Stress Work Effects in Laminar Natural Convection Flow About a Vertical Rectangular Inclined Plane Surface.

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By

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None of the materials contained in this thesis will be submitted in support of any other degree or diploma at any other university or institution other than publications.

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Abstract

Both viscous and pressure stress work effects are taken into accounts in three-dimensional laminar natural convection flow about a vertical rectangular inclined plane surface. Previous investigators dealt with those effects in the two-dimensional natural convectional flow. We focused mainly here for flow configuration in three exterior situations among the variation of the properties of the fluid exterior to the boundary layers. We discussed here only the one case to predict essential flow parameters involved.
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Chapter - 1

Introduction

For the incompressible flow at a three-dimensional nodal point of attachment, Howarth (1951) showed that the boundary-layer equations yield similarity solutions which are also exact solutions of the Navier-Stokes equations. The external flow was assumed to be irrotational and the components were given by \( \{ax_1, bx_2, -(a+b)x_3\} \). \( x_1, x_2 \) are cartesian co-ordinates of any point on the tangent plane at the stagnation point \( x_1 = x_2 = 0 \) and \( x_3 \) is measured along the normal at the stagnation point. Howarth discussed the properties of these solutions for \( c = \frac{b}{a} = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \) and 1. The limiting values \( c = 0 \) and 1 correspond to the two-dimensional and axially symmetric stagnation points, respectively. Later Davey (1961) showed that similar solutions exit for \( c \geq -1 \). In Davey's terminology the similar solutions for \( -1 < c < 0 \) correspond to the flow near saddle-points of attachment and may in some cases be related to the flow in the vicinity of geometrical saddle-points on the surface.

A transformation, due to Howarth (1948), is applied to the boundary-layer equations for a gas in which the Prandtl number is unity and the viscosity varies linearly with the temperature.

Poole (1965) discussed the compressible solution for the case of a thermally insulated surface. Howarth (1951) and Davey (1961) studied the same case for an incompressible flow also.
Lighthill (1954), investigated the response of an incompressible laminar boundary-layer over an arbitrary cylinder placed in small fluctuations in the external stream. Low and high frequency solutions were obtained by a momentum-integral method. Moore (1951), Moore and Ostrach (1956), Illingworth (1958), Gribben (1961) and King (1966) studied the unsteady laminar compressible boundary-layer flow over two-dimensional bodies by momentum-integral and series-expansion methods.

Sparrow and Gregg (1958) also discussed the similar solutions for free convective flow from isothermal vertical plate.

Telionis and Gupta (1977) investigated the response of the compressible laminar boundary-layer to small fluctuations in the outer flow under more general conditions for both two-dimensional and axisymmetric bodies.

Vimala and Nath (1975) studied the unsteady laminar boundary layers in a compressible stagnation flow problem for a cold wall and solved the governing partial differential equations numerically using an implicit finite-difference scheme.

Kumari and Nath (1978) investigated unsteady laminar compressible boundary-layer flow in the immediate neighbourhood of the stagnation points on a class of a three-dimensional bodies ranging from spheres through cylinders to saddle shapes.

Stewartson (1961) showed with a high degree of certainty that a general compressible laminar boundary-layer can develop a singularity at a point of zero skin-friction only if the heat transfer at that point is zero.
Cooke (1961), considered the three-dimensional compressible boundary layer flow for (i) zero heat transfer, (ii) unit Prandtl number, (iii) linear viscosity-temperature relation, and (iv) small cross-flow. A correlation between a compressible laminar boundary-layer flow and an incompressible flow with different external conditions is established.

Cohen and Reshotko (1956) derived the similar solutions for the compressible laminar boundary layer flow.

Ostrach (1955) studied the compressible laminar boundary and heat transfer for unsteady motions of a flat plate.

Moore and Ostrach (1956) worked on the average properties of compressible laminar boundary layer on a flat plate with unsteady flight velocity.

Davies and Walker (1977) investigated the two-dimensional compressible laminar boundary-layer equations up to the point of separation and presented numerical solution.

Merkin (1989) dealt with the free convection on a heated vertical plate with prescribed power laws heating for small values of Prandtl number using the matching principle of Van Dyke (1964).

Zakerullah and Maleque (1996) studied the possible similarity solutions of combined forced and free convection laminar boundary layer flows on a vertical inclined surface.
Free Convection:
If there is no internally induced flow, but the flow arises “naturally” due to the effect of a density difference resulting from a temperature or concentration deviation in a body force field (such as gravitational field), the process is termed as “natural” or “free” convection. The density difference causes buoyancy effects due to which the flow is generated.

A general statement of energy conservation for a viscous, heat conducting compressible fluid may be deduced as Howarth (1953) by the balance of the rates of convection of specific internal and kinetic energies into an elementary volume of fluid compared with the rate of heat conduction into the volume and the rates of working of the body force, the pressure and the viscous stresses on the volume. Numerous alternative forms of the energy equation follow from this statement and are obtained by replacing internal energy by some other desired property of state. Furthermore, the kinetic energy term usually is eliminated by appeal to the momentum equation. However, the point to be noted is that in all such alternative forms there remain explicitly only certain parts of the terms representing the rates of working of the pressure and viscous stresses.

In the discussion and analysis of natural convection flows, such pressure and viscous stress work effects are generally ignored. However, the influence and importance of viscous stress work effects in laminar flows have been examined by Gebhart (1962) and also, by Gebhart and Mollendorf (1969). Both of these investigation considered special flows over semi-infinite flat surfaces set parallel to the direction of the gravity vector. Whereas Gebhart (1962) considered flows generated by the plate surface temperatures which vary as powers of $\xi$ (the distance along the plate surface from the leading edge), and Mollendorf (1969)
considered flows generated by plate surface temperatures which vary exponentially in $\xi$. In both of these investigations it was scale equal to $\frac{C_p}{G\beta}$, where $\beta$ is the volumetric coefficient of thermal expansion. Since this length scale is usually extremely large for most fluids, it was shown that this led to the conclusion that viscous stress work effects are very small in most situations. For example in the case of constant surface temperature, Gebhart (1962) showed that viscous stress work effects are governed simply by the ratio of $\xi$ and the above large length scale. It is easy to show that this ratio is the Eckert number for this flow.

In reviewing the vast literature on the theoretical study of laminar boundary layer natural convection flows it is observed that a few paid attention in favour of inclusion of both viscous dissipation and pressure work effects in energy equation. In conventional analysis both effect are generally ignored. Under the above discussion, the effects of pressure and viscous stress work are retained in the energy equation.

Gebhart and Mollendorf showed that pressure work effects are at least as important as (and in most cases rather more important than) those of viscous work for both liquids and gases. In the studies of both Gebhart (1962) and Gebhart & Mollendorf (1969) no account was taken of variations of state properties in the fluid outside the moving boundary layer. This neglect coincides with their use of the Boussinesq form of the boundary-layer equations.

Ackroyd (1974) studied the effects of both viscous and pressure stress work in natural convection flow on flat surface.
OH direction is the horizontal direction. OG is the vertical direction. OK is \perp to the plane. \( \xi \text{O} \eta \) rectangular plane is inclined at angle \( \delta \) with vertical plane HOG.
In order to obtain some comparisons with the results of previous investigations, only in three-dimensional laminar natural convection flow about a vertical rectangular inclined plane surface is discussed here. For the flow configuration shown in Fig. A, he showed that pressure and viscous work effects, together with the effects of property variations in the stationary fluid outside the boundary layer, are all governed by the same length scale $\frac{C_p}{G/\beta}$. Thus, one may not include viscous stress work effects and ignore the other two effects, particularly since it will be shown that the latter are generally rather more important than the former. However, it should be remembered that, except possibly for a natural convection flow which occurs over an exceptionally large development lengths $(x, \eta)$, all three effects will usually be of little significance.

In view of the above points, it appears to be inevitable that we consider the fluid to be of variable properties and that the necessary of state and transport properties be examined with the same care. In the present investigation the fluids studied are taken to be either a thermally and calorically perfect gas undergoing small changes in temperature and pressure. Two cases of fluid-property variations outside the boundary layer are discussed. These constant conditions are either for temperature or entropy. Together with these property variations, two reasonably temperature conditions are discussed for natural convection flow about a vertical inclined plane surface.
Chapter 2

Basic Equations and their order analysis

The Navier-Stokes equations for compressible viscous steady flow with the body force term and energy equation in cartesian co-ordinates \((\xi, \eta, \zeta)\) are

\[
\begin{align*}
(\nabla, \rho \overline{\mathbf{q}}) &= 0 \quad & (1-1) \\
(\rho \overline{\mathbf{q}} \cdot \nabla) \overline{\mathbf{q}} &= \mathbf{F} - \nabla p + \nabla (\mu \nabla \overline{\mathbf{q}}) \quad & (1-2)
\end{align*}
\]

and

\[
\rho C_p (\overline{\mathbf{q}} \cdot \nabla) T - (\overline{\mathbf{q}} \cdot \nabla) p = \nabla \cdot (\kappa \nabla T) + \mu \phi \quad & (1-3)
\]

Here, \(\phi\) represents the dissipation function given by

\[
\phi = 2 \left[ \left( \frac{\partial u}{\partial \xi} \right)^2 + \left( \frac{\partial v}{\partial \eta} \right)^2 + \left( \frac{\partial w}{\partial \zeta} \right)^2 \right] + \left( \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi} \right)^2 + \left( \frac{\partial u}{\partial \zeta} + \frac{\partial v}{\partial \eta} \right)^2
\]

\[
+ \left( \frac{\partial u}{\partial \xi} + \frac{\partial w}{\partial \eta} \right)^2 - \frac{2}{3} \left( \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{\partial w}{\partial \zeta} \right)^2
\]

and

\[
\overline{q} = u \hat{i} + v \hat{j} + w \hat{k} = \overline{q}(u, v, w)
\]

and body force

\[
\mathbf{F} = \rho \mathbf{g} = \rho (g_x, g_y, 0)
\]

The above equations (1-1) to (1-3) can be reduced to most simplified forms such as,

Equation of continuity

\[
\frac{\partial}{\partial \xi} (\rho u) + \frac{\partial}{\partial \eta} (\rho v) + \frac{\partial}{\partial \zeta} (\rho w) = 0
\]

\[(2)\]
u-momentum equation
\[ \rho \left( u \frac{\partial u}{\partial \xi} + v \frac{\partial u}{\partial \eta} + w \frac{\partial u}{\partial \zeta} \right) = -\frac{\partial p}{\partial \xi} + \rho g_x + \nabla \cdot (\mu \nabla u) \] (3-1)

v-momentum equation
\[ \rho \left( u \frac{\partial v}{\partial \xi} + v \frac{\partial v}{\partial \eta} + w \frac{\partial v}{\partial \zeta} \right) = -\frac{\partial p}{\partial \eta} + \rho g_\eta + \nabla \cdot (\mu \nabla v) \] (3-2)

w-momentum equation
\[ \rho \left( u \frac{\partial w}{\partial \xi} + v \frac{\partial w}{\partial \eta} + w \frac{\partial w}{\partial \zeta} \right) = -\frac{\partial p}{\partial \zeta} + \rho g_\zeta + \nabla \cdot (\mu \nabla w) \] (3-3)

and also the energy equation
\[ \left[ \rho C_p \left( u \frac{\partial T}{\partial \xi} + v \frac{\partial T}{\partial \eta} + w \frac{\partial T}{\partial \zeta} \right) - \left( u \frac{\partial p}{\partial \xi} + v \frac{\partial p}{\partial \eta} + w \frac{\partial p}{\partial \zeta} \right) \right] \]
\[ = \frac{\partial}{\partial \xi} \left( k \frac{\partial T}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( k \frac{\partial T}{\partial \eta} \right) + \frac{\partial}{\partial \zeta} \left( k \frac{\partial T}{\partial \zeta} \right) + \mu \phi \] (4)

where \( \nabla \cdot (\mu \nabla u) = \frac{\partial}{\partial \xi} \left( \mu \frac{\partial u}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \mu \frac{\partial u}{\partial \eta} \right) + \frac{\partial}{\partial \zeta} \left( \mu \frac{\partial u}{\partial \zeta} \right) \nabla \cdot (\mu \nabla v) = \frac{\partial}{\partial \xi} \left( \mu \frac{\partial v}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \mu \frac{\partial v}{\partial \eta} \right) + \frac{\partial}{\partial \zeta} \left( \mu \frac{\partial v}{\partial \zeta} \right) \nabla \cdot (\mu \nabla w) = \frac{\partial}{\partial \xi} \left( \mu \frac{\partial w}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \mu \frac{\partial w}{\partial \eta} \right) + \frac{\partial}{\partial \zeta} \left( \mu \frac{\partial w}{\partial \zeta} \right) \]

and so on \([\nabla \cdot (\mu \nabla w), \nabla \cdot (k \nabla T)]\).

For convenience, dimensionless quantities have been introduced into the equations (2), (3-1) to (3-3) and (4) in the following way. All length have been referred to a representative length "I". The velocity has been made dimensionless with respect to the resultant free stream velocity, \( U_f = \sqrt{U_o^2 + U_x^2} \), the density with respect to
\[ \rho_u \] and the pressure with reference to \( \rho_0 U_r^2 \). The temperature in the energy equation i.e. equation (4) has been made dimensionless with reference to the temperature difference \( \Delta T = (T_w - T_e) \) between the wall and the fluid at the large distance from the body.

Thus the dimensionless quantities with primes are

\[
\xi = l \xi', \eta = l \eta', \zeta = l \zeta', u = u' U_r, v = v' U_r, w = w' U_r \]

\[
p = \rho_0 \rho', p = p' \rho_0 U_r^2, T - T_e = (T_w - T_e), \mu = \mu_u \mu', k = k_u k' \]

\[
g_{ij} = g_{ij}', g_s = g_{ss}', g_e = g_{ee}' \] and \( C_\rho = C_{\rho_u} C_p' \)

In view of (5) continuity equation becomes

\[
\frac{\partial}{\partial \xi'} (\rho u') + \frac{\partial}{\partial \eta'} (\rho v') + \frac{\partial}{\partial \zeta'} (\rho w') = 0
\]

\[
\Rightarrow \rho_0 U_r \frac{\partial (\rho u')}{\partial \xi'} + \frac{\partial (\rho v')}{\partial \eta'} + \frac{\partial (\rho w')}{\partial \zeta'} = 0
\]

\[
\Rightarrow \frac{\partial (\rho u')}{\partial \xi'} + \frac{\partial (\rho v')}{\partial \eta'} + \frac{\partial (\rho w')}{\partial \zeta'} = 0
\]

\[ (6) \]

\( u \)-momentum equation in dimensionless form is,

\[
\rho \left( u' \frac{\partial u'}{\partial \xi'} + v' \frac{\partial u'}{\partial \eta'} + w' \frac{\partial u'}{\partial \zeta'} \right) = -\frac{\partial p'}{\partial \xi'} + \frac{p' g_{\xi \xi}}{F} + \frac{1}{Re_e} \left[ \frac{\partial}{\partial \xi'} \left( \mu' \frac{\partial u'}{\partial \xi'} \right) + \frac{\partial}{\partial \eta'} \left( \mu' \frac{\partial u'}{\partial \eta'} \right) + \frac{\partial}{\partial \zeta'} \left( \mu' \frac{\partial u'}{\partial \zeta'} \right) \right]
\]

\[ (7-1) \]
Similarly \( w \) -momentum equations are:

\[
\rho \left( u \frac{\partial v'}{\partial \xi'} + v' \frac{\partial v'}{\partial \eta'} + w' \frac{\partial v'}{\partial \zeta'} \right) = -\frac{\partial p'}{\partial \eta'} + \frac{\rho' g' g'_z}{F},
\]

\[
+ \frac{1}{R_n} \left[ \frac{\partial}{\partial \xi'} \left( \mu' \frac{\partial v'}{\partial \eta'} \right) + \frac{\partial}{\partial \eta'} \left( \mu' \frac{\partial v'}{\partial \eta'} \right) + \frac{\partial}{\partial \zeta'} \left( \mu' \frac{\partial v'}{\partial \zeta'} \right) \right]
\]

(7-2)

and

\[
\rho \left( u \frac{\partial w'}{\partial \xi'} + v' \frac{\partial w'}{\partial \eta'} + w' \frac{\partial w'}{\partial \zeta'} \right) = -\frac{\partial p'}{\partial \eta'} + \frac{\rho' g' g'_z}{F},
\]

\[
+ \frac{1}{R_n} \left[ \frac{\partial}{\partial \xi'} \left( \mu' \frac{\partial w'}{\partial \eta'} \right) + \frac{\partial}{\partial \eta'} \left( \mu' \frac{\partial w'}{\partial \eta'} \right) + \frac{\partial}{\partial \zeta'} \left( \mu' \frac{\partial w'}{\partial \zeta'} \right) \right]
\]

(7-3)

**Energy equation**

\[
\rho C_p \left( \frac{\partial T}{\partial \xi} + u \frac{\partial T}{\partial \eta} + w \frac{\partial T}{\partial \zeta} \right) = \left( \frac{\partial p}{\partial \xi} + v \frac{\partial p}{\partial \eta} + w \frac{\partial p}{\partial \zeta} \right) - \frac{\partial}{\partial \xi} \left( k \frac{\partial T}{\partial \xi} \right) - \frac{\partial}{\partial \eta} \left( k \frac{\partial T}{\partial \eta} \right) - \frac{\partial}{\partial \zeta} \left( k \frac{\partial T}{\partial \zeta} \right)
\]

\[
+ \mu \left[ 2 \left( \frac{\partial u}{\partial \xi} \right)^2 + \left( \frac{\partial v}{\partial \eta} \right)^2 \right] + \left( \frac{\partial w}{\partial \zeta} \right)^2 + \left( \frac{\partial v + \partial w}{\partial \eta} \right)^2 + \left( \frac{\partial v + \partial w}{\partial \zeta} \right)^2
\]

\[
+ \left( \frac{\partial w}{\partial \xi} \right)^2 - \frac{2}{3} \left( \frac{\partial w}{\partial \eta} \right)^2 \left( \frac{\partial w}{\partial \zeta} \right)^2
\]

(8)

Substituting the dimensionless quantities with primes the non-dimensional energy equation becomes

\[
\rho_0 \rho' C_p' C_r' \frac{U}{l} \left[ u' \left\{ \Delta T \frac{\partial \theta}{\partial \xi} + \theta \frac{\partial \Delta T}{\partial \xi} \right\} + v' \left\{ \Delta T \frac{\partial \theta}{\partial \eta} + \theta \frac{\partial \Delta T}{\partial \eta} \right\} + w' \left\{ \Delta T \frac{\partial \theta}{\partial \zeta} + \theta \frac{\partial \Delta T}{\partial \zeta} \right\} \right]
\]
\[
- \frac{\rho_s U}{l} \left\{ \frac{\partial \theta'}{\partial \xi'} + v \frac{\partial \phi'}{\partial \eta'} + w \frac{\partial \phi'}{\partial \zeta'} \right\} = \frac{k_v}{l^2} \left[ \frac{\partial}{\partial \xi'} \left( k' \left( \frac{\partial T}{\partial \xi'} + \theta \frac{\partial \Delta T}{\partial \xi'} \right) \right) \right]
\]

\[
+ \frac{\partial}{\partial \eta'} \left[ k' \left( \frac{\partial T}{\partial \eta'} + \theta \frac{\partial \Delta T}{\partial \eta'} \right) \right] + \frac{\partial}{\partial \zeta'} \left[ k' \left( \frac{\partial T}{\partial \zeta'} + \theta \frac{\partial \Delta T}{\partial \zeta'} \right) \right] \right]
\]

\[
+ \frac{\mu_0 \mu' U^2}{l^2} \left[ 2 \left( \frac{\partial u'}{\partial \eta'} \right)^2 + \left( \frac{\partial v'}{\partial \eta'} \right)^2 + \left( \frac{\partial w'}{\partial \eta'} \right)^2 \right] + \left( \frac{\partial \theta'}{\partial \xi'} + \frac{\partial \phi'}{\partial \xi'} \right)^2 + \left( \frac{\partial \phi'}{\partial \eta'} + \frac{\partial \phi'}{\partial \zeta'} \right)^2 \right]
\]

