GROUP-THEORY METHOD ON SIMILARITY SOLUTION OF UNSTEADY FREE-CONVECTION FLOW FROM A MOVING VERTICAL SURFACE WITH SUCTION AND INJECTION

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Abstract

Two-parameter transformation group-theory method is applied to present the analysis and analytic solution of the problem of unsteady free-convection flow from a moving vertical porous surface considering suction and injection. The application of two-parameter groups reduces the number of independent variables by one, and consequently the system of governing partial differential equations with boundary conditions reduces to a system of ordinary differential equations with appropriate boundary conditions. The possible forms of difference between wall temperature and the ambient fluid temperature variations are derived. The reduced ordinary differential equations obtained from the mathematical model of the present problem are then solved numerically using the shooting method known as Runge-Kutta-Butcher initial value problem solver together with the Nachtsheim-Swigert iteration scheme described by Nachtsheim and Swigert. Programming codes have been written in FORTRAN 90 to implement shooting method for the present problem. The post processing software TECPLOT has been used to display the numerical results graphically. The effects of different values of Prandtl number Pr, suction and injection parameter F_w, viscosity variation parameters σ and τ on velocity and temperature profiles have been discussed with the help of graphs and tables. Comparisons of suction, injection and neutral position (no suction or injection) have been analyzed graphically and in tabular form.

Author's Declaration

I am hereby declaring that the works in this dissertation have been carried out in accordance with the regulations of Bangladesh University of Engineering and Technology (BUET), Dhaka, Bangladesh. The work is also original except where indicated by and attached with special reference in the context and no part of it has been submitted for any attempt to get other degrees or diplomas.

All views expressed in the dissertation are those of the author and in no way or by no means represent those of Bangladesh University of Engineering and Technology, Dhaka. This dissertation has not been submitted to any other University for examination either in home or abroad.

(Dipika Rani Dhar)

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Contents

Nomenclature

Symbol	Entities
a_1, a_2	Parameters
(a_1^0, a_2^0)	Value of (a_1, a_2) to find out identity element of the group
C^S	Real-number
F_w	Suction parameter
f	Function of y and ξ
G	Two parameter group
g	Gravitational acceleration
g_1, g_2, g_3, g_4, g_5	Absolute invariant
K^S	Real- number
Pr	Prandtl number
S	Represent the symbol x, y, t, u, v, u_w, ΔT and θ
S_i	Derivative of S with respect to i stands for x, y, t
S_{ij}	Derivative of S with respect to i and j when i, j = x, y, t
$\bar{S}$	Transformed form of S
$\bar{S}_{ij}$	Transformed form of S_{ij}
T	Temperature of the fluid
T_w	Wall temperature
T_∞	Constant temperature of the ambient fluid
u	Non-dimensional velocity
u'	Velocity components in the boundary layer along the sheet

u_w	Velocity at the wall
v'	Velocity components in the boundary layer normal to the sheet
v	Non-dimensional velocity
x	Distance along the direction of the moving sheet
y	Distance normal to the sheet.

Greek Symbols

Symbol	Entities
α	Thermal diffusivity
$\alpha_1, \alpha_2, ..., \alpha_{16}, \beta_1, \beta_2, ..., \beta_{16}$	Real- number
β	Co-efficient of thermal expansion
η	Similarity variable
θ	Non-dimensional temperature
λ_{ij}	Real- number can be defined by α andβ
v	Kinematics viscosity
ξ	Constant function of x, t
σ, r	Viscosity variation parameter

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Chapter 1

1. Introduction

Applications of group-theory in fluid mechanics and boundary layer flow have received much attention by many researchers as the concepts of group theory are extensively used in similarity and non-similarity related problems. Group-theory method provides a powerful tool to nonlinear differential models. The transformation group theory approach is applied to present an analysis of the similarity problem of unsteady free-convection flow from a moving vertical surface with suction and injection. The application of two-parameter groups reduces the number of independent variables and consequently the system of governing partial differential equations with the boundary conditions reduces to the system of ordinary differential equations with appropriate boundary conditions. The advantages of the group theory method are obvious. The first is that the method is rather simple to apply. There is no concern about boundary conditions, choices for various functions, etc. Secondly, in reducing the number of independent variables by two it is possible to obtain a new system of partial differential equations without continuing to obtain ordinary differential equations. The possible advantage of group-theory method is that it transforms the partial differential equations into ordinary differential equations that makes possible to solve wider variety of problems in this manner. It would be very interesting to explore this possibility in solving the boundary layer equations. Fortunately, the types of group employed in the examples seem to be adequate for yielding the classes of similarity transformations obtained by other methods. However, the group- theory methods should yield complete results with less effort.

The thermal boundary layer induced within a vertical semi-infinite layer of Boussinseq fluid by a constantly heated plate. Heat transfer problem in the boundary layer on a moving surface has instances of implementation in fluid mechanics. Recently, unsteady conditions of motion and heating of bodies in fluids have become important in certain applications for some engineering fields of aerodynamics and hydrodynamics.

The problem of group theory method on similarity solutions for unsteady free-convection flow from a moving vertical surface has also gained different dimensions in the manufacturing processes in industries.

Similarity system is preferred for the measurements over the British Engineering system. Similarity analyses determine the conditions under which the model experiments are to be performed and the essential parameters involving in the processes are predicated. Similarity analyses leads to have the idea of dimensionless numbers. Dimensionless analyses yield a functional relationship between the dimensional products. Dimensionless analyses also reduce the experimental cost highly in most of the solution of the problem in comparison with cost required for the variables of the phenomenon separately.

Hydraulic characteristics of a fluid change with time if the flow is unsteady. If water flows at a changing rate, as is the case when tap is just opened, the flow is unsteady. The introduction of time as the third independent variable in the unsteady problem increases the complexity of the problem.

The Phenomenon of free convection arises in the fluid when temperature changes cause density variation leading to buoyancy forces acting on the fluid elements. This can be seen in everyday life in the atmospheric flow, which is driven by temperature differences. When the vertical plate is being heat, the air layer adjacent to the wall expands and arises due to buoyancy. A free convection boundary layer is formed adjacent to the wall surface.

Suction or injection is necessary to control the boundary layer flows by injecting or withdrawing fluid through a heat boundary layer wall. The plate from which the fluid enters into the flow region is known as plate with injection and the plate from which the fluid leaves out the flow region is known as plate with suction. Since this can enhance heating or cooling of the boundary layer system, it can help delay the transition from the laminar to turbulent flow. Boundary layer suction is used to control laminar and turbulent separation removing flow of the low momentum. The technique is used in air wings and in some wind tunnels to control the laminar flow. Blowing (injection) on boundary layer of high temperature components can maintain a thin layer of colder flow that allows the system to function with very high fluid velocity.

Many attempts were made to find analytical and numerical solutions applying certain special conditions and using different mathematical approaches but for stationary vertical plate. Illingworth (1950) studied the problem of unsteady laminar flow of gas near to an infinite flat plate. He obtained solutions, which are available only with Prandtl number unity and under transient conditions of step change in the surface temperature.

Siegel (1958) investigated the transient free convection from a vertical plate. Free convection flow past vertical plate has been studied extensively by Ostrach (1953, 1954) and many others in various solution technique. Possible similarity solutions for laminar free convection on vertical plates and cylinders have been studied by Yang (1960). He established some necessary and sufficient conditions for which similarity solutions are possible.

Soundalgekar (1977) studied the unsteady free convection flow past an infinite vertical plate with constant suction and mass transfer. It was assumed that the plate temperature oscillates in such a way that its amplitude is small.

Williams et al. (1987) obtained semi-similar solutions for the unsteady free-convective boundary-layer flow on a vertical plate using an implicit finite difference method. In 1990 and 1991 four works concerning the same problem were introduced applying the group-theory method.

Abd-el-Malek and Badran (1991, 1990) analysed the steady and the unsteady free-convective laminar boundary-layer flow on a non isothermal vertical circular cylinder. Recently, many studies have been made on the steady free convective boundary-layer flow on moving vertical plates considering the effect of buoyancy forces on the boundary layer.

Some of these studies are presented in the work of Chen and Strobel (1980), Moutsolglou and Chen (1980), Ramachandran et al. (1987), and Lee and Tsai (1990), Kumari et al. (1996) investigated the unsteady free convection flow over a continuous moving vertical plate, the Keller box method and Nakamura's method were used to solve the differential equations governing the flow.

Zakerullah (2001) derived similarity solutions of some of possible cases of unsteady mixed convection by group theory without suction. He also investigated steady natural convection by group theory method without suction.

Finally, Abd-el-Malek et al. (2004) investigated the unsteady free convection flow over a continuously moving vertical surface.

The mathematical technique used in the present analysis is two-parameter group transformation that leads to a similarity representation of the problem. Morgan (1952) presented a theory that led to improvements over earlier similarity methods.

Michal (1952) extended Morgan's theory. Group methods, as a class of methods which lead to a reduction of the number of independent variables, were first introduced by Birkoff (1948,1960) He made use of one parameter group transformations to reduce a system of partial differential equations in two independent variables to a system of ordinary differential equations in one independent variable, the similarity variable.

Morgan and Gaggioli (1966, 1968) presented general systematic group formalism for similarity analysis, where a given system of partial differential equations was reduced to a system of ordinary differential equations. They utilized elementary group theory for the purpose of reducing a given system of partial differential equations to a system of ordinary differential equation in a single variable. Similarity analysis has been applied intensively by Gabbert (1967).

In this work, the effect of suction and injection on unsteady free-convection flow over a moving vertical plate has been investigated. Problems are solved analytically using group methods and then numerically by Runge-Kutta shooting method. Under the application of two-parameter group, the governing partial differential equations are reduced to system of ordinary differential equations with the appropriate boundary conditions and the resulting nonlinear system of differential equations are solved analytically and then numerically using the sixth order Runge-Kutta shooting method known as Runge-Kutta-Butcher initial value problem solver of Butcher (1974) together with the Nachtsheim-Swigert iteration scheme described by Nachtsheim and Swigert (1965). Programming codes have been written in FORTRAN 90 to implement shooting method for the present problem.

Attention has been taken on the evaluation of the velocity profiles as well as temperature profiles for selected values of parameters consisting of suction and injection parameter F_w, Prandtl number Pr and the parameters σ and τ due to viscosity variation.

The numerical results of the velocity profiles as well as temperature profiles are displayed graphically for different values of Prandtl number, suction and injection parameter and viscosity variation parameters. The post processing software TECPLOT has been used to display the numerical results graphically.

In chapter-2, a thin non-isothermal porous continuously moving flat sheet has been considered which is heat. The physical model with co-ordinate system for unsteady free-convection flow from a moving vertical surface with suction or injection has been shown in figure. The governing equations of the above physical problem with appropriate boundary conditions are obtained by mathematical modeling.

Then the momentum and energy equations are transformed into non-dimensional form. Three independent variables have been reduced to one similarity variable and governing equations are transformed into ordinary differential equations. Then group systematic formulation and invariance analysis are used to find out two- parameter group The complete sets of absolute invariant are also discussed.

In chapter-3, the effect of suction and injection on unsteady free-convection flow from a moving vertical surface for the case of "Unsteady free-convection with surface temperature varying inversely as a linear combination of x and t, the flow velocity at wall varying directly with In function of (x, t) and the suction and injection velocity varying inversely as a square root of the linear combination of x and t" have been investigated. Velocity profiles and temperature profiles have been displayed graphically for the wide range of suction, injection parameter Fw, Prandtl number Pr and viscosity variation parameter σ. Then variation of velocity and temperature profiles due to suction, injection and neutral position (neither suction nor injection) have been presented in tabular form.

In chapter-4, the effect of suction and injection on unsteady free-convection flow from a moving vertical surface for the case of "Unsteady free-convection with surface temperature varying directly with a function x and inversely with the square of a function of t, the flow velocity at wall varying directly with a function x and inversely with a function of t and suction and injection velocity varying inversely with a square root of a function of t" have been investigated. Velocity profiles and temperature profiles have been displayed graphically for the wide range of suction and injection parameter F_w, Prandtl number *Pr* and viscosity variation parameters τ. Then variations of velocity and temperature profiles due to suction, injection and neutral position (no suction or injection) have been presented in tabular form.

Chapter 2

Mathematical formulation, Invariance analysis and Group investigation

2.1 Introduction

Unsteady free-convection flow from a moving vertical surface with suction and injection is discussed in this chapter. A thin non-isothermal porous sheet has been considered as a continuous moving flat sheet .The moving flat sheet has been taken vertically. The non-isothermal porous moving flat sheet is heated and boundary layer equations with boundary conditions are governed. Then the velocity and temperature of the governing equations are transformed in non-dimensional form. Three independent variables have been reduced to one similarity variable and governed equations are transformed into ordinary differential equations. Then Group systematic formulation and invariance analysis are used to find out two- parameter group. The complete set of absolute invariant is also discussed here.

2.2 Mathematical formulation of the problem and the governing equations

Unsteady laminar flow of an incompressible fluid caused by a continuous moving flat sheet illustrated in Fig.2.1 issues from a thin slit at $x = y = 0$ and is subsequently stretched vertically. The positive x coordinate is measured along the direction of the moving sheet with the slot as the origin and the direction 'y' is measured normal to the sheet. The non-isothermal porous sheet is heated which gives rise to a buoyancy force and the temperature distribution over the plate T_w will be a function of the vertical distance x and the time t. The fluid is isothermal of constant temperature T_∞ far from the sheet such that $T_w > T_\infty$ .The velocity of the moving sheet is time dependent. The physical configuration considered is as shown in Fig. 2.1:

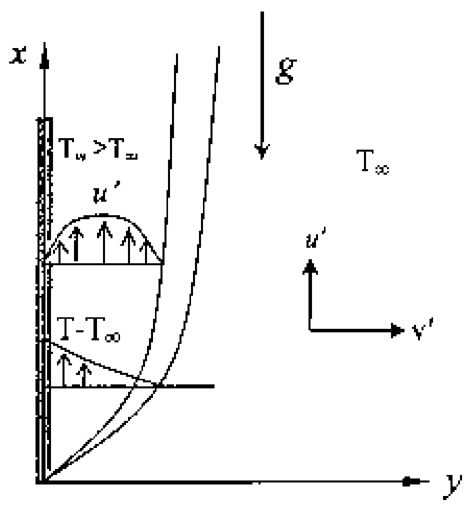

Figure 2.1: Physical model and co-ordinate system

Under the boundary layer approximation the free convection flow over the moving sheet is governed by the following boundary layer equations and boundary conditions,

Continuity equation:

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0 \tag{2.1}$$

Momentum equation:

$$\frac{\partial u'}{\partial t}+u'\frac{\partial u'}{\partial x}+v'\frac{\partial u'}{\partial y}=\nu\frac{\partial^2 u'}{\partial y^2}+g\beta\left(T-T_\infty\right) \tag{2.2}$$

Energy equation:

$$\frac{\partial T}{\partial t}+u'\frac{\partial T}{\partial x}+v'\frac{\partial T}{\partial y}=\frac{\nu}{\Pr}\frac{\partial^2 T}{\partial y^2} \tag{2.3}$$

Boundary conditions:

$$u'(x,0,t) = u_w(x,t),\ v'(x,0,t) = v_w(x,t),\ T(x,0,t) = T_w(x,t) \quad \text{at } y = 0$$
$$u'(x,y,t) = 0, \qquad T(x,y,t) = T_\infty \qquad \text{as } y \to \infty \tag{2.4}$$

where u' and v' are the velocity components along the x and y directions respectively, T is the temperature; g is the acceleration due to gravity, Pr is the Prandtl number which is the ratio of the kinematics viscosity 'ν' to the thermal diffusivity 'α' (i.e. $Pr = \nu/\alpha$) β is the co-efficient of thermal expansion; u_w is the flow velocity at the wall and T_w is the temperature of the moving sheet.

2.3 The transformed equations and methods of solution

Now, non-dimensional velocity and temperature are introduced as

$u = u'/u_w$, $v = v'$, $\theta = (T - T_\infty)/(T_w - T_\infty)$

$\Delta T = (T_w - T_\infty)$ implies $\theta\,\Delta T = (T - T_\infty)$

The non-dimensional forms of the equations (2.1) to (2.3) are

$$u\frac{\partial u_w}{\partial x} + u_w\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.5}$$

$$u\frac{\partial u_w}{\partial t} + u_w\frac{\partial u}{\partial t} + u_w^2 u\frac{\partial u}{\partial x} + u_w u^2\frac{\partial u_w}{\partial x} + v u_w\frac{\partial u}{\partial y} = \nu u_w\frac{\partial^2 u}{\partial y^2} + g\beta\Delta T\theta \tag{2.6}$$

$$\frac{\partial \theta}{\partial t} + \theta\left[\frac{\partial}{\partial t}(\ln\Delta T) + u u_w\frac{\partial}{\partial x}(\ln\Delta T)\right] + u u_w\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{\nu}{\Pr}\frac{\partial^2 \theta}{\partial y^2} \tag{2.7}$$

The associated boundary conditions are transformed to

$$u(x,0,t)=1,\ v(x,0,t)=v_w(x,t),\ \theta(x,0,t)=1 \quad \text{at } y=0$$
$$u(x,y,t)=0, \qquad \theta(x,y,t)=0 \quad \text{as } y\to\infty \tag{2.8}$$

The method of solution depends on the application of two-parameter group transformation to the system of partial differential equations (2.5)-(2.7). Under this transformation, the three independent variables x, y, t will be reduced to one similarity variable 'η' and the differential equations (2.5)-(2.8) will be transformed to a system of ordinary differential equations in only one independent variable, which is the similarity variable 'η'.

2.3.1 The group systematic formulation

The procedure is initiated with the group G, a class of transformation of two parameters (a_1, a_2) of the form

$$G: \bar{S} = C^S(a_1,a_2)S + K^S(a_1,a_2) \tag{2.9}$$

S stands for x, y, t, u, v, u_w, $\Delta T, \theta$ and the C^S and K^S are real-valued and at least differentiable in each real argument.

