisfy the equation of continuity (30a). Guided by the idea of similarity procedure of Hansen (1964) we are allowed to write

$$\int_0^{\overline{\Phi}} \frac{u}{U(X,Y)}\, d\overline{\Phi} = F(X,Y,\overline{\Phi})$$

and $$\int_0^{\overline{\Phi}} \frac{v}{V(X,Y)}\, d\overline{\Phi} = S(X,Y,\overline{\Phi}). \tag{34a,b}$$

In attempting seperation of variables of $F(X,Y,\overline{\Phi})$, $S(X,Y,\overline{\Phi})$ and $\theta(X,Y,\overline{\Phi})$. Let it be assumed that

$$F(X,Y,\overline{\Phi}) = L(X,Y)\,\overline{F}(\overline{\Phi}),$$
$$S(X,Y,\overline{\Phi}) = M(X,Y)\,\overline{S}(\overline{\Phi}), \tag{35(a,b,c)}$$
$$\theta(X,Y,\overline{\Phi}) = N(X,Y)\,\overline{\vartheta}(\overline{\Phi}),$$

where $\overline{F}$, $\overline{S}$, and $\overline{\vartheta}$ are the functions of the single variable $\overline{\Phi}$. From (34), and 35 (a,b) it is found that

$$\frac{u}{U} = F_{\phi} = L\overline{F}_{\overline{\phi}} \quad \therefore \quad u = UL\overline{F}_{\overline{\phi}},$$

36(a,b)

$$\frac{v}{V} = S_{\phi} = M\overline{S}_{\overline{\phi}} \quad \therefore \quad v = VM\overline{S}_{\overline{\phi}}.$$

Again, from 33(a)

$$\psi_{\zeta} = h_2 u \Rightarrow \frac{\partial\psi}{\partial\zeta} = h_2 u \Rightarrow \frac{u}{U} = \frac{1}{h_2}\frac{\partial}{\partial\zeta}\left(\frac{\psi}{U(\xi,\eta)}\right)$$

$$\Rightarrow \frac{u}{U} = \frac{1}{h_2\gamma(X,Y)}\cdot\frac{\partial}{\partial\overline{\phi}}\left(\frac{\psi}{U}\right)$$

$$= \frac{\partial}{\partial\overline{\phi}}\left(\frac{\psi}{h_2\gamma U}\right).$$

$$\therefore \int_0^{\overline{\phi}} \frac{u}{U}\, d\overline{\phi} = \frac{1}{h_2\gamma U}\left[\psi(X,Y,\overline{\phi}) - \psi(X,Y,0)\right] . \tag{37}$$

From 34(a) and (37) expression for $F(X,Y,\overline{\phi})$ and $\psi(X,Y,\overline{\phi})$ are

$$F(X,Y,\overline{\phi}) = \frac{1}{h_2\gamma U}\left[\psi(X,Y,\overline{\phi}) - \psi(X,Y,0)\right]$$

$$\therefore \psi(X,Y,\overline{\phi}) = h_2\gamma UL\overline{F}(\overline{\phi}) + \psi(X,Y,0) \tag{38}$$

Similarly,

$$\varphi(X,Y,\overline{\phi}) = h_1\gamma VM\overline{S}(\overline{\phi}) + \varphi(X,Y,0) \tag{39}$$

and

$$h_1 h_2 w = -(h_2\gamma UL\overline{F})_X - \psi_X(X,Y,0) + \frac{\overline{\phi}}{\gamma}\gamma_X h_2\gamma UL\overline{F}_{\overline{\phi}}$$

$$-(h_1\gamma VM)_Y\overline{S}(\overline{\phi}) - \varphi_Y(X,Y,0) + \frac{\overline{\phi}}{\gamma}\gamma_Y h_1\gamma VM\overline{S}_{\overline{\phi}} \tag{40}$$

If Lim $\overline{\Phi} \rightarrow 0$, then.

$$w_0(X,Y,0) = -\frac{1}{h_1 h_2}\left[\psi_X(X,Y,0) + \phi_Y(X,Y,0)\right] \tag{41}$$

When the surface is porous, w_0 represents the suction or injection velocity normal to the surface, Since U_e and V_e are independent of ζ, Hence $(U_e)_{\overline{\Phi}} = (V_e)_{\overline{\Phi}} = 0$.

So equation (40) becomes,

$$\begin{aligned} h_1 h_2 w &= -(h_2 \gamma UL)_X \overline{F} + \overline{\Phi}\gamma_X h_2 UL\overline{F}_{\overline{\Phi}} - (h_1 \gamma VM)_Y \overline{S} \\ &\quad + \overline{\Phi}\gamma_Y h_1 VM\overline{S}_{\overline{\Phi}} + h_1 h_2 w(X,Y,0) . \end{aligned} \tag{42}$$

The convective operator

$$\frac{d}{dt} = \frac{1}{h_1 h_2}\left[h_2 u\frac{\partial}{\partial \xi} + h_1 v\frac{\partial}{\partial \eta} + h_1 h_2 w\frac{\partial}{\partial \zeta}\right] \tag{43}$$

in terms of new set of variables X, Y and $\overline{\Phi}$ may be derived. Thus the convective operator is ,

$$\begin{aligned} h_1 h_2 \frac{d}{dt} &= h_2 UL\overline{F}_{\overline{\Phi}}\frac{\partial}{\partial X} + h_1 MV\overline{S}_{\overline{\Phi}}\frac{\partial}{\partial Y} \\ &\quad - \frac{1}{\gamma}\left[(h_2 \gamma UL)_X \overline{F} + (h_1 \gamma VM)_Y \overline{S} + h_1 h_2 w_o\right]\frac{\partial}{\partial \overline{\Phi}} . \end{aligned} \tag{44}$$

Using equation (44), Equations 30(b,c,d) become,

u - momentum equation:

$$\nu \bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{\gamma(\gamma h_2 UL)_X}{h_1 h_2} \bar{F} \bar{F}_{\bar{\phi}\bar{\phi}} + \frac{\gamma(\gamma h_1 VM)_Y}{h_1 h_2} \bar{S} \bar{F}_{\bar{\phi}\bar{\phi}} + \gamma w_o \bar{F}_{\bar{\phi}\bar{\phi}}$$

$$- \frac{\gamma^2}{h_1}(UL)_X \bar{F}_{\bar{\phi}}^2 - \frac{MV}{h_2}\gamma^2 \left[\frac{(UL)_Y}{UL} + \frac{h_{1Y}}{h_1}\right] \bar{F}_{\bar{\phi}} \bar{S}_{\bar{\phi}}$$

$$+ \frac{\gamma^2}{h_1 h_2 UL} V^2 M^2 h_{2X} \bar{S}_{\bar{\phi}}^2 - \frac{\gamma^2}{UL} g_X \beta_T N \theta (\Delta T) + \frac{\gamma^2}{h_1 UL} U_e U_{eX}$$

$$+ \frac{\gamma^2}{h_2 UL} V_e U_{eY} + \frac{\gamma^2}{h_1 h_2 UL} U_e V_e h_{1Y} - \frac{\gamma^2}{h_1 h_2 UL} h_{2X} V_e^2 = 0 , \qquad (45)$$

v - momentum equation:

$$\nu \bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{\gamma}{h_1 h_2} (h_1 \gamma MV)_Y \bar{S} \bar{S}_{\bar{\phi}\bar{\phi}} + \frac{\gamma}{h_1 h_2} (h_2 \gamma UL)_X \bar{F} \bar{S}_{\bar{\phi}\bar{\phi}}$$

$$+ \gamma w_o \bar{S}_{\bar{\phi}\bar{\phi}} - \frac{\gamma^2}{h_2} (MV)_Y \bar{S}_{\bar{\phi}}^2 - \frac{\gamma^2 UL}{h_1} \left[\frac{(MV)_X}{MV} + \frac{h_{2X}}{h_2}\right] \bar{F}_{\bar{\phi}} \bar{S}_{\bar{\phi}}$$

$$+ \frac{\gamma^2}{h_1 h_2 MV} L^2 U^2 h_{1Y} \bar{F}_{\bar{\phi}}^2 - \frac{\gamma^2}{MV} g_Y \beta_T N \theta (\Delta T) + \frac{\gamma^2}{h_1 MV} U_e V_{eX}$$

$$+ \frac{\gamma^2}{h_1 MV} V_e V_{eY} + \frac{\gamma^2}{h_1 h_2 MV} U_e V_e h_{2X} - \frac{\gamma^2 h_{1Y}}{h_1 h_2 MV} U_e^2 = 0 , \qquad (46)$$

Energy equation:

$$\frac{\nu}{P_r}\,\bar{\vartheta}_{\bar{\Phi}\bar{\Phi}} + \frac{\gamma}{h_1 h_2}\,(h_2 \gamma\, UL)_X \bar{F}\,\bar{\vartheta}_{\bar{\Phi}} + \frac{\gamma}{h_1 h_2}\,(h_1 \gamma\, VM)_Y\, \bar{S}\,\bar{\vartheta}_{\bar{\Phi}}$$

$$- \gamma w_o \bar{\vartheta}_{\bar{\Phi}} - \frac{\gamma^2 UL}{h_1}\,[(\ln N)_X + (\ln \Delta T)_X]\,\bar{\vartheta}\,\bar{F}_{\bar{\Phi}}$$

$$- \frac{\gamma^2}{h_2} MV[(\ln N)_Y + (\ln \Delta T)_Y]\,\bar{\vartheta}\,\bar{S}_{\bar{\Phi}} = 0. \qquad (47)$$

Here the boundary conditions which must be imposed to determine the solutions of the equations (45), (46) and (47) are,

(a) The fluid adheres to the transformed surface and this surface must be a stream surface. However, if the developable surface be porous then mathematically on this surface,

$$u(X, Y, 0) = 0 = \bar{F}_{\bar{\Phi}}(0),$$

$$v(X, Y, 0) = 0 = \bar{S}_{\bar{\Phi}}(0),$$

$$w(X, Y, 0) = -w_o,$$

where w_o is considered here to be the surface suction or injection velocity for the developable surface, then

$$\theta(X, Y, 0) = N(X, Y)\,\bar{\vartheta}(0) = 1 \rightarrow N(X, Y) = 1 \quad and \quad \bar{\vartheta}(0) = 1.$$

(b) The fluid at a large distance from the said surface must be undisturbed by the presence of the boundary layer. Then

$$\operatorname*{Lt}_{\bar{\phi}\to\infty} u(X,Y,\bar{\phi}) = U_e = UL\bar{F}_{\bar{\phi}}(\infty) \rightarrow UL = U_e,\ \bar{F}_{\bar{\phi}}(\infty) = 1\ ,$$

$$\operatorname*{Lt}_{\bar{\phi}\to\infty} v(X,Y,\bar{\phi}) = V_e = VM\bar{S}_{\bar{\phi}}(\infty) \rightarrow VM = V_e;\quad \bar{S}_{\bar{\phi}}(\infty) = 1\ ,$$

$$\operatorname*{Lt}_{\bar{\phi}\to\infty} \theta(X,Y,\bar{\phi}) = 0 = N(X,Y)\,\bar{\vartheta}(\infty) \rightarrow \bar{\vartheta}(\infty) = 0,\ N \neq 0\ .$$

In conditions (a) and (b) if general boundary conditions $[\bar{F}(0) = \bar{S}(0) = 0,\ \ \bar{\vartheta}(0) = 1$ and $\bar{F}_{\bar{\phi}}(\infty) = \bar{S}_{\bar{\phi}}(\infty) = 1,\ \ \bar{\vartheta}(\infty) = 0\,]$ be introduced, without loss of generality it may be written that

$UL = U_e$ and $VM = V_e$.

Then the two momentum equations (45), (46) and the energy equation (47) take the following forms:

u- momentum euqtion:

$$\nu\bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{\gamma\,(\gamma h_2 U_e)_X}{h_1 h_2}\bar{F}\bar{F}_{\bar{\phi}\bar{\phi}} + \frac{\gamma\,(\gamma h_1 V_e)_Y}{h_1 h_2}\bar{S}\bar{F}_{\bar{\phi}\bar{\phi}}$$

$$+\ \gamma w_o \bar{F}_{\bar{\phi}\bar{\phi}} - \frac{\gamma^2}{h_1}U_{eX}\bar{F}^2_{\bar{\phi}} - \frac{V_e}{h_1}\gamma^2\left[\frac{U_{eY}}{U_e} + \frac{h_{1Y}}{h_1}\right]\bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}}$$

$$+\ \frac{\gamma^2}{h_1 h_2}\frac{V_e^2}{U_e}h_{2X}\bar{S}^2_{\bar{\phi}} - \frac{\gamma^2}{U_e}g_X\,\beta_T(\Delta T)\,\bar{\vartheta} + \frac{\gamma^2}{h_1}U_{eX}$$

$$+\ \frac{\gamma^2}{h_2 U_e}V_e U_{eY} + \frac{\gamma^2}{h_1 h_2}V_e h_{1Y} - \frac{\gamma^2}{h_1 h_2}\frac{V_e^2}{U_e}h_{2X} = 0,$$

or, $$\nu \bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{\gamma\,(\gamma h_2\, U_e)_X}{h_1 h_2}\, \bar{F}\, \bar{F}_{\bar{\phi}\bar{\phi}} + \frac{\gamma\,(\gamma h_1\, V_e)_Y}{h_1 h_2}\, \bar{S}\, \bar{F}_{\bar{\phi}\bar{\phi}}$$

$$+ \gamma w_o \bar{F}_{\bar{\phi}\bar{\phi}} + \frac{\gamma^2\, U_{eX}}{h_1}(1 - \bar{F}^2_{\bar{\phi}}) - \frac{\gamma^2}{h_1 h_2}\, \frac{V_e^2}{U_e}\, h_{2X}(1 - \bar{S}^2_{\bar{\phi}})$$

$$+ \frac{\gamma^2 V_e}{h_2}\left(\frac{U_{eY}}{U_e} + \frac{h_{1Y}}{h_1}\right)(1 - \bar{F}_{\bar{\phi}}\, \bar{S}_{\bar{\phi}}) - \frac{\gamma^2}{U_e}\, g_X \beta_T\, (\Delta T)\, \bar{\vartheta} = 0 \qquad 48(a)$$

v-momentum equation:

$$\nu \bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{\gamma\,(\gamma\, h_1\, V_e)_Y}{h_1\, h_2}\, \bar{S}\, \bar{S}_{\bar{\phi}\bar{\phi}} + \frac{\gamma}{h_1\, h_2}\,(h_2\, \gamma\, U_e)_X\, \bar{F}\, \bar{S}_{\bar{\phi}\bar{\phi}}$$

$$+ \gamma w_o \bar{S}_{\bar{\phi}\bar{\phi}} + \frac{\gamma^2}{h_2}\, V_{eY}\, (1 - \bar{S}^2_{\bar{\phi}}) + \frac{\gamma^2\, U_e}{h_1}\left(\frac{V_{eX}}{V_e} + \frac{h_{2X}}{h_2}\right)(1 - \bar{F}_{\bar{\phi}}\, \bar{S}_{\bar{\phi}})$$

$$- \frac{\gamma^2\, U_e^2}{h_1\, h_2\, V_e}\, h_{1Y}(1 - \bar{F}^2_{\bar{\phi}}) - \frac{\gamma^2}{V_e}\, g_Y\, \beta_T\, (\Delta T)\, \bar{\vartheta} = 0 \qquad 48(b)$$

Energy equation:

$$\frac{\nu}{P_r}\, \bar{\vartheta}_{\bar{\phi}\bar{\phi}} + \frac{\gamma\,(h_2\, \gamma\, U_e)_X}{h_1\, h_2}\, \bar{F}\, \bar{\vartheta}_{\bar{\phi}} + \frac{\gamma}{h_1\, h_2}\,(\gamma h_1\, V_e)_Y\, \bar{S}\, \bar{\vartheta}_{\bar{\phi}}$$

$$+ \gamma w_o\, \bar{\vartheta}_{\bar{\phi}} - \frac{\gamma^2\, U_e}{h_1}\,(\ln \Delta T)_X\, \bar{\vartheta}\, \bar{F}_{\bar{\phi}} - \frac{\gamma^2\, V_e}{h_2}\,(\ln \Delta T)_Y\, \bar{\vartheta}\, \bar{S}_{\bar{\phi}} = 0 \qquad 48(c)$$

But

$$\frac{\gamma\,(\gamma h_2 U_e)_X}{h_1 h_2} = \frac{1}{2}\,[\,(\frac{U_e\,\gamma^2}{h_1})_X + \frac{\gamma^2\,(U_e h_2)_X}{h_1 h_2} - \gamma^2 h_2 U_e (\frac{1}{h_1 h_2})_X]$$

and

$$\frac{\gamma\ (\gamma h_1\ V_e)_Y}{h_1\ h_2} = \frac{1}{2}\,[\,(\frac{\gamma^2 V_e}{h_2})_Y + \frac{\gamma^2\,(h_1 V_e)_Y}{h_1 h_2} - \gamma^2 h_1 V_e (\frac{1}{h_1 h_2})_Y] \,.$$

Therefore, the momentum and energy equations become,

u- momentum equation:

$$\nu\,\overline{F}_{\Phi\Phi\Phi} + \frac{1}{2}\,(a_0 + a_1 - a_2)\,\overline{F}\,\overline{F}_{\Phi\Phi} + \frac{1}{2}(a_3 + a_4 - a_5)\,\overline{S}\,\overline{F}_{\Phi\Phi} + a_6\,\overline{F}_{\Phi\Phi}$$

$$+\ a_7(1-\overline{F}_{\Phi}^2) + (a_8 + a_9)\,(1-\overline{F}_{\Phi}\,\overline{S}_{\Phi}) - a_{10}(1-\overline{S}_{\Phi}^2) + a_{11}\,\overline{\vartheta} = 0\,, \quad (49)$$

v-momentum euqation:

$$\nu\,\overline{S}_{\Phi\Phi\Phi} + \frac{1}{2}\,(a_0 + a_1 - a_2)\,\overline{F}\,\overline{S}_{\Phi\Phi} + \frac{1}{2}\,(a_3 + a_4 - a_5)\,\overline{S}\,\overline{S}_{\Phi\Phi}$$

$$+\ a_6\,\overline{S}_{\Phi\Phi} + a_{12}(1 - \overline{S}_{\Phi}^2) + (a_{13} + a_{14})\,(1 - \overline{F}_{\Phi}\,\overline{S}_{\Phi})$$

$$-\ a_{15}\,(1 - \overline{F}_{\Phi}^2) + a_{16}\,\overline{\vartheta} = 0\,, \quad (50)$$

energy equation:

$$\frac{\nu}{P_r}\,\overline{\vartheta}_{\Phi\Phi} + \frac{1}{2}\,(a_0 + a_1 - a_2)\,\overline{F}\,\overline{\vartheta}_{\Phi} + \frac{1}{2}(a_3 + a_4 - a_5)\,\overline{S}\,\overline{\vartheta}_{\Phi} + a_6\,\overline{\vartheta}_{\Phi}$$

$$-(a_{17}\,\overline{\vartheta}\,\overline{F}_{\Phi} + a_{18}\,\overline{\vartheta}\,\overline{S}_{\Phi}) = 0\,, \quad (51)$$

where a_o, a_1, a_2 ..... a_{18} are the function of X, and Y given by,

(i) $a_0 = \left(\frac{\gamma^2 U_e}{h_1}\right)_X$,

(ii) $a_1 = \gamma^2 (h_2 U_e)_X \frac{1}{h_1 h_2}$,

(iii) $a_2 = \gamma^2 (h_2 U_e) \left(\frac{1}{h_1 h_2}\right)_X$,

(iv) $a_3 = \left(\frac{\gamma^2 V_e}{h_2}\right)_Y$,

(v) $a_4 = \gamma^2 (h_1 V_e)_Y \cdot \frac{1}{h_1 h_2}$,

(vi) $a_5 = \gamma^2 h_1 V_e \left(\frac{1}{h_1 h_2}\right)_Y$,

(vii) $a_6 = \gamma w_o$,

(viii) $a_7 = \frac{1}{h_1} \gamma^2 U_{eX}$,

(ix) $a_8 = \frac{1}{h_2 U_e} V_e \gamma^2 U_{eY}$,

(x) $a_9 = \frac{1}{h_1 h_2} V_e \gamma^2 h_{1Y}$,

(xi) $a_{10} = \frac{1}{h_1 h_2} \frac{\gamma^2 V_e^2 h_{2X}}{U_e}$,

(xii) $a_{11} = -\frac{\gamma^2}{U_e} g_X \beta_T (\Delta T)$,

(xiii) $a_{12} = \frac{1}{h_2} \gamma^2 V_{eY}$,

(xiv) $a_{13} = \frac{1}{h_1} \cdot \frac{U_e \gamma^2 V_{eX}}{V_e}$,

(xv) $a_{14} = \gamma^2 U_e \frac{h_{2X}}{h_1 h_2}$, (52)

(xvi) $a_{15} = \frac{\gamma^2 U_e^2}{V_e} \cdot \frac{1}{h_1 h_2} \cdot h_{1Y}$,

(xvii) $a_{16} = -\frac{\gamma^2}{V_e} g_Y \beta_T (\Delta T)$,

(xviii) $a_{17} = \frac{1}{h_1} \gamma^2 U_e (\ln \Delta T)_X$,

(xix) $a_{18} = \frac{1}{h_2} \gamma^2 V_e (\ln \Delta T)_Y$.

