

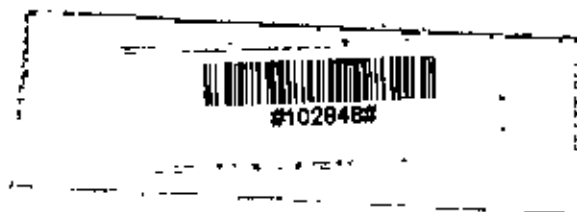
**BEHAVIOUR OF THE TRIGONOMETRIC INTERPOLATION IN  
THE SOLUTION OF AN EIGENVALUE PROBLEM BY USING  
FINITE ELEMENT METHOD**

BY  
**MAIN UDDIN AHAMMAD**

A Thesis submitted to the Department of Mathematics, Bangladesh University of Engineering and Technology, Dhaka in partial fulfillment of the requirements for the Degree of **Master of Philosophy** in Mathematics.



**BANGLADESH UNIVERSITY OF ENGINEERING AND  
TECHNOLOGY, DHAKA-1000  
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**A THESIS ON**

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has been accepted as satisfactory in partial fulfillment of the requirements for the Degree of **Master of Philosophy** in Mathematics and certify that the student demonstrated a satisfactory knowledge of the field covered by this thesis in an Oral Examination held on 25<sup>th</sup> September 2006.

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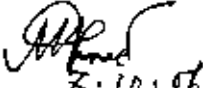
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I hereby declare that the work which is being presented in the thesis entitled "Behaviour of the Trigonometric Interpolation in the solution of an Eigenvalue problem by using Finite Element Method." submitted in partial fulfillment of the requirement for the award of the degree of Master of Philosophy in Mathematics, in the department of Mathematics, Bangladesh University of Engineering and Technology, Dhaka is an authentic record of my own work.. I have not submitted the matter presented in this thesis for the award of any other degree in this or any other university.

25<sup>th</sup> September, 2006

  
7.10.06  
(Main Uddin Ahammad)

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## ABSTRACT

Finite element method is an efficient method for solving ordinary and partial differential equations in both linear and nonlinear cases that arise in different branches of applied sciences such as heat transfer, fluid flow, solid mechanics, quantum mechanics. All kinds of problems such as initial and boundary value problems and eigenvalue problems are solved by using finite element method. In all these cases algebraic polynomial or Lagrange interpolation function is used to approximate the field variable.

In our present study we have replaced the Lagrange interpolation function by the trigonometric interpolations namely sine and tangent interpolation in solving an eigenvalue problem by finite element method. The result shows that eigenvalues obtained by using sine and tangent interpolation agree well with those of Lagrange interpolation.

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## CHAPTER 1

### REVIEW OF THE EARLIER WORKS AND OBJECTIVES

The conception of finite element method has come from the idea of removing the difficulties faced by Weighted residual methods where the approximation function was derived from intuitive idea and the body was considered as a whole body. In the finite element method the body is discretized into sub-domains and the systematic function is derived for each sub-domain.

The use of piecewise-continuous functions defined over a subdomain to approximate an unknown function can be found in the work of Courant [1], who used an assemblage of triangular elements and the principle of minimum total potential energy to study the St. venant torsion problem. Although certain key features of the finite element method can be found in the works of Hrenikoff [2] and Courant [1], its formal presentation is attributed to Argyris and Kelsey [3] and Turner, Clough, Martin, and Topp[4]. The term 'finite element' was first used by Clough [5] in 1960, in his paper titled "The Finite Element Method in Plane Stress Analysis." After its introduction it has continually developed and improved. Since its inception the literature on finite element application has grown exponentially. While in early days the contributors have been almost engineers, today a large of them come from the field of mathematics.

Various types of problems such as Hyperbolic Problems [6], Transient analysis [7, 8], solid mechanics, [9] and nonlinear problems [10,11] are solved by using finite element method. Initial value problems and boundary value problems having Dirichlet, Neumann and mixed boundary conditions are solved by this method. Different eigenvalue problems [12-14] are to solved in connections with various applications.

The eigenvalue problem solved by Rammohan et al [14] is

$$-\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{d\psi}{dx} \right) - \frac{2}{x} \psi = \lambda \psi \quad (1.1)$$

with domain having limit 0 to 20 taking 20 elements.

The sample eigenvalue problem discussed in the previous chapter was with constant coefficient. So the element stiffness equation was same for all elements. But for this case the coefficient is variable. So the element stiffness matrix will be different for different element.

The element stiffness matrix for the first two elements and 20-th element are shown below :

Multiply the equation (1.1) by  $x^2$  to obtain

$$-\frac{d}{dx}\left(x^2 \frac{d\psi}{dx}\right) - 2x\psi = \lambda x^2\psi \quad (1.2)$$

Now multiply the equation (1.2) by the weight function  $w$  and integrate from 0 to 20. An integration by parts leads to the equation

$$\int_0^{20} \left( x^2 \frac{dw}{dx} \frac{d\psi}{dx} - 2xw\psi \right) dx = \int_0^{20} \lambda w\psi dx$$

where the boundary condition  $\psi(20) = 0$  is applied,

writing  $\psi = N_1\psi_1 + N_2\psi_2$  and considering the Galerkin approach  $w = N_1$  and  $w = N_2$ , we get the element of the stiffness matrix as

$$k_{ij} = \int_{x_1}^{x_2} \left( x^2 \frac{dN_i}{dx} \frac{dN_j}{dx} - 2xN_iN_j \right) dx$$

Using the substitution

$$\begin{aligned} x &= \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2} \xi \\ &= x_1 \frac{1}{2}(1 - \xi) + x_2 \frac{1}{2}(1 + \xi) \end{aligned}$$

$$dx = \frac{h}{2} d\xi \quad \therefore \frac{d\xi}{dx} = \frac{2}{h}$$

$$x_2 - x_1 = h = \text{length of an element}$$

The shape functions are taken from Lagrange interpolation function as

$$N_1 = \frac{x_2 - x}{x_2 - x_1} = \frac{x_2 - \frac{x_1 + x_2}{2} - \frac{x_2 - x_1}{2} \xi}{x_2 - x_1} = \frac{1}{2}(1 - \xi)$$

$$\text{similarly, } N_2 = \frac{1}{2}(1 + \xi)$$



$$k_{ij} = \int_{-1}^1 \left[ \left( \frac{x_A + x_B}{2} + \frac{x_B - x_A}{2} \xi \right)^2 \frac{dN_i}{d\xi} \frac{d\xi}{dx} \frac{dN_j}{d\xi} \frac{d\xi}{dx} - 2 \left( \frac{x_A + x_B}{2} + \frac{x_B - x_A}{2} \xi \right) N_i N_j \right] \frac{h}{2} d\xi$$

$$= \int_{-1}^1 \left[ \left( \frac{x_A + x_B}{2} + \frac{x_B - x_A}{2} \xi \right)^2 \frac{dN_i}{d\xi} \frac{2}{h} \frac{dN_j}{d\xi} \frac{2}{h} - 2 \left( \frac{x_A + x_B}{2} + \frac{x_B - x_A}{2} \xi \right) N_i N_j \right] \frac{h}{2} d\xi$$