2.3.2 The invariance analysis

To transform the differential equations, transformation of the derivatives are obtained from G via chain-rule operations

$$\bar{S}_{\bar{i}} = \left(\frac{C^S}{C^i}\right) S_i \qquad i = x, y, t$$

$$\bar{S}_{\bar{i}\bar{j}} = \left(\frac{C^S}{C^i C^j}\right) S_{ij} \qquad j = x, y, t \tag{2.10}$$

Where S stands for u, v, u_w, ΔT and θ.

Now equation (2.5) is said to be invariantly transformed whenever

$$\bar{u}\frac{\partial \bar{u}_w}{\partial \bar{x}} + \bar{u}_w\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = H_1(a_1, a_2)\left[u\frac{\partial u_w}{\partial x} + u_w\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] \tag{2.11}$$

for some function $H_1(a_1,a_2)$ which may be a constant.

Substitution from (2.9) and (2.10) into (2.11), the functions and their partial derivatives yields

$$\left[C^u C^{u_w}/C^x\right]u\frac{\partial u_w}{\partial x} + \left[C^{u_w} C^u/C^x\right]u_w\frac{\partial u}{\partial x} + \left[C^v/C^y\right]\frac{\partial v}{\partial y} + R_1 = H_1(a_1,a_2)\left[u\frac{\partial u_w}{\partial x} + u_w\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] \tag{2.12}$$

where $R_1 = \left[C^{u_w}K^u/C^x\right]\left(\partial u_w/\partial x\right) + \left[C^u K^{u_w}/C^x\right]\left(\partial u/\partial x\right)$

The invariance of (2.12) implies $R_1 = 0$. This is satisfied by putting

$$K^u = K^{u_w} = 0 \tag{2.13}$$

and

$$[C^u C^{u_w}/C^x] = [C^v/C^y] = H_1(a_1,a_2) \tag{2.14}$$

where $H_1(a_1,a_2)$ = constant

Equation (2.6) is invariantly transformed giving

$$K^{u}=K^{u_w}=K^{v}=K^{\Delta T}=K^{\theta}=0 \tag{2.15}$$

and

$$[C^{u}C^{u_w}/C^{t}]=[(C^{u_w})^{2}(C^{u})^{2}/C^{x}]=[C^{v}C^{u_w}C^{u}/C^{y}]=[C^{u_w}C^{u}/(C^{y})^{2}]=[C^{\Delta T}C^{\theta}]=H_2(a_1,a_2) \tag{2.16}$$

Where $H_2(a_1,a_2)$=constant.

In a similar manner, the invariant transform of (2.7) gives

$$K^{\theta}=K^{\Delta T}=K^{u}=K^{u_w}=K^{v}=0 \tag{2.17}$$

and

$$[C^{\theta}/C^{t}]=[C^{\theta}C^{u}C^{u_w}/C^{x}]=[C^{v}C^{\theta}/C^{y}]=[C^{\theta}/(C^{y})^{2}]=H_3(a_1,a_2) \tag{2.18}$$

where $H_3(a_1,a_2)$ = constant.

Moreover, the boundary conditions (2.8) are also invariant in form, whenever

$$K^{y}=0 \tag{2.19}$$

and

$$C^{u}=C^{\theta}=1 \tag{2.20}$$

Now, combining equations (2.14), (2.16), (2.18) and invoking the result (2.20) yields

$$C^{t}=(C^{y})^{2}, C^{v}=1/C^{y}, C^{u_w}=C^{\Delta T}(C^{y})^{2}, C^{x}=C^{\Delta T}(C^{y})^{4} \tag{2.21}$$

So, two-parameter group G is found which transforms invariantly the differential equations (2.5), (2.6), (2.7) and the boundary conditions (2.8). The group G is of the form

$$
G: \begin{cases} G_s \begin{cases} \bar{x} = \left[C^{\Delta T}(C^y)^4\right]x + K^x \\ \bar{y} = \left[C^y\right]y \\ \bar{t} = \left[(C^y)^2\right]t + K^t \end{cases} \\ \bar{u} = u \\ \bar{\mathrm{v}} = \left[1/C^y\right]\mathrm{v} \\ \bar{u}_w = \left[C^{\Delta T}(C^y)^2\right]u_w \\ \overline{\Delta T} = C^{\Delta T}\Delta T \\ \bar{\theta} = \theta \end{cases} \qquad (2\ 22)
$$

2.3.3 The complete set of absolute invariants

Use of group methods to represent the problem in the form of an ordinary differential equation (Similarity representation) in a single independent variable (Similarity variable) has taken as an aim.

The complete set of absolute invariants is:

a) $\eta = \eta(x, y, t)$ is the absolute invariant of the independent variables x, y, t.

b) $g_j(x, y, t; u, v, u_w, \Delta T, \theta) = F_j(\eta(x, y, t))$, $j = 1,2,3,4,5$ are the five absolute invariants

Corresponding to the five dependent variables u, v, u_w, ΔT, θ.

The application of a basic theorem in group theory by Moran and Gaggioli (1968) states that; a function $g(x, y, t; u, v, u_w, \Delta T, \theta)$ is an absolute invariant of a two-parameter group if it satisfies the two first- order linear differential equations

$$\left(\alpha_1 x+\alpha_2\right)\frac{\partial g}{\partial x}+\left(\alpha_3 y+\alpha_4\right)\frac{\partial g}{\partial y}+\left(\alpha_5 t+\alpha_6\right)\frac{\partial g}{\partial t}+\left(\alpha_7 u+\alpha_8\right)\frac{\partial g}{\partial u}+\left(\alpha_9 \mathrm{v}+\alpha_{10}\right)\frac{\partial g}{\partial \mathrm{v}}$$
$$+\left(\alpha_{11} u_w+\alpha_{12}\right)\frac{\partial g}{\partial u_w}+\left(\alpha_{13}\Delta T+\alpha_{14}\right)\frac{\partial g}{\partial(\Delta T)}+\left(\alpha_{15}\theta+\alpha_{16}\right)\frac{\partial g}{\partial\theta}=0 \tag{2.23}$$

and

$$\left(\beta_1 x+\beta_2\right)\frac{\partial g}{\partial x}+\left(\beta_3 y+\beta_4\right)\frac{\partial g}{\partial y}+\left(\beta_5 t+\beta_6\right)\frac{\partial g}{\partial t}+\left(\beta_7 u+\beta_8\right)\frac{\partial g}{\partial u}+\left(\beta_9 \mathrm{v}+\beta_{10}\right)$$
$$\frac{\partial g}{\partial \mathrm{v}}+\left(\beta_{11} u_w+\beta_{12}\right)\frac{\partial g}{\partial u_w}+\left(\beta_{13}\Delta T+\beta_{14}\right)\frac{\partial g}{\partial(\Delta T)}+\left(\beta_{15}\theta+\beta_{16}\right)\frac{\partial g}{\partial\theta}=0 \tag{2.24}$$

where $\alpha_1=\left({\partial C^x}/{\partial a_1}\right)(a_1^0,a_2^0)$

$$\alpha_2=\left({\partial k^x}/{\partial a_1}\right)(a_1^0,a_2^0)$$

$$\beta_1=\left({\partial C^x}/{\partial a_2}\right)(a_1^0,a_2^0)$$

$$\beta_2=\left({\partial k^x}/{\partial a_2}\right)(a_1^0,a_2^0),\ \text{etc.}$$

and (a^0_1, a^0_2) indicates the value of (a_1,a_2) which yields the identity element of the group.

By observation of (2.23) and (2.24), it is apparent that independent variable $(x$, y, t) has an absolute invariant $\eta(x$, y, t) when it satisfies two first –order partial differential equations.

Two possible cases are derived considering none of the co-efficient of these first-order partial differential equations vanishes and only one of the co-efficient vanishes identically. In chapter 3, the first case "Unsteady free-convection with surface temperature varying inversely as a linear combination of x and t, the flow velocity at wall varying directly with log function of (x, t) and the suction velocity varying inversely as a square root of the linear combination of x and t " has been investigated and in chapter-4, the second case "Unsteady free convection with surface temperature varying directly with x and inversely with the square of a function of t, the flow velocity at wall varying directly with x and inversely with a function of t and suction velocity varying inversely with square root of a function of t" has been investigated.

Chapter 3

$\Delta T \propto (a\,x + bt + c)^{-1}, u_w \propto \log(ax + b\,t + c)$ and $v_w \propto (ax + b\,t + c)^{-1/2}$

i.e. Unsteady free-convection with surface temperature varying inversely as a linear combination of x and t, the flow velocity at wall varying directly with log function of (x, t) and the suction and injection velocity varying inversely as a square root of the linear combination of x and t

3.1 Introduction

In this chapter, analysis of the two- dimensional unsteady free convection boundary layer flow with suction and injection on moving vertical plate in the case of "Unsteady free-convection with surface temperature varying inversely as a linear combination of x and t, the flow velocity at wall varying directly with log function of (x, t) and the suction and injection velocity varying inversely as a square root of the linear combination of x and t" has been studied. For independent and dependent variables absolute invariants have been found and then governing non-dimensional partial differential equation and boundary conditions have been reduced to ordinary differential equations with boundary conditions. Then it is numerically solved by shooting method known as Runge-Kutta-Butcher (Butcher (1974)) initial value solver together with the Nachtsheim-Swigert iteration scheme described by Nachtsheim and Swigert (1965). Here attention has been given on the evaluation of the velocity distribution as well as temperature distribution for a selection of parameters set consisting of suction and injection parameter, viscosity variation parameter and the Prandlt number. Variations of velocity and temperature profiles due to suction, injection and neutral position (no suction or injection) have been presented graphically and in tabular form.

3.2 Transformation of variables

3.2.1 Independent variables as absolute invariants

Owing to equations (2.23) and (2.24), $\eta(x, y, t)$ is an absolute invariant of the independent variables (x, y, t) if it satisfies the two first –order partial differential equations:

$$(\alpha_1 x+\alpha_2)\frac{\partial\eta}{\partial x}+(\alpha_3 y+\alpha_4)\frac{\partial\eta}{\partial y}+(\alpha_5 t+\alpha_6)\frac{\partial\eta}{\partial t}=0 \tag{3.1}$$

and

$$(\beta_1 x+\beta_2)\frac{\partial\eta}{\partial x}+(\beta_3 y+\beta_4)\frac{\partial\eta}{\partial y}+(\beta_5 t+\beta_6)\frac{\partial\eta}{\partial t}=0 \tag{3.2}$$

Since $K^y=0$ so $\alpha_4=\beta_4=0$.

Now (3.1) and (3.2) can be written as

$$(\lambda_{31}x+\lambda_{32})\frac{\partial\eta}{\partial x}+(\lambda_{35}t+\lambda_{36})\frac{\partial\eta}{\partial t}=0 \tag{3.4}$$

$$(\lambda_{31}x+\lambda_{32})y\frac{\partial\eta}{\partial y}-(\lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26})\frac{\partial\eta}{\partial t}=0 \tag{3.5}$$

where $\lambda ij = \alpha_i \beta_j - \alpha_j \beta_i$ $\quad i, j = 1, 2, 3, 4, 5, 6$

Now the transformation group G given by (2.22) and making use of the definition of α's and β's and invoking $\alpha_5 = 2\alpha_3, \beta_5 = 2\beta_3$ implies

$$\lambda_{35}=\alpha_3\beta_5-\alpha_5\beta_3=0 \tag{3.6}$$

For this case, considering none of the co-efficient in (3.4) and (3.5) vanishes and

$\lambda_{31} = 0$, $\lambda_{35} = 0$, $\lambda_{32} \neq 0$, $\lambda_{36} \neq 0$

So, equations (3.4) and (3.5) reduce to

$$\lambda_{32}\frac{\partial\eta}{\partial x}+\lambda_{36}\frac{\partial\eta}{\partial t}=0 \tag{3.7}$$

$$\lambda_{32}y\frac{\partial\eta}{\partial y}-(\lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26})\frac{\partial\eta}{\partial t}=0 \tag{3.8}$$

Equation (3.7) has the general solution of the following form according to a standard technique for linear partial differential equations

$$\eta=f\ (y,\xi\ (x,\ t)) \tag{3.9}$$

Where f is an arbitrary function and ξ is any function such that $\xi(x,\ t)$ = Constant.

$\xi(x,\ t)$ = Constant is a solution of the equation

$$\frac{dx}{\lambda_{32}}=\frac{dt}{\lambda_{36}} \tag{3.10}$$

which yields

$$\xi\ (x,\ t)=\lambda_{36}x-\lambda_{32}t=\text{Constant} \tag{3.11}$$

The solution of (3.7) also satisfies equation (3.8). Now (3.8) and (3.9) gives,

$$y\frac{\partial f}{\partial y}-\left(\frac{\lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26}}{\lambda_{32}}\frac{\partial\xi}{\partial t}\right)\frac{\partial f}{\partial\xi}=0 \tag{3.12}$$

Since ξ is independent of y, so $\left(\frac{\lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26}}{\lambda_{32}}\right)\frac{\partial\xi}{\partial t}$ co-efficient of $\partial f/\partial\xi$ is also independent of y.

So, the equation (3.12) can be written as

$$y\frac{\partial f}{\partial y}-h(\xi)\frac{\partial f}{\partial \xi}=0 \tag{3.13}$$

where $h(\xi)=(\frac{\lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26}}{\lambda_{32}})\frac{\partial \xi}{\partial t}$ (3.14)

The solution of equation (3.13) and consequently of equations (3.7) and (3.8) is of the form

$$f=\phi\left(y\,H(\xi)\right) \tag{3.15}$$

where $H(\xi)$ is given by the ordinary differential equation

$$h(\xi)\frac{d}{d\xi}(\ln H)=1 \tag{3.16}$$

obtained substitution of (3.15) into (3.13)

Now (3.16) takes the form

$$H(\xi)=\exp(\int\frac{d\xi}{h(\xi)}) \tag{3.17}$$

We know that

$$h(\xi)=\left(\frac{\lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26}}{\lambda_{32}}\right)\frac{\partial \xi}{\partial t}$$

$$=\frac{\lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26}}{\lambda_{32}}(-\lambda_{32})$$

$$h(\xi)=-(\lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26}) \tag{3.18}$$

Since h is determined by ξ alone,

$$\left.\frac{\partial h}{\partial x}\right]_{\xi} = \left.\frac{\partial h}{\partial x}\right]_{t} + \left.\frac{\partial h}{\partial t}\right]_{x} \left.\frac{\partial t}{\partial x}\right]_{\xi} = 0 \tag{3.19}$$

Using (3.11) and (3.18) , equation (3.19) can be written as

$$(\lambda_{15}t + \lambda_{16}) + (\lambda_{15}x + \lambda_{25})\frac{\lambda_{36}}{\lambda_{32}} = 0 \tag{3.20}$$

The conditions necessary for (3.20) to be satisfied are $\lambda_{15} = 0$, and $\lambda_{36}\lambda_{25} = -\lambda_{16}\lambda_{32}$ Since α's and β's and consequently the λ's can be chosen arbitrarily, so the above conditions can easily be satisfied. Now using (3.11), (3.18) implies

$$h(\xi) = -\left(\frac{\lambda_{16}}{\lambda_{36}}\xi + \lambda_{26}\right) \tag{3.21}$$

Then (3.17) gives

$$H(\xi) = \exp\left(-\int \frac{d\xi}{\frac{\lambda_{16}}{\lambda_{36}}\xi + \lambda_{26}}\right) \tag{3.22}$$

Integration of (3.22) yields

$$H(\xi) = \left(\lambda_{16}x + \lambda_{25}t + \lambda_{26}\right)^{\frac{-\lambda_{36}}{\lambda_{16}}} \tag{3.23}$$

where $\lambda_{16} \neq 0$

The absolute invariant η can be obtained from equations (3.9) and (3.15) as

$$\eta = \phi\left(y(a\,x + b\,t + c)^{-1/2}\right) \tag{3.24}$$

where the constants a, b and c stand for λ_{16},λ_{25} and λ_{26} respectively and the exponent

$(-\lambda_{36}/\lambda_{16}) = -1/2$.