Similar solutions for (49), (50) and (51) exist only when all the a's are finite and independent of X and Y; that is to say that all a's must be constants. Thus the boundary layer momentum equations and the energy equation will become non-linear ordinary differential equations, if $U_e(X,Y)$, $V_e(X,Y)$, $h_1(X,Y)$, $h_2(X,Y)$ and $\gamma(X,Y)$. satisfy the equations (52).

To find U_e, V_e and γ in different situations for simplicity we first ignore the injection or suction effects i.e, $a_6 = 0$.
From the expressions for a's, one have

$$a_1 + a_2 = \gamma^2 \left(\frac{U_e}{h_1}\right)_X \tag{53(a)}$$

Similarly,

$$a_4 + a_5 = \gamma^2 \left(\frac{V_e}{h_2}\right)_Y \tag{53(b)}$$

From 52(i),

$$a_0 = \left(\frac{\gamma^2 U_e}{h_1}\right)_X = \gamma^2 \left(\frac{U_e}{h_1}\right)_X + 2\gamma\gamma_X \frac{U_e}{h_1}$$

$$\Rightarrow 2\gamma\gamma_X = \frac{h_1}{U_e}(a_0 - a_1 - a_2) \tag{53(c)}$$

Similarly,

$$2\gamma\gamma_Y = \frac{h_2}{V_e}(a_3 - a_4 - a_5) \tag{53(d)}$$

By virtue of equation 52(i),

$$\left(\frac{\gamma^2 U_e}{h_1}\right)_X = a_0$$

which, when integrated gives

$$\frac{\gamma^2 U_e}{h_1} = a_0 X + A(Y) \, , \qquad 54(i)$$

where A(Y) is either constant or function of Y only.

Differentiating 54(i) with respect to Y, and in view of similarity requirements (54)

$$\frac{dA}{dY} = \left(\frac{\gamma^2 U_e}{h_1}\right)_Y$$

$$= \gamma^2 \left(\frac{U_e}{h_1}\right)_Y + \frac{U_e}{h_1} 2 \gamma \gamma_Y$$

$$= \gamma^2 \left(\frac{h_1 U_{eY} - U_e h_{1Y}}{h_1^2}\right) + \frac{U_e}{h_1} 2 \gamma \gamma_Y$$

$$= \gamma^2 \frac{h_2 U_e}{h_1 V_e \gamma^2} a_8 - \frac{\gamma^2 U_e h_1 h_2}{h_1^2 V_e \gamma^2} a_9 + \frac{U_e}{h_1} \cdot \frac{h_2}{V_e} (a_3 - a_4 - a_5)$$

$$\therefore \frac{dA}{dY} = \frac{h_2}{h_1} \cdot \frac{U_e}{V_e} (a_3 - a_4 - a_5 + a_8 - a_9) \qquad 54(ii)$$

In view of equation 52(iv)

$$\frac{\gamma^2 V_e}{h_2} = a_3 Y + B(X) \qquad 54(iii)$$

where B(X) is either constant or function of X only. Differentiating 54(iii) with respect to X, it is found that

$$\frac{dB}{dX} = \frac{V_e}{U_e} \cdot \frac{h_1}{h_2}(a_0 - a_1 - a_2 + a_{13} - a_{14}) . \qquad (55)$$

Taking the product of 54(ii) and (55) it is found that

$$\frac{dA}{dY} \cdot \frac{dB}{dX} = (a_3 - a_4 - a_5 + a_8 - a_9)\,(a_0 - a_1 - a_2 + a_{13} - a_{14}) . \qquad (55a)$$

The form of similarity solution, the scale factors $U_e(X,Y)$, $V_e(X,Y)$ and $\gamma(X,Y)$ depend wholly on the equation (55a). This situation leads to the following four possibilities:

A. Both $\frac{dA}{dY}$ and $\frac{dB}{dX}$ are equal to zero

B. $\frac{dA}{dY} \neq 0$ but $\frac{dB}{dX} = 0$,

C. $\frac{dA}{dY} = 0$ but $\frac{dB}{dX} \neq 0$

D. Both $\frac{dA}{dY}$ and $\frac{dB}{dX}$ are finite and constant separately.

CHAPTER 4

Study of different cases

(For rectangular surfaces)

CHAPTER 4

STUDIES OF DIFFERENT CASES

$[\ h_1 = h_2 = 1\ ]$

For rectangular surfaces equations (52) become ,

$$
\begin{array}{lll}
\text{(i)} & a_0 = (\gamma^2 U_e)_X , & \\
\text{(ii)} & a_1 = \gamma^2 U_{eX} , & \\
\text{(iii)} & a_2 = 0 , & \\
\text{(iv)} & a_3 = (\gamma^2 V_e)_Y , & \\
\text{(v)} & a_4 = \gamma^2 V_{eY} , & \\
\text{(vi)} & a_5 = 0 , & \\
\text{(vii)} & a_6 = 0 , & \\
\text{(viii)} & a_7 = a_1 , & (56) \\
\text{(ix)} & a_8 = \gamma^2 \dfrac{V_e}{U_e} U_{eY} , & \\
\text{(x)} & a_9 = 0 , & \\
\text{(xi)} & a_{10} = 0 , & \\
\text{(xii)} & a_{11} = - \dfrac{1}{U_e} \gamma^2 g_X \beta_T (\Delta T) , & \\
\text{(xiii)} & a_{12} = a_4 . &
\end{array}
$$

(xiv) $a_{13} = \frac{U_e}{V_e}\gamma^2 V_{eX}$,

(xv) $a_{14} = 0$,

(xvi) $a_{15} = 0$,

(xvii) $a_{16} = -\frac{1}{V_e}\gamma^2 g_Y \beta_T \Delta T$,

(xviii) $a_{17} = \gamma^2 U_e (\ln \Delta T)_X$,

(xix) $a_{18} = \gamma^2 V_e (\ln \Delta T)_Y$.

CASE-A(i)

Let $\frac{dA}{dY}$ and $\frac{dB}{dX}$ both be zero, then,

A = constant and B = constant.

And equations 54(ii) and (55) become,

$$\frac{U_e}{V_e}\,(a_3 - a_4 + a_8) = 0$$

57(i,ii)

and $$\frac{V_e}{U_e}\,(a_0 - a_1 + a_{13}) = 0$$

But $\frac{U_e}{V_e}$ being not equal to zero, so

$$a_3 - a_4 + a_8 = 0$$

58(i,ii)

$$a_0 - a_1 + a_{13} = 0$$

From equations 54(i) and 54(iii) it is found that

$$\gamma^2 = \frac{1}{U_e}\,(a_0 X + A)$$

and 59(i,ii)

$$\gamma^2 = \frac{1}{V_e}\,(a_3 Y + B)$$

From equation (59),

$$\frac{U_e}{V_e} = \frac{(a_0 X + A)}{(a_3 Y + B)} \tag{60}$$

In view of 56(ii) and 59(i)

$$a_1 = \gamma^2 U_{ex} = \frac{1}{U_e} U_{ex} (a_0 X + A)$$

or $$\frac{U_{ex}}{U_e} = \frac{a_1}{a_0 X + A}$$

i.e, $$U_e = K_1(Y) (a_0 X + A)^{\frac{a_1}{a_0}} \qquad 61(i)$$

Similarly, from equations 56(v) and 59(ii)

$$V_e = K_2(X) (a_3 Y + B)^{\frac{a_4}{a_3}} \qquad 61(ii)$$

where $K_1(Y)$ and $K_2(X)$ are integrating constants,

From equations, (60) and (61)

$$\frac{U_e}{V_e} = \frac{(a_0 X + A)}{(a_3 Y + B)} = \frac{K_1(Y) (a_0 X + A)^{\frac{a_1}{a_0}}}{K_2(X) (a_3 Y + B)^{\frac{a_4}{a_3}}}.$$

Comparing with (60), one obtains

$$\frac{K_2(X)}{K_1(Y)} = \frac{(a_0 X + A)^{\frac{a_1}{a_0} - 1}}{(a_3 Y + B)^{\frac{a_4}{a_3} - 1}}. \tag{62}$$

Assuming $\frac{a_1}{a_0}=m$, $\frac{a_4}{a_3}=n$, it can be shown that,

$$K_1(Y) = (a_3 Y + B)^{n-1},$$

and $K_2(X) = (a_0 X + A)^{m-1}.$

Hence,

$$U_e = (a_0 X + A)^m (a_3 Y + B)^{n-1} \qquad 63(a)$$

and $V_e = (a_0 X + A)^{m-1} (a_3 Y + B)^n.$ 63(b)

From equations (58)

$$a_8 = (n-1)a_3 \quad \text{and} \quad a_{13} = (m-1)a_o.$$

The similarity requirements furnish the relation between constants (a's). These relations are,

$$a_2=a_5=a_6=a_9=a_{10}=a_{14}=a_{15}=0,$$

$$a_1=ma_o, a_4=na_3, a_7=ma_o, a_8=(n-1)a_3,$$

$$a_{12}=na_3, a_{13}=(m-1)a_o, a_{17}=(2m-1)a_o,$$

$$a_{18}=(2n-1)a_3,$$

where a_o and a_3 are arbitrary.

Hence the general transformed equations take the form

u-momentum:

$$\nu \bar{F}_{\Phi\Phi\Phi} + \frac{a_o}{2}(m+1) \bar{F} \bar{F}_{\Phi\Phi} + \frac{a_3}{2}(n+1) \bar{S} \bar{F}_{\Phi\Phi} + ma_o(1 - \bar{F}_{\Phi}^2)$$

$$+ a_3(n-1)(1 - \bar{F}_{\Phi} \bar{S}_{\Phi}) + a_{11} \bar{\vartheta} = 0, \qquad 64(a)$$

v-momentum:

$$\nu \bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{1}{2} a_o (m+1)\ \bar{F}\, \bar{S}_{\bar{\phi}\bar{\phi}} + \frac{1}{2}(n+1)\, a_3\, \bar{S}\, \bar{S}_{\bar{\phi}\bar{\phi}} + n a_3 (1 - \bar{S}^2_{\bar{\phi}})$$

$$+\ a_o(m-1)\ (1 - \bar{F}_{\bar{\phi}}\, \bar{S}_{\bar{\phi}}) + a_{16}\, \bar{\vartheta} = 0, \qquad 64(b)$$

energy equation:

$$\frac{\nu}{P_r}\, \bar{\vartheta}_{\bar{\phi}\bar{\phi}} + \frac{1}{2}\, a_o(m+1)\ \bar{F}\, \bar{\vartheta}_{\bar{\phi}} + \frac{1}{2}\ (n+1)\ a_3\, \bar{S}\, \bar{\vartheta}_{\bar{\phi}} + a_0\ (1-2m)\ \bar{\vartheta}\, \bar{F}_{\bar{\phi}}$$

$$+\ a_3\,(1-2n)\ \bar{S}_{\bar{\phi}}\, \bar{\vartheta} = 0, \qquad 64(c)$$

subject to the boundary conditions.

$$\bar{F}(0) = \bar{F}_{\bar{\phi}}(0) = 0, \qquad \bar{F}_{\bar{\phi}}(\infty) = 1,$$

$$\bar{S}(0) = \bar{S}_{\bar{\phi}}(0) = 0, \qquad \bar{S}_{\bar{\phi}}(\infty) = 1, \qquad (65)$$

$$\bar{\vartheta}(0) = 1, \qquad \bar{\vartheta}(\infty) = 0.$$

Substitution of $\bar{F} = \alpha_1 f$, $\bar{S} = \alpha_2 s$, $\bar{\vartheta} = \vartheta$ and $\bar{\phi} = \alpha_3 \phi$ in the above equations we have

u-momentum equation,

$$f_{\phi\phi\phi} + \left(\frac{m+1}{2}\right) \frac{\alpha_1 \alpha_3\, a_o}{\nu}\, f f_{\phi\phi} + \left(\frac{n+1}{2}\right) \frac{a_3\, \alpha_2\, \alpha_3}{\nu}\, s\, f_{\phi\phi}$$

$$+\ m\left(\frac{a_o\, \alpha_3^3}{\alpha_1 \nu} - \frac{\alpha_1 \alpha_3}{\nu}\, a_o f_\phi^2\right) + (n-1)\ \left(\frac{\alpha_3^3}{\alpha_1} - \alpha_2 \alpha_3\, f_\phi\, s_\phi\right) \frac{a_3}{\nu}$$

$$+\ \frac{a_{11}}{a_o}\ .\ a_o\, \frac{\alpha_3^3}{\nu \alpha_1}\, \vartheta = 0, \qquad 66(a)$$

v-momentum equation,

$$s_{\phi\phi\phi} + \frac{1}{2}(m+1)\frac{a_o\,\alpha_1\,\alpha_3}{\nu} f s_{\phi\phi} + \frac{1}{2}(n+1)\frac{a_3\,\alpha_2\,\alpha_3}{\nu} s s_{\phi\phi}$$

$$+ \quad \frac{na_3}{\nu}\left(\frac{\alpha_3^3}{\alpha_2} - \alpha_2\,\alpha_3\,s_\phi^2\right) + \frac{a_o}{\nu}(m-1)\left(\frac{\alpha_3^3}{\alpha_2} - \alpha_1\,\alpha_3\,f_\phi\,s_\phi\right)$$

$$+ \quad \frac{\alpha_3^3}{\alpha_2 \nu}\, a_o\, \frac{a_{16}}{a_o}\,\vartheta = 0; \qquad 66(b)$$

and energy equation,

$$P_r^{-1}\,\vartheta_{\phi\phi} + \left(\frac{m+1}{2}\right)\frac{a_0\,\alpha_1\,\alpha_3}{\nu} f\,\vartheta_\phi + \left(\frac{n+1}{2}\right)\frac{a_3\,\alpha_2\,\alpha_3}{\nu} s\,\vartheta_\phi$$

$$+ \quad (1-2m)\frac{a_o\alpha_1\,\alpha_3}{\nu} f_\phi\,\vartheta + (1-2n)\frac{a_3\,\alpha_2\,\alpha_3}{\nu} s_\phi\,\vartheta = 0. \qquad 66(c)$$

If it is assumed that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, $\frac{a_o\alpha^2}{\nu} = 1$ and $\frac{a_3}{a_o} = \beta$, the

equations (66) take the form

u-momentum equation

$$f_{\phi\phi\phi} + \left(\frac{m+1}{2}\right) f f_{\phi\phi} + \left(\frac{n+1}{2}\right)\beta\, s\, f_{\phi\phi} + m(1-f_\phi^2)$$

$$+ \quad \beta\,(n-1)\,(1-f_\phi\,s_\phi) + \frac{U_P^2}{U_e^2}\,\vartheta = 0, \qquad 67(a)$$

v-momentum equation

$$s_{\phi\phi\phi} + \frac{1}{2}(m+1)\, f s_{\phi\phi} + \frac{1}{2}(n+1)\, \beta\, s s_{\phi\phi} + n\beta(1 - s_\phi^2)$$

$$+ (m-1)(1 - f_\phi\, s_\phi) + \frac{V_F^2}{V_e^2}\, \theta = 0, \qquad 67(b)$$

and energy equation

$$P_r^{-1}\, \theta_{\phi\phi} + \left(\frac{m+1}{2}\right) f\, \theta_\phi + \beta\left(\frac{n+1}{2}\right) s\, \theta_\phi + (1-2m)\, f_\phi\, \theta$$

$$+ \beta(1-2n)\, s_\phi\, \theta = 0, \qquad 67(c)$$

subject to the boundary conditions;

$$\begin{aligned} f(0) &= f_\phi(0) = 0 & f_\phi(\infty) &= 1 \\ s(0) &= s_\phi(0) = 0 & s_\phi(\infty) &= 1 \\ \theta(0) &= 1 & \theta(\infty) &= 0 \end{aligned} \qquad (68)$$

where,

$$\frac{a_{11}}{a_0} = -\gamma^2\, g_x\, \beta_T\, (\Delta T) \cdot \frac{1}{a_0\, U_e}$$

$$= -g_x\, \beta_T\, (\Delta T) \left(\frac{X + X_0}{U_e^2}\right),$$

$$\therefore \frac{a_{11}}{a_0} = \frac{U_F^2}{U_e^2} \text{ (say).}$$

Here $U_F^2 = -g_x\, \beta_T\, (\Delta T)\, (X + X_0)$

$= -g_x\, \beta_T\, (\Delta T) \times$ (characteristic length).

where characterisitc length = $(X + X_0)$.

Similarly,

$$\frac{a_{16}}{a_0} = \frac{V_F^2}{V_e^2} \quad \text{(say)},$$

here $V_F^2 = -\beta g_Y \beta_T (\Delta T) \times$ (characteristic length)

where characteristic length = $(Y + Y_0)$.

In this case $X_0 = \dfrac{A}{a_0}$ and $Y_0 = \dfrac{B}{a_0}$ are set.

The terms $\dfrac{U_F^2}{U_e^2}\theta$ and $\dfrac{V_F^2}{V_e^2}\theta$ in the momentum equations represent bouyancy effects compared with the forced flow effect.

The controlling parameters are P_r, ß, $\dfrac{U_F^2}{U_e^2}$, $\dfrac{V_F^2}{V_e^2}$, m and n.

where $\beta = \dfrac{(\gamma^2 V_e)_Y}{(\gamma^2 U_e)_X} = \dfrac{C_2}{C_1}$, where C_2 and C_1 are given by equations (69.f) and (69.g).