Taking  $x_A = 0, x_B = 1, h = 1$

$$k_{11} = \int_{-1}^1 \left[ \frac{1}{4} (1 + \xi)^2 (-1)(-1) - \frac{1}{2} (1 + \xi) \frac{1}{2} (1 - \xi) \frac{1}{2} (1 - \xi) \right] \frac{1}{2} d\xi$$

$$= .1666$$

Similarly other elements of the matrix are obtained as

$$k_{12} = k_{21} = -.500 \quad k_{22} = -.1666$$

Thus for element 1  $k^{(1)} = \begin{bmatrix} .166 & -.500 \\ -.500 & -.166 \end{bmatrix}$

Taking  $x_A = 1, x_B = 2, h = 1$

For element 2  $k^{(2)} = \begin{bmatrix} 1.50 & -2.83 \\ -2.83 & 1.166 \end{bmatrix}$

Also Taking  $x_A = 19, x_B = 20, h = 1$

For element 20  $k^{(20)} = \begin{bmatrix} 367.75 & -386.83 \\ -386.83 & 367.17 \end{bmatrix}$

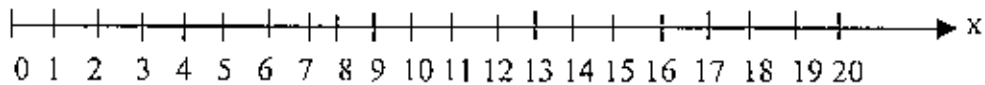


Fig.1.1 Discretization of the body into 20 linear elements

The elements of the mass matrix are

$$M_{ij} = \int_{x_A}^{x_B} x^2 N_i N_j dx$$

$$M_{yy} = \int_{-1}^1 \left( \frac{x_A + x_B}{2} + \frac{x_B - x_A}{2} \xi \right)^2 N_i N_i \frac{h}{2} d\xi$$

$$M_{11} = \int_{-1}^1 \frac{1}{4} (1 + \xi)^2 \frac{1}{2} (1 - \xi) \frac{1}{2} (1 - \xi) \frac{1}{2} d\xi$$

$$= .0333$$

Similarly,  $M_{12} = M_{21} = .05$        $M_{22} = 0.2$

Thus for element 1

$$M^{(1)} = \begin{bmatrix} .033 & .05 \\ .05 & 0.2 \end{bmatrix}$$

For element 2

$$M^{(2)} = \begin{bmatrix} .533 & .383 \\ .383 & 1.033 \end{bmatrix}$$

For element 20

$$M^{(20)} = \begin{bmatrix} 123.53 & 63.383 \\ 63.383 & 130.03 \end{bmatrix}$$

The global stiffness matrix **K** and the global mass matrix **M** will be a 21x21 matrix.

The matrix eigenvalue equation to be solved is  $K\Psi = \lambda M\Psi$

with  $\Psi = [\psi_1, \psi_2, \dots, \psi_{19}, \psi_{20}, \psi_{21}]^T$

and  $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_{19}, \lambda_{20}]^T$

The equation is solved by using computer programming to find the eigenvalues which are given in chapter 7.

## **OBJECTIVES :**

The main object of this study is to investigate the effect on the solution of the eigenvalue problem by using finite element method, if the Lagrange interpolation is replaced by the trigonometric interpolation. Here the eigenvalue problem will be solved by taking trigonometric interpolation instead of Lagrange interpolation.

## CHAPTER 2

### DIFFERENT WEIGHTED-RESIDUAL METHODS (WRM)

To have a clear idea about FEM we first discuss the solution of differential equation by Weighted-residual methods.

Suppose the differential equation is

$$Au = f \text{ in } \Omega \text{ where } A \text{ is the operator.}$$

The function  $u$  is not only required to satisfy the equation, it is also required to satisfy the boundary condition (b.c.) associated with the equation.

In the Weighted-residual method the solution  $u$  is approximated as

$$u_n = \sum_{j=1}^n c_j \phi_j + \phi_0 \quad (1)$$

$\phi_0$  is required to satisfy all specified boundary condition, and  $\phi_j$  are required to satisfy homogeneous form of all specified boundary conditions of the problem.

Now

$$R = A(u_n) - f = A\left(\sum_{j=1}^n c_j \phi_j + \phi_0\right) - f \neq 0.$$

The residual  $R$  is a function of position as well as of the parameter  $c_j$ . In the Weighted-residual method as the name suggests the parameter  $c_j$  are determined by requiring the residual  $R$  to vanish in the weighted integral sense.

$$\begin{aligned} \int_{\Omega} w R dx &= 0 \\ \int_{\Omega} w (A(u_n) - f) dx &= 0 \\ \int_{\Omega} w (A(\sum c_j \phi_j + \phi_0) - f) dx &= 0 \\ \int_{\Omega} w A \sum c_j \phi_j dx &= \int_{\Omega} w (f - A(\phi_0)) dx \\ \sum [ \int_{\Omega} w A(\phi_j) dx ] c_j &= \int_{\Omega} w (f - A(\phi_0)) dx \end{aligned}$$

Various Weighted-residual methods are available depending on the choice of different weight functions.

Different Weighted-residual methods are

1. The Petrov – Galerkin Method
2. The Galerkin Method
3. The Least square method
4. The Collocation method

### 1. The Petrov – Galerkin Method

The WRM is known as Petrov –Galerkin method when  $w = \Psi_i \neq \phi_i$ ,

$$\sum_{i=1}^N \left[ \int_{\Omega} \Psi_i A(\phi_i) dx \right] c_i = \int_{\Omega} \Psi_i [f - A(\phi_0)] dx$$

$$\sum_{i=1}^N A_{ij} c_j = F_i$$

$i = 1, 2, 3, \dots, N$

### 2. The Galerkin Method

For the choice of weight function  $w$  equal to the approximate function  $\phi_i$ , the Weighted-residual method is better known as Galerkin method.

