Compressible Boundary Layer for a Developable Surface in Mixed Convection Flow.

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By

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Declaration

None of the materials contained in this thesis will be submitted in support of any other degree or diploma at any other university or institution other than publications.

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Abstract

Mixed convective laminar compressible boundary layer flow about a vertical developable surface is studied theoretically. Modified forms of Howarth-Drodnistyn transformations are applied to avoid nsual Boussinesq approximation in order to have the transformed equations in almost incompressible forms. The additional parameters owed to surface geometry and compressibility show many interesting effects on velocity and temperature profiles. This suggests that the present analysis may provide accurate predictions for boundary layer characteristics.

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CONTENTS

Chapter 1

Introduction

The approximations and simplifying assumptions involved in boundary layer equations for two dimensional incompressible flow have been thoroughly discussed in many text books. **Cohen and Reshotko** (1956) derived similar solutions for two dimensional compressible boundary layer and **Stewartson** (1964) gave an account of certain aspects of theory of boundary layer in compressible fluids. In some cases analytical and in most of the cases numerical solutions were displayed to predict essential flow parameters. In the present theoretical study the surface geometry and the curving of the external flow play vital role, for which the mathematical and physical analysis of the three dimensional boundary layer becomes considerably tough. Attention is concentrated mainly to the situation where the forced and free convection act simultaneously and both the convections are of comparable order.

Heat transfer, a branch of applied thermodynamics, estimates the rate at which heat is transferred across the boundaries subjected to specific temperature differences and the temperature distribution of the system during the process whereas classical thermodynamics deals the amount of heat transferred during the process. Heat transfer is concerned with temperature difference and we live in a world full of such difference, owing to either natural or artificial causes.

We know that externally induced flow is provided and the flow arises naturally simply due to the effect of a density difference, resulting from a temperature difference, in a body force field, such as gravitational field, the process is termed as "natural" or "free" convection.

If the motion is maintained by an externally applied pressure difference as in the case of flow through a pipe, we use the term forced convection .

Developable flow surface: Developable flow surface defines as the envelope of a one parameter family of planes: a surface that can be developed or rolled out on a plane without stretching or shrinking: i.e. a surface for which the total curvature vanishes identically. That is for a developable surface following equations,

$$K = \frac{1}{h_1 h_2}\left[\frac{\delta}{\delta\xi}(h_2 k_2) + \frac{\delta}{\delta\eta}(h_1 k_1)\right] = 0 \qquad (A)$$

where, $k_2 = \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \eta}$; for ξ = constant,

and

$k_1 = \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \xi}$; for η = constant.

$$\frac{\partial}{\partial\xi}\left[\frac{\partial}{\partial\eta}\{\log h_1\}\right] + \frac{\partial}{\partial\eta}\left[\frac{\partial}{\partial\xi}\{\log h_2\}\right] = 0$$

(K is the total curvature, $h_1(\xi,\eta)$ and $h_2(\xi,\eta)$ be the length functions, k_1 and k_2 are known as the geodesic curvatures for the co-ordinate lines.) hold good.

Stewartson (1949) showed that if it is assumed in two-dimensional flow that (a) the surface is thermally insulating, (b) viscosity varies as the absolute temperature and (c) the Prandtl number is unity, it is possible to transform the coordinates so that the boundary layer equation for a compressible fluid with a given main-stream velocity become identical with those for an incompressible fluid with a different main-stream velocity. In general, for three-dimensional flow it is not possible to find a transformation to correlate compressible and incompressible boundary layers in this way. However, there is one case which is amenable, namely the case of small cross-flow which has been fairly extensively studied for incompressible fluids. Here it is assumed that the velocity in the boundary layer normal to the external streamlines and certain of its derivatives are small. For the incompressible flow at a three-dimensional nodal point of attachment, **Howarth** (1951) has shown that the boundary layer equations yield similarity solution which are also exact solutions of the Navier-Stoke's equations. The external flow is assumed to be irrotational and given by $\{ax_1, bx_2, -(a+b)x_3\}$, where x_1, x_2, are cartesian co-ordinates of any point on the tangent plane at the stagnation point $x_1 = x_2 = 0$ and x_3 is measured along the normal at the stagnation point. Howarth discussed the properties of these solutions for $c = b/a = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1; the limiting values c=0 and 1 corresponding to the two-dimensional and axially symmetric stagnation point, respectively. Later **Davey** (1961) showed that similar solutions exist for $c \geq -1$. In Davey's terminology the similar solutions for $-1 \leq c < 0$ correspond to the flow near saddle-points of attachment and in some case be related to the flow in the vicinity of geometrical saddle-point on the surface.

Moore (1951), Moore and **Ostrach** (1956), **Illingworth** (1958), **Gribben** (1961) and **King** (1966) studied the unsteady laminar compressible boundary layer flow over two-dimensional bodies by momentum-integral or series expansion methods.

Lighthill (1954) investigated the response of an incompressible laminar boundary layer over an arbitrary cylinder to small fluctuations in the external stream. Low and high frequency solutions were obtained by a momentum-integral method.

Telionis and **Gupta** (1977) investigated the response of the compressible laminar boundary layer to small fluctuations in the outer flow under more general conditions for both two-dimensional and axisymmetric bodies. **Vimala** and **Nath** (1975) studied the above problem as well as a constantly accelerating flow problem for a cold wall and solved the governing equation numerically using an implicit finite-difference scheme.

Many natural convection processes encountered are not sufficiently vigorous to the result that includes viscous dissipation effect (i.e. a thermalization of energy through the mechanism of viscous stresses) which is a appreciable compared to other energy flows on the convection region. However, it is clear (**Gebhart,** 1962) that the natural convection flow fields of extreme size, or extremely low-temperatures, or in high gravity, the viscous generation of heat will affect the flow. In that case external boundary layer flow over a vertical plate surface was considered. The dissipation effect is non-similar for the most common boundary conditions and a dissipation parameter. $\in(x) = g\beta\ x/c_p$ arose in a perturbation analysis of the uniform temperature and the uniform heat flux surface conditions. The effect was calculated for a Prandtl number range from 10^{-2} to 10^4, and was found to increase over that range.

Other studies of this effect and the application of external two-dimensional flow results are discussed in **Gebhart** (1962). **Roy** (1968) found an asymptotic solutions for large Prandtl number for external flow on an isothermal surface, using a double boundary layer concept.

Cheng (1977) investigated the combined forced and free convection boundary layer flow along inclined surfaces embedded in porous media. It was found that when both the wall temperature distribution of the plate and the velocity parallel to the plate outside of the boundary layer vary according to the some power function of distance, ie 'x', then the similarity solutions exist. **Hassanien** (1997) studied also the combined forced and free convection in boundary layer.

Merkin and **Ingham** (1987) dealt with the case for a wall temperature prescribed as an inverse square root of the distance from the leading edge. In a related problem of similar solutions for mixed convection on a horizontal plate **Hossain and Afzal** (1988) showed that the solutions were dual with a turning point where the shear stress was still finite. This conclusion was also supported by **De Hoog et al.** (1984) and **Raju et al** (1984).

The problem of mixed convection boundary layer flow on a vertical plate has been discussed in some detail. by **Wilks** (1974), **Hunt** and **Wilks** (1980), **Gryzagoridus** (1975) and **Carey** and **Gebhart** (1982).

The problem of the mixed convection on a horizontal cylinder has been studied by **Merkin** (1977). The occurrence of dual solutions in mixed convection similarity solutions was shown by **Wilks** and **Bramley** (1980). They also considered the eigen value problem arising out of a linear stability analysis of these solutions. **Raju et al.** (1984) and **Schneider** and **Wasel** (1985) studied the mixed convection on a horizontal plate integrating the boundary layer equations by finite difference schemes.

Hassanien and **Gorla** (1990) analyzed the combined forced and free convection in micropolar boundary layer fluid flow over a horizontal cylinder. Also, **Gorla** (1995) studied the unsteady mixed convection in micropolar boundary layer flow on a vertical plate.

More recently Quiser (1998), studied mixed convection flow about the vertical developable surfaces with transpiration effects in the case of incompressible flow.

So far the author's knowledge goes no attempt has yet been made for compressible boundary layer for developable surface in mixed convection flow with a special attention to the similarity solution.

Chapter 2

Geometrical figures of the flow configuration.

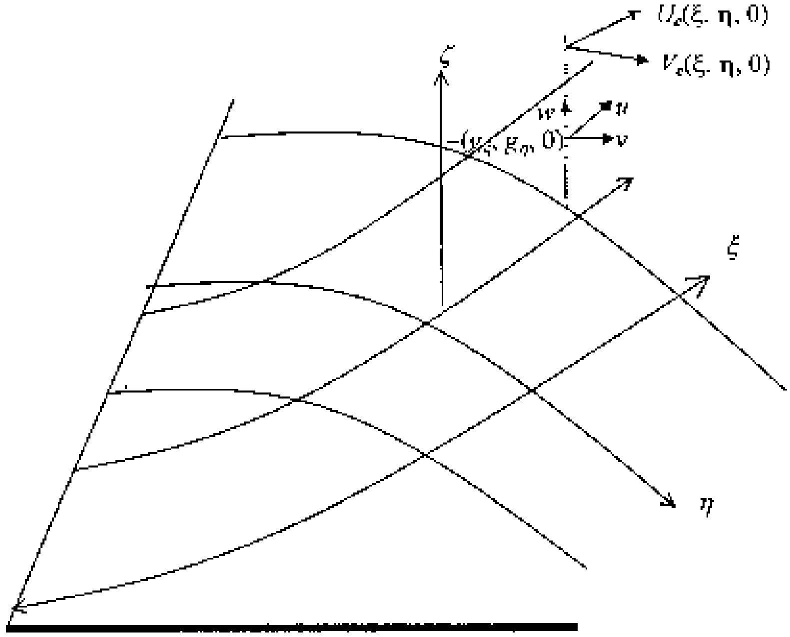


Fig (1)

A heated (or cooled) body with curvilinear surface and ζ-axis in upward directon is moving vertically in a compressible fluid. The physical model of this problem in orthogonal curvilinear co-ordinates is schematically shown in Fig. (1). The fluid flow near the body is caused by the combination of density difference (free convection) and external forcing velocity (forced convection). The boundary layer flow is thus influenced by the surface geometry as well as by the orthogonal curvilinear co-ordinates (h_1, h_2, h_3). For simplicity we consider here $h_3 = 1$. According to Prandtl boundary-layer concept, boundary layer occurs next to regularly curved surface; where the radius of curvature were large in comparison to the boundary-layer thickness. It has been suggested by **Kemp[1951]** that the flow in the regions where large or irregularly varying surface curvature called boundary region flow is distinguished from conventional boundary layer flow.

It is clear form the foregoing remarks that one must now re-evaluate the original simplifications of the boundary layer theory, and re-examine which terms may, and which may not, be neglected in writing the conservation laws.

For this reason we consider a developable surface satisfying curvature equation (A) associated with the differential are length of the ξ, η and co-ordinate system on the boundary surface by

$$(ds)^2 = h_1^2 d\xi^2 + h_2^2 d\eta^2 + d\zeta^2 \tag{B}$$

On the basis of dimensional analysis followed by Quiser (1998) a few terms in the boundary layer equation are ignored and the following forms of the equations are subject to our study onward for a compressible flow situation.

Equation of continuity

$$\frac{\partial}{\partial \xi}(\rho u h_2)+\frac{\partial}{\partial \eta}(\rho v h_1)+\frac{\partial}{\partial \zeta}(\rho w h_1 h_2)=0 \tag{1}$$

u-momentum equation

$$\rho\left[\frac{u}{h_1}\frac{\partial u}{\partial \xi}+\frac{v}{h_2}\frac{\partial u}{\partial \eta}+w\frac{\partial u}{\partial \zeta}+\frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta}-\frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right]=-\frac{1}{h_1}\frac{\partial P}{\partial \xi}+\frac{\rho g_\xi}{h_1}+\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial u}{\partial \zeta}\right) \tag{2}$$

v-momentum equation

$$\rho\left[\frac{u}{h_1}\frac{\partial v}{\partial \xi}+\frac{v}{h_2}\frac{\partial v}{\partial \eta}+w\frac{\partial v}{\partial \zeta}+\frac{vu}{h_1 h_2}\frac{\partial h_1}{\partial \eta}-\frac{u^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right]=-\frac{1}{h_2}\frac{\partial P}{\partial \eta}+\frac{\rho g_\eta}{h_2}+\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial v}{\partial \zeta}\right) \tag{3}$$

w-momentum equation

$$\frac{\partial P}{\partial \zeta}=0 \tag{4}$$

Energy equations

$$\rho C_p\left(\frac{u}{h_1}\frac{\partial T}{\partial \xi}+\frac{v}{h_2}\frac{\partial T}{\partial \eta}+w\frac{\partial T}{\partial \zeta}\right)=\frac{\partial}{\partial \zeta}\left(k\frac{\partial \theta}{\partial \zeta}\right)$$
$$+\left(\frac{u}{h_1}\frac{\partial P}{\partial \xi}+\frac{v}{h_2}\frac{\partial P}{\partial \eta}\right)+\mu\left\{\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial v}{\partial \zeta}\right)^2\right\} \tag{5}$$

Here u,v,w are boundary layer velocity components in $\xi-,\eta-$ and ζ-directions; h_1 h_2 are related to the body surface by the equation (B). ρ is the density, p is the pressure. We shall consider the body force to be caused by gravitational acceleration. The components of the gravitational acceleration are denoted here by g_ξ and g_η in ξ and η directions. μ is the viscosity, k is thermal conductivity. C_p is the specific heat at constant pressure and T is the temperature of the fluid.

The potential flow in the main stream outside the boundary layer is the function of (ξ, η). Hence external velocity components U_e , V_e and external temperature T_e be independent of ζ.

To eliminate pressure tenus in equations (2), (3) and (5), the velocity component out side the boundary layer are imposed. Imposing the boundary conditions $u \to U_e$; $v \to V_e$; $\rho \to \rho_e$; $T \to T_e$; $\frac{\partial}{\partial \zeta} \to 0$ we have the following equations :

$$\rho_e\left[\frac{U_e}{h_1}\frac{\partial U_e}{\partial \xi}+\frac{V_e}{h_2}\frac{\partial U_e}{\partial \eta}+\frac{U_e V_e}{h_1 h_2}\frac{\partial h_1}{\partial \eta}-\frac{V_e^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right]=-\frac{1}{h_1}\frac{\partial P}{\partial \xi}+\frac{\rho_e}{h_1}g_\xi \quad (6)$$

$$\rho_e\left[\frac{U_e}{h_1}\frac{\partial V_e}{\partial \xi}+\frac{V_e}{h_2}\frac{\partial V_e}{\partial \eta}+\frac{U_e V_e}{h_1 h_2}\frac{\partial h_1}{\partial \eta}-\frac{U_e^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right]=-\frac{1}{h_2}\frac{\partial P}{\partial \eta}+\frac{\rho_e}{h_2}g_\eta \quad (7)$$

And
$$\rho_e\left[\frac{U_e}{h_1}\frac{\partial T_e}{\partial \xi}+\frac{V_e}{h_2}\frac{\partial T_e}{\partial \eta}\right]=0 \quad (8)$$

$T = T_e$ (constant) is one of the solutions of (8). In view of (6), (7), (8) and $\frac{T-T_e}{T_w-T_e}=\theta(\xi,\eta,\zeta)$; $T_w - T_e = \Delta T$, the boundary layer equations with the elimination of pressure terms from (2) and (3) become

<u>Equation of continuity</u>

$$\frac{\partial}{\partial \xi}(\rho u h_2)+\frac{\partial}{\partial \eta}(\rho v h_2)+\frac{\partial}{\partial \zeta}(\rho w h_1 h_2)=0 \quad (9)$$

<u>u-momentum equation</u>

$$\rho\left[\frac{u}{h_1}\frac{\partial u}{\partial \xi}+\frac{v}{h_2}\frac{\partial u}{\partial \eta}+w\frac{\partial u}{\partial \zeta}+\frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta}-\frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right]=\frac{1}{h_1}g_\xi(\rho-\rho_e)$$

$$+\rho_e\left[\frac{U_e}{h_1}\frac{\partial U_e}{\partial \xi}+\frac{V_e}{h_2}\frac{\partial U_e}{\partial \eta}+\frac{U_e V_e}{h_1 h_2}\frac{\partial h_1}{\partial \eta}-\frac{V_e^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right]+\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial u}{\partial \zeta}\right) \quad (10)$$

v-momentum equation

$$\rho\left[\frac{u}{h_1}\frac{\partial v}{\partial \xi}+\frac{v}{h_2}\frac{\partial v}{\partial \eta}+w\frac{\partial v}{\partial \zeta}+\frac{vu}{h_1 h_2}\frac{\partial h_2}{\partial \eta}-\frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \xi}\right]=\frac{1}{h_2}g_\eta(\rho-\rho_e)$$

$$\rho_e\left[\frac{U_e}{h_1}\frac{\partial V_e}{\partial \xi}+\frac{V_e}{h_2}\frac{\partial V_e}{\partial \eta}+\frac{U_e V_e}{h_1 h_2}\frac{\partial h_1}{\partial \eta}-\frac{U_e^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right]+\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial u}{\partial \zeta}\right) \tag{11}$$

w-momentum equation

$$\frac{\partial P}{\partial \zeta}=0 \tag{12}$$

Energy equations

$$\rho C_p\left(\frac{u}{h_1}\frac{\partial T}{\partial \xi}+\frac{v}{h_2}\frac{\partial T}{\partial \eta}+w\frac{\partial T}{\partial \zeta}\right)-\left(\frac{u}{h_1}\frac{\partial P}{\partial \xi}+\frac{v}{h_2}\frac{\partial P}{\partial \eta}\right)$$

$$=\frac{\partial}{\partial \zeta}\left(k\frac{\partial \theta}{\partial \zeta}\right)+\mu\left\{\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial v}{\partial \zeta}\right)^2\right\} \tag{13}$$

We confine our attention to solve the above boundary layer equation governing the flow field around a vertically inclined developable surface. P-elimination is considered in the momentum equation only, but not in energy equation at present stage.

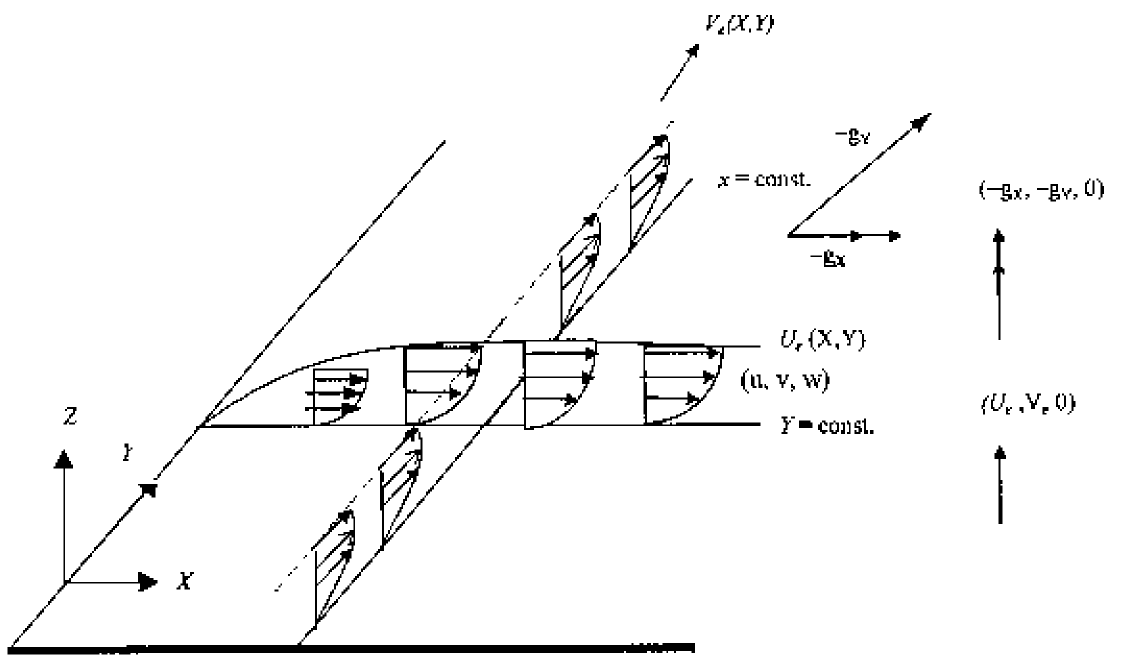


Fig 2

Chapter – 3

Boundary Layer Equations and Transformations

Let us now transform the boundary layer equations, (10), (11) and (13) from physical co-ordinates (ξ,η,ζ) to a new system of co-ordinates (X, Y, Φ). The appropriate transformation is based on the work initiated by **Illingwarth(58).** A modified following form of transformations, a bit different from those of Howarth-Dorodnitsyn are used.

$$\left.\begin{aligned} X &= \int_0^{\xi} h_1 d\xi \\ Y &= \int_0^{\eta} h_2 d\eta \\ &\text{and} \\ \Phi &= \frac{\sqrt{u_e + \alpha v_e}}{\sqrt{\rho_e \mu_e (X + CY)}} \int_0^{\zeta} \rho \, d_{\zeta} \end{aligned}\right\} \tag{14}$$

The transformed boundary layer flow configuration can be visualized in (X.Y.$\zeta = Z$) co-ordinates by [Fig.-2]. Here α is constant and C is additive length scale parameter, the suffix e refers to any convenient exterior conditions, Φ represents non-dimensional similarity variable. ρ_e, u_e , v_e and μ_e are function of X and Y. Guided by the idea of non-dimensionalization and similarity technique together we substitute u and v by the following equations.

$$\frac{u}{u_e} = F_{\Phi}(X,Y,\Phi) \quad \text{and} \quad \frac{v}{v_e} = S_{\Phi}(X,Y,\Phi) \tag{15}$$

The independent variable transformations given by equations (14) must be incorporated in terms of derivatives, because terms involving ξ and η in the original boundary layer equations, (10), (11) and (13) are derivative terms. From the chain rule, we have

$$\frac{\partial}{\partial \xi} = \frac{\partial X}{\partial \xi}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial \xi}\frac{\partial}{\partial Y} + \frac{\partial \Phi}{\partial \xi}\frac{\partial}{\partial \Phi} \qquad \text{Thus} \quad \left.\begin{aligned} \frac{\partial X}{\partial \xi} &= \frac{\partial}{\partial \xi}\int_0^{\xi} h_1 d\xi = h_1 \\ \text{and} \quad \frac{\partial Y}{\partial \eta} &= \frac{\partial}{\partial \eta}\int_0^{\eta} h_2 d\eta = h_2 \end{aligned}\right\} \tag{15.1}$$

N.B. u_e and v_e are treated onward instead of U_e and V_e.

$$\frac{\partial\Phi}{\partial X} = -\frac{1}{2}\frac{\Phi}{(X+CY)} + \frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_X \tag{15.2}$$

$$\frac{\partial\Phi}{\partial Y} = -\frac{c}{2}\frac{\Phi}{(X+CY)} + \frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y \tag{15.3}$$

$$\frac{\partial}{\partial\xi} = h_1\frac{\partial}{\partial X} + \left\{\int_0^\eta \frac{\partial h_2}{\partial\xi}d\eta\right\}\frac{\partial}{\partial Y} - \frac{1}{2}\frac{h_1\Phi}{(X+cY)}\frac{\partial}{\partial\Phi} + \frac{h_1\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_X\frac{\partial}{\partial\Phi} \tag{16}$$

$$\frac{\partial}{\partial\eta} = \frac{\partial X}{\partial\eta}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial\eta}\frac{\partial}{\partial Y} + \frac{\partial\Phi}{\partial\eta}\frac{\partial}{\partial\Phi}$$

$$\frac{\partial}{\partial\eta} = \int_0^\xi \frac{\partial h_1}{\partial\eta}d\xi\frac{\partial}{\partial X} + h_2\frac{\partial}{\partial Y} - \frac{c}{2}\frac{h_2\Phi}{(X+cY)}\frac{\partial}{\partial\Phi} + \frac{h_2\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y\frac{\partial}{\partial\Phi} \tag{17}$$

or, $\frac{\partial}{\partial\zeta} = \frac{\partial X}{\partial\zeta}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial\zeta}\frac{\partial}{\partial Y} + \frac{\partial\Phi}{\partial\zeta}\frac{\partial}{\partial\Phi}$

or, $\frac{\partial}{\partial\zeta} = 0.\frac{\partial}{\partial X} + 0.\frac{\partial}{\partial Y} + \frac{\partial\Phi}{\partial\zeta}\frac{\partial}{\partial\Phi}$

$$\therefore \frac{\partial}{\partial\zeta} = \frac{\partial\Phi}{\partial\zeta}\frac{\partial}{\partial\Phi} \tag{18}$$

$$\frac{\partial\Phi}{\partial\xi} = \frac{\partial\Phi}{\partial X}\frac{\partial X}{\partial\xi} = -\frac{1}{2}\frac{h_1\Phi}{(X+CY)} + \frac{h_1\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_X \tag{19}$$

$$\frac{\partial\Phi}{\partial\eta} = \frac{\partial\Phi}{\partial X}\frac{\partial Y}{\partial\eta} = -\frac{C}{2}\frac{h_2\Phi}{(X+CY)} + \frac{h_2\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y \tag{20}$$

and $\frac{\partial\Phi}{\partial\zeta} = \frac{\sqrt{u_e+\alpha v_e}}{\sqrt{\rho_e\mu_e(X+CY)}}\rho$

$$\text{or, } \frac{\partial\Phi}{\partial\zeta} = \frac{1}{k_0}\rho \tag{21}$$

where,

$$k_0(X,Y) = \sqrt{\frac{\rho_e\mu_e(X+cY)}{u_e+\alpha v_e}} = N(\Phi)\sqrt{\frac{\rho_0\mu_0(X+cY)}{u_e+\alpha v_e}} \tag{21a}$$

where $N(\Phi) = \sqrt{\frac{\rho_e\mu_e}{\rho_0\mu_0}}$.

The laminar boundary layer equations and the energy equation in combined convection flow are governed by the following equations, where U_e and V_e are replaced by u_e and v_e respectively.

Continuity equations

$$\frac{\partial}{\partial \xi}(h_2 \rho u)+\frac{\partial}{\partial \eta}(h_1 \rho v)+\frac{\partial}{\partial \zeta}(h_1 h_2 w \rho)=0 \tag{22}$$

u-mementum equation along the ξ-direction:

$$\frac{u}{h_1}\frac{\partial u}{\partial \xi}+\frac{v}{h_2}\frac{\partial u}{\partial \eta}+w\frac{\partial u}{\partial \zeta}+\frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta}-\frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}=\frac{1}{h_1}g_\xi\left(\frac{\rho-\rho_e}{\rho}\right)$$

$$+\frac{\rho_e}{\rho}\left\{\frac{u_e}{h_1}\frac{\partial u_e}{\partial \xi}+\frac{v_e}{h_2}\frac{\partial u_e}{\partial \eta}+\frac{u_e v_e}{h_1 h_2}\frac{\partial h_1}{\partial \eta}-\frac{v_e^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right\}+\frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial u}{\partial \zeta}\right) \tag{23}$$

v-mementum equation along the η-direction :

$$\frac{u}{h_1}\frac{\partial v}{\partial \xi}+\frac{v}{h_2}\frac{\partial v}{\partial \eta}+w\frac{\partial v}{\partial \zeta}+\frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi}-\frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta}=\frac{1}{h_2}g_\xi\left(\frac{\rho-\rho_e}{\rho}\right)$$

$$+\frac{\rho_e}{\rho}\left\{\frac{u_e}{h_1}\frac{\partial v_e}{\partial \xi}+\frac{v_e}{h_2}\frac{\partial v_e}{\partial \eta}+\frac{u_e v_e}{h_1 h_2}\frac{\partial h_2}{\partial \xi}-\frac{u_e^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta}\right\}+\frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial v}{\partial \zeta}\right) \tag{24}$$

and w-momentum equation along the ζ direction is zero so that the have

$$\frac{\partial P}{\partial \zeta}=0 \tag{25}$$

Energy equation :

$$\rho C_p\left(\frac{u}{h_1}\frac{\partial T}{\partial \xi}+\frac{v}{h_2}\frac{\partial T}{\partial \eta}+\frac{w}{h_3}\frac{\partial T}{\partial \zeta}\right)-\left(\frac{u}{h_1}\frac{\partial P}{\partial \xi}+\frac{v}{h_2}\frac{\partial P}{\partial \eta}\right)=\frac{\partial}{\partial \zeta}\left(\frac{k}{h_3}\frac{\partial T}{\partial \zeta}\right)+\Phi \tag{26}$$

The terms $g_\xi\left(\frac{\rho-\rho_e}{\rho}\right)$ and $g_\eta\left(\frac{\rho-\rho_e}{\rho}\right)$ represent the body force components exerted on fluid particle, where $\Phi=\mu\left\{\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial v}{\partial \zeta}\right)^2\right\}$ is known as the viscous dissipation function. Φ represents that part of the viscous work necessary for the correct balance of energy in this particular form of the energy equation. The term $\left(\frac{u}{h_1}\frac{\partial P}{\partial \xi}+\frac{v}{h_2}\frac{\partial P}{\partial \eta}\right)$ represents the corresponding part of the pressure work. At very slow motion of fluid flow where natural convection also exists, the heat due to natural convection and compression must be taken into account.