The similarity functions $f(\phi)$ and $s(\phi)$, the similarity variable ϕ, the velocity component (= u, v, w), the skin-frictions ($=\tau_w$'s), heat flux ($=q_w$) and Nusselt numbers ($=N_u$'s) associated with the equations (67) are

$$\phi = \left(\frac{Z}{X+X_0}\right)\sqrt{\frac{U_e(X+X_0)}{\nu}} = \left(\frac{Z}{X+X_0}\right) R_{eX}^{1/2}$$

or, $$\phi = \left(\frac{Z}{Y+Y_0}\right)\sqrt{\frac{V_e(Y+Y_0)}{\beta\,\nu}} = \left(\frac{Z}{Y+Y_0}\right)\sqrt{\frac{R_{eY}}{\beta}}$$

where $R_{eX} = \dfrac{U_e(X+X_0)}{\nu}$ and $R_{eY} = \dfrac{V_e(Y+Y_0)}{\nu}$

$u = U_e f_\phi$, $v = V_e s_\phi$ and

$$w = -\left(\frac{U_e \nu}{X+X_0}\right)^{\frac{1}{2}} \left[\left(\frac{m+1}{2}\right) f + \left(\frac{m-1}{2}\right) \phi f_\phi + \beta \left(\frac{n+1}{2}\right) s + \left(\frac{n-1}{2}\right) \beta \phi s_\phi\right]$$

Skin-frictions are,

$$\tau_{w_1} = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0}$$

$$\therefore \tau_{w_1} = \mu U_e \left(\frac{R_{ex}^{\frac{1}{2}}}{X+X_0}\right) f_{\phi\phi}(0)$$

and $$\tau_{w_2} = \mu\left(\frac{\partial v}{\partial Z}\right)_{z=0}$$

$$\therefore \tau_{w_2} = \mu V_e\left(\frac{R_{ex}^{\frac{1}{2}}}{X+X_0}\right) s_{\phi\phi}(0) .$$

Heat flux,

$$q_w = -\kappa\left(\frac{\partial T}{\partial Z}\right)_{z=0}$$

here κ is the thermal conductivity of heat.

$$\therefore q_w = -\kappa(\Delta T)\left(\frac{U_e}{\nu(X+X_0)}\right)^{\frac{1}{2}} \vartheta_\phi(0)$$

$$= -\kappa(\Delta T)\left(\frac{R_{eX}^{\frac{1}{2}}}{X+X_0}\right) \vartheta_\phi(0) .$$

Nusselt number's are

$$N_{uX} = \frac{q_w(X+X_0)}{\kappa(\Delta T)}$$

$$\therefore N_{uX} = -R_{eX}^{\frac{1}{2}} \vartheta_\phi(0)$$

and

$$N_{uY} = \frac{q_w(Y+Y_0)}{\kappa(\Delta T)}$$

$$\therefore N_{uY} = -\left(\frac{R_{eY}}{\beta}\right)^{\frac{1}{2}} \vartheta_\phi(0) .$$

Hence similarity variable,

$$\phi = \left(\frac{Z}{X+X_0}\right) R_{eX}^{\frac{1}{2}} = \frac{Z}{X_1} R_{eX_1}^{\frac{1}{2}} \qquad (69.a)$$

or

$$\phi = \left(\frac{Z}{Y+Y_0}\right)\left(\frac{R_{eY}}{\beta}\right)^{\frac{1}{2}} = \frac{Z}{Y_1}\left(\frac{R_{eY_1}}{\beta}\right)^{1/2} \qquad (69.b)$$

$$u = U_e f_\phi, \qquad v = V_e s_\phi , \tag{69.c.d}$$

$$w = -\left(\frac{U_e \nu}{X_1}\right)^{\frac{1}{2}} \left[\left(\frac{m+1}{2}\right) f + \left(\frac{m-1}{2}\right) \phi f_\phi + \beta \left(\frac{n+1}{2}\right) s \right.$$

$$\left. + \left(\frac{n-1}{2}\right) \beta \phi s_\phi \right] , \tag{69.e}$$

$$U_e = c_1 (X + X_0)^m (Y + Y_0)^{n-1} = c_1 X_1^m Y_1^{n-1}, \tag{69.f}$$

$$V_e = c_2 (X + X_0)^{m-1} (Y + Y_0)^n = c_2 X_1^{m-1} Y_1^n, \tag{69.g}$$

$$\Delta T = c_3 (X + X_0)^{2m-1} (Y + Y_0)^{2n-2} = c_3 X_1^{2m-1} Y_1^{2n-2}, \tag{69.h}$$

$$\gamma = c_4 (X + X_0)^{\frac{1-m}{2}} (Y + Y_0)^{\frac{1-n}{2}} = c_4 X_1^{\frac{1-m}{2}} Y_1^{\frac{1-n}{2}},$$

and stream functions are,

$$\psi (X_1, Y_1, Z) = \sqrt{\nu c_1} X_1^{\frac{m+1}{2}} Y_1^{\frac{n-1}{2}} f(\phi) \tag{69.i}$$

and $\varphi (X_1, Y_1, Z) = \beta \sqrt{\nu c_1} (X_1)^{\frac{m-1}{2}} (Y_1)^{\frac{n+1}{2}} s(\phi) ,$ (69.j)

$$T - T_0 = \Delta T \vartheta(\phi) \tag{69.k}$$

$$= c_3 (X_1)^{2m-1} (Y_1)^{2n-2} \vartheta(\phi) ,$$

where $(X+X_0) = X_1$,

$(Y+Y_0) = Y_1$,

$c_1 = a_0^m a_3^{n-1}$,

$$c_2 = a_0^{m-1}\, a_3^{n},$$

$$c_3 = \frac{a_{11}}{g_X\, \beta_T} a_0^{2m-1}\, a_3^{2n-2},$$

and $$c_4 = a_0^{\frac{1-m}{2}}\, a_3^{\frac{1-n}{2}}.$$

Skin frictions,

$$\tau_{w_1} = \mu\, U_e \left(\frac{R_{eX_1}^{\frac{1}{2}}}{X_1}\right) f_{\phi\phi}(0) \qquad (69.l)$$

and $$\tau_{w_2} = \mu V_e \left(\frac{R_{eX_1}^{\frac{1}{2}}}{X_1}\right) s_{\phi\phi}(0)\ . \qquad (69.m)$$

Heat - flux,

$$q_w = -\kappa\,(\Delta T)\left[\frac{U_e}{\nu X_1}\right]^{\frac{1}{2}} \theta_\phi(0)\ . \qquad (69.n)$$

Nusselt Numbers,

$$Nu_X = -R_{eX}^{\frac{1}{2}}\, \theta_\phi(0) \qquad (69.p)$$

$$Nu_Y = -\left(\frac{R_{eY}}{\beta}\right)^{\frac{1}{2}} \vartheta_\phi(0) \; . \qquad (69.q)$$

The physical behaviour of the controlling parameters $\left(= P_r, \; \beta, \; \frac{U_p^2}{U_e^2}; \; \frac{V_p^2}{V_e^2}\right)$ used in equations 67(a,b,c) are now discussed.

The Prandtle number $P_r (= \frac{\mu C_p}{K})$ depends on the properties of the media. For air at room temperature its value is 0.72, for water at temperature 62^{0}F its value is 7.0, where for oil this value is of the order 1000. (ΔT is equal to $c_3 (X_1)^{2m-1} (Y_1)^{2n-2}$). For m = n = 1 and ß = 1, f = s and this gives the flow at an axi-symmetrical stagnation point. When ß = 0, c_2 = 0 i.e. v = 0 and the flow is in the neighbourhood of a two-dimensional stagnetion point. The solutions were obtained by Howarth for ß = 0.25, 0.50, 0.75 in absence of bouyancy effects. For intermediate values of ß the velocity profiles and temperature profile given by $f_\phi(\phi)$, $s_\phi(\phi)$ and $\vartheta(\phi)$ are of boundary layer type.

The third and fourth parameters are the coupling agent between the momentum and energy equations. These parameters have been replaced by $\frac{G_{rX}}{R_{eX}^2}$ and $\frac{G_{rY}}{R_{eY}^2}$ by Sparow, Eichorn and Gregg (1959). Where G_{rX} and G_{rY} are the local Grashop numbers based on ΔT, X, and Y; where R_{eX} and R_{eY} are the Reynold numbers based on the components of outer by $\frac{G_{rX}}{R_{eX}^2}$ and $\frac{G_{rY}}{R_{eY}^2}$ by Sparow, Eichorn and Gregg (1959). Where G_{rX} and G_{rY} are the local Grashop numbers based on ΔT, X, and Y; where R_{eX} and R_{eY} are the Reynold numbers based on the components of outer stream velocity, X and Y. However ΔT as well as U_e and V_e are functions of X and Y and neither $\frac{G_{rX}}{R_{eX}^2}$ nor $\frac{G_{rY}}{R_{eY}^2}$ can appropriately represent this term. The use of Grashop numbers here are rather misleading. Since it introduces μ^2, which is eleminated by using ${R_{eX}}^2$ or ${R_{eY}}^2$ in the denominator. The non-dimensional solutions should not depend on R_{eX} and R_{eY}. The use of U_F^2 and V_F^2 are preferred here, since U_F and V_F defined by Ostrach (1953) give the measure of order of magnitude of velocity caused solely by buoyancy effects. More precisely, the parameters $\frac{U_F^2}{U_e^2}$ and $\frac{V_F^2}{V_e^2}$ are defined as the square of the ratio between the velocity caused by the buoyancy effects, and the external velocity components for the forced flow. The flow is said to be aided when $\frac{U_F^2}{U_e^2}$ or $\frac{V_F^2}{V_e^2}$ is greater than zero and is called to be the opposing flow when the parameters are less then zero.

When $U_F^2 \ll U_e^2$ *and* $V_F^2 \ll V_e^2$ the flow becomes forced flow; whereas as $U_F^2 \gg U_e^2$ *and* $V_F^2 \gg V_e^2$ the flow becomes free convection flow. In absence of bouyancy effect the equations 67 (a,b,c) (= the forced convection) are dealt with Davey (1961) when m = n = 1.

CASE A(ii)

The case A (i) is characterised by arbitrary constant in the similarity requirement namely a_0 and a_3. Setting the arbitrary constant a_0 equal to zero but $a_3 \neq 0$ the problem changes to different type. In addition to the traditional conditions of case A, if one sets $a_0 = 0$ the equation for γ^2 reduces to

$$\gamma^2 = \frac{A}{U_e}. \tag{70}$$

Setting the value of γ^2 in 56(ii),

$$U_e = K_1(Y)\, e^{\frac{a_1}{A}X}. \qquad 70(a)$$

From 56(iv) it is found that

$$\gamma^2 = \frac{a_3 Y+B}{V_e} \qquad 70(b)$$

From 56(v) and 70(b) the following equation is found as

$$V_e = K_2(X)\,(a_3 Y+B)^{\frac{a_4}{a_3}}. \qquad 70(c)$$

From (70) and 70(b) the following relationship is obtained

$$\frac{U_e}{V_e} = \frac{A}{a_3 Y+B}. \qquad 70(d)$$

Here $K_1(Y)$ and $K_2(X)$ are constant of integrations.

Again, from 70(a) and 70(c)

$$\frac{U_e}{V_e} = \frac{K_1(Y)\, e^{\frac{a_1}{A}X}}{K_2(X)\,(a_3Y+B)^{\frac{a_4}{a_3}}} \, . \qquad 70(e)$$

From 70(d) and 70(e) it can be found that

$$\frac{C_1(Y)}{C_2(X)} = \frac{A(a_3Y+B)^{\frac{a_4}{a_3}-1}}{e^{\frac{a_1}{A}X}} \qquad 70(f)$$

Hence from 70(f)

$$K_2(X) = e^{\frac{a_1}{A}X} \quad \text{and} \quad K_1(Y) = A(a_3Y+B)^{\frac{a_4}{a_3}-1}$$

The equations 70(a) and 70(c) take the following form

$$U_e = A(a_3Y+B)^{\frac{a_4}{a_3}-1} e^{\frac{a_1}{A}X}$$

$$V_e = e^{\frac{a_1}{A}X}(a_3Y+B)^{\frac{a_4}{a_3}} \, ,$$

Hence

$$U_e = c_1 e^{mX}(Y+Y_o)^{n-1}$$

$$V_e = c_2 e^{mX}(Y+Y_o)^{n} \qquad 71(a,b,c)$$

$$\gamma^2 = c_3 e^{-mX}(Y+Y_o)^{1-n}$$

and $\Delta T \propto e^{2mX} (Y + Y_o)^{2n-1}$

where $c_1 = A(a_3)^{\frac{a_4}{a_3}-1}$, $c_2 = a_3^{a_4/a_3}$, $m = \frac{a_1}{A}$, $n = \frac{a_4}{a_3}$ and $Y_0 = \frac{B}{a_3}$.

Setting the expression of U_e, V_e and γ^2 in equation (56),

$$a_0 = a_2 = a_5 = a_6 = a_9 = a_{10} = a_{15} = a_{14} = 0$$

Let a_1 and a_3 are arbitrary, then

$$a_4 = a_{12} = na_3, \quad a_7 = a_{13} = a_1 \quad a_8 = (n-1)a_3, \quad a_{17} = 2a_1$$

$$a_{18} = (2n-1)a_3,$$

Hence the general transformed equations (49), (50) and (51) take the following forms:

u-momentum equation,

$$\nu \overline{F}_{\overline{\phi}\overline{\phi}\overline{\phi}} + \frac{a_1}{2}\overline{F}\,\overline{F}_{\overline{\phi}\overline{\phi}} + \frac{1}{2}a_3(n+1)\overline{S}\,\overline{F}_{\overline{\phi}\overline{\phi}} + a_1(1 - \overline{F}^2_{\overline{\phi}})$$

$$+ \quad a_3(n-1)(1-\overline{F}_{\overline{\phi}}\overline{S}_{\overline{\phi}}) + a_{11}\overline{\theta} = 0 \qquad 72(a)$$

v-mementum equation,

$$\nu \overline{S}_{\overline{\phi}\overline{\phi}\overline{\phi}} + \frac{1}{2}a_1\overline{F}\,\overline{S}_{\overline{\phi}\overline{\phi}} + \frac{1}{2}a_3(n + 1)\overline{S}\,\overline{S}_{\overline{\phi}\overline{\phi}}$$

$$+ \quad na_3(1 - \overline{S}^2_{\overline{\phi}}) + a_1(1 - \overline{F}_{\overline{\phi}}\overline{S}_{\overline{\phi}}) + a_{16}\overline{\theta} = 0 \qquad 72(b)$$

Energy equation,

$$\frac{\nu}{P_r}\overline{\vartheta}_{\overline{\phi}\overline{\phi}} + \frac{1}{2}a_1\overline{F}\,\overline{\vartheta}_{\overline{\phi}} + \frac{1}{2}(n+1)a_3\overline{S}\,\overline{\vartheta}_{\overline{\phi}} - 2a_1\overline{\vartheta}\,\overline{F}_{\overline{\phi}}$$

$$+ \quad a_3(1-2n)\overline{\vartheta}\,\overline{S}_{\overline{\phi}} = 0 \qquad 72(c)$$

subject to the boundary conditions:

$$\overline{F}(0) = \overline{F}_{\overline{\phi}}(0) = 0 \qquad \overline{F}_{\overline{\phi}}(\infty) = 1$$

$$\overline{S}(0) = \overline{S}_{\overline{\phi}}(0) = 0 \qquad \overline{S}_{\overline{\phi}}(\infty) = 1 \qquad (73)$$

$$\overline{\vartheta}(0) = 1, \qquad \overline{\vartheta}(\infty) = 0 ,$$

Substituting $\overline{F} = \alpha f$, $\overline{S} = \alpha s$, $\overline{\phi} = \alpha\phi$ and $\overline{\vartheta} = \vartheta$ in the above equations and setting $(\frac{n+1}{2})\frac{a_3\alpha^2}{\nu} = 1$ and $\frac{A}{a_3} = 2c$.

The above equations (72) take the form

$$f_{\phi\phi\phi} + sf_{\phi\phi} + \frac{2mc}{n+1}ff_{\phi\phi} + \frac{4cm}{n+1}(1 - f_\phi^2)$$

$$+ \quad (2 - \frac{4}{n+1})(1 - f_\phi s_\phi) + (\frac{2}{n+1})\frac{U_P^2}{U_e^2}\vartheta = 0 ,$$

$$s_{\phi\phi\phi} + ss_{\phi\phi} + \frac{2mc}{n+1}fs_{\phi\phi} + \frac{2n}{n+1}(1 - s_\phi^2) \qquad 74(a,b,c)$$

$$+ \quad \frac{4mc}{n+1}(1 - f_\phi s_\phi) + \frac{2}{n+1}\frac{V_P^2}{V_e^2}\vartheta = 0 ,$$

$$P_r^{-1}\vartheta_{\phi\phi} + s\vartheta_\phi + \frac{2cm}{n+1}f\vartheta_\phi - \frac{8mc}{n+1}\vartheta f_\phi + \frac{2(1-2n)}{n+1}\vartheta s_\phi = 0 .$$

The boundary conditions applicable for the above equations are,

$$f(0) = f_\phi(0) = 0, \quad f_\phi(\infty) = 1$$
$$s(0) = s_\phi(0) = 0, \quad s_\phi(\infty) = 1 \qquad (75)$$
$$\theta(0) = 1, \quad \theta(\infty) = 0,$$

where, $\frac{a_{11}}{a_3} = \frac{U_F^2}{U_e^2}, \quad \frac{a_{16}}{a_3} = \frac{V_F^2}{V_e^2},$

$U_F^2 = -\ cg_x\beta_T(\Delta T)$, c plays the role of characteristic length,

and $V_F^2 = -\ (Y+Y_0)\, g_Y \beta_T \Delta T$.

The similarity functions f(ϕ), s (ϕ), the similarity variable ϕ, the velocity components u, v, w ; the skin-frictions τ_w's heat flux q_w and Nusselt number (Nu) associated with the equations (74) are

$$\psi(X, Y, \phi) = \left(\frac{4cv}{n+1}\right)^{\frac{1}{2}} U_e^{\frac{1}{2}}\, f(\phi),$$

$$\varphi(X, Y, \phi) = \left(\frac{2\nu Y_1}{n+1}\right)^{\frac{1}{2}} V_e^{\frac{1}{2}}\, s(\phi),$$

and $\phi = \frac{Z}{Y_1}\left(\frac{n+1}{2}\right)^{1/2} R_{eY_1}^{1/2},$

where $R_{eY_1} = \frac{V_e Y_1}{\nu}$ and $Y_1 = Y+Y_0$.

$$u = U_e f_\phi, \quad v = V_e s_\phi,$$

$$w = -\left(\frac{V_e \nu}{Y_1}\right)^{\frac{1}{2}}\left(\frac{2}{n+1}\right)^{\frac{1}{2}}\left[mcf + mc\phi f_\phi + \left(\frac{n+1}{2}\right)s + \left(\frac{n-1}{2}\right)\phi s_\phi\right]$$

Skin-frictions are,

$$\tau_{w_1} = \mu\left(\frac{n+1}{2}\right)^{\frac{1}{2}} U_e\left(\frac{R_{eY_1}^{\frac{1}{2}}}{Y_1}\right) f_{\phi\phi}(0),$$

and $$\tau_{w_2} = \mu\left(\frac{n+1}{2}\right)^{\frac{1}{2}} V_e\left(\frac{R_{eY_1}^{\frac{1}{2}}}{Y_1}\right) s_{\phi\phi}(0).$$

Heat flux,

$$q_w = -\kappa\left(\frac{\partial T}{\partial Z}\right)_{z=0} = -\kappa(\Delta T)\left[\left(\frac{n+1}{2}\right)\frac{V_e}{\nu Y_1}\right]^{\frac{1}{2}} \vartheta_\phi(0)$$

$$\therefore q_w = -\frac{\kappa(\Delta T)}{Y_1}\left(\frac{n+1}{2}\right)^{\frac{1}{2}} R_{eY_1}^{\frac{1}{2}} \vartheta_\phi(0).$$

Nusselt Number,

$$N_{uY_1} = -\left(\frac{n+1}{2}\right)^{\frac{1}{2}} R_{eY_1}^{\frac{1}{2}} \vartheta_\phi(0).$$

In this case, if c = 0 $\Longrightarrow$ A = 0, i.e, $U_e = 0 \Longrightarrow u = 0$. Then the fluid moves with two velocity components (= v, w). The velocity components are independent of X. Therefore, the similarity functions 'f' is negligible, let $\frac{2n}{n+1} = \beta_1$

The equations (74) take the form,

$$S_{\phi\phi\phi}+SS_{\phi\phi}+\beta_1(1-s_\phi^2)+(2-\beta_1)\frac{V_P^2}{V_e^2}\vartheta = 0 ,$$

$$\vartheta_{\phi\phi} + P_r [s\vartheta_\phi + (2-3\beta_1)\vartheta s_\phi] = 0 . \qquad 76(a,b)$$

The boundary conditions are

$$s(0) = s_\phi(0) = 0, \qquad s_\phi(\infty) = 1 ,$$

$$\theta(0) = 1 , \qquad \theta(\infty) = 0 .$$

Here, $V_P^2 = - g_Y\beta_T(\Delta T)$ x characteristic length,

where characteristic length = Y_1

$$V_e \propto Y_1^{\frac{\beta_1}{(2-\beta_1)}} \quad and \quad \Delta T \propto (Y_1)^{\frac{3\beta_1-2}{2-\beta_1}} .$$

Hence the non-dimensional stream function, similarity variable, the velocity components (=v, w) within the boundary layer and skin friction (= τ_w), heat transfer co-efficient (=q_w) are given by following equations.

$$\varphi = [(2-\beta_1)\nu V_e Y_1]^{\frac{1}{2}} s(\phi),$$

$$\phi = \frac{Z}{\sqrt{(2-\beta_1)}}\left[\frac{V_e}{Y_1 \nu}\right]^{\frac{1}{2}} = \frac{1}{\sqrt{2-\beta_1}}\frac{Z}{Y_1}R_{eY_1}^{\frac{1}{2}}$$

$$v = V_e s_\phi$$

$$w = -\frac{1}{\sqrt{(2-\beta_1)}}\left(\frac{V_e \nu}{Y_1}\right)^{\frac{1}{2}}[s+(\beta_1-1)\phi s_\phi]$$

$$\tau_w = \frac{\mu}{\sqrt{(2-\beta_1)}}\frac{V_e}{Y_1}R_{eY_1}^{\frac{1}{2}} s_{\phi\phi}(0)$$

$$q_w = -\frac{\kappa(\Delta T)}{Y_1\sqrt{2-\beta_1}} R_{eY_1}^{\frac{1}{2}} \theta_\phi(0)$$

and Nusselt Number,

$$N_u = -\frac{1}{\sqrt{(2-\beta_1)}}R_{eY_1}^{\frac{1}{2}} \theta_\phi(0)$$

The above case is a steady flow problem for combined forced and free convection. The parameter β_1 is well known wedge flow parameter (infact $\pi\beta_1$ is the wedge angle) related to the exponent of V_e and ΔT.