Dividing throughout by \( \frac{\rho_s C_p U}{l} \Delta T \) we have

\[
\rho C_p' \left[ \frac{\partial \theta'}{\partial \xi'} + v \frac{\partial \phi'}{\partial \eta'} + w \frac{\partial \phi'}{\partial \zeta'} + \theta \left[ \frac{\partial (\log \Delta T)}{\partial \xi'} + \frac{\partial (\log \Delta T)}{\partial \eta'} + \frac{\partial (\log \Delta T)}{\partial \zeta'} \right] \right]
\]

\[
- \frac{U^2}{C_p' \Delta T} \left[ \frac{\partial \phi'}{\partial \xi'} + v \frac{\partial \phi'}{\partial \eta'} + w \frac{\partial \phi'}{\partial \zeta'} \right] = \frac{k_v}{\rho_s C_p U_l \Delta T} \left[ \frac{\partial}{\partial \xi'} \left( k' \left( \frac{\partial T}{\partial \xi'} + \theta \frac{\partial \Delta T}{\partial \xi'} \right) \right) \right]
\]

\[
+ \frac{\partial}{\partial \eta'} \left[ k' \left( \frac{\partial T}{\partial \eta'} + \theta \frac{\partial \Delta T}{\partial \eta'} \right) \right] + \frac{\partial}{\partial \zeta'} \left[ k' \left( \frac{\partial T}{\partial \zeta'} + \theta \frac{\partial \Delta T}{\partial \zeta'} \right) \right] \right]
\]

\[
+ \frac{\mu_0 \mu' U^2}{\rho_s C_p' \Delta T} \left[ 2 \left( \frac{\partial u'}{\partial \eta'} \right)^2 + \left( \frac{\partial v'}{\partial \eta'} \right)^2 + \left( \frac{\partial w'}{\partial \eta'} \right)^2 \right] + \left( \frac{\partial \theta'}{\partial \xi'} + \frac{\partial \phi'}{\partial \xi'} \right)^2 + \left( \frac{\partial \phi'}{\partial \eta'} + \frac{\partial \phi'}{\partial \zeta'} \right)^2 \right]
\]

\[
+ \left( \frac{\partial \phi'}{\partial \xi'} + \frac{\partial \phi'}{\partial \xi'} \right)^2 - \frac{2}{3} \left( \frac{\partial \phi'}{\partial \eta'} + \frac{\partial \phi'}{\partial \zeta'} \right)^2 \right]
\]
or \( \rho' C' \left[ u' \frac{\partial \theta}{\partial \xi'} + v' \frac{\partial \theta}{\partial \eta'} + w' \frac{\partial \theta}{\partial \zeta'} + \theta \left( u' \log(\Delta T)_x + v' \log(\Delta T)_y \right) \right] - E_c \left[ u' \frac{\partial p'}{\partial \xi'} + v' \frac{\partial p'}{\partial \eta'} + w' \frac{\partial p'}{\partial \zeta'} \right]

= \frac{1}{\rho_o R_a \Delta T} \left[ \frac{\partial}{\partial \eta'} \left[ k \left( \Delta T \frac{\partial \theta}{\partial \xi'} + \theta \frac{\partial \Delta T}{\partial \xi'} \right) \right] + \frac{\partial}{\partial \eta'} \left[ k' \left( \Delta T \frac{\partial \theta}{\partial \eta'} + \theta \frac{\partial \Delta T}{\partial \eta'} \right) \right] \right]

+ \frac{\partial}{\partial \xi'} \left[ k \left( \Delta T \frac{\partial \theta}{\partial \eta'} + \theta \frac{\partial \Delta T}{\partial \eta'} \right) \right] + \frac{E_c u' \phi'}{R_a}

or \( \rho' C' \left[ u' \frac{\partial \theta}{\partial \xi'} + v' \frac{\partial \theta}{\partial \eta'} + w' \frac{\partial \theta}{\partial \zeta'} + \theta \left( u' \log(\Delta T)_x + v' \log(\Delta T)_y \right) \right] - E_c \left[ u' \frac{\partial p'}{\partial \xi'} + v' \frac{\partial p'}{\partial \eta'} + w' \frac{\partial p'}{\partial \zeta'} \right]

= \frac{1}{\rho_o R_a \Delta T} \left[ \frac{\partial}{\partial \xi'} \left[ k \left( \Delta T \frac{\partial \theta}{\partial \xi'} + \theta \frac{\partial \Delta T}{\partial \xi'} \right) \right] + \frac{\partial}{\partial \xi'} \left[ k' \left( \Delta T \frac{\partial \theta}{\partial \xi'} + \theta \frac{\partial \Delta T}{\partial \xi'} \right) \right] \right]

+ \frac{\partial}{\partial \zeta'} \left[ k \left( \Delta T \frac{\partial \theta}{\partial \xi'} + \theta \frac{\partial \Delta T}{\partial \xi'} \right) \right] + \frac{E_c u' \phi'}{R_a}, \quad (9)

Where the Eckart number \( E_c = \frac{U_x^2}{C' \Delta T} \)

The Reynolds number \( R_a = \frac{U_x L}{v_0} \), \( u_0 = \frac{H_o}{\rho_0} \)

The Froude number \( F_r = \frac{U_x^2}{g L} \), and The Prandtl number \( P_s = \frac{H_o C' \rho_0}{k_0} \)

The variations in \( \Delta T \) normal to the surface is considered negligible. That is \( (\log(\Delta T))_z = 0 \).
Here it is concluded that the pressure is constant across the boundary layer, its value there being equal to its value in the main stream just exterior to the boundary layer. The equation of state is unaltered and the dissipation function $\phi'$ reduces to

$$\phi' = \left( \frac{\partial u'}{\partial \xi'} \right)^2 + \left( \frac{\partial v'}{\partial \xi'} \right)^2$$

If $\delta$ be the boundary layer thickness, then the dimensionless boundary layer thickness is $\delta' = \frac{\delta}{l} << 1$, since $l >> 1$.

Order of magnitude of each of the terms in equations (6), (7-1) to (7-3) and (9) are estimated, so that very small terms can be neglected.

Since $\frac{\partial u'}{\partial \xi'} \sim O(1)$, $\frac{\partial v'}{\partial \xi'} \sim O(1)$, $\frac{\partial u'}{\partial \eta'} \sim O(1)$, $\frac{\partial v'}{\partial \eta'} \sim O(1)$.

Also $\frac{\partial w'}{\partial \xi'} \sim O(1)$, since $\xi'$ is of order $\delta'$, similarly $w' \sim O(\delta')$.

Therefore $\frac{\partial^2 u'}{\partial \xi'^2} \sim O(1)$, $\frac{\partial^2 v'}{\partial \xi'^2} \sim O(1)$, $\frac{\partial^2 u'}{\partial \eta'^2} \sim O(1)$, $\frac{\partial^2 v'}{\partial \eta'^2} \sim O(1)$.

Since the maximum values of each $u'$ and $v'$ is 1

$$\frac{\partial^2 u'}{\partial \xi'^2} \sim O\left(\frac{1}{\delta'^2}\right), \quad \frac{\partial^2 v'}{\partial \xi'^2} \sim O\left(\frac{1}{\delta'^2}\right), \quad \frac{\partial^2 w'}{\partial \xi'^2} \sim O\left(\frac{1}{\delta'}\right), \quad \frac{\partial w'}{\partial \xi'} \sim O(1)$$

$$\frac{\partial w'}{\partial \eta'} \sim O(\delta'), \quad \frac{\partial w'}{\partial \eta'} \sim O(\delta'), \quad R_a \sim O\left(\frac{1}{\delta'_{ii}}\right), \quad F_r \sim O(1)$$

and $E_s \sim O(1)$ and

$$\phi = \left( \frac{\partial u'}{\partial \xi'} \right)^2 + \left( \frac{\partial v'}{\partial \eta'} \right)^2 + \left( \frac{\partial w'}{\partial \xi'} \right)^2 + \ldots \sim O\left(\frac{1}{\delta'^2}\right)$$

Also $\frac{\partial \theta}{\partial \xi'} \sim O(1)$, $\frac{\partial \theta}{\partial \eta'} \sim O(1)$, $\frac{\partial \theta}{\partial \xi'} \sim O\left(\frac{1}{\delta'_{ii}}\right)$, $u' \sim O(1)$, $v' \sim O(1)$. 

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\[ w ' \sim O(\delta'), \quad \frac{\Delta T'_e}{\Delta T} \sim O(1), \quad \frac{\rho}{\Delta T'_e} \sim O(1). \]

Now, since \( \delta' \ll 1 \)

\[ \frac{\partial^2 w'}{\partial \xi'^2}, \frac{\partial^2 w'}{\partial \eta'^2}, \frac{\partial w'}{\partial \xi'}, \frac{\partial w'}{\partial \eta'} \text{ and } \frac{\partial w'}{\partial \zeta'} \text{ can be neglected.} \]

Again \( \frac{\partial \rho'}{\partial \xi'} \sim O(1), \quad \frac{\partial \rho'}{\partial \eta'} \sim O(1), \quad \text{and } \frac{\partial \rho'}{\partial \zeta'} \sim O \left( \frac{1}{\delta'} \right). \]

Let \( \delta_r \) be the thermal boundary layer thickness, the conduction term becomes of the same order of magnitude as the convectional term, only if the thickness of the thermal boundary layer is of the order of \( \left( \frac{\delta_r}{L} \right) \sim \frac{1}{R_e F_r} \). Again we know

\[ \delta \sim \frac{1}{\sqrt{R_e}}. \]

Therefore \( \frac{\delta}{\delta} \sim \frac{1}{\sqrt{P_r}}. \)

Now setting the order of magnitude in each terms of equations (6), (7-1) to (7-3) and (9) one obtains.

**Equation of continuity**

\[ \frac{\partial}{\partial \xi'} \left( \rho' u' \right) + \frac{\partial}{\partial \eta'} \left( \rho' v' \right) + \frac{\partial}{\partial \zeta'} \left( \rho' w' \right) = 0 \]

order \( 1 \quad 1 \quad 1 \quad 1 \quad \frac{1}{\delta'} \quad \delta' \)

**u-momentum equation**

\[ \rho' \left( u' \frac{\partial u'}{\partial \xi'} + v' \frac{\partial u'}{\partial \eta'} + w' \frac{\partial u'}{\partial \zeta'} \right) = - \frac{\partial \rho'}{\partial \xi'} + \frac{\rho' g'_x}{F_r} + \frac{1}{R_e} \left[ \frac{\partial}{\partial \zeta'} \left( \mu' \frac{\partial u'}{\partial \zeta'} \right) \right] \]

order \( 1 \quad 1 \quad 1 \quad \delta' \quad \frac{1}{\delta'} \quad 1 \quad 1 \quad \delta'^2 \quad 1 \quad 1 \)
\[ + \frac{\partial}{\partial \eta'} \left( \mu' \frac{\partial u'}{\partial \eta'} \right) + \frac{\partial}{\partial \zeta'} \left( \mu' \frac{\partial u'}{\partial \zeta'} \right) \]

\[ \frac{1}{\delta'} \quad \frac{1}{\delta'} \]

(11-1)

**v-momentum equation**

\[ \rho' \left( u' \frac{\partial v'}{\partial \xi'} + v' \frac{\partial v'}{\partial \eta'} + w' \frac{\partial v'}{\partial \zeta'} \right) = \frac{\partial p'}{\partial \eta'} + \frac{\rho' g'_z}{F_r} \]

order \[ 1 \quad 1 \quad 1 \quad \delta' \quad \frac{1}{\delta'} \quad 1 \quad 1 \]

\[ + \frac{1}{R_n} \left[ \frac{\partial}{\partial \xi'} \left( \mu' \frac{\partial v'}{\partial \xi'} \right) + \frac{\partial}{\partial \eta'} \left( \mu' \frac{\partial v'}{\partial \eta'} \right) + \frac{\partial}{\partial \zeta'} \left( \mu' \frac{\partial v'}{\partial \zeta'} \right) \right] \]

\[ \delta'^2 \quad 1 \quad 1 \quad 1 \quad ! \quad \frac{1}{\delta'} \quad \frac{1}{\delta'} \]

(11-2)

**w-momentum equation**

\[ \rho' \left( u' \frac{\partial w'}{\partial \xi'} + v' \frac{\partial w'}{\partial \eta'} + w' \frac{\partial w'}{\partial \zeta'} \right) = \frac{\partial p'}{\partial \zeta'} + \frac{\rho' g'_z}{F_r} \]

order \[ 1 \quad \delta' \quad 1 \quad \delta' \quad \delta' \quad 1 \quad \frac{1}{\delta'} \quad 1 \]

\[ + \frac{1}{R_n} \left[ \frac{\partial}{\partial \xi'} \left( \mu' \frac{\partial w'}{\partial \xi'} \right) + \frac{\partial}{\partial \eta'} \left( \mu' \frac{\partial w'}{\partial \eta'} \right) + \frac{\partial}{\partial \zeta'} \left( \mu' \frac{\partial w'}{\partial \zeta'} \right) \right] \]

\[ \delta'^2 \quad 1 \quad \delta' \quad 1 \quad \delta' \quad \frac{1}{\delta'} \quad 1 \]

(11-3)

**Energy equations**

\[ \rho' C'_p \left[ u' \frac{\partial \theta}{\partial \xi'} + v' \frac{\partial \theta}{\partial \eta'} + w' \frac{\partial \theta}{\partial \zeta'} + \theta \left( u' \left( \log \Delta T \right)_x + v' \left( \log \Delta T \right)_y \right) \right] - E \left[ u' \frac{\partial \rho'}{\partial \xi'} + v' \frac{\partial \rho'}{\partial \eta'} + w' \frac{\partial \rho'}{\partial \zeta'} \right] \]

order \[ 1 \quad 1 \quad 1 \quad 1 \quad \delta' \quad \frac{1}{\delta'} \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \delta' \quad \frac{1}{\delta'} \]
\[
\frac{1}{P_o R_o} \frac{\Delta T_0}{\Delta T} \left[ \frac{\partial}{\partial \xi} \left\{ k \left( \frac{\Delta T}{\Delta T_0} \frac{\partial \theta}{\partial \xi} + \theta \frac{\partial}{\partial \xi} \left( \frac{\Delta T}{\Delta T_0} \right) \right) \right\} + \frac{\partial}{\partial \eta} \left\{ k \left( \frac{\Delta T}{\Delta T_0} \frac{\partial \theta}{\partial \eta} + \theta \frac{\partial}{\partial \eta} \left( \frac{\Delta T}{\Delta T_0} \right) \right) \right\} \right] \\
\delta_T^{12} 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\]

\[
+ \frac{\partial}{\partial \zeta'} \left\{ k \left( \frac{\Delta T}{\Delta T_0} \frac{\partial \theta}{\partial \zeta'} + \theta \frac{\partial}{\partial \zeta'} \left( \frac{\Delta T}{\Delta T_0} \right) \right) \right\} + \frac{E_o \mu' \phi'}{R_o} \\
\frac{1}{\delta_T'} \quad 1 \quad \frac{1}{\delta_T'} \quad \frac{1}{\delta_T'} \quad 1 \quad \delta_T^{12} \quad \frac{1}{\delta_T'}
\]

Order of each term of equation (11-3) when multiplied by \( \delta' \) then order of all the terms in equation (11-3) becomes above \( \delta' \) except \( \frac{\partial \rho'}{\partial \zeta'} \). Hence equation (11-3) can be neglected compared to the equations (13) i.e. \( \frac{\partial \rho'}{\partial \zeta'} = 0 \).

For \( \delta' \) and \( \delta_T' \) are small and also \( \frac{\Delta T'}{\Delta T_0} \) is constant. Here \( \delta' = \delta_T' \) (considered).

Neglecting the terms higher than order of \( \delta' \) and \( \delta_T' \) and omitting the dashes, we have the three-dimensional boundary layer and energy equations for natural convection flow.

**Equation of continuity**

\[
\frac{\partial}{\partial \xi} (\rho u) + \frac{\partial}{\partial \eta} (\rho v) + \frac{\partial}{\partial \zeta'} (\rho w) = 0
\]  

(12)

**u-momentum equation**

\[
\rho \left[ u \frac{\partial u}{\partial \xi} + v \frac{\partial u}{\partial \eta} + w \frac{\partial u}{\partial \zeta'} \right] = - \frac{\partial p}{\partial \xi} + \rho g_z + \frac{\partial}{\partial \zeta'} \left( \mu \frac{\partial u}{\partial \zeta'} \right)
\]  

(13)
v-momentum equation

\[ \rho \left( u \frac{\partial v}{\partial \xi} + v \frac{\partial v}{\partial \eta} + w \frac{\partial v}{\partial \zeta} \right) = -\frac{\partial p}{\partial \eta} + \rho g_y + \frac{\partial}{\partial \zeta} \left( \mu \frac{\partial v}{\partial \zeta} \right) \]  \hfill (14)

w-momentum equation

\[ \frac{\partial p}{\partial \zeta} = 0 \]  \hfill (15)

Energy equations

\[ \rho C_p \left( u \frac{\partial T}{\partial \xi} + v \frac{\partial T}{\partial \eta} + w \frac{\partial T}{\partial \zeta} \right) - \left( u \frac{\partial p}{\partial \xi} + v \frac{\partial p}{\partial \eta} \right) = \frac{\partial}{\partial \zeta} \left( k \frac{\partial \theta}{\partial \zeta} \right) + \mu \left( \frac{\partial u}{\partial \zeta} \right)^2 + \left( \frac{\partial v}{\partial \zeta} \right)^2 \]  \hfill (16)
Chapter 3

Boundary-layer equations and transformations

We consider a steady three dimensional in laminar natural convection fluid flow at high Reynolds number based on free convective velocity $U_F$ about a vertical rectangular inclined plane surface. Cartesian co-ordinates $(\xi, \eta, \zeta)$ are chosen. The co-ordinates $\xi$ and $\eta$ are considered to lie and be defined in the surface over which the boundary layer is flowing, while $\zeta$ extends into the boundary layer. Here $\zeta$ represents an actual distance measured along a straight normal from the surface. In the $\xi$ and $\eta$ directions, the components of the body force vector are $\rho g \xi$ and $\rho g \eta$ respectively where the gravitational force $-\mathbf{G} = (-g_x, -g_y, 0)$ and body force $\mathbf{F} = (\rho g_x, \rho g_y, 0)$.

The laminar boundary equations and the energy equation in natural convection flow about a vertical rectangular inclined plane surface are governed by the following equations:

Continuity equation

$$\frac{\partial}{\partial \xi} (\rho u) + \frac{\partial}{\partial \eta} (\rho v) + \frac{\partial}{\partial \zeta} (\rho w) = 0 \quad (3-1)$$

$u$-momentum equation along the $\xi$-direction

$$\rho \left( \frac{\partial u}{\partial \xi} + v \frac{\partial u}{\partial \eta} + w \frac{\partial u}{\partial \zeta} \right) = - \frac{\partial p}{\partial \xi} + \rho g_x + \frac{\partial}{\partial \zeta} \left( \mu \frac{\partial u}{\partial \zeta} \right) \quad (3-2)$$

$v$-momentum equation along the $\eta$-direction

$$\rho \left( \frac{\partial v}{\partial \xi} + v \frac{\partial v}{\partial \eta} + w \frac{\partial v}{\partial \zeta} \right) = - \frac{\partial p}{\partial \eta} + \rho g_y + \frac{\partial}{\partial \zeta} \left( \mu \frac{\partial v}{\partial \zeta} \right) \quad (3-3)$$

and
The terms $\rho g_\xi$, $\rho g_\eta$ represent the body force components exerted on fluid particle. The boundary conditions for the present problem is determined as follows

(a) The fluid must adhere to the surface (the no slip condition).