In general the pressure term in the energy equation is of order Eckert number, so it is neglected. We want to retain these in our flow analysis, since for slow flow E_c/T_r is of same order. Hence the elimination of pressure term like momentum equations holds good.

Let two stream functions $\psi(\xi,\eta,\zeta)$ and $\chi(\xi,\eta,\zeta)$ be defined as the mass flow components with in the boundary layer for the case of compressible flow. To satisfy the equation of continuity,

$$\frac{\partial}{\partial \xi}(h_2 \rho u)+\frac{\partial}{\partial \eta}(h_1 \rho v)+\frac{\partial}{\partial \zeta}(h_1 h_2 w \rho)=0 \tag{27}$$

we may introduce the component of the mass flow relations in the following way :-

Let $\psi_\zeta=(h_2\rho u)$ and $\chi_\zeta=(h_1\rho v)$. Hence

$$\psi=\int_0^\zeta (h_2\rho u)d\zeta=\int_0^\Phi h_2\rho\, u_e F_\Phi(X,Y,\Phi)d\Phi\frac{\partial \zeta}{\partial \Phi}$$

$$=h_2\rho u_e F(X,Y,\Phi)\frac{\sqrt{\rho_e\mu_e(X+CY)}}{\sqrt{u_e+\alpha v_e}}\frac{1}{\rho} \qquad \text{[using the equation (15)]}$$

$$\therefore \quad \psi=\frac{\sqrt{\rho_e\mu_e(Y+CY)}}{\sqrt{u_e+\alpha v_e}}(u_e h_2)F(X,Y,\Phi)$$

$$\therefore \quad \psi=k_0(X,Y)\,u_e h_2 F(X,Y,\Phi) \tag{28}$$

Similarly $\chi(X,Y,\Phi)$ is found to be

$$\therefore \quad \chi(X,Y,\Phi)=k_0(X,Y)u_e h_1 S(X,Y,\Phi) \tag{29}$$

where $k_0(X,Y)=\sqrt{\dfrac{\rho_e\mu_e(X+cY)}{u_e+\alpha v_e}}=N(\Phi)\sqrt{\dfrac{\rho_0\mu_0(X+cY)}{u_e+\alpha v_e}}$. Now,

$$\frac{\partial\psi}{\partial\xi}=\psi_\xi=\left[h_1\frac{\partial}{\partial x}+\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\frac{\partial}{\partial Y}-\frac{1}{2}\frac{h_1\Phi}{(X+cY)}\frac{\partial}{\partial\Phi}+\frac{h_1\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_X\frac{\partial}{\partial\Phi}\right](k_0\,u_e\,h_2\,F(X,Y,\Phi))$$

$$\frac{\partial\psi}{\partial\xi}=\psi_\xi=h_1\frac{\partial}{\partial X}(k_0u_eh_2F(X,Y,\Phi))+\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial}{\partial Y}(k_0u_eh_2F(X,Y,\Phi))$$

$$-\frac{1}{2}\frac{h_1\Phi}{(X+cY)}\frac{\partial}{\partial\Phi}(k_0u_eh_2F(X,Y,\Phi))+\frac{h_1\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_X\frac{\partial}{\partial\Phi}(k_0u_eh_2F(X,Y,\Phi))$$

$$=h_1\frac{\partial}{\partial X}(k_0u_eh_2F(X,Y,\Phi))+\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial}{\partial Y}(k_0u_eh_2F(X,Y,\Phi))-\frac{1}{2}\frac{h_1\Phi}{(X+cY)}\frac{\partial}{\partial\Phi}(k_0\,u_e\,h_2\,F(X,Y,\Phi))$$

$$+\frac{h_1\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_X\frac{\partial}{\partial\Phi}(k_0\,u_e\,h_2\,F(X,Y,\Phi))=h_1\frac{\partial}{\partial X}(k_0u_e\,h_2\,F(X,Y,\Phi))$$

$$+\int_0^{\eta}\frac{\partial h_2}{\partial \xi}d\eta\frac{\partial}{\partial Y}\left(k_0u_e h_2 F(X,Y,\Phi)\right)-\frac{1}{2}\frac{h_1\Phi}{(X+cY)}\left(k_0 u_e h_2 F_{\Phi}(X,Y,\Phi)\right)$$

$$+\frac{h_1\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_X\left(k_0 u_e h_2 F_{\Phi}(X,Y,\Phi)\right)$$

$$\therefore \Psi_\xi=\left[(h_1h_2)k_0u_e\frac{\partial F}{\partial X}+(h_1h_2)k_0F(X,Y,\Phi)\frac{\partial u_e}{\partial X}+k_0h_1u_eF(X,Y,\Phi)\frac{\partial h_2}{\partial X}+(h_1h_2)u_eF(X,Y,\Phi)(k_0)_X\right.$$

$$+\left(\int_0^{\eta}\frac{\partial h_2}{\partial \xi}d\eta\right)\left\{k_0h_2u_e\frac{\partial F}{\partial Y}+k_0h_2F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}+k_0u_eF(X,Y,\Phi)\frac{\partial h_2}{\partial Y}+h_2u_eF(X,Y,\Phi)(k_0)_Y\right\}$$

$$\left.-(h_1h_2)\frac{\Phi}{2(X+cY)}k_0u_eF(X,Y,\Phi)+(h_1h_2)\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_X k_0u_eF_{\Phi}(X,Y,\Phi)\right] \tag{30}$$

$$\frac{\partial\psi}{\partial\eta}=\psi_\eta=\left[\int_0^{\xi}\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial}{\partial X}+h_2\frac{\partial}{\partial Y}-\frac{c}{2}\frac{h_2\Phi}{(X+cY)}\frac{\partial}{\partial\Phi}+\frac{h_2\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y\frac{\partial}{\partial\Phi}\right]\left(k_0u_e h_1 S(X,Y,\Phi)\right)$$

$$\chi_\eta=\left[\left(\int_0^{\xi}\frac{\partial h_1}{\partial\eta}d\xi\right)\frac{\partial}{\partial X}+h_2\frac{\partial}{\partial Y}-\frac{c}{2}\frac{h_2\Phi}{(X+cY)}\frac{\partial}{\partial\Phi}+\frac{h_2\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y\frac{\partial}{\partial\Phi}\right]k_0h_1v_eS(X,Y,\Phi)$$

$$\chi_\eta=\left(\int_0^{\xi}\frac{\partial h_1}{\partial\eta}d\xi\right)\left\{k_0h_1v_e\frac{\partial S}{\partial X}+k_0v_eS(X,Y,\Phi)\frac{\partial h_1}{\partial X}+k_0h_1S(X,Y,\Phi)\frac{\partial v_e}{\partial X}+v_eh_1S(X,Y,\Phi)(k_0)_X\right\}$$

$$+(h_1h_2)k_0v_e\frac{\partial S}{\partial Y}+(h_1h_2)k_0S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}+h_2k_0S(X,Y,\Phi)v_e\frac{\partial h_1}{\partial Y}+(h_1h_2)v_eS(X,Y,\Phi)(k_0)_Y$$

$$-(h_1h_2)\frac{c\Phi}{2(X+cY)}k_0v_eS_{\Phi}(X,Y,\Phi)+(h_1h_2)\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y k_0v_eS_{\Phi}(X,Y,\Phi) \tag{31}$$

But we have,

$$w=-\frac{1}{\rho h_1h_2}\left(\psi_\xi+\chi_\eta\right) \tag{32}$$

Substituting ψ_ξ and χ_η from the (30) and (31) in to the equation (32), we get

$$\therefore w=-\frac{1}{\rho h_1h_2}\left[(h_1h_2)k_0u_e\frac{\partial F}{\partial X}+(h_1h_2)k_0F(X,Y,\Phi)\frac{\partial u_e}{\partial X}+k_0h_1u_eF(X,Y,\Phi)\frac{\partial h_2}{\partial X}+(h_1h_2)u_eF(X,Y,\Phi)(k_0)_X\right.$$

$$+\left(\int_0^{\eta}\frac{\partial h_2}{\partial\xi}d\eta\right)\left\{k_0h_2u_e\frac{\partial F}{\partial Y}+k_0h_2F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}+k_0u_eF(X,Y,\Phi)\frac{\partial h_2}{\partial Y}+h_2u_eF(X,Y,\Phi)(k_0)_Y\right\}_Y$$

$$-(h_1h_2)\frac{\Phi}{2(X+cY)}k_0u_eF(X,Y,\Phi)+(h_1h_2)\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y k_0u_eF_\Phi(X,Y,\Phi)$$

$$+\left(\int_0^\zeta\frac{\partial h_1}{\partial\eta}d\xi\right)\left\{k_0h_1v_e\frac{\partial S}{\partial X}+k_0v_eS(X,Y,\Phi)\frac{\partial h_1}{\partial X}+k_0h_1S(X,Y,\Phi)\frac{\partial v_e}{\partial X}+v_eh_1S(X,Y,\Phi)(k_0)_X\right\}$$

$$+(h_1h_2)k_0v_e\frac{\partial S}{\partial Y}+(h_1h_2)k_0S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}+h_2k_0S(X,Y,\Phi)v_e\frac{\partial h_1}{\partial Y}+(h_1h_2)v_eS(X,Y,\Phi)(k_0)_Y$$

$$\left.-(h_1h_2)\frac{c\Phi}{2(X+cY)}k_0v_eS_\Phi(X,Y,\Phi)+(h_1h_2)\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y k_0v_eS_\Phi(X,Y,\Phi)\right] \quad (33)$$

or, $$w=-\frac{1}{\rho}\left[k_0u_e\frac{\partial F}{\partial X}+k_0F(X,Y,\Phi)\frac{\partial u_e}{\partial X}+\frac{k_0}{h_2}u_eF(X,Y,\Phi)\frac{\partial h_2}{\partial X}+u_eF(X,Y,\Phi)(k_0)_X+\left(\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\right)*\right.$$

$$\left\{\frac{k_0}{h_1}u_e\frac{\partial F}{\partial Y}+\frac{k_0}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}+\frac{k_0}{h_1h_2}u_eF(X,Y,\Phi)\frac{\partial h_2}{\partial Y}+\frac{1}{h_1}u_eF(X,Y,\Phi)(k_0)_Y\right\}$$

$$-\frac{\Phi}{2(X+cY)}k_0u_eF_\Phi(X,Y,\Phi)+\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y k_0u_eF_\Phi(X,Y,\Phi)+\left(\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\right)\left\{\frac{k_0}{h_2}v_e\frac{\partial S}{\partial X}\right.$$

$$\left.+\frac{k_0}{h_1h_2}v_eS(X,Y,\Phi)\frac{\partial h_1}{\partial X}+\frac{k_0}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}+\frac{1}{h_2}v_eS(X,Y,\Phi)(k_0)_X\right\}+k_0v_e\frac{\partial S}{\partial Y}+k_0S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}$$

$$+\frac{k_0}{h_1}S(X,Y,\Phi)v_e\frac{\partial h_1}{\partial Y}+v_eS(X,Y,\Phi)(k_0)_Y-\frac{c\Phi}{2(X+cY)}k_0v_eS_\Phi(X,Y,\Phi)$$

$$+\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y k_0v_eS_\Phi(X,Y,\Phi)$$

$$\therefore\quad w\frac{\partial}{\partial\zeta}=-\frac{1}{\rho}\left[k_0u_e\frac{\partial F}{\partial X}+k_0F(X,Y,\Phi)\frac{\partial u_e}{\partial X}+\frac{k_0}{h_2}u_eF(X,Y,\Phi)\frac{\partial h_2}{\partial X}+u_eF(X,Y,\Phi)(k_0)_X+\left(\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\right)*\right.$$

$$\left\{\frac{k_0}{h_2}u_e\frac{\partial F}{\partial Y}+\frac{k_0}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}+\frac{k_0}{h_1h_2}u_eF(X,Y,\Phi)\frac{\partial h_2}{\partial Y}+\frac{1}{h_2}u_eF(X,Y,\Phi)(k_0)_Y\right\}$$

$$-\frac{\Phi}{2(X+cY)}k_0u_eF_\Phi(X,Y,\Phi)+\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y k_0u_eF_\Phi(X,Y,\Phi)$$

$$+\left(\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\right)\left\{\frac{k_0}{h_2}v_e\frac{\partial S}{\partial X}+\frac{k_0}{h_1h_2}v_eS(X,Y,\Phi)\frac{\partial h_1}{\partial X}+\frac{k_0}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}+\frac{1}{h_2}v_eS(X,Y,\Phi)(k_0)_Y\right\}$$

$$+k_0v_e\frac{\partial S}{\partial Y}+k_0S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}+\frac{k_0}{h_1}S(X,Y,\Phi)v_e\frac{\partial h_1}{\partial Y}+v_eS(X,Y,\Phi)(k_0)_Y$$

$$-\frac{c\Phi}{2(X+cY)}k_0 v_e S_\Phi(X,Y,\Phi)+\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e \mu_e}\right)\right\}_Y k_0 v_e S_\Phi(X,Y,\Phi)\Bigg]\frac{1}{k_0}\rho\frac{\partial}{\partial\Phi}$$

$$\Rightarrow w\frac{\partial}{\partial\zeta}=-\Bigg[u_e\frac{\partial F}{\partial X}+F(X,Y,\Phi)\frac{\partial u_e}{\partial X}+\frac{k_0}{h_2}u_e F(X,Y,\Phi)\frac{\partial h_2}{\partial X}+u_e F(X,Y,\Phi)\frac{(k_0)_X}{k_0}+\left(\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\right)*$$

$$\left\{\frac{1}{h_1}u_e\frac{\partial F}{\partial Y}+\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}+\frac{1}{h_1h_2}u_e F(X,Y,\Phi)\frac{\partial h_2}{\partial Y}+\frac{1}{h_1}u_e F(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\right\}$$

$$-\frac{\Phi}{2(X+cY)}u_e F_\Phi(X,Y,\Phi)+\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e \mu_e}\right)\right\}_X u_e F_\Phi(X,Y,\Phi)$$

$$+\left(\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\right)\left\{\frac{1}{h_2}v_e\frac{\partial S}{\partial X}+\frac{1}{h_1h_2}v_e S(X,Y,\Phi)\frac{\partial h_1}{\partial X}+\frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}+\frac{1}{h_2}v_e S(X,Y,\Phi)\frac{(k_0)_X}{k_0}\right\}$$

$$+v_e\frac{\partial S}{\partial Y}+S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}+\frac{1}{h_1}S(X,Y,\Phi)v_e\frac{\partial h_1}{\partial Y}+v_e S(X,Y,\Phi)\frac{(k_0)_Y}{k_0}$$

$$-\frac{c\Phi}{2(X+cY)}v_e S_\Phi(X,Y,\Phi)+\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e \mu_e}\right)\right\}_Y v_e S_\Phi(X,Y,\Phi)\Bigg]\frac{\partial}{\partial\Phi} \qquad (34)$$

$$\therefore \frac{u}{h_1}\frac{\partial}{\partial\xi}=\frac{u_e}{h_1}F_\phi(X,Y,\Phi)\left[h_1\frac{\partial}{\partial X}+\left(\int_0^\eta\frac{\partial h_2}{\partial\zeta}d\eta\right)\frac{\partial}{\partial Y}-\frac{\Phi h_1}{2(X+cY)}\frac{\partial}{\partial\Phi}+\frac{\Phi h_1}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e \mu_e}\right)\right\}_X\frac{\partial}{\partial\Phi}\right] \qquad (35)$$

And

$$\therefore \frac{v}{h_2}\frac{\partial}{\partial\eta}=\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\left[\left(\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\right)\frac{\partial}{\partial X}+h_2\frac{\partial}{\partial Y}-\frac{c\Phi h_2}{2(X+cY)}\frac{\partial}{\partial\Phi}+\frac{\Phi h_2}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e \mu_e}\right)\right\}_Y\frac{\partial}{\partial\Phi}\right] \qquad (36)$$

The convective operators are as follows :-

$$\left(\frac{u}{h_1}\frac{\partial}{\partial\xi}+\frac{v}{h_2}\frac{\partial}{\partial\eta}+w\frac{\partial}{\partial\zeta}\right)=u_e F_\phi(X,Y,\Phi)\frac{\partial}{\partial X}+u_e\frac{F_\Phi(X,Y,\Phi)}{h_1}\left(\int_0^\eta\frac{\partial h_2}{\partial\zeta}d\eta\right)\frac{\partial}{\partial Y}-\frac{\Phi}{2(X+cY)}u_e F_\phi(X,Y,\Phi)\frac{\partial}{\partial\Phi}$$

$$+\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e \mu_e}\right)\right\}_X u_e F_\phi(X,Y,\Phi)\frac{\partial}{\partial\Phi}+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\left(\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\right)\frac{\partial}{\partial X}+v_e S_\Phi(X,Y,\Phi)\frac{\partial}{\partial Y}$$

$$-\frac{c\Phi}{2(X+cY)}v_e S_\Phi(X,Y,\Phi)\frac{\partial}{\partial\Phi}+\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e \mu_e}\right)\right\}_Y v_e S_\Phi(X,Y,\Phi)\frac{\partial}{\partial\Phi}-u_e\frac{\partial F}{\partial X}\frac{\partial}{\partial\Phi}-F(X,Y,\Phi)\frac{\partial u_e}{\partial X}\frac{\partial}{\partial\Phi}$$

$$-\frac{1}{h_2}u_e F(X,Y,\Phi)\frac{\partial h_2}{\partial X}\frac{\partial}{\partial\Phi}-u_e F(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial\Phi}-\int_0^\eta\frac{\partial h_2}{\partial\zeta}d\eta\left\{\frac{1}{h_1}u_e\frac{\partial F}{\partial Y}\frac{\partial}{\partial\Phi}+\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}\frac{\partial}{\partial\Phi}\right.$$

$$+\frac{1}{h_1h_2}F(X,Y,\Phi)u_e\frac{\partial h_2}{\partial Y}\frac{\partial}{\partial\Phi}+\frac{1}{h_1}u_eF(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial\Phi}\Bigg\}+\frac{\Phi}{2(X+cY)}u_eF_\Phi(X,Y,\Phi)\frac{\partial}{\partial\Phi}$$

$$-\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y u_eF_\Phi(X,Y,\Phi)\frac{\partial}{\partial\Phi}+\frac{\Phi}{2(X+cY)}u_eF_\Phi(X,Y,\Phi)\frac{\partial}{\partial\Phi}-\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y u_eF_\Phi(X,Y,\Phi)\frac{\partial}{\partial\Phi}$$

$$-\int_0^\eta\frac{\partial h_2}{\partial\eta}d\xi\left\{\frac{1}{h_2}v_e\frac{\partial S}{\partial X}\frac{\partial}{\partial\Phi}+\frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}\frac{\partial}{\partial\Phi}+\frac{1}{h_1h_2}S(X,Y,\Phi)v_e\frac{\partial h_2}{\partial X}\frac{\partial}{\partial\Phi}+\frac{1}{h_2}v_eS(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial\Phi}\right\}$$

$$-v_e\frac{\partial S}{\partial Y}\frac{\partial}{\partial\Phi}-S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}\frac{\partial}{\partial\Phi}-\frac{1}{h_2}v_eSF(X,Y,\Phi)\frac{\partial h_2}{\partial Y}\frac{\partial}{\partial\Phi}-v_eS(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial\Phi}$$

$$+\frac{c\Phi}{2(X+cY)}v_eS_\Phi(X,Y,\Phi)\frac{\partial}{\partial\Phi}-\frac{\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y v_eS_\Phi(X,Y,\Phi)\frac{\partial}{\partial\Phi}\left(\frac{u}{h_1}\frac{\partial}{\partial\xi}+\frac{v}{h_2}\frac{\partial}{\partial\eta}+w\frac{\partial}{\partial\zeta}\right)$$

$$=u_eF_\Phi(X,Y,\Phi)\frac{\partial}{\partial X}+u_e\frac{F_\Phi(X,Y,\Phi)}{h_1}\left(\int_0^\eta\frac{\partial h_2}{\partial\zeta}d\eta\right)\frac{\partial}{\partial Y}+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\left(\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\right)\frac{\partial}{\partial X}+v_eS_\Phi(X,Y,\Phi)\frac{\partial}{\partial Y}$$

$$-u_e\frac{\partial F}{\partial X}\frac{\partial}{\partial\Phi}-F(X,Y,\Phi)\frac{\partial u_e}{\partial X}\frac{\partial}{\partial\Phi}-\frac{1}{h_2}u_eF(X,Y,\Phi)\frac{\partial h_2}{\partial X}\frac{\partial}{\partial\Phi}-u_eF(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial\Phi}$$

$$-\int_0^\eta\frac{\partial h_2}{\partial\zeta}d\eta\left\{\frac{1}{h_1}u_e\frac{\partial F}{\partial Y}\frac{\partial}{\partial\Phi}+\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}\frac{\partial}{\partial\Phi}+\frac{1}{h_1h_2}F(X,Y,\Phi)u_e\frac{\partial h_2}{\partial Y}\frac{\partial}{\partial\Phi}+\frac{1}{h_1}u_eF(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial\Phi}\right\}$$

$$-\int_0^\eta\frac{\partial h_2}{\partial\eta}d\xi\left\{\frac{1}{h_2}v_e\frac{\partial S}{\partial X}\frac{\partial}{\partial\Phi}+\frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}\frac{\partial}{\partial\Phi}+\frac{1}{h_1h_2}S(X,Y,\Phi)v_e\frac{\partial h_2}{\partial X}\frac{\partial}{\partial\Phi}+\frac{1}{h_2}v_eS(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial\Phi}\right\}$$

$$-v_e\frac{\partial S}{\partial Y}\frac{\partial}{\partial\Phi}-S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}\frac{\partial}{\partial\Phi}-\frac{1}{h_2}v_eSF(X,Y,\Phi)\frac{\partial h_1}{\partial Y}\frac{\partial}{\partial\Phi}-v_eS(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial\Phi}$$

or.

$$\left(\frac{u}{h_1}\frac{\partial}{\partial\xi}+\frac{v}{h_2}\frac{\partial}{\partial\eta}+w\frac{\partial}{\partial\zeta}\right)=u_eF_\Phi(X,Y,\Phi)\frac{\partial}{\partial X}+v_eS_\Phi(X,Y,\Phi)\frac{\partial}{\partial Y}-u_e\frac{\partial F}{\partial X}\frac{\partial}{\partial\Phi}-F(X,Y,\Phi)\frac{\partial u_e}{\partial X}\frac{\partial}{\partial\Phi}$$

$$-u_eF(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial\Phi}-v_e\frac{\partial S}{\partial Y}\frac{\partial}{\partial\Phi}-S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}\frac{\partial}{\partial\Phi}-v_eS(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial\Phi}$$

$$+u_e\frac{F_\Phi(X,Y,\Phi)}{h_1}\left(\int_0^\eta\frac{\partial h_2}{\partial\zeta}d\eta\right)\frac{\partial}{\partial Y}+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\left(\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\right)\frac{\partial}{\partial X}-\frac{1}{h_2}u_eF(X,Y,\Phi)\frac{\partial h_2}{\partial X}\frac{\partial}{\partial\Phi}$$

$$-\frac{1}{h_1}v_eS(X,Y,\Phi)\frac{\partial h_1}{\partial Y}\frac{\partial}{\partial\Phi}-\int_0^\eta\frac{\partial h_2}{\partial\zeta}d\eta\left\{\frac{1}{h_1}u_e\frac{\partial F}{\partial Y}\frac{\partial}{\partial\Phi}+\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}\frac{\partial}{\partial\Phi}\right.$$

$$\left.+\frac{1}{h_1h_2}F(X,Y,\Phi)u_e\frac{\partial h_2}{\partial Y}\frac{\partial}{\partial\Phi}+\frac{1}{h_1}u_eF(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial\Phi}\right\}-\int_0^\eta\frac{\partial h_2}{\partial\eta}d\xi\left\{\frac{1}{h_2}v_e\frac{\partial S}{\partial X}\frac{\partial}{\partial\Phi}\right.$$

$$\left.+\frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}\frac{\partial}{\partial\Phi}+\frac{1}{h_1h_2}S(X,Y,\Phi)v_e\frac{\partial h_1}{\partial X}\frac{\partial}{\partial\Phi}+\frac{1}{h_2}v_eS(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial\Phi}\right\}$$

$$\therefore\left(\frac{u}{h_1}\frac{\partial}{\partial\xi}+\frac{v}{h_2}\frac{\partial}{\partial\eta}+w\frac{\partial}{\partial\zeta}\right)u=u_eF_\Phi(X,Y,\Phi)\frac{\partial}{\partial X}\left(u_eF_\Phi(X,Y,\Phi)\right)+v_eS_\Phi(X,Y,\Phi)\frac{\partial}{\partial Y}\left(u_eF_\Phi(X,Y,\Phi)\right)$$

$$-u_e\frac{\partial F}{\partial X}\frac{\partial}{\partial\Phi}\left(u_eF_\Phi(X,Y,\Phi)\right)-F(X,Y,\Phi)\frac{\partial u_e}{\partial X}\frac{\partial}{\partial\Phi}\left(u_eF_\Phi(X,Y,\Phi)\right)-u_eF(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial\Phi}\left(u_eF_\Phi(X,Y,\Phi)\right)$$

$$-v_e\frac{\partial S}{\partial Y}\frac{\partial}{\partial\Phi}\left(u_eF_\Phi(X,Y,\Phi)\right)-S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}\frac{\partial}{\partial\Phi}\left(u_eF_\Phi(X,Y,\Phi)\right)-v_eS(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial\Phi}\left(u_eF_\Phi(X,Y,\Phi)\right)$$

$$+u_e\frac{F_\Phi(X,Y,\Phi)}{h_1}\left(\int_0^\eta\frac{\partial h_2}{\partial\zeta}d\eta\right)\frac{\partial}{\partial Y}\left(u_eF_\Phi(X,Y,\Phi)\right)+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\left(\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\right)\frac{\partial}{\partial X}\left(u_eF_\Phi(X,Y,\Phi)\right)$$

$$-\frac{1}{h_2}u_eF(X,Y,\Phi)\frac{\partial h_2}{\partial X}\frac{\partial}{\partial\Phi}\left(u_eF_\Phi(X,Y,\Phi)\right)-\frac{1}{h_1}v_eS(X,Y,\Phi)\frac{\partial h_1}{\partial Y}\frac{\partial}{\partial\Phi}\left(u_eF_\Phi(X,Y,\Phi)\right)-\int_0^\eta\frac{\partial h_2}{\partial\zeta}d\eta\,*$$

$$\left\{\frac{1}{h_1}u_e\frac{\partial F}{\partial Y}\frac{\partial}{\partial\Phi}+\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}\frac{\partial}{\partial\Phi}+\frac{1}{h_1h_2}F(X,Y,\Phi)u_e\frac{\partial h_2}{\partial Y}\frac{\partial}{\partial\Phi}+\frac{1}{h_1}u_eF(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial\Phi}\right\}$$

$$\left(u_eF_\Phi(X,Y,\Phi)\right)-\int_0^\eta\frac{\partial h_2}{\partial\eta}d\xi\left\{\frac{1}{h_2}v_e\frac{\partial S}{\partial X}\frac{\partial}{\partial\Phi}+\frac{1}{h_{12}}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}\frac{\partial}{\partial\Phi}\right.$$

$$\left.+\frac{1}{h_1h_2}S(X,Y,\Phi)v_e\frac{\partial h_2}{\partial X}\frac{\partial}{\partial\Phi}+\frac{1}{h_2}v_eS(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial\Phi}\right\}\left(u_eF_\Phi(X,Y,\Phi)\right)$$

$$\therefore\left(\frac{u}{h_1}\frac{\partial}{\partial\xi}+\frac{v}{h_2}\frac{\partial}{\partial\eta}+w\frac{\partial}{\partial\zeta}\right)u=u_e^{\,2}F_\Phi(X,Y,\Phi)\frac{\partial F_\Phi}{\partial X}+u_e\frac{\partial u_e}{\partial X}F_{\Phi\Phi}^2(X,Y,\Phi)+u_ev_eS_\Phi(X,Y,\Phi)\frac{\partial F_\Phi}{\partial Y}$$