Without loss of generality, the function φ can be taken to be the identity function. So,

$$\eta = y\pi_1(x, t) \tag{3.25}$$

where

$$\pi_1(x, t) = (a\,x + b\,t + c)^{-1/2} \tag{3.26}$$

3.2.2 Dependent variables as absolute invariants

In this step, absolute invariants have to obtain corresponding to the dependent variables u, v, u_w, ΔT and θ. From the group transformation (2.22) it can be written as

$$g_1(x, y, t\,; u) = u(\eta) \tag{3.27}$$

and

$$g_2\ (x, y, t, \theta) = \theta(\eta) \tag{3.28}$$

According to (2.23) and (2.24) any function g_3 (x, t; u_w) satisfies

$$(\alpha_1 x + \alpha_2)\frac{\partial g_3}{\partial x} + (\alpha_5 t + \alpha_6)\frac{\partial g_3}{\partial t} + (\alpha_{11} u_w + \alpha_{12})\frac{\partial g_3}{\partial u_w} = 0 \tag{3.29}$$

and

$$(\beta_1 x + \beta_2)\frac{\partial g_3}{\partial x} + (\beta_5 t + \beta_6)\frac{\partial g_3}{\partial t} + (\beta_{11} u_w + \beta_{12})\frac{\partial g_3}{\partial u_w} = 0 \tag{3.30}$$

The solution of equations (3.29) and (3.30) gives

$$g_3(x, t\,; u_w) = \phi_1(u_w / \omega(x, t)) = E(\eta) \tag{3.31}$$

In a similar manner, $g_4(x, t; v)$ and $g_5(x, t; \Delta T)$ are obtained

$$g_4 (x, t ;v) = \varphi_2(v/\Gamma(x, t)) = F (\eta) \tag{3.32}$$

$$g_5 (x, t; \Delta T) = \varphi_3(\Delta T/\gamma(x, t)) = G (\eta) \tag{3.33}$$

where $\omega (x, t)$, $\Gamma(x, t)$, $\gamma (x, t)$, $E(\eta)$, $F(\eta)$ and $G(\eta)$ are functions to be determined. Without loss of generality, φ's in (3.31) to (3.33) can be selected to identity functions. Then we can express the functions $u_w(x,t)$,$v(x, y, t)$ and $\Delta T(x, t)$ in terms of the absolute invariants

$E (\eta)$, $F(\eta)$ and $G(\eta)$ respectively, in the form

$$u_w(x, t) = \omega (x, t) E(\eta) \tag{3.34}$$

$$v(x, y, t) = \Gamma(x, t) F(\eta) \tag{3.35}$$

$$\Delta T(x, t) = \gamma(x, t) G (\eta) \tag{3.36}$$

Since $\omega(x, t)$, $u_w(x ,t)$ and $\Delta T(x, t)$, $\gamma (x ,t)$ are independent of y, whereas η depends on y, it follows that $E(\eta)$ and $G(\eta)$ must be equal to constant E_0 and G_0 respectively .Without loss of generality it can be taken $E_0=1$ and $G_0 =1$. Then (3.34) and (3.36) become

$$u_w(x, t) = \omega(x, t) \tag{3.37}$$

$$\Delta T(x, t) = \gamma (x, t) \tag{3.38}$$

3.3 The reduction to the ordinary differential equations

Substituting from (3.25) to (3.28), (3.35), (3.37) & (3.38) into equations (2.5) ,(2.6) and (2.7) yields,

$$\frac{dF}{d\eta}+\left(\frac{y\omega}{\pi_1\Gamma}\frac{\partial\pi_1}{\partial x}\right)\frac{du}{d\eta}+\left(\frac{1}{\pi_1\Gamma}\frac{\partial\omega}{\partial x}\right)u=0 \tag{3.39}$$

$$\frac{d^2u}{d\eta^2}-\left(\frac{y}{\nu\pi_1^2}\frac{\partial\pi_1}{\partial t}+\frac{F\Gamma}{\nu\pi_1}\right)\frac{du}{d\eta}-\left(\frac{\omega y}{\nu\pi_1^2}\frac{\partial\pi_1}{\partial x}\right)u\frac{du}{d\eta}-\left(\frac{1}{\nu\pi_1^2}\frac{\partial\omega}{\partial x}\right)u^2 -\left(\frac{1}{\nu\omega\pi_1^2}\frac{\partial\omega}{\partial t}\right)u+\frac{g\beta\theta\gamma}{\nu\omega\pi_1^2}=0 \tag{3.40}$$

and

$$\frac{1}{\Pr}\frac{d^2\theta}{d\eta^2}-\left(\frac{y}{\nu\pi_1^2}\frac{\partial\pi_1}{\partial t}+\frac{y\omega}{\nu\pi_1^2}\frac{\partial\pi_1}{\partial x}u+\frac{F\Gamma}{\nu\pi_1}\right)\frac{d\theta}{d\eta}-\left(\frac{1}{\nu\pi_1^2\gamma}\frac{\partial\gamma}{\partial t}+\frac{\omega}{\gamma\nu\pi_1^2}\frac{\partial\gamma}{\partial x}u\right)\theta=0 \tag{3.41}$$

The co-efficient of the functions u (η), θ(η) and F (η) and their derivatives, be constants or functions of η only to reduce (3.39) to (3.41) as a system of equations in a single variable η. Since $\pi_1(x, t)$, $\omega(x, t)$ and γ (x, t) are independent of y, so

$$C_1=\frac{\omega}{\pi_1^2\Gamma}\frac{\partial\pi_1}{\partial x} \tag{3.42}$$

$$C_2=\frac{1}{\pi_1\Gamma}\frac{\partial\omega}{\partial x} \tag{3.43}$$

$$C_3 = \frac{1}{\nu\pi_1^3}\frac{\partial\pi_1}{\partial t} \tag{3.44}$$

$$C_4 = \frac{\Gamma}{\nu\pi_1} \tag{3.45}$$

$$C_5 = \frac{\omega}{\nu\pi_1^3}\frac{\partial\pi_1}{\partial x} \tag{3.46}$$

$$C_6 = \frac{1}{\nu\pi_1^2}\frac{\partial\omega}{\partial x} \tag{3.47}$$

$$C_7 = \frac{1}{\nu\omega\pi_1^2}\frac{\partial\omega}{\partial t} \tag{3.48}$$

$$C_8 = \frac{g\beta\gamma}{\nu\omega\pi_1^2} \tag{3.49}$$

$$C_9 = \frac{1}{\nu\pi_1^2\gamma}\frac{\partial\gamma}{\partial t} \tag{3.50}$$

$$C_{10} = \frac{\omega}{\nu\pi_1^2\gamma}\frac{\partial\gamma}{\partial x} \tag{3.51}$$

Where C's are constants and to be determined corresponding to each set of absolute invariants. The equations (3.39) to (3.41) take the forms

$$\frac{dF}{d\eta}+C_1\eta\frac{du}{d\eta}+C_2u=0 \tag{3.52}$$

$$\frac{d^2u}{d\eta^2}-(C_3\eta+C_4F)\frac{du}{d\eta}-C_5\eta u\frac{du}{d\eta}-C_6u^2-C_7u+C_8\theta=0 \tag{3.53}$$

and

$$\frac{1}{\Pr}\frac{d^2\theta}{d\eta^2}-(C_3\eta+C_5\eta u+C_4F)\frac{d\theta}{d\eta}-(C_9+C_{10}u)\theta=\theta \tag{3.54}$$

The boundary conditions are

$$\begin{aligned} &u(0)=1, \theta(0)=1,\ F(0)=F_w(0)\neq 0 && at\ \eta\to 0 \\ &u(\infty)=0\ ,\ \theta(\infty)=0 && at\ \eta\to\infty \end{aligned} \tag{3.55}$$

The parameter F_w = F (0) related to the suction parameter v_w when $\eta = 0$

$$v_w=\Gamma(x,t)F(0)=\nu(ax+bt+c)^{-1/2}F(0) \tag{3.56}$$

Taking C_6 = 1 in (3.47) implies,

$$\omega=(\nu/a)\ \ln(ax+bt+c) \tag{3.57}$$

In a similar manner, considering

$$C_4=1 \text{ and } C_8=1 \tag{3.58}$$

equations (3.43),(3.45) and (3.47) implies

$$C_2=1 \tag{3.59}$$

Using (3.26), (3.44) becomes

$C_3 = -b/2\nu$ (3.60)

Now considering as $C_{10} = 1$ then (3.51) using (3.26) implies

$\gamma(x, t) = -(\nu/a)(ax+bt+c)^{-1}$ (3.61)

Using (3.26), (3.57) and (3.61), it can be shown from (3.48) that

$C_7 = -b/\nu$ (3.62)

Again $C_4 = 1$ in (3.58) implies

$\Gamma(x, t) = \nu\pi_1 = \nu(ax+bt+c)^{-1/2}$ (3.63)

Applying (3.26), (3.56),(3.60), (3.62) in(3.42),(3.46) and (3.50) implies

$C_1 = C_5 = 1/2$ (3.64)

$C_9 = -b/\nu$ (3.65)

Substituting the above obtained values into equations (3.52) to (3.54) and taking $\sigma = -b/\nu$, we get

$$\frac{dF}{d\eta} + \frac{\eta}{2}\frac{du}{d\eta} + u = 0 \quad (3.66)$$

$$\frac{d^2u}{d\eta^2} - \left(\frac{\sigma}{2}\eta + F + \frac{\eta}{2}u\right)\frac{du}{d\eta} - u^2 - \sigma u + \theta = 0 \quad (3.67)$$

$$\frac{1}{\Pr}\frac{d^2\theta}{d\eta^2}-\left(\frac{\eta}{2}+\frac{\eta}{2}u+F\right)\frac{d\theta}{d\eta}-(\sigma+u)\theta=0 \tag{3.68}$$

with the boundary conditions are

$$u(0)=1,\ \theta(0)=1,\ F(0)=F_w\neq 0 \quad \text{at}\ \eta=0$$
$$u(\infty)=0,\ \theta(\infty)=0 \ \text{at}\ \eta\to\infty \tag{3.69}$$

$$v_w=\Gamma(x,t)\,F(0)=\nu\,(ax+bt+c)^{-1/2}F(0) \tag{3.70}$$

$v_w < 0$ Signifies suction.

$v_w > 0$ Signifies injection.

The variations of ΔT, u_w and suction v_w are proportional to $(a\,x+b\,t+c)^{-1}$, $\ln(a\,x+b\,t+c)$ and $(a\,x+b\,t+c)^{-1/2}$ respectively.

For the above case, the boundary-layer characteristic (Surface heat flux) is

Surface heat flux,

$$q=-\frac{\nu}{a(ax+bt+c)^{3/2}}[-\theta'(0)]$$

3.4. Results and discussion

Similarity solution for unsteady free convection flow from a moving vertical surface with suction and injection has been investigated analytically using group theory method and then numerically by sixth order Runge-Kutta shooting method. The case "Unsteady free-convection with surface temperature varying inversely as a linear combination of x and t, the flow velocity at wall varying directly with log function of (x, t) and the suction and injection velocity varying inversely as a square root of the linear combination of x and t" is considered in this chapter. After making proper use of similarity transformations, three independent variables reduced to one similarity variable η. Analytical solutions have been derived for the present case and then numerical solution have been obtained using Runge-Kutta shooting method. numerical results of the problem in terms of u-velocity, v-velocity and temperature profiles have been presented graphically against the similarity variable η, for different values of suction and injection parameter, viscosity variation parameter and Prandtl number.

The influences of different values of suction and injection parameter F_w on the u-velocity, v-velocity and temperature profiles are shown in figs. 3.1, 3.3 and 3.5 respectively while Prandtl number $Pr = 0.72$, viscosity variation parameter $\sigma = 1.0$. The results are obtained for $F_w = -0.22, -0.20, -0.15, -0.10, -0.05$ and 0.0. u-velocity, v-velocity and temperature profiles for three situations when F_w takes the values of suction, injection and neutral position (no suction or injection) while Prandtl number $Pr = 0.72$, viscosity variation parameter $\sigma = 1.0$ are presented by the figures 3.2, 3.4 and 3.6. In figs. 3.7-3.9, u-velocity, v-velocity and temperature profiles are shown for different values of Prandtl number Pr while $F_w = 0.1$, $\sigma = 1.0$. For viscosity variation parameter σ, u-velocity, v-velocity and temperature profiles are displayed in figs.3.10-3.12 while $Pr = 0.72$, $F_w = 0.1$.

It has been observed from the fig. 3.1 that u-velocity increases slightly with the increasing values of suction parameter F_w while Prandtl number $Pr = 0.72$, viscosity variation parameter $\sigma = 1.0$. The maximum values of the dimensionless u-velocities have been found as 1.0 at the boundary wall $\eta = 0.0$ for all F_w and then the u- velocity decrease along η and finally approach to zero (the asymptotic value).

In fig. 3.3, v-velocity profiles become negative in the boundary layer for different values of suction parameter F_w while Prandtl number $Pr = 0.72$, viscosity variation parameter $\sigma = 1.0$. It has been found that v-velocity decreases for the increasing of suction. v-velocity profiles decrease slowly for increasing values of η and converge to different asymptotic values.

For the temperature profiles, figure 3.5 shows that the temperature increases slightly with increasing values of suction parameter F_w. The maximum values of dimensionless temperature is 1.0 for F_w = -0.22, -0.20, -0.15, -0.10, -0.05 and 0.0. The change of temperature profiles in the η direction also shows the typical temperature profile for free convection boundary layer flow that is the value of temperature profile is 1.0 (one) at the boundary wall then the temperature profile decreases gradually along η direction to the asymptotic value.

In fig 3.2 u-velocities for two suction values, two injection values and neutral value of F_w (no suction or injection) are represented. The maximum velocity has been found as 1.0 at the same point $\eta = 0.0$ and after then velocity decreases owing to increasing values of η for F_w =0.40, 0.10, 0.0,-0.10 and -0.15. From the figure it has been found that u-velocity for neutral position (no suction or injection) is larger than that of due to suction but smaller than u-velocity due to injection. It is observed that at $\eta = 5.0$ the velocity increases by 24.599% as the suction and injection parameter F_w changes from, -0.15 to 0.40

v-velocities for two suction values, two injection values and neutral value of F_w (no suction or injection) are presented in fig.3.4. It has been found that v-velocity for neutral position (no suction or injection) is larger than that of due to suction but smaller than v-velocity due to injection.

In figure 3.6 temperature profiles are represented for suction, injection and neutral value of F_w (no suction or injection). It has been found that temperature profiles decrease due to increasing values of suction and increase due to increasing values of injection. From maximum and common value 1.0 of temperature profiles decrease along η-axis for all the values of F_w.

Figures 3.7-3.9 display results for the u-velocity, v-velocity and temperature profiles, for different values of Prandtl number Pr while suction and injection parameter $F_w = 0.1$ viscosity variation parameter $\sigma = 1.0$. Fig. 3.7and 3.9 indicate u-velocity boundary-layer thickness and temperature boundary-layer thickness decrease with increasing values of Prandtl number Pr. v- velocity in the fig.3.8 shows that the v-velocity increases corresponding to increasing values of Pr. For different values of Pr, dimensionless v-velocity profiles have unit value near the edge of the boundary layer and difference between the velocity profiles increases gradually from $\eta = 0.5$ onwards.

The influence of viscosity variation on the velocity and temperature profiles while Prandtl number Pr = 0.72 and suction and injection parameter $F_w = 0.1$ are shown in figures 3.10-3.12.

It appears from the figure 3.10 that the u- velocity decreases in the vicinity of the plate with increasing values of viscosity variation parameter σ. The maximum value of the u-velocity has been found as 1.0 at the boundary wall then the u-velocity decreases as η increases. It can easily be understood from the figure 3.11 that the v-velocity and the corresponding boundary layer thickness increase owing to increasing values of viscosity variation parameter σ. The v-velocity profiles in the fig. 3.11 show that v- velocity becomes positive in a certain region of the boundary layer and negative in the rest of the region for varying values of σ.

From figure 3.12 shows that as the viscosity variation parameter σ increases, the temperature profiles decrease. The changes of temperature profiles show the typical profiles of temperature for natural convection boundary layer flow that is the value of temperature profile which is 1.0 (one) at the boundary wall and then the temperature profile decreases gradually for increasing values of η.

Numerical values of u- velocity, v- velocity and temperature profiles, for different values of F_w (Suction, injection or neutral position of suction and injection) are depicted in Table 3.1.u-velocity, v-velocity and temperature increase with increasing F_w are found to the table.

Table 3.1 Dimensionless u-velocity, v-velocity profiles and temperature profiles against η for different values of Suction and injection parameter F_w with other controlling parameters $Pr = 0.72$, $\sigma = 1.0$

η	u-velocity for different values of F_w			v-velocity for different values of F_w			Temperature profile for different values of F_w		
	$F_w = -0.10$	$F_w = 0.0$	$F_w = 0.10$	$F_w = -0.10$	$F_w = 0.0$	$F_w = 0.10$	$F_w = -0.10$	$F_w = 0.0$	$F_w = 0.10$
0.0000	1.0000	1.0000	1.0000	-0.1000	0.0000	0.1000	1.0000	1.0000	1.0000
0.1000	0.9156	0.9191	0.9224	-0.1936	-0.0939	0.0059	0.8947	0.8975	0.9002
0.2000	**0.8405**	**0.8468**	**0.8528**	**-0.2757**	**-0.1767**	**-0.0776**	**0.8024**	**0.8074**	**0.8122**
0.3000	0.7736	0.7820	0.7902	-0.3480	-0.2500	-0.1519	0.7214	0.7280	0.7344
0.4000	0.7136	0.7239	0.7337	-0.4119	-0.3151	-0.2182	0.6500	0.6578	0.6655
0.5000	**0.6599**	**0.6714**	**0.6825**	**-0.4684**	**-0.3730**	**-0.2775**	**0.5870**	**0.5957**	**0.6043**
0.6000	0.6114	0.6240	0.6362	-0.5186	-0.4247	-0.3306	0.5313	0.5406	0.5498
0.7000	0.5677	0.5811	0.5940	-0.5634	-0.4710	-0.3784	0.4819	0.4916	0.5012
0.8000	0.5282	0.5421	0.5556	-0.6033	-0.5125	-0.4215	0.4379	0.4479	0.4578
0.9000	0.4923	0.5066	0.5205	-0.6391	-0.5498	-0.4603	0.3988	0.4089	0.4189
1.0000	**0.4596**	**0.4742**	**0.4883**	**-0.6711**	**-0.5835**	**-0.4955**	**0.3639**	**0.3740**	**0.3840**