That is, mathematically for the surface

$$u(\xi, \eta, 0) = v(\xi, \eta, 0) = 0$$  \hspace{1cm} (3-6a)

(b) The temperature of the fluid at the surface must be function of $\xi$ and $\eta$ (non-isothermal surface):

$$T(\xi, \eta, 0) = T_\infty(\xi, \eta)$$  \hspace{1cm} (3-6b)

(c) The fluid at large distances from the surface must remain undisturbed:

$$u(\xi, \eta, \infty) = v(\xi, \eta, \infty) = 0$$  \hspace{1cm} (3-6c)

(d) The temperature at large distances from the surface must be equal to the undisturbed fluid

$$L_{\xi \rightarrow \infty} T(\xi, \eta, \zeta) = T_\infty(\xi, \eta, \zeta) = \text{constant}$$  \hspace{1cm} (3-6d)

For thermally perfect gas

$$\beta = \frac{1}{T} \text{ so that } \beta T = 1 \text{ and } \beta = -\frac{1}{\rho} \frac{\partial p}{\partial T}$$  \hspace{1cm} (3-6e)

Conditions exterior to the boundary layer (denoted by suffix e), at which the exterior fluid is at rest. Therefore for hydrostatic conditions

$$\rho e g_\xi \frac{\partial p}{\partial \xi} = 0 \text{ implies that } \rho e g_\xi = \frac{\partial p}{\partial \xi}$$  \hspace{1cm} (3-7)

$$\rho e g_\eta \frac{\partial p}{\partial \eta} = 0 \text{ implies that } \rho e g_\eta = \frac{\partial p}{\partial \eta}$$  \hspace{1cm} (3-7)
Therefore the modified three dimensional governing equations for the compressible steady flows are [using (3-6e) and (3-7)]

**Continuity equation**

\[
\frac{\partial}{\partial \xi} (\rho u) + \frac{\partial}{\partial \eta} (\rho v) + \frac{\partial}{\partial \zeta} (\rho w) = 0
\]

(3-8)

**u-momentum and v-momentum equation**

\[
\rho \left( u \frac{\partial u}{\partial \xi} + v \frac{\partial u}{\partial \eta} + w \frac{\partial u}{\partial \zeta} \right) = (\rho - \rho_0) g_z + \frac{\partial}{\partial \zeta} \left( \mu \frac{\partial u}{\partial \zeta} \right)
\]

(3-9)

and

\[
\rho \left( u \frac{\partial v}{\partial \xi} + v \frac{\partial v}{\partial \eta} + w \frac{\partial v}{\partial \zeta} \right) = (\rho - \rho_0) g_z + \frac{\partial}{\partial \zeta} \left( \mu \frac{\partial v}{\partial \zeta} \right)
\]

(3-10)

where the pressure gradient terms eliminated by the hydrostatics conditions.

**The energy equations**

\[
\rho C_p \left( u \frac{\partial T}{\partial \xi} + v \frac{\partial T}{\partial \eta} + w \frac{\partial T}{\partial \zeta} \right) - T \beta \left( u \frac{\partial \rho}{\partial \xi} + v \frac{\partial \rho}{\partial \eta} \right) = \frac{\partial}{\partial \zeta} \left( k \frac{\partial T}{\partial \zeta} \right) + \mu \left[ \left( \frac{\partial u}{\partial \zeta} \right)^2 + \left( \frac{\partial v}{\partial \zeta} \right)^2 \right]
\]

(3-11)

Where \( \phi = \mu \left[ \left( \frac{\partial u}{\partial \zeta} \right)^2 + \left( \frac{\partial v}{\partial \zeta} \right)^2 \right] \) is known as the viscous dissipation function. Whilst \( \phi \) represents that part of the viscous work necessary for the correct balance of energy in this particular form of the energy equation, the term \( T \beta \left[ u \frac{\partial \rho}{\partial \xi} + v \frac{\partial \rho}{\partial \eta} \right] \) represents the corresponding part of the pressure work. It is the latter term which is ignored by Gebhart (1962, 1969) and Gebhart and Mollendorf (1969). Ackroyd (1974), Count the pressure work and viscous work terms for two dimensional cases. Zakerullah (1982) considered the pressure work and viscous work terms for the axi-symmetric natural convection flow.
Now here we considered a non-dimensional temperature function

\[ \phi(\xi, \eta, \zeta) = \frac{T(\xi, \eta, \zeta) - T_{e}(\xi, \eta)}{T_{e}(\xi, \eta) - T_{e}(\xi, \eta)} \]

(3-12)

Where \( \Delta T(\xi, \eta) = T_{e}(\xi, \eta) - T_{e}(\xi, \eta) \)

(3-13)

We change the independent variables \( (\xi, \eta, \zeta) \) to a new set of variable \( (X, Y, \Phi) \), where relations between two sets of variable are given by:

\[ \xi = X, \quad \eta = Y \quad \text{and} \quad \Phi = \left[ \frac{\rho_{r}^{3}(-G(\Delta T)_{r})\beta_{r}}{16\mu_{r}^{2}(X + CY)} \right]^{\frac{1}{3}} \int_{0}^{\xi} \frac{\rho}{\rho_{r}} d\zeta \]

(3-14)

Here \( C \) is the additive length scale parameter and the suffix \( \text{r} \) refers to any convenient reference condition and also \( (\Delta T)_{r} \) is any convenient reference temperature difference. \( \Phi \) represents non-dimensional similarity variable. From equation (3-14) we have (by chain rule) the following expressions:

\[ \frac{\partial}{\partial \xi} = \frac{\partial}{\partial X} - \frac{\Phi}{4(X + CY)} \frac{\partial}{\partial \Phi} \]

(3-15)

\[ \frac{\partial}{\partial \eta} = \frac{\partial}{\partial Y} - \frac{C \Phi}{4(X + CY)} \frac{\partial}{\partial \Phi} \]

(3-16)

\[ \frac{\partial}{\partial \zeta} = \frac{\rho}{\rho_{r}} \left[ \frac{\rho_{r}^{3}(-G(\Delta T)_{r})\beta_{r}}{16\mu_{r}^{2}(X + CY)} \right]^{\frac{1}{3}} \frac{\partial}{\partial \Phi} \]

(3-17)

\[ \frac{\partial}{\partial \Phi} = \frac{\rho}{\rho_{r}} \left[ \frac{16\mu_{r}^{2}(X + CY)}{\rho_{r}^{2}(-G(\Delta T)_{r})\beta_{r}} \right]^{\frac{1}{3}} \]

(3-18)

Let two stream functions \( \psi(\xi, \eta, \zeta) \) and \( \chi(\xi, \eta, \zeta) \) be defined as the mass flow components within the boundary layer for the case of compressible viscous flow. To satisfy the equation of continuity, we may introduce the components of the mass flow in the following way,

\[ \rho u = \frac{\partial \psi}{\partial \zeta}, \quad \rho v = \frac{\partial \chi}{\partial \zeta} \]

\[ \rho w = \left\{ \psi_{i} + \chi_{i} \right\} \]

(3-19)
In order to seek the similarity functions, we introduce the following equations,

\[ J \frac{dW}{\rho U_f} = F(X, Y, \Phi) \]  

(3-20)

and

\[ \int_0^\Phi \frac{dV}{\rho U_f} = S(X, Y, \Phi) \]  

(3-21)

where \( U_f = -\sqrt{G(\Delta T)} \beta_0 (X + CY) \) which represent the characteristic velocity (maximum) generated by the buoyancy effect and \( (X + CY) \) denotes some characteristic length.

Now

\[ \int_0^\Phi \frac{d\mu}{\rho U_f} = F(X, Y, \Phi) \]  

which implies that \( \frac{\rho u}{\rho U_f} = F_0 (X, Y, \Phi) \)  

(3-22)

Similarly,

\[ \int_0^\Phi \frac{d\nu}{\rho U_f} = S(X, Y, \Phi) \]  

can be written \( \frac{\rho v}{\rho U_f} = S_0 (X, Y, \Phi) \)  

(3-23)

Again \( \frac{\partial \psi}{\partial \zeta} = \rho u \) means

\[ \psi = \int_0^\zeta \rho u d\zeta = \rho U_f \int_0^\Phi \frac{d\zeta}{\rho U_f} = \rho U_f \int_0^\Phi \frac{dF(X, Y, \Phi)}{d\Phi} d\zeta \]

\[ = \rho U_f \left[ \frac{16 \mu_i^2 (X + CY)}{\rho \beta_r (1 - G(\Delta T))} \right] \int_0^\Phi \frac{dF(X, Y, \Phi)}{d\Phi} d\zeta \]

\[ = \rho U_f \left[ \frac{16 \mu_i^2 (X + CY)}{\rho \beta_r (1 - G(\Delta T))} \right] \int_0^\Phi \frac{dF(X, Y, \Phi)}{d\Phi} d\zeta \]

\[ = \rho U_f \left[ \frac{16 \mu_i^2 (X + CY)}{\rho \beta_r (1 - G(\Delta T))} \right] \int_0^\Phi \frac{dF(X, Y, \Phi)}{d\Phi} d\zeta \]

\[ \psi = \int_0^\Phi \frac{d\mu}{\rho U_f} = F(X, Y, \Phi), \text{ when } \Phi = 0 \text{ then } F(X, Y, \Phi) = F(X, Y, 0) = 0 \]

using (3-19) and also integration by parts.

Here \( \int_0^\Phi \frac{d\mu}{\rho U_f} = F(X, Y, \Phi) \) when \( \Phi = 0 \) then \( F(X, Y, \Phi) = F(X, Y, 0) = 0 \)
Here the variation of $\frac{\rho_x}{\rho}$ is considered negligible although it contains a term $\theta(\kappa, \gamma, \phi)$ multiplied by a smaller quantity $\frac{\Delta T}{T_x}$. Later it will be shown that this assumption contradicts the approximation expressed by the equation (3-79g).

Therefore,

$$\psi = \rho U \left[ \frac{16 \mu^2 (X + CY)}{\rho^2 (-G(\Delta T),)} \right]^{\frac{1}{2}} \left( \frac{\rho_x F(X, Y, \Phi)}{\rho} \right)$$

that is

$$\psi(x, y, \Phi) = \left[ 16 \mu^2 \rho^2 (-G(\Delta T),) \beta, (X + CY) \right]^{\frac{1}{2}} \left( \frac{\rho_x F(X, Y, \Phi)}{\rho} \right) \quad (3-24)$$

Similarly we have

$$\chi(x, y, \Phi) = \left[ 16 \mu^2 \rho^2 (-G(\Delta T),) \beta, (X + CY) \right]^{\frac{1}{2}} \left( \frac{\rho_x S(X, Y, \Phi)}{\rho} \right) \quad (3-25)$$

$$\frac{\partial \psi}{\partial x} = \left[ \frac{\partial}{\partial X} - \frac{\Phi}{4(X + CY)} \frac{\partial}{\partial \Phi} \right] \left[ 16 \mu^2 \rho^2 (-G(\Delta T),) \beta, (X + CY) \right]^{\frac{1}{2}} \left( \frac{\rho_x F(X, Y, \Phi)}{\rho} \right)$$

$$= \left[ \frac{16 \mu^2 \rho^2 (-G(\Delta T),) \beta, (X + CY)}{4(X + CY)} \right]^{\frac{1}{2}} \left( \frac{\rho_x F(X, Y, \Phi)}{\rho} \right) + \left[ 16 \mu^2 \rho^2 (-G(\Delta T),) \beta, (X + CY) \right]^{\frac{1}{2}} \frac{\partial}{\partial X} \left( \frac{\rho_x F(X, Y, \Phi)}{\rho} \right)$$

Again introducing (3-16) and (3-25), \( \frac{\partial \chi}{\partial \eta} \) can be written as

$$\frac{\partial \chi}{\partial \eta} = \left[ \frac{\partial}{\partial Y} - \frac{C\Phi}{4(X + CY)} \frac{\partial}{\partial \Phi} \right] \left[ 16 \mu^2 \rho^2 (-G(\Delta T),) \beta, (X + CY) \right]^{\frac{1}{2}} \left( \frac{\rho_x S(X, Y, \Phi)}{\rho} \right)$$

$$= \left[ \frac{16 \mu^2 \rho^2 (-G(\Delta T),) \beta, (X + CY)}{4(X + CY)} \right]^{\frac{1}{2}} \left( \frac{\rho_x S(X, Y, \Phi)}{\rho} \right) + \left[ 16 \mu^2 \rho^2 (-G(\Delta T),) \beta, (X + CY) \right]^{\frac{1}{2}} \frac{\partial}{\partial Y} \left( \frac{\rho_x S(X, Y, \Phi)}{\rho} \right)$$
In view of equation (3-26) and (3-27), the equation (3-19) becomes

\begin{align*}
  p &= \frac{\Phi C}{4(X+CY)} \left[ 16 \mu^2 \rho (-G(\Delta T,)) \beta \frac{1}{\rho} \frac{\partial}{\partial \xi} \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right) \right. \\
  &\left. + C \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right) \right] \\
\end{align*}

(3-27)

With the help of (3-17) and (3-24), \( \rho u \) can be written as,

\begin{align*}
  \rho u &= \partial_x \rho = \rho \left[ \frac{\partial}{\partial \xi} \left( -G(\Delta T,)) \beta \frac{1}{\rho} \frac{\partial}{\partial \Phi} \left( 16 \mu^2 \rho \beta \frac{1}{\rho} \frac{\partial}{\partial \xi} \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right) \right) \right. \\
  &\left. - \frac{\partial}{\partial \eta} \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right) + \frac{\partial}{\partial \xi} \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right) \right] \\
  &= \rho \left[ -G(\Delta T,)) \beta \frac{1}{\rho} \frac{\partial}{\partial \Phi} \left( 16 \mu \rho \beta \frac{1}{\rho} \frac{\partial}{\partial \xi} \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right) \right) \right. \\
  &\left. - \frac{\partial}{\partial \eta} \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right) + \frac{\partial}{\partial \xi} \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right) \right] \\
  &= \rho U \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right) \\
\end{align*}

(3-29)

Similarly

\begin{align*}
  \rho v &= \rho U \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right) \\
\end{align*}

(3-30)

Therefore,

\begin{align*}
  u &= U \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right) \\
  v &= U \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right) \\
\end{align*}

(3-31)

The convective operator for steady case stands as

\begin{align*}
  \rho \frac{D}{Dt} &= \rho u \frac{\partial}{\partial \xi} + \rho v \frac{\partial}{\partial \eta} + \rho w \frac{\partial}{\partial \zeta} \\
\end{align*}
\[
= \rho U_r \left( \frac{\rho_r}{\rho} F(X, Y, \Phi) \right) \frac{\partial}{\partial X} \left[ \frac{\Phi}{4(X + CY)} \right] + \rho U_r \left( \frac{\rho_r}{\rho} S(X, Y, \Phi) \right) \frac{\partial}{\partial Y} \left[ \frac{C \Phi}{4(X + CY)} \right] \\
- \frac{3 \rho U_r}{4(X + CY)} \left[ \frac{\rho_r}{\rho} F(X, Y, \Phi) + C \frac{\rho_r}{\rho} S(X, Y, \Phi) \right] \frac{\partial}{\partial \Phi} - \rho U_r \left[ \frac{\partial}{\partial X} \left( \frac{\rho_r}{\rho} F(X, Y, \Phi) \right) \right] \\
+ C \frac{\partial}{\partial Y} \left( \frac{\rho_r}{\rho} S(X, Y, \Phi) \right) \frac{\partial}{\partial \Phi} + \rho U_r \left( \frac{\rho_r}{\rho} \frac{\Phi \partial U_r}{4(X + CY)} \right) \frac{\partial}{\partial \Phi} \\
+ C \frac{\partial}{\partial Y} \left( \frac{\rho_r}{\rho} S(X, Y, \Phi) \right) \frac{\partial}{\partial \Phi} + \rho U_r \left( \frac{\rho_r}{\rho} \frac{\Phi \partial U_r}{4(X + CY)} \right) \frac{\partial}{\partial \Phi}
\]

\[
\frac{\partial u}{\partial \zeta} - \frac{\rho}{\rho_r} \left[ \frac{\beta^3_r}{16 \mu_r^2} \left( \frac{\partial}{\partial \Phi} \right) \left[ \frac{\beta^3_r}{16 \mu_r^2} \left( \frac{\partial}{\partial \Phi} \right) \right] \frac{\partial}{\partial \Phi} \right]
\]

\[
= \frac{\rho}{\rho_r} \left[ \frac{\beta^3_r(-G(\Delta T))_r}{16 \mu_r^2} \left( \frac{\partial}{\partial \Phi} \right) \right] \left( \frac{\partial}{\partial \Phi} \right) \left[ \frac{\beta^3_r(-G(\Delta T))_r}{16 \mu_r^2} \left( \frac{\partial}{\partial \Phi} \right) \right] \left( \frac{\partial}{\partial \Phi} \right) \left[ \frac{\partial}{\partial \Phi} \right]
\]

\[
\frac{\partial}{\partial \zeta} \left( \mu \frac{\partial u}{\partial \zeta} \right) = \frac{\rho U_r^2}{4(X + CY)} \left( \mu \frac{\partial}{\partial \zeta} \left( \mu \frac{\partial u}{\partial \zeta} \right) \right)
\]

Similarly

\[
\frac{\partial}{\partial \zeta} \left( \mu \frac{\partial v}{\partial \zeta} \right) = \frac{\rho U_r^2}{4(X + CY)} \left( \mu \frac{\partial}{\partial \zeta} \left( \mu \frac{\partial v}{\partial \zeta} \right) \right)
\]

Now introducing (3-31), (3-32) and (3-33) in the u-momentum equation (3-9), we have
\[
\rho u \frac{\partial u}{\partial \xi} + \rho v \frac{\partial u}{\partial \eta} + \rho w \frac{\partial u}{\partial \zeta} = (\rho - \rho_e) g_z + \frac{\partial}{\partial \zeta} \left( \mu \frac{\partial u}{\partial \zeta} \right)
\]

or \[
\left( \rho u \frac{\partial}{\partial \xi} + \rho v \frac{\partial}{\partial \eta} + \rho w \frac{\partial}{\partial \zeta} \right) u = (\rho - \rho_e) g_z + \frac{\partial}{\partial \zeta} \left( \mu \frac{\partial u}{\partial \zeta} \right)
\]

or
\[
\rho U_r \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) - \frac{\partial}{\partial \zeta} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) + \frac{\partial}{\partial \Phi} \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right) + \frac{\partial}{\partial Y} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right)
\]

\[
- \frac{3 \rho U_r}{4(X + CY)} \left[ \frac{\rho}{\rho} F(X, Y, \Phi) + C \frac{\rho}{\rho} S(X, Y, \Phi) \right] - \frac{\rho}{\rho} U_r \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) + \frac{\partial}{\partial \Phi} \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right)
\]

\[
= (\rho - \rho_e) g_z + \frac{\rho U_r^2}{4(X + CY)} \left[ \frac{\rho}{\rho} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \right]
\]

or
\[
\rho U_r \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) - \frac{\partial}{\partial \zeta} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) + \frac{\partial}{\partial \Phi} \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right) + \frac{\partial}{\partial Y} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right)
\]

\[
- \frac{3 \rho U_r}{4(X + CY)} \left[ \frac{\rho}{\rho} F(X, Y, \Phi) + C \frac{\rho}{\rho} S(X, Y, \Phi) \right] - \frac{\rho}{\rho} U_r \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) + \frac{\partial}{\partial \Phi} \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right)
\]

\[
= (\rho - \rho_e) g_z + \frac{\rho U_r^2}{4(X + CY)} \left[ \frac{\rho}{\rho} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \right]
\]
\[ + \frac{C \rho U_f^2}{2(X + CY)} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right) + \rho U_f \left( \frac{\partial}{\partial Y} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \right) \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right) \]

\[ - \frac{3 \rho U_f^3}{4(X + CY)} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) - \frac{3C \rho U_f^2}{4(X + CY)} \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right) \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \]

\[ - \rho U_f^2 \left( \frac{\partial}{\partial X} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \right) - \frac{\partial}{\partial X} \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right) \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \]

\[ = (\rho - \rho_e) g_i + \frac{\rho U_f^2}{4(X + CY)} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \]