Now,  $w = \phi_i$

$$\sum_{i=1}^N \left[ \int_{\Omega} \phi_i A(\phi_i) dx \right] c_i = \int_{\Omega} \phi_i [f - A(\phi_0)] dx$$

$$\sum_{i=1}^N A_{ij} c_j = F_i$$

$i = 1, 2, 3, \dots, N$

### 3. The Least square method

In this method we determine the parameters  $c_j$  by minimizing the integral of the square of the residual .

$$\begin{aligned} \frac{\partial}{\partial c_i} \int_{\Omega} R^2(x, c_i) dx &= 0 \\ \text{or, } \int_{\Omega} \frac{\partial R}{\partial c_i} R(x, c_i) dx &= 0 \\ \int_{\Omega} A(\phi_i) [A(\sum c_j \phi_j + \phi_0) - f] dx &= 0 \\ \sum_{j=1}^N \left[ \int_{\Omega} A(\phi_j) A(\phi_i) dx \right] c_j &= \int_{\Omega} A(\phi_i) [f - A(\phi_0)] dx \\ \sum_{j=1}^N A_{ij} c_j &= F_i \\ \left[ \begin{array}{l} \text{Here, } (Au - f) = R \\ A(\sum c_j \phi_j + \phi_0) - f = R \\ A(\phi_i) = \frac{\partial R}{\partial c_i} \end{array} \right] \end{aligned}$$

#### 4. The Collocation Method

In the Collocation method, we seek an approximate solution in the form of (1) by requiring the residual in the equation to be identically zero at  $N$  selected points  $x_i$  in the domain  $\Omega$ .

$$\text{i.e. } R(x_i, c_i) = 0 \quad i = 1, 2, \dots, N$$

In this case,

$$\begin{aligned} w = \psi_i &= \delta(x - x_i) \\ &= \text{Dirac } \delta \text{ function} \end{aligned}$$

$$\text{Now, } \int w R dx = 0$$

$$\Rightarrow \int \delta(x - x_i) R(x, c_i) dx = 0$$

$$\Rightarrow R(x_i, c_i) = 0 \quad \because \int f(x) \delta(x - x_i) dx = f(x_i)$$

#### ILLUSTRATION

Let us illustrate the above methods by considering an example

$$-\frac{d^2 u}{dx^2} - u + x^2 = 0 \quad u(0) = 0, u'(1) = 1$$

The exact solution is given by

$$u(x) = \frac{2\cos(1-x) - \sin x}{\cos 1} + x^2 - 2$$

For a weighted residual method  $\phi_0$  and  $\phi_1$  should satisfy the following conditions :

$$\phi_0(0) = 0, \quad \phi_0'(1) = 1 \quad (\text{satisfy the actual bc.})$$

$$\phi_1(0) = 0, \quad \phi_1'(1) = 0 \quad (\text{satisfy the homogeneous form of actual bc.})$$

We take  $N = 2$ . Let the functions satisfying the above conditions be

$$\phi_0(x) = x, \quad \phi_1 = -x(2-x), \quad \phi_2 = x^2(1-\frac{2}{3}x)$$

The residual in the approximation

$$\begin{aligned} R &= -\left(0 + \sum_{i=1}^N c_i \frac{d^2 \phi_i}{dx^2}\right) - \left(\phi_0 + \sum_{i=1}^N c_i \phi_i\right) + x^2 \\ &= -c_1(2-2x+x^2) + c_2(-2+4x-x^2+\frac{2}{3}x^3) - x + x^2 \end{aligned}$$

### Petrov - Galerkin method

Let the weight function be

$$w = \psi_1 = x, \quad \psi_2 = x^2$$

$$\text{Then } \int_0^1 x R dx = 0, \quad \text{and } \int_0^1 x^2 R dx = 0$$

$$\frac{7}{12}c_1 + \frac{13}{60}c_2 - \frac{1}{12} = 0, \quad \frac{11}{30}c_1 + \frac{11}{45}c_2 - \frac{1}{20} = 0$$

$$c_1 = \frac{103}{682}, \quad c_2 = -\frac{15}{682}$$

$$\text{The solution is } u_{\text{Pet}} = \phi_0 + c_1 \phi_1 + c_2 \phi_2$$

$$= 1.302053x - 0.173021x^2 - .014663x^3$$

### The Galerkin Method

The weight function in this case is

$$w = \phi_1 = -x(2-x) \quad \text{and } w = \phi_2 = x^2\left(1-\frac{2}{3}x\right)$$

Then

$$\int_0^1 x(2-x)R dx = 0, \quad \int_0^1 x^2(1-\frac{2}{3}x)R dx = 0$$

$$\frac{4}{5}c_1 + \frac{28}{45}c_2 - \frac{7}{60} = 0, \quad \frac{17}{90}c_1 + \frac{29}{315}c_2 - \frac{1}{30} = 0$$

$$c_1 = \frac{623}{4306}, \quad c_2 = \frac{21}{4306}$$

$$u_G = 1.2894x - 0.1398x^2 - .00325x^3$$

### The Least square method

Taking  $w = \psi_1 = \frac{\partial R}{\partial c_1}$ ,  $w = \psi_2 = \frac{\partial R}{\partial c_2}$

We have,

$$\int_0^1 (2-2x+x^2)R dx = 0, \quad \int_0^1 (2-4x+x^2-\frac{2}{3}x^3)R dx = 0$$

$$\frac{28}{15}c_1 - \frac{47}{90}c_2 - \frac{13}{60} = 0, \quad -\frac{47}{90}c_1 + \frac{253}{315}c_2 + \frac{1}{36} = 0$$

$$c_1 = \frac{1292}{9935}, \quad c_2 = \frac{991}{19870}$$

$$u_{LS} = 1.2601x - 0.08017x^2 - 0.03325x^3$$

### The Collocation method

Choosing the points  $x = \frac{1}{3}$  and  $x = \frac{2}{3}$  as the collocation points, we evaluate the residuals at these points and set them equal to zero:

$$R(1/3)=0; \quad 117c_1-61c_2=18$$

$$R(2/3)=0; \quad 90c_1+34c_2=18$$

The solution is given by ( $c_1=1710/9468$  and  $c_2=486/9468$ )

$$u_c = 1.3612x - 0.12927x^2 - 0.03422x^3$$

The four approximate solutions are compared in Table 1.1 with the exact solution. For this problem, the Petrov-Galerkin method gives the most accurate solution.



**Table 1.1**

Comparison of the Weighted-residual, and exact solutions of the boundary value problem in  $-\frac{d^2u}{dx^2} - u + x^2 = 0, \quad u(0) = 0, \quad u'(1) = 1$

x	Solution, u(x)				
	u <sub>exact</sub>	u <sub>PG</sub>	u <sub>G</sub>	u <sub>LS</sub>	u <sub>C</sub>
0.0	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.1262	0.1285	0.1275	0.1252	0.1348
0.2	0.2513	0.2536	0.2523	0.2485	0.2668
0.3	0.3742	0.3754	0.3741	0.3699	0.3958
0.4	0.4943	0.4941	0.4932	0.4891	0.5216
0.5	0.6112	0.6096	0.6093	0.6058	0.6440
0.6	0.7244	0.7221	0.7226	0.7200	0.7628
0.7	0.8340	0.8317	0.8329	0.8314	0.8778
0.8	0.9402	0.9384	0.9404	0.9397	0.9887
0.9	1.0433	1.0424	1.0448	1.0449	1.0954
1.0	1.1442	1.1437	1.1463	1.1467	1.1977

### Difficulties in Weighted-residual methods :

From the above discussion of different WRM it is clear that one is to find an approximation function for the whole domain, that satisfies the boundary conditions. The difficulty is that there is no definite procedure of finding the approximation function. One is just to suppose it.