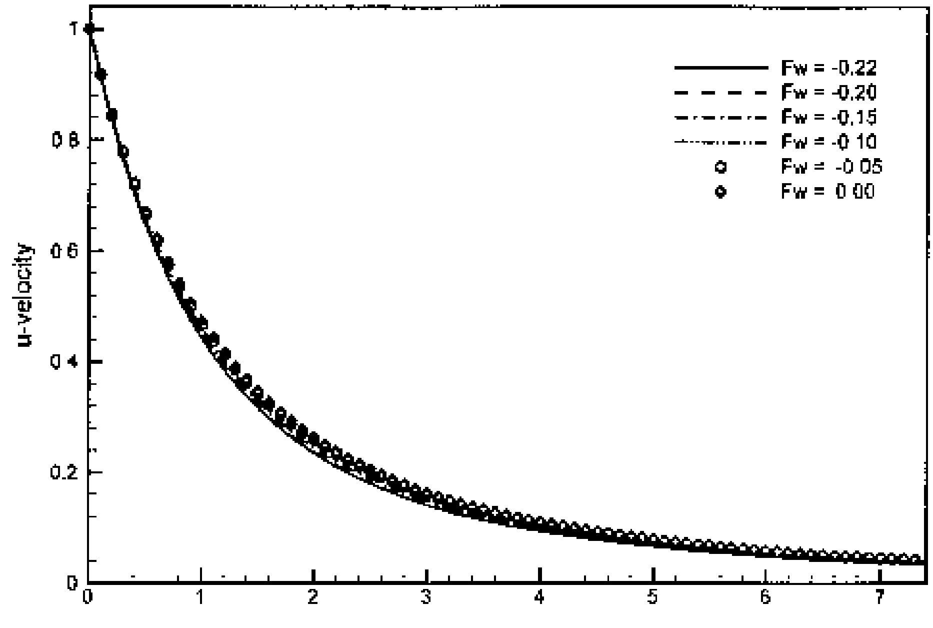

Figure 3.1: Dimensionless u-velocity profiles against similarity variable η for different values of F_w (suction) while $Pr = 0.72$, $\sigma = 1.0$

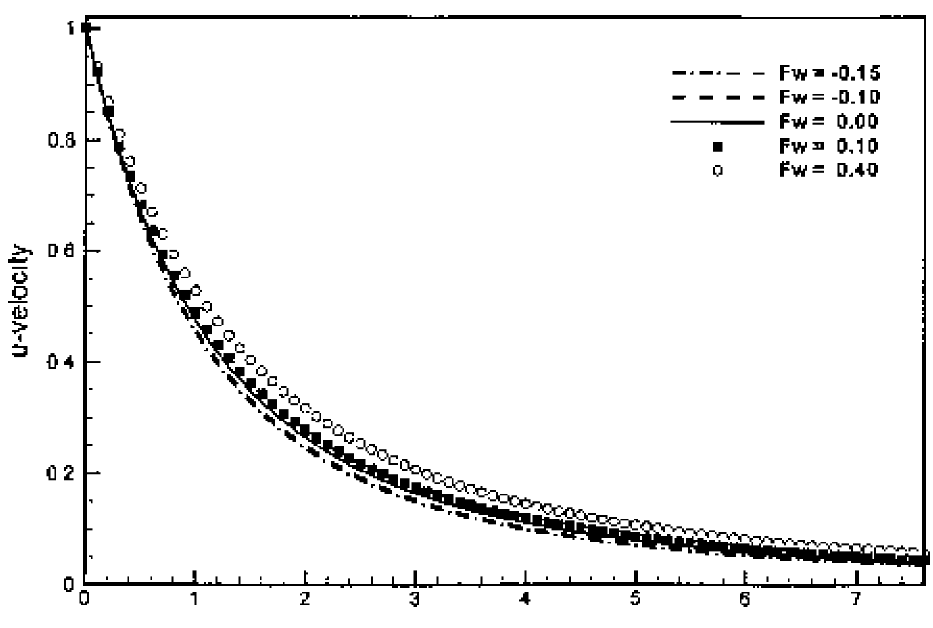

Figure 3.2: Dimensionless u-velocity profiles against similarity variable η for different values of F_w (suction and injection) while $Pr = 0.72$, $\sigma = 1.0$

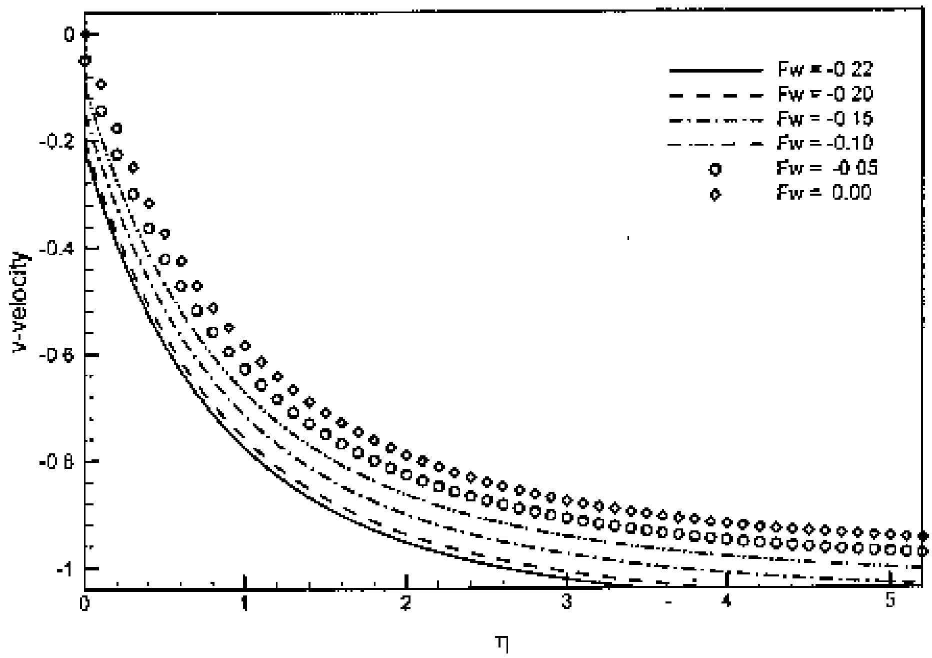

Figure 3.3: Dimensionless v-velocity profiles against similarity variable η for different values of F_w (suction) while $Pr = 0.72$, $\sigma = 1.0$

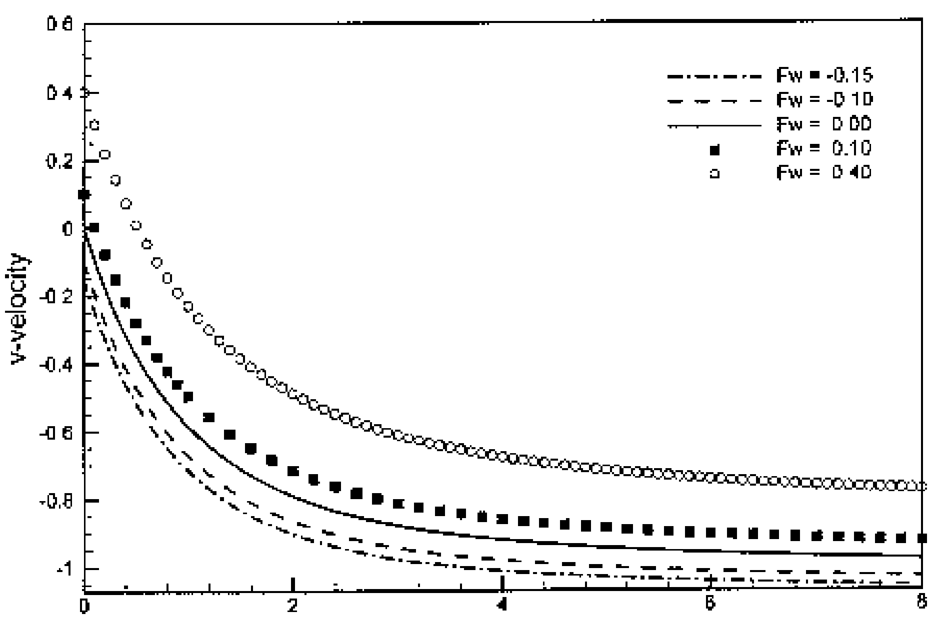

Figure 3.4: Dimensionless v-velocity profiles against similarity variable η for different values of F_w (suction and injection) while $Pr = 0.72$, $\sigma = 1.0$

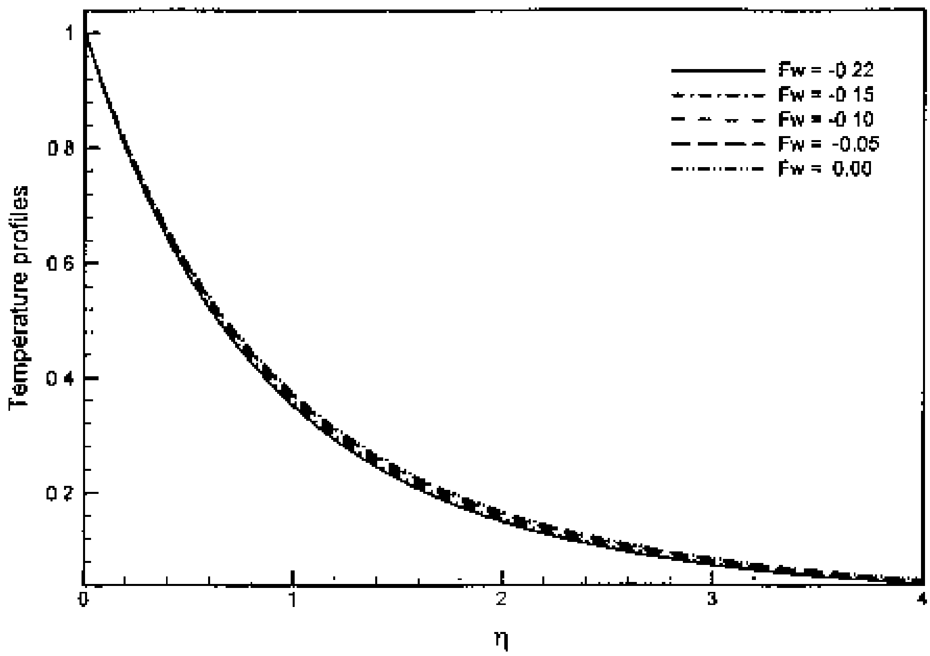

Figure 3.5: Dimensionless temperature profiles against similarity variable η for different values of F_w (suction) while $Pr = 0.72$, $\sigma = 1.0$

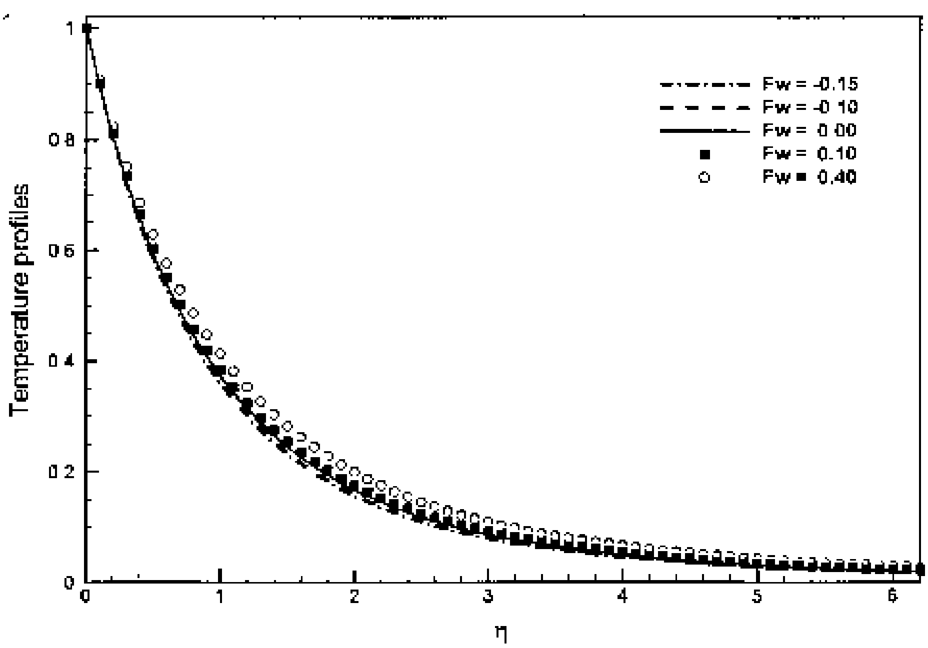

Figure 3.6: Dimensionless temperature profiles against similarity variable η for different values of F_w (suction and injection) while $Pr = 0.72$, $\sigma = 1.0$

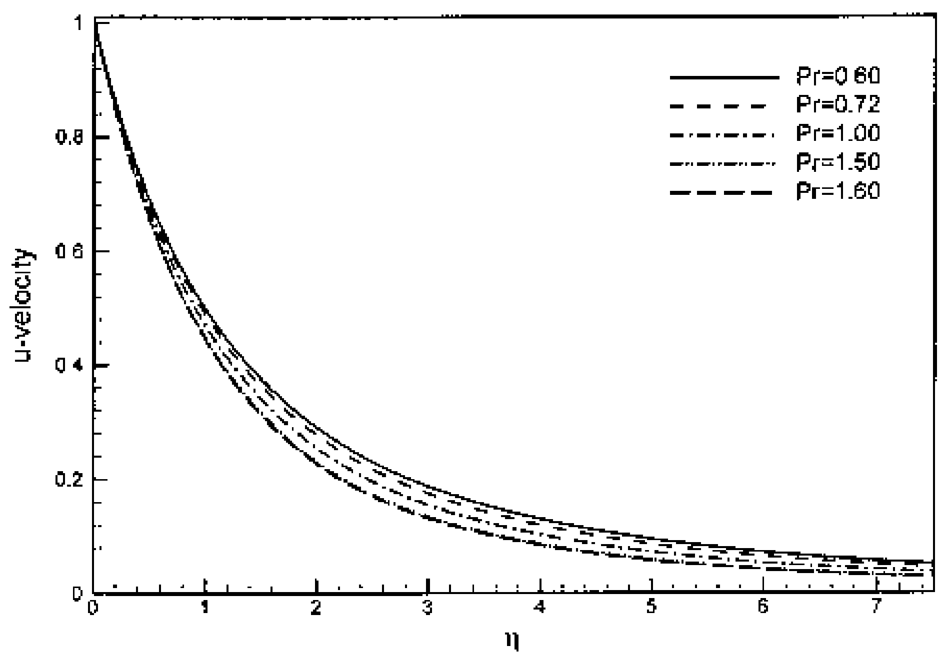

Figure 3.7: Dimensionless u-velocity profiles against similarity variable η for different values of Pr with $F_w = 0.1$, $\sigma = 1.0$

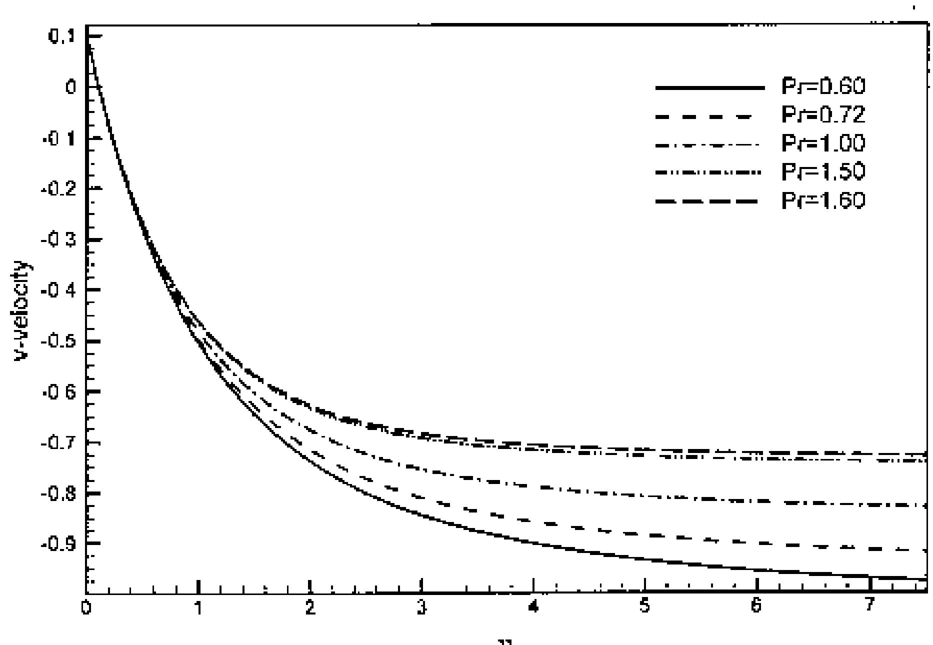

Figure 3.8: Dimensionless v-velocity profiles against similarity variable η for different values of Pr with $F_w = 0.1$, $\sigma = 1.0$

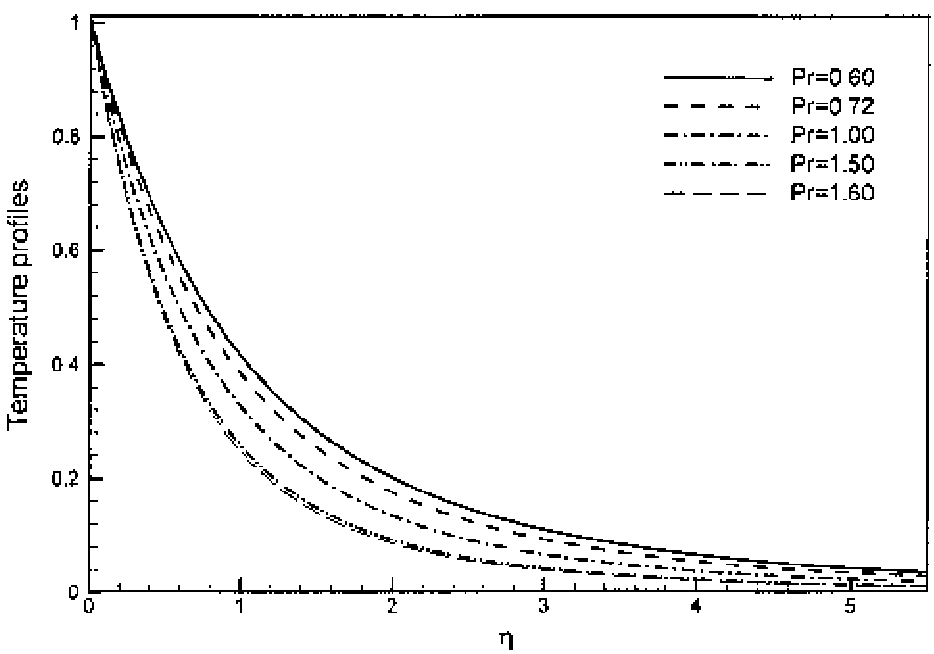

Figure 3.9: Dimensionless temperature profiles against similarity variable η for different values of Pr with $F_w = 0.1$, $\sigma = 1.0$