Now dividing both sides of (3-35) by \( \rho U_f^2 \) and simplifying we have

\[ \left\{ \frac{\rho}{\rho} F(X, Y, \Phi) \right\}_{\infty} + 4(\rho - \rho_e) g_i - 2 \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \]

\[ - 2C \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right) + 3 \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \]

\[ + 3C \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right) \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \]

\[ - \frac{\partial}{\partial X} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \]

\[ + \frac{\partial}{\partial Y} \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right) \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \]

\[ - C \frac{\partial}{\partial Y} \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right) \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \]

(3-36)
Here,

\[
\frac{4(\rho - \rho_\infty)(X + CY)g_z}{\rho U_z^2} = \frac{4(\rho - \rho)(X + CY)g_z}{\rho G(\Delta T), \beta, (X + CY)} = \frac{4 \left( \frac{\rho_e}{\rho} - 1 \right) g_z}{G(\Delta T), \beta}.
\]

Here u-momentum equation is expressed in the following way by using similarity transformation,

\[
\left\{ \frac{\rho \mu}{\rho \mu_X} \left( \frac{\rho_e}{\rho} F(X, Y, \Phi) \right) \right\}_{o} + \frac{4 \left( \frac{\rho_e}{\rho} - 1 \right) g_x}{G(\Delta T), \beta} - 2 \left( \frac{\rho_e}{\rho} F(X, Y, \Phi) \right)_o
\]

\[-2C \left( \frac{\rho_e}{\rho} F(X, Y, \Phi) \right) \left( \frac{\rho_e}{\rho} S(X, Y, \Phi) \right)_{o\Phi} + 3 \left( \frac{\rho_e}{\rho} F(X, Y, \Phi) \right) \left( \frac{\rho_e}{\rho} F(X, Y, \Phi) \right)_{o\Phi}
\]

\[+ 3C \left( \frac{\rho_e}{\rho} S(X, Y, \Phi) \right) \left( \frac{\rho_e}{\rho} F(X, Y, \Phi) \right)_{o\\Phi} = 4(X + CY) \left[ \left( \frac{\rho_e}{\rho} F(X, Y, \Phi) \right) \frac{\partial}{\partial X} \left( \frac{\rho_e}{\rho} F(X, Y, \Phi) \right)_{o} \right.
\]

\[- \frac{\partial}{\partial X} \left( \frac{\rho_e}{\rho} F(X, Y, \Phi) \right) \left( \frac{\rho_e}{\rho} F(X, Y, \Phi) \right)_{o\Phi} \left[ \left( \frac{\rho_e}{\rho} S(X, Y, \Phi) \right) \frac{\partial}{\partial Y} \left( \frac{\rho_e}{\rho} F(X, Y, \Phi) \right)_{o} \right.
\]

\[-C \frac{\partial}{\partial Y} \left( \frac{\rho_e}{\rho} S(X, Y, \Phi) \right) \left( \frac{\rho_e}{\rho} F(X, Y, \Phi) \right)_{o\phi} \]  

Again introducing (3-31), (3-32) and (3-34) in equation (3-10) and then simplifying we have the \( v \)-momentum equation in the following form

\[
\left\{ \frac{\rho \mu}{\rho \mu_X} \left( \frac{\rho_e}{\rho} S(X, Y, \Phi) \right) \right\}_{o\Phi} + \frac{4 \left( \frac{\rho_e}{\rho} - 1 \right) g_\eta}{G(\Delta T), \beta} - 2C \left( \frac{\rho_e}{\rho} S(X, Y, \Phi) \right)_{o\phi}^2
\]
\[-2\left(\frac{\rho_r}{\rho} F(X,Y,\Phi)\right)_{\partial\Phi} \left(\frac{\rho_r}{\rho} S(X,Y,\Phi)\right)_{\partial\Phi} + 3\left(\frac{\rho_r}{\rho} F(X,Y,\Phi)\right)_{\partial\Phi} \left(\frac{\rho_r}{\rho} S(X,Y,\Phi)\right)_{\partial\Phi}\]

\[+3c\left(\frac{\rho_r}{\rho} S(X,Y,\Phi)\right)_{\partial\Phi} \left(\frac{\rho_r}{\rho} S(X,Y,\Phi)\right)_{\partial\Phi} = 4(X + CY)\left[\left(\frac{\rho_r}{\rho} F(X,Y,\Phi)\right)_{\partial\Phi} \frac{\partial}{\partial X} \left(\frac{\rho_r}{\rho} S(X,Y,\Phi)\right)_{\partial\Phi}\right]\]

\[-\frac{\partial}{\partial X} \left(\frac{\rho_r}{\rho} F(X,Y,\Phi)\right)_{\partial\Phi} \left(\frac{\rho_r}{\rho} S(X,Y,\Phi)\right)_{\partial\Phi} + \left[\left(\frac{\rho_r}{\rho} S(X,Y,\Phi)\right)_{\partial\Phi} \frac{\partial}{\partial Y} \left(\frac{\rho_r}{\rho} S(X,Y,\Phi)\right)_{\partial\Phi}\right]\]

\[-c\frac{\partial}{\partial Y} \left(\frac{\rho_r}{\rho} S(X,Y,\Phi)\right)_{\partial\Phi} \left(\frac{\rho_r}{\rho} S(X,Y,\Phi)\right)_{\partial\Phi}\]

\[(3-38)\]

The boundary conditions for \(F(X,Y,\Phi)\) and \(S(X,Y,\Phi)\) transformed

\[F(X,Y,0) = F_{\Phi}(X,Y,0) = \Phi_{\Phi}(X,Y,\omega) = 0\]

\[S(X,Y,0) = S_{\Phi}(X,Y,0) = S_{\Phi}(X,Y,\omega) = 0\]

From (3-12), we have

\[T(\xi,\eta,\zeta) = T_{\Phi}(\xi,\eta) + \Delta T, \theta(\xi,\eta,\zeta)\]

So that

\[\frac{\partial T}{\partial \xi} = \frac{\partial T_{\Phi}}{\partial \xi} + \theta \frac{\partial \Delta T}{\partial \xi} + C_{\Phi} \frac{\partial T}{\partial \xi} - \Phi_{\Delta T} \frac{\partial \theta}{\partial \Phi}\]

\[(3-40)\]

Similarly

\[\frac{\partial T}{\partial \eta} = \frac{\partial T_{\Phi}}{\partial \eta} + \theta \frac{\partial \Delta T}{\partial \eta} + C_{\Phi} \frac{\partial T}{\partial \eta} - \Phi_{\Delta T} \frac{\partial \theta}{\partial \Phi}\]

\[(3-41)\]

And

\[\frac{\partial T}{\partial \zeta} = \rho_r \left[\frac{\rho_r}{\rho} \left(\frac{\rho_r}{\rho} \left(-G(\Delta T, \theta) \right) \right) \right] \frac{1}{\Delta T} \frac{\partial \theta}{\partial \Phi}\]

\[(3-42)\]
Also

\[
\frac{\partial}{\partial \zeta} \left( \kappa \frac{\partial T}{\partial \zeta} \right) = \rho \left[ \frac{\rho_r^4(-G(\Delta T)\beta_r)}{16 \mu_r^2(X+CY)} \right]^{\frac{1}{4}} \left[ \kappa \frac{\Delta T \beta_r}{16 \mu_r^2(X+CY)} \right] \frac{\partial \Delta T}{\partial \Phi} \left[ \rho \frac{\partial \Delta T}{\partial \Phi} \right] \frac{\partial \Phi}{\partial \Delta T}
\]

\[
= \frac{\rho U_x}{4(X+CY)} \frac{\partial}{\partial \Phi} \left[ \frac{\rho \mu C_p}{\rho, \mu, \mu} \right] \frac{\partial \Phi}{\partial \Delta T}
\]

\[
= \frac{\rho U_x \Delta T C_p}{4(X+CY)} \left[ \frac{\rho \mu C_p}{\rho, \mu, \mu} \right] \frac{1}{\partial \Phi} \frac{\partial \Phi}{\partial \Delta T}
\]

Similarly

\[
\frac{\partial v}{\partial \zeta} = \frac{\rho U_x}{\rho} \left[ \frac{\rho_r^4(-G(\Delta T)\beta_r)}{16 \mu_r^2(X+CY)} \right]^{\frac{1}{4}} \frac{\partial}{\partial \Phi} \left[ \frac{\rho_r F(X,Y,\Phi)}{\rho} \right]_{\Phi}
\]

\[
T \beta U \frac{\partial p}{\partial \Phi} = T \beta U \left( \frac{\rho_r F(X,Y,\Phi)}{\rho} \right)_{\Phi} \cdot \rho_e g_z
\]

\[
T \beta V \frac{\partial p}{\partial \eta} = T \beta U \left( \frac{\rho_r S(X,Y,\Phi)}{\rho} \right)_{\Phi} \cdot \rho_e g_z
\]

Now substituting (3-40) to (3-47) into the energy equation (3-11), we have
\[
\rho C_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) - T \beta \left( u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} \right) = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial y} \right) + \mu \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2
\]

or \[\rho C_p U_p \left[ \left( \frac{\partial T}{\partial x} + \Theta \frac{\partial \Delta T}{\partial x} + \Delta T \frac{\partial \Theta}{\partial x} \right) \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right)_{\Theta} + \left( \frac{\partial T}{\partial y} + \Theta \frac{\partial \Delta T}{\partial y} + \Delta T \frac{\partial \Theta}{\partial y} \right) \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right)_{\Theta} \right] - \frac{\rho C_p U_p}{4(X+CY)} \frac{\partial \theta}{\partial x} \left[ \frac{\rho}{\rho} F(X,Y,\Phi) \right]_{\Theta} + C \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right)_{\Theta} - \frac{3 C_p U_p \Delta T}{4(X+CY)} \frac{\partial \theta}{\partial x} \left[ \frac{\rho}{\rho} F(X,Y,\Phi) \right]_{\Theta} + C \frac{\partial}{\partial y} \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right)_{\Theta}
\]

\[\rho C_p U_p \left[ \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right)_{\Theta} + C \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right)_{\Theta} \right] - T \beta \rho U_p \left[ g_{\theta} \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right)_{\Theta} \right] + \frac{\mu T^3 U_p^3}{4 \mu \rho_0 (X+CY)} \left[ \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right)^2_{\Theta} + \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right)^2_{\Theta} \right]
\]

or \[\frac{C_p}{C_r} \left[ \frac{\rho u}{\rho} C_p \frac{1}{\rho} \frac{\partial \Theta}{\partial x} + \frac{3 \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right)_{\Theta} + C \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right)_{\Theta} \right] \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x}
\]

\[\frac{4(X+CY) \frac{\partial T}{\partial x} + \Theta \frac{\partial \Delta T}{\partial x} \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right)_{\Theta} - \frac{4(X+CY) \frac{\partial T}{\partial y} + \Theta \frac{\partial \Delta T}{\partial y} \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right)_{\Theta} \right] - \frac{G \beta \rho_0 (X+CY)}{C_r} \left[ \left( \frac{\rho u}{\rho} (\Delta T)_{\Theta} \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right)^2_{\Theta} + \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right)^2_{\Theta} \right] \right]
\]

\[\frac{-4 \beta \rho_0 T \beta}{G \rho \beta \Delta T} \left( \frac{\rho}{\rho} F(X,Y,\Phi) \right)_{\Theta} - \frac{4 \beta \rho_0 T \beta}{G \rho \beta \Delta T} \left( \frac{\rho}{\rho} S(X,Y,\Phi) \right)_{\Theta}
\]

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\[ 4(X + CY) \left[ \frac{\partial \theta}{\partial X} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right)_x - \frac{\partial}{\partial X} \left( \frac{\rho}{\rho} F(X, Y, \Phi) \right) \theta \right] \]

\[ + \left[ \frac{\partial \theta}{\partial Y} \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right)_y - \frac{\partial}{\partial Y} \left( \frac{\rho}{\rho} S(X, Y, \Phi) \right) \theta \right] \]  

(3-48)

**The boundary conditions for energy equation:**

\[ \theta(\xi, \eta, 0) = 1, \quad \theta(\xi, \eta, \infty) = 0 \]  

Therefore the u-momentum, v-momentum and energy equations i.e. (3-37), (3-38) and (3-48) can be written in a most simplified forms as follows.

**u-momentum equation:**

\[ \left\{ \frac{\rho \mu}{\rho, \mu} \left( \frac{\rho}{\rho} F \right)^{10} \right\}^{10} + \left( \frac{\rho}{\rho} - 1 \right) \left( \frac{\rho}{\rho} F \right)^{10} - 2 \left( \frac{\rho}{\rho} F \right)^{10} - 2 \left( \frac{\rho}{\rho} S \right) \left( \frac{\rho}{\rho} F \right)^{10} + 3 \left( \frac{\rho}{\rho} S \right) \left( \frac{\rho}{\rho} F \right)^{10} \]

\[ + 2 \left( \frac{\rho}{\rho} F \right)^{10} \left( \frac{\rho}{\rho} F \right)^{10} = 4(X + CY) \left[ \left( \frac{\rho}{\rho} F \right) \frac{\partial}{\partial X} \left( \frac{\rho}{\rho} F \right) - \left( \frac{\rho}{\rho} F \right) \frac{\partial}{\partial X} \left( \frac{\rho}{\rho} F \right) \right] \]

\[ + \left[ \left( \frac{\rho}{\rho} S \right) \frac{\partial}{\partial Y} \left( \frac{\rho}{\rho} F \right) - \mathcal{C} \left( \frac{\rho}{\rho} F \right) \right] \]  

(3-50)

**v-momentum equation:**
\[
\left\{ \frac{\rho_\mu}{\rho_{\mu,\rho}} \left( \frac{\rho_{\lambda}}{\rho} \right) \right\}' + \frac{4(\rho_{\cdot 1} - 1)}{G(\Delta T)_{\rho,\rho}} + 2C \left( \frac{\rho_{\cdot 2}}{\rho} \right)' - 2 \left( \frac{\rho_{\cdot 2}}{\rho} \right) \left( \frac{\rho_{\cdot 2} F}{\rho} \right) + 3C \left( \frac{\rho_{\cdot 2}}{\rho} \right) \left( \frac{\rho_{\cdot 2} S}{\rho} \right)''
\]

+ 3 \left( \frac{\rho_{\cdot 2} F}{\rho} \right) \left( \frac{\rho_{\cdot 2} S}{\rho} \right)'' = 4(X + CY) \left[ \left( \frac{\rho_{\cdot 2} F}{\rho} \right)' \right]

\]

\[
+ \left[ \left( \frac{\rho_{\cdot 2} S}{\rho} \right)' - C \left( \frac{\rho_{\cdot 2} S}{\rho} \right)'' \right]
\]  

\[
(3-51)
\]

**The energy equation:**

\[
\frac{C_{\rho}}{C_{\rho}} \left( \frac{\rho_{\mu} C_{\rho}}{\rho_{\mu}, C_{\rho}} \right)' + 3 \left( \frac{\rho_{\cdot 2} F}{\rho} \right) + C \left( \frac{\rho_{\cdot 2} S}{\rho} \right)' - \frac{4(X + CY)}{\Delta T} \left( \frac{\rho_{\cdot 2} F}{\rho} \right)'
\]

\[
- \frac{4(X + CY)}{\Delta T} \left( \frac{\rho_{\cdot 2} S}{\rho} \right)' + (X + CY)C_{\rho} \left[ \left( \frac{\rho_{\mu}}{\rho_{\mu}} \right) (\Delta T)_{\rho} \left( \frac{\rho_{\cdot 2} F}{\rho} \right) + \left( \frac{\rho_{\cdot 2} S}{\rho} \right)'' \right]
\]

\[
- \frac{4g_{\cdot 2} \rho_{\cdot 2}}{G} \beta \Delta T \left( \frac{\rho_{\cdot 2} F}{\rho} \right)' - \frac{4g_{\cdot 2} \rho_{\cdot 2}}{G} \beta \Delta T \left( \frac{\rho_{\cdot 2} S}{\rho} \right)' = 4(X + CY) \left[ \left( \frac{\theta'}{\Delta X} \left( \frac{\rho_{\cdot 2} F}{\rho} \right) - \frac{\theta'}{\Delta X} \left( \frac{\rho_{\cdot 2} F}{\rho} \right) \right]'
\]

\[
+ \left[ \left( \frac{\theta'}{\Delta Y} \left( \frac{\rho_{\cdot 2} S}{\rho} \right) - \frac{\theta'}{\Delta Y} \left( \frac{\rho_{\cdot 2} S}{\rho} \right) \right]'
\]  

\[
(3-52)
\]

where \( F(X,Y,\Phi) = F \), \( S(X,Y,\Phi) = S \) and \( T(X,Y,\Phi) = T \)
The boundary conditions of the above equations are

\[
\begin{align*}
F'(\xi, \eta, 0) &= F'(\xi, \eta, 0) = F'(\xi, \eta, \infty) = 0 \\
S'(\xi, \eta, 0) &= S'(\xi, \eta, 0) = S'(\xi, \eta, \infty) = 0 \\
\text{and} & \quad \psi(\xi, \eta, 0) = 1, \quad \psi(\xi, \eta, \infty) = 0 
\end{align*}
\]  

(3-53)

Here primes denote differentiation with respect to \( \Phi \) and the parameter \( L \) (which has the dimensions of length) is defined as

\[
L = \frac{C_p}{G} \quad \text{where} \quad G = \text{absolute value of } \bar{G} 
\]  

(3-54)

In left hand side of (3-52), the effect of viscous work and pressure work are found to be proportional to the first and second, third terms in square brackets respectively. Whilst the first term is of order \( \left( \frac{\rho_r}{\rho} F \right)^{\frac{1}{2}} + \left( \frac{\rho_r}{\rho} S \right)^{\frac{1}{2}} \), the 2nd and 3rd terms are of order \( \left( \frac{\rho_r}{\rho} F \right) \frac{T_r}{(\Delta T)_r} \) and \( \left( \frac{\rho_r}{\rho} S \right) \frac{T_r}{(\Delta T)_r} \). So that these would appear that for both liquids and gases the effect of pressure work is not necessarily small in comparison with the effect of viscous work. Furthermore, both stress work terms are seen to be multiplied by \( \frac{X + CY}{L} \).

Thus we note that the importance of the two stress work terms (relative to those terms representing convection and diffusion of heat) is determined largely by the nature of the \( T_r \) and \( \Delta T \) variation. These points will be examined in detail in the following sections.

Now \( \Phi \) can be written from (3-14) as

\[
\Phi = \frac{1}{2} \frac{R_c^{\frac{1}{2}}}{(X + CY)} \int \frac{\rho}{\rho_r} d\xi, 
\]  

(3-55)

Here \( R_c = \frac{\rho_r U_r (X + CY)}{\mu_r} \) is a Reynolds number based on the characteristic free convection velocity \( U_r \) (Ostrach 1964), are given by
Further from (3-54) and (3-56) we can write,

\[ \frac{X + CY}{L} = \frac{-G(\Delta T) \beta_r}{C_p} \frac{X + CY}{C_p(\Delta T)} \]

That is

\[ \frac{X + CY}{L} = \frac{U^2}{C_p(\Delta T)} \]  

(3-57)

which is the Eckert number based on \( U_r \).