$$+v_eF_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\frac{\partial u_\Phi}{\partial Y}-u_e^{\,2}\frac{\partial F}{\partial X}F_{\Phi\Phi}(X,Y,\Phi)-u_e\frac{\partial u_e}{\partial X}F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)$$

$$-u_e^{\,2}\left(\log k_0\right)_X F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)-u_ev_e\frac{\partial S}{\partial Y}F_{\Phi\Phi}(X,Y,\Phi)-u_eS(X,Y,\Phi)\frac{\partial v_\Phi}{\partial Y}F_{\Phi\Phi}(X,Y,\Phi)$$

$$-u_ev_e\left(\log k_0\right)_Y S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)+u_e\frac{F_\Phi(X,Y,\Phi)}{h_1}\left(\int_0^\eta\frac{\partial h_2}{\partial\zeta}d\eta\right)\frac{\partial}{\partial Y}\left(u_eF_\Phi(X,Y,\Phi)\right)$$

$$+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\left(\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\right)\frac{\partial}{\partial X}\left(u_eF_\Phi(X,Y,\Phi)\right)-\frac{1}{h_2}u_e^{\,2}\frac{\partial h_2}{\partial X}F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)$$

$$-\frac{1}{h_1}u_ev_e\frac{\partial h_1}{\partial X}S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)-\int_0^\eta\frac{\partial h_2}{\partial\zeta}d\eta\left\{\frac{1}{h_1}u_e^{\,2}\frac{\partial F}{\partial Y}\frac{\partial}{\partial\Phi}F_{\Phi\Phi}(X,Y,\Phi)\right.$$

$$+\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}\frac{\partial}{\partial\Phi}F_{\Phi\Phi}(X,Y,\Phi)+\frac{1}{h_1h_2}u_e^{\,2}\frac{\partial h_2}{\partial Y}F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)$$

$$\left.+\frac{1}{h_1}u_e^{\,2}\left(\log k_0\right)_Y F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)\right\}-\int_0^\xi\frac{\partial h_2}{\partial\eta}d\xi\left\{\frac{1}{h_2}u_ev_e\frac{\partial S}{\partial X}F_{\Phi\Phi}(X,Y,\Phi)\right.$$

$$+\frac{1}{h_{21}}u_e\frac{\partial v_e}{\partial X}S_\Phi(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)+\frac{1}{h_1h_2}u_e^2\frac{\partial h_1}{\partial X}S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)$$

$$+\frac{1}{h_2}u_ev_e(\log k_0)_\lambda S(X,Y,\Phi)F_{\Phi\Phi0}(X,Y,\Phi)\Bigg\} \tag{37}$$

$$\frac{uv}{h_1h_2}\frac{\partial h_1}{\partial\eta}=\frac{u_ev_e}{h_1h_2}F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\frac{\partial h_1}{\partial Y}\left[\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial}{\partial X}\right.$$
$$\left.+h_2\frac{\partial}{\partial Y}-\frac{c}{2}\frac{h_2\Phi}{(X+cY)}\frac{\partial}{\partial\Phi}+\frac{h_2\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y\frac{\partial}{\partial\Phi}\right]h_1$$

$$\therefore\ \frac{uv}{h_1h_2}\frac{\partial h_1}{\partial\eta}=\frac{u_ev_e}{h_1h_2}F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\frac{\partial h_1}{\partial Y}$$
$$+\frac{u_ev_e}{h_1h_2}F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial h_1}{\partial X} \tag{38}$$

$$\therefore\ \frac{v^2}{h_1h_2}\frac{\partial h_2}{\partial\xi}=\frac{v_e^2}{h_1h_2}S_\Phi^2(X,Y,\Phi)\left[h_1\frac{\partial}{\partial X}+\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial}{\partial Y}\right.$$
$$\left.-\frac{1}{2}\frac{h_1\Phi}{(X+cY)}\frac{\partial}{\partial\Phi}+\frac{h_1\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_\lambda\frac{\partial}{\partial\Phi}\right]h_2$$

$$\therefore\ \frac{v^2}{h_1h_2}\frac{\partial h_2}{\partial\xi}=\frac{v_e^2}{h_2}S_\Phi^2(X,Y,\Phi)\frac{\partial h_2}{\partial X}+\frac{v_e^2}{h_1h_2}S_\Phi^2(X,Y,\Phi)\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta \tag{39}$$

$$\frac{u_e}{h_1}\frac{\partial u_e}{\partial\xi}=u_e\frac{\partial u_e}{\partial X}+\frac{u_e}{h_1}\frac{\partial u_e}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta \tag{40}$$

$$\frac{v_e}{h_2}\frac{\partial u_e}{\partial\eta}=v_e\frac{\partial u_e}{\partial Y}+\frac{v_e}{h_2}\frac{\partial u_e}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi \tag{41}$$

$$\therefore\ \frac{u_ev_e}{h_1h_2}\frac{\partial h_1}{\partial\eta}=\frac{u_ev_e}{h_1h_2}\left[\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial}{\partial X}+h_2\frac{\partial}{\partial Y}-\frac{c}{2}\frac{h_2\Phi}{(X+cY)}\frac{\partial}{\partial\Phi}+\frac{h_2\Phi}{2}\left\{\ln\left(\frac{u_e+\alpha v_e}{\rho_e\mu_e}\right)\right\}_Y\frac{\partial}{\partial\Phi}\right]h_1$$

$$\therefore\ \frac{u_ev_e}{h_1h_2}\frac{\partial h_1}{\partial\eta}=\frac{u_ev_e}{h_1}\frac{\partial h_1}{\partial Y}+\frac{u_ev_e}{h_1h_2}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial h_1}{\partial X} \tag{42}$$

$$\therefore \quad \frac{v_e^{\,4}}{h_1 h_2}\frac{\partial h_2}{\partial \xi} = \frac{v_e^{\,2}}{h_1 h_2} S_\Phi^{\,2}(X,Y,\Phi)\left[h_1 \frac{\partial}{\partial X} + \int_0^\eta \frac{\partial h_2}{\partial \xi} d\eta \frac{\partial}{\partial Y} - \frac{1}{2}\frac{h_1 \Phi}{(X+cY)}\frac{\partial}{\partial \Phi} + \frac{h_1 \Phi}{2}\left\{ \ln\left(\frac{u_e + \alpha v_e}{\rho_e \mu_e}\right)\right\}_1 \frac{\partial}{\partial \Phi}\right] h_2$$

$$\therefore \quad \frac{v_e^{\,2}}{h_1 h_2}\frac{\partial h_2}{\partial \xi} = \frac{v_e^{\,2}}{h_2}\frac{\partial h_2}{\partial X} + \frac{v_e^{\,2}}{h_1 h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta \frac{\partial h_2}{\partial \xi} d\eta \tag{43}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\mu \frac{\partial u}{\partial \zeta}\right) = \frac{1}{\rho}\frac{\partial \Phi}{\partial \zeta}\frac{\partial}{\partial \Phi}\left\{\mu \frac{\partial \Phi}{\partial \zeta}\frac{\partial}{\partial \Phi}\left(u_e F_\Phi(X,Y,\Phi)\right)\right\}$$

$$= \frac{1}{\rho}\frac{\sqrt{(u_e + \alpha v_e)}}{\sqrt{\rho_e \mu_e (X+CY)}}\rho \frac{\partial}{\partial \Phi}\left\{\mu \frac{\sqrt{(u_e+\alpha v_e)}}{\sqrt{\rho_e \mu_e (X+CY)}}\rho u_e F_{\Phi\Phi}(X,Y,\Phi)\right\}$$

$$= \frac{(u_e + \alpha v_e)}{\rho_e \mu_e (X+CY)}\frac{\partial}{\partial \Phi}\left\{\frac{\rho \mu}{\rho_e \mu_e} u_e F_{\Phi\Phi}(X,Y,\Phi)\right\}$$

$$\therefore \quad \frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\mu \frac{\partial u}{\partial \zeta}\right) = \frac{(u_e + \alpha v_e)}{(X+CY)} u_e \left(\frac{\rho \mu}{\rho_e \mu_e} F_{\Phi\Phi}(X,Y,\Phi)\right)_\Phi \tag{44}$$

Inserting (37) to (44) into the u-momentum equation (23), we have,

$$u_e^2 F_\Phi(X,Y,\Phi)\frac{\partial F_\Phi}{\partial X} + u_e \frac{\partial u_e}{\partial X} F_\Phi^2(X,Y,\Phi) + u_e v_e S_\Phi(X,Y,\Phi)\frac{\partial F_\Phi}{\partial Y} + v_e F_\Phi(X,Y,\Phi) S_\Phi(X,Y,\Phi)\frac{\partial u_e}{\partial Y}$$

$$- u_e^2 \frac{\partial F}{\partial X} F_{\Phi\Phi}(X,Y,\Phi) - u_e \frac{\partial u_e}{\partial X} F(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi) - u_e^2 \left(\log k_0\right)_X F(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi)$$

$$- u_e v_e \frac{\partial S}{\partial Y} F_{\Phi\Phi}(X,Y,\Phi) - u_e S(X,Y,\Phi)\frac{\partial v_e}{\partial Y} S(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi) - u_e v_e \left(\log k_0\right)_1 S(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi)$$

$$+ u_e \frac{F_\Phi(X,Y,\Phi)}{h_1}\left(\int_0^\eta \frac{\partial h_2}{\partial \zeta} d\eta\right)\frac{\partial}{\partial Y}\left(u_e F_\Phi(X,Y,\Phi)\right) + \frac{v_e}{h_2} S_\Phi(X,Y,\Phi)\left(\int_0^\xi \frac{\partial h_1}{\partial \eta} d\xi\right)\frac{\partial}{\partial X}\left(u_e F_\Phi(X,Y,\Phi)\right)$$

$$- \frac{1}{h_2} u_e^2 \frac{\partial h_2}{\partial X} F(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi) - \frac{1}{h_1} u_e v_e \frac{\partial h_1}{\partial X} S(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi)$$

$$- \int_0^\eta \frac{\partial h_2}{\partial \zeta} d\eta \left\{\frac{1}{h_1} u_e^2 \frac{\partial F}{\partial Y} F_{\Phi\Phi}(X,Y,\Phi) + \frac{1}{h_1} F(X,Y,\Phi) u_e \frac{\partial u_e}{\partial Y} F_{\Phi\Phi}(X,Y,\Phi)\right.$$

$$\left. + \frac{1}{h_1 h_2} u_e^2 \frac{\partial h_2}{\partial Y} F(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi) + \frac{1}{h_1} u_e^2 \left(\log k_0\right)_Y F(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi)\right\}$$

$$- \int_0^\xi \frac{\partial h_1}{\partial \eta} d\xi \left\{\frac{1}{h_2} u_e v_e \frac{\partial S}{\partial X} F_{\Phi\Phi}(X,Y,\Phi) + \frac{1}{h_2} u_e \frac{\partial v_e}{\partial X} S(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi)\right.$$

$$+\frac{1}{h_1h_2}v_e\frac{\partial h_1}{\partial X}S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)+\frac{1}{h_2}u_ev_e\left(\log k_0\right)_1 S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)\Bigg\}$$

$$+\frac{u_ev_e}{h_1h_2}F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\frac{\partial h_1}{\partial Y}+\frac{u_ev_e}{h_1h_2}F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial h_1}{\partial X}$$

$$-\frac{{v_e}^2}{h_2}S_\Phi^2(X,Y,\Phi)\frac{\partial h_2}{\partial X}-\frac{{v_e}^2}{h_1h_2}S_\Phi^2(X,Y,\Phi)\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta=\frac{(u_e+\alpha v_e)}{(X+CY)}u_e\left(\frac{\rho\mu}{\rho_e\mu_e}F_{\Phi\Phi}(X,Y,\Phi)\right)_\Phi$$

$$+\frac{1}{h_1}g_\xi\left(\frac{\rho-\rho_e}{\rho}\right)+\frac{\rho_e}{\rho}\left\{u_e\frac{\partial u_e}{\partial X}+\frac{u_e}{h_1}\frac{\partial u_e}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta+v_e\frac{\partial u_e}{\partial Y}+\frac{v_e}{h_2}\frac{\partial u_e}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi+\frac{u_ev_e}{h_1}\frac{\partial h_1}{\partial Y}\right.$$

$$\left.+\frac{u_ev_e}{h_1h_2}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial h_1}{\partial X}-\frac{v_e^2}{h_2}\frac{\partial h_2}{\partial X}-\frac{v_e^2}{h_1h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\right\}$$

$$=\frac{(u_e+\alpha v_e)}{(X+CY)}u_e\left(\frac{\rho\mu}{\rho_e\mu_e}F_{\Phi\Phi}(X,Y,\Phi)\right)_\Phi+\frac{1}{h_1}g_\xi\left(\frac{\rho-\rho_e}{\rho}\right)+\frac{\rho_e}{\rho}\left\{u_e\frac{\partial u_e}{\partial X}+\frac{u_e}{h_1}\frac{\partial u_e}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\right.$$

$$\left.+v_e\frac{\partial u_e}{\partial Y}+\frac{v_e}{h_2}\frac{\partial u_e}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi+\frac{u_ev_e}{h_1}\frac{\partial h_1}{\partial Y}+\frac{u_ev_e}{h_1h_2}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial h_1}{\partial X}-\frac{v_e^2}{h_2}\frac{\partial h_2}{\partial X}-\frac{v_e^2}{h_1h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\right\}$$

$$\Rightarrow \frac{(u_e+\alpha v_e)}{(X+CY)} u_e \left(\frac{\rho\mu}{\rho_e \mu_e} F_{\Phi\Phi}(X,Y,\Phi) \right)_{\Phi} + \frac{1}{h_1} g_{\xi} \left(\frac{\rho-\rho_e}{\rho} \right) + \frac{\rho_e}{\rho} u_e \frac{\partial u_e}{\partial X} + \frac{\rho_e}{\rho} v_e \frac{\partial u_e}{\partial Y}$$

$$- u_e \frac{\partial u_e}{\partial X} F_{\Phi}^2(X,Y,\Phi) - v_e \frac{\partial u_e}{\partial X} F_{\Phi}(X,Y,\Phi) S_{\Phi}(X,Y,\Phi)$$

$$+ u_e \frac{\partial u_e}{\partial X} F(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi) + u_e \frac{\partial v_e}{\partial Y} S(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi)$$

$$+ u_e^2 (\log k_0)_X F(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi) + u_e v_e (\log k_0)_Y S(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi)$$

$$= u_e v_e \frac{\partial F_{\Phi}}{\partial X} S_{\Phi}(X,Y,\Phi) - u_e^2 \frac{\partial F}{\partial X} F_{\Phi\Phi}(X,Y,\Phi) - u_e v_e \frac{\partial S}{\partial Y} F_{\Phi\Phi}(X,Y,\Phi)$$

$$+ u_e^2 \frac{\partial F_{\Phi}}{\partial X} F_{\Phi}(X,Y,\Phi) + \Bigg[\frac{u_e}{h_1} F_{\Phi}(X,Y,\Phi) \int_0^{\eta} \frac{\partial h_2}{\partial \xi} d\eta \frac{\partial}{\partial Y} \left(u_e F_{\Phi}(X,Y,\Phi) \right)$$

$$+ \frac{v_e}{h_2} S_{\Phi}(X,Y,\Phi) \int_0^{\xi} \frac{\partial h_1}{\partial \eta} d\xi \frac{\partial}{\partial X} \left(u_e F_{\Phi}(X,Y,\Phi) \right) - \frac{1}{h_2} u_e^2 \frac{\partial h_2}{\partial X} F(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi)$$

$$- \frac{1}{h_1} u_e v_e \frac{\partial h_1}{\partial Y} S(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi) - \left(\int_0^{\eta} \frac{\partial h_2}{\partial \xi} d\eta \right) \Bigg\{ \frac{1}{h_1} u_e^2 \frac{\partial F}{\partial Y} F_{\Phi\Phi}(X,Y,\Phi)$$

$$+ \frac{1}{h_1} u_e \frac{\partial u_e}{\partial Y} F(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi) + \frac{1}{h_1 h_2} u_e^2 \frac{\partial h_1}{\partial Y} F(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi)$$

$$+ \frac{1}{h_1} u_e^2 (\log k_0)_Y F(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi) \Bigg\} - \left(\int_0^{\xi} \frac{\partial h_1}{\partial \eta} d\xi \right) \Bigg\{ \frac{1}{h_2} u_e v_e \frac{\partial S}{\partial X} F_{\Phi\Phi}(X,Y,\Phi)$$

$$+ \frac{1}{h_2} u_e \frac{\partial v_e}{\partial X} S(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi) + \frac{1}{h_1 h_2} u_e v_e \frac{\partial h_1}{\partial X} S(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi)$$

$$+ \frac{1}{h_2} u_e v_e (\log k_0)_X S(X,Y,\Phi) F_{\Phi\Phi}(X,Y,\Phi) \Bigg\} + \frac{u_e v_e}{h_1} \frac{\partial h_1}{\partial Y} S_{\Phi}(X,Y,\Phi) F_{\Phi}(X,Y,\Phi)$$

$$+ \frac{u_e v_e}{h_1 h_2} \frac{\partial h_1}{\partial X} \int_0^{\xi} \frac{\partial h_1}{\partial \eta} d\xi \, S_{\Phi}(X,Y,\Phi) F_{\Phi}(X,Y,\Phi) - \frac{1}{h_2} v_e^2 \frac{\partial h_1}{\partial X} S_{\Phi}^2(X,Y,\Phi) - \frac{v_e^2}{h_1 h_2} \frac{\partial h_2}{\partial X} \int_0^{\eta} \frac{\partial h_2}{\partial \xi} d\eta \, S_{\Phi}^2(X,Y,\Phi)$$

$$- \frac{\rho_e}{\rho} \left\{ \frac{u_e}{h_1} \frac{\partial u_e}{\partial Y} \int_0^{\eta} \frac{\partial h_2}{\partial \xi} d\eta + \frac{v_e}{h_2} \frac{\partial u_e}{\partial X} \int_0^{\xi} \frac{\partial h_1}{\partial \eta} d\xi + \frac{u_e v_e}{h_1} \frac{\partial h_1}{\partial Y} + \frac{u_e v_e}{h_1 h_2} \frac{\partial h_1}{\partial X} \int_0^{\xi} \frac{\partial h_1}{\partial \eta} d\xi - \frac{v_e^2}{h_2} \frac{\partial h_2}{\partial X} - \frac{v_e^2}{h_1 h_2} \frac{\partial h_2}{\partial Y} \int_0^{\eta} \frac{\partial h_2}{\partial \xi} d\eta \right\} \Bigg]$$

$$\Rightarrow \left(\frac{\rho \mu}{\rho_e \mu_e} F_{\Phi\Phi}(X,Y,\Phi) \right)_{\Phi} + \frac{(X+cY)}{h_1 u_e (u_e + \alpha v_e)} g_{\xi} \left(\frac{\rho-\rho_e}{\rho} \right)$$

$$+\frac{u_e(X+cY)}{(u_e+\alpha v_e)}\left\{\frac{1}{u_e}\frac{\partial u_e}{\partial X}+(\log k_0)_X\right\}F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)$$

$$+\frac{v_e(X+cY)}{(u_e+\alpha v_e)}\left\{\frac{1}{v_e}\frac{\partial v_e}{\partial Y}+(\log k_0)_Y\right\}S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)+\frac{u_e(X+cY)}{(u_e+\alpha v_e)}(\log u_e)_X\left(\frac{\rho_e}{\rho}-F_\Phi^2(X,Y,\Phi)\right)$$

$$+\frac{v_e(X+cY)}{(u_e+\alpha v_e)}(\log u_e)_Y\left(\frac{\rho_e}{\rho}-F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\right)$$

$$=\frac{u_e(X+cY)}{(u_e+\alpha v_e)}\left(F_\Phi(X,Y,\Phi)\frac{\partial F_\Phi}{\partial X}-F_{\Phi\Phi}(X,Y,\Phi)\frac{\partial F}{\partial X}\right)+\frac{v_e(X+cY)}{(u_e+\alpha v_e)}\left(S_\Phi(X,Y,\Phi)\frac{\partial F_\Phi}{\partial Y}-F_{\Phi\Phi}(X,Y,\Phi)\frac{\partial S}{\partial Y}\right)+H_u$$

where

$$H_u=\frac{(X+cY)}{u_e(u_e+\alpha v_e)}\left[\frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial}{\partial Y}\left(u_eF_\Phi(X,Y,\Phi)\right)\right.$$

$$+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial}{\partial X}\left(u_eF_\Phi(X,Y,\Phi)\right)-\frac{1}{h_2}u_e^2\frac{\partial h_2}{\partial X}F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)$$

$$-\frac{1}{h_1}u_ev_e\frac{\partial h_1}{\partial Y}S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)-\left(\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\right)\left\{\frac{1}{h_1}u_e^2\frac{\partial F}{\partial Y}F_{\Phi\Phi}(X,Y,\Phi)\right.$$

$$+\frac{1}{h_1}u_e\frac{\partial u_e}{\partial Y}F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)+\frac{1}{h_1h_2}u_e^2\frac{\partial h_2}{\partial Y}F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)$$

$$\left.+\frac{1}{h_1}u_e^2\left(\log k_0\right)_Y F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)\right\}-\left(\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\right)\left\{\frac{1}{h_2}u_ev_e\frac{\partial S}{\partial X}F_{\Phi\Phi}(X,Y,\Phi)\right.$$

$$+\frac{1}{h_2}u_e\frac{\partial v_e}{\partial X}S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)+\frac{1}{h_1h_2}u_ev_e\frac{\partial h_1}{\partial X}S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)$$

$$\left.+\frac{1}{h_2}u_ev_e\left(\log k_0\right)_X S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)\right\}+\frac{u_ev_e}{h_1}\frac{\partial h_1}{\partial Y}S_\Phi(X,Y,\Phi)F_\Phi(X,Y,\Phi)$$

$$+\frac{u_ev_e}{h_1h_2}\frac{\partial h_1}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi S_\Phi(X,Y,\Phi)F_\Phi(X,Y,\Phi)-\frac{1}{h_2}v_e^2\frac{\partial h_1}{\partial X}S_\Phi^2(X,Y,\Phi)-\frac{v_e^2}{h_1h_2}\frac{\partial h_2}{\partial X}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta S_\Phi^2(X,Y,\Phi)$$

$$\left.-\frac{\rho_e}{\rho}\left\{\frac{u_e}{h_1}\frac{\partial u_e}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta+\frac{v_e}{h_2}\frac{\partial u_e}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi+\frac{u_ev_e}{h_1}\frac{\partial h_1}{\partial Y}+\frac{u_ev_e}{h_1h_2}\frac{\partial h_1}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi-\frac{v_e^2}{h_2}\frac{\partial h_2}{\partial X}-\frac{v_e^2}{h_1h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\right\}\right].$$

Finally u-momentum equation takes the form.

$$\Rightarrow\quad\left(\frac{\rho\mu}{\rho_e\mu_e}F_{\Phi\Phi}(X,Y,\Phi)\right)_\Phi+\left(P_1+\Omega_0\right)F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)+c\left(P_1+\Omega_0\right)S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)$$

$$+P_1\left(\frac{\rho_e}{\rho}-F_\Phi^2(X,Y,\Phi)\right)+cQP_1\left(\frac{\rho_e}{\rho}-F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\right)+\frac{U_F^2}{U_e^2}\theta$$

$$= \left[J\left(\frac{(F, F_{\Phi})}{(X,\Phi)}\right) + J\left(\frac{(S,S_{\Phi})}{(Y,\Phi)}\right)\right] + H_u \tag{45}$$

where, $U_r^{\,2} = g_{\xi}\beta_T \Delta T \times$ characteristic length, characteristic length $= \frac{1}{h_1}(X + CY)$,

$$H_u = \frac{(X + cY)}{u_e(u_e + \alpha v_e)}\left[\frac{u_e}{h_1}F_{\Phi}(X,Y,\Phi)\int_0^{\eta}\frac{\partial h_2}{\partial \xi}d\eta \frac{\partial}{\partial Y}(u_e F_{\Phi}(X,Y,\Phi))\right.$$

$$+\frac{v_e}{h_2}S_{\Phi}(X,Y,\Phi)\int_0^{\xi}\frac{\partial h_1}{\partial \eta}d\xi\frac{\partial}{\partial X}(u_e F_{\Phi}(X,Y,\Phi)) - \frac{1}{h_2}u_e^{\,2}\frac{\partial h_2}{\partial X}F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)$$

$$-\frac{1}{h_1}u_e v_e\frac{\partial h_1}{\partial Y}S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi) - \left(\int_0^{\eta}\frac{\partial h_2}{\partial \xi}d\eta\right)\left\{\frac{1}{h_1}u_e^{\,2}\frac{\partial F}{\partial Y}F_{\Phi\Phi}(X,Y,\Phi)\right.$$

$$+\frac{1}{h_1}u_e\frac{\partial u_e}{\partial Y}F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi) + \frac{1}{h_1h_2}u_e^{\,2}\frac{\partial h_2}{\partial Y}F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)$$

$$\left.+\frac{1}{h_1}u_e^{\,2}(\log k_0)_Y F(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)\right\} - \left(\int_0^{\xi}\frac{\partial h_1}{\partial \eta}d\xi\right)\left\{\frac{1}{h_2}u_e v_e\frac{\partial S}{\partial X}F_{\Phi\Phi}(X,Y,\Phi)\right.$$

$$+\frac{1}{h_2}u_e\frac{\partial v_e}{\partial X}S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi) + \frac{1}{h_1h_2}u_e v_e\frac{\partial h_1}{\partial X}S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)$$

$$\left.+\frac{1}{h_2}u_e v_e(\log k_0)_X S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi)\right\} + \frac{u_e v_e}{h_1}\frac{\partial h_1}{\partial Y}S_{\Phi}(X,Y,\Phi)F_{\Phi}(X,Y,\Phi)$$

$$+\frac{u_e v_e}{h_1h_2}\frac{\partial h_1}{\partial X}\int_0^{\xi}\frac{\partial h_1}{\partial \eta}d\xi\, S_{\Phi}(X,Y,\Phi)F_{\Phi}(X,Y,\Phi) - \frac{1}{h_2}v_e^2\frac{\partial h_1}{\partial X}S_{\Phi}^2(X,Y,\Phi) - \frac{v_e^2}{h_1h_2}\frac{\partial h_2}{\partial X}\int_0^{\eta}\frac{\partial h_2}{\partial \xi}d\eta\, S_{\Phi}^2(X,Y,\Phi)$$

$$-\frac{\rho_e}{\rho}\left\{\frac{u_e}{h_1}\frac{\partial u_e}{\partial Y}\int_0^{\eta}\frac{\partial h_2}{\partial \xi}d\eta + \frac{v_e}{h_2}\frac{\partial u_e}{\partial X}\int_0^{\xi}\frac{\partial h_1}{\partial \eta}d\xi + \frac{u_e v_e}{h_1}\frac{\partial h_1}{\partial Y}\right.$$

$$\left.\left.+\frac{u_e v_e}{h_1h_2}\frac{\partial h_1}{\partial X}\int_0^{\xi}\frac{\partial h_1}{\partial \eta}d\xi - \frac{v_e^{\,2}}{h_2}\frac{\partial h_2}{\partial X} - \frac{v_e^{\,2}}{h_1h_2}\frac{\partial h_2}{\partial Y}\int_0^{\eta}\frac{\partial h_2}{\partial \xi}d\eta\right\}\right] \tag{45a}$$

and J denotes the the Jaccobian

The boundary conditions are,

F(X,Y,0)= F_Φ(X,Y,0)=0, F_Φ(X,Y, ∝)=1, S(X,Y,0) = S_Φ(X,Y,0) = 0, S_Φ(X,Y, ∝)=1 (46)
θ (X, Y, 0) = 1 , θ (X, Y, ∝) = 0

Coefficients of dependent variables (F, S, θ and their derivatives w.r.to Φ) in u-momentum equation are as follows:-

If the coefficients are treated as constants as similarity requirements for a incompressible case in addition to $\frac{u_e}{v_e} = Q$, the coefficient becomes $\frac{n}{1+\alpha Q}$ where n is given by an equation

$$u_e = u_0(X + CY)^n. \tag{C}$$

(1) $\frac{u_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial}{\partial X}\{\log u_e\}$

$$= \frac{u_e}{(u_e+\alpha v_e)} \frac{\frac{\partial}{\partial X}\{\log u_e\}}{\frac{\partial}{\partial X}\{\log(X+cY)\}}$$

$= \frac{n}{1+\alpha Q} = P_1$, the common co-efficient for all the terms except 1st and last term for u-momentum equation.