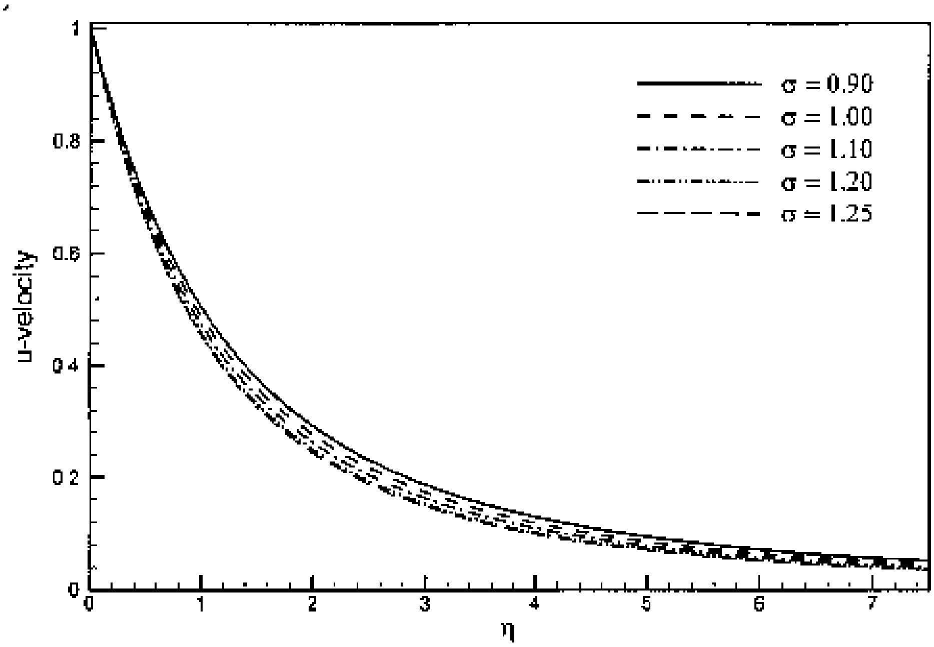

Figure 3.10: Dimensionless u-velocity profiles against similarity variable η for different values of σ with Pr =0.72, $F_w = 0.1$

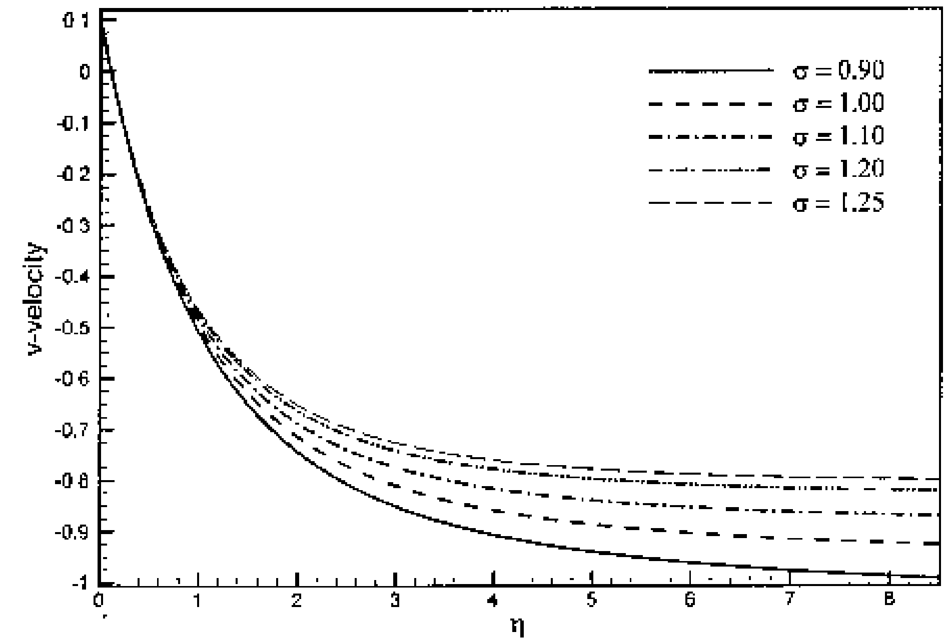

Figure 3.11: Dimensionless v-velocity profiles against similarity variable η for different values of σ with Pr =0.72, F_w = 0.1

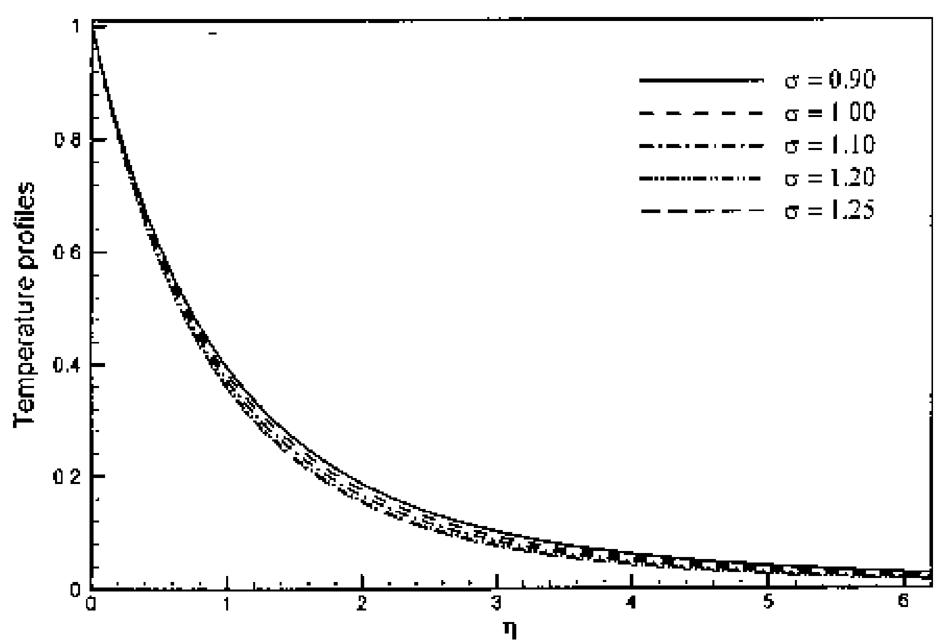

Figure 3.12: Dimensionless temperature profiles against similarity variable η for different values of σ with Pr =0.72, F_w = 0.1

3.5 Conclusion

The effect of suction and injection parameter on unsteady free-convection flow from a moving vertical surface has been investigated for different values of relevant physical parameters. The case "Unsteady free-convection with surface temperature varying inversely as a linear combination of x and t, the flow velocity at wall varying directly with log function of (x, t) and the suction velocity varying inversely as a square root of the linear combination of x and t" has been studied numerically. The u-velocity profile, v-velocity and temperature profile are presented for finite values of suction and injection parameter, viscosity variation parameter and Prandtl number.

- The increases values of suction and injection parameter F_w lead to increase of the u-velocity profile, v-velocity profile and the temperature profile while Pr =0.72 and σ =1.0. Variations of velocity and temperature profiles due to suction, injection and neutral position (neither suction nor injection) have been presented here graphically and in tabular form. It has been observed that velocity and temperature due to injection are larger than velocity and temperature due to suction.
- For increasing values of Prandtl number Pr leads to increase of the v-velocity profiles but decrease of the u-velocity and the temperature profiles for F_w =0.1 and σ =1.0.
- The effects of viscosity variation parameter σ on the u- velocity profile, v-velocity profile and temperature profile are also found here. For increasing values of viscosity variation parameter σ leads to decrease of the u -velocity profile and the temperature profile but increase of v-velocity profile while Pr =0.72, F_w = 0.1.

Chapter 4

$\Delta T \propto x(a_1\, t + b_1)^{-2}$, $u_w \propto x(a_1\, t + b_1)^{-1}$ and $v_w \propto (a_1 t + b_1)^{-1/2}$

i. e. Unsteady free-convection with surface temperature varying directly with a function x and inversely with the square of a function of t, the flow velocity at wall varying directly with a function x and inversely with a function of t and snction and injection velocity varying inversely with a square root of a function of t

4.1. Introduction

This chapter describes the two- dimensional unsteady free convection boundary layer flow with suction and injection on moving vertical plate for the case of "Unsteady free-convection with surface temperature varying directly with a function x and inversely with the square of a function of t, the flow velocity at wall varying directly with a function x and inversely with a function of t and suction and injection velocity varying inversely with a square root of a function of t." Using the group derived in chapter 2 absolute invariants have been found for independent and dependent variables and then governing non-dimensional partial differential equation and boundary conditions are reduced to ordinary differential equations with boundary conditions. For numerical solution we have taken the help of shooting method known as Runge-Kutta-Butcher initial value problem solver together with the Nachtsheim-Swigert iteration scheme described by Nachtsheim and Swigert (1965). Here attention has been given on the evaluation of the u- velocity profiles, v-velocity profiles as well as temperature profiles for some selected values of parameters set consisting of suction and injection parameter F_w, viscosity variation parameter τ and the Prandlt number Pr. Variations of velocity and temperature profiles due to suction, injection and neutral position (no suction or injection) have been presented graphically and in tabular form.

4.2 Transformation of variables

4.2.1 Independent variables as absolute invariants

Owing to equations (2.23) and (2.24), $\eta(x, y, t)$ is an absolute invariant of the independent Variables (x, y, t) if it satisfies the two first –order partial differential equations:

$$(\alpha_1 x+\alpha_2)\frac{\partial\eta}{\partial x}+(\alpha_3 y+\alpha_4)\frac{\partial\eta}{\partial y}+(\alpha_5 t+\alpha_6)\frac{\partial\eta}{\partial t}=0 \tag{4.1}$$

and

$$(\beta_1 x+\beta_2)\frac{\partial\eta}{\partial x}+(\beta_3 y+\beta_4)\frac{\partial\eta}{\partial y}+(\beta_5 t+\beta_6)\frac{\partial\eta}{\partial t}=0 \tag{4.2}$$

Since $K^y = 0$ so $\alpha_4 = \beta_4 = 0$.

Now (4.1) and (4.2) can be written as

$$(\lambda_{31}x+\lambda_{32})\frac{\partial\eta}{\partial x}+(\lambda_{35}t+\lambda_{36})\frac{\partial\eta}{\partial t}=0 \tag{4.3}$$

and

$$(\lambda_{31}x+\lambda_{32})y\frac{\partial\eta}{\partial y}-(\lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26})\frac{\partial\eta}{\partial t}=0 \tag{4.4}$$

where $\lambda_{ij}=\alpha_i\beta_j-\alpha_j\beta_i$ $\quad$ i, j=1,2,3,4,5,6

Now the transformation group G given by (2.22) and making use of the definition of the α's and β's and invoking that $\alpha_5=2\alpha_3$,$\beta_5=2\beta_3$,implies

$$\lambda_{35}=\alpha_3\beta_5-\alpha_5\beta_3=0 \tag{4.5}$$

By considering only one of the co-efficient in (4.3) and (4.4) vanishes and taking

$\lambda_{31}x + \lambda_{32} \neq 0$, $\lambda_{35}t + \lambda_{36} = 0$, $\lambda_{15}x\, t + \lambda_{16}x + \lambda_{25}t + \lambda 26 \neq 0$ shows

$$\frac{\partial \eta}{\partial x} = 0 \tag{4.6}$$

which shows that η is only a function of y and t.

Using (4.3) and (4.4) in (4.1) and (4.2), we get

$$\alpha_3 y \frac{\partial \eta}{\partial y} + (\alpha_5 t + \alpha_6)\frac{\partial \eta}{\partial t} = 0 \tag{4.7}$$

and

$$\beta_3 y \frac{\partial \eta}{\partial y} + (\beta_5 t + \beta_6)\frac{\partial \eta}{\partial t} = 0 \tag{4.8}$$

Applying the standard technique for linear partial differential equations the solution of (4.6) of the form

$$\eta = y\pi_2(t) \tag{4.9}$$

Where

$$\pi_2(t) = I\,(a_1\, t + b_1)^{-1/2} \tag{4.10}$$

and $a_1 = \alpha_5 = \beta_5$

$b_1 = \alpha_6 = \beta_6$

$-\alpha_3/\alpha_5 = -\beta_3/\beta_5 = -1/2$

Without loss of generality, the value of I can be taken as unity. Then the absolute invariant for this case will be

$$\eta = y\,(a_1\, t + b_1)^{-1/2} \tag{4.11}$$

where

$$\pi_2(t) = (a_1 t + b_1)^{-1/2} \tag{4.12}$$

4.2.2 Dependent variables as absolute invariants

In this step , absolute invariants have to be obtained corresponding to the dependent variables u, v, u_w, ΔT and θ . From the group transformation (2.22) it can be written as

$$g_1(x, y, t ; u) = u(\eta) \tag{4.13}$$

and

$$g_2(x, y, t;\theta) = \theta(\eta) \tag{4.14}$$

According to (2.23) and (2.24) any function g_3 (x, t; u_w) satisfies

$$(\alpha_1 x + \alpha_2)\frac{\partial g_3}{\partial x} + (\alpha_5 t + \alpha_6)\frac{\partial g_3}{\partial t} + (\alpha_{11} u_w + \alpha_{12})\frac{\partial g_3}{\partial u_w} = 0 \tag{4.15}$$

and

$$(\beta_1 x + \beta_2)\frac{\partial g_3}{\partial x} + (\beta_5 t + \beta_6)\frac{\partial g_3}{\partial t} + (\beta_{11} u_w + \beta_{12})\frac{\partial g_3}{\partial u_w} = 0 \tag{4.16}$$

The solution of equations (4.15) and (4.16) gives

$$g_3(x, t ;u_w) = \varphi_1(u_w / \omega (x, t)) = E(\eta) \tag{4.17}$$

In a similar manner, g_4 $(x, t ;v)$ and g_5 $(x, t ;\Delta T)$ are obtained

$$g_4(x, t ;v) = \varphi_2 (v/\Gamma(x, t)) = F (\eta) \tag{4.18}$$

$$g_5 (x, t ; \Delta T) = \varphi_3(\Delta T/\gamma(x, t)) = G (\eta) \tag{4.19}$$

where ω (x, t) , Γ(x, t) , γ (x, t), E(η), F(η) and G(η) are functions to be determined. Without loss of generality, the φ's in (4.17) to (4.19) are selected to be the identity functions.

Then the functions can be expressed as $u_w(x, t)$, $v(x, y, t)$ and $\Delta T(x, t)$ in terms of the absolute invariants $E(\eta)$, $F(\eta)$ and $G(\eta)$ respectively, in the form

$$u_w(x, t) = \omega(x, t)\, E(\eta) \tag{4.20}$$

$$v(x, y, t) = \Gamma(x, t)\, F(\eta) \tag{4.21}$$

$$\Delta T(x, t) = \gamma(x, t)\, G(\eta) \tag{4.22}$$

Since $\omega(x, t)$, $u_w(x, t)$ and $\Delta T(x, t)$, $\gamma(x, t)$ are independent of y, whereas η depends on y, it follows that $E(\eta)$ and $G(\eta)$ must be equal to constant E_0 and G_0 respectively. Without loss of generality it can also be taken $E_0 = 1$ and $G_0 = 1$. Then (4.20) and (4.22) become

$$u_w(x, t) = \omega(x, t) \tag{4.23}$$

$$\Delta T(x, t) = \gamma(x, t) \tag{4.24}$$

4.3 The reduction to the ordinary differential equations

Substitution from (4.11) to (4.14), (4.21), (4.23) & (4.24) into equations (2.5), (2.6) and (2 7) yields,

$$\frac{dF}{d\eta} + \left(\frac{y\omega}{\pi_2\Gamma}\frac{\partial \pi_2}{\partial x}\right)\frac{du}{d\eta} + \left(\frac{1}{\pi_2\Gamma}\frac{\partial \omega}{\partial x}\right)u = 0 \tag{4.25}$$

$$\frac{d^2u}{d\eta^2}-\left(\frac{y}{\nu\pi_2{}^2}\frac{\partial\pi_2}{\partial t}+\frac{F\Gamma}{\nu\pi_2}\right)\frac{du}{d\eta}-\left(\frac{\omega y}{\nu\pi_2{}^2}\frac{\partial\pi_2}{\partial x}\right)u\frac{du}{d\eta}-\left(\frac{1}{\nu\pi_2{}^2}\frac{\partial\omega}{\partial x}\right)u^2$$
$$-\left(\frac{1}{\nu\omega\pi_2{}^2}\frac{\partial\omega}{\partial t}\right)u+\frac{g\beta\theta\gamma}{\nu\omega\pi_2{}^2}=0 \tag{4.26}$$

$$\frac{1}{\Pr}\frac{d^2\theta}{d\eta^2}-\left(\frac{y}{\nu\pi_2^2}\frac{\partial\pi_2}{\partial t}+\frac{y\omega}{\nu\pi_2{}^2}\frac{\partial\pi_2}{\partial x}u+\frac{F\Gamma}{\nu\pi_2}\right)\frac{d\theta}{d\eta}-\left(\frac{1}{\nu\pi_2^2\gamma}\frac{\partial\gamma}{\partial t}+\frac{\omega}{\gamma\nu\pi_2^2}\frac{\partial\gamma}{\partial x}u\right)\theta=0 \tag{4.27}$$