Since \( \frac{C_p}{G\beta_r} \) typically has values of order \( 10^4 \) m and \( 10^6 \) m for air and water respectively, at atmospheric temperature and pressure conditions, it is to be appreciated that in most cases of natural convection flow about a vertical rectangular inclined plane surface such that \( \frac{X + CY}{L} \ll 1 \). Therefore, in the following study, we shall obtain solutions of (3-50), (3-51) and (3-52) to first order is only.

**Equations of state and transport properties**

The two equations of state to be used are, first, that for a thermally perfect gas and second that for a general fluid undergoing small changes in temperature and pressure. Before proceeding to study the solution of a particular flow problem, we must specify the necessary thermodynamic relations and determine suitable approximations for the transport properties of the fluid concerned. In general, these are complicated functions in temperature and pressure. The following expansion is valid for a function of two variables.

i.e.

\[ \rho(T, p) = \rho_r(T, p) + \left( h \cdot \frac{\partial}{\partial T} + k \cdot \frac{\partial}{\partial p} \right) \rho_r(T, p) + \frac{1}{2!} \left( h \cdot \frac{\partial}{\partial T} + k \cdot \frac{\partial}{\partial p} \right)^2 \rho_r(T, p) + \ldots \]  

(3-58a)
Where $h = (T - T_r)$ and $k = (p - p_r)$ are small quantities, and suffix 'r' denotes some reference condition.

For thermally perfect gas,

$$p \propto \rho T, \quad \beta = \frac{1}{T}$$  \hspace{1cm} (3-58b)

Since $p$ is constant across any cross-section of the boundary layer, we obtain without any additional approximation

$$\frac{\rho_r - 1}{\rho} = \frac{T}{T_r} - 1$$

$$\frac{1}{T_r (\Delta T)_r} \frac{(T - T_r) T_r}{(T - T_r)} = \frac{(T - T_r)}{\Delta T} \frac{\Delta T}{(\Delta T)_r} \frac{T_r}{T} = \frac{\Delta T}{(\Delta T)_r} \frac{T_r}{T}$$  \hspace{1cm} (3-58c)

and

$$\frac{T}{\Delta T} \frac{\rho_r \beta_r \beta}{\rho} = \frac{T}{\Delta T} \frac{T_r}{T} = \frac{T}{\Delta T} \frac{(T + \theta \Delta T)}{\theta T} \frac{T_r}{T} = \frac{T_r}{\Delta T} + \theta \frac{T_r}{T}$$  \hspace{1cm} (3-58d)

using $\frac{\rho_r}{\rho} = \frac{T}{T_r}$ and $\beta_r = \frac{1}{T}$.

We know

$$d\rho = -\rho \beta_r dT + \rho K dp$$

or

$$\frac{d\rho}{\rho} = -\beta_r dT + K dp$$

or

$$\int \frac{d\rho}{\rho} = -\beta_r (T - T_r) + K (p - p_r)$$

or

$$\log \frac{\rho_r}{\rho} = -\beta_r T_r T' + K p_r p'$$

\[ \therefore \] $\frac{\rho_r}{\rho} = e^{-\beta_r T_r T' - K p_r p'}$

i.e

$$\frac{\rho_r}{\rho} = -\beta_r T_r T' + K p_r p'$$  \hspace{1cm} (3-58e)

where
\[ \beta_r = \left\{ \frac{1}{\rho} \left( \frac{\partial p}{\partial T} \right)_T \right\}, \quad \theta_r = \left\{ \frac{1}{\rho} \left( \frac{\partial p}{\partial T} \right)_T \right\} \]

and

\[ T' = \frac{T_0 - T_r}{T_r}, \quad \rho' = \frac{\rho_0 - \rho_r}{\rho_r}, \quad \frac{T_0 - T_r}{T_r} = \frac{\Delta T}{T_r} \theta + T' \]

Here \( T', \rho' \) and \( \frac{\Delta T}{T_r} \) are all small compared with unity, and \( \theta \) is at most of order unity in the boundary layer but \( \theta = 0 \) in the exterior fluid, we shall find in the following section that the leading terms in the expressions for \( T' \) and \( \rho' \) are of order \( \frac{X + CY}{L} \). Again using (3.58a) and (3.58e) also we have

\[
\frac{\rho_r}{\rho} - \frac{\rho_0}{\rho} = \left[ 1 - \beta \frac{T_r}{T} + K_r p_r \rho' \right] + \beta \frac{T_0 - T_r}{T_r} \left[ 1 + \frac{1}{\beta^2} \left( \frac{\partial \rho}{\partial T} \right)_T \right] - \rho_r p_r' \beta T_r \left[ K + \frac{1}{\beta} \left( \frac{\partial K}{\partial T} \right)_T \right],
\]

\[ + T' \frac{\beta^2 T_r}{6} \left[ 1 + \frac{1}{\beta^2} \left( \frac{\partial \rho}{\partial T} \right)_T \right] + O \left( \frac{\Delta T}{T_r} \right)^2 \]

And

\[
\frac{\rho_r}{\rho} \left[ \frac{\beta - 1}{\beta (\Delta T)_T} \right] \sim \frac{\Delta T}{(\Delta T)_T} \theta + \frac{\Delta T}{(\Delta T)_T} \left[ 1 + \frac{\beta T_r}{T} \frac{1}{2} \left( \frac{\partial \rho}{\partial T} \right)_T \right] + O \left( \frac{\Delta T}{T_r} \right)^2
\]

\[ + T' \left[ \frac{\beta \left( \frac{\partial \rho}{\partial T} \right)_T}{\theta} + O \left( \frac{\Delta T}{T_r} \right)^2 \right] - \rho_r \left[ \frac{\beta \left( \frac{\partial K}{\partial T} \right)_T}{\theta} + O \left( \frac{\Delta T}{T_r} \right)^2 \right] \]

Since for a thermally perfect gas \( \beta = \frac{1}{T} \) where \( \beta = \beta(T, p) \)

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\[
\left( \frac{\partial \beta}{\partial T} \right)_\rho = -\frac{1}{T^2} \quad \text{and} \quad \left( \frac{\partial \kappa}{\partial T} \right)_\rho = 0 \quad (3-59)
\]

We expand the later to first order in small quantities and obtain, to zeroth order in \( T' \) and \( p' \),

\[
\frac{T}{\Delta T} \frac{\rho_v \beta}{\beta_r} = \frac{T_r}{\Delta T} + \theta \left[ 1 + (1 + \frac{1}{\beta^2 T^2}) \right] + \mathcal{O} \left( \frac{\Delta T}{T} \right) \quad (3-60)
\]

Using (3.59) in equation (3.60), we have

\[
\frac{T}{\Delta T} \frac{\rho_v \beta}{\beta_r} = \frac{T_r}{\Delta T} + \theta \left[ 1 + (1 + \frac{1}{\beta^2 T^2}) \right] + \mathcal{O} \left( \frac{\Delta T}{T} \right)
\]

\[
= \frac{T_r}{\Delta T} + \theta = \frac{T_r}{\Delta T} + \theta \frac{T_r}{T_r}
\]

(3-61)

[ since by zeroth order in \( \frac{X + CY}{L} \), \( \frac{T_r}{T} = 1 \).]

It is to be noted that there are three variables \( T' \), \( p' \) and \( \frac{\Delta T}{T} \), which have been used to obtain (3.58g). This equation (3.58g) can be written in simplified form with the help of (3.59)

\[
\frac{\rho_v - 1}{\beta_r (\Delta T)_r} = \frac{\Delta T}{(\Delta T)_r} \theta \left[ 1 - T' + \mathcal{O} \left( \frac{\Delta T}{T_r} \right)^2 + T' \theta \left( \frac{\Delta T}{T_r} \right) + p' \theta \left( \frac{\Delta T}{T_r} \right) \right]
\]

Neglecting the order \( \left( \frac{\Delta T}{T_r} \right) \) and order \( \left( \frac{\Delta T}{T_r} \right)^2 \)

\[
\frac{\rho_v - 1}{\beta_r (\Delta T)_r} = \frac{\Delta T}{(\Delta T)_r} \theta \left[ 1 - T' \right]
\]

\[
= \frac{\Delta T}{(\Delta T)_r} \theta \left[ 1 + \frac{T_r}{T_r} \right] = \frac{\Delta T}{(\Delta T)_r} \theta \frac{T_r}{T_r}
\]

(3-62)
We see that (3-58g) and (3-61) as the same as (3-58c) and (3-58d). However, for a liquid such useful simplifications are not always possible. For example, for water at atmospheric pressure and 293°K (Reynolds 1971)

$$\beta T_r \left(1 + \frac{1}{\beta^2} \left( \frac{\partial \beta}{\partial T} \right)_p \right) \approx \beta T_r \left( \frac{1}{\beta^2} \left( \frac{\partial \beta}{\partial T} \right)_p \right) \approx 15.2$$

(3-63)

Except for \( T_r \) and \( \Delta T \) variations, which will be discussed in the following section, the remaining terms to be determined in the \( \mu \)-momentum, \( \nu \)-momentum and energy equations, where \( \mu, C_p \) and \( K \) involve. In describing these properties for a gas, we shall assume for simplicity that

$$\mu \propto T^\omega, \quad C_p = \text{const}, \quad P_r = \text{const}.$$  

(3-64)

Thus, without additional approximation we obtain

$$\frac{\rho \mu}{\rho_r \mu_r} \approx 1 - \left( \frac{\Delta T}{T_r} + \theta \right) \left(1 - \omega \right) + p'$$

$$K_r = \frac{1}{P_r}$$

$$\approx 1 + T' + \left( \frac{\Delta T}{T_r} \theta \right)^{\left(1-\omega \right)} + p'$$

$$\approx 1 + T' + \left( \frac{\Delta T}{T_r} \theta \right)^{\left(\omega-1 \right)} + p'$$

$$\approx 1 + T' + \left( \frac{\Delta T}{T_r} \theta \right)^{\left(\omega-1 \right)} \left(1 + p' \right) - O \left( p'T', (\omega - 1) \frac{\Delta T}{T_r} \theta p' \right)$$

Neglecting

$$O \left( p'T', (\omega - 1) \frac{\Delta T}{T_r} \theta p' \right)$$

$$\frac{\rho \mu}{\rho_r \mu_r} \approx \left(1 + \frac{T_r - T}{T_r} + \frac{\Delta T}{T_r} \theta \right)^{\left(\omega-1 \right)} \left(1 + \frac{P_r - P_r'}{P_r} \right)$$

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\[
\left( \frac{T_r + \Delta T}{T_r} \right)^{(a-1)} \left( \frac{p_r}{p_r} \right) \quad \text{but } p_r = p \quad \text{for exterior conditions.}
\]

\[
\left( \frac{T_r}{T_r} \right)^{(a-1)} \left( 1 + \frac{\Delta T}{T_r} \theta \frac{T_r}{T_r} \right)^{(a-1)} \left( \frac{p_r}{p_r} \right)
\]

(3-65)

In similar way we have

\[
\frac{\rho \rho^*}{\rho} \approx 1 + \frac{\Delta T}{T_r} + T' - \left( 1 + \frac{\Delta T}{T_r} \theta \right) p'
\]

\[
\approx \left( \frac{T_r}{T_r} + \frac{\Delta T}{T_r} \theta \right) \left( \frac{p_r}{p_r} \right)
\]

(3-66)

For the general fluid undergoing small changes in properties, we assume \( \mu = \mu(T) \), \( C_v = C_v(T) \) and \( K = K(T) \). In keeping with the approach resulting in (3-58g) and (3-60), expanding about the reference condition \( r \) to first order in small quantities. We may assume

\[
a = \left( \frac{T d \mu}{\mu d T} \right), \quad b = \left( \frac{T d C_v}{C_v d T} \right), \quad c = \left( \frac{T d K}{K d T} \right),
\]

(3-67)

To first order in \( T' \) and \( P' \) we obtain

\[
\frac{\rho \mu}{\rho, \mu} \approx 1 - \frac{\Delta T}{T_r} \theta (\beta, T_r - a) + O\left( \frac{\Delta T}{T_r} \right)^2 - T' \left( \beta, T_r - a + O\left( \frac{\Delta T}{T_r} \right) \right)
\]

\[
+ P' \left( \kappa, p_r + O\left( \frac{\Delta T}{T_r} \right) \right)
\]

(3-68i)

\[
\frac{C_v}{C_v} \approx 1 - \frac{\Delta T}{T_r} b \theta + O\left( \frac{\Delta T}{T_r} \right)^2 - T' \left( b + O\left( \frac{\Delta T}{T_r} \right) \right)
\]

(3-68ii)
Exterior Conditions and surface temperature variations

As mentioned earlier, exterior property variations contribute terms in (3-50 to 3-52) which can be comparable with or even greater than the stress work terms in the energy equation (3-52).

Two simple cases are studied here of representative exterior property variations which contribute effects comparable with those of stress work. The cases are those for which, first, temperature and second, entropy are constants. Thermally and calorically perfect gases and general fluids undergoing small changes in state are considered for each of these two cases.

For convenience in the following analysis, we write

$$\frac{X + Cy}{L} = x + Cy$$  \hspace{1cm} (3-69)

The reference condition \( r \) are taken to be those at the plane surface leading edge in the exterior fluid. We consider first the case of the thermally and colorically perfect gas. Using the hydrostatic condition (3-7), together with the equation of state (3-6), we obtain two cases.

$$\frac{T_e}{T_r} = 1, \quad \frac{p}{p_r} = \frac{\rho_r T_e}{\rho T_r} = e^{\frac{-γ}{γ-1} (x + Cy)}$$

$$= 1 - \frac{γ}{γ-1} (x + Cy) + O(x + Cy)^2$$
For absolute value of \((x+Cy) \ll 1\)

Again,

\[
\frac{T_r}{T} = 1 - (x + Cy), \quad \frac{\rho_r}{\rho} = \left[1 - (x + Cy)\right]^\frac{1}{r-1}
\]

\[
\frac{P}{\rho_r} = \left(\frac{P}{\rho_r}\right)^\gamma = \left[1 - (x + Cy)\right]^\frac{\gamma}{r-1}
\]

(3.71)

For \(S_e = \text{constant and } p = \kappa \rho'\)

We require the following thermodynamic relation

\[
Tds = C_p dT - \beta T \frac{dp}{\rho}
\]

(3.72)

The results (3.70) and (3.72) for gases are valid for all values of \((x+Cy)\). Such accuracy may not be achieved for more general fluids. However, we recall that we require exterior property variations to first order in \((x+Cy)\) only. Thus, on repeating the procedure used earlier to obtain (3.58g), together with the use of (3.7) and (3.58f), we obtain for a general fluid undergoing small changes in state,

\[
T' = \frac{T_2}{T_r} - 1 = 0 \text{ and } K_r p_r p' = \rho' = -k(x + Cy) + O(x + Cy)^2
\]

(3.73)

Where

\[
T_r = \text{constant}, K_r p_r = 1 \text{ and } \rho' = \frac{P}{\rho_r} - 1
\]

Again,

\[
\frac{T_2}{T_r} = 1 - (x + Cy), T' = -(x + Cy) + O(x + Cy)^2; \quad \rho' = \left(\beta T_r - \alpha\right)(x + Cy) + O(x + Cy)^2
\]

(3.74)

\[
K_r p_r p' = \alpha(x + Cy) + O(x + Cy)^2 \quad \text{for } S_e = \text{constant}
\]
Here $\rho' = \frac{\rho_x - 1}{\rho_r}$, so that for this fluid, to order $(x+C_y)$, equations (3-70) and (3-71), (3-73) and (3-74) are in agreement.

For the calorically and thermally perfect gas $\alpha = \frac{\gamma}{\gamma - 1}$, so that for this fluid, to order $(x+C_y)$, equations (3-70) and (3-71), (3-73) and (3-74) are in agreement.

The two considered cases (i) the constant exterior temperature and (ii) constant exterior entropy result in exterior property variations which are governed solely by the length scale $L$ where one of the state properties is chosen to be constant. The remaining relationship needed to specify the exterior variations is provided by the purely thermodynamics relationship between two properties of state which results from the choice of a third state property being constant. An alternative to the above constant state property approach, the exterior property variations can be determined if one of them is specified. As for example, a temperature stratification may be chosen as a function of $\frac{X+CY}{l}$ (Where the length scale $l$ need not be of the same order as $L$). For the more likely practical situation in which $I \ll L$, we see from (3-50) to (3-52) that temperatures stratification effects will dominate completely the stress work effects. On the other hand, unlikely circumstances arise in which $I \gg L$. Here hydrostatic condition will still introduce $L$ as a scale height for pressure (although $p$ will depend on $l$). Thus, the important point emerges that, unless imposed temperature completely dominate stress work, these will in general be at least an exterior pressure variation which will produce effects comparable with those of stress work and which must be included in any considerations of stress work.

The above constant entropy fluids are therefore neutrally stable and it is easy to show that the calorically and thermally perfect gas at constant temperature is stable. No such general result can be demonstrated for the general fluid. There remain a number of other common functions of state which might be subjected to the constant state property approach. Pressure being constant must be excluded since the hydrostatic condition (3-7) ensures a pressure variation. Density might be held constant, but it is easy to show that
for the calorically and thermally perfect gas this leads to instability. Consequently, conditions of constant temperature and constant entropy appear reasonably satisfactory from the stability viewpoint.

In addition to the exterior property variations discussed above, two simple and representative cases of surface temperature variations are considered. These are, first, a surface temperature variation which results in $\Delta T$ being maintained constant and, second, a constant surface temperature. For these two cases

$$\frac{\Delta T}{T_r} = \frac{(\Delta T)}{T_r} \quad \text{for } \Delta T = \text{constant}$$

(3-75)

and

$$\frac{\Delta T}{T_r} = \frac{(\Delta T)}{T_r} - \left(\frac{T_x}{T_r} - 1\right) \quad \text{for } T_s = \text{constant}$$

(3-76)

Thus for $T_s$ constant, $\Delta T$ is constant, whereas when $S_s$ is constant, both of the above $\Delta T$ variations must be considered.

**Expansions for $(x+C\gamma) << 1$**

Following the discussions of the previous section, three cases of exterior properties and $\Delta T$ variations are to be considered for both perfect gases and general fluids. These are

1. $T_s = T_r = \text{constant}$, $T_w = \text{constant}$ so that $\Delta T = (\Delta T)_r = \text{constant}$

2. $S_s = \text{constant}$, so that $\Delta T = (\Delta T)_s = \text{constant}$

3. $S_s = \text{constant}$, $T_u = \text{constant}$ so that $\Delta T = (\Delta T)_u = -T_x \left(\frac{T_x}{T_r} - 1\right) = \text{constant}$

(3-77)

Since we have chosen for simplicity to consider exterior properties and $\Delta T$ variations which are governed by the single length scale $L$ (which is also the length scale governing stress work) we see that (3-50) to (3-52) can be written entirely in terms of the independent variables ($\Phi, x, y$), the latter two variables being generally small compared
with unity. Here \( x + Cy = \frac{X + CY}{L} \) which implies that \( x = \frac{X}{L} \) and \( y = \frac{Y}{L} \), but \((x + Cy) << 1\). So that \( x = \frac{X}{L} \) is \( << 1 \) and also \( y = \frac{Y}{L} \) \( << 1 \).

Now we have to expand the dependent variables \( F, S \) and \( \theta \) in straight forward expansion in integer powers of \((x + Cy)\), where the co-efficient are function of \( \Phi \) alone.