Detail of physical significance of β was explained by Falkner-Skan (1931). This problem has been explained previously by Sparrow, Eichorn and Gregg (1959) with new similarity function $s(\lambda)$ and similarity variable ϕ related to our present function and variable.

$$s(\phi) = \frac{S(\lambda)}{\sqrt{(2-\beta_1)}}, \quad \phi = \frac{2\lambda}{\sqrt{(2-\beta_1)}}$$

The equation (76) is transformed pair of equations dealt with by Sparrow, Eichorn and Gregg(1959). They tabulated skin-friction and heat transfer factors ($=s_{\lambda\lambda}(0)$; $\vartheta_\lambda(0)$) for uniform wall temperature (i.e, $\beta=\frac{3}{2}$) and uniform heat flux (i.e, $\beta_1=\frac{3}{4}$) in the range $\frac{G_{rY}}{R_{eY}^2}$ = 100 down to the point of speration for air $P_r = 0.7$.

CASE-A (iii)

In this case, setting the arbitary constant a_3 equal to zero and $a_0 \neq 0$, the equation for γ^2 reduces to,

$$\gamma^2 = \frac{B}{V_e} \; . \tag{77}$$

Setting the value of γ^2 in 56(v), one obtains.

$$V_e = K_1(X)\, e^{\frac{a_4}{B}Y}. \qquad 78(a)$$

From 56(i),

$$\gamma^2 = \frac{1}{U_e}(a_0X+A) \quad . \qquad 78(b)$$

Substituting γ^2 in 56(ii),

$$U_e = K_2(Y)\,(a_0X+A)^{\frac{a_1}{a_0}} \; , \qquad 78(c)$$

where $K_1(X)$ and $K_2(Y)$ are constant of integrations.

From (77) and 78(b),

$$\frac{U_e}{V_e} = \frac{a_0X+A}{B} \qquad 79(a)$$

and from 78(a) and 78(c) we get,

$$\frac{U_e}{V_e} = \frac{K_2(Y)\,(a_0X+A)^{\frac{a_1}{a_0}}}{K_1(X)\, e^{\frac{a_4}{B}Y}} \; , \qquad 79(b)$$

From 79(a) and 79(b), it may be written that

$$\frac{a_0X+A}{B} = \frac{K_2(Y)\,(a_0X+A)^{\frac{a_1}{a_0}}}{K_1(X)\,e^{\frac{a_4}{B}Y}}$$

$$==> \quad \frac{K_1(X)}{K_2(Y)} = \frac{B(a_0X+A)^{\frac{a_1}{a_0}-1}}{e^{\frac{a_4}{B}Y}} \; . \qquad (80)$$

From equation (80),

$$K_1(X) = B(a_0X+A)^{\frac{a_1}{a_0}-1} \qquad 81(a)$$

$$\text{and} \quad K_2(Y) = e^{\frac{a_4}{B}Y} \; . \qquad 81(b)$$

Substituting the expressions of $K_1(X)$ and $K_2(Y)$ in equation 78(a,c) one obtains,

$$U_e = (a_0X+A)^n e^{mY} = c_1(X+X_0)^n e^{mY} \; , \qquad 82(a)$$

$$V_e = B(a_0X+A)^{n-1} e^{mY} = c_2(X+X_0)^{n-1} e^{mY} \; , \qquad 82(b)$$

and,

$$\gamma^2 = (a_0X+A)^{1-n} e^{-mY} = c_3(X+X_0)^{1-n} e^{-mY} \; , \qquad 82(c)$$

where, $c_1 = a_0^n$, $c_2 = a_0^{n-1}B$, $c_3 = a_0^{1-n}$, $\frac{a_1}{a_0} = n$ and $\frac{a_4}{B} = m$.

The similarity requirements furnish the relationship among the constants a_1, a_2, a_3 . . . . a_{17} and a_{18}.

Here a_0 and a_4 are arbitrary,

These relations are,

$$a_2 = a_3 = a_5 = a_6 = a_9 = a_{10} = a_{14} = a_{15} = 0$$

$$a_1 = a_7 = na_0, \quad a_8 = a_{12} = a_4, \quad a_{13} = (n-1)a_0,$$

$$a_{17} = (2n-1)a_0 \quad \text{and} \quad a_{18} = 2a_4 .$$

Hence the general transformed equations (49), (50), and (51) take the form,

$$\nu\overline{F}_{\Phi\Phi\Phi} + \left(\frac{n+1}{2}\right)a_0\overline{F}\,\overline{F}_{\Phi\Phi} + \frac{a_4}{2}\overline{S}\,\overline{F}_{\Phi\Phi} + na_0(1-\overline{F}_{\Phi}^2)$$

$$+ \quad a_4(1-\overline{F}_{\Phi}\overline{S}_{\Phi}) + a_{11}\overline{\theta} = 0 \, , \qquad 83(a)$$

$$\nu\overline{S}_{\Phi\Phi\Phi} + \left(\frac{n+1}{2}\right)a_0\overline{F}\,\overline{S}_{\Phi\Phi} + \frac{a_4}{2}\overline{S}\,\overline{S}_{\Phi\Phi} + a_4(1-\overline{S}_{\Phi}^2)$$

$$+ (n-1) \quad a_0(1-\overline{F}_{\Phi}\overline{S}_{\Phi}) + a_{16}\overline{\theta} = 0 \qquad 83(b)$$

and

$$\nu P_r^{-1}\,\overline{\theta}_{\Phi\Phi} + \left(\frac{n+1}{2}\right)a_0\overline{F}\,\overline{\theta}_{\Phi} + \frac{a_4}{2}\overline{S}\,\overline{\theta}_{\Phi}$$

$$- \quad (2n-1)a_0\overline{\theta}\,\overline{F}_{\Phi} - 2a_4\overline{\theta}\,\overline{S}_{\Phi} = 0 \, . \qquad 83(c)$$

The boundary conditions are

$$\overline{F}(0) = \overline{F}_{\phi}(0) = \overline{S}(0) = \overline{S}_{\phi}(0) = 0,$$

$$\overline{F}_{\phi}(\infty) = \overline{S}_{\phi}(\infty) = 1 , \qquad 83(d)$$

$$\overline{\vartheta}(0) = 1, \quad \overline{\vartheta}(\infty) = 0 .$$

Substituting $\overline{F} = \alpha f$, $\overline{S} = \alpha s$, $\overline{\phi} = \alpha\phi$ and $\overline{\vartheta} = \vartheta$ in above equation and setting

$$(\frac{n+1}{2})\frac{a_0\alpha^2}{\nu} = 1, \quad a_4 = Bm \quad \text{and} \quad \frac{B}{a_0} = 2c,$$

the above equations (83) become,

$$f_{\phi\phi\phi} + ff_{\phi\phi} + (\frac{2cm}{n+1})sf_{\phi\phi} + (\frac{2n}{n+1})(1 - f_{\phi}^2)$$

$$+ \quad (\frac{4cm}{n+1})(1-f_{\phi}s_{\phi}) + (\frac{2}{n+1})\frac{U_F^2}{U_e^2}\vartheta = 0, \qquad 84(a)$$

$$s_{\phi\phi\phi} + fs_{\phi\phi} + (\frac{2cm}{n+1})ss_{\phi\phi} + (\frac{4cm}{n+1})(1 - s_{\phi}^2)$$

$$+ \quad (2-\frac{4}{n+1})(1-f_{\phi}s_{\phi}) + (\frac{2}{n+1})\frac{V_F^2}{V_e^2}\vartheta = 0, \qquad 84(b)$$

and

$$P_r^{-1}\vartheta_{\phi\phi} + f\vartheta_{\phi} + \frac{2cm}{n+1}s\vartheta_{\phi} - \frac{2(2n-1)}{n+1}\vartheta f_{\phi}$$

$$- \quad \frac{8cm}{n+1}\vartheta s_{\phi} = 0. \qquad 84(c)$$

The boundary conditions are,

$$
\begin{aligned}
&f(0) = f_\phi(0) = 0, \quad f_\phi(\infty) = 1\,, \\
&s(0) = s_\phi(0) = 0, \quad s_\phi(\infty) = 1\,, \\
&\vartheta(0) = 1, \quad \vartheta(\infty) = 0\,.
\end{aligned}
\tag{85}
$$

Here,

$$\frac{a_{11}}{a_0} = \frac{U_F^2}{U_e^2} \quad \text{(say)} \quad \text{and} \quad \frac{a_{16}}{a_0} = \frac{V_F^2}{V_e^2}\,;$$

where $U_F^2 = -g_X \beta_T (\Delta T)(X+X_0)\,,$

$V_F^2 = -2 g_Y \beta_T (\Delta T)\, c\,,$

and $\Delta T \propto (X+X_0)^{2n-1} e^{2nY}\,.$

The similarity functions $f(\phi)$, $s(\phi)$; similarity variable ϕ, the velocity components u, v, w, the skin frictions τ_w's ; Heat transfer co-efficient q_w and Nusselt number Nu associated with the equations (84) are

$$\psi(X,Y,\phi) = \left(\frac{2\nu X_1}{n+1}\right)^{\frac{1}{2}} U_e^{\frac{1}{2}} f(\phi)\,,$$

$$\varphi(X,Y,\phi) = \left(\frac{4c\nu}{n+1}\right)^{\frac{1}{2}} V_e^{\frac{1}{2}} s(\phi)\,,$$

$$\phi = \frac{Z}{X_1}\left(\frac{n+1}{2}\right)^{\frac{1}{2}} R_{eX_1}^{\frac{1}{2}}, \quad u = U_e f_\phi, \quad v = V_e S_\phi$$

and

$$w = -\left[\frac{\nu U_e}{X_1}\cdot\frac{2}{(n+1)}\right]^{\frac{1}{2}}\left[\left(\frac{n+1}{2}\right)f + \left(\frac{n-1}{2}\right)\phi f_\phi + cms + cm\phi s_\phi\right],$$

where $R_{eX_1} = \frac{U_e X_1}{\nu}$ and $X_1 = (X + X_0)$.

Skin-frictions are,

$$\tau_{w_1} = \mu\left(\frac{n+1}{2}\right)^{\frac{1}{2}}\frac{U_e}{X_1}R_{eX_1}^{\frac{1}{2}} f_{\phi\phi}(0),$$

and $$\tau_{w_2} = \mu\left(\frac{n+1}{2}\right)^{\frac{1}{2}}\frac{V}{X_1}R_{eX_1}^{\frac{1}{2}} s_{\phi\phi}(0).$$

Heat flux, $$q_w = \kappa\frac{(\Delta T)}{X_1}\cdot\left(\frac{n+1}{2}\right)^{\frac{1}{2}}R_{eX_1}^{\frac{1}{2}}\vartheta_\phi(0),$$

where κ is the conductivity of heat and μ is co-efficient of viscosity.

And Nusselt number, $$N_{uX_1} = -\left(\frac{n+1}{2}\right)R_{eX_1}^{\frac{1}{2}}\vartheta_\phi(0).$$

It is interesting to note that if c=0 i.e, B=0 then $V_e = 0 \Longrightarrow v=0$. That is the flow moves with two velocity component u and w, and the flow becomes 2-dimensional. Hence the other component of similarity function s is negligible and $\frac{2n}{n+1} = \beta$ (say).

Therefore, equations (84) become,

$$f_{\phi\phi\phi}+ff_{\phi\phi}+\beta(1-f_\phi^2)+(2-\beta)\frac{U_F^2}{U_e^2}\,\vartheta = 0 \, , \qquad 86(a)$$

$$\vartheta_{\phi\phi} + P_r\,[f\vartheta_\phi + (2-3\beta)\,f_\phi\,\vartheta] = 0 \, . \qquad 86(b)$$

The bounary conditions are,

$$\begin{aligned} &f(0) = f_\phi(0) = 0, && f_\phi(\infty) = 1\,, \\ &s(0) = s_\phi(0) = 0\,, && s_\phi(\infty) = 1\,, \\ &\vartheta(0) = 1, && \vartheta(\infty) = 0\,. \end{aligned} \qquad 86(c)$$

In this case $U_F^2 = -g_X\beta_T(\Delta T)X_1$, $U_e \propto X_1^{\frac{\beta}{2-\beta}}$ and $\Delta T \propto (X_1)^{\frac{3\beta-2}{2-\beta}}$.

Hence the non-dimensional stream function, similarity variable and velocity components are

$$\psi(X,Z) = [(2-\beta)\nu U_e X_1]^{\frac{1}{2}} f(\phi)\,,$$

$$\phi = \frac{Z}{X_1\sqrt{2-\beta}} R_{eX_1}^{\frac{1}{2}}\,,$$

$$u = U_e f_\phi\,, \qquad V = 0$$

and $$w = -\frac{1}{\sqrt{2-\beta}}\left(\frac{\nu U_e}{X_1}\right)^{\frac{1}{2}}[f(\phi) + (\beta-1)\phi f_\phi]\,.$$

Skin-friction,

$$\tau_w = \frac{\mu}{\sqrt{2-\beta}} \frac{U_e}{X_1} R_{eX_1}^{\frac{1}{2}} f_{\phi\phi}(0).$$

heat flux,

$$q_w = -\frac{\kappa}{\sqrt{2-\beta}} \left(\frac{\Delta T}{X_1}\right) R_{eX_1}^{\frac{1}{2}} \vartheta_\phi(0) \, ,$$

and Nusselt number,

$$N_{uX_1} = -\frac{1}{\sqrt{2-\beta}} R_{eX_1}^{\frac{1}{2}} \vartheta_\phi(0).$$

This case is a steady flow problem for combined forced and free convection. The parameter β is well known wedge parameter related to the exponent of U_e and ΔT.

Detail of physical significance of β was explained by Falkner-Skan (1931). This problem has been explained previously by Sparrow, Eichorn and Gregg (1959) and is same as case A(ii).

CASE A (IV)

Setting the arbitrary constants (a_0, a_3) to zero in addition to the traditional conditions of case A, the equation for γ^2 reduces to

$$\gamma^2 = \frac{A}{U_e} \quad \text{and} \quad \gamma^2 = \frac{B}{V_e} \tag{87(a,b)}$$

From (87), $$\frac{U_e}{V_e} = \frac{A}{B} = \text{constant} \tag{88}$$

Hence it can be said that U_e is proportional to V_e,
From equations 56(ii) and 56(v),

$$U_e = C_1(Y)\, e^{\frac{a_1}{A}X} \quad \text{and} \quad V_e = c_2(X)\, e^{\frac{a_4}{B}Y} \tag{89(a,b)}$$

where $c_1(Y)$ and $c_2(X)$ are the constant of integrations.
Therefore,

$$\frac{U_e}{V_e} = \frac{C_1(Y)\, e^{\frac{a_1}{A}X}}{c_2(X)\, e^{\frac{a_4}{b}Y}} \tag{90}$$

From equations (88) and (90), the resulsts is, $\dfrac{c_1(Y)\, e^{\frac{a_1}{A}X}}{c_2(X)\, e^{\frac{a_4}{B}Y}} = \dfrac{A}{B}$

$$\Longrightarrow \quad \frac{c_2(X)}{c_1(Y)} = \frac{Be^{\frac{a_1}{A}X}}{Ae^{\frac{a_4}{B}Y}} \tag{91}$$

From equation (91), expression for $C_2(X)$ and $C_1(Y)$ are

$$c_2(X) = Be^{\frac{a_1}{A}X} \quad \text{and} \quad c_1(Y) = Ae^{\frac{a_4}{B}Y}$$

Substitution of c_1 and c_2 in equation (89) gives

$$U_e = Ae^{(mX+nY)} \quad V_e = Be^{(mX+nY)} \quad \gamma^2 = e^{-(mX+nY)} \qquad 92(a,b,c)$$

and $\Delta T \propto e^{2(mX+nY)}$ (93)

where $\frac{a_1}{A} = m$, and $\frac{a_4}{B} = n$

The similarity requirements furnish the relations among the constants a's. These relations are,

$$a_0 = a_2 = a_3 = a_5 = a_6 = a_9 = a_{10} = a_{14} = a_{15} = 0$$

$$a_7 = a_{13} = a_1, \quad a_8 = a_{12} = a_4, \quad a_{17} = 2a_1,$$

and $a_{18} = 2a_4$.

where a_1 and a_4 are arbitrary,

Hence the general transform equations (49),(50),(51) take the form,

$$\nu \overline{F}_{\overline{\Phi}\overline{\Phi}\overline{\Phi}} + \frac{a_1}{2}\overline{F}\,\overline{F}_{\overline{\Phi}\overline{\Phi}} + \frac{a_4}{2}\overline{S}\,\overline{F}_{\overline{\Phi}\overline{\Phi}} + a_1(1 - \overline{F}_{\overline{\Phi}}^{2})$$

$$+ a_4(1 - \overline{F}_{\overline{\Phi}}\,\overline{S}_{\overline{\Phi}}) + a_{11}\overline{\theta} = 0, \qquad 94(a)$$

$$\nu \bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{a_1}{2}\bar{F}\bar{S}_{\bar{\phi}\bar{\phi}} + \frac{a_4}{2}\bar{S}\bar{S}_{\bar{\phi}\bar{\phi}}$$

$$+ a_4(1 - \bar{S}_{\bar{\phi}}^2) + a_1(1 - \bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}}) + a_{16}\bar{\vartheta} = 0, \qquad 94(b)$$

$$\frac{\nu}{P_r}\bar{\vartheta}_{\bar{\phi}\bar{\phi}} + \frac{a_1}{2}\bar{F}\bar{\vartheta}_{\bar{\phi}} + \frac{1}{2}a_4\bar{S}\bar{\vartheta}_{\bar{\phi}} - 2a_1\bar{\vartheta}\bar{F}_{\bar{\phi}}$$

$$- 2a_4\bar{\vartheta}\bar{S}_{\bar{\phi}} = 0, \qquad 94(c)$$

subject to the boundary conditions,

$$\begin{aligned} &\bar{F}(0) = \bar{F}_{\bar{\phi}}(0) = 0, \quad \bar{F}_{\bar{\phi}}(\infty) = 1, \\ &\bar{S}(0) = \bar{S}_{\bar{\phi}}(0) = 0, \quad \bar{S}_{\bar{\phi}}(\infty) = 1, \\ &\bar{\vartheta}(0) = 1, \quad \bar{\vartheta}(\infty) = 0 \end{aligned} \qquad (95)$$

Let us now substitute $\bar{F} = \alpha f$, $\bar{S} = \alpha s$, $\bar{\phi} = \alpha\phi$ and $\bar{\vartheta} = \vartheta$ in the above equations and choose $\frac{\alpha^2 a_1}{2\nu} = 1$ and $\frac{a_4}{a_1} = \beta$ the above equations (94) are further simplified to

$$f_{\phi\phi\phi} + ff_{\phi\phi} + \beta s f_{\phi\phi} + 2(1 - f_{\phi}^2)$$

$$+ 2\beta(1 - f_{\phi}s_{\phi}) + 2\frac{U_F^2}{U_e^2}\vartheta = 0, \qquad 96(a)$$

$$s_{\phi\phi\phi} + fs_{\phi\phi} + \beta s s_{\phi\phi} + 2\beta(1 - s_{\phi}^2)$$

$$+ 2(1 - f_{\phi}s_{\phi}) + 2\frac{V_F^2}{V_e^2}\vartheta = 0, \qquad 96(b)$$

$$P_r^{-1}\,\vartheta_{\phi\phi} + f\vartheta_\phi + \beta s\vartheta_\phi - 4\vartheta f_\phi - 4\beta\vartheta s_\phi = 0. \qquad 97(c)$$

The boundary conditions are,

$$\begin{aligned} &f(0) = f_\phi(0) = 0, && f_\phi(\infty) = 1, \\ &s(0) = s_\phi(0) = 0, && s_\phi(\infty) = 1, \\ &\vartheta(0) = 1, && \vartheta(\infty) = 0, \end{aligned} \qquad (98)$$

where, $\frac{a_{11}}{a_1} = \frac{U_F^2}{U_e^2}$ and $\frac{a_{16}}{a_1} = \frac{V_F^2}{V_e^2}$.

Here, $U_F^2 = -\frac{1}{m} g_X \beta_T(\Delta T)$,

$V_F^2 = -\frac{1}{n}\beta g_Y \beta_T(\Delta T)$,

$\frac{1}{m}, \frac{1}{n}$ are the respective characteristic lengths.

and $\beta = \frac{a_4}{a_1} = \frac{nB}{mA} = \frac{n}{m}\beta'$, where $\beta' = \frac{B}{A}$.