(2) $\frac{v_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial}{\partial Y}\{\log u_e\}$

$$= \frac{v_e}{(u_e+\alpha v_e)} \frac{\frac{\partial}{\partial Y}\{\log u_e\}}{\frac{\partial}{\partial Y}\{\log(X+cY)\}} c$$

$$= \frac{nQ}{1+\alpha Q} c$$

$$= cQP_1$$

(3) $\frac{u_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial}{\partial X}\{\log(u_e k_0)\}$

$$= \frac{u_e}{(u_e+\alpha v_e)} \left[\frac{\frac{\partial}{\partial X}\{\log u_e\}}{\frac{\partial}{\partial X}\{\log(X+cY)\}} + \frac{\frac{\partial}{\partial X}\{\log k_0\}}{\frac{\partial}{\partial X}\{\log(X+cY)\}} \right]$$

$$= \frac{u_e}{(u_e+\alpha v_e)} \frac{\frac{\partial}{\partial X}\{\log u_e\}}{\frac{\partial}{\partial X}\{\log(X+cY)\}} + \frac{u_e}{(u_e+\alpha v_e)} \frac{\frac{\partial}{\partial X}\{\log k_0\}}{\frac{\partial}{\partial X}\{\log(X+cY)\}},$$ considering $\frac{\rho\mu}{\rho_e\mu_e} = N(\Phi)$

$= \frac{n}{1+\alpha Q}[n + N(\Phi)n_0]$, where n is given by the expression (C) and n_0 is related as

$$n_0 = 1 - n \tag{D}$$

If u_e and v_e are proportional, so that $u_e=Qv_e$ or $\frac{u_e}{v_e}=Q$ where Q→proportionally constant, $\frac{u_e}{(u_e+\alpha v_e)}=\frac{1}{1+\alpha Q}$

and $\dfrac{\frac{\partial}{\partial X}(\log u_e)}{\frac{\partial}{\partial X}\{\log(X+cY)\}}=n(\text{cont}),\quad \dfrac{\frac{\partial}{\partial X}(\log k_0)}{\frac{\partial}{\partial X}\{\log(X+cY)\}}=N(\Phi)\Omega_0$

Hence, $\frac{u_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial}{\partial X}\{\log(u_e k_0)\}=(P_1+\Omega_0)$

where, $P_1=\frac{n}{1+\alpha Q}$ and $\Omega_0=\frac{n_0}{1+\alpha Q}$

(4) $\frac{v_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial}{\partial Y}\{\log(v_e k_0)\}$

$$=\frac{v_e}{(u_e+\alpha v_e)}\left[\frac{\frac{\partial}{\partial Y}\{\log v_e\}}{\frac{\partial}{\partial Y}\{\log(X+cY)\}}c+\frac{\frac{\partial}{\partial Y}\{\log k_0\}}{\frac{\partial}{\partial Y}\{\log(X+cY)\}}c\right]$$

$$=\frac{v_e}{(u_e+\alpha v_e)}\frac{\frac{1}{Q}\frac{\partial}{\partial Y}\{\log v_e\}}{\frac{\partial}{\partial Y}\{\log(X+cY)\}}c+\frac{v_e}{(u_e+\alpha v_e)}\frac{\frac{\partial}{\partial Y}\{\log k_0\}}{\frac{\partial}{\partial Y}\{\log(X+cY)\}}c$$

$$=c\frac{1}{1+\alpha Q}n+N(\Phi)c\frac{Q}{1+\alpha Q}n_0$$

$$=c(P_1+N(\Phi)\Omega_0)$$

$$=\frac{c}{1+\alpha Q}[n+N(\Phi)n_0 Q]$$

v-momentum equation:

$$\frac{u}{h_1}\frac{\partial v}{\partial \xi}+\frac{v}{h_2}\frac{\partial v}{\partial \eta}+\frac{w}{h_3}\frac{\partial v}{\partial \zeta}+\frac{uv}{h_1h_2}\frac{\partial h_2}{\partial \xi}-\frac{u^2}{h_1h_2}\frac{\partial h_1}{\partial \eta}=\frac{1}{h_2}g_\eta\left(\frac{\rho-\rho_e}{\rho}\right)$$
$$+\frac{\rho_e}{\rho}\left\{\frac{u_e}{h_1}\frac{\partial v_e}{\partial \xi}+\frac{v_e}{h_2}\frac{\partial v_e}{\partial \eta}+\frac{u_e v_e}{h_1h_2}\frac{\partial h_2}{\partial \xi}-\frac{u_e^2}{h_1h_2}\frac{\partial h_1}{\partial \eta}\right\}+\frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\frac{\mu}{h_3}\frac{\partial v}{\partial \zeta}\right)$$

i.e. Convective operators (for v-momentum equations) are as follows :-

$$\left(\frac{u}{h}\frac{\partial}{\partial \xi}+\frac{v}{h_2}\frac{\partial}{\partial \eta}+w\frac{\partial}{\partial \zeta}\right)v=\left[u_e F_\Phi(X,Y,\Phi)\frac{\partial}{\partial X}+v_e S_\Phi(X,Y,\Phi)\frac{\partial}{\partial y}-u_e\frac{\partial F}{\partial X}\frac{\partial}{\partial \Phi}\right.$$

$$- F(X,Y,\Phi)\frac{\partial u_e}{\partial X}\frac{\partial}{\partial \Phi} - u_e\frac{(k_0)_X}{k_0}F(X,Y,\Phi)\frac{\partial}{\partial \Phi} - v_e\frac{\partial S}{\partial Y}\frac{\partial}{\partial \Phi} - S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}\frac{\partial}{\partial \Phi}$$

$$- v_e\frac{(k_0)_Y}{k_0}S(X,Y,\Phi)\frac{\partial}{\partial \Phi} + \frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta \frac{\partial h_2}{\partial \xi}d\eta\frac{\partial}{\partial Y} + \frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi \frac{\partial h_1}{\partial \eta}d\xi\frac{\partial}{\partial X}$$

$$-\frac{u_e}{h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial X}\frac{\partial}{\partial \Phi} - \frac{v_e}{h_1}S(X,Y,\Phi)\frac{\partial h_1}{\partial Y}\frac{\partial}{\partial \Phi} - \int_0^\eta \frac{\partial h_2}{\partial \xi}d\eta\left\{\frac{u_e}{h_1}\frac{\partial F}{\partial Y} + \frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}\right.$$

$$\left. + \frac{u_e}{h_1 h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial Y} + \frac{u_e}{h_1}F(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\right\}\frac{\partial}{\partial \Phi} - \int_0^\xi \frac{\partial h_1}{\partial \eta}d\xi\left\{\frac{v_e}{h_2}\frac{\partial S}{\partial X} + \frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}\right.$$

$$\left.\left. + \frac{v_e}{h_1 h_2}S(X,Y,\Phi)\frac{\partial h_1}{\partial X} + \frac{v_e}{h_2}S(X,Y,\Phi)\frac{(k_0)_X}{k_0}\right\}\frac{\partial}{\partial \Phi}\right] v_e S_\Phi(X,Y,\Phi)$$

$$or.\left(\frac{u}{h}\frac{\partial}{\partial \xi} + \frac{v}{h_2}\frac{\partial}{\partial \eta} + w\frac{\partial}{\partial \zeta}\right)v = \left[u_e F_\Phi(X,Y,\Phi)\frac{\partial}{\partial X}(v_e S_\Phi) v_e S_\Phi(X,Y,\Phi)\frac{\partial}{\partial Y}(v_e S_\Phi)\right.$$

$$- u_e v_e\frac{\partial F}{\partial X}S_{\Phi\Phi}(X,Y,\Phi) - v_e\frac{\partial u_e}{\partial X}F(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi) - u_e v_e(\log k_0)_X F(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi)$$

$$- v_e^2\frac{\partial S}{\partial Y}S_{\Phi\Phi}(X,Y,\Phi) - v_e\frac{\partial v_e}{\partial Y}S(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi) - v_e^2(\log k_0)_Y S(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi)$$

$$+ \frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta \frac{\partial h_2}{\partial \xi}d\eta\frac{\partial}{\partial Y}(v_e S_\Phi) + \frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi \frac{\partial h_1}{\partial \eta}d\xi\frac{\partial}{\partial X}(v_e S_\Phi) - \frac{u_e v_e}{h_2}\frac{\partial h_2}{\partial X} *$$

$$F(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi) - \frac{v_e^2}{h_1}\frac{\partial h_1}{\partial Y}S(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi) - \int_0^\eta \frac{\partial h_2}{\partial \xi}d\eta\left\{\frac{u_e}{h_1}\frac{\partial F}{\partial Y}\right.$$

$$+ \frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y} + \frac{u_e}{h_1 h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial Y}$$

$$\left. + \frac{u_e}{h_1}F(X,Y,\Phi)(\log k_0)_Y\right\}v_e S_{\Phi\Phi}(X,Y,\Phi) - \int_0^\xi \frac{\partial h}{\partial \eta}d\xi\left\{\frac{v_e}{h_2}\frac{\partial S}{\partial X}\frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}\right.$$

$$\left.\left. + \frac{v_e}{h_1 h_2}S(X,Y,\Phi)\frac{\partial h_1}{\partial X} + \frac{v_e}{h_2}S(X,Y,\Phi)(\log k_0)_X\right\}v_e S_{\Phi\Phi}(X,Y,\Phi)\right] \tag{47}$$

Now,

$$\frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi} = \frac{u_e v_e}{h_1 h_2}F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\left[h_1\frac{\partial}{\partial X} + \int_0^\eta \frac{\partial h_2}{\partial \xi}d\eta\frac{\partial}{\partial Y} - \frac{1}{2}\frac{h_1\Phi}{(X+cY)}\frac{\partial}{\partial \Phi} + \frac{h_1\Phi}{2}\left\{\ln\frac{u_e + \alpha v_e}{\rho_e \mu_e}\right\}_X'\frac{\partial}{\partial \Phi}\right]h_2$$

$$\frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi} = \frac{u_e v_e}{h_2}F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\frac{\partial h_2}{\partial X} + \frac{u_e v_e}{h_1 h_2}F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\int_0^\eta \frac{\partial h_2}{\partial \xi}d\eta\frac{\partial h_2}{\partial Y} \tag{48}$$

$$\frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta} = \frac{u_e^{\ 2}}{h_1 h_2} F_\Phi^{\ 2}(X,Y,\Phi)\left[\int_0^\xi \frac{\partial h_1}{\partial \eta} d\xi \frac{\partial}{\partial X} + h_2 \frac{\partial}{\partial Y} - \frac{c}{2}\frac{h_2 \Phi}{(X+cY)}\frac{\partial}{\partial \Phi} + \frac{h_2 \Phi}{2}\left\{\ln \frac{u_e + \alpha v_e}{\rho_e \mu_e}\right\}_Y \frac{\partial}{\partial \Phi}\right] h_1$$

$$\frac{u_e^{\ 2}}{h_1 h_2}\frac{\partial h_1}{\partial \eta} = \frac{u_e^{\ 2}}{h_1} F_\Phi^{\ 2}(X,Y,\Phi)\frac{\partial h_1}{\partial Y} + \frac{u_e^{\ 2}}{h_1 h_2} F_\Phi^{\ 2}(X,Y,\Phi)\int_0^\xi \frac{\partial h_1}{\partial \eta} d\xi \frac{\partial h_1}{\partial X} \tag{49}$$

i.e. $$\frac{u_e}{h_1}\frac{\partial v_e}{\partial \xi} = u_e \frac{\partial v_e}{\partial X} + \frac{u_e}{h_1}\int_0^\eta \frac{\partial h_2}{\partial \xi} d\eta \frac{\partial v_e}{\partial Y} \tag{50}$$

$$\frac{v_e}{h_2}\frac{\partial v_e}{\partial \eta} = \frac{v_e}{h_2}\frac{\partial v_e}{\partial X}\int_0^\xi \frac{\partial h_1}{\partial \eta} d\xi + v_e \frac{\partial v_e}{\partial Y} \tag{51}$$

$$\frac{u_e v_e}{h_1 h_2}\frac{\partial h_2}{\partial \xi} = \frac{u_e v_e}{h_2}\frac{\partial h_2}{\partial X} + \frac{u_e v_e}{h_1 h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta \frac{\partial h_2}{\partial \xi} d\eta \tag{52}$$

$$\frac{u_e^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta} = \frac{u_e^2}{h_1}\frac{\partial h_1}{\partial Y} + \frac{u_e^2}{h_1 h_2}\frac{\partial h_1}{\partial X}\int_0^\xi \frac{\partial h_1}{\partial \eta} d\xi \tag{53}$$

$$\therefore \quad \frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\mu \frac{\partial v}{\partial \zeta}\right) = \frac{1}{\rho}\frac{\partial \Phi}{\partial \zeta}\frac{\partial}{\partial \Phi}\left\{\mu \frac{\partial \Phi}{\partial \zeta}\frac{\partial}{\partial \Phi} v_e S_\Phi(X,Y,\Phi)\right\}$$

or.

$$\frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\mu \frac{\partial v}{\partial \zeta}\right) = \frac{1}{\rho}\frac{\sqrt{(u_e + \alpha v_e)}}{\sqrt{\rho_e \mu_e (X+CY)}} \rho \frac{\partial}{\partial \Phi}\left\{\mu \frac{\sqrt{(u_e + \alpha v_e)}}{\sqrt{\rho_e \mu_e (X+CY)}} \rho v_e S_{\Phi\Phi}(X,Y,\Phi)\right\}$$

$$\therefore \quad \frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\mu \frac{\partial v}{\partial \zeta}\right) = \frac{(u_e + \alpha v_e)}{(X+CY)h_3} v_e \left(\frac{\rho \mu}{\rho_e \mu_e} S_{\Phi\Phi}(X,Y,\Phi)\right)_\Phi \tag{54}$$

Substitute equations (47) to (54) substitute into the equation (24); we have the

v-momentum equation in compressible flow as follows:

$$v_e \frac{(u_e + \alpha v_e)}{(X+CY)}\left(\frac{\rho \mu}{\rho_e \mu_e} S_{\Phi\Phi}(X,Y,\Phi)\right)_\Phi + \frac{1}{h_2} g_\eta \left(\frac{\rho - \rho_e}{\rho}\right) + \frac{\rho_e}{\rho}\left(u_e \frac{\partial v_e}{\partial X} + v_e \frac{\partial v_e}{\partial Y}\right)$$

$$- u_e F_\Phi(X,Y,\Phi)\left(v_e \frac{\partial S_\Phi}{\partial X} + S_\Phi(X,Y,\Phi)\frac{\partial v_e}{\partial X}\right) - v_e S_\Phi(X,Y,\Phi)\left(v_e \frac{\partial S_\Phi}{\partial Y} + S_\Phi(X,Y,\Phi)\frac{\partial v_e}{\partial Y}\right)$$

$$+ u_e v_e \frac{\partial F}{\partial X} S_{\Phi\Phi}(X,Y,\Phi) + v_e \frac{\partial u_e}{\partial X} F(X,Y,\Phi) S_{\Phi\Phi}(X,Y,\Phi) + u_e v_e (\log k_0)_X F(X,Y,\Phi) S_{\Phi\Phi}(X,Y,\Phi)$$

$$+ v_e^2 \frac{\partial S}{\partial Y} S_{\Phi\Phi}(X,Y,\Phi) + v_e \frac{\partial v_e}{\partial Y} S(X,Y,\Phi) S_{\Phi\Phi}(X,Y,\Phi) + v_e^2 (\log k_0)_Y S(X,Y,\Phi) S_{\Phi\Phi}(X,Y,\Phi) = H_v$$

Where,

$$H_1=\frac{(X+cY)}{v_e(u_e+\alpha v_e)}\left[\frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial}{\partial Y}\left(v_eS_\Phi(X,Y,\Phi)\right)+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial}{\partial X}\left(v_eS_\Phi(X,Y,\Phi)\right)\right.$$

$$-\frac{1}{h_2}\frac{\partial h_2}{\partial X}u_ev_eF(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi)-\frac{1}{h_1}\frac{\partial h_1}{\partial Y}v_e^2S(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi)-\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\left\{\frac{1}{h_1}u_e\frac{\partial F}{\partial Y}\right.$$

$$\left.-\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}+\frac{1}{h_1h_2}u_eF(X,Y,\Phi)\frac{\partial h_2}{\partial Y}+\frac{1}{h_1}u_eF(X,Y,\Phi)(\log k_0)_Y\right\}v_eS_{\Phi\Phi}(X,Y,\Phi)$$

$$-\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\left\{\frac{v_e}{h_2}\frac{\partial S}{\partial X}+\frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}+\frac{1}{h_1h_2}v_eS(X,Y,\Phi)\frac{\partial h_1}{\partial X}+\frac{1}{h_2}v_eS(X,Y,\Phi)(\log k_0)_X\right\}v_eS_{\Phi\Phi}(X,Y,\Phi)$$

$$+\frac{1}{h_2}u_ev_eF_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\frac{\partial h_2}{\partial X}+\frac{1}{h_1h_2}u_ev_eF_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial h_2}{\partial Y}$$

$$-\frac{u_e^2}{h_1}F_\Phi^2(X,Y,\Phi)\frac{\partial h_1}{\partial Y}+\frac{u_e^2}{h_1h_2}F_\Phi^2(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial h_1}{\partial X}-\frac{\rho_e}{\rho_1}\left\{\frac{u_e}{h_1}\frac{\partial v_e}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\right.$$

$$\left.\left.+\frac{v_e}{h_2}\frac{\partial v_e}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi+\frac{u_ev_e}{h_2}\frac{\partial h_2}{\partial X}+\frac{u_ev_e}{h_1h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta-\frac{u_e^2}{h_1}\frac{\partial h_1}{\partial Y}-\frac{u_e^2}{h_1h_2}\frac{\partial h_1}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\right\}\right]$$

$$\Rightarrow\left(\frac{\rho\mu}{\rho_e\mu_e}S_{\Phi\Phi}(X,Y,\Phi)\right)_\Phi^{\cdot}+\frac{(X+cY)}{v_e(u_e+\alpha v_e)}\frac{1}{h_2}g_\eta\left(\frac{\rho-\rho_e}{\rho}\right)+\frac{(X+cY)}{v_e(u_e+\alpha v_e)}\left\{v_e\frac{\partial v_e}{\partial Y}+v_e^2(\log k_0)_Y\right\}*$$

$$S(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi)+\frac{(X+cY)}{v_e(u_e+\alpha v_e)}\left\{v_e\frac{\partial u_e}{\partial X}+u_ev_e(\log k_0)_X\right\}F(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi)$$

$$\frac{(X+cY)}{v_e(u_e+\alpha v_e)}v_e\frac{\partial v_e}{\partial Y}\left(\frac{\rho_e}{\rho}-S_\Phi^2(X,Y,\Phi)\right)+\frac{(X+cY)}{v_e(u_e+\alpha v_e)}u_e\frac{\partial v_e}{\partial X}\left(\frac{\rho_e}{\rho}-F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\right)$$

$$=\frac{(X+cY)}{v_e(u_e+\alpha v_e)}\left\{u_ev_e\frac{\partial S_\Phi}{\partial X}F_\Phi(X,Y,\Phi)+v_e^2\frac{\partial S_\Phi}{\partial Y}S_\Phi(X,Y,\Phi)-u_ev_e\frac{\partial F}{\partial X}S_{\Phi\Phi}(X,Y,\Phi)\right.$$

$$\left.-v_e^2\frac{\partial S_\Phi}{\partial Y}S_{\Phi\Phi}(X,Y,\Phi)\right\}+H_v$$

$$\Rightarrow\left(\frac{\rho\mu}{\rho_e\mu_e}S_{\Phi\Phi}(X,Y,\Phi)\right)_\Phi+\frac{(X+cY)}{v_e(u_e+\alpha v_e)}\left\{v_e\frac{\partial v_e}{\partial Y}+v_e^2(\log k_0)_Y\right\}S(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi)$$

$$+\frac{(X+cY)}{v_e(u_e+\alpha v_e)}\left\{v_e\frac{\partial u_e}{\partial X}+u_ev_e(\log k_0)_X\right\}F(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi)+\frac{(X+cY)}{v_e(u_e+\alpha v_e)}v_e\frac{\partial v_e}{\partial Y}\left(\frac{\rho_e}{\rho}-S_\Phi^2(X,Y,\Phi)\right)$$

$$+\frac{(X+cY)}{v_e(u_e+\alpha v_e)}u_e\frac{\partial v_e}{\partial X}\left(\frac{\rho_e}{\rho}-F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\right)+\frac{V_\Gamma^2}{V_B^2}\theta$$

$$= \frac{(X+cY)}{v_e(u_e+\alpha v_e)}\frac{\delta(F,S_\Phi)}{\delta(X,\Phi)} + \frac{(X+cY)}{v_e(u_e+\alpha v_e)}\frac{\delta(S,S_\Phi)}{\delta(Y,\Phi)} + H,$$

$$\Rightarrow \left(\frac{\rho\mu}{\rho_e\mu_e} S_{\Phi\Phi}(X,Y,\Phi)\right)_\Phi + c(P_1+\Omega_0)S(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi) + (P_1+\Omega_0)F(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi)$$

$$+cP_1\left(\frac{\rho_e}{\rho} - S_\Phi^{\,2}(X,Y,\Phi)\right) + cQP_1\left(\frac{\rho_e}{\rho} - S_\Phi(X,Y,\Phi)F_\Phi(X,Y,\Phi)\right)$$

$$+\frac{V_F^{\,2}}{V_F}\theta = \left[J\frac{(F,S_\Phi)}{(X,\Phi)} + J\frac{(S,S_\Phi)}{(Y,\Phi)}\right] + H, \tag{55}$$

where

$$H_v = \frac{(X+cY)}{v_e(u_e+\alpha v_e)}\Bigg[\frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta \frac{\partial h_2}{\partial \xi}d\eta \frac{\partial}{\partial Y}(v_e S_\Phi(X,Y,\Phi)) + \frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi \frac{\partial h_1}{\partial \eta}d\xi \frac{\partial}{\partial X}(v_e S_\Phi(X,Y,\Phi$$

$$-\frac{1}{h_2}\frac{\partial h_2}{\partial X}u_e v_e F(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi) - \frac{1}{h_1}\frac{\partial h_1}{\partial Y}v_e^{\,2}S(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi) - \int_0^\eta \frac{\partial h_2}{\partial \xi}d\eta\Bigg\{\frac{1}{h_1}u_e\frac{\partial F}{\partial Y}$$

$$-\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y} + \frac{1}{h_1h_2}u_e F(X,Y,\Phi)\frac{\partial h_2}{\partial Y} + \frac{1}{h_1}u_e F(X,Y,\Phi)(\log k_0)_Y\Bigg\}v_e S_{\Phi\Phi}(X,Y,\Phi)$$

$$-\int_0^\xi \frac{\partial h_1}{\partial \eta}d\xi\Bigg\{\frac{v_e}{h_2}\frac{\partial S}{\partial X} + \frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X} + \frac{1}{h_1h_2}v_e S(X,Y,\Phi)\frac{\partial h_1}{\partial X} + \frac{1}{h_2}v_e S(X,Y,\Phi)(\log k_0)_X\Bigg\}v_e S_{\Phi\Phi}(X,Y,\Phi)$$

$$+\frac{1}{h_2}u_e v_e F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\frac{\partial h_2}{\partial X} + \frac{1}{h_1h_2}u_e v_e F_\Phi(X,Y,\Phi)S_\Phi(X,Y,\Phi)\int_0^\eta \frac{\partial h_2}{\partial \xi}d\eta\frac{\partial h_2}{\partial Y}$$

$$-\frac{u_e^{\,2}}{h_1}F_\Phi^{\,2}(X,Y,\Phi)\frac{\partial h_1}{\partial Y} + \frac{u_e^{\,2}}{h_1h_2}F_\Phi^{\,2}(X,Y,\Phi)\int_0^\xi \frac{\partial h_1}{\partial \eta}d\xi\frac{\partial h_1}{\partial X} - \frac{\rho_e}{\rho_1}\Bigg\{\frac{u_e}{h_1}\frac{\partial v_e}{\partial Y}\int_0^\eta \frac{\partial h_2}{\partial \xi}d\eta$$

$$+\frac{v_e}{h_2}\frac{\partial v_e}{\partial X}\int_0^\xi \frac{\partial h_1}{\partial \eta}d\xi + \frac{u_e v_e}{h_2}\frac{\partial h_2}{\partial X} + \frac{u_e v_e}{h_1h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta \frac{\partial h_2}{\partial \xi}d\eta - \frac{u_e^{\,2}}{h_1}\frac{\partial h_1}{\partial Y} - \frac{u_e^{\,2}}{h_1h_2}\frac{\partial h_1}{\partial X}\int_0^\xi \frac{\partial h_1}{\partial \eta}d\xi\Bigg\}\Bigg] \tag{55a}$$

where, $V_F^2 = -g_\xi \beta_T \Delta T \times$ characteristic length and characteristic length, $= \frac{1}{h_2}(X+CY)$.

Equation (55) is the transformed boundary layer v-momentum equation for a three dimensional compressible flow in curvilinear co-ordinates.