Thus the assumed expansions are,

\[
\begin{align*}
F(\Phi; x + Cy) &= F_0(\Phi) + (x + Cy)F_1(\Phi) + O((x + Cy)^2) \\
S(\Phi; x + Cy) &= S_0(\Phi) + (x + Cy)S_1(\Phi) + O((x + Cy)^2) \\
\theta(\Phi; x + Cy) &= \theta_0(\Phi) + (x + Cy)\theta_1(\Phi) + O((x + Cy)^2)
\end{align*}
\]

Expressions for the additional dependent variables contained in (3-50) to (3-52) are found from (3-58c), (3-58d) and (3-65) to zeroth and first orders in \((x + Cy)\) as required. We obtain for these

\[
\begin{align*}
\frac{\rho_e - 1}{\beta, (\Delta T)} &= \theta_0 + (x + Cy)(\theta_1 + A(\Phi)) + O((x + Cy)^2) \\
\rho \mu \rho_e &= B(\Phi)\left[(x + Cy)D(\Phi) + O((x + Cy)^2)\right] \\
\frac{1}{\Delta T}\left[C_p, \rho, T, \beta, \frac{\partial T}{\partial x} + \theta \frac{\partial T}{\partial x}\right] &= \theta_0 + E(\Phi) + O((x + Cy)) \\
\frac{1}{\Delta T}\left[C_p, \rho, T, \beta, \frac{\partial T}{\partial y} + \theta \frac{\partial T}{\partial y}\right] &= \theta_0 + H(\Phi) + O((x + Cy))
\end{align*}
\]
Also \( \frac{\Delta T}{\Delta T} = 1 + O(x + Cy) \) \( (3-79e) \)

\[
B(\Phi) = \left[ 1 + \left( \frac{\Delta T}{T} \right) \right] \theta^{(\phi-1)} \quad \text{(3-79f)}
\]

\[
\frac{p_r}{\rho} \approx \left[ 1 + \left( \frac{\Delta T}{T} \right) \right] \theta \quad \text{where} \quad \theta = 1 + \epsilon \theta \quad \text{(3-79g)}
\]

And

\[
= \left\{ 1 + \frac{\Delta T}{T} (1 + \epsilon \theta) \right\}
\]

Here \( p_r = P_r, p_x = \mu \), and \( C_p = C_r \) are considered.

We applying \( x + Cy = \frac{X + CY}{L} \) in (3-50) to (3-52), then we have the \( u \)-momentum, \( v \)-momentum and energy equations by the following forms.

**u-momentum equation:**

\[
\left\{ \frac{\rho u}{\rho, u, (\rho, F)} \right\}'' + \frac{4}{G(\Delta T, \beta)} \left( \frac{\rho, F}{\rho} \right)' - 2 \left( \frac{\rho, F}{\rho} \right)'' - 2C \left( \frac{\rho, S}{\rho} \right) \left( \frac{\rho, F}{\rho} \right)' + 3C \left( \frac{\rho, S}{\rho} \right) \left( \frac{\rho, F}{\rho} \right)''
\]

\[
+ 2 \left( \frac{\rho, F}{\rho} \right) \left( \frac{\rho, F}{\rho} \right)'' = 4(x + Cy) \left\{ \left( \frac{\rho, F}{\rho} \right) \frac{\partial}{\partial x} \left( \frac{\rho, F}{\rho} \right) - \left( \frac{\rho, F}{\rho} \right) \frac{\partial}{\partial x} \left( \frac{\rho, S}{\rho} \right),
\]

\[
+ \left\{ \left( \frac{\rho, S}{\rho} \right) \frac{\partial}{\partial y} \left( \frac{\rho, F}{\rho} \right) - \left( \frac{\rho, F}{\rho} \right) \frac{\partial}{\partial y} \left( \frac{\rho, S}{\rho} \right) \right\}\right\}
\]

(3-80a)
\[ \left\{ \frac{\rho \mu}{\rho, \mu} \left( \frac{\rho - S}{\rho} \right) \right\}^{v} + \frac{4 \left( \frac{\rho \mu}{\rho} \right) \left( \frac{\rho - S}{\rho} \right)}{G(\Delta T) \beta} - 2C \left( \frac{\rho - S}{\rho} \right)^{v} - 2 \left( \frac{\rho - S}{\rho} \right) \left( \frac{\rho}{\rho} \right)^{v} + 3 \left( \frac{\rho - S}{\rho} \right) \left( \frac{\rho}{\rho} \right) \]

\[ + 3 \left( \frac{\rho - F}{\rho} \right) \left( \frac{\rho}{\rho} \right) = 4(x + Cy) \left[ \left( \frac{\rho - F}{\rho} \right) \frac{\partial}{\partial x} \left( \frac{\rho - S}{\rho} \right) - \left( \frac{\rho - S}{\rho} \right) \frac{\partial}{\partial x} \left( \frac{\rho - F}{\rho} \right) \right] \]

\[ \left( \frac{\rho - S}{\rho} \right) \frac{\partial}{\partial y} \left( \frac{\rho - S}{\rho} \right) - C \left( \frac{\rho - S}{\rho} \right) \frac{\partial}{\partial y} \left( \frac{\rho - S}{\rho} \right) \]  \hspace{1cm} (3-80b)

**The energy equation:**

\[ \frac{C_p}{C_p} \left( \frac{\rho \mu}{\rho, \mu} \left( \frac{\rho}{\rho} \right) \left( \frac{\rho - S}{\rho} \right) \right) + \frac{3}{C_p} \left( \frac{\rho - F}{\rho} \right) + \left( \frac{\rho - S}{\rho} \right) \frac{\partial}{\partial x} \left( \frac{\rho - S}{\rho} \right) - \frac{4(x + Cy)}{C_p} \frac{\partial T}{\partial x} + \frac{\partial \Delta T}{\partial x} \]

\[ + \frac{C_p \rho \mu}{C_p} \left( \frac{\rho - S}{\rho} \right) \frac{\partial}{\partial y} \left( \frac{\rho - S}{\rho} \right) - \frac{4(x + Cy)}{C_p} \frac{\partial T}{\partial y} + \frac{\partial \Delta T}{\partial y} + \frac{C_p \rho \mu}{C_p} \frac{\partial}{\partial y} \left( \frac{\rho - S}{\rho} \right) \]

\[ + \frac{(x + Cy)C_p}{C_p} \left[ \left( \frac{\rho \mu}{\rho, \mu} \left( \frac{\rho}{\rho} \right) \left( \frac{\rho - S}{\rho} \right) \right)^{v} + \left( \frac{\rho - S}{\rho} \right)^{v} \right] = 4(x + Cy) \left[ \left( \frac{\rho - F}{\rho} \right) \frac{\partial}{\partial x} \left( \frac{\rho - S}{\rho} \right) - \frac{\partial}{\partial x} \left( \frac{\rho - F}{\rho} \right) \right] \]

\[ + \left( \frac{\rho - S}{\rho} \right) \frac{\partial}{\partial y} \left( \frac{\rho - S}{\rho} \right) \]  \hspace{1cm} (3-80c)

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Here,  

\[
\begin{align*}
\left( \frac{\rho_u}{\rho} F \right) &= \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right] F_0 (\Phi) + (x + Cy)^2 \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right] F_i (\Phi) + O(x + Cy)^2 \\
\left( \frac{\rho_u}{\rho} \right)' &= \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right]^2 F_0 (\Phi) + (x + Cy)^2 \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right]^2 F_i (\Phi) \\
\left( \frac{\rho_u S}{\rho} \right)' &= \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right]^2 S_0 (\Phi) + (x + Cy)^2 \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right]^2 S_i (\Phi) \\
&\quad + 2(x + Cy) \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right]^2 S_0 (\Phi) S_i (\Phi) + O(x + Cy)^2
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial x} \left( \frac{\rho_u}{\rho} F \right)' &= F_0 \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right] + 2(x + Cy) F_i \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right] + O \left( \frac{\Delta T}{T_r} (x + Cy)^2 \right) \\
\frac{\partial}{\partial x} \left( \frac{\rho_u}{\rho} F' \right) &= F_0 \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right] + 2(x + Cy) F_i \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right] + O \left( \frac{\Delta T}{T_r} (x + Cy)^2 \right) \\
\frac{\partial}{\partial y} \left( \frac{\rho_u}{\rho} F \right)' &= CF_0 \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right] + 2C(x + Cy) F_i \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right] + O \left( \frac{\Delta T}{T_r} (x + Cy)^2 \right) \\
\frac{\partial}{\partial y} \left( \frac{\rho_u}{\rho} F' \right) &= CF_0 \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right] + 2C(x + Cy) F_i \left[ 1 + \frac{\Delta T}{T_r} (1 + e \theta) \right] + O \left( \frac{\Delta T}{T_r} (x + Cy)^2 \right)
\end{align*}
\]
Now calculating the values of terms by terms of equations (3-80a) to (3-80e) are given below with the help of (3-79) and (3-81).

\[ \frac{\partial}{\partial x} \left( \frac{\partial \rho}{\partial F} \right) = S_x \left( 1 + \frac{\Delta T}{T_r} \right) \left\{ 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right\} + 2(x + Cy) S_x \left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) + O \left( \frac{\epsilon \Delta T}{T_r}, (x + Cy)^2 \right) \]

\[ \frac{\partial}{\partial y} \left( \frac{\partial \rho}{\partial F} \right) = S_y \left( 1 + \frac{\Delta T}{T_r} \right) \left\{ 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right\} + 2(x + Cy) S_y \left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) + O \left( \frac{\epsilon \Delta T}{T_r}, (x + Cy)^2 \right) \]

\[ \frac{\partial}{\partial z} \left( \frac{\partial \rho}{\partial F} \right) = S_z \left( 1 + \frac{\Delta T}{T_r} \right) \left\{ 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right\} + 2(x + Cy) S_z \left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) + O \left( \frac{\epsilon \Delta T}{T_r}, (x + Cy)^2 \right) \]

\[ \frac{\partial}{\partial x} \left( \frac{\partial \rho}{\partial S} \right) = S_x \left( 1 + \frac{\Delta T}{T_r} \right) \left\{ 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right\} + 2(x + Cy) S_x \left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) + O \left( \frac{\epsilon \Delta T}{T_r}, (x + Cy)^2 \right) \]

\[ \frac{\partial}{\partial y} \left( \frac{\partial \rho}{\partial S} \right) = S_y \left( 1 + \frac{\Delta T}{T_r} \right) \left\{ 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right\} + 2(x + Cy) S_y \left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) + O \left( \frac{\epsilon \Delta T}{T_r}, (x + Cy)^2 \right) \]

\[ \frac{\partial}{\partial z} \left( \frac{\partial \rho}{\partial S} \right) = S_z \left( 1 + \frac{\Delta T}{T_r} \right) \left\{ 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right\} + 2(x + Cy) S_z \left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) + O \left( \frac{\epsilon \Delta T}{T_r}, (x + Cy)^2 \right) \]

(3-81e)

Now calculating the values of terms by terms of equations (3-80a) to (3-80e) are given below with the help of (3-79) and (3-81).

\[
\left[ \frac{\partial^2 \rho}{\partial \rho \partial F} \right] = \left[ \phi(\Phi) F_{0} \left\{ 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right\} \right] + (x + Cy) \left\{ B(\Phi) F_{0} \right\}

- B(\Phi) D(\Phi) F_{0} \left\{ 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right\} + O \left( \frac{\epsilon \Delta T}{T_r}, (x + Cy)^2 \right)
\] (3-82a)

\[
\frac{4 \left( \frac{\rho_{x} - 1}{\rho} \right) g_{x}}{G(\Delta T) \beta_{r}} = 4 \theta_{e} \sin \delta + 4(x + Cy)(\theta_{e} + A(\Phi)) \sin \delta + 4 \sin \delta \left. O(x + Cy)^2 \right)
\] (3-82b)

Where \( g_{x} = \sin \delta \).

\[
2 C \left( \frac{\partial \rho}{\partial S} \right) \left( \frac{\partial \rho}{\partial F} \right) = 2 CF_{0} S_{0} \left\{ 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right\} + 2 C(x + Cy)
\]

\[
\left[ S_{0} F_{0} + F_{0} S_{0} \right] \left\{ 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right\} + O \left( \frac{\epsilon \Delta T}{T_r}, (x + Cy)^2 \right)
\] (3-82c)

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\[
3 \left( \frac{\rho_L F}{\rho} \right) \left( \frac{\rho_F}{\rho} \right) = 3 F_0 F_0' \left( 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right)^2 + 3 (x + Cy)
\]

\[
\{F_1 F_0'' + F_0 F_1'' \left( 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right)^2 + O \left( \varepsilon, \frac{\Delta T}{T_r}, (x + Cy)^2 \right) \} = (3-82d)
\]

\[
3 C \left( \frac{\rho_L S}{\rho} \right) \left( \frac{\rho_F}{\rho} \right) = 3 C S_0 F_0' \left( 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right)^2 + 3 C (x + Cy)
\]

\[
\{S_0 F_0'' + S_1 F_0'' \left( 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right)^2 + O \left( \varepsilon, \frac{\Delta T}{T_r}, (x + Cy)^2 \right) \} = (3-82e)
\]

\[
2 \left( \frac{\rho_L F}{\rho} \right)'' = 2 F_0'' \left( 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right)^2 + 4 F_0' F_1 (x + Cy)
\]

\[
\left( 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right)^2 + O \left( \varepsilon, \frac{\Delta T}{T_r}, (x + Cy)^2 \right) = (3-82f)
\]

\[
\left( \frac{\rho_L F}{\rho} \right) \frac{\partial}{\partial x} \left( \frac{\rho_F}{\rho} \right) = F_0' F_1 \left( 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right)^2 + (x + Cy)
\]

\[
\{2 F_0'' F_1' + F_1'' \left( 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right)^2 + O \left( \varepsilon, \frac{\Delta T}{T_r}, (x + Cy)^2 \right) \} = (3-82g)
\]

\[
\frac{\partial}{\partial x} \left( \frac{\rho_L F}{\rho} \right) \left( \frac{\rho_F}{\rho} \right)'' = F_0' F_1 \left( 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right)^2 + (x + Cy)
\]
\begin{equation}
\left(2F_0F_0 + F_1F_1\right)\left(1 + \frac{\Delta T}{T_r}(1 + \varepsilon \theta)\right)^2 + O\left(\varepsilon, \frac{\Delta T}{T_r}, (x + Cy)^2\right) \tag{3-82h}
\end{equation}

\begin{equation}
\left(\frac{\rho_x S}{\rho} \right) \frac{\partial}{\partial y} \left(\frac{\rho_x F}{\rho} \right) = C S_x F_x \left(1 + \frac{\Delta T}{T_r}(1 + \varepsilon \theta)\right)^2 + C(x + Cy) \tag{3-82i}
\end{equation}

\begin{equation}
\left(2S_0F_1' + S_1F_1'\right)\left(1 + \frac{\Delta T}{T_r}(1 + \varepsilon \theta)\right)^2 + O\left(\varepsilon, \frac{\Delta T}{T_r}, (x + Cy)^2\right) \tag{3-82j}
\end{equation}

\begin{equation}
\frac{\partial}{\partial y} \left(\frac{\rho_x S}{\rho} \right) \left(\frac{\rho_x F}{\rho} \right) = C S_x F_x \left(1 + \frac{\Delta T}{T_r}(1 + \varepsilon \theta)\right)^2 + C(x + Cy) \tag{3-83a}
\end{equation}

Again
\begin{equation}
\left(\frac{\rho_x F}{\rho} \right)' = \left[ B(\Phi)S_x' \left(1 + \frac{\Delta T}{T_r}(1 + \varepsilon \theta)\right)\right]' + (x + Cy)[B(\Phi)S_x' \left(1 + \frac{\Delta T}{T_r}(1 + \varepsilon \theta)\right)]' + O\left(\varepsilon, \frac{\Delta T}{T_r}, (x + Cy)^2\right) \tag{3-83b}
\end{equation}

\begin{equation}
\frac{4\left(\frac{\rho_x}{\rho} - 1\right)}{G(\Delta T), \beta} = 4\theta_0 \cos \delta + 4(x + Cy)(\theta_1 + A(\Phi)) \cos \delta + 4 \cos \delta O(x + Cy)^2 \tag{3-83b}
\end{equation}

Where \( \frac{g_\theta}{G} = \cos \delta \).

\begin{equation}
2\left(\frac{\rho_x F}{\rho} \right)' = 2F_0S_x' \left(1 + \frac{\Delta T}{T_r}(1 + \varepsilon \theta)\right)^2 + 2(x + Cy) \tag{3-83b}
\end{equation}
\[
\left\{ S^0_0 F^0_1 + F^0_0 S^0_1 \right\} \left( 1 + \frac{\Delta T}{T_r} \right)^2 \approx O \left( \frac{\Delta T}{T_r} (x + Cy)^2 \right)
\]

(3-83c)

\[
3 \left( \frac{\rho_x}{\rho} F^0 \right) \left( \frac{\rho_x}{\rho} S^0 \right) = 3 F^0_0 S^0_0 \left( 1 + \frac{\Delta T}{T_r} (1+ \theta) \right)^2 + 3(x + Cy)
\]

(3-83d)

\[
\left\{ F_1 S^0_0 + F_0 S^0_1 \right\} \left( 1 + \frac{\Delta T}{T_r} (1+ \theta) \right)^2 + O \left( \frac{\Delta T}{T_r}, (x + Cy)^2 \right)
\]

(3-83e)

\[
3C \left( \frac{\rho_x}{\rho} S^0 \right) - \left( \frac{\rho_x}{\rho} S^0 \right) = 3C S^0_0 \left( 1 + \frac{\Delta T}{T_r} (1+ \theta) \right)^2 + 3C(x + Cy)
\]

(3-83f)

\[
\left\{ S_0 S^0_1 + S_1 S^0_0 \right\} \left( 1 + \frac{\Delta T}{T_r} (1+ \theta) \right)^2 + O \left( \frac{\Delta T}{T_r}, (x + Cy)^2 \right)
\]

(3-83g)

\[
2C \left( \frac{\rho_x}{\rho} S^0 \right) = 2C S^0_0 \left( 1 + \frac{\Delta T}{T_r} (1+ \theta) \right)^2 + 4S^0_0 (x + Cy)
\]

\[
\left\{ 1 + \frac{\Delta T}{T_r} (1+ \theta) \right\} + O \left( \frac{\Delta T}{T_r}, (x + Cy)^2 \right)
\]

(3-83h)

\[
\left( \frac{\rho_x}{\rho} F^0 \right) \frac{\partial}{\partial x} \left( \frac{\rho_x}{\rho} S^0 \right) = F^0_0 S^0_0 \left( 1 + \frac{\Delta T}{T_r} (1+ \theta) \right)^2 + (x + Cy)
\]

(3-83i)

\[
\left\{ 2F^0_0 S^0_1 + 2F^0_1 S^0_0 \right\} \left( 1 + \frac{\Delta T}{T_r} (1+ \theta) \right)^2 + O \left( \frac{\Delta T}{T_r}, (x + Cy)^2 \right)
\]

(3-83j)

\[
\frac{\partial}{\partial x} \left( \frac{\rho_x}{\rho} P^0 \right) \left( \frac{\rho_x}{\rho} S^0 \right) = S^0_0 \left( 1 + \frac{\Delta T}{T_r} (1+ \theta) \right)^2 + (x + Cy)
\]

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\[
\{2F_sS_{n} + F_tS_{1}\left(1 + \frac{\Delta T}{T_r}(1 + \epsilon \theta)\right)^2 + O\left(\epsilon, \frac{\Delta T}{T_r}, (x + Cy)^3\right) \tag{3-83h}
\]

\[
\left(\frac{\rho_r}{\rho}S\right)'' \frac{\partial}{\partial y}\left(\frac{\rho_r}{\rho}S\right) = CS_{n}S_{t}\left(1 + \frac{\Delta T}{T_r}(1 + \epsilon \theta)\right)^2 + C(x + Cy) \tag{3-83i}
\]

\[
\{2S_{n}S_{t} + S_{1}^2\left(1 + \frac{\Delta T}{T_r}(1 + \epsilon \theta)\right)^2 + O\left(\epsilon, \frac{\Delta T}{T_r}, (x + Cy)^3\right) \tag{3-83j}
\]

\[
\frac{\partial}{\partial y}\left(\frac{\rho_r}{\rho}S\right)\frac{\rho_r}{S}'' = CS_{n}S_{t}\left(1 + \frac{\Delta T}{T_r}(1 + \epsilon \theta)\right)^2 + C(x + Cy) \tag{3-84a}
\]