The similarity functions $f(\phi)$ and $s(\phi)$, the similarity variable ϕ, the velocity components (= u, v, w), the skin frictions (=τ_w's), heat transfer co-efficient (=q_w) associated with the equations (97) are

$$\psi(X, Y, \phi) = \left(\frac{2\nu U_e}{m}\right)^{\frac{1}{2}} f(\phi), \quad \varphi = \left(\frac{2\beta\nu V_e}{n}\right)^{\frac{1}{2}} s(\phi),$$

$$\phi = Z\sqrt{\frac{m}{2X}}.R_{ex}^{\frac{1}{2}}, \qquad u = U_e f_\phi, \quad v = V_e s_\phi,$$

$$w = -\left(\frac{U_e \nu}{2m}\right)^{\frac{1}{2}}[mf + m\phi f_\phi + n\beta' s + n\beta'\phi s_\phi] \quad ,$$

$$\tau_{w_1} = \mu U_e \sqrt{\frac{m}{2X}} R_{ex}^{\frac{1}{2}} f_{\phi\phi}(0) \quad ,$$

$$\tau_{w_2} = \mu V_e \sqrt{\frac{m}{2X}} R_{ex}^{\frac{1}{2}} s_{\phi\phi}(0)$$

and $q_w = -\kappa(\Delta T)\sqrt{\frac{m}{2X}} \cdot R_{ex}^{\frac{1}{2}} \vartheta_\phi(0)$ where $R_{ex} = \frac{U_e X}{\nu}$.

The variation of β' would be effectively limited to the value ranging from 0 to 1. If $\beta' = 0$ then B = 0 and $A \neq 0$, So that $V_e = 0$ => v = 0. When $\beta' = 1$ then A=B so that the $U_e = V_e$ => f = s and the flow is two dimensional.

CASE B (i)

In view of the equations 54(ii) and (55) the condition imposed on the case B may be written as

$$\frac{dA(Y)}{dY} = \frac{U_e}{V_e}(a_3 - a_4 + a_8) \neq 0$$

99(a,b)

$$\frac{dB(X)}{dX} = \frac{V_e}{U_e}(a_0 - a_1 + a_{13}) = 0$$

here $a_2 = a_5 = a_9 = a_{14} = 0$, because $h_1 = h_2 = 1$.
Since $\frac{U_e}{V_e}$ is nonzero and finite, the equation 99(a) suggests that U_e and V_e are either function of Y or constant. Hence equation 99(b) yeilds,

$$a_0 - a_1 + a_{13} = 0$$

$$\Longrightarrow \quad a_{13} = a_1 - a_0$$

and $B(X)$ = constant = B.

Thus γ^2 becomes for this case

$$\gamma^2 = \frac{1}{V_e}(a_3 Y + B). \tag{100}$$

Substituting the value of γ^2 in 54(i) $[\gamma^2 U_e = a_0 X + A(Y)]$ one obtains

$$\frac{U_e}{V_e}(a_3\, Y + B) = a_0\, X + A(Y), \tag{101}$$

To maintain the consistency on both sides of the above relation, we have to put $a_0 = 0$ in equation (101) then

$$\frac{U_e}{V_e}(a_3 Y + B) = A(Y). \tag{102}$$

From equation 99(a) and equation (102) it is found that

$$\frac{d}{dY}\left\{\frac{U_e}{V_e}(a_3 Y + B)\right\} = \frac{U_e}{V_e}(a_3 - a_4 + a_8)$$

$$\Longrightarrow \quad a_3 \frac{U_e}{V_e} + (a_3 Y + B)\frac{d}{dY}\left(\frac{U_e}{V_e}\right) = \frac{U_e}{V_e}(a_3 - a_4 + a_8)$$

$$\Longrightarrow \quad \frac{d}{dY}\left(\frac{U_e}{V_e}\right) = \frac{U_e}{V_e}\frac{(a_8 - a_4)}{(a_3 Y + B)} .$$

Integrating both sides gives

$$\frac{U_e}{V_e} = \frac{C_1 (Y + Y_0)^{\frac{a_8}{a_3}}}{C_2 (Y + Y_0)^{\frac{a_4}{a_3}}} .$$

If we choose $U_e = C_1 (Y+Y_0)^m$ and $V_e = C_2 (Y+Y_0)^n$,

where $\frac{a_8}{a_3} = m$, $\frac{a_4}{a_3} = n$, $C_1 = a_3^m$, $C_2 = a_3^n$ and $\frac{B}{a_3} = Y_0$;

we have $\gamma^2 = C_3 (Y+Y_0)^{1-n}$, where $C_3 = a_3^{1-n}$

and $\Delta T \propto (Y+Y_0)^{2n-1}$.

Substituting U_e, V_e and γ^2 in the similarity requirements, equation (56) gives

$$a_0 = a_1 = a_2 = a_5 = a_6 = a_7 = a_9 = a_{10} = a_{13} = a_{14} = a_{15} = a_{17} = 0$$

$$a_8 = ma_3, \quad a_{12} = a_4 = na_3, \quad a_{18} = (2n-1)a_3 ,$$

where a_3 is arbitrary.

Hence the general transform equations (49, 50, 51) take the form,

$$\nu \bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{1}{2}(n+1)a_3\bar{S}\,\bar{F}_{\bar{\phi}\bar{\phi}} + ma_3(1 - \bar{F}_{\bar{\phi}}\,\bar{S}_{\bar{\phi}}) + a_{11}\bar{\vartheta} = 0 \qquad 103(a)$$

$$\nu \bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{1}{2}(n+1)a_3\bar{S}\,\bar{S}_{\bar{\phi}\bar{\phi}} + na_3(1 - \bar{S}_{\bar{\phi}}^2) + a_{16}\bar{\vartheta} = 0 \qquad 103(b)$$

$$\frac{\nu}{P_r}\bar{\vartheta}_{\bar{\phi}\bar{\phi}} + \frac{1}{2}(n+1)a_3\bar{S}\,\bar{\vartheta}_{\bar{\phi}} + a_3(1-2n)\bar{\vartheta}\,\bar{S}_{\bar{\phi}} = 0 \qquad 103(c)$$

subject to the boundary conditions

$$\begin{aligned} &\bar{F}(0) = \bar{F}_{\bar{\phi}}(0) = 0, && \bar{F}_{\bar{\phi}}(\infty) = 1, \\ &\bar{S}(0) = \bar{S}_{\bar{\phi}}(0) = 0, && \bar{S}_{\bar{\phi}}(\infty) = 1, \qquad (104)\\ &\bar{\vartheta}(0) = 1, && \vartheta(\infty) = 0. \end{aligned}$$

Let now substitute $\bar{F} = \alpha f$, $\bar{S} = \alpha s$, $\bar{\phi} = \alpha\phi$ and $\bar{\vartheta} = \vartheta$; and choose $\left(\frac{n+1}{2\nu}\right)\alpha^2 a_3 = 1$, $\frac{2m}{n+1} = \beta$ and $\frac{2n}{n+1} = \beta'$ in the above equations (103) and (104).

Thus the above equations are changed to

$$f_{\phi\phi\phi} + sf_{\phi\phi} + \beta(1-f_\phi s_\phi) + (2-\beta')\frac{U_F^2}{U_e^2}\vartheta = 0,$$

$$s_{\phi\phi\phi} + ss_{\phi\phi} + \beta'(1-s_\phi^2) + (2-\beta')\frac{V_F^2}{V_e^2}\vartheta = 0, \qquad 105(a,b,c)$$

$$P_r^{-1}\vartheta_{\phi\phi} + s\vartheta_\phi + (2-3\beta')\vartheta s_\phi = 0.$$

The boundary conditions are

$$\begin{aligned} &f(0) = f_\phi(0) = 0, \quad && f_\phi(\infty) = 1, \\ &s(0) = s_\phi(0) = 0, && s_\phi(\infty) = 1, \qquad (106) \\ &\vartheta(0) = 1, && \vartheta(\infty) = 0, \end{aligned}$$

where $\frac{a_{11}}{a_3} = \frac{U_F^2}{U_e^2}$, and $\frac{a_{16}}{a_3} = \frac{V_F^2}{V_e^2}$,

here $U_F^2 = -\frac{A(Y)}{a_3}(\Delta T)\beta_T g_x$ and $V_F^2 = -g_Y\beta_T(\Delta T)(Y+Y_o)$.

To have the meaningful characteristic length we have to put either m=n or g_X=0 resulting the equations 105(a) and 105(b) identical implying $\beta=\beta'$ or 105(a) becomes a force flow equation where $U_F^2=0$.

The similarity functions f(ϕ) and s(ϕ), the similarity variable ϕ, the velocity components (= u, v, w) the skin frictions, the heat transfer co-efficient and the Nusselt number associated with the equations (105) are,

$$\phi = \frac{Z}{Y_1\sqrt{2-\beta'}} R_{eY}^{\frac{1}{2}}, \quad \text{where } (Y + Y_0) = Y_1 \text{ (say) and } R_{eY} = \frac{V_e Y_1}{\nu}$$

$U_e = c_1 Y_1^m$, $V_e = c_2 Y_1^n$, $\Delta T \propto Y_1^{2n-1}$,

$u = U_e f_\phi$, $v = V_e s_\phi$

and $w = (\sqrt{2-\beta'})\left(\frac{V_e\nu}{Y_1}\right)^{\frac{1}{2}}\left[s + \left(\frac{\beta'-1}{2-\beta'}\right)\phi s_\phi\right].$

Stream functions,

$$\psi(X, Y, \phi) = \left[\frac{2\nu \cdot Y_1^{1-n}}{c_2(n+1)}\right]^{\frac{1}{2}} U_e f(\phi),$$

$$\varphi(X, Y, \phi) = \left(\frac{2\nu Y_1}{n+1}\right)^{\frac{1}{2}} V_e^{\frac{1}{2}} s(\phi).$$

Skin frictions are

$$\tau_{w_1} = \frac{\mu U_e}{Y_1\sqrt{(2-\beta')}} R_{eY}^{\frac{1}{2}} f_{\phi\phi}(0),$$

and $$\tau_{w_2} = \frac{\mu V_e}{Y_1\sqrt{(2-\beta')}} R_{eY}^{\frac{1}{2}} s_{\phi\phi}(0).$$

Heat flux, $$q_w = -\frac{\kappa(\Delta T)}{Y_1\sqrt{(2-\beta')}} R_{eY}^{\frac{1}{2}} \vartheta_\phi(0),$$

and Nusselt number $$Nu_Y = -\frac{R_{eY}^{\frac{1}{2}}}{\sqrt{(2-\beta')}} \vartheta_\phi(0),$$

when $\beta = \beta'$ then $m = n$, $c_1 = c_2$ and $U_e = V_e$ => $f = s$. Therefore equations 105(a) and 105(b) are identical. If $\beta = 0$ then $c_1 = 0$, => $U_e = 0$ => $u = 0$. The flow moves with the velocities v and w having no velocity along x-axis in the boundary layer. Again if $\beta' = 0$ then $c_2 = 0$ => $V_e = 0$ => $v=0$, then the flow moves with the velocities u and w like 2-dimensional flow.

CASE B(ii)

Case B(i) is characterised by the arbitrary constant a_3 in the similarity requirement. The possibility case B(i) reduces to special form when $\beta = 0$ *or* $\beta' = 0$. Setting the arbitrary constant a_3 to zero, the problem changes to a different type. In addition to the traditional conditions of case B, if one sets a_3=0 the equation for γ^2 reduces to

$$\gamma^2 = \frac{B}{V_e}.$$

In view of similarity requirements Eq. 56(v) and Eq. 56(ix) the following two equations are obtained

$$V_e = c_1 e^{\frac{a_4}{B}Y} \qquad \text{and} \qquad U_e = c_2 e^{\frac{a_8}{B}Y},$$

where c_1 and c_2 are constant of integrations. Thus for this type of scale factors, the relation among constants are

$$a_0 = a_1 = a_2 = a_3 = a_5 = a_6 = a_7 = a_9 = a_{10} = a_{13} = a_{14} = a_{15} = a_{17} = 0$$

$$a_{12} = a_4, \quad a_{18} = 2a_4,$$

when a_4 and a_8 are arbitrary.

Hence the general equations (49),(50), (51) take the form

$$\nu \bar{F}_{\bar{\Phi}\bar{\Phi}\bar{\Phi}} + \frac{1}{2}a_4\bar{S}\,\bar{F}_{\bar{\Phi}\bar{\Phi}} + a_8(1 - \bar{F}_{\Phi}\,\bar{S}_{\bar{\Phi}}) + a_{11}\bar{\theta} = 0, \qquad 107(a)$$

$$\nu \bar{S}_{\bar{\Phi}\bar{\Phi}\bar{\Phi}} + \frac{1}{2}a_4\bar{S}\,\bar{S}_{\bar{\Phi}\bar{\Phi}} + a_4(1 - \bar{S}^2_{\bar{\Phi}}) + a_{16}\bar{\theta} = 0, \qquad 107(b)$$

$$\frac{\nu}{P_r}\,\bar{\vartheta}_{\bar{\Phi}\bar{\Phi}} + \frac{1}{2}a_4\bar{S}\,\bar{\vartheta}_{\bar{\Phi}} - 2a_4\bar{\vartheta}\,\bar{S}_{\bar{\Phi}} = 0\,, \qquad 107(c)$$

subject to the boundary conditions

$$\begin{aligned} &\bar{F}(0) = \bar{F}_{\bar{\Phi}}(0) = 0, \quad \bar{F}_{\bar{\Phi}}(\infty) = 1\,, \\ &\bar{S}(0) = \bar{S}_{\bar{\Phi}}(0) = 0, \quad \bar{S}_{\bar{\Phi}}(\infty) = 1\,, \qquad (108) \\ &\bar{\vartheta}(0) = 1, \qquad \vartheta(\infty) = 0\,. \end{aligned}$$

Setting $\bar{F} = \alpha f$, $\bar{S} = \alpha s$, $\bar{\Phi} = \alpha\phi$, $\bar{\vartheta} = \vartheta$ in the above equations and assuming $\frac{\alpha^2 a_4}{2\nu} = 1$, $\frac{2a_8}{a_4} = \beta$,

the above equations are changed to

$$f_{\phi\phi\phi} + s f_{\phi\phi} + \beta(1 - f_\phi\, s_\phi) + 2\frac{U_F^2}{U_*^2}\,\vartheta = 0\,,$$

$$s_{\phi\phi\phi} + s\, s_{\phi\phi} + 2(1 - s_\phi^2) + 2\frac{V_F^2}{V_*^2}\,\vartheta = 0\,, \qquad 109(a,b,c)$$

$$P_r^{-1}\,\vartheta_{\phi\phi} + s\,\vartheta_\phi - 4\,\vartheta\, s_\phi = 0\,,$$

subject to the boundary conditions,

$$\begin{aligned} &f(0) = f_\phi(0) = 0, \quad f_\phi(\infty) = 1\,, \\ &s(0) = s_\phi(0) = 0, \quad s_\phi(\infty) = 1\,, \qquad (110) \\ &\vartheta(0) = 1, \qquad \vartheta(\infty) = 0\,. \end{aligned}$$

where $\dfrac{a_{11}}{a_4} = \dfrac{U_F^2}{U_e^2}$, and $\dfrac{a_{16}}{a_4} = \dfrac{V_F^2}{V_e^2}$;

here $U_F^2 = -\dfrac{A}{a_4} g_X (\Delta T) \beta_T$ and $V_F^2 = -\dfrac{B}{a_4} g_Y (\Delta T) \beta_T$.

ϕ, U_e, V_e, ΔT and the velocity components within the boundary layer for the above equations are,

$$\phi = Z\sqrt{\frac{n}{2Y}}\; R_{eY}^{\frac{1}{2}}, \qquad U_e = c_2 e^{mY}, \qquad V_e = c_1 e^{nY},$$

$$\Delta T \propto e^{2nY}, \qquad u = U_e f_\phi, \qquad v = V_e s_\phi\,,$$

$$w = -\left(\frac{V_e \nu n}{2}\right)^{\frac{1}{2}} (s + \phi\, s_\phi)\,,$$

$$\psi(X, Y, \phi) = \left(\frac{2\nu}{n c_1} e^{-nY}\right)^{\frac{1}{2}} U_e f(\phi)\,,$$

$$\varphi(X, Y, \phi) = \left(\frac{2\nu V_e}{n}\right)^{\frac{1}{2}} s(\phi)\,.$$

Skin frictions are,

$$\tau_{w_1} = \mu U_e \sqrt{\frac{n}{2Y}} \cdot R_{eY}^{\frac{1}{2}} f_{\phi\phi}(0)\,,$$

$$\tau_{w_2} = \mu V_e \sqrt{\frac{n}{2Y}} \cdot R_{eY}^{\frac{1}{2}} s_{\phi\phi}(0)\,.$$

Heat flux, $\quad q_w = -\kappa(\Delta T)\sqrt{\dfrac{n}{2Y}}\, R_{eY}^{\frac{1}{2}}\, \theta_\phi(0)$

where $\dfrac{a_4}{B} = n$ and $\dfrac{a_8}{B} = m$ are the inverse of length scale respectively.

CASE C

Similar to case B, two possible situation arise again in dealing with the case 'C'. These are,

CASE C(i)

In view of equations 54(ii) and (55), the condition imposed on the case C may be written as

$$\frac{dA(Y)}{dY} = \frac{U_e}{V_e}(a_3-a_4+a_8) = 0$$

and $$\frac{dB(x)}{dx} = \frac{V_e}{U_e}(a_0-a_1+a_{13}) \neq 0, \qquad 111(a,b)$$

Since $\frac{U_e}{V_e} \neq 0 \Rightarrow U_e$ and V_e are either constant or function of X only, $\frac{U_e}{V_e}$ = constant, will be found to be a special form of the present case. Hence from 54(i) and (55).

$$\gamma^2 = \frac{1}{U_e}(a_0X+A) \quad \text{and} \quad \gamma^2 = \frac{B(X)}{V_e} \qquad 112(a,b)$$

From (112) it is found that

$$\frac{V_e}{U_e} = \frac{B(X)}{a_0X+A} \qquad 112(c)$$

Setting $\frac{V_e}{U_e}$ in 111(b), and integrating, the following equation results

$$B(X) = (a_0X+A)^{1-\frac{a_1}{a_0}+\frac{a_{13}}{a_0}} + C \tag{113}$$

here C is integrating constant, let C=0. Substituting B(X) in 112(C), expressed for $\frac{U_e}{V_e}$ is

$$\frac{U_e}{V_e} = \frac{c_2(X+X_o)^m}{c_1(X+X_o)^n} \tag{114}$$

where $\frac{a_1}{a_0} = m$, $\frac{a_{13}}{a_0} = n$, $c_1 = a_0^n$, $c_2 = a_0^m$ and $\frac{A}{a_0} = X_0$.

If from Eq(114), it is assumed that

$$U_e = c_2X_1^m, \quad V_e = c_1X_1^n \quad \text{and} \quad \gamma = c_3X_1^{\frac{1-m}{2}} \qquad 115(a,b,c)$$

where $c_3 = a_o^{\frac{1-m}{2}}$ and $(X + X_0) = X_1$

Substituting γ^2, U_e and V_e in the similarity requirements are obtained,

$a_2 = a_3 = a_4 = a_5 = a_6 = a_8 = a_9 = a_{10} = a_{12} = a_{14} = a_{15} = a_{18} = 0$

$a_7 = a_1 = ma_0$, $a_{13} = na_0$, $a_{17} = (2m-1)a_0$ and a_o is arbitrary.