The finite element method overcomes the difficulty of the Weighted-residual methods by providing a systematic procedure for derivation of approximation function over sub-regions of the domain. The function is  $u = \sum u_i \phi_i$  where  $u_i$  are the nodal values of the function and  $\phi_i$  are the Lagrange interpolation function.

So finite element method is a piecewise application of WRM in which the approximation function is an algebraic polynomial that are obtained from interpolation theory.

## CHAPTER 3

### INTERPOLATION AND ITS USE IN THE FINITE ELEMENT METHOD

#### 3.1 LAGRANGE INTERPOLATION

##### 3.1.1 INTRODUCTION

In the finite element method there is a systematic method of finding approximation function. In this case for each element we derive an approximation function. For the linear element i.e. the element having two nodes the approximation function is  $u = u_1\phi_1 + u_2\phi_2$ , where  $u_1$  and  $u_2$  are the values of  $u$  at the nodes.

For the quadratic element i.e. the element having three nodes the approximation function is  $u = u_1\phi_1 + u_2\phi_2 + u_3\phi_3$ , where  $u_1$ ,  $u_2$  and  $u_3$  are the values of  $u$  at the nodes and so on for the higher order element. Here we derive the approximation function for the linear element.

##### 3.1.2 LINEAR ONE DIMENSIONAL ELEMENT

DERIVATION OF APPROXIMATION FUNCTION FOR ONE DIMENSIONAL LINEAR ELEMENT FOR A FUNCTION  $u = u(x)$ .

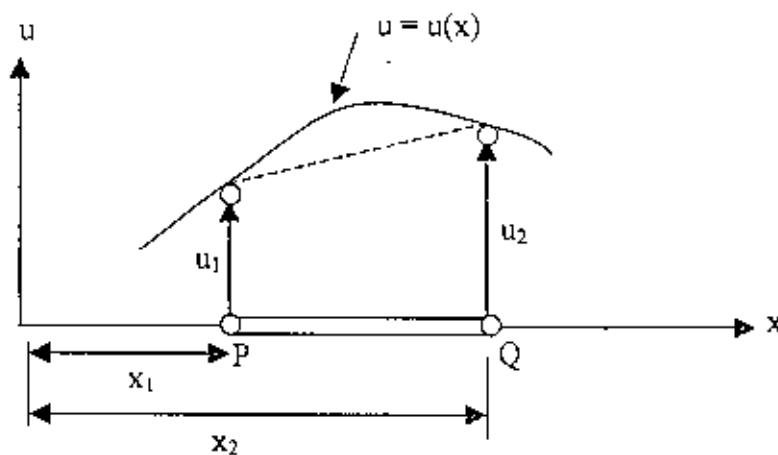


Fig 3.1 The simplest possible one-dimensional element PQ having nodes P and Q

The function  $u(x)$  is shown in figure 3.1. A linear equation that would approximate  $u(x)$  between  $u_1$  and  $u_2$  is assumed as

$$u = A + Bx \quad (3.1)$$

where  $A$  and  $B$  are constants.

Substituting the boundary conditions  $u(x_1) = u_1$  and  $u(x_2) = u_2$  gives two equations that can be solved for  $A$  and  $B$ .

$$A = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1} \quad B = \frac{u_2 - u_1}{x_2 - x_1}$$

$$u = u_1 \frac{x_2 - x}{x_2 - x_1} + u_2 \frac{x - x_1}{x_2 - x_1} \quad (3.2)$$

$$u = N_1 u_1 + N_2 u_2,$$

$$\text{where } N_1 = \frac{x_2 - x}{x_2 - x_1} \quad \text{and} \quad N_2 = \frac{x - x_1}{x_2 - x_1} \quad (3.3)$$

are called element shape functions.

The equation (3.2) is the simplest form of Lagrange interpolation function.

It is of interest to evaluate how the element shape functions  $N_1^e(x)$  and  $N_2^e(x)$  vary with position  $x$ , and figure 3.2 shows this dependence. Here superscript  $e$  represents the arbitrary element number. From equation (3.3) it is obvious that  $N_1^e(x)$  and  $N_2^e(x)$  vary linearly with position  $x$ . Also from equation (3.3) and figure 3.2 it follows that the element shape functions possess the following important properties:

$$\begin{aligned} N_1^e(x_1) &= 1: N_1^e(x_2) = 0 \\ N_2^e(x_1) &= 0: N_2^e(x_2) = 1 \end{aligned} \quad (3.4)$$

These properties are valid for the more complex elements.

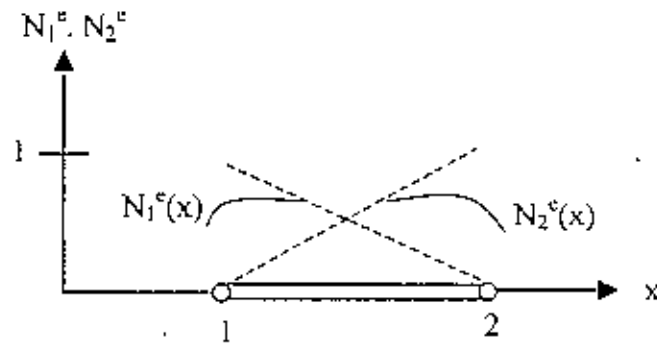


Fig.3.2 Variation of element shape functions with position  $x$

Suppose the function  $u(x)$  represents the temperature function of the element, we write it in terms of  $T$ . Then the expression  $T(x) = N_1^e(x)T_1 + N_2^e(x)T_2$  shows clearly that the approximate temperature is expressed as a suitable interpolation between the temperatures at the nodal points. This interpolation is given by the element shape functions, which in the present case are linear functions. Moreover the approximate temperature  $T$  depends linearly on the temperatures  $T_1$  and  $T_2$  at the nodal points, i.e. the approximation of the temperature can be illustrated as shown in figure 3.3.