Boundary conditions are,

$$\left.\begin{array}{ll} F(X,Y,0) = F_\Phi(X,Y,0) = 0, & F_\Phi(X,Y,\infty) = 1 \\ S(X,Y,0) = S_\Phi(X,Y,0) = 0. & S_\Phi(X,Y,\infty) = 1 \\ \theta(X,Y,0) = 1, & \theta(X,Y,\infty) = 0 \end{array}\right\} \tag{56}$$

Energy equation :

$$\rho C_p\left(\frac{u}{h_1}\frac{\partial T}{\partial \xi}+\frac{v}{h_2}\frac{\partial T}{\partial \eta}+w\frac{\partial T}{\partial \zeta}\right)-\left(\frac{u}{h_1}\frac{\partial P}{\partial \xi}+\frac{v}{h_2}\frac{\partial P}{\partial \eta}\right)=\frac{\partial}{\partial \zeta}\left(k\frac{\partial T}{\partial \zeta}\right)+\mu\left\{\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial v}{\partial \zeta}\right)^2\right\} \tag{57}$$

Convective operator (for energy equations) are as follows : -

$$\left(\frac{u}{h_1}\frac{\partial}{\partial \xi}+\frac{v}{h_2}\frac{\partial}{\partial \eta}+w\frac{\partial}{\partial \zeta}\right)T=\left[u_eF_\Phi(X,Y,\Phi)\frac{\partial}{\partial X}+v_eS_\Phi(X,Y,\Phi)\frac{\partial}{\partial Y}-u_e\frac{\partial F}{\partial X}\frac{\partial}{\partial \Phi}-F(X,Y,\Phi)\frac{\partial u_e}{\partial X}\frac{\partial}{\partial \Phi}\right.$$
$$-u_eF(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial \Phi}-v_e\frac{\partial S}{\partial Y}\frac{\partial}{\partial \Phi}-S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}\frac{\partial}{\partial \Phi}-v_eS(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial \Phi}$$
$$-\frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\frac{\partial}{\partial Y}+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\frac{\partial}{\partial X}-\frac{u_e}{h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial X}\frac{\partial}{\partial \Phi}-\frac{v_e}{h_1}S(X,Y,\Phi)\frac{\partial h_1}{\partial Y}\frac{\partial}{\partial \Phi}$$
$$-\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\left\{\frac{u_e}{h_1}\frac{\partial F}{\partial Y}\frac{\partial}{\partial \Phi}+\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}\frac{\partial}{\partial \Phi}+\frac{u_e}{h_1h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial Y}\frac{\partial}{\partial \Phi}+\frac{u_e}{h_1}F(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial \Phi}\right\}$$
$$-\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\left\{\frac{v_e}{h_2}\frac{\partial F}{\partial Y}\frac{\partial}{\partial \Phi}+\frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}\frac{\partial}{\partial \Phi}+\frac{v_e}{h_1h_2}S(X,Y,\Phi)\frac{\partial h_1}{\partial X}\frac{\partial}{\partial \Phi}\right.$$
$$\left.\left.+\frac{v_e}{h_2}S(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial \Phi}\right\}\right]T \tag{58}$$

But we have , $\frac{T-T_e}{\Delta T}=\theta$ or $T-T_e=\Delta T\theta \Rightarrow T=T_e+\Delta T\theta$

i.e. convective operators are :-

$$\left(\frac{u}{h_1}\frac{\partial}{\partial \xi}+\frac{v}{h_2}\frac{\partial}{\partial \eta}+w\frac{\partial}{\partial \zeta}\right)(T_e+\Delta T\theta)=\left[u_eF_\Phi(X,Y,\Phi)\frac{\partial}{\partial X}+v_eS_\Phi(X,Y,\Phi)\frac{\partial}{\partial Y}-u_e\frac{\partial F}{\partial X}\frac{\partial}{\partial \Phi}\right.$$
$$-F(X,Y,\Phi)\frac{\partial u_e}{\partial X}\frac{\partial}{\partial \Phi}-u_eF(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial \Phi}-v_e\frac{\partial S}{\partial Y}\frac{\partial}{\partial \Phi}-S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}\frac{\partial}{\partial \Phi}$$
$$-v_eS(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial \Phi}-\frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\frac{\partial}{\partial Y}+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\frac{\partial}{\partial X}$$
$$-\frac{u_e}{h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial X}\frac{\partial}{\partial \Phi}-\frac{v_e}{h_1}S(X,Y,\Phi)\frac{\partial h_1}{\partial Y}\frac{\partial}{\partial \Phi}-\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\left\{\frac{u_e}{h_1}\frac{\partial F}{\partial Y}\frac{\partial}{\partial \Phi}+\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}\frac{\partial}{\partial \Phi}\right.$$
$$\left.+\frac{u_e}{h_1h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial Y}\frac{\partial}{\partial \Phi}+\frac{u_e}{h_1}F(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial \Phi}\right\}-\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\left\{\frac{v_e}{h_2}\frac{\partial S}{\partial Y}\frac{\partial}{\partial \Phi}\right.$$
$$\left.\left.+\frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}\frac{\partial}{\partial \Phi}+\frac{v_e}{h_1h_2}S(X,Y,\Phi)\frac{\partial h_1}{\partial X}\frac{\partial}{\partial \Phi}+\frac{v_e}{h_2}S(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial \Phi}\right\}\right](T_e+\Delta T\theta)$$

Or,

$$\left(\frac{u}{h_1}\frac{\partial}{\partial\xi}+\frac{v}{h_2}\frac{\partial}{\partial\eta}+w\frac{\partial}{\partial\zeta}\right)(T_e+\Delta T\theta)=\Bigg[u_eF_\Phi(X,Y,\Phi)\frac{\partial}{\partial X}(T_e+\Delta T\theta)+v_eS_\Phi(X,Y,\Phi)\frac{\partial}{\partial Y}(T_e+\Delta T\theta)$$

$$-u_e\frac{\partial F}{\partial X}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)-F(X,Y,\Phi)\frac{\partial u_e}{\partial X}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)-u_eF(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)$$

$$-v_e\frac{\partial S}{\partial Y}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)-S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)-v_eS(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)$$

$$-\frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial}{\partial Y}(T_e+\Delta T\theta)+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial}{\partial X}(T_e+\Delta T\theta)$$

$$-\frac{u_e}{h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial X}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)-\frac{v_e}{h_1}S(X,Y,\Phi)\frac{\partial h_1}{\partial Y}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)-\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\Bigg\{\frac{u_e}{h_1}\frac{\partial F}{\partial Y}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)$$

$$+\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)+\frac{u_e}{h_1h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial Y}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)$$

$$+\frac{u_e}{h_1}F(X,Y,\Phi)\frac{(k_0)_Y}{k_0}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)\Bigg\}-\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\Bigg\{\frac{v_e}{h_2}\frac{\partial S}{\partial Y}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)++\frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)$$

$$+\frac{v_e}{h_1h_2}S(X,Y,\Phi)\frac{\partial h_1}{\partial X}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)+\frac{v_e}{h_2}S(X,Y,\Phi)\frac{(k_0)_X}{k_0}\frac{\partial}{\partial\Phi}(T_e+\Delta T\theta)\Bigg\}\Bigg]$$

$$\Rightarrow\left(\frac{u}{h_1}\frac{\partial}{\partial\xi}+\frac{v}{h_2}\frac{\partial}{\partial\eta}+w\frac{\partial}{\partial\zeta}\right)(T_e+\Delta T\theta)=\Bigg[u_eF_\Phi(X,Y,\Phi)\frac{\partial}{\partial X}(\Delta T\theta)+v_eS_\Phi(X,Y,\Phi)\frac{\partial}{\partial Y}(\Delta T\theta)$$

$$-u_e\frac{\partial F}{\partial X}(\Delta T)\theta_\Phi-F(X,Y,\Phi)\frac{\partial u_e}{\partial X}(\Delta T)\theta_\Phi-u_eF(X,Y,\Phi)(\log k_0)_X(\Delta T)\theta_\Phi-v_e\frac{\partial S}{\partial Y}(\Delta T)\theta_\Phi$$

$$-S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}(\Delta T)\theta_\Phi-v_eS(X,Y,\Phi)(\log k_0)_Y(\Delta T)\theta_\Phi-\frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial}{\partial Y}(\Delta T\theta)$$

$$+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial}{\partial X}(\Delta T\theta)-\frac{u_e}{h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial X}(\Delta T)\theta_\Phi-\frac{v_e}{h_1}S(X,Y,\Phi)\frac{\partial h_1}{\partial Y}(\Delta T)\theta_\Phi$$

$$-\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\left\{\frac{u_e}{h_1}\frac{\partial F}{\partial Y}+\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}+\frac{u_e}{h_1h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial Y}+\frac{u_e}{h_1}F(X,Y,\Phi)(\log k_0)_Y\right\}(\Delta T)\theta_\Phi$$

$$-\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\Bigg\{\frac{v_e}{h_2}\frac{\partial S}{\partial Y}+\frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}+\frac{v_e}{h_1h_2}S(X,Y,\Phi)\frac{\partial h_1}{\partial X}$$

$$+\frac{v_e}{h_2}S(X,Y,\Phi)(\log k_0)_X\Bigg\}(\Delta T)\theta_\Phi\Bigg] \qquad (59a)$$

Eliminating pressure term of the energy equation by (6) and (7), we have

$$\frac{1}{h_1}\frac{\partial P}{\partial \xi}=\frac{\rho_e}{h_1}g_\xi-\rho_e\left[\frac{u_e}{h_1}\frac{\partial u_e}{\partial \xi}+\frac{v_e}{h_2}\frac{\partial u_e}{\partial \eta}+\frac{u_e v_e}{h_1 h_2}\frac{\partial h_1}{\partial \eta}-\frac{v_e^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right]$$

or. $$\frac{u}{h_1}\frac{\partial P}{\partial \xi}=\frac{\rho_e g_\xi u_e F_\Phi}{h_1}-\rho_e u_e F_\Phi\left[u_e\frac{\partial u_e}{\partial X}+v_e\frac{\partial u_e}{\partial Y}+u_e\frac{u_e v_e}{h_1}\frac{\partial h_1}{\partial Y}-\frac{v_e^2}{h_2}\frac{\partial h_2}{\partial X}+\frac{u_e}{h_1}\frac{\partial u_e}{\partial Y}\right.$$

$$\left.\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta+\frac{v_e}{h_2}\frac{\partial u_e}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi+\frac{u_e v_e}{h_1 h_2}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi+\frac{\partial h_1}{\partial X}-\frac{v_e^2}{h_1 h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\right] \tag{59b}$$

and

$$\frac{1}{h_2}\frac{\partial P}{\partial \eta}=\frac{\rho_e}{h_2}g_\eta-\rho_e\left[\frac{u_e}{h_1}\frac{\partial v_e}{\partial \xi}+\frac{v_e}{h_2}\frac{\partial v_e}{\partial \eta}+\frac{u_e v_e}{h_1 h_2}\frac{\partial h_2}{\partial \xi}-\frac{u_e^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta}\right]$$

or. $$\frac{v}{h_2}\frac{\partial P}{\partial \eta}=\frac{\rho_e g_\eta v_e S_\Phi}{h_2}-\rho_e v_e S_\Phi\left[u_e\frac{\partial v_e}{\partial X}+\frac{u_e}{h_1}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\frac{\partial v_e}{\partial Y}+v_e\frac{\partial v_e}{\partial Y}+\frac{v_e}{h_2}\frac{\partial v_e}{\partial X}\right.$$

$$\left.\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi+\frac{u_e v_e}{h_2}\frac{\partial h_2}{\partial X}+\frac{u_e v_e}{h_1 h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta-\frac{u_e^2}{h_1}\frac{\partial h_1}{\partial Y}-\frac{u_e^2}{h_1 h_2}\frac{\partial h_1}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\right]$$

or,

$$\frac{v}{h_2}\frac{\partial P}{\partial \eta}=\frac{\rho_e g_\eta v_e S_\Phi}{h_2}-\rho_e v_e S_\Phi\left[u_e\frac{\partial v_e}{\partial X}+v_e\frac{\partial v_e}{\partial Y}+\frac{u_e v_e}{h_2}\frac{\partial h_2}{\partial X}-\frac{u_e^2}{h_1}\frac{\partial h_1}{\partial Y}+\frac{u_e}{h_1}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\frac{\partial v_e}{\partial Y}\right.$$

$$\left.+\frac{v_e}{h_2}\frac{\partial v_e}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi+\frac{u_e v_e}{h_1 h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta-\frac{u_e^2}{h_1 h_2}\frac{\partial h_1}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\right] \tag{59c}$$

Adding (59b) and (59c)

$$\frac{u}{h_1}\frac{\partial P}{\partial \xi}+\frac{v}{h_2}\frac{\partial P}{\partial \eta}=\frac{\rho_e g_\eta v_e F_\Phi}{h_1}+\frac{\rho_e g_\eta v_e S_\Phi}{h_2}-\rho_e u_e F_\Phi\left[u_e\frac{\partial u_e}{\partial X}+v_e\frac{\partial u_e}{\partial Y}+\frac{u_e v_e}{h_1}\frac{\partial h_1}{\partial Y}-\frac{v_e^2}{h_2}\frac{\partial h_2}{\partial X}\right.$$

$$\left.+\frac{u_e}{h_1}\frac{\partial u_e}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta+\frac{v_e}{h_2}\frac{\partial u_e}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi+\frac{u_e v_e}{h_1 h_2}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\frac{\partial h_1}{\partial X}-\frac{v_e^2}{h_1 h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\right]$$

$$-\rho_e v_e S_\Phi\left[u_e\frac{\partial v_e}{\partial X}+v_e\frac{\partial v_e}{\partial Y}+\frac{u_e v_e}{h_2}\frac{\partial h_2}{\partial X}-\frac{u_e^2}{h_1}\frac{\partial h_1}{\partial Y}+\frac{u_e}{h_1}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\frac{\partial v_e}{\partial Y}\right.$$

$$\left.+\frac{v_e}{h_2}\frac{\partial v_e}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi+\frac{u_e v_e}{h_1 h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta-\frac{u_e^2}{h_1 h_2}\frac{\partial h_1}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\right]$$

or

$$\frac{u}{h_1}\frac{\partial P}{\partial \xi}+\frac{v}{h_2}\frac{\partial P}{\partial \eta}=-\left(\frac{\rho_e}{P}\right)E_c\left\{F_\Phi\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{\partial u_e}{\partial X}+QS_\Phi\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{\partial v_e}{\partial X}+QF_\Phi\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{\partial u_e}{\partial Y}\right.$$

$$\left.+S_\Phi\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{\partial v_e}{\partial Y}\right\}+\left\{\frac{\rho_e}{\rho}\frac{g_\xi F_\Phi}{u_e(u_e+\alpha v_e)}\frac{(X+CY)}{h_1}+\frac{\rho_e}{\rho}\frac{g_\eta S_\Phi}{v_e(u_e+\alpha v_e)}\frac{(X+CY)}{h_2}\right\}E_c$$

$$-\frac{1}{\rho C_P}\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{1}{\Delta T}\left[\rho_e u_e F_\Phi\left\{\frac{u_e v_e}{h_1}\frac{\partial h_1}{\partial Y}-\frac{v_e^2}{h_2}\frac{\partial h_2}{\partial X}+\frac{u_e}{h_1}\frac{\partial u_e}{\partial Y}\int_0^\xi\frac{\partial h_2}{\partial \xi}d\eta+\frac{v_e}{h_2}\frac{\partial u_e}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\right.\right.$$

$$\left.+\frac{u_e v_e}{h_1 h_2}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\frac{\partial h_1}{\partial X}-\frac{v_e^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\right\}+\rho_e v_e S_\Phi\left\{\frac{u_e v_e}{h_2}\frac{\partial h_2}{\partial X}-\frac{u_e^2}{h_1}\frac{\partial h_1}{\partial Y}\right.$$

$$\left.\left.+\frac{u_e}{h_1}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\frac{\partial v_e}{\partial Y}+\frac{v_e}{h_2}\frac{\partial v_e}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi+\frac{u_e v_e}{h_1 h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta-\frac{u_e^2}{h_1 h_2}\frac{\partial h_1}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\right]$$

or

$$\frac{u}{h_1}\frac{\partial P}{\partial \xi}+\frac{v}{h_2}\frac{\partial P}{\partial \eta}=-E_c(P_1F_\Phi+QP_1S_\Phi+QCP_1F_\Phi+CP_1S_\Phi+E_c\left\{\left(\frac{\rho_e}{P}\right)\frac{g_\xi F_\Phi(X+CY)}{u_e(u_e+\alpha v_e)h_1}\right.$$

$$\left.+\left(\frac{\rho_e}{P}\right)\frac{g_\xi F_\Phi(X+CY)}{u_e(u_e+\alpha v_e)h_1}\right\}-\left(\frac{\rho_e}{P}\right)E_c\left[F_\Phi\left\{\frac{v_e(X+CY)}{(u_e+\alpha v_e)h_1}\frac{\partial h_1}{\partial Y}-\frac{u_e(X+CY)}{(u_e+\alpha v_e)h_2}\frac{\partial h_2}{\partial X}\right.\right.$$

$$+\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{\partial u_e}{\partial Y}\frac{1}{h_1}\int_0^\xi\frac{\partial h_2}{\partial \xi}d\eta+Q\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{\partial u_e}{\partial X}\frac{1}{h_2}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi+\frac{v_e}{h_1h_2}\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{\partial h_1}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi$$

$$\left.-\frac{u_e}{h_1h_2}\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{\partial h_2}{\partial \xi}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\right\}+S_\Phi\left\{\frac{u_e(X+CY)}{(u_e+\alpha v_e)}\frac{\partial h_2}{\partial X}\frac{1}{h_2}-\frac{v_e(X+CY)}{h_1(u_e+\alpha v_e)}\frac{\partial h_1}{\partial Y}\frac{1}{h_2}\right.$$

$$+Q\frac{u_e(X+CY)}{(u_e+\alpha v_e)}\frac{1}{h_1}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta\frac{1}{v_e}\frac{\partial v_e}{\partial Y}+\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{1}{h_2}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\frac{\partial v_e}{\partial X}$$

$$\left.\left.+\frac{u_e(X+CY)}{(u_e+\alpha v_e)}\frac{1}{h_1h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial \xi}d\eta-\frac{v_e(X+CY)}{(u_e+\alpha v_e)}\frac{1}{h_1h_2}\frac{\partial h_1}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial \eta}d\xi\right\}\right] \tag{60}$$

Now,

$$\frac{\partial}{\partial \zeta}\left(k\frac{\partial T}{\partial \zeta}\right)=\frac{\partial \Phi}{\partial \zeta}\frac{\partial}{\partial \Phi}\left\{k\frac{\partial \Phi}{\partial \zeta}\frac{\partial}{\partial \Phi}\left(T_e+(\Delta T)\theta_\Phi\right)\right\}$$

or,

$$\frac{\partial}{\partial \zeta}\left(k\frac{\partial T}{\partial \zeta}\right)=\frac{\sqrt{u_e+\alpha v_e}}{\sqrt{\rho_e\mu_e(X+CY)}}\rho\frac{\partial}{\partial \Phi}\left\{k\frac{\sqrt{u_e+\alpha v_e}}{\sqrt{\rho_e\mu_e(X+CY)}}\rho(\Delta T)\theta_\Phi\right\}$$

$$\therefore \frac{\partial}{\partial \zeta}\left(k\frac{\partial T}{\partial \zeta}\right)=\frac{u_e+\alpha v_e}{\rho_e\mu_e(X+CY)}\rho\frac{\partial}{\partial \Phi}\{k\rho\,\Delta T\theta_\Phi\} \tag{61}$$

or,

$$\therefore \frac{1}{\rho C_p}\frac{\partial}{\partial \zeta}\left(k\frac{\partial T}{\partial \zeta}\right)=\rho\frac{u_e+\alpha v_e}{(X+CY)}\left\{\frac{1}{\rho C_p}k\frac{\rho}{\rho_e\mu_e}\Delta T\theta_\Phi\right\}_\Phi \frac{1}{\rho C_p}\frac{\partial}{\partial \zeta}\left(k\frac{\partial T}{\partial \zeta}\right)$$

$$=\frac{u_e+\alpha v_e}{(X+CY)}(\Delta T)\left\{\frac{k}{\mu C_p}\frac{\rho\mu}{\rho_e\mu_e}\theta_\Phi\right\}_\Phi \frac{1}{\rho C_p}\frac{\partial}{\partial \zeta}\left(k\frac{\partial T}{\partial \zeta}\right)=\frac{u_e+\alpha v_e}{(X+CY)}(\Delta T)\left\{\frac{1}{P_r}\frac{\rho\mu}{\rho_e\mu_e}\theta_\Phi\right\}_\Phi \tag{62}$$

where, $P_r=\frac{\mu C_p}{k}$ [Prandtl number]

Now, $\mu\left\{\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial v}{\partial \zeta}\right)^2\right\}=\mu\left[\left\{\frac{\partial \Phi}{\partial \zeta}\frac{\partial}{\partial \Phi}u_e F_\Phi(X,Y,\Phi)\right\}^2+\left\{\frac{\partial \Phi}{\partial \zeta}\frac{\partial}{\partial \Phi}v_e S_\Phi(X,Y,\Phi)\right\}^2\right]$

$$\mu\left\{\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial v}{\partial \zeta}\right)^2\right\}=\mu\left[\frac{(u_e+\alpha v_e)}{\rho_e\mu_e(X+CY)}\rho^2u_e^2F_{\Phi\Phi}^2(X,Y,\Phi)+\frac{(u_e+\alpha v_e)}{\rho_e\mu_e(X+CY)}\rho^2v_e^2S_{\Phi\Phi}^2(X,Y,\Phi)\right]$$

or,

$$\frac{1}{\rho C_p}\mu\left\{\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial v}{\partial \zeta}\right)^2\right\}=\frac{1}{\rho C_p}\mu\left[\frac{(u_e+\alpha v_e)}{\rho_e\mu_e(X+CY)}\rho^2u_e^2F_{\Phi\Phi}^2(X,Y,\Phi)+\frac{(u_e+\alpha v_e)}{\rho_e\mu_e(X+CY)}\rho^2v_e^2S_{\Phi\Phi}^2(X,Y,\Phi)\right]$$

or.

$$\frac{\mu}{\rho C_p}\left\{\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial v}{\partial \zeta}\right)^2\right\}=\frac{1}{\rho C_p}\frac{(u_e+\alpha v_e)}{(X+CY)}\left[\frac{\rho\mu}{\rho_e\mu_e}u_e^2F_{\Phi\Phi}^2(X,Y,\Phi)+\frac{\rho\mu}{\rho_e\mu_e}v_e^2S_{\Phi\Phi}^2(X,Y,\Phi)\right]$$

$$\frac{\mu}{\rho C_p}\left\{\left(\frac{\partial u}{\partial \zeta}\right)^2+\left(\frac{\partial v}{\partial \zeta}\right)^2\right\}=\frac{(u_e+\alpha v_e)}{(X+CY)}\left[\frac{\rho\mu}{\rho_e\mu_e}\frac{u_e^2}{C_p}F_{\Phi\Phi}^2(X,Y,\Phi)+\frac{\rho\mu}{\rho_e\mu_e}\frac{v_e^2}{C_p}S_{\Phi\Phi}^2(X,Y,\Phi)\right] \tag{63}$$

From the equations (58) to (62) substitute into the equation (26); we have the transform energy equation in compressible flow are as follows :

Or,

$$\frac{(u_e+\alpha v_e)}{(X+CY)}\Delta T\left(\frac{1}{P_r}\frac{\rho\mu}{\rho_e\mu_e}\theta_\Phi\right)_\Phi+\frac{(u_e+\alpha v_e)}{(X+CY)}\left\{\frac{\rho\mu}{\rho_e\mu_e}\frac{u_e^2}{C_p}F_{\Phi\Phi}^2(X,Y,\Phi)+\frac{\rho\mu}{\rho_e\mu_e}\frac{v_e^2}{C_p}S_{\Phi\Phi}^2(X,Y,\Phi)\right\}$$

$$=\Bigg[u_e F_\Phi(X,Y,\Phi)\frac{\partial}{\partial X}(\Delta T\theta)+v_e S_\Phi(X,Y,\Phi)\frac{\partial}{\partial Y}(\Delta T\theta)-u_e\frac{\partial F}{\partial X}(\Delta T)\theta_\Phi-F(X,Y,\Phi)\frac{\partial u_e}{\partial X}(\Delta T)\theta_\Phi$$

$$-u_e F(X,Y,\Phi)(\log k_0)_X(\Delta T)\theta_\Phi-v_e\frac{\partial S}{\partial Y}(\Delta T)\theta_\Phi-S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}(\Delta T)\theta_\Phi-v_e S(X,Y,\Phi)(\log k_0)_Y(\Delta T)\theta_\Phi$$

$$-\frac{v_e}{h_1}S(X,Y,\Phi)\frac{\partial h_1}{\partial Y}(\Delta T)\theta_\Phi-\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\Bigg\{\frac{u_e}{h_1}\frac{\partial F}{\partial Y}+\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}+\frac{u_e}{h_1h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial Y}$$

$$+\frac{u_e}{h_1}F(X,Y,\Phi)(\log k_0)_Y\Bigg\}(\Delta T)\theta_\Phi-\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\Bigg\{\frac{v_e}{h_2}\frac{\partial S}{\partial Y}+\frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}+\frac{v_e}{h_1h_2}S(X,Y,\Phi)\frac{\partial h_1}{\partial X}$$

$$+\frac{v_e}{h_2}S(X,Y,\Phi)(\log k_0)_Y\Bigg\}(\Delta T)\theta_\Phi\Bigg]-\frac{1}{\rho c_p}\Bigg[u_e F_\Phi(X,Y,\Phi)\frac{\partial P}{\partial X}+v_e S_\Phi(X,Y,\Phi)\frac{\partial P}{\partial Y}$$

$$+\frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial P}{\partial Y}+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial P}{\partial X}\Bigg]\left(\frac{1}{P_r}\frac{\rho\mu}{\rho_e\mu_e}\theta_\Phi\right)_\Phi$$

$$+E_c\left\{\frac{\rho\mu}{\rho_e\mu_e}\left(F_{\Phi\Phi}^2(X,Y,\Phi)+S_{\Phi\Phi}^2(X,Y,\Phi)\right)\right\}$$

$$=\frac{(X+cY)}{(u_e+\alpha v_e)}\frac{1}{\Delta T}\Bigg[u_e F_\Phi(X,Y,\Phi)\frac{\partial}{\partial X}(\Delta T\theta)+v_e S_\Phi(X,Y,\Phi)\frac{\partial}{\partial Y}(\Delta T\theta)-u_e\frac{\partial F}{\partial X}(\Delta T)\theta_\Phi$$

$$-F(X,Y,\Phi)\frac{\partial u_e}{\partial X}(\Delta T)\theta_\Phi-u_e F(X,Y,\Phi)(\log k_0)_X(\Delta T)\theta_\Phi-v_e\frac{\partial S}{\partial Y}(\Delta T)\theta_\Phi-S(X,Y,\Phi)\frac{\partial v_e}{\partial Y}(\Delta T)\theta_\Phi$$

$$-v_e S(X,Y,\Phi)(\log k_0)_Y(\Delta T)\theta_\Phi-\frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial}{\partial Y}(\Delta T\theta)+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial}{\partial X}(\Delta T\theta)$$

$$-\frac{u_e}{h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial X}(\Delta T)\theta_\Phi-\frac{v_e}{h_1}S(X,Y,\Phi)\frac{\partial h_1}{\partial Y}(\Delta T)\theta_\Phi-\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\Bigg\{\frac{u_e}{h_1}\frac{\partial F}{\partial Y}+\frac{1}{h_1}F(X,Y,\Phi)\frac{\partial u_e}{\partial Y}$$

$$+\frac{u_e}{h_1h_2}F(X,Y,\Phi)\frac{\partial h_2}{\partial Y}+\frac{u_e}{h_1}F(X,Y,\Phi)(\log k_0)_Y\Bigg\}(\Delta T)\theta_\Phi-\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\Bigg\{\frac{v_e}{h_2}\frac{\partial S}{\partial Y}+\frac{1}{h_2}S(X,Y,\Phi)\frac{\partial v_e}{\partial X}$$

$$+\frac{v_e}{h_1h_2}S(X,Y,\Phi)\frac{\partial h_1}{\partial X}+\frac{v_e}{h_2}S(X,Y,\Phi)(\log k_0)_Y\Bigg\}(\Delta T)\theta_\Phi\Bigg]-\frac{1}{\rho c_p}\frac{(X+cY)}{(u_e+\alpha v_e)}\frac{1}{\Delta T}\Bigg[u_e F_\Phi(X,Y,\Phi)\frac{\partial P}{\partial X}$$

$$+v_e S_\Phi(X,Y,\Phi)\frac{\partial P}{\partial Y}+\frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial P}{\partial Y}+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial P}{\partial X}\Bigg]$$

$$\Rightarrow\left(\frac{1}{P_r}\frac{\rho\mu}{\rho_e\mu_e}\theta_\Phi\right)_\Phi+E_c\left\{\frac{\rho\mu}{\rho_e\mu_e}\left(F_{\Phi\Phi}^2(X,Y,\Phi)+S_{\Phi\Phi}^2(X,Y,\Phi)\right)\right\}+\left\{u_e\frac{\partial F}{\partial X}+\frac{\partial u_e}{\partial X}F(X,Y,\Phi)\right.$$

$$+u_e(\log k_0)_X F(X,Y,\Phi)+v_e\frac{\partial S}{\partial Y}+\frac{\partial v_e}{\partial Y}S(X,Y,\Phi)+v_e(\log k_0)_Y S(X,Y,\Phi)\Bigg\}\frac{(X+cY)}{(u_e+\alpha v_e)}\theta_\Phi$$

$$=\frac{(X+cY)u_e}{\Delta T(u_e+\alpha v_e)}\frac{\partial}{\partial X}(\Delta T\theta)F_\Phi(X,Y,\Phi)+\frac{(X+cY)v_e}{\Delta T(u_e+\alpha v_e)}\frac{\partial}{\partial Y}(\Delta T\theta)S_\Phi(X,Y,\Phi)+H_e$$

Or, $$\left(\frac{1}{P_r}\frac{\rho\mu}{\rho_e\mu_e}\theta_\Phi\right)_\Phi+E_c\left\{\frac{\rho\mu}{\rho_e\mu_e}\left(F^2_{\Phi\Phi}(X,Y,\Phi)+S^2_{\Phi\Phi}(X,Y,\Phi)\right)\right\}+\Bigg\{\frac{u_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial F}{\partial X}$$

$$+\frac{(X+cY)}{(u_e+\alpha v_e)}\frac{\partial u_e}{\partial X}F(X,Y,\Phi)+\frac{v_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial S}{\partial Y}+\frac{(X+cY)}{(u_e+\alpha v_e)}\frac{\partial v_e}{\partial Y}S(X,Y,\Phi)$$

$$+\frac{u_e(X+cY)}{(u_e+\alpha v_e)}(\log k_0)_X F(X,Y,\Phi)+\frac{v_e(X+cY)}{(u_e+\alpha v_e)}(\log k_0)_Y S(X,Y,\Phi)\Bigg\}\theta_\Phi$$

$$=\frac{(X+cY)u_e}{\Delta T(u_e+\alpha v_e)}\left(\Delta T\frac{\partial\theta}{\partial X}+\theta\frac{\partial(\Delta T)}{\partial X}\right)F_\Phi(X,Y,\Phi)+\frac{(X+cY)v_e}{\Delta T(u_e+\alpha v_e)}\left(\Delta T\frac{\partial\theta}{\partial Y}+\theta\frac{\partial(\Delta T)}{\partial Y}\right)S_\Phi(X,Y,\Phi)+H_E$$

where $E_c=\dfrac{u_e^2}{c_p\Delta T}$ [Eckert number]

or, $$\left(\frac{1}{P_r}\frac{\rho\mu}{\rho_e\mu_e}\theta_\Phi\right)_\Phi+E_c\left\{\frac{\rho\mu}{\rho_e\mu_e}\left(F^2_{\Phi\Phi}(X,Y,\Phi)+S^2_{\Phi\Phi}(X,Y,\Phi)\right)\right\}+\Bigg\{\frac{u_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial F}{\partial X}$$

$$+\frac{v_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial S}{\partial Y}+\frac{u_e(X+cY)}{(u_e+\alpha v_e)}\frac{1}{u_e}\frac{\partial u_e}{\partial X}F(X,Y,\Phi)+\frac{v_e(X+cY)}{(u_e+\alpha v_e)}\frac{1}{v_e}\frac{\partial v_e}{\partial Y}S(X,Y,\Phi)$$

$$+\frac{u_e(X+cY)}{(u_e+\alpha v_e)}(\log k_0)_X F(X,Y,\Phi)+\frac{v_e(X+cY)}{(u_e+\alpha v_e)}(\log k_0)_Y S(X,Y,\Phi)\Bigg\}\theta_\Phi$$

$$-\left\{\frac{(X+cY)u_e}{\Delta T(u_e+\alpha v_e)}\frac{\partial(\Delta T)}{\partial X}F_\Phi(X,Y,\Phi)+\frac{(X+cY)v_e}{\Delta T(u_e+\alpha v_e)}\frac{\partial(\Delta T)}{\partial Y}S_\Phi(X,Y,\Phi)\right\}\theta$$

$$=\frac{(X+cY)u_e}{(u_e+\alpha v_e)}\frac{\partial(\theta)}{\partial X}F_\Phi(X,Y,\Phi)+\frac{(X+cY)v_e}{(u_e+\alpha v_e)}\frac{\partial\theta}{\partial Y}S_\Phi(X,Y,\Phi)+H_E$$