Also for \( C_r = C_s \),

\[
\frac{C_r}{C_s}\left(\frac{C_s}{C_r} \frac{\rho_\mu}{\rho_\mu} \frac{\vartheta'}{\vartheta'}\right)' = \left(\frac{\rho_\mu}{\rho_\mu} \frac{\vartheta'}{\vartheta'}\right)' = \left[\frac{B(\Phi)\vartheta'}{P_r}\right]' + \left[\frac{B(\Phi)[\vartheta'_t - \vartheta'_t D(\Phi)]}{P_r}\right]'(x + Cy) \tag{3-84a}
\]

\[
+ O(B(\Phi)D(\Phi), (x + Cy)^2) \tag{3-84a}
\]

\[
3\left(\frac{\rho_r}{\rho}F_t + \rho_c \frac{\rho_r}{\rho}S\right)\vartheta' = 3\left[F_s(\Phi) + CS_{s}(\Phi)\right]\vartheta_t\left(1 + \frac{\Delta T}{T_r}(1 + \epsilon \theta)\right) + 3\left[F_t(\Phi) + CS_{t}(\Phi)\right]\vartheta'_t \tag{3-84b}
\]

\[
+ \{F_s(\Phi) + CS_{s}(\Phi)\}\vartheta_t\left(1 + \frac{\Delta T}{T_r}(1 + \epsilon \theta)\right) + O\left(\epsilon, \frac{\Delta T}{T_r}, (x + Cy)^2\right) \tag{3-84b}
\]
\[
\frac{4(x + C_y)}{\Delta T} \left[ \frac{\partial \theta}{\partial x} + \theta \frac{\partial \Delta T}{\partial x} + \frac{C_n}{C_p} \frac{\partial \rho}{\partial x} \frac{\partial \theta}{\partial \rho} \left( \frac{\rho_c}{\rho} \right) \right] = 4(x + C_y)\left[ \theta_0 + E(\Phi) \right]
\]

\begin{equation}
\left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) F_0'(\Phi) + O\left( \epsilon, \frac{\Delta T}{T_r}, (x + C_y)^2 \right) \tag{3-84c}
\end{equation}

\[
\frac{4(x + C_y)}{\Delta T} \left[ \frac{\partial \theta}{\partial y} + \theta \frac{\partial \Delta T}{\partial y} + \frac{C_n}{C_p} \frac{\partial \rho}{\partial y} \frac{\partial \theta}{\partial \rho} \left( \frac{\rho_c}{\rho} \right) \right] = 4(x + C_y)\left[ \theta_0 + H(\Phi) \right]
\]

\begin{equation}
\left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) S_0'(\Phi) + O\left( \epsilon, \frac{\Delta T}{T_r}, (x + C_y)^2 \right) \tag{3-84d}
\end{equation}

\[
\frac{(x + C_y) C_n}{C_p} \left[ \frac{\rho_c}{\rho, \mu, \Delta T} \left( \frac{\rho_c}{\rho} \right)^{n^2} + \left( \frac{\rho_c}{\rho} \right)^{n^2} \right] = (x + C_y) \left[ B(\Phi) \left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) \right]
\]

\begin{equation}
\left[ F_0^{n^2} + S_0^{n^2} \right] + O\left( \epsilon, \frac{\Delta T}{T_r}, (x + C_y)^2 \right) \tag{3-84c}
\end{equation}

\[
\frac{\partial}{\partial x} \left( \frac{\rho_c}{\rho} F \right) \theta' = \left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) F_1(\Phi) \theta_0'(\Phi) + \left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) (x + C_y)
\]

\begin{equation}
\left( F_1(\Phi) \theta_0'(\Phi) + 2F_2(\Phi) \theta_0'(\Phi) \right) + O\left( \epsilon, \frac{\Delta T}{T_r}, (x + C_y)^2 \right) \tag{3-84d}
\end{equation}

\[
\frac{\partial \theta}{\partial x} \left( \frac{\rho_c}{\rho} F \right) = \left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) F_0''(\Phi) \theta_1'(\Phi) + \left( 1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta) \right) (x + C_y)
\]
\[
\{f_s'(\Phi)\theta_s(\Phi) + 2F_0'(\Phi)\theta_s(\Phi)\} + O\left(\varepsilon, \frac{\Delta T}{T_r}, (x+Cy)^2\right)
\] (3-84g)

\[
\frac{\partial}{\partial y} \left( \frac{\rho_r}{\rho} S'_r \right)' = C \left( 1 + \frac{\Delta T}{T_r} \right) S'_r(\Phi)\theta_s(\Phi) + C \left( 1 + \frac{\Delta T}{T_r} \right) (x+Cy)
\] (3-84h)

\[
\{S'_r(\Phi)\theta_s(\Phi) + 2S'_s(\Phi)\theta_s(\Phi)\} + O\left(\varepsilon, \frac{\Delta T}{T_r}, (x+Cy)^2\right)
\] (3-84i)

By using the expressions (3-82), (3-83) and (3-84), in the u-momentum, v-momentum and energy equations i.e. (3-80a), (3-80b) and (3-80c) and also we have from them in reduced form for the zeroth and Ist order of \((x+Cy)\).

For zeroth order of \((x+Cy)^0\)

\[
\left[ BF_0 \left( 1 + \frac{\Delta T}{T_r}, (1+e\theta) \right) \right] ^2 + 4\theta_0 \sin \delta - 2F_0^2 \left( 1 + \frac{\Delta T}{T_r} (1+e\theta) \right) ^2 - 2CF_0 S'_s \left( 1 + \frac{\Delta T}{T_r} (1+e\theta) \right) ^2
\]

\[
+ 3F_0 F'_s \left( 1 + \frac{\Delta T}{T_r} (1+e\theta) \right) ^2 + 3CS'_s F'_s \left( 1 + \frac{\Delta T}{T_r} (1+e\theta) \right) ^2 = 0
\] (3-83a)

Therefore zeroth order of \(x\) and \(y\) are given below,

For zeroth order of \(x\) is

\[
\left[ BF_0 \left( 1 + \frac{\Delta T}{T_r} (1+e\theta) \right) \right] ^2 + 4\theta_0 \sin \delta - 2F_0^2 \left( 1 + \frac{\Delta T}{T_r} (1+e\theta) \right) ^2 - 2CF_0 S'_s \left( 1 + \frac{\Delta T}{T_r} (1+e\theta) \right) ^2
\]

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For zeroth order of $y$

\[
C F_O \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right] + 4 C b_o \sin \delta - 2 C F_O^2 \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right]^2 - 2 C^2 F_O S_O \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right]^2
\]

\[+ 3 C F_O \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right]^2 + 3 C^2 S_O F_O \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right]^2 = 0 \tag{3-85a_2}
\]

The equation (3-85a_2) is same as equation (3-85a_1) if we cancel common $C$ from (3-85a_1).

Similarly the zeroth order of $x$ and $y$ are satisfied the following equations

\[
B S_O \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right] + 4 \theta_o \cos \delta - 2 C S_O^2 \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right]^3 - 2 F_O S_O \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right]^2
\]

\[+ 3 F_O S_O \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right]^2 + 3 C S_O S_O \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right]^2 = 0 \tag{3-85b}
\]

\[4 \left[ \frac{B \theta_o'}{P_r} \right] + 3 \{ F_O + C S_O \} \theta_o \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right] = 0 \tag{3-85c}
\]

Here the boundary conditions are

\[F_O(0) = F_O'(0) = F_O'(\infty) = 0, \quad S_O(0) = S_O'(0) = S_O'(\infty) = 0 \quad \text{and} \quad \theta_o(0) = 1, \quad \theta_o(\infty) = 0. \tag{3-85d}
\]

For the order of $(x + C y)^1$

\[
B(F^* - F^* \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right] - 4 F^* F^* \left[ 1 + \frac{\Delta T}{T_r} (1 + \varepsilon \theta) \right]^2 - 2 C \{ S_O F_O' + F_O S_O' \}
\]

\[= 0 \tag{3-85e}
\]
\[
\left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2 + 3\left(F_0^* + 3F_0^r\right) \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2 + 3C\left[S_0F_0^r + S_iF_i^r\right] \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2 \\
+ 4(\theta_i + A(\Phi)) \sin \delta = \left(F_0^r - F_0^s\right) + C\left[S_0F_0^r - S_iF_i^r\right] \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2
\]

Therefore first order of \(x\) and \(y\) are given below.

For first order of \(x\) is

\[
\left[B\left(F_0^r - DF_0^r\right) \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)\right] - 4F_0^r \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2 - 2C\left[S_0F_0^r + F_0^s\right]
\]

\[
\left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2 + 3\left(F_0^* + 3F_0^r\right) \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2 + 3C\left[S_0F_0^r + S_iF_i^r\right] \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2 \\
+ 4(\theta_i + A(\Phi)) \sin \delta = \left(F_0^r - F_0^s\right) + C\left[S_0F_0^r - S_iF_i^r\right] \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2
\]

For first order of \(y\)

\[
\left[CB\left(S_0^* - DS_0^r\right) \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)\right] - 4C\left[S_0F_0^r\right] \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2 - 2C\left[S_0F_0^r + F_0^s\right]
\]

\[
\left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2 + 3\left(F_0^* + 3F_0^r\right) \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2 + 3C\left[S_0F_0^r + S_iF_i^r\right] \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2 \\
+ 4C(\theta_i + A(\Phi)) \sin \delta = [C\left(F_0^r - S_0^r\right) + C^2\left(S_0^r - CS_0^r\right)] \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon \theta)\right)^2
\]
The equation (3-86a2) is same as equation (3-86a1) if we cancel common \(C\) from (3-86a2).

Similarly the first order of \(x\) and \(y\) are satisfied the following equations

\[
\left[\beta (\theta' - D\theta')\right]' + 2\{F_0 + CS_0\}\theta' \left(1 + \frac{\Delta T}{T_e} (1 + \epsilon e)\right) - 4\{F_0' + CS_0'\}\theta' \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon e)\right)
\]

\[
+ 3\{F_0 + CS_0\} \theta \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon e)\right) - 4\{\theta_0 + E\} F_0' \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon e)\right)
\]

\[- 4(\theta_0 + H) S_0' \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon e)\right) + B[F_0'' + S_0''] \left(1 + \frac{\Delta T}{T_r} (1 + \epsilon e)\right) = 0 \quad (3-86c)
\]

Now here the boundary conditions are

\[
F_i(0) = F_i(\infty) = 0, \quad S_i(0) = S_i'(\infty) = 0 \quad \text{and} \quad \theta_0(0) = \theta_0(\infty) = 0.
\]

\[
(3-86d)
\]

Now here to find the values of \(A, D, E\) and \(H\), we take the relations among

(3-79a) to (3-79g) and (3-58c) or (3-58d).

For the value of \(A\), here we using (3-58c) and (3-79a)

\[
\frac{\Delta T}{T_r (\Delta T)_{r_0}} = \frac{\beta}{\beta_0 (\Delta T)_{r_0}} \left[\theta_0 + (x + Cy)\theta_i + O(x + Cy)^2\right] = \theta_0 + (x + Cy)\theta_i + A + O(x + Cy)^2
\]

\[
(\text{Case - i}) \quad T_r = T_{r_0} = \text{constant}, \quad T_{r_0} = \text{constant} \quad \text{so that} \quad \Delta T = (\Delta T)_{r_0} = \text{constant}
\]

\[
\text{so that} \quad \theta_0 + (x + Cy)\theta_i + O(x + Cy)^2 = \theta_0 + (x + Cy)\theta_i + A\quad \text{therefore} \quad A = 0
\]

Equate the co-efficient of \((x + Cy)\), we have

\[
\theta_i = \theta_i + A(\Phi) \quad \text{therefore} \quad A(\Phi) = 0
\]

\[
(\text{Case - ii}) \quad S_r = \text{constant}, \quad \Delta T = (\Delta T)_{r_0} = \text{constant}, \quad \frac{T_r}{T_{r_0}} = 1 - (x + Cy), \quad \frac{T_{r_0}}{T_r} = 1 + (x + Cy)
\]

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\[ \frac{\Delta T}{(\Delta T)_r} \theta_0 \frac{T_x}{T_e} = \theta_0 + (x + Cy) \theta_1 + O(x + Cy)^2 = \theta_0 + (x + Cy)(\alpha + A) + O(x + Cy)^2 \]

Again we have for the co-efficient of \(x + Cy\)

Therefore, \(A(\Phi) = \theta_0\)

[Case-ii] \(S_e = \text{constant}, T_e = \text{constant}, \frac{\Delta T}{(\Delta T)_r} = 1 - \frac{T_x}{(\Delta T)_r} \left[ \frac{T_x}{T_r} - 1 \right] \)

\(T_x = 1 - (x + Cy), \quad (x + Cy) \)

Using these and equating the co-efficient of \(x + Cy\) from both sides, we have

\[ \theta_0 + \frac{T_x}{(\Delta T)_r} - \theta_0 + \theta_1 = \theta_0 + A(\Phi) \]

that is \(A(\Phi) = \theta_0 \left[ 1 + \frac{T_x}{(\Delta T)_r} \right] \)

We calculate the values of \(D, H\), and \(E\) for three cases by similar way. These values are shown in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Case-i</th>
<th>Case-ii</th>
<th>Case-iii</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>(\theta_e)</td>
<td>(\theta_0 \left[ 1 + \frac{T_x}{(\Delta T)_r} \right] )</td>
</tr>
<tr>
<td>D</td>
<td>(\frac{\omega - 1}{\gamma - 1} \frac{T_x}{T_e} \theta_0)</td>
<td>(\frac{\omega - 1}{\gamma - 1} \frac{T_x}{T_e} \theta_0)</td>
<td>(\frac{\omega - 1}{\gamma - 1} \frac{T_x}{T_e} \theta_0)</td>
</tr>
<tr>
<td>E</td>
<td>(\frac{T_x}{(\Delta T), G} + \theta_0 \left( \frac{g_x}{G} - 1 \right))</td>
<td>(\frac{T_x}{(\Delta T), G} + \theta_0 \left( \frac{g_x}{G} - 1 \right))</td>
<td>(\frac{T_x}{(\Delta T), G} + \theta_0 \left( \frac{g_x}{G} - 1 \right))</td>
</tr>
<tr>
<td>H</td>
<td>(\frac{T_x}{(\Delta T), G} + \theta_0 \left( \frac{g_x}{G} - 1 \right))</td>
<td>(\frac{T_x}{(\Delta T), G} + \theta_0 \left( \frac{g_x}{G} - 1 \right))</td>
<td>(\frac{T_x}{(\Delta T), G} + \theta_0 \left( \frac{g_x}{G} - 1 \right))</td>
</tr>
</tbody>
</table>

For isothermal cases, in equations (3-85) and (3-86) we considered \(B(\Phi) = 1\) and \(A(\Phi) = D(\Phi) = 0\). With the approximation \(\omega = 1\), it implies that \(B(\Phi) = 1\), but from
table we see that \( A \) and \( D \) is not equal to zero. It will be appreciated that the finite values of the variables \( A \) and \( D \) arise because of variations in the exterior and surface conditions and these effects are ignored by Gebhart (1962). According to the discussions of the previous section, we see that it is not possible to choose a combination of exterior conditions such that both \( A \) and \( D \) are zero simultaneously. The pressure work effects in eqns. (3-86) for
\[
4(\theta_0 + E(\Phi))F'(1 + \frac{\Delta T}{T_c}(1 + \varepsilon \theta)) \quad \text{and} \quad 4(\theta_0 + H(\Phi))S'(1 + \frac{\Delta T}{T_c}(1 + \varepsilon \theta))
\]
whose modified forms are neglected in Gebhart analysis. It is seen from table-I that, in most cases, the values of \( A, D, E \) and \( H \) are not small compared with viscous work effects.

Inherent in both the analysis of Gebhart (1962) and that of Gebhart and Mollendorf (1969) is the assumption that \( \frac{(\Delta T)}{T_c} \ll 1 \). For convenience, we write \( \varepsilon = \frac{(\Delta T)}{T_c} \) and also
\[
\left(1 + \frac{\Delta T}{T_c}(1 + \varepsilon \theta)\right) \approx \left(1 + \frac{(\Delta T)}{T_c}(1 + \varepsilon \theta)\right) = (1 + (1 + \varepsilon \theta))
\]
Substituting (3-87) in (3-85) and (3-86) and then equate the co-efficient of \( \varepsilon^0 \) from both sides.

For zeroth order of \( x \) and \( y \) equations are same, so that the equations of \( x^0 \) and \( y^0 \) are given below.

\[
\begin{align*}
\left[ BF''_o \right] & - 2F'_o + 3F_oF'_o + 3C_S oF'' + 4\theta_0 \sin \delta = 0 \\
\left[ BS''_o \right] & - 2C_S^2 - 2F'_oS'_o + 3F_oS'' + 3C_S oS'' + 4\theta_0 \cos \delta = 0 \\
\left[ B\theta'_o \right] & + 3\{F'_o + C_S o\} \theta'_o = 0
\end{align*}
\]
boundary conditions are given below
\[
F_o(0) = F_o'(0) = F_o'(\infty) = 0, \quad S_o(0) = S_o'(0) = S_o'(\infty) = 0
\]
And
\[
\theta_o(0) = 1, \quad \theta_o'(\infty) = 0.
\]
Similarly the first order of $x$ and $y$,

\[
\begin{align*}
\left[ B(F' - DF''_0) \right] - 8F'_0F'' + 7F''_0F' - 6CS''_0F'' + 7CS_1F''_0 \\
+ 3(F'_0 + CS_0)F'' + 2CF_0S'_1 + 4(\theta_1 + A)\sin \delta = 0
\end{align*}
\]

(3-89a)

\[
\begin{align*}
\left[ B(S'_0 - DS''_0) \right] - 8CS''_0S''_1 + C(3 + 4C)S''_0S''_1 - 6S''_0F''_0 + 7F''_0S''_0 \\
+ 3(F'_0 + CS_0)S''_0 - 2F''_0S''_0 + 4(\theta_1 + A)\cos \delta = 0
\end{align*}
\]

(3-89b)

\[
\begin{align*}
\left[ \frac{B(\theta'_0 - D\theta''_0)}{P} \right] + 7(F'_0 + CS''_0)\theta''_0 + 3(F'_0 + CS''_0)\theta''_0 - 4(F''_0 + CS''_0)\theta''_0 \\
- 4(\theta_0 + E)F''_0 - 4(\theta_0 + H)S''_0 + B\left\{ F''_0 + S''_0 \right\} = 0
\end{align*}
\]

(3-89c)

The boundary conditions for the above first order $u$-momentum, $v$-momentum and energy equations are shown below.

\[
F'_0(0) = F'_0(\infty) = F'_0(\infty) = 0, \quad S'_0(0) = S'_0(\infty) = S'_0(\infty) = 0
\]

And

\[
\theta'_0(0) = 0, \quad \theta'_0(\infty) = 0.
\]

(3-89d)

The transformed equations can be solved with the help of the controlling parameters $Pr, 8, A, B, C, D, E, H, (\Delta\Gamma/T)$. 