The co-efficient of the functions u (η), θ(η) and F (η) and their derivatives, be constants or functions of η only, to reduce (4.25) to (4.27) as a system of equations in a single variable η. Since $\pi_2(x, t)$, $\omega(x, t)$ and $\gamma(x, t)$ are independent of y, so

$$C_1=\frac{\omega}{\pi_2^2\Gamma}\frac{\partial\pi_2}{\partial x} \tag{4.28}$$

$$C_2=\frac{1}{\pi_2\Gamma}\frac{\partial\omega}{\partial x} \tag{4.29}$$

$$C_3=\frac{1}{\nu\pi_2^3}\frac{\partial\pi_2}{\partial t} \tag{4.30}$$

$$C_4=\frac{\Gamma}{\nu\pi_2} \tag{4.31}$$

$$C_5=\frac{\omega}{\nu\pi_2^3}\frac{\partial\pi_2}{\partial x} \tag{4.32}$$

$$C_6=\frac{1}{\nu\pi_2^2}\frac{\partial\omega}{\partial x} \tag{4.33}$$

$$C_7 = \frac{1}{\nu\omega\pi_2^2}\frac{\partial\omega}{\partial t} \tag{4.34}$$

$$C_8 = \frac{g\beta\gamma}{\nu\omega\pi_2^2} \tag{4.35}$$

$$C_9 = \frac{1}{\nu\pi_2^2\gamma}\frac{\partial\gamma}{\partial t} \tag{4.36}$$

$$C_{10} = \frac{\omega}{\nu\pi_2^2\gamma}\frac{\partial\gamma}{\partial x} \tag{4.37}$$

Where C's are constants and to be determined corresponding to each set of absolute invariants. The equations (4.25) to (4.27) take the forms as follows:

$$\frac{dF}{d\eta} + C_1\eta\frac{du}{d\eta} + C_2u = 0 \tag{4.38}$$

$$\frac{d^2u}{d\eta^2} - (C_3\eta + C_4F)\frac{du}{d\eta} - C_5\eta u\frac{du}{d\eta} - C_6u^2 - C_7u + C_8\theta = 0 \tag{4.39}$$

and

$$\frac{1}{\Pr}\frac{d^2\theta}{d\eta^2} - (C_3\eta + C_5\eta u + C_4F)\frac{d\theta}{d\eta} - (C_9 + C_{10}u)\theta = 0 \tag{4.40}$$

The boundary conditions are

$$\begin{aligned} &u(0) = 1,\ \theta(0) = 1,\ F(0) = F_w(0) \neq 0 && \text{at } \eta \to 0 \\ &u(\infty) = 0,\ \theta(\infty) = 0 && \text{as } \eta \to \infty \end{aligned} \tag{4.41}$$

The parameter $F_w = F(0)$ related to the suction parameter v_w when $\eta = 0$

Taking $C_6 = 1$ in (4.33) yields

$$\omega = \nu\,(a_1 t + b_1)^{-1} x \qquad (4.42)$$

In a similar manner, we can take

$$C_4 = 1 \text{ and } C_8 = 1 \qquad (4.43)$$

Now (4.29), (4.31) ,(4.33) and (4.43) implies

$$C_2 = 1 \qquad (4.44)$$

With the help of (4.11) the equation (4.30) implies

$$C_3 = -a_1/2\nu \qquad (4.45)$$

(4.12), (4.34) and (4.42) gives

$$C_7 = -a_1/\nu \qquad (4.46)$$

In (4.43) $C_8 = 1$ shows that

$$\gamma = \nu\, x\,(a_1 t + b_1)^{-2} \qquad (4.47)$$

(4.36), (4 42) and (4.47) implies

$$C_9 = -2\,a_1/\nu \qquad (4.48)$$

Substituting (4.12), (4.42), (4.47) in (4.37) implies

$$C_{10} = 1 \tag{4.49}$$

Using (4.12) in (4.28) and (4.32) implies

$$C_1 = 0 \tag{4.50}$$

$$C_5 = 0 \tag{4.51}$$

Substituting the above obtained values into equations (4.38) to (4.40) and takingτ τ =-$a_{1/v}$ yields

$$\frac{dF}{d\eta} + u = 0 \tag{4.52}$$

$$\frac{d^2u}{d\eta^2} - \left(\frac{\tau}{2}\eta + F\right)\frac{du}{d\eta} - u^2 - \tau u + \theta = 0 \tag{4.53}$$

$$\frac{1}{\Pr}\frac{d^2\theta}{d\eta^2} - \left(\frac{\tau}{2}\eta + F\right)\frac{d\theta}{d\eta} - (2\tau + u)\theta = 0 \tag{4.54}$$

with the boundary conditions:

$$\begin{aligned} &u(0) = 1,\ \theta(0) = 1\ ,\ F(0) = F_w(0) \neq 0 && \text{at } \eta = 0 \\ &u(\infty) = 0\ ,\theta(\infty) = 0 && \text{as } \eta \to \infty \end{aligned} \tag{4.55}$$

$v_w = \Gamma(x,t)\ F(0) = \nu\ (a_1 t + b_1)^{-1/2}\ F(0)$

$v_w < 0$ signifies suction

$v_w > 0$ signifies injection

For the above case, the boundary-layer characteristics (Surface heat flux) is

$$q = \nu x\left(a_1 t + b_1\right)^{-5/2}\left[-\theta'(0)\right]$$

4.4. Results and discussion

Similarity solution for unsteady free convection flow from a moving vertical surface with suction and injection has been investigated analytically using group theory method and then numerically by sixth order Runge-Kutta shooting method. The case "Unsteady free convection with surface temperature varying directly with a function x and inversely with the square of a function of t, the flow velocity at wall varying directly with a function x and inversely with a function of t and suction velocity varying inversely with a square root of a function of t" is considered in this chapter. After making proper use of similarity transformations, three independent variables reduced to one similarity variable η. Analytical solutions have been derived for the present case and then numerical solution have been obtained using Runge-Kutta shooting method. numerical results of the problem in terms of u-velocity, v-velocity and temperature profiles have been presented graphically against the similarity variable η, for different values of suction and injection parameter, viscosity variation parameter and Prandtl number.

The effects of suction parameter F_w on u-velocity, v-velocity and temperature profiles while Prandtl number $Pr = 0.72$, viscosity variation parameter $\tau = 1.0$ are displayed in figures 4.1, 4.3 and 4.5 respectively. Different values of suction parameter $F_w = -1.15, -1.0, -0.8, -0.5, -0.2$ and 0.0 have been considered.

u-velocity, v-velocity and temperature profiles for three situations when F_w takes the values of suction, injection and neutral position (no suction or injection) while Prandtl number $Pr = 0.72$, viscosity variation parameter $\tau = 1.0$ are represented by the figures 4.2, 4.4 and 4.6.

In figs. 4.7-4.9, u-velocity, v-velocity and temperature profiles are shown for different values of Prandtl number Pr while $F_w = 0.5$, $\tau = 1.5$. For different values of viscosity variation parameter τ, u-velocity, v-velocity and temperature profiles are displayed in figs. 4.10-4.12 while $Pr = 0.72$, $F_w = 0.5$.

In fig. 4.1, it is observed that u-velocity profiles increase with the increasing values of suction parameter F_w. The maximum values of the u-velocities have been found as 1.0 at the boundary wall for F_w = -1.15, -1.0, -0.8, -0.5, -0.2 and 0.0 which occur at the same point η = 0.0 and then the velocities decrease with the increasing values of η and finally velocity profiles approach to zero (the asymptotic value).

In fig. 4.3 v-velocity becomes negative in a certain region of the boundary layer for different values of suction parameter F_w while Prandtl number Pr = 0.72, viscosity variation parameter τ = 1.0. Slightly ups and downs of v-velocity profiles for different values of suction parameter F_w are found in the boundary layer.

For the temperature profiles, figure 4.5 shows that boundary layer thickness increases with increasing values of suction parameter F_w. The maximum values of temperature profiles are found as 1.000 for F_w = -1.15 -1.0, -0.8, -0 5, -0.2 and 0.0 that occur at the same point The change of temperature profiles in the η direction also shows the typical temperature profile for free convection boundary layer flow that is the value of temperature profile is 1.0 (one) at the boundary wall then the temperature profile decreases gradually along η direction to the asymptotic value.

u-velocities for three suction values, two injection values and neutral value of F_w (neither suction nor injection) are represented in fig. 4.2. It has been found that u-velocity takes the values -0.0204, -0.0094, -0.0089, -0.0075,-0.0081,-0.0015 and 0.0108 at η = 5.000 for F_w = -1.0, -0.50, -0.20, -0.10, 0.0, 0.5, 1.0 respectively i.e. velocity increases with the increasing value of F_w. From the figure it has been found that u-velocity boundary layer thickness for neutral (no suction or injection) position is larger than boundary layer thickness due to suction but smaller than boundary layer thickness due to injection. It is observed that at η = 5.000 the velocity increases as the suction and injection parameter F_w changes from, -1.0 to 1.0.

v-velocity for two suction values, two injection values and neutral values of F_w are represented by fig.4.4. It has been found from the figure that v-velocity for injection is larger than that of due to suction. v-velocity takes the values-1.0, -0.50, -0.20, 0.00, 0.50, 1.00 for F_w =-1.0, -0.50, -0.20, , 0.0,, 0.5, 1.0 respectively at starting point η = 0.0 then velocity decreases gradually.

In figure 4.6 temperature profiles are presented for suction, injection and neutral position. Temperature for neutral is larger than temperature due to suction and smaller than temperature due to injection. From maximum and common value 1.0 temperature profile decreases for increasing values of η while F_w = -1.0, -0.50, -0.20, 0.0, 0.5, 1.0.

The variation of the u-velocity, v-velocity and temperature profiles for different values of Prandtl number Pr while suction parameter F_w = 0.5 and viscosity variation parameter τ =1.5 are shown in the figures 4.7 to 4.9. For the velocity profile, figures 4.7 and 4.8 indicate the occurrence of the increase in u-velocity and decrease in v-velocity respectively with increasing Prandtl number Pr. It can also be seen that u-velocity increases and v-velocity decreases with increasing values of Prandtl number Pr except Pr = 0.2. It is observed that the value of u-velocity and v-velocity ups and downs against some values of η while Pr = 0.2.This phenomenon is known as velocity defect. In the figure 4.9, it can be shown that temperature profiles decrease with increasing values of the Prandtl number Pr.

The effect for different values of viscosity variation parameter τ, the velocity and temperature profiles while Prandtl number Pr = 0.72 and suction parameter F_w =0.5 are shown in the figures 4.10 to 4.12.

The increasing values of the viscosity variation parameter τ in figure 4.10 represent that the u-velocity profiles decrease near the surface of the moving vertical plate but the values of the u-velocity profiles don't decrease smoothly for = 0.7, sometimes the velocity profiles increase for decreasing τ while η < 2.10 they meet and then cross and increase for increasing τ after η > 2.15.

It is observed from the figure 4.11 that the v-velocity increases gradually with increasing value of viscosity variation parameter τ. Starting with the fixed value the v-velocity profile increases slowly near the surface of the moving vertical plate and increases rapidly far from the plate.

The changes of temperature profiles in figure 4.12 show the typical temperature profile for natural convection boundary layer flow that is the value of dimensionless temperature is 1.0 at the boundary wall then the temperature profile decreases gradually along η direction and approaches to zero, the asymptotic value. Numerical values of u-velocity, v-velocity and temperature profiles, for different values of F_w (Suction, injection or neutral position of suction and injection) have been shown in Table 4.1. All the u-velocity, v-velocity and temperature are observed to increase for increasing F_w from the table.

Table 4.1: Dimensionless u-velocity, v-velocity profiles and temperature profiles against η for different values of suction and injection parameter F_w with other controlling parameters $Pr = 0.72$, $\tau=1.0$

η	u-velocity for different values of F_w			v-velocity for different values of F_w			Temperature profile for different values of F_w		
	F_w= -0.10	F_w= 0.0	F_w= 0.10	F_w= -0.10	F_w= 0.0	F_w= 0.10	F_w= -0.10	F_w= 0.0	F_w= 0.10
0.0000	1.0000	1.0000	1.0000	-0.5000	0.0000	0.5000	1.0000	1.0000	1.0000
0.1000	0.8073	0.8599	0.9224	-0.5900	-0.0915	0.4072	0.8575	0.8723	0.8854
0.2000	0.6516	0.7406	0.8528	-0.6626	-0.1680	0.3274	0.7375	0.7627	0.7855
0.3000	0.5257	0.6389	0.7902	-0.7213	-0.2320	0.2585	0.6360	0.6686	0.6983
0.4000	0.4238	0.5520	0.7337	-0.7685	-0.2856	0.1991	0.5500	0.5874	0.6219
0.5000	0.3410	0.4777	0.6825	-0.8066	-0.3305	0.1477	0.4769	0.5172	0.5550
0.6000	0.2738	0.4139	0.6362	-0.8372	-0.3682	0.1032	0.4145	0.4564	0.4961
0.7000	0.2191	0.3592	0.5940	-0.8618	-0.3998	0.0647	0.3612	0.4036	0.4443
0.8000	0.1745	0.3121	0.5556	-0.8814	-0.4263	0.0312	0.3155	0.3577	0.3986
0.9000	0.1383	0.2715	0.5205	-0.8970	-0.4485	0.0020	0.2762	0.3176	0.3582
1.0000	0.1087	0.2365	0.4883	-0.9093	-0.4672	-0.0233	0.2423	0.2825	0.3224

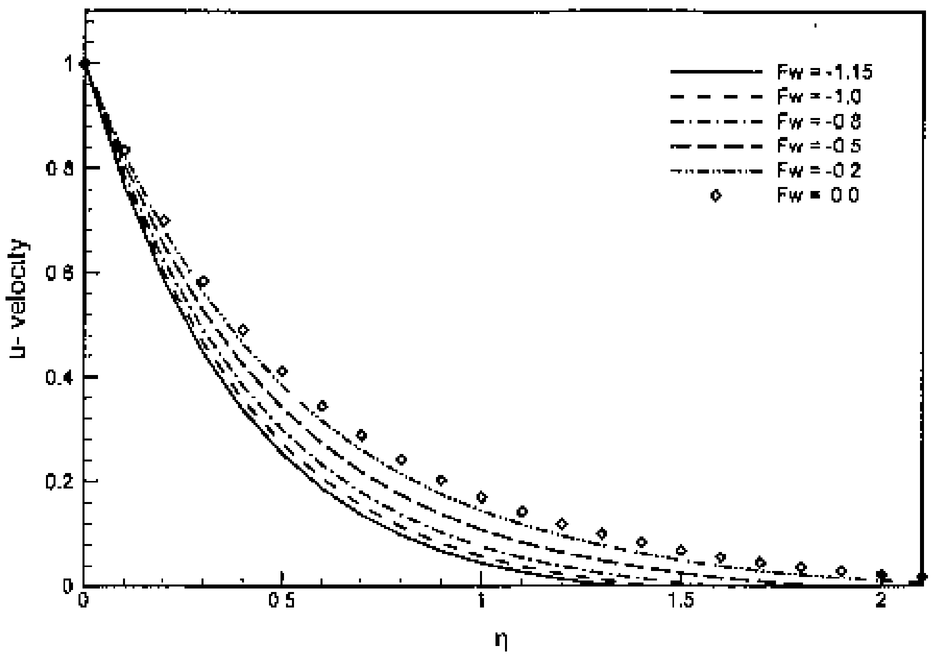

Figure 4.1: Dimensionless u-velocity profiles against similarity variable η for different values of F_w (suction) while Pr = 0.72, τ=1.0

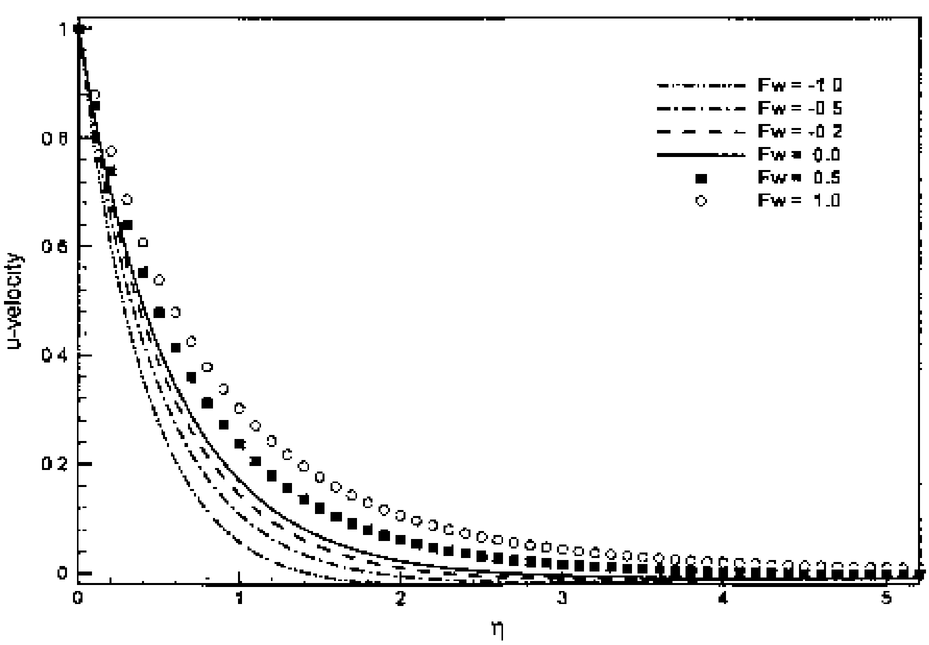

Figure 4.2: Dimensionless u-velocity profiles against similarity variable η for different values of F_w (suction and injection) while Pr = 0.72, τ=1.0