The general transformed equations (49, 50, 51) take the form,

$$\nu \bar{F}_{\bar\phi\bar\phi\bar\phi} + \frac{1}{2}(m+1)a_0\bar{F}\,\bar{F}_{\bar\phi\bar\phi} + ma_0(1-\bar{F}_{\bar\phi}^2) + a_{11}\bar{\vartheta} = 0,$$

$$\nu \bar{S}_{\bar\phi\bar\phi\bar\phi} + \frac{1}{2}(m+1)a_0\bar{F}\,\bar{S}_{\bar\phi\bar\phi} + na_0(1-\bar{F}_{\bar\phi}\,\bar{S}_{\bar\phi}) + a_{16}\bar{\vartheta} = 0, \qquad 116(a,b,c)$$

$$\frac{\nu}{P_r}\bar{\vartheta}_{\bar\phi\bar\phi} + \frac{1}{2}(m+1)a_0\bar{F}\,\bar{\vartheta}_{\bar\phi} + (1-2m)\;a_0\;\bar{\vartheta}\;\bar{F}_{\bar\phi} = 0,$$

subject to the boundary conditions:

$$\begin{aligned} &\bar{F}(0) = \bar{F}_{\bar\phi}(0) = 0, && \bar{F}_{\bar\phi}(\infty) = 1,\\ &\bar{S}(0) = \bar{S}_{\bar\phi}(0) = 0, && \bar{S}_{\bar\phi}(\infty) = 1, \qquad (117)\\ &\bar{\vartheta}(0) = 1, && \vartheta(\infty) = 0. \end{aligned}$$

As before substituting $\bar{F} = \alpha f$, $\bar{S} = \alpha s$, $\bar{\phi} = \alpha\phi$, $\bar{\vartheta} = \vartheta$ and writing $\frac{\alpha^2 a_0(m+1)}{2\nu} = 1$, $\frac{2m}{m+1} = \beta$ and $\frac{2n}{m+1} = \beta'$,

the above equations (116) are changed to

$$f_{\phi\phi\phi} + ff_{\phi\phi} + \beta(1-f_\phi^2) + (2-\beta)\frac{U_F^2}{U_e^2}\vartheta = 0,$$

$$s_{\phi\phi\phi} + fs_{\phi\phi} + \beta'(1-f_\phi s_\phi) + (2-\beta)\frac{V_F^2}{V_e^2}\vartheta = 0, \qquad 118(a,b,c)$$

$$P_r^{-1}\vartheta_{\phi\phi} + f\vartheta_\phi + (2-3\beta)f_\phi\vartheta = 0,$$

the boundary conditions become

$$f(0) = f_\phi(0) = 0, \qquad f_\phi(\infty) = 1,$$
$$s(0) = s_\phi(0) = 0, \qquad s_\phi(\infty) = 1, \tag{119}$$
$$\vartheta(0) = 1, \qquad \vartheta(\infty) = 0.$$

where $\frac{a_{11}}{a_0} = \frac{U_F^2}{U_e^2}, \qquad \frac{a_{16}}{a_0} = \frac{V_F^2}{V_e^2},$

$$U_F^2 = -g_1\beta_T(\Delta T)X_1$$

and $\quad V_F^2 = -\frac{1}{a_0}g_2B(X_1)\beta_T(\Delta T).$

To have the meaningful characteristic length we have to put either m=n or g_Y=0 resulting the equation 118(a) and 118(b) identical implying $\beta = \beta'$ or 118(b) becomes a force flow equation where V_F^2=0.

here $\quad \phi = \frac{Z}{X_1}\left(\frac{1}{2-\beta}\right)^{\frac{1}{2}} R_{ex}^{\frac{1}{2}} \qquad$ where $\quad R_{ex} = \frac{U_e X_1}{\nu},$

$$\Delta T \propto (X_1)^{\frac{2\beta-2}{2-\beta}}, \quad U_e = c_2 X_1^m, \quad V_e = c_1 X_1^n$$

$$u = U_e f_\phi, \quad v = V_e s_\phi,$$

and $\quad w = -\left(\frac{U_e \nu}{X_1}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2-\beta}} [f - (1-\beta)\phi f_\phi].$

Stream functions are,

$$\psi(X, Y, \phi) = \left(\frac{2\nu}{m+1} . X_1 U_e\right)^{\frac{1}{2}} f(\phi)$$

and $\varphi(X, Y, \phi) = [\frac{2\nu X_1^{1-m}}{C_2(m+1)}]^{\frac{1}{2}} V_e s(\phi)$

The skin frictions are

$$\tau_{w_1} = \frac{\mu}{\sqrt{2-\beta}} \frac{R_{ex}^{\frac{1}{2}}}{X_1} U_e f_{\phi\phi}(0)$$

and $\tau_{w_2} = \frac{\mu V_e}{X_1\sqrt{2-\beta}} R_{ex}^{\frac{1}{2}} s_{\phi\phi}(0)$

The heat flux, $q_w = -\kappa \frac{(\Delta T)}{X_1}(2 - \beta)^{-\frac{1}{2}} \; R_{ex}^{\frac{1}{2}} \; \theta_\phi(0)$

and Nusselt number

$$Nu = - \frac{1}{\sqrt{2-\beta}} R_{ex}^{\frac{1}{2}} \theta_\phi(0)$$

Since U_e, V_e and ΔT are independent of Y, the case C(i) is a two dimensional (X,Z) problem for combined forced and free convective flow. The parameter ß and ß' are well known wedge flow parameter related to the exponent of U_e, V_e and ΔT. If $\beta = \beta'$ then $m = n \Longrightarrow U_e = V_e$ and equations 118(a) and 118(b) are identical. Details of physical significance of U_e, V_e, ΔT and ß are explained by Sparrow, Eichorn and Gregg (1959).

CASE C(ii)

Similar to the case B(ii), another situation arises if one of the constants considered in C(ii) be zero. Thus setting $a_o=0$ and in view of equations (111) one may derive the following expression for γ^2, U_e and V_e. These are

$$\gamma^2 = \frac{A}{U_e}, \quad U_e = c_1 e^{\frac{a_1}{A}x} \quad \text{and} \quad V_e = c_2 e^{\frac{a_{13}}{A}x}$$

where c_1 and c_2 are constant of integrations. Substituting the values of γ^2, U_e and V_e in similarity requirements, the relations between the constant a's become

$$a_0 = a_2 = a_3 = a_4 = a_5 = a_6 = a_8 = a_9 = a_{10} = a_{12} = a_{14} = a_{15} = a_{18} = 0$$

$$a_7 = a_1, \quad a_{17} = 2a_1$$

a_{13} and a_1 are arbitrary.

Hence the general transform equations (49, 50, 51) take the form,

$$v\overline{F}_{\overline{\Phi}\overline{\Phi}\overline{\Phi}} + \frac{a_1}{2}\overline{F}\overline{F}_{\overline{\Phi}\overline{\Phi}} + a_1(1 - \overline{F}^2_{\overline{\Phi}}) + a_{11}\overline{\vartheta} = 0$$

$$v\overline{S}_{\overline{\Phi}\overline{\Phi}\overline{\Phi}} + \frac{a_1}{2}\overline{F}\overline{S}_{\overline{\Phi}\overline{\Phi}} + a_{13}(1 - \overline{F}_{\overline{\Phi}}\,\overline{S}_{\overline{\Phi}}) + a_{16}\overline{\vartheta} = 0 \qquad 120(a,b,c)$$

$$\frac{v}{P_r}\overline{\vartheta}_{\overline{\Phi}\overline{\Phi}} + \frac{1}{2}a_1\overline{F}\,\overline{\vartheta}_{\overline{\Phi}} - 2a_1\overline{\vartheta}\,\overline{F}_{\overline{\Phi}} = 0$$

subject to the boundary conditions,

$$\begin{aligned} &\bar{F}(0) = \bar{F}_{\phi}(0) = 0, && \bar{F}_{\phi}(\infty) = 1, \\ &\bar{S}(0) = \bar{S}_{\phi}(0) = 0, && \bar{S}_{\phi}(\infty) = 1, \\ &\bar{\vartheta}(0) = 1, && \bar{\vartheta}(\infty) = 0. \end{aligned} \tag{121}$$

Setting $\bar{F} = \alpha f$, $\bar{S} = \alpha s$, $\bar{\phi} = \alpha\phi$, $\bar{\vartheta} = \vartheta$ in the above equation and choosing $\frac{a_1\alpha^2}{2\nu} = 1$ and $\frac{2a_{13}}{a_1} = \beta$ the above equations (120) are changed to

$$f_{\phi\phi\phi} + ff_{\phi\phi} + 2(1-f_\phi^2) + 2\frac{U_F^2}{U_e^2}\vartheta = 0,$$

$$s_{\phi\phi\phi} + fs_{\phi\phi} + \beta(1-f_\phi s_\phi) + 2\frac{V_F^2}{V_e^2}\vartheta = 0, \qquad 122(a,b,c)$$

$$P_r^{-1}\vartheta_{\phi\phi} + f\vartheta_\phi - 4\vartheta f_\phi = 0.$$

And the boundary conditions are

$$\begin{aligned} &f(0) = f_\phi(0) = 0, && f_\phi(\infty) = 1, \\ &s(0) = s_\phi(0) = 0, && s_\phi(\infty) = 1, \\ &\vartheta(0) = 1, && \vartheta(\infty) = 0. \end{aligned} \tag{123}$$

Where $\frac{a_{11}}{a_1} = \frac{U_F^2}{U_e^2}$ and $\frac{a_{16}}{a_1} = \frac{V_F^2}{V_e^2}$.

here $$U_F^2 = -\frac{g_X(\Delta T)\,\beta_T A}{a_1},$$

$$V_F^2 = -g_Y(\Delta T)\,\beta_T \cdot \frac{B}{a_1} \cdot$$

Here,

$$\phi = Z\sqrt{\frac{m}{2X}}\,R_{eX}^{\frac{1}{2}}, \quad U_e = c_1 e^{mX}, \quad V_e = c_2 e^{nX},$$

$$\Delta T \propto e^{2mX}, \quad u = U_e f_\phi, \quad v = V_e s_\phi$$

$$w = -\left(\frac{mU_e \nu}{2}\right)^{\frac{1}{2}}(f + \phi f_\phi),$$

$$\psi(X, Y, \phi) = \left(\frac{2\nu U_e}{m}\right)^{\frac{1}{2}} f(\phi)$$

and $$\varphi(X, Y, \phi) = \left(\frac{2\nu}{c_1 m}\,e^{-mX}\right)^{\frac{1}{2}} V_e s(\phi),$$

where $\frac{a_1}{A} = m$, $\frac{a_{13}}{A} = n$ (say) and $R_{eX} = \frac{U_e X}{\nu}$.

Skin frictions are,

$$\tau_{w_1} = \mu\left(\frac{m}{2X}R_{eX}\right)^{\frac{1}{2}} U_e f_{\phi\phi}(0)$$

and $$\tau_{w_1} = \mu\left(\frac{mR_{eX}}{2X}\right)^{\frac{1}{2}} V_e s_{\phi\phi}(0) \cdot$$

Heat flux, $q_w = -\kappa(\Delta T)\left(\frac{m}{2X}R_{eX}\right)^{\frac{1}{2}} \theta_\phi(0)$.

CASE D

In view of the equations 54(ii), (55) and the condition stated for this case it is obtained as

$$\frac{U_e}{V_e} = \text{constant},$$

$$A(Y) = \frac{U_e}{V_e}(a_3 - a_4 + a_8)Y + d_1, \qquad 124(a)$$

$$B(X) = \frac{V_e}{U_e}(a_0 - a_1 + a_{13})X + d_2, \qquad 124(b)$$

where d_1 and d_2 are two integrating constants.

Substituting A(Y) and B(X) in equations 54(i) and 54(iii), γ^2 is found to be

$$\gamma^2 = \frac{1}{U_e}a_0 X + \frac{1}{V_e}(a_3 - a_4 + a_8)Y + \frac{d_1}{U_e} \qquad 125(a)$$

and $$\gamma^2 = \frac{1}{V_e}a_3 Y + \frac{1}{U_e}(a_0 - a_1 + a_{13})X + \frac{d_2}{V_e} \qquad 125(b)$$

Comparing above equations of γ^2, the following relation can be set

$$\frac{d_1}{d_2} = \frac{U_e}{V_e} = K_1 \text{ (say)}, \implies U_e = K_1 V_e, \qquad (126)$$

$a_8 = a_4$, and $a_{13} = a_1$.

Again from equations 54(i), 54(iii) and (126),

$$\frac{U_e}{V_e} = \frac{a_0 X + A(Y)}{a_3 Y + B(X)} = K_1 \text{ (= constant)}.$$

$$\Longrightarrow a_0 X + A(Y) = K_1 [a_3 Y + B(X)] . \qquad (127)$$

From the above equation (127), it can be written as

$$A(Y) = K_1 a_3 Y, \quad \text{and} \quad B(X) = \frac{1}{K_1} a_0 X.$$

Then equations 54(i) and 54(iii) take the form,

$$\gamma^2 = \frac{1}{U_e}(a_0 X + K_1 a_3 Y)$$

$$\text{and} \quad \gamma^2 = \frac{1}{V_e}\left(\frac{1}{K_1} a_0 X + a_3 Y\right) . \qquad 128(a,b)$$

Substituting γ^2 from 128(a) in 56(ii),

$$U_e = c_1 (aX + bY)^m ,$$

$$V_e = c_2 (aX+bY)^m \qquad 129(a,b,c)$$

$$\text{and} \quad \gamma^2 = c_3 (aX + bY)^{1-m},$$

where c_1 is a constant of integration,

$c_2 = \frac{c_1}{K_1}$, $c_3 = \frac{1}{c_1}$, $\frac{a_1}{a_0} = m$ (Say), $a_0 = a$ and $a_3 K_1 = b$.

Similarity requirements furnish the relations among the constants (=a's) as

$$a_2 = a_5 = a_6 = a_9 = a_{10} = a_{14} = a_{15} = 0 ,$$

$$a_1 = a_7 = a_{13} = ma_0 , \quad a_8 = a_4 = a_{12} = ma_3 ,$$

$$a_{17} = (2m - 1)a_0 , \quad a_{18} = (2m-1)a_3 ,$$

where a_0 and a_3 are arbitrary.

Hence the general transform equations (49,50, 51) take the form,

$$\nu \bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \left(\frac{m+1}{2}\right) a_0 \bar{F} \bar{F}_{\bar{\phi}\bar{\phi}} + \left(\frac{m+1}{2}\right) a_3 \bar{S} \bar{F}_{\bar{\phi}\bar{\phi}} + ma_0(1 - \bar{F}_{\bar{\phi}}^2)$$

$$+ \quad ma_3(1-\bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}}) + a_{11}\bar{\vartheta} = 0 , \qquad 130(a)$$

$$\nu \bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \left(\frac{m+1}{2}\right) a_0 \bar{F} \bar{S}_{\bar{\phi}\bar{\phi}} + \left(\frac{m+1}{2}\right) a_3 \bar{S} \bar{S}_{\bar{\phi}\bar{\phi}} + a_3 m(1 - \bar{S}_{\bar{\phi}}^2)$$

$$+ \quad ma_0(1-\bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}}) + a_{16}\bar{\vartheta} = 0 , \qquad 130(b)$$

$$\frac{\nu}{P_r} \bar{\vartheta}_{\bar{\phi}\bar{\phi}} + \left(\frac{m+1}{2}\right) a_0 \bar{F} \bar{\vartheta}_{\bar{\phi}} + \frac{1}{2}(m+1) a_3 \bar{S} \bar{\vartheta}_{\bar{\phi}}$$

$$- \quad (2m - 1) a_0 \bar{\vartheta} \bar{F}_{\bar{\phi}} - (2m - 1) a_3 \bar{\vartheta} \bar{S}_{\bar{\phi}} = 0 , \qquad 130(c)$$

subject to the boundary conditions,

$$\begin{aligned} &\bar{F}(0) = \bar{F}_{\bar{\phi}}(0) = 0, \quad \bar{F}_{\bar{\phi}}(\infty) = 1 , \\ &\bar{S}(0) = \bar{S}_{\bar{\phi}}(0) = 0, \quad \bar{S}_{\bar{\phi}}(\infty) = 1 , \\ &\bar{\vartheta}(0) = 1, \quad \vartheta(\infty) = 0 . \end{aligned} \qquad (131)$$

Let now the following substitution be made $\overline{F} = \alpha f$, $\overline{S} = \alpha s$, $\overline{\phi} = \alpha\phi$, $\overline{\vartheta} = \vartheta$ along with the assumptions

$$\left(\frac{m+1}{2}\right)\frac{a_0\alpha^2}{\nu} = 1, \qquad \frac{a_3}{a_0} = C \qquad and \qquad \frac{2m}{m+1} = \beta .$$

Hence equations (130) become,

$$f_{\phi\phi\phi} + ff_{\phi\phi} + csf_{\phi\phi} + \beta(1-f_\phi^2) + c\beta(1-f_\phi s_\phi) + (2-\beta)\frac{U_F^2}{U_e^2}\vartheta = 0 , \qquad 132(a)$$

$$s_{\phi\phi\phi} + fs_{\phi\phi} + css_{\phi\phi} + c\beta(1-s_\phi^2) + \beta(1-f_\phi s_\phi) + (2-\beta)\frac{V_F^2}{V_e^2}\vartheta = 0, \qquad 132(b)$$

$$P_r^{-1}\vartheta_{\phi\phi} + f\vartheta_\phi + cs\vartheta_\phi + (2-3\beta)[\vartheta f_\phi + c\vartheta s_\phi] = 0 , \qquad 132(c)$$

subject to the boundary conditions,

$$\begin{aligned} &f(0) = f_\phi(0) = 0, && f_\phi(\infty) = 1 , \\ &s(0) = s_\phi(0) = 0, && s_\phi(\infty) = 1 , \\ &\vartheta(0) = 1, && \vartheta(\infty) = 0 , \end{aligned} \qquad (133)$$

where $\dfrac{a_{11}}{a_0} = \dfrac{U_F^2}{U_e^2}$ and $\dfrac{a_{16}}{a_0} = \dfrac{V_F^2}{V_e^2}$,

here $U_F^2 = -g_x\left(\dfrac{aX + bY}{a}\right)\beta_T(\Delta T)$, and $V_F^2 = -\frac{c}{b} g_Y (\Delta T)\ (aX + bY) \beta_T$.

If c = 0 $\Longrightarrow$ a_3 = 0, then the velocity components are independent of Y. The equations (132) are similar to the Sparrow, Eichorn and Gregg (1957)'s equation.

The similarity functions f(ϕ), and s(ϕ), the similarity variable ϕ, the velocity component u, v,w, the skin frictions(=τ_w's), heat flux (=q_w) and Nuselt number (=Nu) associated with the equations (132) are

$$\psi(X, Y, \phi) = \left(\frac{2\nu U_e}{m+1}\right)^{\frac{1}{2}} \left(\frac{aX+bY}{a}\right)^{\frac{1}{2}} f(\phi),$$

$$\varphi(X, Y, \phi) = \left(\frac{2\nu V_e}{m+1}\right)^{\frac{1}{2}} \left(\frac{aX+bY}{aK_1}\right)^{\frac{1}{2}} s(\phi),$$

$$\phi = \left[\frac{m+1}{2} . a\ R_e\right]^{\frac{1}{2}} . \frac{Z}{(aX + bY)} ,$$

where $$R_e = \frac{U_e(aX + bY)}{\nu} ,$$

$$u = U_e f_\phi, \qquad v = V_e s_\phi,$$

$$w = -\left(\frac{U_e a\nu}{aX + bY}\right)^{\frac{1}{2}} \left(\frac{m+1}{2}\right)^{\frac{1}{2}} \left[f - \phi \frac{1-m}{1+m} f_\phi + cs - \frac{1-m}{1+m} \phi s_\phi\right],$$

$U_e = c_1(aX + bY)^m, \quad V_e = c_2(aX + bY)^m,$

$\Delta T \propto (aX + bY)^{2m-1},$

Skin friction are,

$$\tau_{w_1} = \mu\left[\frac{m+1}{2}.aR_e\right]^{\frac{1}{2}} \frac{U_e}{(aX + bY)} f_{\phi\phi}(0) \;,$$

and $$\tau_{w_2} = \mu\left[\frac{m+1}{2}.aR_e\right]^{\frac{1}{2}} \frac{V_e}{(aX + bY)} s_{\phi\phi}(0) \;.$$

Heat flux, $$q_w = -\kappa(\Delta T)\left(\frac{m+1}{2}.a\right)^{\frac{1}{2}} \frac{R_e^{\frac{1}{2}}}{(aX + bY)} \vartheta_\phi(0) \,,$$

and Nusselt number

$$Nu = -\left(\frac{m+1}{2}.a\right)^{\frac{1}{2}} R_e^{\frac{1}{2}} \vartheta_\phi(0) \;.$$

where μ and κ are co-effecient of viscosity and thermal conductivity of heat respectively.