Or,

$$\left(\frac{1}{P_r}\frac{\rho\mu}{\rho_e\mu_e}\theta_\Phi\right)_\Phi+E_c\left\{\frac{\rho\mu}{\rho_e\mu_e}\left(F^2_{\Phi\Phi}(X,Y,\Phi)+S^2_{\Phi\Phi}(X,Y,\Phi)\right)\right\}+\left(P_1F(X,Y,\Phi)+3cQP_1S(X,Y,\Phi)\right)\theta_\Phi$$

$$-\left(P_1F_\Phi(X,Y,\Phi)+cQP_1S_\Phi(X,Y,\Phi)\right)\theta$$

$$+P_1\left[\left(\frac{E_c}{F_{rx}}+\frac{U_F^2}{U_e^2}E_c\right)F_\Phi(X,Y,\Phi)+\left(\frac{E_c}{F_{rY}}+\frac{V_F^2}{V_e^2}E_c\right)S_\Phi(X,Y,\Phi)\right]=H_F \qquad (64)$$

where $\frac{U_e^2}{g_X L}=F_{rX}$ and $\frac{V_e^2}{g_Y L}=F_{rY}$ [Froude number]

$$H_F=\frac{(X+cY)}{(u_e+\alpha v_e)}\frac{1}{\Delta T}\left[\frac{u_e}{h_1}F_\Phi(X,Y,\Phi)\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial(\Delta T\theta)}{\partial Y}+\frac{v_e}{h_2}S_\Phi(X,Y,\Phi)\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial(\Delta T\theta)}{\partial X}\right.$$

$$-\left(\frac{u_e}{h_2}\frac{\partial h_2}{\partial X}F(X,Y,\Phi)+\frac{v_e}{h_1}\frac{\partial h_1}{\partial Y}S(X,Y,\Phi)\right)(\Delta T)\theta_\Phi-\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\left\{\frac{u_e}{h_1}\frac{\partial F}{\partial Y}+\frac{F}{h_1}\frac{\partial u_e}{\partial Y}\right.$$

$$\left.+\frac{u_e}{h_1h_2}\frac{\partial h_2}{\partial Y}F(X,Y,\Phi)+\frac{u_e}{h_1}(\log k_0)_Y F(X,Y,\Phi)\right\}(\Delta T)\theta_\Phi-\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\left\{\frac{v_e}{h_2}\frac{\partial S}{\partial X}\right.$$

$$\left.+\frac{1}{h_2}\frac{\partial v_e}{\partial X}S(X,Y,\Phi)+\frac{v_e}{h_1h_2}\frac{\partial h_1}{\partial X}S(X,Y,\Phi)+\frac{v_e}{h_2}(\log k_0)_X S(X,Y,\Phi)\right\}(\Delta T)\theta_\Phi$$

$$-\frac{1}{\rho c_p}\left\{\frac{u_e}{h_1}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{\partial P}{\partial Y}F_\Phi(X,Y,\Phi)+\frac{v_e}{h_2}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial P}{\partial X}S_\Phi(X,Y,\Phi)\right\}$$

$$-\left(\frac{\rho_e}{P}\right)E_c\left(F_\Phi\left\{\frac{v_e(X+CY)}{(u_e+\alpha v_e)h_1}\frac{\partial h_1}{\partial Y}-\frac{u_e(X+CY)}{(u_e+\alpha v_e)h_2}\frac{\partial h_2}{\partial X}+\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{\partial u_e}{\partial Y}\frac{1}{h_1}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\right.\right.$$

$$+Q\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{\partial u_e}{\partial X}\frac{1}{h_2}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi+\frac{v_e}{h_1h_2}\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{\partial h_1}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi$$

$$\left.-\frac{u_e}{h_1h_2}\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{\partial h_2}{\partial\xi}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\right\}+S_\Phi\left\{\frac{u_e(X+CY)}{(u_e+\alpha v_e)}\frac{\partial h_2}{\partial X}\frac{1}{h_2}-\frac{v_e(X+CY)}{h_1(u_e+\alpha v_e)}\frac{\partial h_1}{\partial Y}\frac{1}{h_2}\right.$$

$$+Q\frac{u_e(X+CY)}{(u_e+\alpha v_e)}\frac{1}{h_1}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta\frac{1}{v_e}\frac{\partial v_e}{\partial Y}+\frac{(X+CY)}{(u_e+\alpha v_e)}\frac{1}{h_2}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\frac{\partial v_e}{\partial X}$$

$$\left.\left.\left.+\frac{u_e(X+CY)}{(u_e+\alpha v_e)}\frac{1}{h_1h_2}\frac{\partial h_2}{\partial Y}\int_0^\eta\frac{\partial h_2}{\partial\xi}d\eta-\frac{v_e(X+CY)}{(u_e+\alpha v_e)}\frac{1}{h_1h_2}\frac{\partial h_1}{\partial X}\int_0^\xi\frac{\partial h_1}{\partial\eta}d\xi\right\}\right)\right] \qquad (64a)$$

The boundary condition for (64) are the traditional boundary condition (46).

The co-efficients of the dependent variables in the energy equation are.

(1) $$\frac{u_e(X+cY)}{(u_e+\alpha v_e)}\frac{1}{u_e}\frac{\partial u_e}{\partial X}=\frac{1}{1+\alpha Q}n=P_1$$

(2) $$\frac{v_e(X+cY)}{(u_e+\alpha v_e)}\frac{1}{v_e}\frac{\partial v_e}{\partial Y}=\frac{cQ}{1+\alpha Q}n=cQP_1$$

(3) $$\frac{u_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial}{\partial X}\left(\log k_0\right)=\frac{1}{1+\alpha Q}n=N(\Phi)\Omega_0$$

(4) $$\frac{v_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial}{\partial Y}\left(\log k_0\right)=\frac{cQ}{1+\alpha Q}n=N(\Phi)c\,\Omega_0$$

(5) $$\frac{u_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial}{\partial X}\left(\log \Delta T\right)=\frac{1}{1+\alpha Q}n=P_1$$

(6) $$\frac{v_e(X+cY)}{(u_e+\alpha v_e)}\frac{\partial}{\partial Y}\left(\log \Delta T\right)=\frac{cQ}{1+\alpha Q}n=cQP_1$$

Chapter 4

STUDY OF DIFERENT CASES FOR A DEVELOPABLE SURFACE

We shall discuss different situation for different values of h_1 and h_2 satisfying the following equation for a developable surface.

$$K = -\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \xi}(h_2 k_2) + \frac{\partial}{\partial \eta}(h_1 k_1)\right] \qquad \text{(A)}$$

where K is the total Gaussian curvature,

substituting the values of $k_1 = \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \eta}$ and $k_2 = \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \xi}$ the equation (A) takes the form.

$$\frac{\partial}{\partial \xi}\left(\frac{1}{h_1}\frac{\partial h_2}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{h_2}\frac{\partial h_1}{\partial \eta}\right) = 0 \qquad \text{(4-1)}$$

To derive different nature of of h_1 and h_2 we are to use generalised values of h_1 and h_2 as to satisfy the equation (4.1).

__Case I__

$h_1 = f_1(\xi)$ and $h_2 = f_2(\eta)$ are the easiest solutions of (4-1). For simplicity we may take for the above case as

$h_1 = a\xi^m, \quad h_2 = a'\eta^{m'}$

$h_1 = be^{n\xi}; \quad h_2 = b'e^{n'\eta}$

$h_1 = c\sin p\xi; \quad h_2 = c'\sin(p'\eta)$

$h_1 = c\cos p\xi; \quad h_2 = c'\cos(p'\eta)$

$h_1 = ae^{n\xi}\xi^m; \quad h_2 = a'e^{n'\eta}\eta^{n'}$ etc.

__Case II__

If $\quad h_1 = (\xi + \beta\eta)^m$ and $h_2 = (\xi + \beta\eta)^n$, then by (A) we have

$$n\frac{\partial}{\partial\xi}\left[\frac{1}{(\xi+\beta\eta)^m}(\xi+\beta\eta)^{n-1}\right]+\beta\frac{\partial}{\partial\eta}\left[\frac{1}{(\xi+\beta\eta)^n}(\xi+\beta\eta)^{m-1}\right]=0$$

$$\text{or, } n\frac{\partial}{\partial\xi}\left[(\xi+\beta\eta)^{n-m-1}\right]+\beta\frac{\partial}{\partial\eta}\left[(\xi+\beta\eta)^{m-n-1}\right]=0$$

$$n(n-m-1)(\xi+\beta\eta)^{n-m-2}+\beta^2(m-n-1)(\xi+\beta\eta)^{m-n-2}=0$$

$$n(n-m-1)(\xi+\beta\eta)^{n-m}+\beta^2(m-n-1)(\xi+\beta\eta)^{m-n}=0$$

$$n(n-m-1)+\beta^2(m-n-1)(\xi+\beta\eta)^{2m-2n}=0$$

If $m=n$,

$$-n-\beta^2=0\Rightarrow\beta=\pm i\sqrt{n},\ \pm i\sqrt{m}$$

for real values of m or n, β becomes imaginary. In this situation say for $m = n = 1$. $h_1 = \xi+i\eta$ which really absurd. Hence we have to put $m = n = 0$ implying $h_1=1=h_2$. This is the situation of rectangular Cartesian system, one of the developable situations. These cases are extensively discussed by Mager ('64).

Case III

Suppose $\beta=0 \Rightarrow$ $n(n-m-1)=0$, $n\neq 0\Rightarrow n=m+1$

i.e., $h_1=\xi^m$ and $h_2=\xi^{m+1}$ represents one of the developable surface .

Similarly $h_1=\eta^{m+1}$, $h_2=\eta^m$ represent also a developable surface .

Case IV

Treating if $h_1=\xi^m\eta^n$ and $h_2=\xi^{m'}\eta^{n'}$, we have from (4-1)

$$\frac{\partial}{\partial\xi}\left[\frac{1}{\xi^m\eta^n}\frac{\partial}{\partial\xi}\left(\xi^{m'}\eta^{n'}\right)\right]+\frac{\partial}{\partial\eta}\left[\frac{1}{\xi^{m'}\eta^{n'}}\frac{\partial}{\partial\eta}\left(\xi^m\eta^n\right)\right]=0$$

$$\eta^{n'-n}m'(m'-m-1)\xi^{m'-m-2}+\xi^{m-m'}(n-n'-1)\eta^{n-n'-2}=0$$

$$\{m'(m'-m-1)+n(n-n'-1)\}\xi^{2(m-m'-1)}\eta^{2(n-n'-1)}=0$$

$\Rightarrow$ (i) $m = m' - 1, \quad n = n' + 1$

or, $m + 1 = m', \quad n = n'$

(ii) $m'(m' - m - 1) + n(n - n' - 1) = 0$

$(m+1).0 + n(n - n + 1 - 1) = 0$

$0 \quad + \quad 0 \qquad = 0$

Hence $h_1 = \xi^m \eta^n$, then $h_2 = \xi^{m+1}\eta^{n-1}$ represents an another developable surface.

Case V

$h_1 = e^{p\xi + q\eta}$ and $h_2 = e^{p'\xi + q'\eta}$

$= e^{p\xi}.e^{q\eta}$ and $\quad e^{p'\xi}.e^{q'\eta}$

$$e^{(q+q')\eta} p' \frac{\partial}{\partial \xi}\left\{e^{-p\xi}.e^{p'\xi}\right\} + e^{(p+p')\xi} q \frac{\partial}{\partial \eta}\left\{e^{-q'\eta}.e^{q\eta}\right\} = 0$$

or, $p'(p'-p)e^{(p'-p)\xi + (q+q')\eta} + q(q-q')e^{(q-q')\eta + (p+p')\xi} = 0$

or, $p'(p'-p) + q(q-q')e^{2(p\xi - q'\eta)} = 0$

This implies $p = 0, \quad q' = 0$. That is, $h_1 = e^{q\eta}$ and $h_2 = e^{p'\xi}$.

This is similar to the one of the situation of case I.

Case VI

Again, let us consider

$h_1 = ae^{p\xi + q\eta}\left(\xi^m \eta^n\right)\left(a'\xi + b'\eta\right)$ ---------------------- (i)

and $h_2 = be^{p'\xi + q'\eta}\left(\xi^{m'} \eta^{n'}\right)\left(a'\xi + b'\eta\right)$ ------------- (ii)

or, $h_1 = e^{p\xi + q\eta}\left(\xi^m \eta^n\right)\left(aa'\xi + ab'\eta\right)$

or, $h_1 = e^{p\xi + q\eta}\left(aa'\xi^{m+1}\eta^n + ab'\xi^m \eta^{n+1}\right)$

or, $h_1 = e^{p\xi + q\eta}\left(c_1\xi^{m+1}\eta^n + c_2\xi^m \eta^{n+1}\right)$ where $c_1 = aa'$ and $c_2 = ab'$.

Similarly,

$h_2 = e^{p'\xi + q'\eta}\left(c_3\xi^{m'+1}\eta^{n'} + c_4\xi^{m'}\eta^{n'+1}\right)$

or, $c_3 = ba'$ and $c_4 = bb'$.

Substitute h_1 and h_2 in (4-1), we get

$$\frac{\partial}{\partial\xi}\left[\frac{e^{p'\xi+q'\eta}\left\{p'\left(c_1\xi^{m'+1}\eta^{n'}+c_4\xi^{m'}\eta^{n'+1}\right)+c_3(m'+1)\xi^{m'}\eta^{n'}+c_4m'\xi^{m'-1}\eta^{n'+1}\right\}}{e^{p\xi+q\eta}\left(c_1\xi^{m+1}\eta^{n}+c_2\xi^{m}\eta^{n+1}\right)}\right]$$

$$+\frac{\partial}{\partial\eta}\left[\frac{e^{p\xi+q\eta}\left\{p\left(c_1\xi^{m+1}\eta^{n}+c_2\xi^{m}\eta^{n-1}\right)+c_2(m+1)\xi^{m}\eta^{n}+c_2m\xi^{m-1}\eta^{n+1}\right\}}{e^{p'\xi+q'\eta}\left(c_3\xi^{m+1}\eta^{n'}+c_3\xi^{m'}\eta^{n'+1}\right)}\right]$$

From equation (4-1) it is clear that if we consider $h_1 = h_1(\xi)$ and $h_2 = h_2(\eta)$ the equation remain valid. Hence we may chose ($a = 1$, $b = 1$, $a' = 1$, $b' = 0$, $m = \alpha - 1$, $n = 0$, $m' = -1$, $n' = \gamma$, $p = \beta$, $q = 0$, $p' = 0$, $q' = \delta$.

Generally $h_1 = \xi^{\alpha} e^{\beta\xi}$ and $h_2 = \eta^{\gamma} e^{\delta\eta}$; $h_1 \neq h_2$.

Case VII

If $h_1 = h_2$: then the equation (4-1) becomes,

$$\frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\xi}\log h_1\right)+\frac{\partial}{\partial\eta}\left(\frac{\partial}{\partial\eta}\log h_1\right)=0$$

or, $\frac{\partial}{\partial\xi}(\log h_1)_\xi + \frac{\partial}{\partial\eta}(\log h_1)_\eta = 0$ ---------------------------------- (i)

Let, $(\log h_1) = u$ equation (i) becomes $\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} = 0$ -------- (ii)

Now suppose $u = X(\xi)Y(\eta)$ (ii) becomes

$$\frac{1}{X}\frac{\partial^2 X}{\partial\xi^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial\eta^2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial\xi^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial\eta^2} = -\lambda^2 \quad \text{(say)}$$

$\therefore\ X = A\cos\lambda x + B\sin\lambda x$

$Y = Ce^{\lambda t} + De^{-\lambda t}$

$\therefore\ u = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda t} + De^{-\lambda t})$

i.e. $u = (A\cos\lambda x + B\sin\lambda x)De^{-\lambda t}$

$\therefore \ \log h_1 = (A\cos\lambda x + B\sin\lambda x)De^{-\lambda y}$

$\Rightarrow \quad h_1 = \exp(A\cos\lambda x + B\sin\lambda x)De^{-\lambda y}$ ------------------- (iii)

$= e^{-\gamma\eta}\left[A'\cos\lambda\ X + B'\sin\lambda\ X\right]$

or $= e^{-\gamma\xi}\left[A''\cos\lambda'\ Y + B''\sin\lambda'\ Y\right]$

avoiding exponential increasing function for a practical purpose.

λ and λ' are the eigenvalues. Details of Laplace's equations are discussed in many text books with different boundary conditions.

STUDY OF CASE I

We shall now use $h_1 = \eta^{\bar{m}}$ and $h_2 = \xi^{\bar{n}}$ (i.e. one of situation of case I) into the equation (45a), (55a) and (64a). Since $h_1 = \eta^{\bar{m}}$ and $h_2 = \xi^{\bar{n}}$ satisfied the condition of the developable surface and conveniently using $\bar{m} = \bar{n} = 1$, the equations (45), (55) and (64) for u-momentum equation, v-momentum equation and energy equation become

$$\Rightarrow \quad (N(\Phi)F_{\Phi\Phi\Phi}(X,Y,\Phi))_{\Phi} + (P_1 + \Omega_0 + k_1)F_{\Phi}(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi) + \{c(P_1+\Omega_0)+k_2\}$$

$$S(X,Y,\Phi)F_{\Phi\Phi}(X,Y,\Phi) + P_1(1+k_6)\left(\frac{\rho_e}{\rho} - P_2F_{\Phi}^2(X,Y,\Phi)\right)$$
$$+ cQP_1\left(\frac{\rho_e}{\rho} - P_3F_{\Phi}(X,Y,\Phi)S_{\Phi}(X,Y,\Phi)\right) + k_5S_{\Phi}^2(X,Y,\Phi) + \frac{U_F^2}{U_e^2}\theta = 0 \qquad (65a)$$

$$\Rightarrow \quad (N(\Phi)S_{\Phi\Phi\Phi}(X,Y,\Phi))_{\Phi} + \{c(P_1+\Omega_0)+k_7\}S(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi) + (P_1 + \Omega_0 + k_8)$$

$$F(X,Y,\Phi)S_{\Phi\Phi}(X,Y,\Phi) + cP_1\left(\frac{\rho_e}{\rho} - P_4S_{\Phi}^2(X,Y,\Phi)\right)$$
$$+ cQP_1\left(\frac{\rho_e}{\rho} - P_5S_{\Phi}(X,Y,\Phi)F_{\Phi}(X,Y,\Phi)\right) + k_{11}F_{\Phi}^2(X,Y,\Phi) + \frac{V_F^2}{V_e^2}\theta = 0 \qquad (65b)$$

$$\Rightarrow \quad (N(\Phi)\theta_{\Phi}(X,Y,\Phi))_{\Phi} + E_c\{N(F_{\Phi\Phi}^2(X,Y,\Phi) + S_{\Phi\Phi}^2(X,Y,\Phi))\} + P_1\left\{\left(\frac{E_c}{F_r} + \frac{U_f^2}{U_e^2}\right)F_{\Phi}(X,Y,\Phi)\right.$$

$$+\left(\frac{E_c}{F_{rV}}+\frac{V_F^2}{V_e^2}\right)S_\Phi(X,Y,\Phi)+\{(P_1+k_{12})F(X,Y,\Phi)+(cQP_1+k_{11})S(X,Y,\Phi)\}\theta_\Phi(X,Y,\Phi)$$

$$-\{(P_1+k_3+k_{14})F_\Phi(X,Y,\Phi)+(cQP_1+k_5+k_{15})S_\Phi(X,Y,\Phi)\}\theta(X,Y,\Phi)=0 \qquad (65c)$$

The boundary conditions are,

$$\left.\begin{array}{l} F(X,Y,0)=F_\Phi(X,Y,0)=0, \quad F_\Phi(X,Y,\infty)=1 \\ S(X,Y,0)=S_\Phi(X,Y,0)=0, \quad S_\Phi(X,Y,\infty)=1 \\ \theta(X,Y,0)=1, \theta(X,Y,\infty)=0 \end{array}\right\} \qquad (66)$$

where $\frac{U_e^2}{g_X L}=F_{rX}$ and $\frac{V_e^2}{g_Y L}=F_{rY}$ [Froude number]

$k_1=2P_1(1+c)+c\Omega_0$

$k_2=QP_1(3+c)+\Omega_0$

$k_3=2cP_1$

$k_4=P_1\{(2+Q)(1+c)\}$

$k_5=2QP_1$

$k_6=P_1(1+Q)$

$k_7=cQP_1$

$k_8=\{P_1(1+3c+3Q)+2\Omega_0\}$

$k_9=2P_1(1+c)$

$k_{10}=QP_1(1-c)$

$k_{11}=P_1(1-Qc)$

$k_{12}=4P_1+c\Omega_0$

$k_{13}=3QP_1+cP_1+Q\Omega_0$

$k_{14}=P_1(2Q+cQ+c-2)$

$k_{15}=P_1(Qc-Q-3)$

$P_2=\left(\frac{1+k_3}{1+k_6}\right)$, $\quad P_3=(1+k_4)$

$P_4=\frac{k_5}{cP_1}$

$P_5=\frac{QcP_1}{1+k_{12}}$

$k_1, k_2 \ldots\ldots\ldots k_{15}$ are the additional effects due to the introduction of curvilinear co-ordinates (i.e. presence of h_1 and h_2) in the boundary layer equation. All p's and k's are all in terms of α, Q, C and η. n mainly represents exponent for h_1, h_2 which are responsible for the different shapes of the developable surfaces connecting with the exponent of external forcing velocity components u_e.

Chapter – 5

Numerical solutions, Results and Discussions.

The boundary value problems described by the equation (65-a-b-c) with boundary conditions. (66) are to be solved. It is difficult to get their solutions analytically in closed forms. Hence we adopt a procedure to get the solutions numerically. Runge-Kutta shooting method in collaboration with Runge-Kutta-Merson and Sweggert iteration techniques are applied here.

We are about to discuss the nature of the flow processes arising from mixed convection flow for the various values of the controlling parameters $\left[P_r, E_c, N, \frac{U_F^2}{u_e^2}, \frac{V_F^2}{v_e^2}, k's, p_1, p_2\right]$. Here k's and p's are the additional effects described in different stages. Numerical results are obtained for different value of k_2 and k_5 and coupling parameter $\frac{U_F^2}{u_e^2}, \frac{V_F^2}{v_e^2}$. Representative velocity and temperature profiles are shown in fig (1-8). The skin friction factors $(f''(0). s''(0))$ and the heat transfer co-efficient $(-\theta'(0))$ are give in table (1-6) for different values of controlling parameters.

The heat transfer and the shear stress characteristics of the problem may be of great practical interest. So the numerical results for $f''(0). s''(0)$ and $-\theta'(0)$ are presented in tabular form. Velocity profiles are also shown graphically. These are taken for the variations of controlling parameter $\left[P_r, E_c, N, \frac{U_F^2}{u_e^2}, \frac{V_F^2}{v_e^2}, k's, p_1 . p_2\right]$ in the range $0 \le c \le 1$. Fig. 1(a) shows that one of the components of velocity profiles $f'(\Phi)$ exhibit its unusual behaviour for the increasing values of $\frac{U_F^2}{u_e^2}$ where as other component $S'(\Phi)$ [Fig. 1(c)] rises with the increase of $\frac{U_F^2}{u_e^2}$ for $\frac{V_F^2}{v_e^2}=0$. When the buoyancy velocity is much higher than forcing velocity $\left[\text{i.e.}\frac{U_F^2}{u_e^2}=\frac{V_F^2}{v_e^2}=100\right]$, the components $(F'(\Phi),S'(\Phi))$ of the velocity profiles show their usual trends in changing the flow configuration from forced to free convection flow, but the effects of $\frac{U_F^2}{u_e^2}$ variation on $F'(\Phi)$ and $\frac{V_F^2}{v_e^2}$ on $S'(\Phi)$ are similar [Fig. 2(a) and 2(b)].

Decrease on heat transfer co-efficient are found with the increase of $\frac{U_F^2}{u_e^2}$ in [Fig. 2(a)]. The velocity profiles $(F'(\Phi),S'(\Phi))$ for the fixed value of controlling parameters $P_r=1.0$; $E_c=0.8$, $N=1$ $p_1=0.5$, $p_2=1.5$, $\left[\frac{U_F^2}{u_e^2}=\frac{V_F^2}{v_e^2}=0\right]$ in the range of $0\le c\le 1$ show their dissimilarity in shapes [Fig. 5(a) and 5(b)]. This indicates that frictional resistance`s on both the edge are not same.

One of the component of skin friction factors is comparable with that of Sparrow, Eichorn and Gregg (1959), but the heat transfer coefficient $(-\theta'(0))$ is not directly comparable due to presence of length function h_1 and h_2 in the governing equations [Table 5-6]. Now the nature of flow process arising from mixed convection with respect to fixed controlling parameters $P_r=1.0$; $E_c=0.8$, $N=1$ $C=0.1$) the velocity profiles $(F'(\Phi),S'(\Phi))$ in figures 7(a) and 7(b) shows that the increase of coupling parameters $\frac{U_F^2}{u_e^2}$ and $\frac{V_F^2}{v_e^2}$, flow processes tendes to coincide with those of free convection.

Discussions

If $N = 1$, $E_c = 0$, $P_1 = \frac{U_F^2}{F_{rX}} = \frac{V_F^2}{F_{rt}} = 0$, $k_{13} = 0$, $p_1 + k_{12} = m + 1$,

$p_1 + k_3 + k_{14} = -(n-2)$ and $k_5 = k_{15} = 0$ then the energy equation coincides with that of Sparrow, Eichorn and Gregg (1959). That is for a incompressible flow,

$N = 1 \Rightarrow \frac{\rho\mu}{\rho_e\mu_e} = 1$ or. $\frac{\rho}{\rho_e} = 1$, then $\frac{\mu}{\mu_e} = 1$ for a Boussinesq fluid.

To compare with the known similarity case of Sparrow , Eichorn and Gregg (1959) we have to set $h_1 = 1 = h_2$. Thus controlling parameters $\frac{U_F^2}{u_e^2}$ becomes

$\frac{-g_a\beta_t\Delta T(X+CY)}{h_1 u_e^2} = \frac{-g_a\beta_T\Delta T\, X}{u_e^2}$ for $C = 0$. As $u_e = u_0(X+CY)^n$ becomes

$u_e = u_0\, X^n$. Writing $\frac{U_f^2}{u_e^2} = Cont.$ we have $\Delta T = K_0 X^{2n-1}$ which agrees with the result of Sparrow , Eichorn and Gregg. Hence we can predict that our equations are true for curvilinear surfaces also.