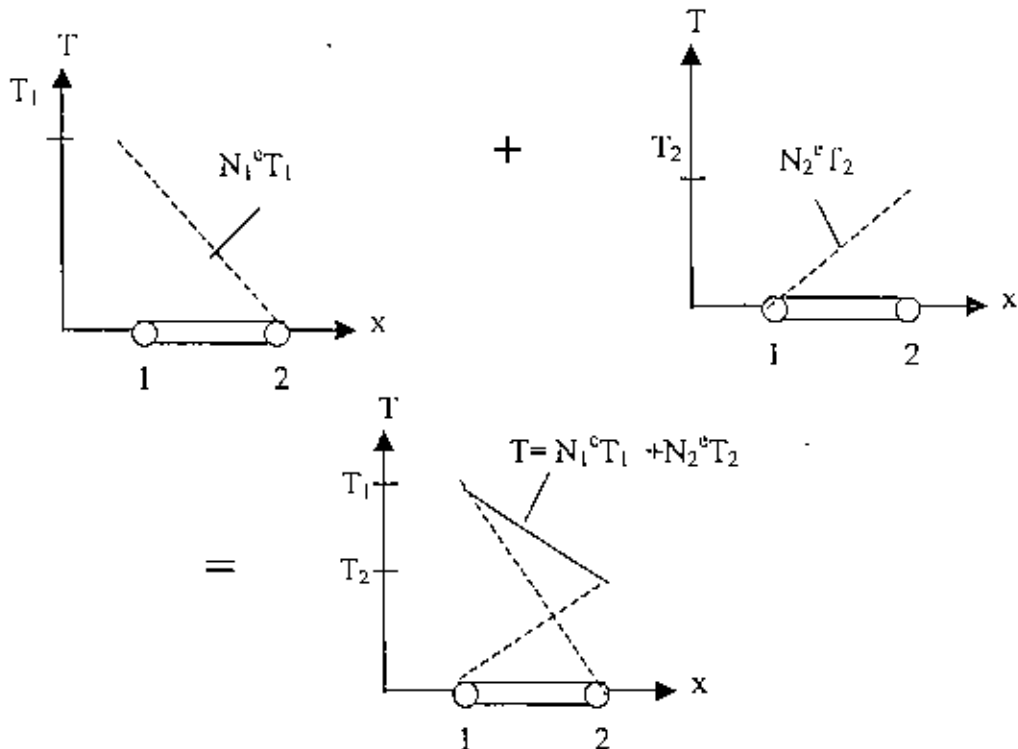


Fig. 3.3 Approximation of temperature in linear element

We shall now derive the approximation function for a body having three elements as shown in figure 2.4. Here 1,2,3,4 are the node numbers and (1), (2), (3) are the element numbers.

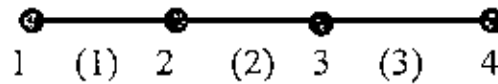


Fig. 3.4

The figures below show the approximation function for three individual elements.

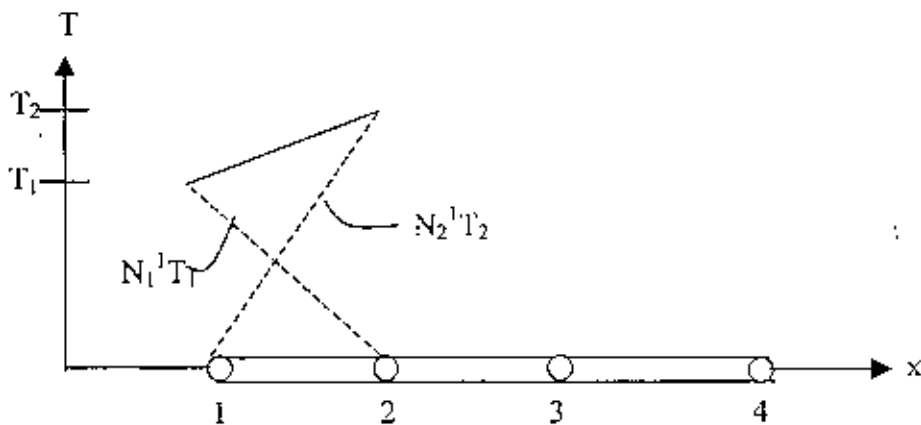


Fig 3.5 Approximation of the temperature of the first element

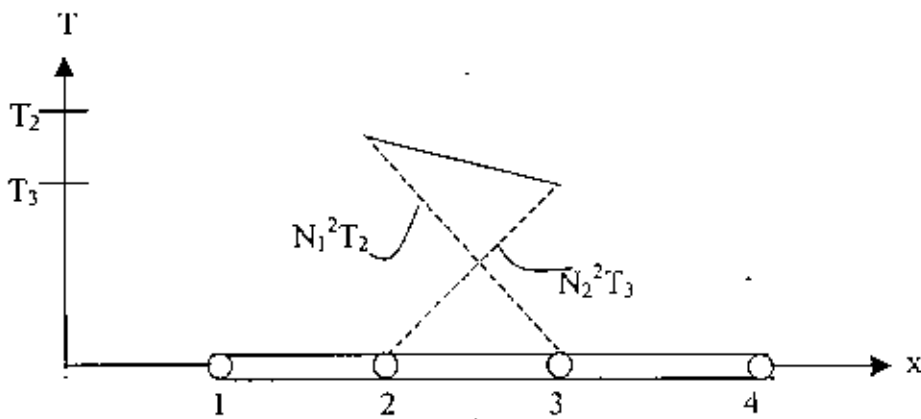


Fig 3.6 Approximation of the temperature of the second element

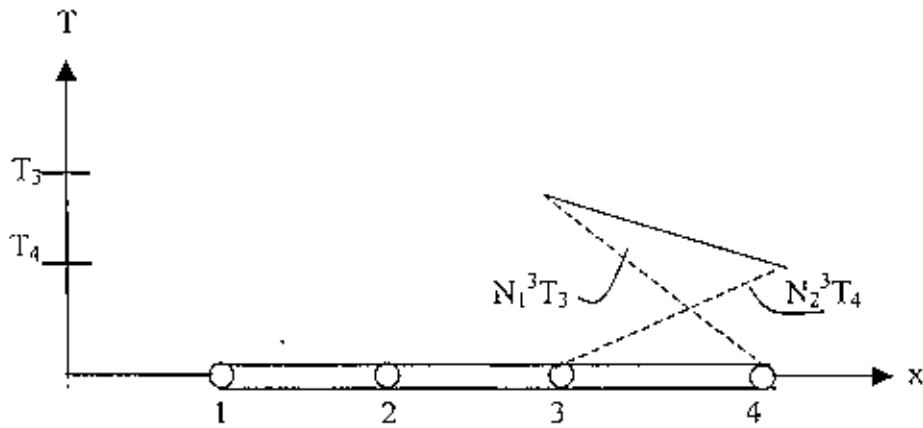


Fig 3.7 Approximation of the temperature of the third element

Figure 3.8 shows how the approximate temperature variation over the entire body is established from the approximation over each element.