CHAPTER 5

The arbitrary surface

CHAPTER 5

The arbitrary surface

In this case, h_1 and h_2 are not constants and not equal to zero

In view of equations 54(ii), (55) and condition stated in case D one obtains

$$\frac{h_2}{h_1}\ \frac{U_e}{V_e} = \text{Constant,} \qquad 134(a)$$

$$A(Y) = \frac{U_e}{V_e}\frac{h_2}{h_1}(a_3 - a_4 - a_5 + a_8 - a_9)Y + d_1, \qquad 134(b)$$

$$B(X) = \frac{V_e h_1}{U_e h_2}(a_0 - a_1 - a_2 + a_{13} - a_{14})X + d_2, \qquad 134(c)$$

where d_1 and d_2 are two integrating constants.
Substituting A(Y) and B(X) in equations 54(i) and 54(iii), γ^2 is found to be

$$\gamma^2 = \frac{h_1}{U_e}a_0 X + \frac{h_2}{V_e}(a_3 - a_4 - a_5 + a_8 - a_9)Y + \frac{d_1 h_1}{U_e} \qquad 135(a)$$

and $$\gamma^2 = \frac{h_2}{V_e}a_3 Y + \frac{h_1}{U_e}(a_0 - a_1 - a_2 + a_{13} - a_{14})X + \frac{d_2 h_2}{V_e}, \qquad 135(b)$$

comparison of above equations for γ^2, gives

$$\begin{aligned} a_4 + a_5 - a_8 + a_9 &= 0, \\ a_1 + a_2 - a_{13} + a_{14} &= 0. \end{aligned} \qquad 136(a,b)$$

From equations 52(xi) and Eq. 52(xv)

$$V_e = K_1 U_e, \quad \text{where} \quad K_1 = \sqrt{\frac{a_{10}}{a_{14}}}, \tag{137(a)}$$

From Eq. 134(a) and Eq. (137), it is found that

$$h_2 = K_2 h_1, \quad \text{where} \quad K_2 = \text{Constant}. \tag{137(b)}$$

From equation 52(viii) and 54(i),

$$U_e = c_1 (a_0 X + A(Y))^{\frac{a_7}{a_0}}, \tag{138(a)}$$

and $$V_e = c_2 (a_0 X + A(Y))^{\frac{a_7}{a_0}}, \tag{138(b)}$$

where c_1 is constant of integration and $c_2 = c_1 k_1$.

Again from 54(i) and 54(iii),

$$\frac{a_0 X + A(Y)}{a_3 Y + B(X)} = K_3 \quad \text{(constant), where} \quad K_3 = \frac{K_2}{K_1}.$$

Hence it can be written that $A(Y) = K_3 a_3 Y$ and $B(X) = \frac{a_0 X}{K_3}$.

Substituting A(Y) and B(X) in Eq. (138)

$$U_e = c_1 (aX + bY)^m, \tag{139(a)}$$

and $$V_e = c_2 (aX + bY)^m, \tag{139(b)}$$

where $a_0 = a, \quad a_3 K_3 = b, \quad \frac{a_8}{a_3} = \frac{a_7}{a_0} = m$.

In view of equation 52(ii) and Eq. 54(i),

$$h_1 = c_3(aX + bY)^{n-m},$$

$$h_2 = c_4(aX + bY)^{n-m} \qquad 140(a,b,c)$$

and $\gamma^2 = c_5(aX + bY)^{n+1-2m},$

where $c_4 = c_3K_2, \quad \frac{a_1}{a_0} = n$ and $c_5 = \frac{c_3}{c_1}$.

Similarity requirements furnish the relations between constants a's. These relations are

$$a_1 = na_0, \quad a_2 = 2(m-n)a_0, \quad a_4 = na_3, \quad a_5 = 2(m-n)a_3,$$

$$a_6 = 0, \quad a_7 = ma_0, \quad a_8 = ma_3, \quad a_9 = (n-m)a_3, \quad a_{10} = K_1^2(n-m)\ a_0,$$

$$a_{12} = ma_3, \quad a_{13} = ma_0, \quad a_{14} = (n-m)a_0, \quad a_{15} = \frac{1}{K_1^2}(n-m)a_3,$$

$$a_{17} = (3m-n-1)a_0, \quad a_{18} = (3m-n-1)a_3,$$

where a_0 and a_3 are arbitrary.

Hence the general transfeorm equations (49), (50), (51) take the form,

$$\nu\overline{F}_{\bar\Phi\bar\Phi\bar\Phi} + \frac{1}{2}(3n - 2m + 1)a_0\ \overline{F}\ \overline{F}_{\bar\Phi\bar\Phi} + \frac{1}{2}(3n - 2m + 1)a_3\overline{F}_{\bar\Phi\bar\Phi}\overline{S}$$

$$+ \quad ma_0(1 - \overline{F}^2_{\bar\Phi}) + na_3(1 - \overline{F}_{\bar\Phi}\ \overline{S}_{\bar\Phi})$$

$$- \quad K_1^2(n - m)\ a_0(1 - \overline{S}^2_{\bar\Phi}) + a_{11}\bar\theta = 0 \quad , \qquad 141(a)$$

$$\nu \bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{1}{2}(3n - 2m + 1)\, a_0 \bar{F}\, \bar{S}_{\bar{\phi}\bar{\phi}}$$

$$+ \quad \frac{1}{2}(3n - 2m + 1)\, a_3 \bar{S}\, \bar{S}_{\bar{\phi}\bar{\phi}} + m a_3 (1 - \bar{S}_{\bar{\phi}}^2)$$

$$+ \quad n a_0 (1 - \bar{F}_{\bar{\phi}}\, \bar{S}_{\bar{\phi}}) - \frac{1}{K_1^2}(n - m)\, a_3 (1 - \bar{F}_{\bar{\phi}}^2) + a_{16}\bar{\vartheta} = 0 , \qquad 141(b)$$

$$\frac{\nu}{P_r}\bar{\vartheta}_{\bar{\phi}\bar{\phi}} + \frac{a_0}{2} \quad (3n - 2m + 1)\, \bar{F}\, \bar{\vartheta}_{\bar{\phi}} + \frac{1}{2}(3n - 2m + 1)\, a_3 \bar{S}\, \bar{\vartheta}_{\bar{\phi}}$$

$$- \quad (3m - n - 1)\, a_0 \bar{\vartheta}\, \bar{F}_{\bar{\phi}} - (3m - n - 1)\, a_3 \bar{\vartheta}\, \bar{S}_{\bar{\phi}} = 0 , \qquad 141(c)$$

subject to the boundary conditions,

$$\begin{aligned} &\bar{F}(0) = \bar{F}_{\bar{\phi}}(0) = 0, && \bar{F}_{\bar{\phi}}(\infty) = 1, \\ &\bar{S}(0) = \bar{S}_{\bar{\phi}}(0) = 0, && \bar{S}_{\bar{\phi}}(\infty) = 1 , \\ &\bar{\vartheta}(0) = 1, && \bar{\vartheta}(\infty) = 0 . \end{aligned} \qquad (142)$$

Substituting $\bar{F} = \alpha f$, $\bar{S} = \alpha s$, $\bar{\phi} = \alpha\phi$, $\bar{\vartheta} = \vartheta$ and

assuming $(3n - 2m + 1)\ \dfrac{\alpha^2 a_0}{2\nu} = 1$, and $\dfrac{a_3}{a_0} = C$.

The equation (141) takes the form,

$$f_{\phi\phi\phi} + f f_{\phi\phi} + c f_{\phi\phi} s + \frac{2m}{3n - 2m + 1}\ (1 - f_\phi^2)$$

$$+ \left(\frac{2nC}{3n - 2m + 1}\right) (1 - f_\phi\, s_\phi) - k_1^2\, 2\left(\frac{n - m}{3n - 2m + 1}\right) (1 - s_\phi^2)$$

$$+ \quad \frac{2}{3n - 2m + 1}\ \frac{U_P^2}{U_e^2}\ \vartheta = 0 , \qquad 143(a)$$

$$s_{\phi\phi\phi} + (f s_{\phi\phi} + c s s_{\phi\phi}) + (\frac{2mc}{3n-2m+1})(1 - s_\phi^2) + \frac{2n}{3n-2m+1}(1 - f_\phi s_\phi)$$

$$- \frac{2c}{K_1^2}(\frac{n-m}{3n-2m+1})(1 - f_\phi^2) + \frac{2}{3n-2m+1}\frac{V_P^2}{V_e^2}\theta = 0, \qquad 143(b)$$

$$P_r^{-1}\theta_{\phi\phi} + (f\theta_\phi + c s \theta_\phi) - 2(\frac{3m-n-1}{3n-2m+1})(\theta f_\phi + c\theta s_\phi) = 0. \qquad 143(c)$$

The boundary conditions are,

$$\begin{aligned} &f(0) = f_\phi(0) = 0, \quad f_\phi(\infty) = 1, \\ &s(0) = s_\phi(0) = 0, \quad s_\phi(\infty) = 1, \\ &\theta(0) = 1, \qquad \theta(\infty) = 0 \end{aligned} \qquad (144)$$

where $\frac{a_{11}}{a_0} = \frac{U_P^2}{U_e^2}, \quad \frac{a_{16}}{a_0} = \frac{V_P^2}{V_e^2},$

here $U_P^2 = -g_X(\Delta T)\beta_T h_1 \times$ characteristic length

and $V_P^2 = -g_Y(\Delta T)\beta_T h_2 \times$ characteristic length,

where characteristic length = $\frac{aX + bY}{a}$.

The similarity variables are,

$$\phi = (\frac{aZ}{aX+bY})\sqrt{\frac{(3n-2m+1)}{2h_1}}\,.\,R_e^{\frac{1}{2}},$$

$$U_e = c_1(aX + bY)^m,$$

$$V_e = c_2(aX + bY)^m,$$

$$h_1 = c_3(aX + bY)^{n-m},$$

$$h_2 = c_4(aX + bY)^{n-m},$$

$$\Delta T \propto (aX + bY)^{2m-n-1}.$$

The velocity components are,

$$u = U_e f_\phi, \qquad v = V_e s_\phi ,$$

$$w = -(3n+1-2m)^{-\frac{1}{2}} \left[\frac{U_e}{2h_1} \cdot \frac{\nu a}{aX+bY}\right]^{\frac{1}{2}} [(3n-2m+1)(f+cs)$$

$$-(n+1-2m)(\phi f_\phi + \phi s_\phi)] .$$

The stream functions are,

$$\psi(X, Y, \phi) = h_2 \sqrt{\frac{U_e h_1 (aX+bY)}{a} \cdot \left(\frac{2\nu}{3n-2m+1}\right)} \cdot f(\phi)$$

and $$\varphi(X, Y, \phi) = h_2 \frac{c}{b} \sqrt{V_e h_1 (aX+bY) \cdot \left(\frac{2\nu a}{3n-2m+1}\right)} \cdot s(\phi) .$$

The skin frictions and heat flux associated with the equations (143) are

$$\tau_{w_1} = \mu\left[\frac{(3n-2m+1)}{2h_1}\right]^{\frac{1}{2}} R_e^{\frac{1}{2}} \frac{aU_e}{(aX+bY)} f_{\phi\phi}(0),$$

$$\tau_{w_2} = \mu\left[\frac{(3n-2m+1)}{2h_1}\right]^{\frac{1}{2}} R_e^{\frac{1}{2}} \frac{aV_e}{(aX+bY)} s_{\phi\phi}(0),$$

and heat flux,

$$q_w = -\kappa(\Delta T)\,.\left[\frac{(3n-2m+1)}{2h_1}\right]^{\frac{1}{2}} R_e^{\frac{1}{2}} \frac{a}{(aX+bY)} \vartheta_{\phi}(0)\,.$$

where κ is conductivity of heat and μ is co-efficient of viscosity and $R_e = \frac{U_e}{\nu}\left(\frac{aX+bY}{a}\right)$

If $c = 0 \Rightarrow a_3 = 0 \Rightarrow b = 0$, then the velocity components are independednt of Y.

Again, if $n = m$ and $\frac{2m}{m+1} = \beta$ then h_1 and h_2 will be constant and equations 143(a,b,c) are similar to equation (132).

CHAPTER 6

Results and Discussions

CHAPTER 6

Result and Discussions:

The ordinary differential equations 67(a,b,c) were solved numerically by Swergert iteration technique. The calculations were carried out for several values of Prandtl numbers (Table-3, 4) and parameter ß (Table 1 and Table 2). Solutions of equations 67(a,b,c) subject to the boundary conditions (Eq.68) were obtained for twenty positive values of ß and in particular for ß=0, 0.25, 0.50, 0.75 and 1.00. For $\frac{G_r}{R_e^2}$ = ·100 and $\frac{G_r}{R_e^2}$ = 100 initial values f"(0), s"(0) and $\vartheta'(0)$ are given in table 1 and table 2. All twenty cases considered and the corresponding quantities for ß=0 are also given for comparison. From the solutions results have been listed in Table 3 and Table 4. The values for f", s" and ϑ' may be obtained at the rectangular body surface (ϕ=0) which are required in evaluating the skin-frictions and heat-transfer calculations, where the primes denote differentiation with respect to ϕ.

It is often advisable to compare, wherever possible, the results obtained for two cases and ß=0.75 only,

i.e, $\frac{G_r}{R_e^2}$ ==> 0.10 (natural convection is negligible) and $\frac{G_r}{R_e^2}$ -> 100 (forced convection is negligible)

where $\frac{G_r}{R_e^2} = \frac{U_F^2}{U_e^2} = \frac{V_F^2}{V_e^2}$

The comparison is shown below,

Case (a): $\frac{G_r}{R_e^2} \longrightarrow 0.100$

In this case components of skin friction factors for the forced convective flow.

$$\frac{\tau_{w_1}}{\varrho U_e^2}\sqrt{R_{ex}} = f''(0) = 1.3315$$

and

$$\frac{\tau_{w_2}}{\varrho V_e^2}\sqrt{R_{ey}} = \frac{s''(0)}{\sqrt{\beta}} = \frac{1.20993}{\sqrt{0.75}} = 1.3971$$

On the other hand, the expression for the Nusselt numbers (=Nu) depend on Prandtl number (=Pr), and it is found that

for $P_r = 0.73$, $\frac{N_{ux}}{\sqrt{R_{ex}}} = -\vartheta_\phi(0) = 0.9030$

and $\frac{N_{uY}}{\sqrt{R_{eY}}} = \frac{-\vartheta_\phi(0)}{\sqrt{\beta}} = 1.0427$

for $P_r = 10$,

$$\frac{N_{uX}}{\sqrt{R_{eX}}} = -\theta_\phi(0) = 2.2771,$$

and $$\frac{N_{uY}}{\sqrt{R_{eY}}} = -\frac{\theta_\phi(0)}{\sqrt{\beta}} = 2.6294$$

for $P_r = 1000$

$$\frac{N_{uX}}{\sqrt{R_{eX}}} = -\theta_\phi(0) = 10.8392$$

and $$\frac{N_{uY}}{\sqrt{R_{eY}}} = -\frac{\theta_\phi(0)}{\sqrt{\beta}} = 12.5164$$

Case (b): $\frac{G_r}{R_e^2} \longrightarrow 100.00$

In the case of the free convection problem it can be shown easily that,

For, $P_r = 0.73$,

$$\frac{\tau_{w_1}}{\varrho U_e^2}\sqrt{R_{eX}} = f''(0) = 22.6893\ ,$$

$$\frac{\tau_{w_2}}{\varrho V_e^2}\sqrt{R_{eY}} = \frac{s''(0)}{\sqrt{\beta}} = 26.7847,$$

$$\frac{N_{uX}}{\sqrt{R_{eX}}} = 2.0532$$

and $$\frac{N_{uY}}{\sqrt{R_{eY}}} = \frac{2.0532}{\sqrt{\beta}} = 2.3709.$$

For $P_r = 10$,

$$\frac{\tau_{w_1}}{\varrho U_e^2}\sqrt{R_{eX}} = 13.8713,$$

$$\frac{\tau_{w_2}}{\varrho V_e^2}\sqrt{R_{eY}} = 16.0634,$$

$$\frac{N_{uX}}{\sqrt{R_{eX}}} = 4.4907$$

and $$\frac{N_{eY}}{\sqrt{R_{eY}}} = 5.1856.$$

For $P_r = 1000$,

$$\frac{\tau_{w_1}}{\varrho U_e^2}\sqrt{R_{eX}} = 5.3831,$$

$$\frac{\tau_{w_2}}{\varrho V_e^2}\sqrt{R_{eY}} = 6.1030,$$

$$\frac{N_{uX}}{\sqrt{R_{eX}}} = 15.9776$$

and $$\frac{N_{eY}}{\sqrt{R_{eY}}} = 18.4499$$

It is observed from Table (1-4) that there are large variation of the skin-frictions and heat transfer rate with the rise buoyancy effect and any Prandtl number.

Representative velocities and temperature profiles for several values of $\frac{G_r}{R_e^2}$ are shown in Fig (1-6). It is seen from Fig.1 that for $\frac{G_{rX}}{R_{eX}^2}$ = 100 and $\frac{G_{rY}}{R_{eY}^2}$ = 0.100,the u-velocity gradient at the wall increases as buoyancy force ($\frac{G_{rX}}{R_{eX}^2}$) increases. Similarly from Fig.5 for $\frac{G_{rX}}{R_{eX}^2}$ = 100 and $\frac{G_{rY}}{R_{eY}^2} > 0$, the v-velocity gradient at the wall increases as buoyancy force ($\frac{G_{rX}}{R_{eX}^2}$) increases.

This is accompanied by an increase in the velocity near the wall, and for $P_r = 0.7$ an overshooting of the velocity beyond its free stream value occurs. The buoyancy force reduces the velocities and the velocity gradients at the wall as compared to those for pure forced convection ($\frac{G_r}{R_e^2}=0$). With regard to the temperature profiles (Fig.3,6), the temperature gradient at the wall is seen to increase as the buoyancy force increases.

A method was presented for calculating the shear stresses and the heat transfer in external flow for combined laminar free and forced convection. The parameter $\frac{U_P^2}{U_e^2} = \frac{G_{rX}}{R_{eX}^2}$ and $\frac{V_P^2}{V_e^2} = \frac{G_{rY}}{R_{eY}^2}$ are of fundamental importance in such problems. Natural convection is negligible as $\frac{G_{rX}}{R_{eX}^2} \to 0$, $\frac{G_{rY}}{R_{eY}^2} \to 0$ and forced convection has little influence as $\frac{G_{rX}}{R_{eX}^2} \to \infty$ and $\frac{G_{rY}}{R_{eY}^2} \to \infty$.

Numerical results are reported in table for the heating of upward flow past a rectangular body surface for several Prandtl numbers. It is found that the mechanisms of forced and free convection are non additive. Moreover the numerical results for the heating problem from a body surface indicate, that the transition from a pure forced to the pure free convection is gradual, especially at high Prandtle numbers. It is very difficult therefore to draw any specific conclutions concerning the values of parameters $\frac{G_{rX}}{R_{eX}^2}$ and $\frac{G_{rY}}{R_{eY}^2}$. It is observed that for the body surface and for $\frac{G_{rX}}{R_{eX}^2} < 0.1$ and $\frac{G_{rY}}{R_{eY}^2} < 0.1$, free convection is negligible whereas for $\frac{G_{rX}}{R_{eX}^2} > 1$ and $\frac{G_{rY}}{R_{eY}^2} > 1$ forced convection has little influence.

Experimental data for three dimentional case are not available in literature to make a comparison with these theoretical predictions.