Table-1

K_2	$F''(0)$	$S''(0)$	$-\theta'(0)$
0.5	1.24816	1.24741	0 63637
1.0	1.24801	1.24703	0.63602
1.5	1.24789	1.24681	0.63577
2.0	1.24760	1.24630	0.63531
2.5	1.24711	1 24591	0.63498
3.0	1.24681	1.24551	0 63453
3.5	1.24634	1.24503	0.63407
4.0	1.24011	1.24476	0.63371
5.0	1.23910	1.24429	0.63349

For Pr = 1.0, P_1 = 1.0, ufe = 0.0, vfe = 0.0

Table-2

K_2	$F''(0)$	$S''(0)$	$-\theta'(0)$
0.10000	11.63750	10.63364	2.22290
0.40000	11.63001	10.64015	2.22297
0.70000	11.62251	10.64724	2.22305
1.00000	11.61549	10.65143	2.22314
1.30000	11.60721	10.65839	2.22323
1.60000	11.60031	10.66153	2.22329
1 90000	11.59280	10 66891	2.22307
2.20000	11.58531	10.67152	2.22313
2.50000	11.57785	10.67854	2.22321
2.80000	11.57003	10 68159	2.22330
3.10000	11.56332	10.68911	2.22338
3.40000	11.55742	10.69661	2.22347
3.70000	11.55002	10.70317	2.22353
4.00000	11.54219	10.71025	2.22361
4.30000	11.53416	10.71832	2.22368
4 60000	11.52823	10.72564	2.22378
4 90000	11.52057	10.73239	2.22385

For Pr = 1.0, P_1 = 1 0, ufe = 0.0, vfe = 0.0

Table-3

K_5	$F''(0)$	$S''(0)$	$-\theta'(0)$
0.10000	5.76944	5.51443	0.82045
0.40000	5.76903	5.51439	0.82044
0.70000	5.76861	5.51435	0.82044
1.00000	5.76820	5.51430	0.82044
1.30000	5.76779	5.51426	0.82043
1.60000	5.76738	5.51422	0.82043
1.90000	5.76697	5.51418	0.82043
2.20000	5.76655	5.51414	0.82042
2.50000	5.76614	5.51410	0.82042
2.80000	5.76573	5.51406	0.82042
3.10000	5.76531	5.51402	0.82041
3.40000	5.76490	5.51398	0.82041
3.70000	5.76449	5.51394	0.82041
4.00000	5.76408	5.51389	0.82040
4.30000	5.76366	5.51385	0.82040
4.60000	5.76325	5.51381	0.82040
4.90000	5.76284	5.51377	0.82039

For Pr = 1.0, P_1=1.0, ufe = 100.0, vfe = 100.0, c = 0.1

Table-4

K_5	$F''(0)$	$S''(0)$	$-\theta'(0)$
0.10000	1.14503	1.14324	0.63847
0.40000	1.14462	1.14320	0.63842
0.70000	1.14420	1.14316	0.63837
1.00000	1.14379	1.14312	0.63832
1.30000	1.14338	1.14308	0.63827
1.60000	1.14297	1.14304	0.63822
1.90000	1.14255	1.14299	0.63817
2.20000	1.14214	1.14295	0.63812
2.50000	1.14173	1.14291	0.63807
2.80000	1.14132	1.14287	0.63802
3.10000	1.14090	1.14283	0.63797
3.40000	1.14049	1.14279	0.63792
3.70000	1.14008	1.14275	0.63787
4.00000	1.13967	1.14271	0.63782
4.30000	1.13925	1.14267	0.63777
4.60000	1.13884	1.14263	0.63772
4.90000	1.13843	1.14259	0.63767

For Pr = 1.0, P_1=1.0, ufe = 0.0, vfe = 0.0, c = 0.1

Table-5

P_1	$F''(0)$	$S''(0)$	$-\theta'(0)$
0.10000	0.87659	0.87817	0.69504
0.40000	0.87672	0.87826	0.69509
0.70000	0.87685	0.87835	0.69514
1.00000	0.87698	0.87844	0.69519
1.30000	0.87711	0.87853	0.69523
1.60000	0.87724	0.87862	0.69527
1.90000	0.87737	0.87871	0.69532
2.20000	0.87750	0.87880	0.69537
2.50000	0.87763	0.87889	0.69542
2.80000	0.87776	0.87898	0.69547

For Pr = 1.0, P_2 =0.0, ufe = 0.0, vfe = 0.0, c = 1.0
N = 1.0, ak = 1.0, k's = 0.0.

Table-6

P_1	$F''(0)$	$S''(0)$	$-\theta'(0)$
0.10000	30.61012	30.59021	2.17944
0.40000	30.61059	30.59050	2.17963
0.70000	30.61096	30.59087	2.17984
1.00000	30.61143	30.59126	2.18003
1.30000	30.61182	30.59163	2.18024
1.60000	30.61229	30.59200	2.18043
1.90000	30.61268	30.59239	2.18064
2.20000	30.61315	30.59276	2.18083
2.50000	30.61354	30.59315	2.18199
2.80000	30.61301	30.59352	2.18219

For Pr = 1.0, P_2=0.0, ufe = 100.0, vfe = 100.0, c = 1.0
N = 1.0, ak = 1.0, k's = 0.0.

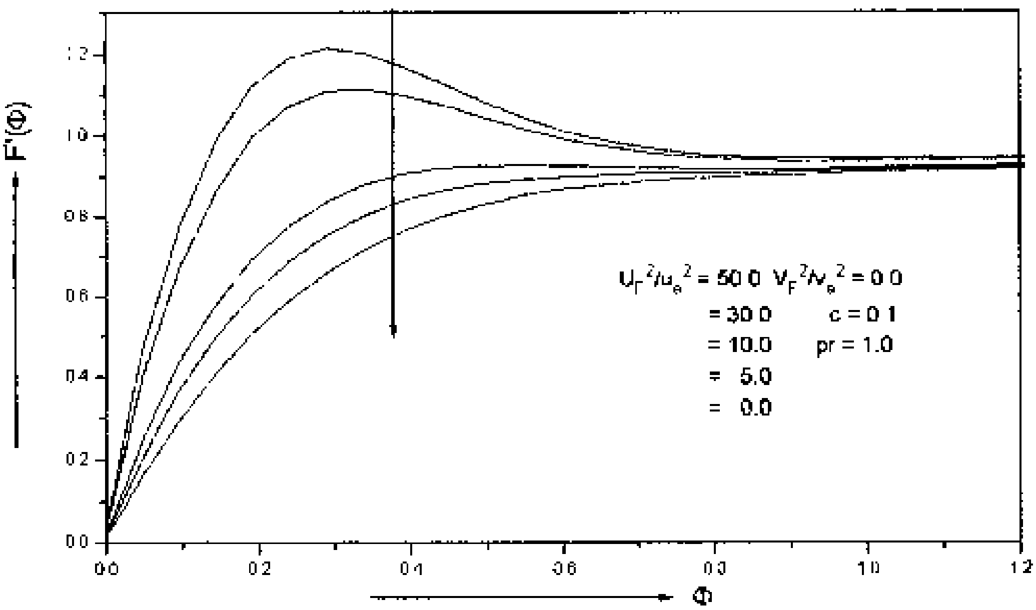


Fig (1.a): Dimension less velocity distribution along u-direction for several values of U_F^2/u_e^2 (=50.0,30.0,10.0,5.0,0.0) for the equation (65)

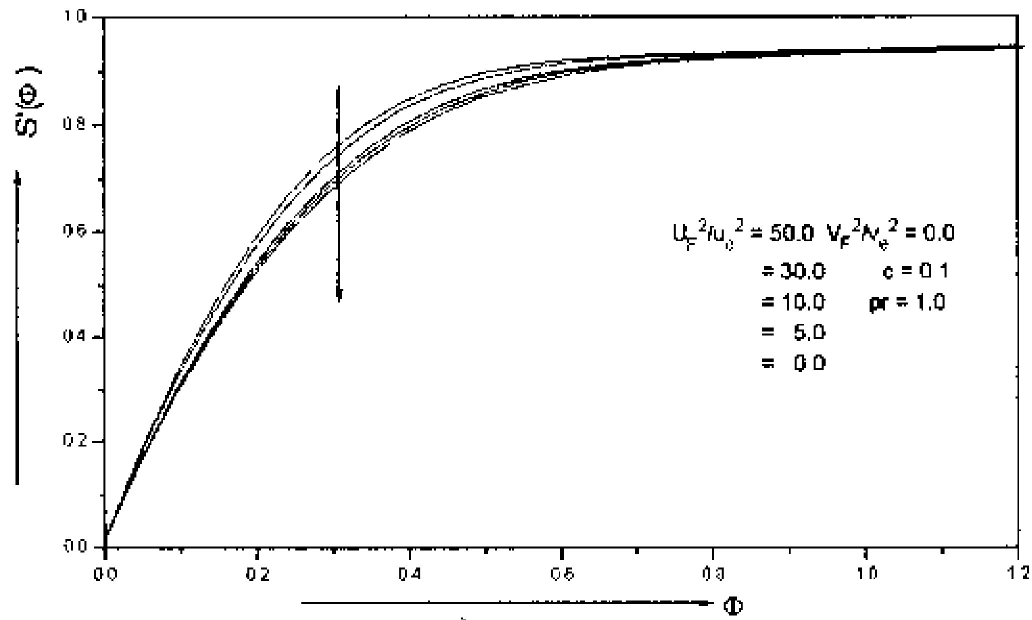
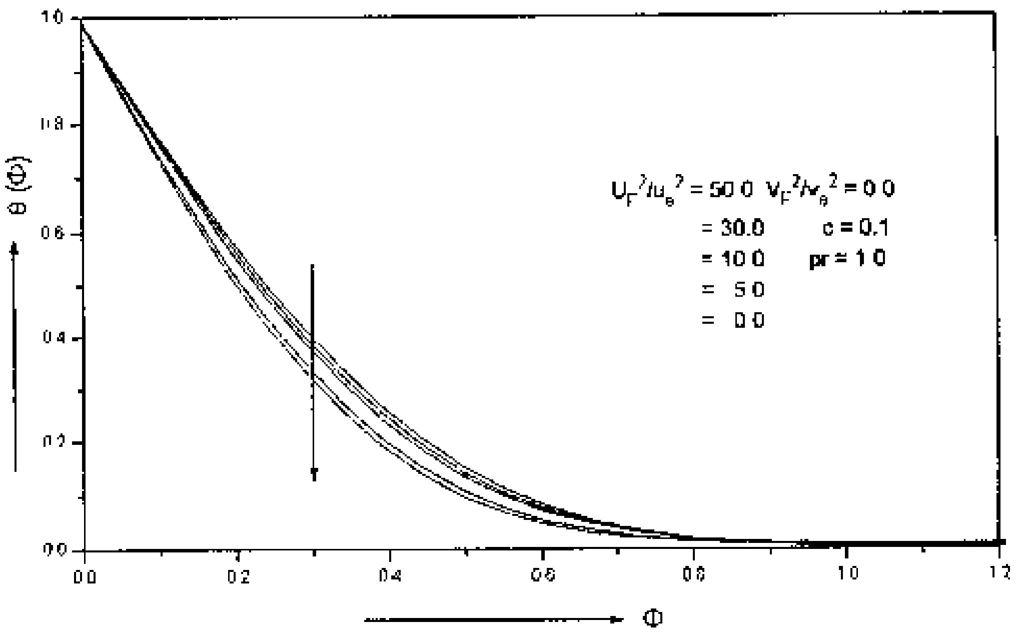
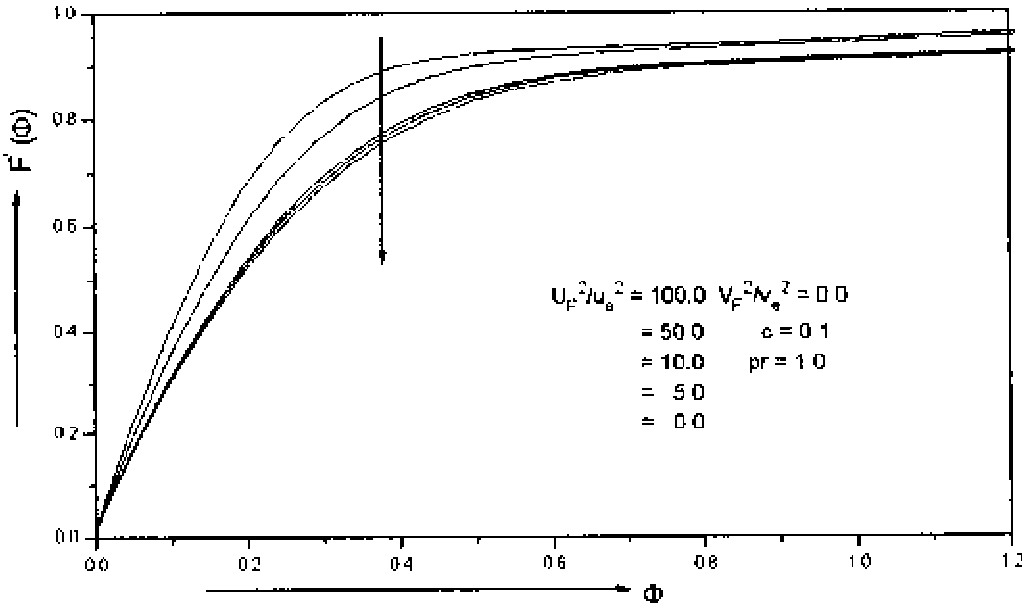


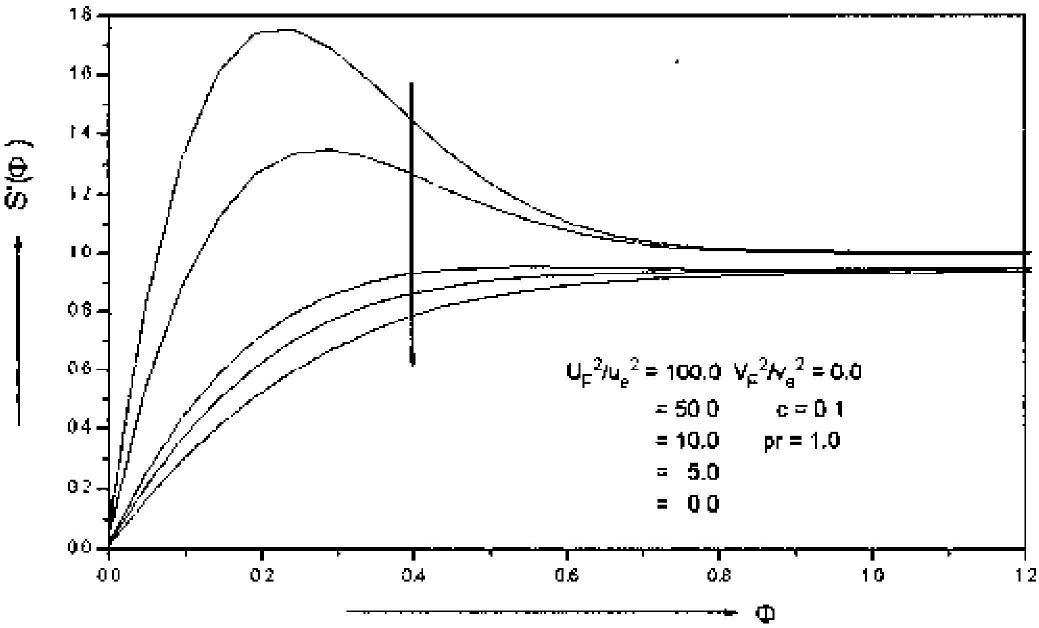
Fig (1 b): Dimension less velocity distribution along v-direction for several values of U_F^2/u_e^2 (=50.0,30.0,10.0,5.0,0.0) for the equation (65)



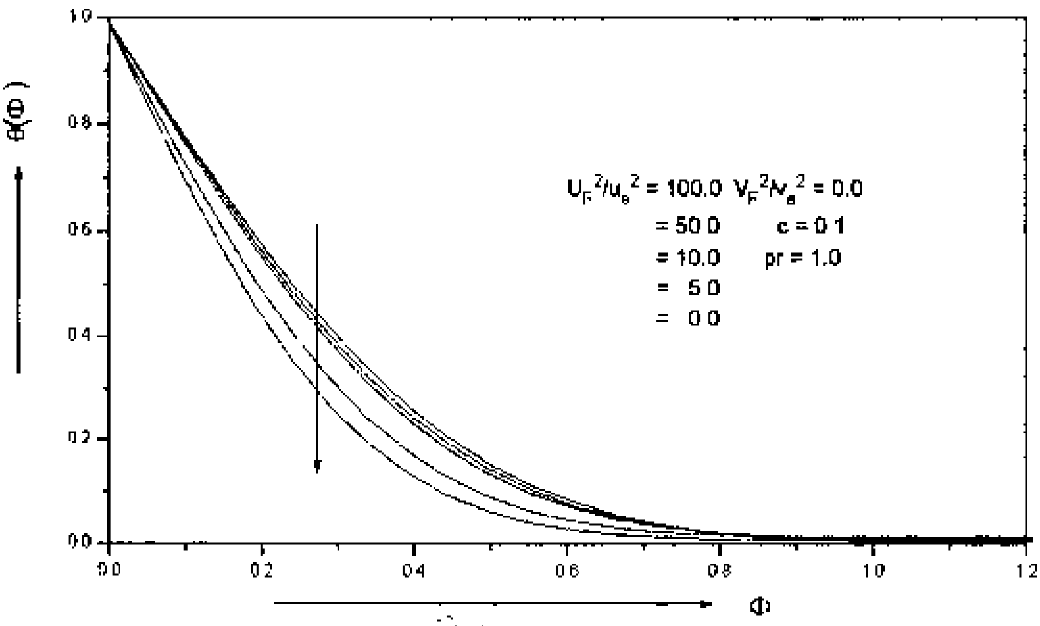
Fig(1 c) Dimensionless temperature distribution for several values of U_F^2/u_e^2 (=50.0,30.0,10.0,5.0,0.0) for the equation (65)



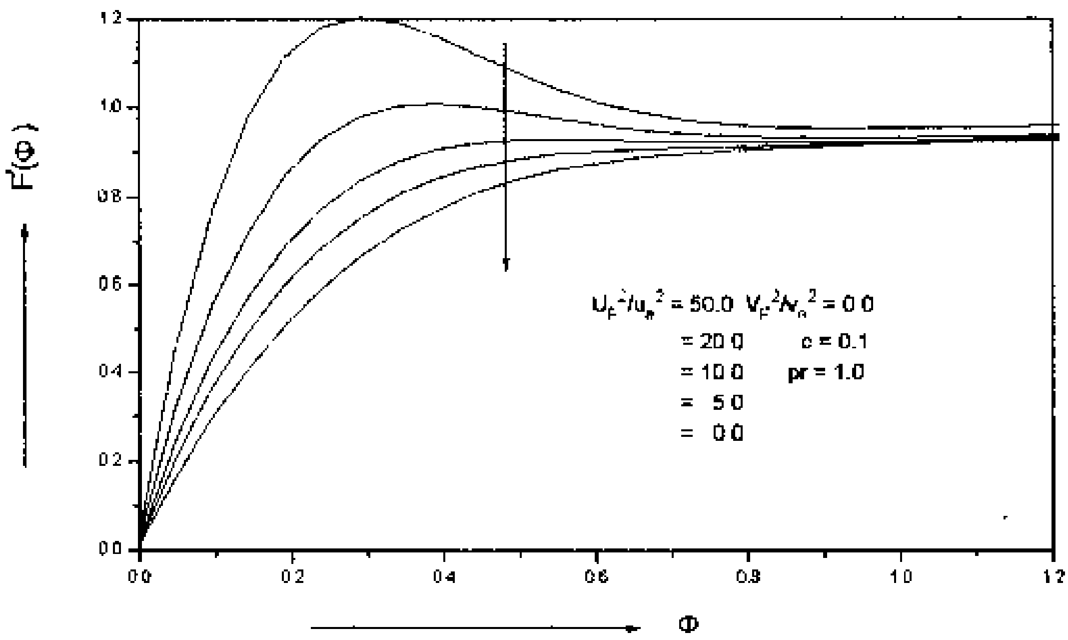
Fig(2.a): Dimensionless velocity distribution along u-direction for several values of V_F^2/v_e^2(=100.0,50.0,10.0,5.0,0.0) for the equation (65)



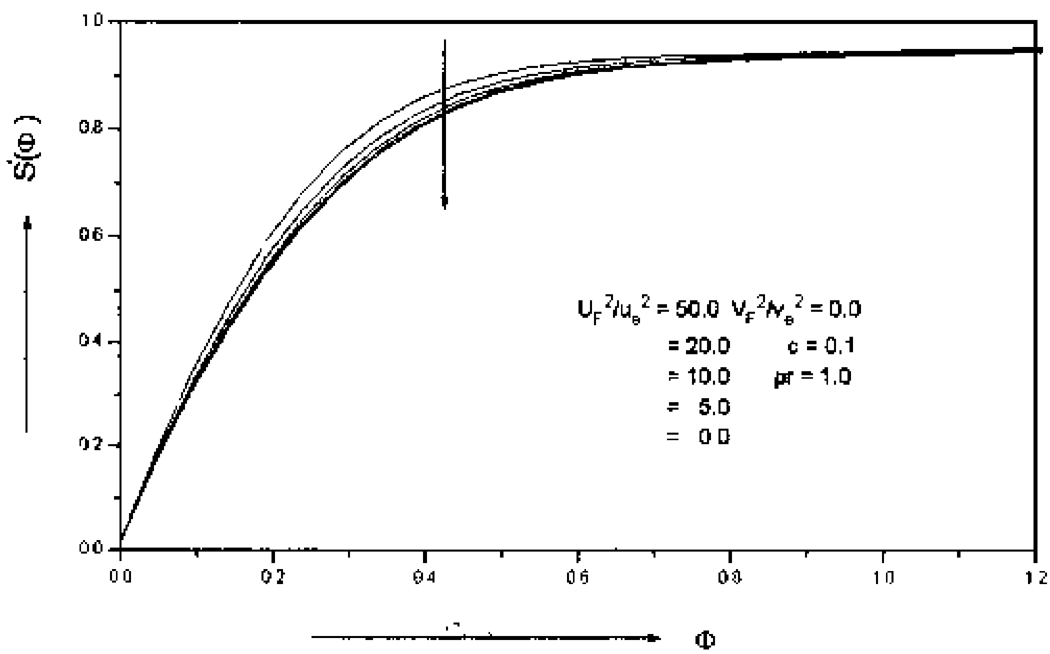
Fig(2 b): Dimensionless velocity distribution along v-direction for several values of V_F^2/v_e^2 (=100.0,50.0,10.0,5.0,0.0) for the equation (65)



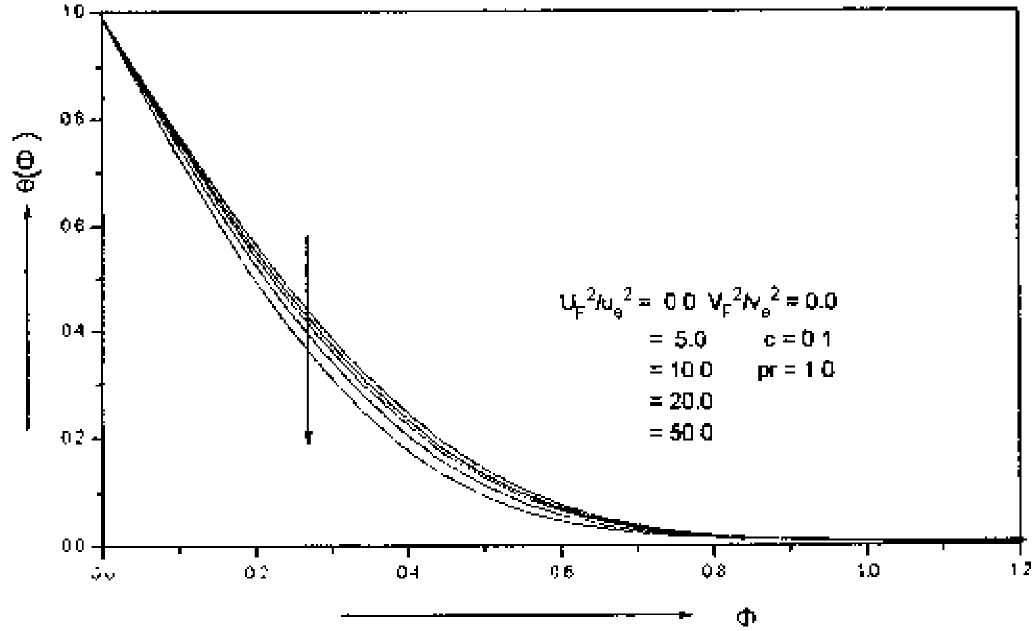
Fig(2.c): Dimensionless temperature distribution for several values of V_F^2/v_e^2 (=100.0,50.0,10.0,5.0,0.0) for the equation (65)



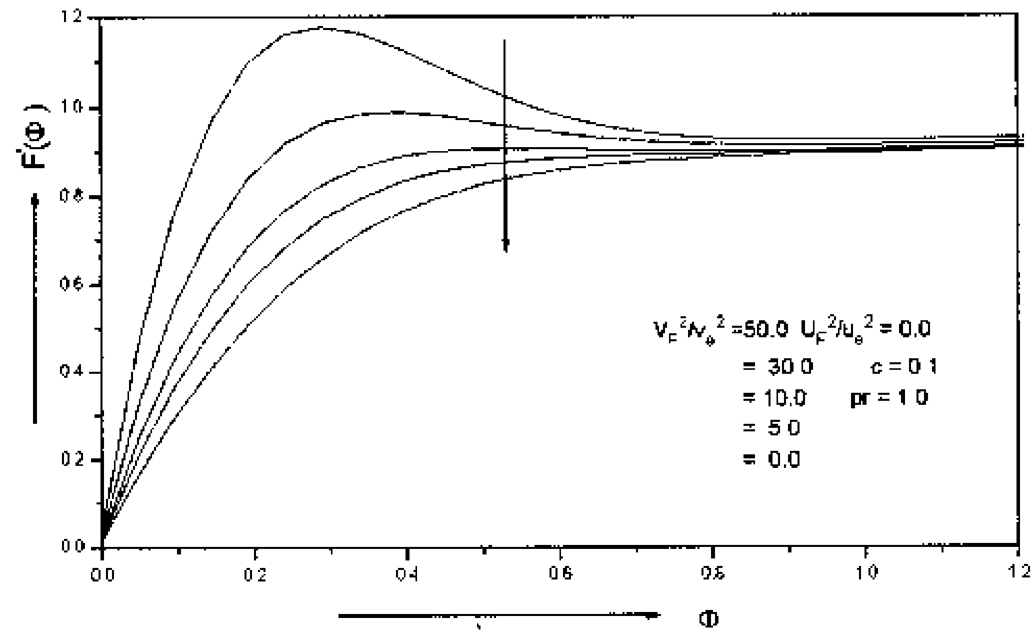
Fig(3.a): Dimensionless velocity distribution along u-direction for several values of U_F^2/u_e^2 (=50.0,20.0,10.0,5.0,0.0) for the equation (65)



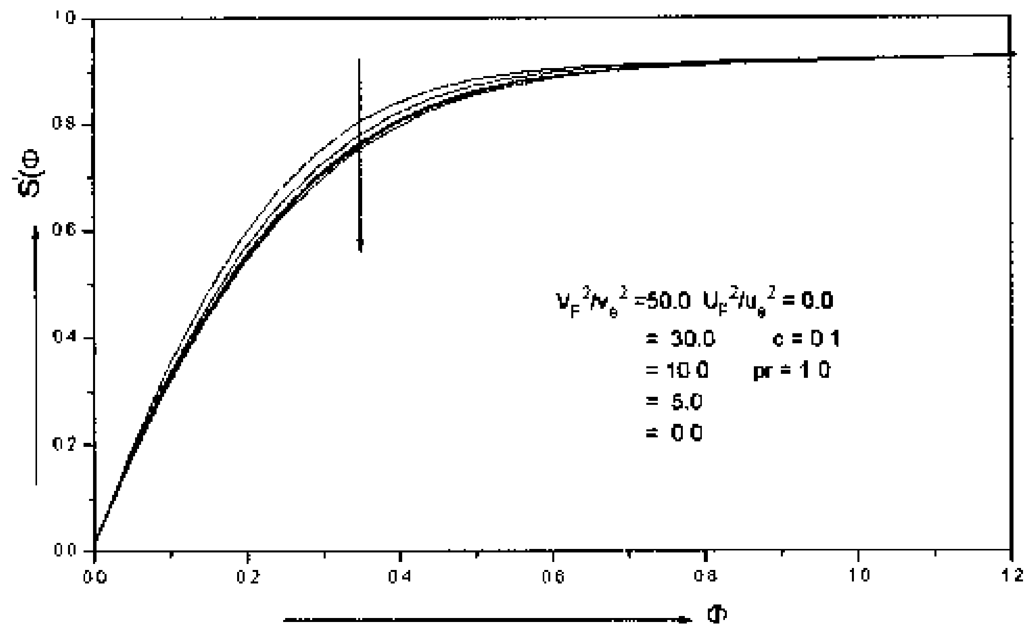
Fig(3.b): Dimensionless velocity distribution along v-direction for several values of U_F^2/u_e^2(=50.0,20.0,10.0,5.0,0.0) for the equation (65)