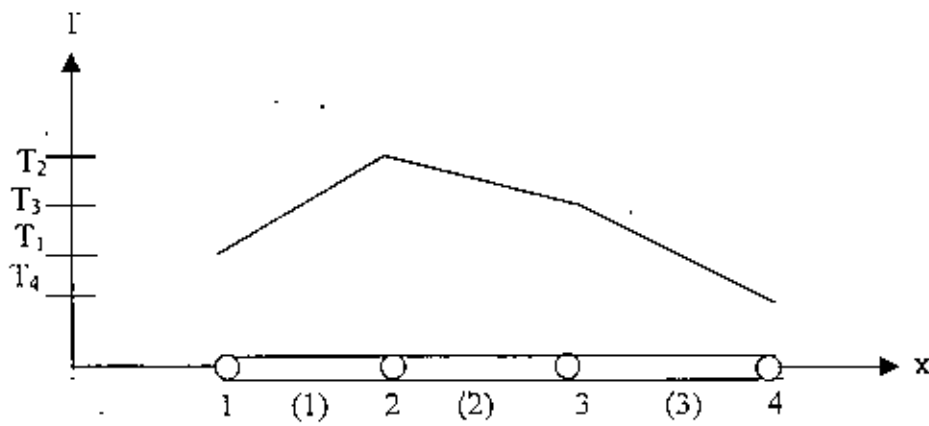


Fig. 3.8 Approximation of the temperature over three elements

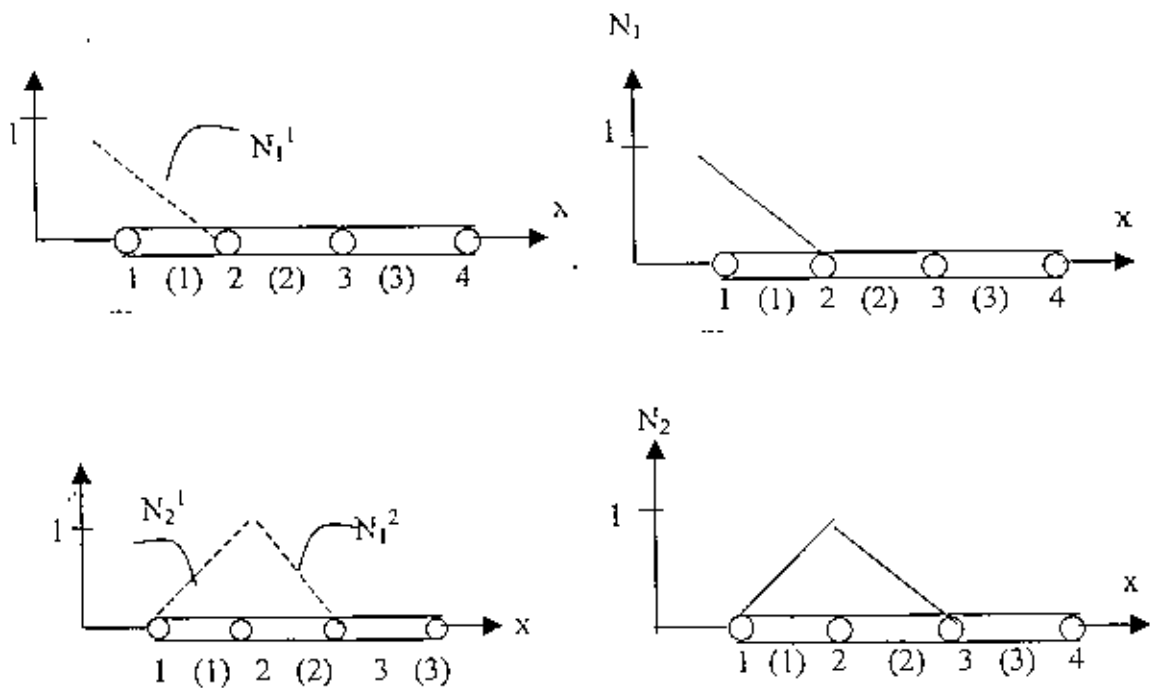
We shall now formulate the approximate temperature variation over the entire body in a slightly different manner. Let us now define global shape functions in such a manner that one global shape function is related to each nodal point. The following definitions are adopted:

$$\begin{aligned}
 N_1 &= \begin{cases} N_1^1 & \text{for } x \text{ in element 1} \\ 0 & \text{otherwise} \end{cases} \\
 N_2 &= \begin{cases} N_2^1 & \text{for } x \text{ in element 1} \\ N_1^2 & \text{for } x \text{ in element 2} \\ 0 & \text{otherwise} \end{cases} \\
 N_3 &= \begin{cases} N_2^2 & \text{for } x \text{ in element 2} \\ N_1^3 & \text{for } x \text{ in element 3} \\ 0 & \text{otherwise} \end{cases} \\
 N_4 &= \begin{cases} N_2^3 & \text{for } x \text{ in element 3} \\ 0 & \text{otherwise} \end{cases}
 \end{aligned} \tag{3.5}$$

These global shape functions, as well as definitions (3.5) are illustrated in figure 3.9.

ELEMENT SHAPE FUNCTIONS

GLOBAL SHAPE FUNCTIONS



ELEMENT SHAPE FUNCTIONS

GLOBAL SHAPE FUNCTIONS

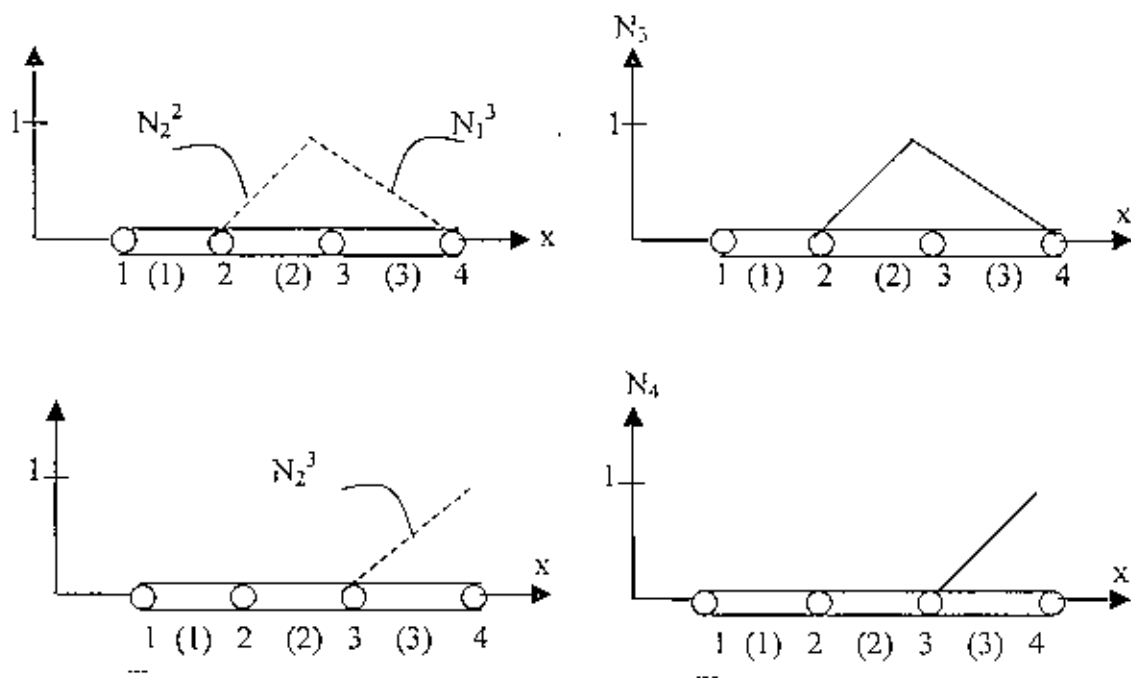


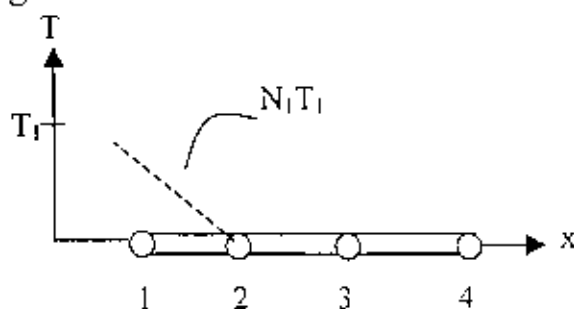
Fig. 3.9 Definition of global shape functions from element shape functions