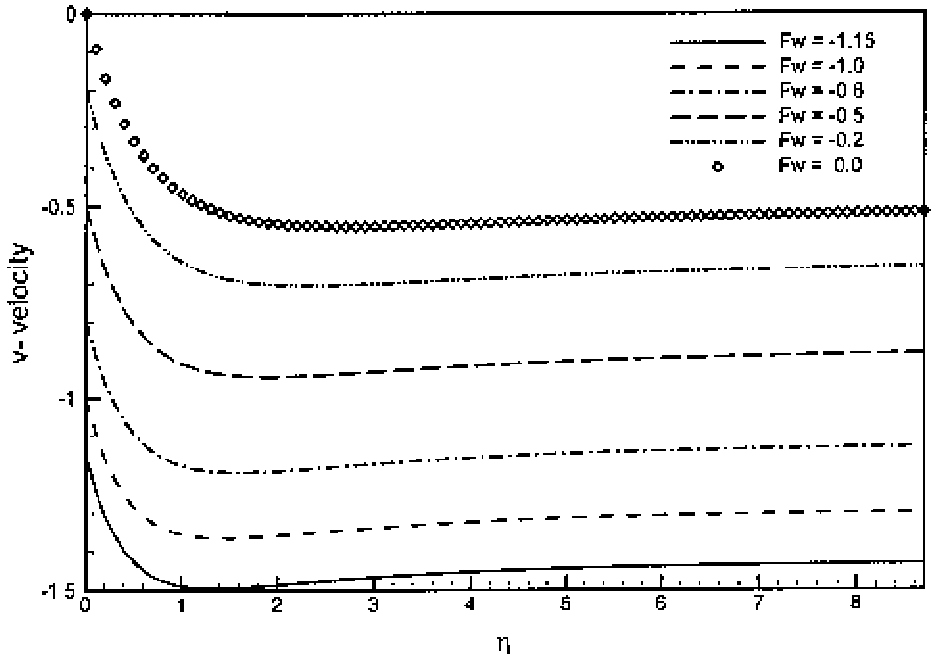

Figure 4.3: Dimensionless v-velocity profiles against similarity variable η for different values of F_w (suction) while Pr = 0.72, τ=1.0

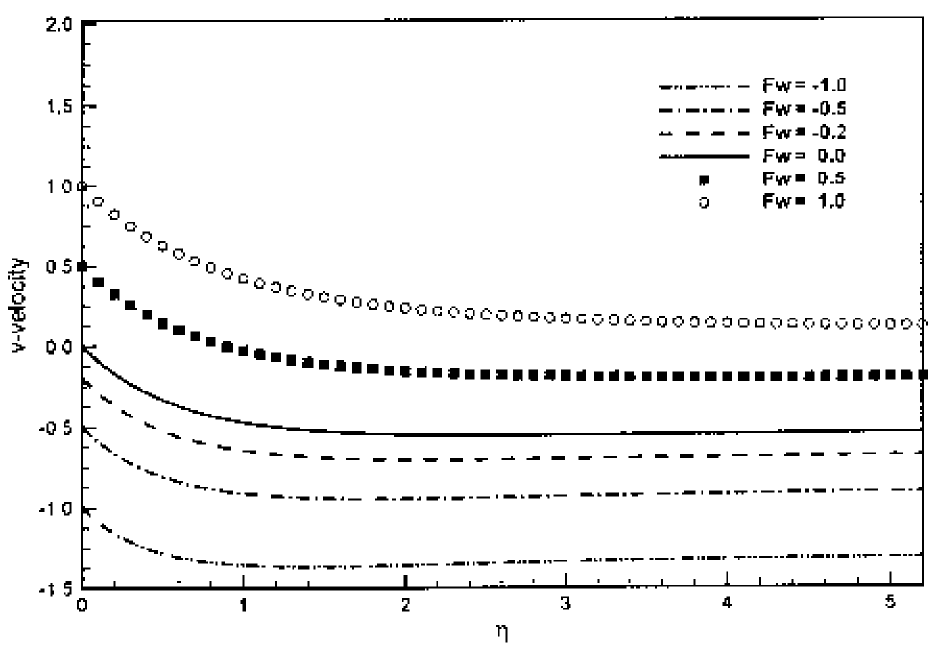

Figure 4.4: Dimensionless v-velocity profiles against similarity variable η for different values of F_w (suction and injection) while Pr = 0.72, τ =1.0

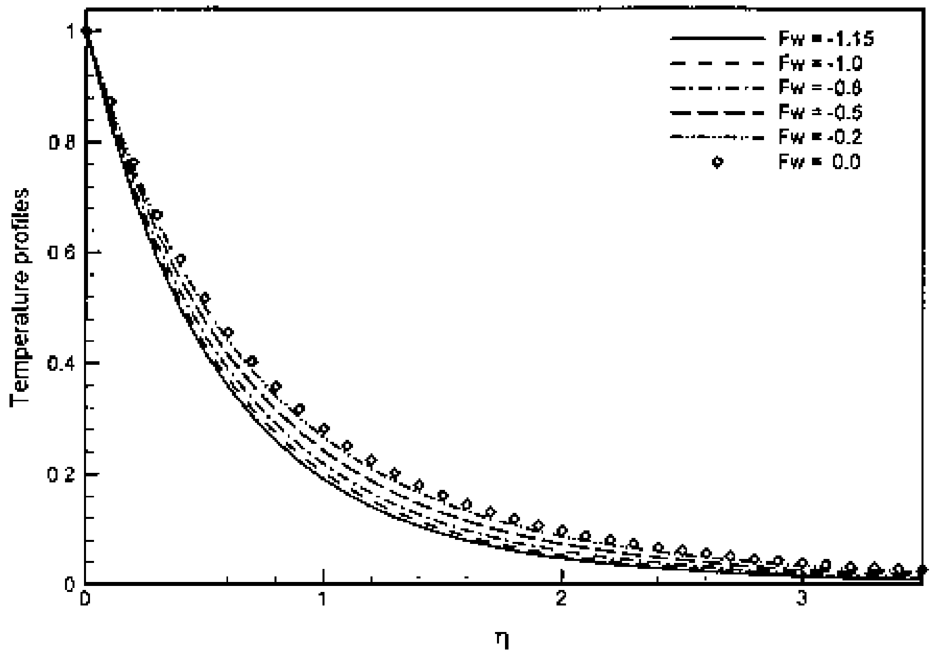

Figure 4.5: Dimensionless temperature profiles against similarity variable η for different values of F_w (suction) while Pr = 0.72, τ=1.0

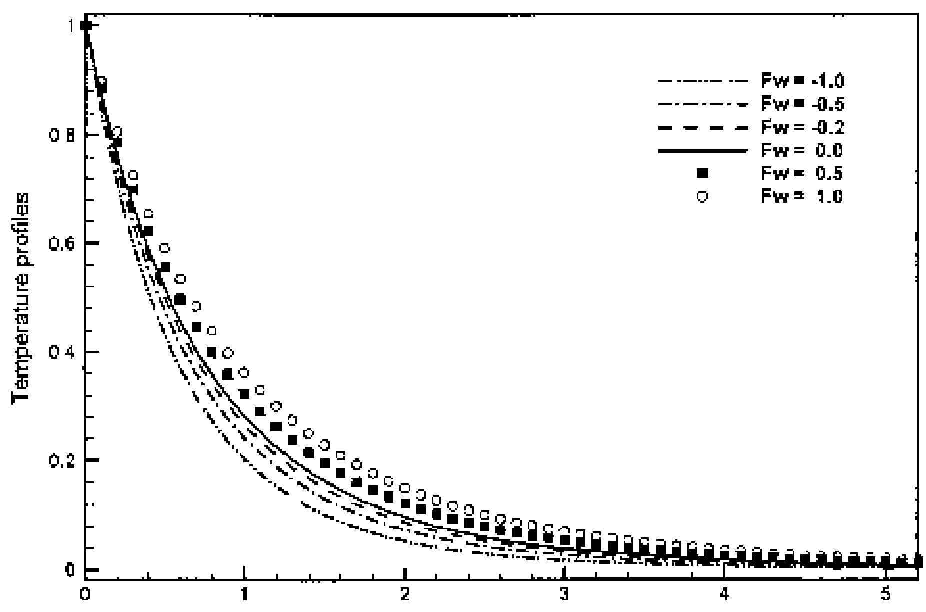

Figure 4.6: Dimensionless temperature profiles against similarity variable η for different values of F_w (suction and injection) while Pr = 0.72, τ=1.0

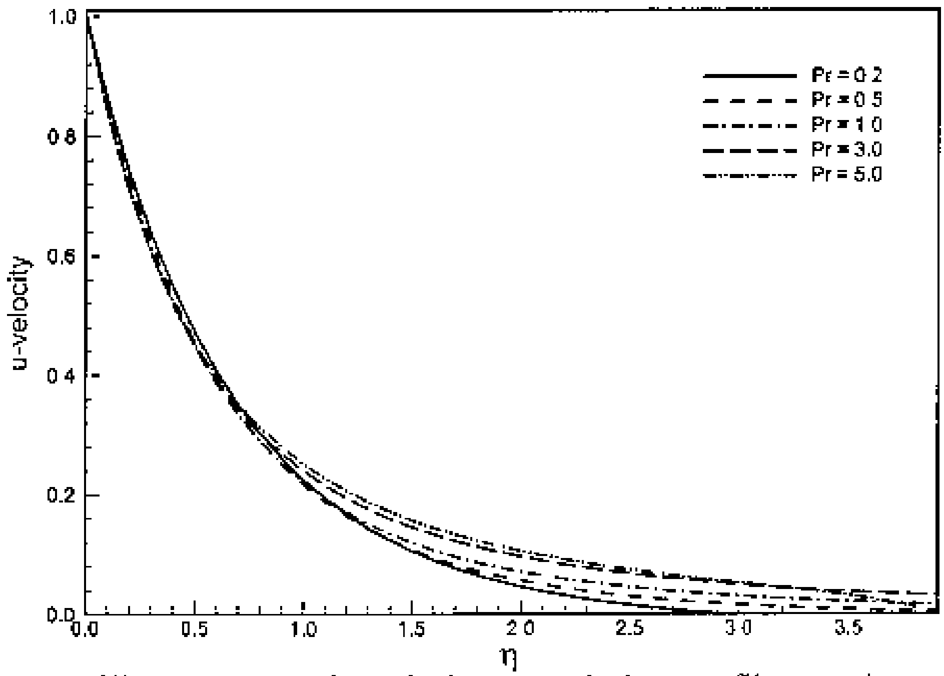

Figure 4.7: Dimensionless u-velocity profiles against similarity variable η for different values of Pr while $F_w = 0.5$, τ=1.5

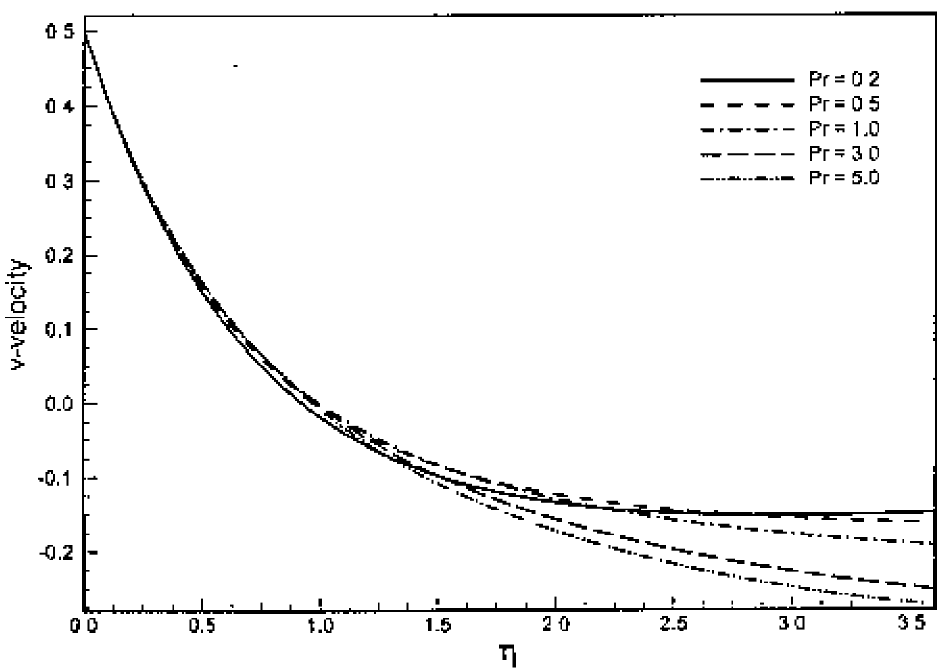

Figure 4.8: Dimensionless v-velocity profiles against similarity variable η for different values of Pr while $F_w = 0.5$, τ=1.5

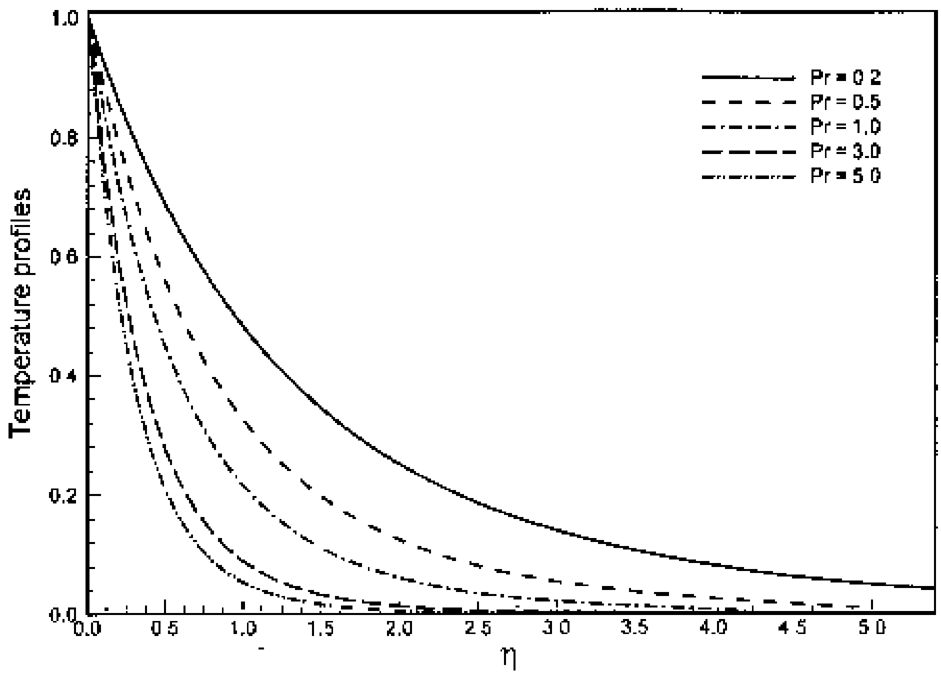

Figure 4.9: Dimensionless temperature profiles against similarity variable η for different values of Pr while $F_w = 0.5$, $\tau=1.5$

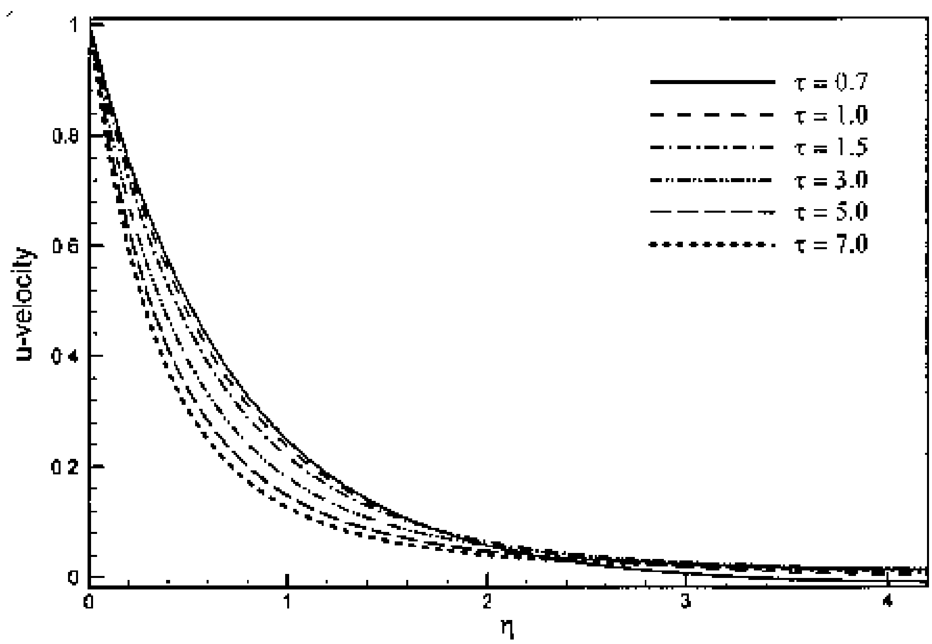

Figure 4.10: Dimensionless u-profiles against similarity variable η for different values of τ while $Pr = 0.72$, $F_w = 0.5$