TABLE - 1

ß	f"(0)	s"(0)	$\theta'(0)$
.00000	1.28151	.63747	-.72623
.05000	1.28375	.68649	-.73637
.10000	1.28620	.73336	-.74700
.15000	1.28887	.77821	-.75807
.20000	1.29172	.82120	-.76949
.25000	1.29475	.86253	-.78117
.30000	1.29793	.90232	-.79306
.35000	1.30125	.94074	-.80510
.40000	1.30470	.97788	-.81724
.45000	1.30827	1.01386	-.82946
.50000	1.31195	1.04878	-.84173
.55000	1.31573	1.08271	-.85401
.60000	1.31960	1.11572	-.86630
.65000	1.32355	1.14790	-.87858
.70000	1.32759	1.17928	-.89082
.75000	1.33169	1.20993	-.90303
.80000	1.33586	1.23988	-.91519
.85000	1.34009	1.26920	-.92730
.90000	1.34438	1.29790	-.93935
.95000	1.34871	1.32603	-.95133
1.00000	1.35310	1.35361	-.96324

For m = n = 1, Pr = .73 and $\frac{U_F^2}{U_e^2} = \frac{V_F^2}{V_e^2} = 0.100$

TABLE - 2

ß	f"(0)	s"(0)	$\theta'(0)$
.00000	24.91593	30.24675	-1.71825
.05000	24.67527	29.19215	-1.75258
.10000	24.46186	28.33895	-1.78338
.15000	24.26708	27.62093	-1.81159
.20000	24.09045	27.00884	-1.83768
.25000	23.92649	26.47300	-1.86209
.30000	23.77310	25.99744	-1.88511
.35000	23.62869	25.57058	-1.90695
.40000	23.49202	25.15385	-1.92776
.45000	23.36214	24.83074	-1.94769
.50000	23.23824	24.50620	-1.96682
.55000	23.11968	24.20622	-1.98525
.60000	23.00593	23.92759	-2.00304
.65000	22.89653	23.66766	-2.02025
.70000	22.79110	23.42427	-2.03694
.75000	22.68931	23.19558	-2.05315
.80000	22.59086	22.98006	-2.06892
.85000	22.49551	22.77638	-2.08427
.90000	22.40303	22.58344	-2.09925
.95000	22.31323	22.40023	-2.11387
1.00000	22.22593	22.22593	-2.12816

For m = n = 1, Pr = .73 and $\frac{U_F^2}{U_e^2} = \frac{V_F^2}{V_e^2} = 100$

TABLE - 3

Pr	f"(0)	s"(0)	θ'(0)
.10000	1.34898	1.22900	-.44169
.50000	1.33528	1.21341	-.78423
.70000	1.33198	1.20973	-.88811
1.00000	1.32807	1.20539	-1.01137
5.00000	1.31467	1.19094	-1.79200
7.00000	1.31229	1.18842	-2.01368
10.00000	1.30993	1.18594	-2.27706
100.00000	1.29897	1.17461	-4.98737
1000.00000	1.29194	1.16933	-10.83923

For m = n = 1, ß = .75 and $\frac{U_F^2}{U_e^2} = \frac{V_F^2}{V_e^2} = .100$

TABLE - 4

Pr	f"(0)	s"(0)	θ'(0)
.10000	29.80584	31.01480	-1.00145
.50000	24.04605	24.65360	-1.80924
.70000	22.82095	23.33331	-2.02514
1.00000	21.51700	21.93623	-2.27406
5.00000	15.98244	16.10094	-3.69433
7.00000	14.96159	15.04165	-4.06831
10.00000	13.87129	13.91091	-4.49069
100.00000	8.67774	8.61242	-8.50366
1000.00000	5.38310	5.28520	-15.97760

For m = n = 1, ß = .75 and $\frac{U_F^2}{U_e^2} = \frac{V_F^2}{V_e^2} = 100$

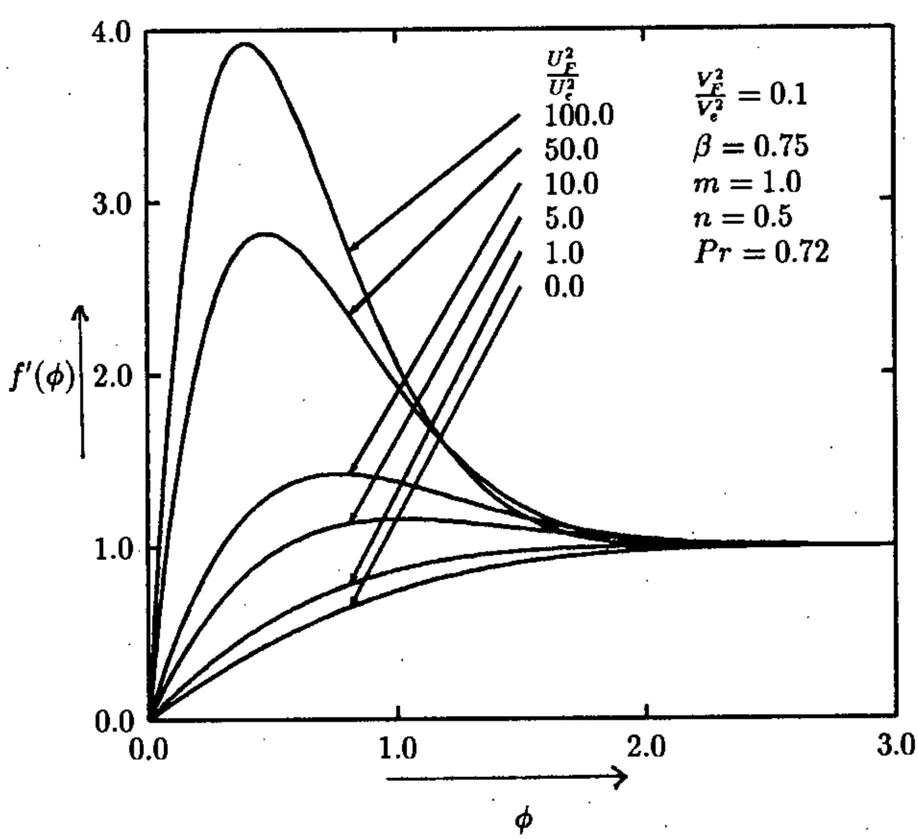


Fig.1 Representative velocity profiles of u-component for different values of $\frac{U_F^2}{U_e^2}$ for the equation (67).

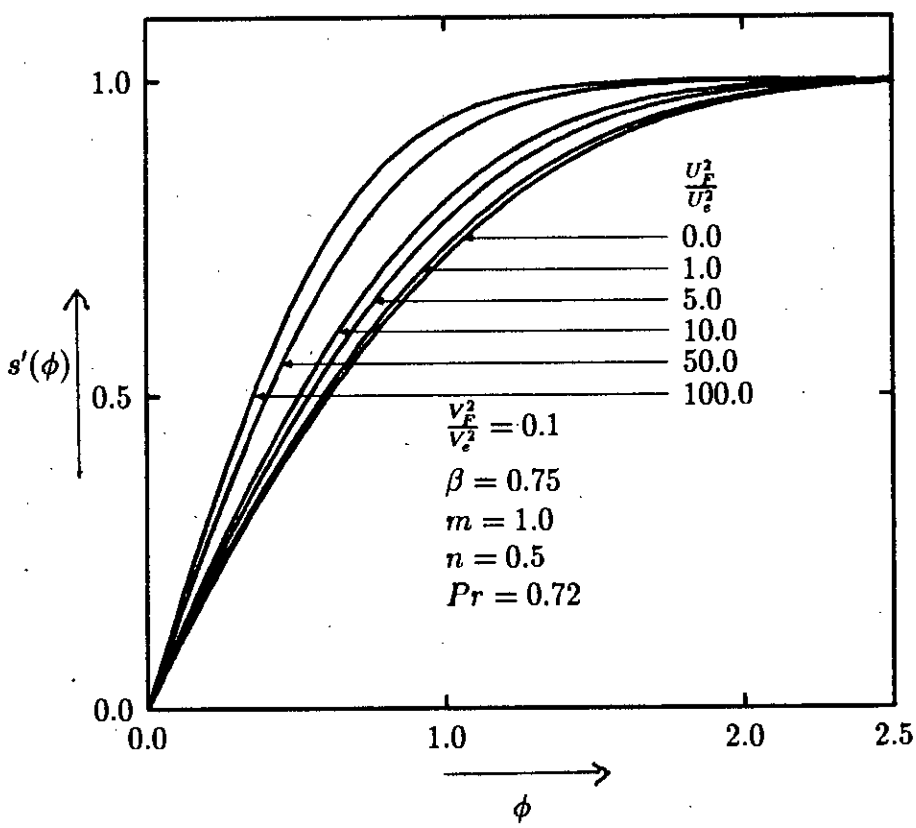


Fig.2 Representative velocity profiles of v-component for different values of $\frac{U_F^2}{U_e^2}$ for the equation (67).

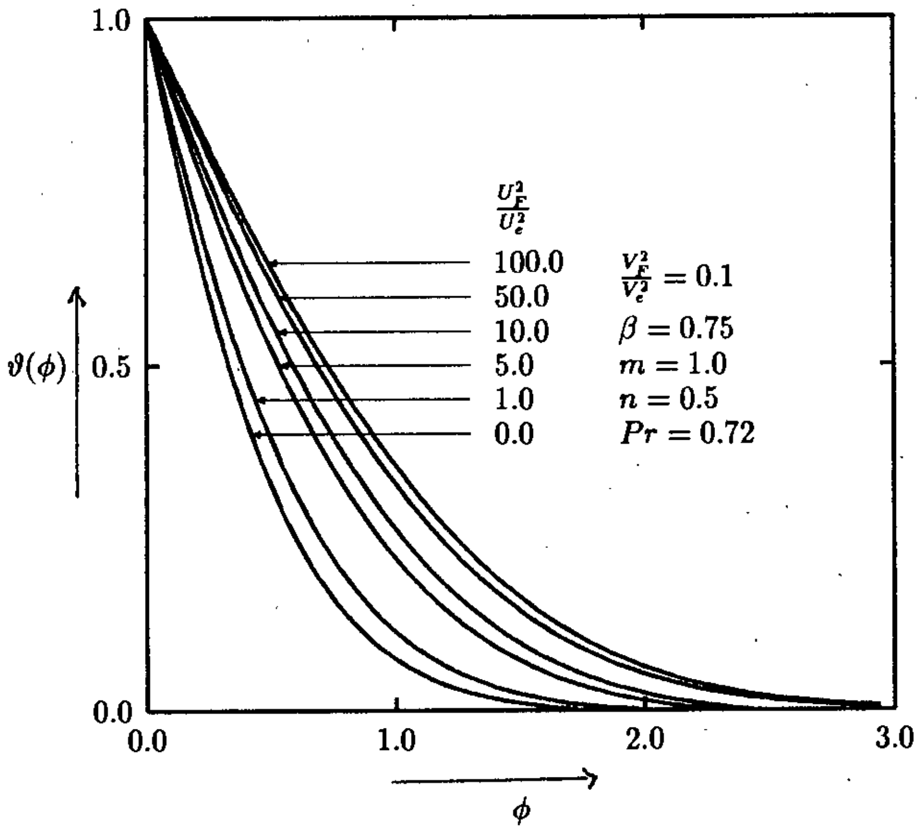


Fig.3 Representative Temperature profiles for different values of $\frac{U_F^2}{U_e^2}$ for the equation (67).

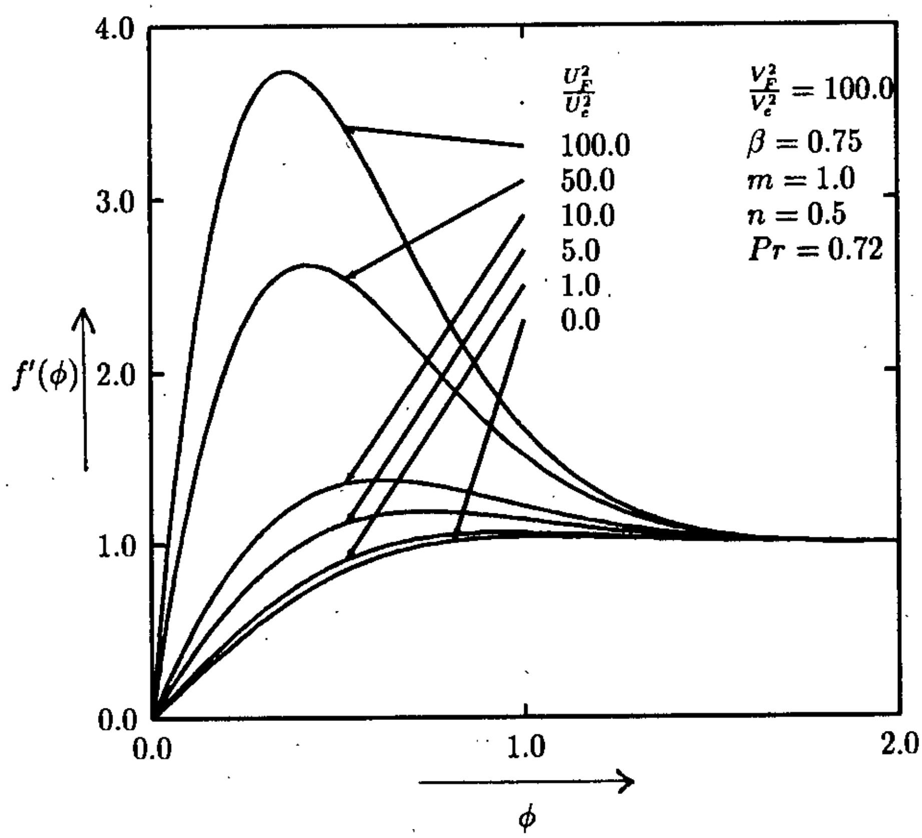


Fig.4 Representative velocity profiles of u-component for different values of $\frac{U_F^2}{U_e^2}$ for the equation (67).

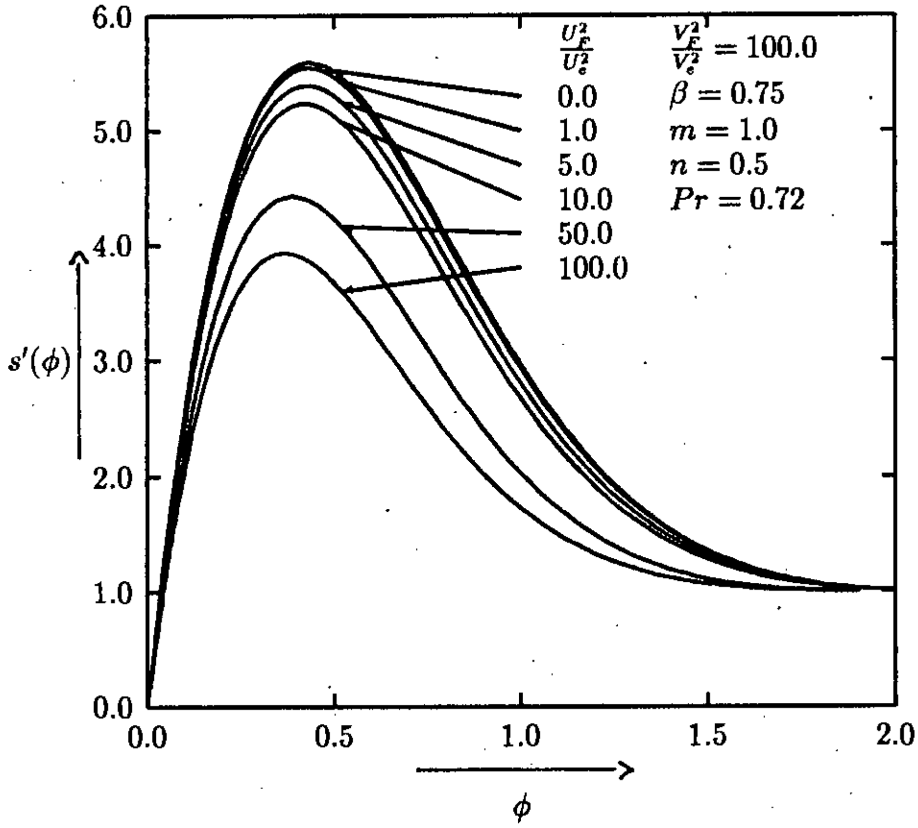


Fig.5 Representative velocity profiles of v-component for different values of $\frac{U_F^2}{U_e^2}$ for the equation (67).

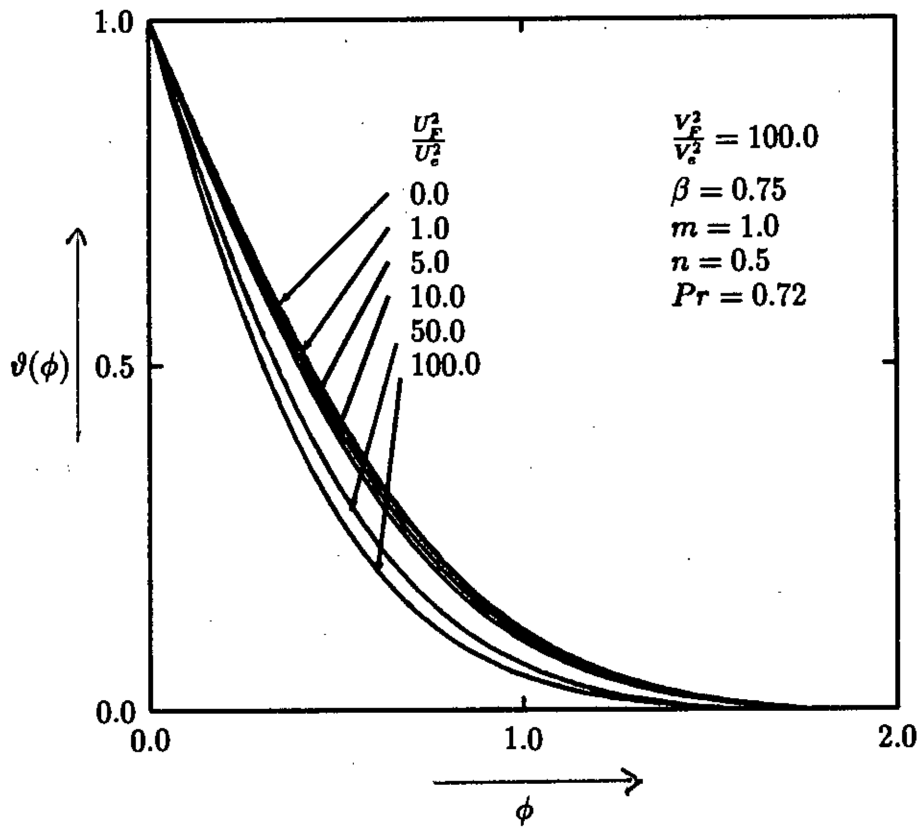


Fig.6 Representative Temperature profiles for different values of $\frac{U_F^2}{U_e^2}$ for the equation (67).

SIMILARITY CASES IN TABULAR FORM

CASES	U_e	V_e	h_1	h_2	$\Delta T \propto$	Similarity Variables ϕ
A(i)	$C_1 X_1^m Y_1^{n-1}$	$C_2 X^{m-1} Y_1^n$	1	1	$X^{2n-1} Y^{2n-2}$	$\frac{Z}{X_1} R_{eX_1}^{\frac{1}{2}}$ or $\frac{Z}{Y_1} \left(\frac{R_{eY_1}}{\beta}\right)^{\frac{1}{2}}$
A(ii)	$C_1 e^{mX} Y_1^{n-1}$	$C_2 e^{mX} Y_1^n$	1	1	$e^{2mX} Y_1^{2n-1}$	$\frac{Z}{Y_1} \left(\frac{n+1}{2}\right)^{\frac{1}{2}} R_{eY_1}^{\frac{1}{2}}$
A(iii)	$C_1 X_1^n e^{mY}$	$C_2 X_1^{n-1} e^{mY}$	1	1	$X_1^{2n-1} e^{2mY}$	$\frac{Z}{X_1} \left(\frac{n+1}{2}\right)^{\frac{1}{2}} R_{eX_1}^{\frac{1}{2}}$
A(iv)	$A\ e^{(mX+nY)}$	$B\ e^{(mX+nY)}$	1	1	$e^{2(mX+nY)}$	$Z\sqrt{\frac{m}{2X}} \cdot R_{eX}^{\frac{1}{2}}$
B(i)	$C_1 Y_1^m$	$C_2 Y_1^n$	1	1	Y_1^{2n-1}	$\frac{Z}{Y_1\sqrt{2-\beta'}} \cdot R_{eY}^{\frac{1}{2}}$
B(ii)	$C_2 e^{mY}$	$C_1 e^{nY}$	1	1	e^{2nY}	$Z\sqrt{\frac{n}{2Y}} R_{eY}^{\frac{1}{2}}$

SIMILARITY CASES IN TABULAR FORM

CASES	U_e	V_e	h_1	h_2	$\Delta T \propto$	Similarity Variables ϕ
C(i)	$C_2 X_1^m$	$C_1 X_1^n$	1	1	$(X_1)^{2m-1}$	$\frac{Z}{X_1}\left(\frac{1}{2-\beta}\right)^{\frac{1}{2}} R_{eX}^{\frac{1}{2}}$
C(ii)	$C_1 e^{mX}$	$C_2 e^{nX}$	1	1	e^{2mX}	$Z\sqrt{\frac{m}{2X}}\, R_{eX}^{\frac{1}{2}}$
D	$C_1(aX+bY)^m$	$C_2(aX+bY)^n$	1	1	$(aX+bY)^{2m-1}$	$\left(\frac{m+1}{2}, aR_e\right)^{\frac{1}{2}} \cdot \left(\frac{Z}{aX+bY}\right)$
	$C_1(aX+bY)^m$	$C_2(aX+bY)^n$	$C_3(aX+bY)^{n-m}$	$C_4(aX+bY)^{n-m}$	$(aX+bY)^{2m-1}$ or $(aX+bY)^{3m-n-1}$	$\left(\frac{aZ}{aX+bY}\right)\sqrt{\frac{(3n-2m+1)}{2h_1}} \cdot R_e^{\frac{1}{2}}$

where $X_1=X+X_o$, $Y_1=Y+Y_o$.

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