In this figure, it appears that the global shape function for a specific nodal point only differs from zero in those elements which contain this nodal point. This feature will turn out to be of fundamental importance in the later finite element formulation and it is characteristic of all types of finite elements.

From these definitions of the global shape functions, it is obvious that the approximation of the temperature over the entire body can be written as

$$T = N_1 T_1 + N_2 T_2 + N_3 T_3 + N_4 T_4 \tag{3.6}$$

as illustrated in figure 3.10.





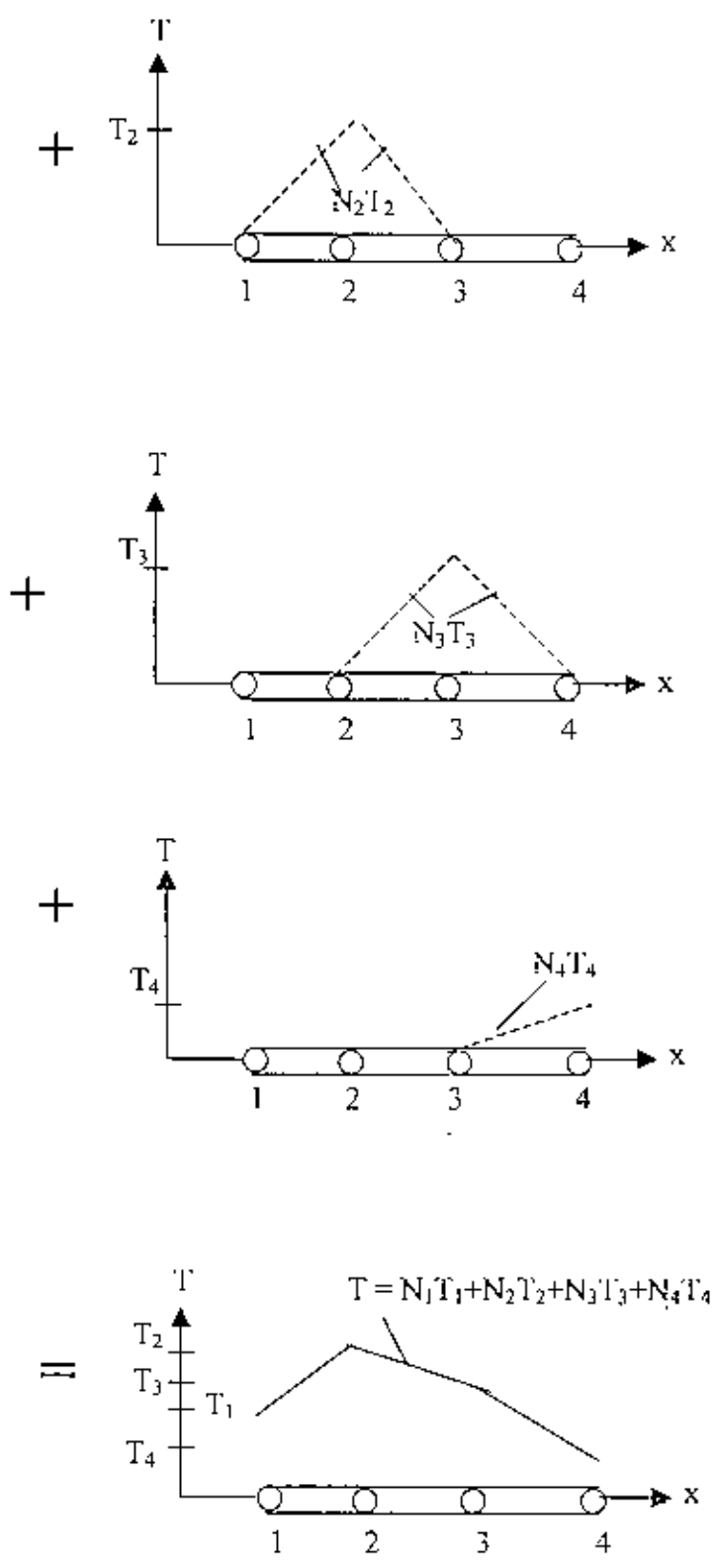


Fig. 3.10 Approximation of the temperature over entire body using global shape functions

### 3.1.3 QUADRATIC ONE DIMENSIONAL ELEMENT

Instead of the simple linear one-dimensional element presented above we may easily construct higher-order elements where the approximation of the temperature contains terms of higher order than the linear ones. As an example consider the quadratic approximation given by

$$T = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \tag{3.7}$$

Again it is advantageous to express the three parameters  $\alpha_1, \alpha_2$  and  $\alpha_3$  by the temperature at the nodal points. That is, there is room for three nodal points as shown in figure 3.11. In order to obtain continuity in the temperature across neighbouring elements, nodal points are located at each end of the element and the third nodal point may be located arbitrarily. In practice, however, the third nodal point is located at the middle of the element as shown in figure 3.11.

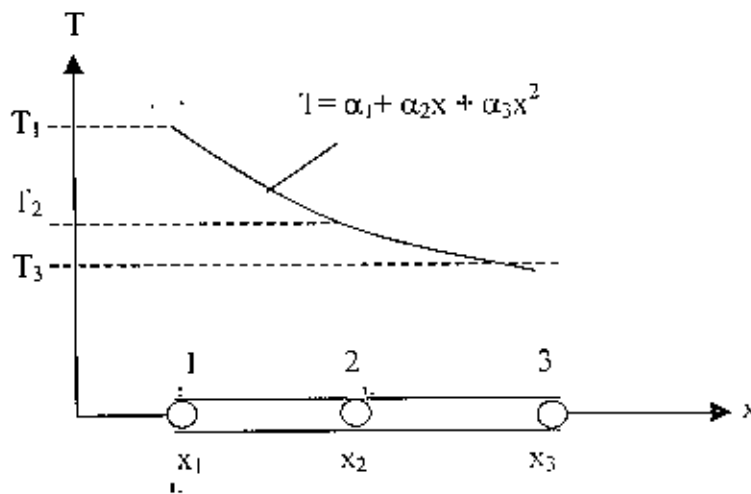


Fig. 3.11 Quadratic one-dimensional element

Let us rewrite equation (3.7) according to

$$T = \tilde{N}\alpha \tag{3.8}$$

where  $\tilde{N} = [1 \quad x \quad x^2]$ ;  $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$