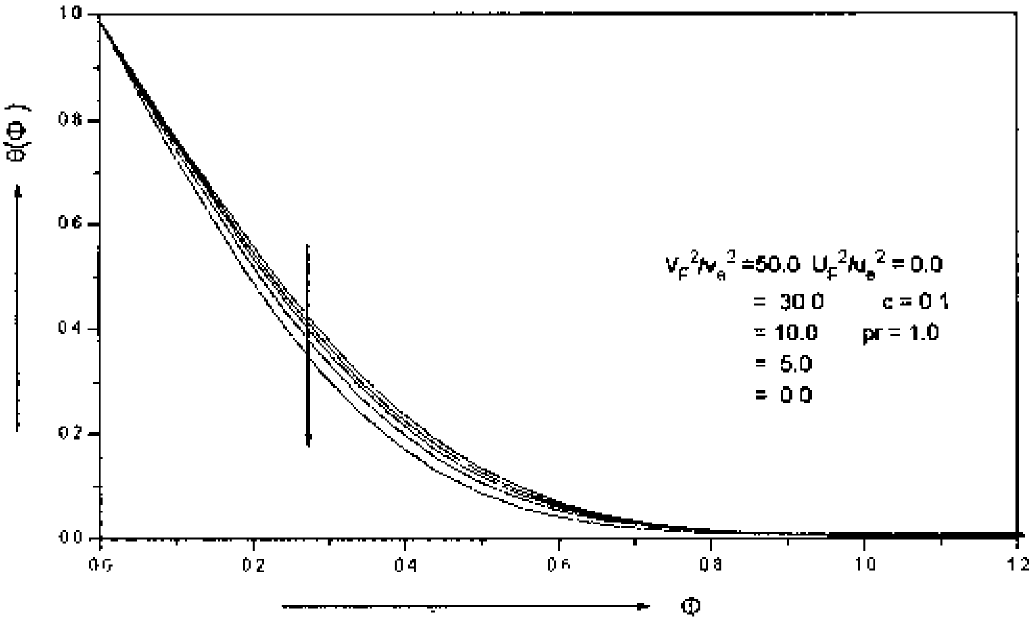
Fig(3 c): Dimensionless temperature distribution for several values of U_F^2/u_e^2(=50.0,20.0,10.0,5.0,0.0) for the equation (65)



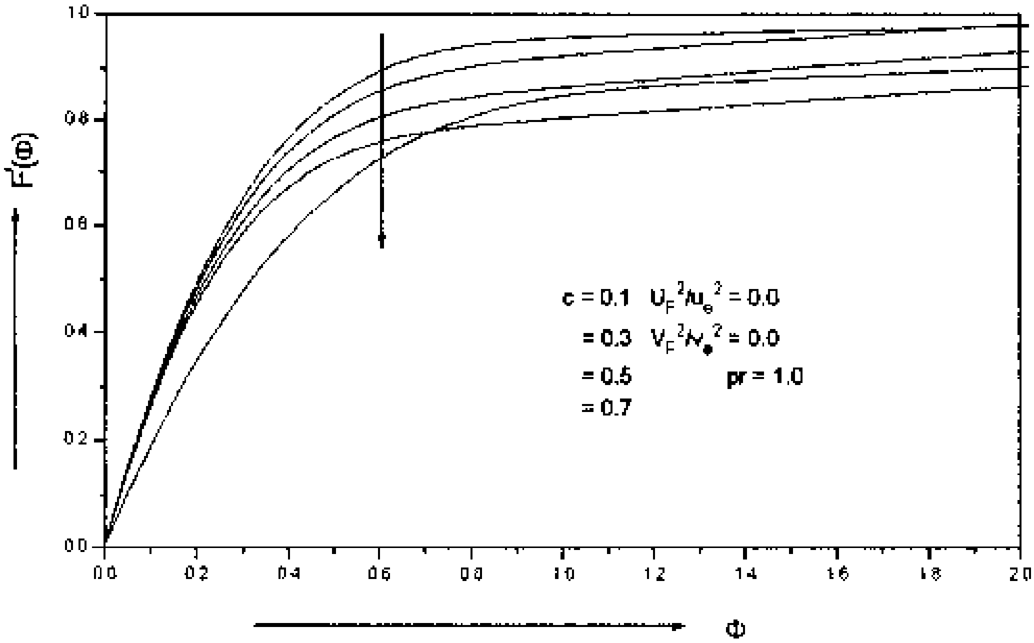
Fig(4 a): Dimensionless velocity distribution along u-direction for several values of V_F^2/v_e^2(=50.0,30.0,10.0,5.0,0.0) for the equation (65)



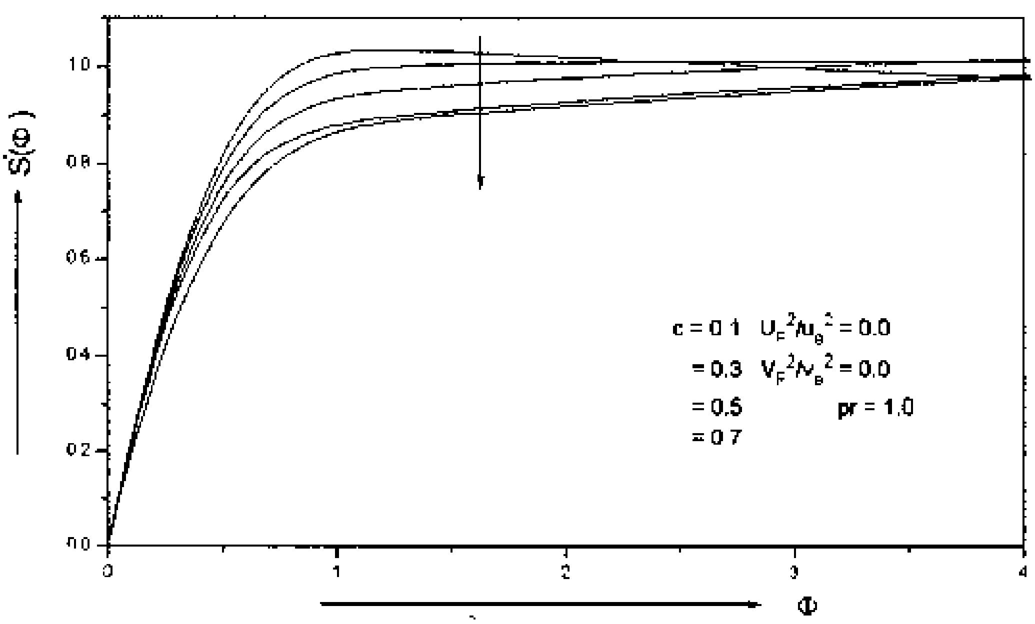
Fig(4,b)' Dimensionless velocity distribution along v-direction for several values of V_F^2/V_e^2(=50.0,30 0,10.0,5 0,0 0) for the equation (65)



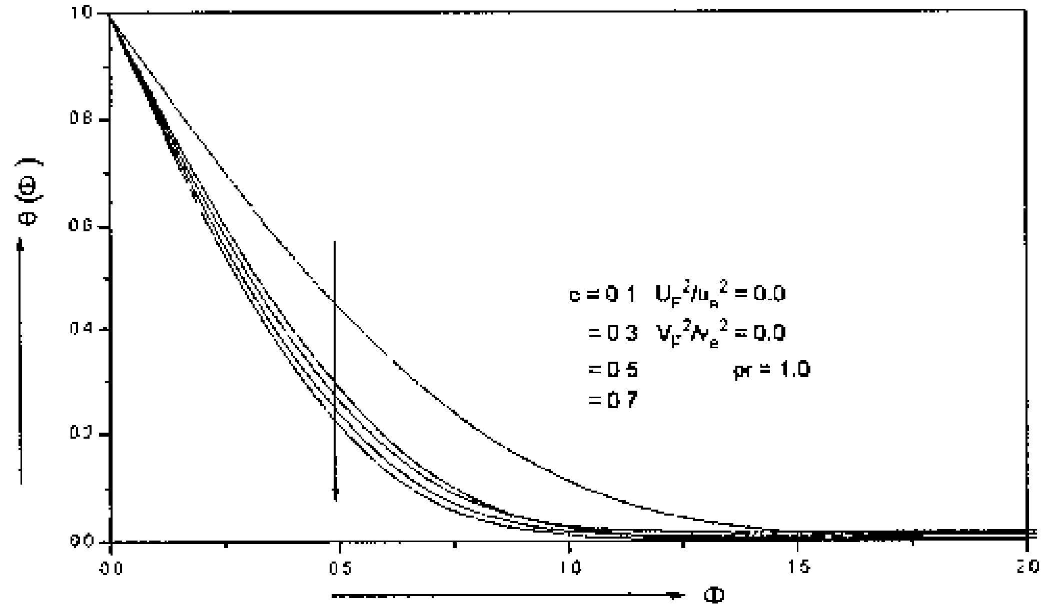
Fig(4 c): Dimensionless temperature distribution for several values of V_F^2/V_e^2(=50 0,30.0,10 0,5 0,0.0) for the equation (65)



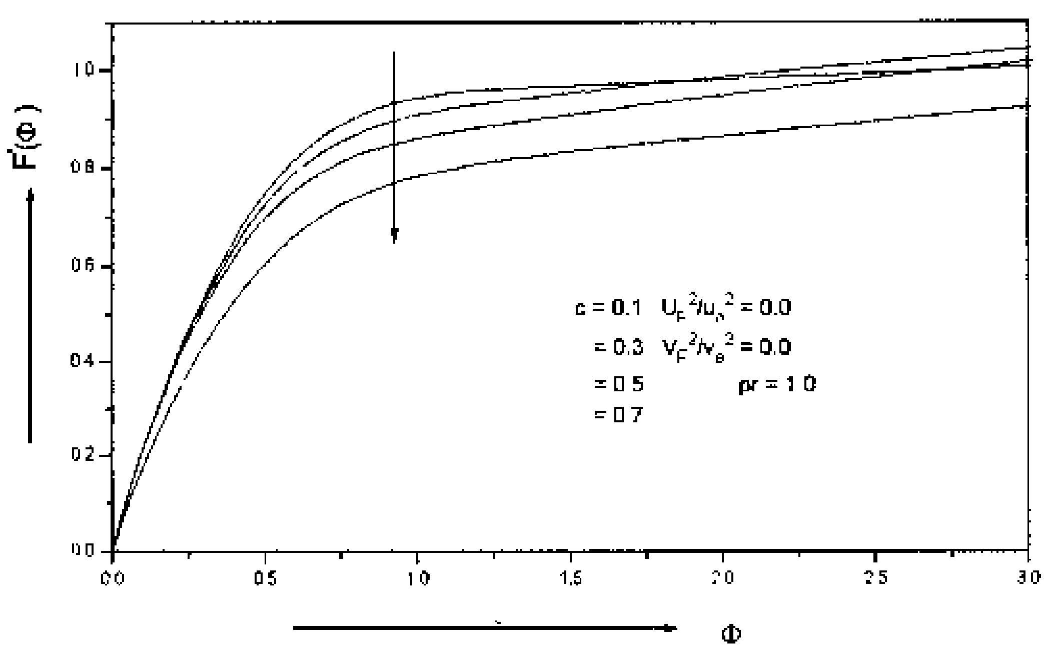
Fig(5.a) Dimensionless velocity distribution along u-direction for several values of c(=0.1,0.3,0.5,0.7) for the equation (65)



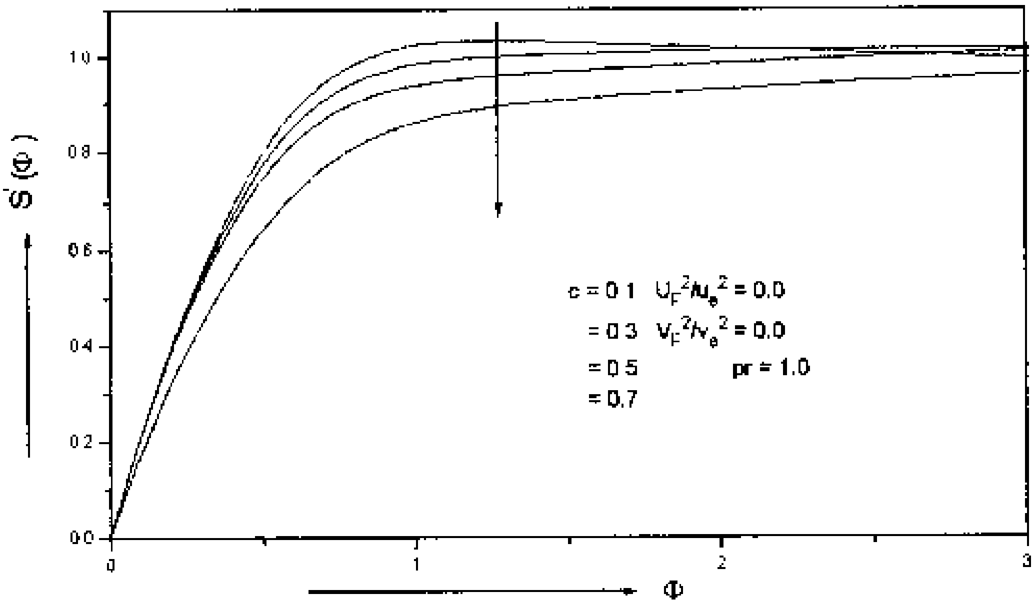
Fig(5.b): Dimensionless velocity distribution along v-direction for several values of c(=0.1,0 3,0 5,0 7) for the equation (65)



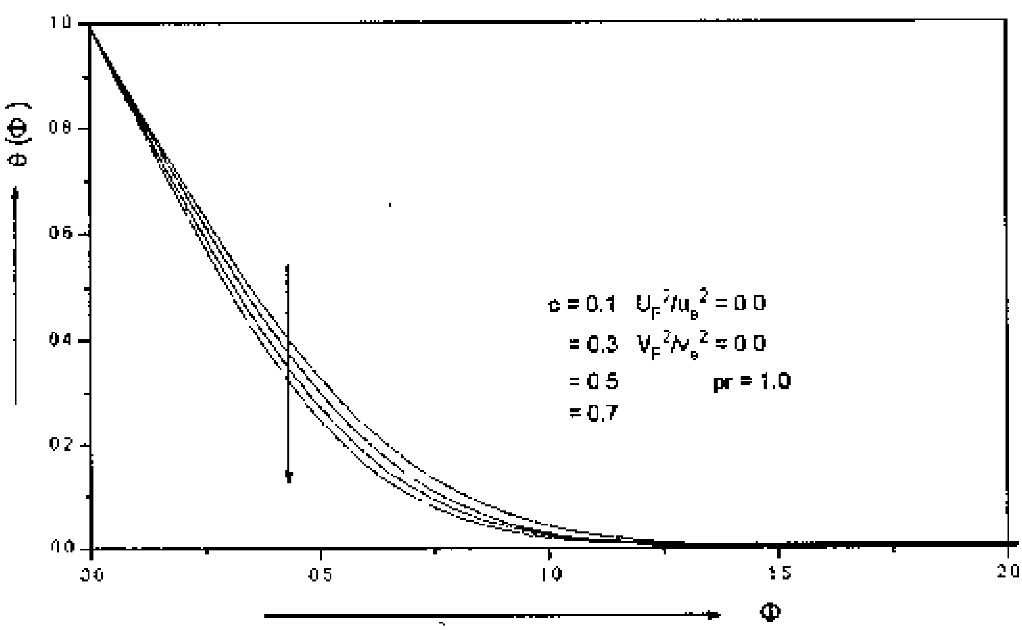
Fig(5.c): Dimensionless temperature distribution for several values of c(=0.1,0.3,0.5,0.7) for the equation (65)



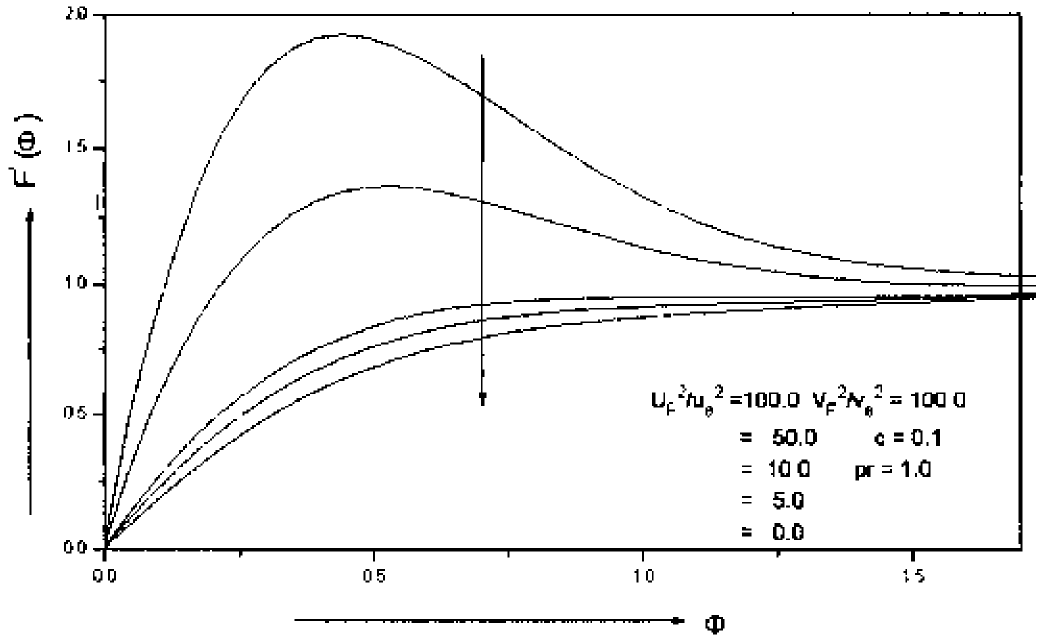
Fig(6.a): Dimensionless velocity distribution along u-direction for several values of c(=0.1,0.3,0.5,0.7) for the equation (65)



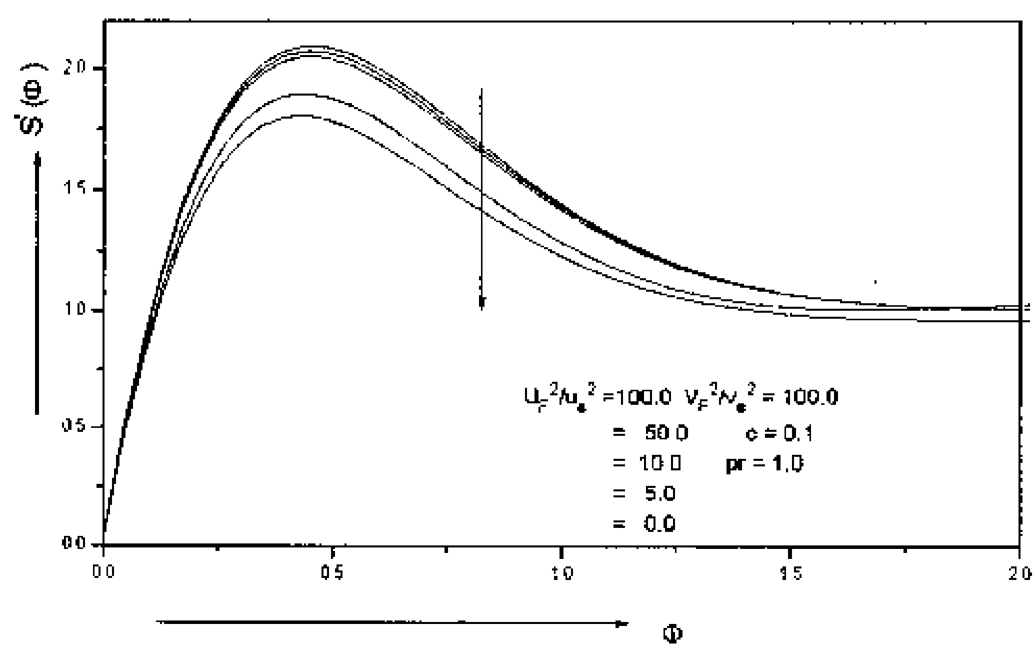
Fig(6 b): Dimensionless velocity distribution along v-direction for several values of c(=0.1,0.3,0.5,0.7) for the equation (65)



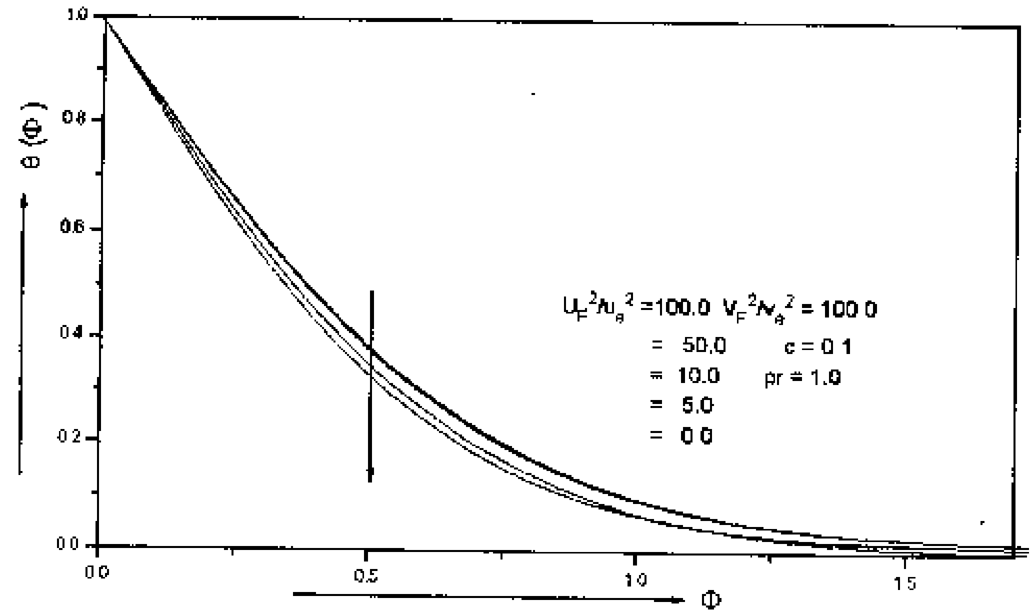
Fig(6 c): Dimensionless temperature distribution for several values of c(=0.1,0.3,0 5,0 7) for the equation (65)



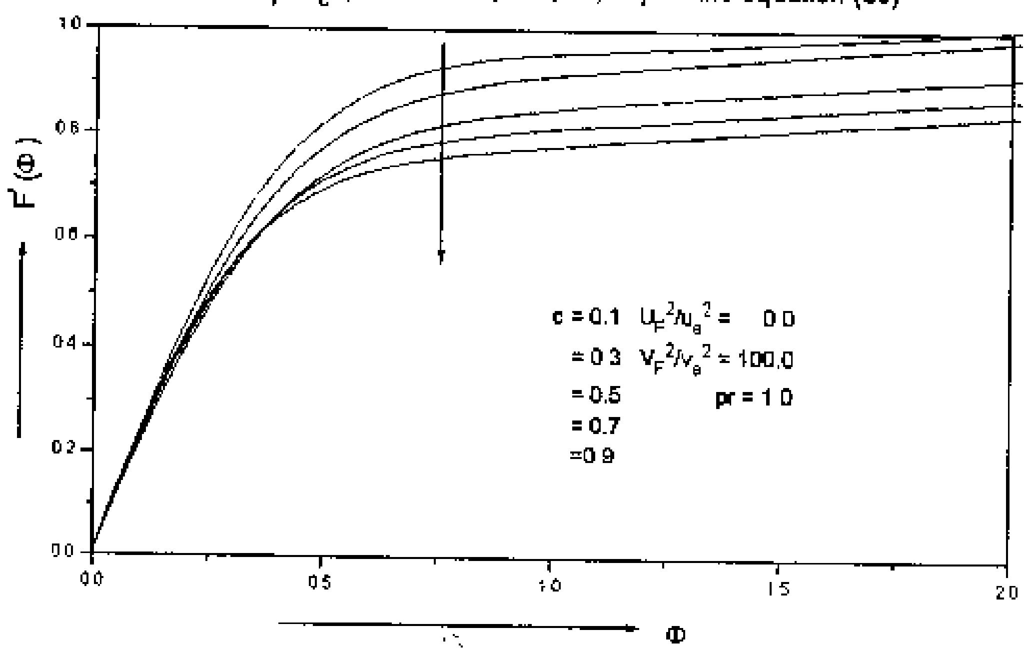
Fig(7 a): Dimensionless velocity distribution along u-direction for several values of U_F^2/u_e^2(=100.0,50.0,10.0,5.0,0.0) for the equation (65)



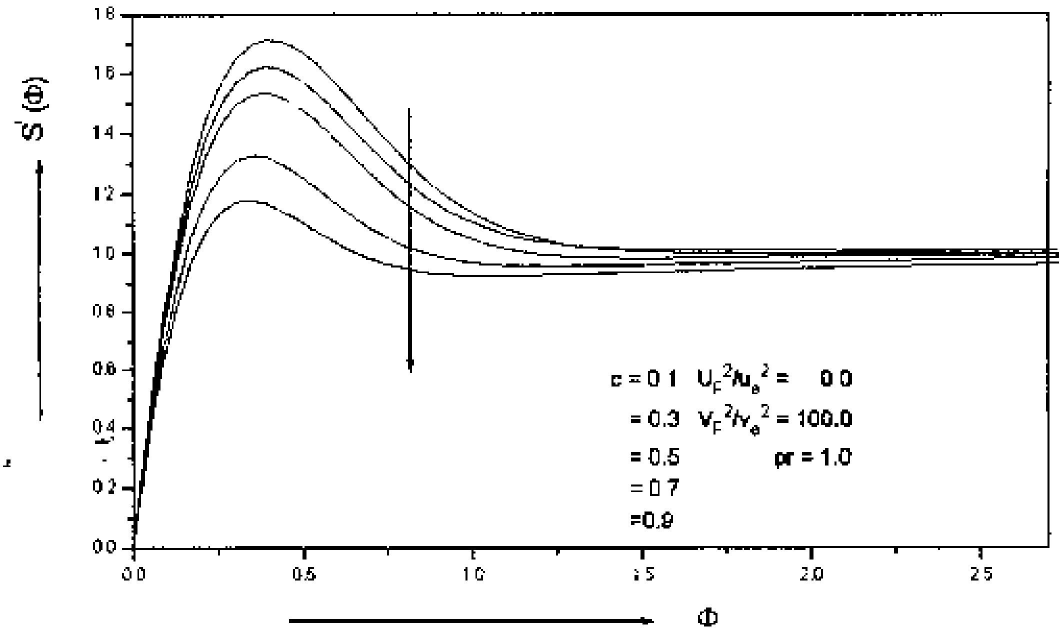
Fig(7 b): Dimensionless velocity distribution along v-direction for several values of U_F^2/u_e^2(=100.0,50.0,10.0,5.0,0.0) for the equation (65)



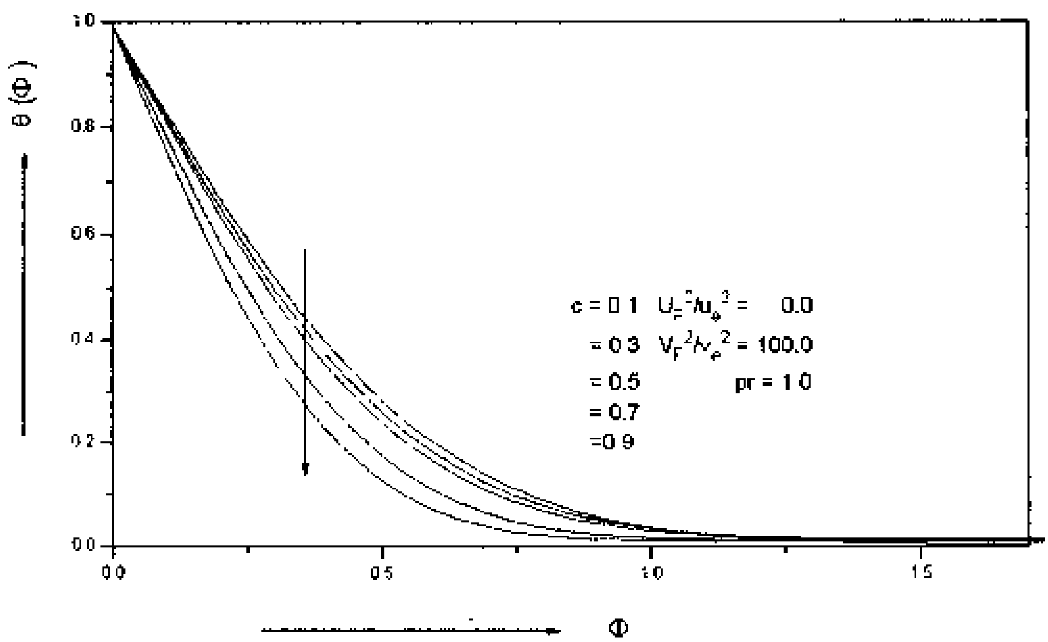
Fig(7.c)· Dimensionless temperature distribution for several values of U_F^2/u_e^2(=100 0,50.0,10 0,5.0,0.0) for the equation (65)



Fig(8 a). Dimensionless velocity distribution along u-direction for several values of c(=0.1,0.3,0.5,0 7,0.9) for the equation (65)



Fig(8.b): Dimensionless velocity distribution along v-direction for several values of c(=0.1,0.3,0.5,0.7,0.9) for the equation (65)



Fig(8.c): Dimensionless temperature distribution for several values of c(=0.1,0.3,0.5,0.7,0.9) for the equation (65)

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