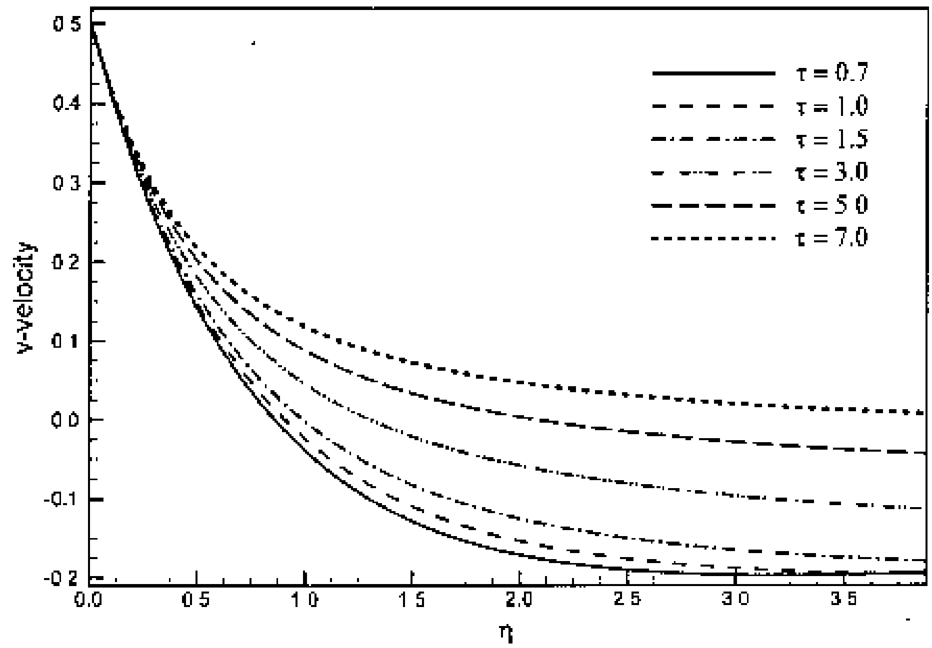

Figure 4.11: Dimensionless v-velocity profiles against similarity variable η for different values of τ while Pr = 0.72, F_w =0.5

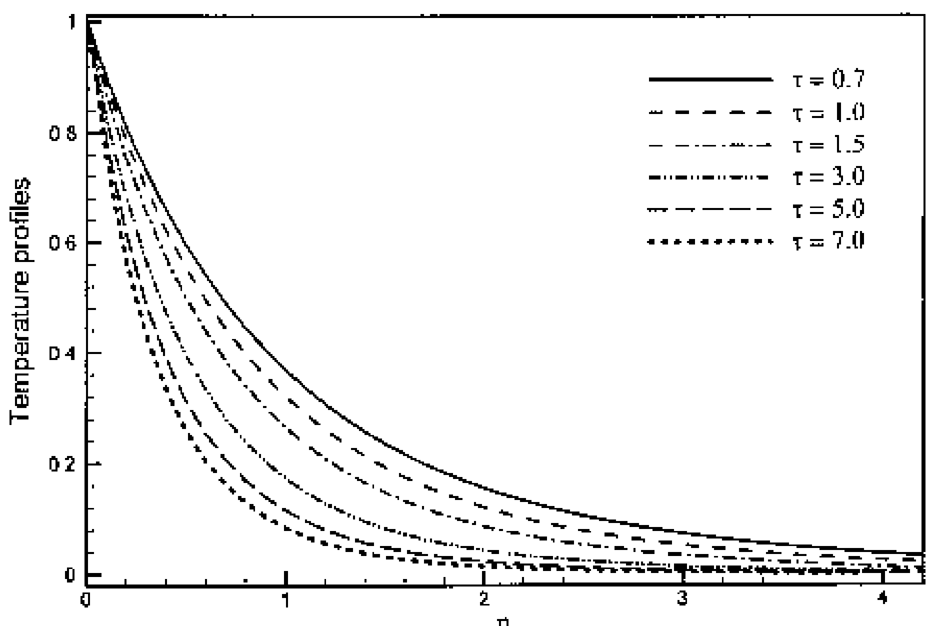

Figure 4.12: Dimensionless temperature profiles against similarity variable η for different values of τ while Pr = 0.72, F_w =0.5

4.5 Conclusion

Group-theory method has been employed to find similarity solution of the unsteady free-convection flow from a moving vertical surface with suction and injection. The effect of suction and injection parameter on unsteady free- convection flow from a moving vertical surface has been investigated for different values of relevant physical parameters. The case "Unsteady free-convection with surface temperature varying directly with a function x and inversely with the square of a function of t, the flow velocity at wall varying directly with a function x and inversely with a function of t and suction and injection velocity varying inversely with a square root of a function of t" has been studied analytically using two parameter group and then numerically. The u-velocity, v-velocity and temperature are presented for different values of suction and injection parameter F_w, viscosity variation parameter τ and Prandtl number Pr.

- Two-parameter group-theory method has been applied to reduce the number of independent variables and then partial differential equations have been transformed to ordinary differential equations. The resulting nonlinear ordinary differential equations have been solved analytically and then numerically using the sixth order Runge-Kutta shooting method.
- The increasing values of suction and injection parameter F_w lead to increase of the u-velocity, v-velocity and the temperature profiles while $Pr = 0.72$ and $\tau = 1.0$.
- Due to increasing values of Prandtl number Pr leads to increase of u-velocity and decrease of v-velocity profiles near the moving vertical plate. But u-velocity and v-velocity ups and downs against some values of η far from the moving vertical surface while $Pr = 0.2$. Thermal boundary layer thickness decreases for increasing values of Prandtl number Pr while $F_w = 0.5$, $\tau = 1.5$.
- The effect of viscosity variation parameter τ on the u- velocity profile, v-velocity profile and temperature profile has been investigated here. The u-velocity profile decreases near the surface of the moving vertical plate but the values of the u-velocity profiles don't decrease smoothly for $\tau = 0.7$. V-velocity increases gradually with increasing viscosity variation parameter τ. Temperature has been observed decreases with increasing values of the viscosity variation parameter τ.

4.6 Extension of this work

1. Consideration of mixed convection flow instead of free-convection flow can extend the problem.
2. Taking sphere or cylinder as a moving surface instead of moving vertical plate can extend the problem
3. Species concentration and mass transfer may be included to extend the problem.

Appendix

Runge-Kutta Shooting Method

In shooting method, the missing (unspecified) initial conditions at the initial point in the interval is guessed and differential equation is then integrated numerically as an initial value problem to the terminal points by Runge-Kutta method. Calculated values are then compared with the given values at the terminal points, if there is any difference (error) found guessed values must be changed before next iteration. This process is repeated until the agreement between the calculated and the given condition at the terminal point which is within the specified degree of accuracy.

The boundary conditions associated with non-dimensional ordinary differential equations of the boundary type are of the two-point asymptotic class. Two-point boundary conditions have values of the dependent variable specified at two different values of the independent variable. Specification of an asymptotic boundary condition implies that u and θ tends to unity as the outer specified value of the independent variable is approached.

The method of numerically integrating a two-point asymptotic boundary-value problem of the boundary layer type, the initial value method, requires that it be recast as an initial value problem. Thus it is necessary to estimate as many boundary conditions at the surface as were given at infinity. The solution has been achieved assuming the required outer boundary conditions are satisfied after the integration of governing differential equations by the assumed surface conditions. If this is not satisfied, another new surface boundary condition is estimated for the next trial integration. But this is not so easy, because selecting a value may result in the divergence of the trial integration or in slow convergence of surface boundary conditions required satisfying the asymptotic outer boundary condition. Moreover, selecting too large a value of the independent variable is expensive in terms of computer time. Nachtsheirn-Swigert developed an iteration method, which overcomes these difficulties. If in boundary equations there are two asymptotic boundary conditions and hence two unknown surface conditions are $u' = g_1$ and $\theta' = g_2$.

Within the context of the initial value method and the Nachtsheim-Swigert iteration technique, the outer boundary conditions may be functionally represented as

$$u(\eta_{max}) = u(g_1, g_2) = \delta_1 \quad \text{(A1)}$$

$$\theta(\eta_{max}) = \theta(g_1, g_2) = \delta_2 \quad \text{(A2)}$$

$$F(\eta_{max}) = F(g_1, g_2) = \delta_3 \quad \text{(A3)}$$

With the asymptotic convergence criteria given by

$$u'(\eta_{max}) = u'(g_1, g_2) = \delta_4 \quad \text{(A4)}$$

$$\theta'(\eta_{max}) = \theta'(g_1, g_2) = \delta_5 \quad \text{(A5)}$$

Expanding the equation (A1)-(A5) in a first order Taylor's series gives

$$u(\eta_{max}) = u_c(\eta_{max}) + \frac{\partial u}{\partial g_1}\Delta g_1 + \frac{\partial u}{\partial g_2}\Delta g_2 = \delta_1 \quad \text{(A6)}$$

$$\theta(\eta_{max}) = \theta_c(\eta_{max}) + \frac{\partial \theta}{\partial g_1}\Delta g_1 + \frac{\partial \theta}{\partial g_2}\Delta g_2 = \delta_2 \quad \text{(A7)}$$

$$F(\eta_{max}) = F_c(\eta_{max}) + \frac{\partial F}{\partial g_1}\Delta g_1 + \frac{\partial F}{\partial g_2}\Delta g_2 = \delta_3 \quad \text{(A8)}$$

$$u'(\eta_{max}) = u'_c(\eta_{max}) + \frac{\partial u'}{\partial g_1}\Delta g_1 + \frac{\partial u'}{\partial g_2}\Delta g_2 = \delta_4 \quad \text{(A9)}$$

$$\theta'(\eta_{max}) = \theta'_c(\eta_{max}) + \frac{\partial \theta'}{\partial g_1}\Delta g_1 + \frac{\partial \theta'}{\partial g_2}\Delta g_2 = \delta_5 \quad \text{(A10)}$$

Where subscript 'C' indicates the value of the function at η_{max} determined from the trial integration. Solution of these equations in a least-squares sense requires determining the minimum value of

$$E = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 + \delta_5^2 \quad \text{(A11)}$$

Differentiating (A11) with respect to g_1, we have

$$2\delta_1 \frac{\partial \delta_1}{\partial g_1} + 2\delta_2 \frac{\partial \delta_2}{\partial g_1} + 2\delta_3 \frac{\partial \delta_3}{\partial g_1} + 2\delta_4 \frac{\partial \delta_4}{\partial g_1} + 2\delta_5 \frac{\partial \delta_5}{\partial g_1} = 0$$

$$\Rightarrow \delta_1 \frac{\partial \delta_1}{\partial g_1} + \delta_2 \frac{\partial \delta_2}{\partial g_1} + \delta_3 \frac{\partial \delta_3}{\partial g_1} + \delta_4 \frac{\partial \delta_4}{\partial g_1} + \delta_5 \frac{\partial \delta_5}{\partial g_1} = 0$$

$$\Rightarrow \left(u_c + \frac{\partial u}{\partial g_1}\Delta g_1 + \frac{\partial u}{\partial g_2}\Delta g_2 \right)\frac{\partial u}{\partial g_1} + \left(\theta_c + \frac{\partial \theta}{\partial g_1}\Delta g_1 + \frac{\partial \theta}{\partial g_2}\Delta g_2 \right)\frac{\partial \theta}{\partial g_1}$$
$$+ \left(F_c + \frac{\partial F}{\partial g_1}\Delta g_1 + \frac{\partial F}{\partial g_2}\Delta g_2 \right)\frac{\partial F}{\partial g_1} + \left(u'_c + \frac{\partial u'}{\partial g_1}\Delta g_1 + \frac{\partial u'}{\partial g_2}\Delta g_2 \right)\frac{\partial u'}{\partial g_1}$$
$$+ \left(\theta'_c + \frac{\partial \theta'}{\partial g_1}\Delta g_1 + \frac{\partial \theta'}{\partial g_2}\Delta g_2 \right)\frac{\partial \theta'}{\partial g_1} = 0$$

$$\left[\left(\frac{\partial u}{\partial g_1}\right)^2 + \left(\frac{\partial \theta}{\partial g_1}\right)^2 + \left(\frac{\partial F}{\partial g_1}\right)^2 + \left(\frac{\partial u'}{\partial g_1}\right)^2 + \left(\frac{\partial \theta'}{\partial g_1}\right)^2 \right]\Delta g_1$$
$$+ \left[\frac{\partial u}{\partial g_1}\frac{\partial u}{\partial g_2} + \frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_2} + \frac{\partial F}{\partial g_1}\frac{\partial F}{\partial g_2} + \frac{\partial u'}{\partial g_1}\frac{\partial u'}{\partial g_2} + \frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_2} \right]\Delta g_2 \qquad \text{(A12)}$$
$$= -\left(u_c \frac{\partial u}{\partial g_1} + \theta_c \frac{\partial \theta}{\partial g_1} + F_c \frac{\partial F}{\partial g_1} + u'_c \frac{\partial u'}{\partial g_1} + \theta'_c \frac{\partial \theta'}{\partial g_1} \right)$$

Differentiating (A11) with respect to g_2, we have

$$2\delta_1 \frac{\partial \delta_1}{\partial g_2} + 2\delta_2 \frac{\partial \delta_2}{\partial g_2} + 2\delta_3 \frac{\partial \delta_3}{\partial g_2} + 2\delta_4 \frac{\partial \delta_4}{\partial g_2} + 2\delta_5 \frac{\partial \delta_5}{\partial g_2} = 0$$

$$\Rightarrow \delta_1 \frac{\partial \delta_1}{\partial g_2} + \delta_2 \frac{\partial \delta_2}{\partial g_2} + \delta_3 \frac{\partial \delta_3}{\partial g_2} + \delta_4 \frac{\partial \delta_4}{\partial g_2} + \delta_5 \frac{\partial \delta_5}{\partial g_2} = 0$$

$$\Rightarrow \left(u_c + \frac{\partial u}{\partial g_1}\Delta g_1 + \frac{\partial u}{\partial g_2}\Delta g_2 \right)\frac{\partial u}{\partial g_2} + \left(\theta_c + \frac{\partial \theta}{\partial g_1}\Delta g_1 + \frac{\partial \theta}{\partial g_2}\Delta g_2 \right)\frac{\partial \theta}{\partial g_2}$$
$$+ \left(F_c + \frac{\partial F}{\partial g_1}\Delta g_1 + \frac{\partial F}{\partial g_2}\Delta g_2 \right)\frac{\partial F}{\partial g_2} + \left(u'_c + \frac{\partial u'}{\partial g_1}\Delta g_1 + \frac{\partial u'}{\partial g_2}\Delta g_2 \right)\frac{\partial u'}{\partial g_2}$$
$$+ \left(\theta'_c + \frac{\partial \theta'}{\partial g_1}\Delta g_1 + \frac{\partial \theta'}{\partial g_2}\Delta g_2 \right)\frac{\partial \theta'}{\partial g_2} = 0$$

$$\left[\frac{\partial u}{\partial g_1}\frac{\partial u}{\partial g_2}+\frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_2}+\frac{\partial F}{\partial g_1}\frac{\partial F}{\partial g_2}+\frac{\partial u'}{\partial g_1}\frac{\partial u'}{\partial g_2}+\frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_2}\right]\Delta g_1$$
$$\left[\left(\frac{\partial u}{\partial g_2}\right)^2+\left(\frac{\partial \theta}{\partial g_2}\right)^2+\left(\frac{\partial F}{\partial g_2}\right)^2+\left(\frac{\partial u'}{\partial g_2}\right)^2+\left(\frac{\partial \theta'}{\partial g_2}\right)^2\right]\Delta g_2 \qquad \text{(A13)}$$
$$=-\left(u_c\frac{\partial u}{\partial g_2}+\theta_c\frac{\partial \theta}{\partial g_2}+F_c\frac{\partial F}{\partial g_2}+u'_c\frac{\partial u'}{\partial g_2}+\theta'_c\frac{\partial \theta'}{\partial g_2}\right)$$

We can write equations (A12) and (A13) in a system of linear equations as follows:

$$a_{11}\Delta g_1 + a_{12}\Delta g_2 = b_1$$

$$a_{21}\Delta g_1 + a_{22}\Delta g_2 = b_2 \qquad \text{(A14)}$$

(A15)

Where

$$a_{11}=\left(\frac{\partial u}{\partial g_1}\right)^2+\left(\frac{\partial \theta}{\partial g_1}\right)^2+\left(\frac{\partial F}{\partial g_1}\right)^2+\left(\frac{\partial u'}{\partial g_1}\right)^2+\left(\frac{\partial \theta'}{\partial g_1}\right)^2$$

$$a_{12}=\frac{\partial u}{\partial g_1}\frac{\partial u}{\partial g_2}+\frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_2}+\frac{\partial F}{\partial g_1}\frac{\partial F}{\partial g_2}+\frac{\partial u'}{\partial g_1}\frac{\partial u'}{\partial g_2}+\frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_2}$$

$$b_1=-\left(u_c\frac{\partial u}{\partial g_1}+\theta_c\frac{\partial \theta}{\partial g_1}+F_c\frac{\partial F}{\partial g_1}+u'_c\frac{\partial u'}{\partial g_1}+\theta'_c\frac{\partial \theta'}{\partial g_1}\right)$$

$$a_{21}=\frac{\partial u}{\partial g_1}\frac{\partial u}{\partial g_2}+\frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_2}+\frac{\partial F}{\partial g_1}\frac{\partial F}{\partial g_2}+\frac{\partial u'}{\partial g_1}\frac{\partial u'}{\partial g_2}+\frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_2}$$

$$a_{22}=\left(\frac{\partial u}{\partial g_2}\right)^2+\left(\frac{\partial \theta}{\partial g_2}\right)^2+\left(\frac{\partial F}{\partial g_2}\right)^2+\left(\frac{\partial u'}{\partial g_2}\right)^2+\left(\frac{\partial \theta'}{\partial g_2}\right)^2$$

$$b_2=-\left(u_c\frac{\partial u}{\partial g_2}+\theta_c\frac{\partial \theta}{\partial g_2}+F_c\frac{\partial F}{\partial g_2}+u'_c\frac{\partial u'}{\partial g_2}+\theta'_c\frac{\partial \theta'}{\partial g_2}\right)$$

From equations (A12) and (A13), we have

$$\Delta g_1 = \frac{\det A_1}{\det A}, \quad \Delta g_2 = \frac{\det A_2}{\det A}$$

Where, $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

$$\det A_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix},$$

and

$$\det A_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

Then we obtain the missing (unspecified) values g_1, g_2 as

$$g_1 = g_1 + \Delta g_1$$
$$g_2 = g_2 + \Delta g_2$$

Thus adopting the numerical technique described above, the solutions of the non-linear differential equations with the boundary conditions are obtained together with the sixth-order implicit Runge-Kutta intial value solver.

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