For numerical solution of the transformed equations (3-88) and (3-89), the values of controlling parameters are given below in case - (i).

\[ A=0, \quad B=1.0, \quad P_r=0.72 \text{ (for air), } 7.0 \text{ (for water), } 0 \leq C \leq 1, \quad \delta=15^\circ, \]

\[ \omega=1.0. \]

where,

\[ \frac{\xi_z}{G} = \sin \delta, \quad \frac{\xi_r}{G} = \cos \delta, \quad \gamma = 1.4 \]

\[ D = \frac{\gamma}{\gamma-1} \left( \frac{\Delta T}{T} \right) \frac{\theta_1}{1 + (\omega - 1) \left( \frac{\Delta T}{T} \right) \theta_0} \]

\[ E = \left( \frac{T}{\Delta T} \right) \frac{\xi_z}{G} + \theta_0 \left( \frac{\xi_z}{G} - 1 \right), \quad H = \left( \frac{T}{\Delta T} \right) \frac{\xi_r}{G} + \theta_0 \left( \frac{\xi_r}{G} - 1 \right) \]
Chapter 4

Graphs & Tables
Fig 1(a): Dimensionless velocity distributions along u direction for several values of $C(=0.3, 0.5, 0.7, 0.9, 1.0)$ for the equations (3-88).

Fig 1(b): Dimensionless velocity distributions along v-direction for the several values of $C(=0.3, 0.5, 0.7, 0.9, 1.0)$ for the equations (3-88).
Fig 1(c): Dimensionless temperature distributions for several values of $C(=0, 0.3, 0.5, 0.7, 0.9, 1.0)$ for the equations (3-88).

Fig 2(a): Dimensionless velocity distributions along u-direction for several values of $C(0.1, 0.3, 0.5, 0.7, 0.9, 1.0)$ for the equations (3-88).
Fig 2(b) Dimensionless velocity distributions along \( \gamma \)-direction for different values of \( C(0.1, 0.3, 0.5, 0.7, 0.9, 1.0) \) for the equations (3.88).

Fig 2(c) Dimensionless temperature distributions for different values of \( C(0.1, 0.3, 0.5, 0.7, 0.9, 1.0) \) for the equations (3.88).
Fig 3(a) Dimensionless velocity distributions along u-direction for several values of
C(=0.25,0.45,0.65,0.75,0.95) for the equations (3-88).

Fig 3(b) Dimensionless velocity distributions along v-direction for several values of
C(0.25,0.45,0.65,0.75,0.95) for the equations (3-88).
Fig. 3(c) Dimensionless temperature distributions for several values of $C(=0.25, 0.45, 0.65, 0.75, 0.95)$ for the equations (3-8c).

Fig. 4(a) Dimensionless velocity distributions along $u$-direction for several values of $C(=0.2, 0.4, 0.6, 0.8, 0.9, 1.0)$ for the equations (3-8c).
Fig. 4(b) Dimensionless velocity distributions along y-direction for several values of $C = 0.2, 0.4, 0.6, 0.8, 0.9, 1.0$ for the equations (3-88).

Fig. 4(c) Dimensionless temperature distributions for several values of $C = 0.4, 0.8, 1.1, 1.4, 1.6, 1.8$ for the equations (3-88).
Fig 5(a) The skin friction coefficient along u-direction against C(0.0 to 1.0) and Prandtl number=7.0 for the equations (3.88).

Fig 5(b) The skin friction factor along v-direction against C(0.0 to 1.0) and Prandtl number=7.0 for the equations (3.88).
Fig. 5(c) The heat transfer coefficient against $C(0.0 \text{ to } 1.0)$ and Prandtl number $= 7.0$
for the equations (3-88).

\[ \text{Pr} = 0.72 \]
\[ \delta = 15^\circ \]
\[ B = 1.0 \]

Fig. 6(a) The skin friction factor along u-direction against $C(0.0 \text{ to } 1.0)$ and Prandtl number $= 0.72$
for the equations (3-88).
$Pr = 0.72$
$\delta = 15^\circ$
$B = 1.0$

**Fig 6(b)** The skin friction factor against $C(0.0 \text{ to } 1.0)$ and Prandtl number $= 7.0$

for the equations (3-88).

$Pr = 0.72$
$\delta = 15^\circ$
$B = 1.0$

**Fig 6(c)** The heat transfer coefficient against $C(0.0 \text{ to } 1.0)$ and Prandtl number $= 0.72$

for the equations (3-88).
Fig. 7(a) Dimensionless velocity distributions along u-direction for several values of 
$C(=0.25, 0.40, 0.55, 0.70, 0.80, 0.95)$ for the equations (3.89a).

$\frac{(\Delta T)_r}{T_r} = 0.6$
$Pr = 0.72$
$B = 1.0$
$\delta = 15^0$

Fig. 7(b) Dimensionless velocity distributions along v-direction for several values of 
$C(=0.25, 0.40, 0.55, 0.70, 0.80, 0.95)$ for the equations (3.89b).

$\frac{(\Delta T)_r}{T_r} = 0.6$
$Pr = 0.72$
$B = 1.0$
$\delta = 15^0$
Fig. 7(c) Dimensionless temperature distributions for several values of \( C = 0.25, 0.40, 0.55, 0.70, 0.80, 0.95 \) for the equations (3-89c).

\[
\frac{(\Delta T)}{T_r} = 0.6
\]
\( \Pr = 0.72 \)
\( \delta = 15^\circ \)
\( B = 1.0 \)

---

Fig. 8(a) Dimensionless velocity distributions along u-direction for several values of \( C = 0.3, 0.5, 0.7, 0.9, 1.0 \) and prandtl number \( \Pr = 7.0 \) for the equations (3-89a).

\[
\frac{(\Delta T)}{T'} = 0.4
\]
\( \Pr = 7.0 \)
\( \delta = 15^\circ \)
\( B = 1.0 \)
Fig. 8(b) Dimensionless velocity distributions along \( \Phi \)-direction for different values of \( C(0.3, 0.5, 0.7, 0.9, 1.0) \) and Prandtl number \( \text{Pr} = 7.0 \) for the equations (3.89b).

Fig. 8(c) Dimensionless temperature distributions for several values of \( C(0.3, 0.5, 0.7, 0.9, 1.0) \) and Prandtl number \( \text{Pr} = 7.0 \) for the equations (3.89c).
Fig. 9(a). Dimensionless velocity distributions along u-direction for several values of
\( \gamma = 0.35, 0.55, 0.65, 0.85, 1.00 \) for the equation (3.89a).

Fig. 9(b). Dimensionless velocity distributions along v-direction for several values of
\( \gamma = 0.35, 0.55, 0.65, 0.85, 1.00 \) for the equation (3.89b).
Fig. 9(c) Dimensionless temperature distributions for several values of $C(0.35, 0.55, 0.65, 0.85, 1.00)$ for the equation (3-89c).

Fig. 10(a) Dimensionless velocity distributions along $\psi$-direction for several values of $C(0.2, 0.4, 0.55, 0.75, 0.8, 1.00)$ and Prandtl number = 7.0 for the equation (3-89a) where $\delta = 15^\circ$.
Fig. 10(b) Dimensionless velocity distributions along y-direction for several values of $C(0.2, 0.4, 0.55, 0.75, 0.90, 1.00)$ and Prandtl number=7.0 for the equation (3.89b) where $\delta=15^\circ$.

Fig. 10(c) Dimensionless temperature distributions for several values of $C(0.2, 0.4, 0.55, 0.75, 0.90, 1.00)$ and Prandtl number=7.0 for the equation (3.89c) where $\delta=15^\circ$. 

\[
\frac{(\Delta T)_{r}}{T_{r}} = 0.4
\]

\[
\begin{align*}
C & = 0.20 \\
 & = 0.40 \\
 & = 0.55 \\
 & = 0.75 \\
 & = 0.90 \\
 & = 1.00
\end{align*}
\]

\[
\begin{align*}
\theta & = 15^\circ \\
Pr & = 7.0 \\
B & = 1.0
\end{align*}
\]
Fig. 11(a) Skin friction factor along u-direction against $C(0.1 \text{ to } 1.0)$ and Prandtl number $= 7.0$ for the equation (3-89a).

Fig. 11(b) Skin friction factor along v-direction against $C(0.1 \text{ to } 1.0)$ and Prandtl number $= 7.0$ for the equation (3-89b).
Fig. 11(c) Heat transfer coefficients against \( \theta(0) \) and Prandtl number=7.0 for the equation (3-89c).

Fig. 12(a) Skin friction factor along u-direction against \( \theta(0) \) and Prandtl number=0.72 for the equation (3-89a).
Fig. 12(b) Skin friction factor along v-direction against $C$ (0.0 to 1.0) and Prandtl = 0.72 for the equation (3.89b).

Fig. 12(c) Heat transfer coefficient against $C$ (0.0 to 1.0) and Prandtl number = 0.72 for the equation (3.89c).
Table-2

<table>
<thead>
<tr>
<th>C</th>
<th>$F_0''(0)$</th>
<th>$S_0'(0)$</th>
<th>$-\theta_0'(0)$</th>
</tr>
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<tbody>
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</tr>
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</table>

For Pr=7.0, B=1.0 and $\delta=15^0$.

Table-3

<table>
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<tr>
<th>C</th>
<th>$F_1''(0)$</th>
<th>$S_1'(0)$</th>
<th>$-\theta_1'(0)$</th>
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<tbody>
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For $B=1.0$, Pr=7.0, $\delta=15^0$ and using reasonable values of other controlling parameters.
Table-4

<table>
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<th>$F_0''(0)$</th>
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For $Pr=0.72$, $B=1.0$ and $\delta=15^\circ$.

Table-5

<table>
<thead>
<tr>
<th>$C$</th>
<th>$F_1''(0)$</th>
<th>$S_1''(0)$</th>
<th>$-\theta_1'(0)$</th>
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<td>1.39738</td>
<td>1.15242</td>
</tr>
<tr>
<td>1.0</td>
<td>0.37598</td>
<td>1.37024</td>
<td>1.17577</td>
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</tbody>
</table>

For $B=1.0$, $Pr=0.72$, $\delta=15^\circ$ and using reasonable values of other controlling parameters.
The Numerical Solution:-

The two point boundary value problems described by the two set of equations with their boundary conditions, [i.e, one set is (3-88a) with boundary conditions (3-88b) and the another set is (3-89a,3-89b,3-89c) with the boundary conditions (3-89d)] are non-linear and coupled. It is difficult to solve them analytically. Hence we adopt a procedure to get the solutions numerically. By applying Runge-Kutta Merson method, in collaboration with Runge-Kutta shooting method and Swigert iteration technique we solve the equations numerically.

The higher order non-linear differential equations are primarily transformed into a set of first order initial value problem with their respective boundary conditions by using the straightforward expansion. Runge-Kutta shooting method, Runge-Kutta-Merson method and Swigert iteration technique (i.e guessing the missing values) are considered to solve the zeroth order equations (3-88) and the first order equations (3-89) with their usual boundary conditions. Our aim is to find the skin friction factors \( (F''_0(0), S''_0(0)) \) and heat transfer coefficient \( (-\theta'_0(0)) \) for different values of controlling parameters \( (\text{Pr}, C) \) considering \( \delta=15^\circ \) and \( B=1.0 \) for the equations (3-88). Also our aim is to find the 1st order correction for skin friction factors \( (F''_1(0), S''_1(0)) \) and heat transfer coefficient \( (-\theta'_1(0)) \) for different values of controlling parameters \( (\text{Pr}, C, A, D, H, E) \) with \( \delta=15^\circ \) and \( B=1.0 \) for the equations (3-89). We consider here the common fluids air and water. \( C \) is also the additive characteristic length parameter.
For zeroth order and first order solutions with $\delta=90^0$ and $C=0$, the present study coincides with those of Ackroyd (1973). Again it yields the same result of Gebhart (1962) when $B=1.0$, $C=0$ and $\delta=90^0$.

Numerical results are obtained for $Pr = 0.72$ and 7.0 in the range $0 < C \leq 1.0$ with fixed values of controlling parameters ($B, A, D, E, H, (dT/T)_s$). Representative velocity and temperature profiles are shown in figures (1(a)-4(c)) for the zeroth order and figures (7(a)-10(c)) are shown for first order differential equations.

The zeroth order skin friction factors $\left( F_0''(0), S_0''(0) \right)$ and the heat transfer coefficient $(-\theta_0'(0))$ are shown in figures (5(a)-6(c)). Their tabulated values also shown in table-2 and table-4 for controlling parameter ($C, B, Pr$).

The 1st order skin friction factors $\left( F_1''(0), S_1''(0) \right)$ and the heat transfer coefficient $(-\theta_1'(0))$ are also displayed in figures 11(a) to 12(c) for the above fixed values of the controlling parameters and their tabulated values also shown in table-3 and table-5.
Result and discussions

We are about to discuss the nature of the flow processes arising from free convection flow for the various values of the controlling parameters ($\text{Pr}, C, B, A, D, E, H, (AT/T)$). Figures 1, 2, 3 and 4 represent the velocity and temperature profiles for the zeroth order transformed similarity equations. Figures 7, 8, 9 and 10 represent profiles for the first order equations. Figures 5, 6 and 11, 12 show the effects of skin friction factors and heat transfer coefficient for the zeroth order and 1st order respectively with the increase of $C$.

The displayed figures 1(a) and 1(b) show that the velocity components increase with the increase of $C$ and vice-versa. But it is remarkable here that both the components of velocity distributions do not decrease or increase in identical manner.

From figure 1(c), it is observed that temperature profiles show the insignificant behaviour for the $C$-variations. This indicates that heat transfer rate does not wholly depend on additive characteristic parameter (length) $C$.

Similar results (from figures 2(a) and 2(b)) are found to be applicable for case of water also ($\text{Pr} = 7.0$). The fixed values of controlling parameters $\delta$ and $B$ are 15° and 1 respectively.

In the case of temperature profiles (figure 2(c)) the additive characteristic parameter (length) $C$ shows its remarkable change for water ($\text{Pr} = 7.0$) than air ($\text{Pr} = 0.72$). Temperature decreases gradually with the increasing values of $C$ when other controlling parameter ($\delta = 15^\circ$ and $B = 1.0$) remain fixed.
Figures 3(a) and 3(b) display the dimensionless velocity distributions along $u$-direction and $v$-direction for several values of $C(0.25, 0.45, 0.65, 0.75, 0.95)$ and fixed values of controlling parameters ($Pr, B, \delta$). Physical meanings of figures 3(a) and 3(b) are same as 1(a) and 1(b). Figure 3(c) is compared with 1(c) and also display the same rule.

From figures 4(a) and 4(b) we see that the velocity profiles along the $u$-direction and $v$-direction decreases when $C$ increases with the controlling parameters $Pr, B$ and $\delta$. These figures also are similar as 2(a) and 2(b). From figure 4(c) we see that the temperature profile is large near the surface of the plate and away from the plate it decreases and finally it becomes asymptotic value. Here also we see that temperature profile decreases with the increases of the $C$.

Dimensionless velocity distributions along $u$-direction and $v$-direction for several values of $C$ and the fixed values of the controlling parameters ($A, B, D, E, H, S, (\Delta T/T)_x$) for the 1st order perturbation are shown in figures 7(a), 8(a), 9(a), 10(a) and 7(b), 8(b), 9(b), 10(b) respectively. These figures obey the same rules as 1(a), 2(a) and 1(b), 2(b).

From figures 7(c), 8(c), 9(c) and 10(c) we see that the effect of the increase of additive characteristic length parameter $C$ leads to decrease the temperature field where $Pr(=0.72, 7.0)$ and the fixed values of $B=1.0, \delta=15^\circ$ and also using other fixed controlling parameters.

The variation of skin frictions ($F_0^\prime\prime(0), S_0^\prime\prime(0); F_1^\prime\prime(0), S_1^\prime\prime(0)$) are displayed in figures 5(a), 5(b) and 11(a), 11(b) respectively, heat transfer coefficients ($-\theta_0^\prime(0)$ and $-\theta_1^\prime(0)$) are shown in figures 5(c) and 11(c) respectively for the
variation of the additive characteristic length $C$ and $Pr = 7.0$. Numerical table-2 and table-3 display the effect of skin friction factors and heat transfer coefficient with variation of $C$ and $Pr = 7.0$, $\delta = 15^\circ$, $B=1.0$ and also using other fixed controlling parameters.

Again the variation of skin frictions $\left( F''_0 (0), S''_0 (0); F'_1 (0), S'_1 (0) \right)$ are displayed in figures 6(a), 6(b) and 12(a), 12(b) respectively, heat transfer coefficients $\left( -\theta_0' (0) \text{ and } -\theta_1' (0) \right)$ are shown in figures 6(c) and 12(c) respectively for the variation of the additive characteristic length parameter $C$ and $Pr = 0.72$. Numerical table-4 and table-5 display the effects of skin friction factors and heat transfer coefficient with variation of the additive characteristic length parameter $C$ and other fixed controlling parameters.
Conclusion

(1) For $\delta = 45^\circ$ and $C = 1.0$, the square side flat surface stands with its diagonal vertical. Here in this natural flow configuration, both $F$ and $S$ becomes identical and also u-momentum, v-momentum equations coincide with each other.

(2) Here $\delta = 15^\circ$ i.e the one of the edges in y-direction makes angle $\delta$ with the vertical, the v-velocity component is more vertical that u-velocity. For this reason $S'$ takes higher value than that of $F'$ for the same values of $C$.

(3) For zeroth order and first order similarity solutions with $\delta = 90^\circ$ and $C = 0$, the present study coincides with those of Ackroyd (1973). Again it yields the same result of Gebhart (1962) when $\theta=1.0$, $C=0$ and $\delta = 90^\circ$.

(4) We find the skin friction factors and heat transfer coefficient for zeroth order and first order are slightly differs from those of Ackroyd (1973) and Gebhart (1962) for the influence of the additive characteristic length $C$ and acute angle $\delta = 15^\circ$. 
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Nomenclature

$A, B, C, D, E, H$  
constants

$a, b, c$  
constants

$C_p$  
specific heat at constant pressure

$C_r$  
specific heat at constant pressure for the reference point.

$F, S$  
dimensionless stream functions

$\bar{G}$  
gravitational force

$\bar{F}$  
body force vector'

$L$  
characteristic length'

$p$  
pressure

$p_r$  
pressure for the reference point

$P_r$  
Prandtl number

$q_w$  
heat flux

$R_e$  
Reynolds number

$T$  
temperature of fluid

$T_e$  
temperature of the fluid for the exterior point

$T_i$  
temperature of the fluid for the reference point

$T_o$  
temperature of ambient fluid

$T_w$  
Surface temperature

$S_e$  
constant entropy for the exterior point

$S_i$  
constant entropy for the reference point

$u, v, w$  
velocity components in the boundary layer

$U_e$  
characteristic velocity generated by buoyancy effects

$X, Y$  
coordinates along the edges of surface

$Z$  
coordinate normal to the surface

$x, y$  
transformed coordinates along the edges of surface

$z$  
transformed coordinate normal to the surface
Greek letters

\(\alpha\) constant

\(\beta_T\) the coefficient of volumetric thermal expansion

\(\beta_r\) the coefficient of volumetric thermal expansion for the reference point

\(\theta\) dimensionless temperature function

\(\Theta\) dimensionless temperature function for special case

\(\theta_s\) dimensionless temperature function at the surface

\(\delta\) boundary layer thickness

\(\delta_t\) thermal boundary layer thickness

\(\psi, \chi\) stream functions

\(\phi\) dissipation function

\(\Phi\) transformation similarity variable

\(\nu\) the kinematic coefficient of viscosity

\(\rho\) the density of ambient fluid

\(\rho_r\) the density of ambient fluid for the reference point

\(\rho_e\) the density of ambient fluid for the exterior point

\(\mu\) coefficient of viscosity

\(\gamma\) the ratio of the specific heat at constant pressure and const. volume

\(\kappa\) the coefficient of thermal diffusivity

\(K\) compressibility constant coefficient

\(\tau_\infty\) nondimensional skin friction

\(\xi, \eta, \zeta\) scaled coordinate defined in equations

\(\gamma\) the square root of the local boundary layer thickness

\(\omega\) angular velocity