In order to express the parameters  $\alpha_1, \alpha_2$  and  $\alpha_3$  in terms of the temperatures at the nodal points, we express the equation (3.7) for  $x = x_1, T = T_1$ , etc. This yields

$$T_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 x_1^2$$

$$T_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 x_2^2$$

$$T_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 x_3^2$$

$$\text{or } \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

which can be written as  $\mathbf{a}^e = \mathbf{C}\alpha$  (3.9)

where  $\mathbf{a}^e = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$ ;  $\mathbf{C} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$  (3.10)

The column matrix  $\mathbf{a}^e$  contains the temperatures at the nodal points of the element. From equation (3.9) it follows that

$$\alpha = \mathbf{C}^{-1} \mathbf{a}^e \quad (3.11)$$

and insertion into equation (3.8) yields

$$T = \bar{\mathbf{N}} \mathbf{C}^{-1} \mathbf{a}^e$$

This expression can be written as

$$T = \mathbf{N}^e \mathbf{a}^e \quad (3.12)$$

where

$$\mathbf{N}^e = [N_1^e \quad N_2^e \quad N_3^e] = \bar{\mathbf{N}} \mathbf{C}^{-1} \quad (3.13)$$

where

$$\begin{aligned} N_1^e &= \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} \\ N_2^e &= \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \\ N_3^e &= \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \end{aligned} \quad (3.14)$$

As before  $N_1^e, N_2^e, N_3^e$  are called element shape functions, whereas  $\mathbf{N}^e$  is termed the element shape function matrix. It appears that the element shape functions are quadratic functions of  $x$ , and figure 3.12 shows their variation with position  $x$ .

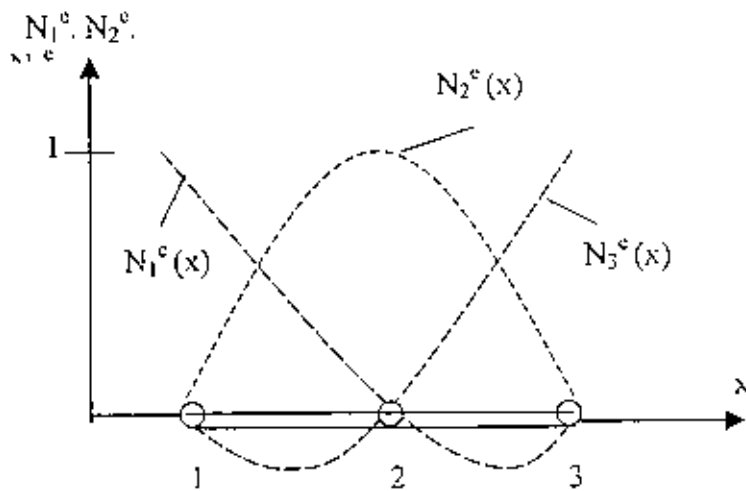


Fig. 3.12 Shape functions for quadratic element

From equation (3.14) and figure 3.12 it follows that the element shape functions possess the following fundamental property

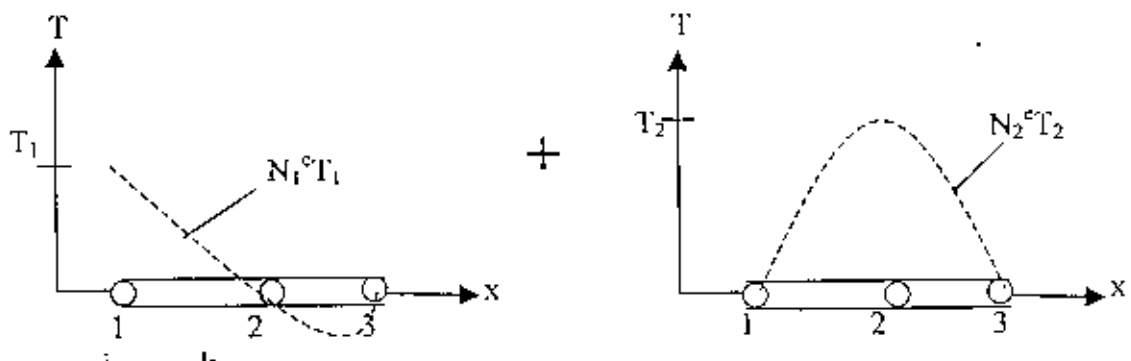
$$N_r^e = \begin{cases} 1 & \text{at nodal point } i \\ 0 & \text{at all other nodal points} \end{cases} \quad (3.15)$$

in complete analogy to equation (3.4)

Expression (3.12) may be written as

$$T = N_1^e T_1 + N_2^e T_2 + N_3^e T_3 \quad (3.16)$$

which shows that the temperature  $T$  depends on the temperatures  $T_1$ ,  $T_2$  and  $T_3$  at the nodal points; that is, the approximation of the temperature can be illustrated as shown in figure 3.13.



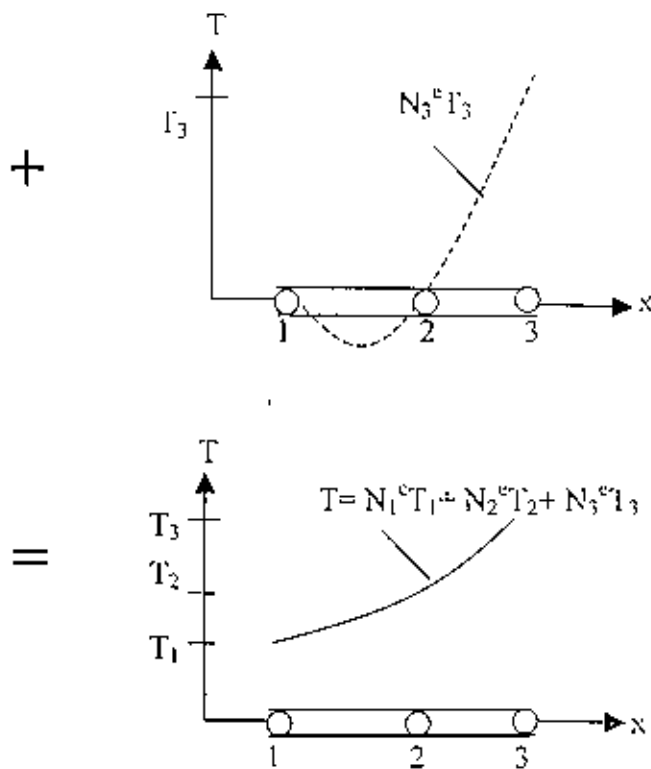


Fig. 3.13 Approximation of temperature in quadratic element

We now consider a body composed of two quadratic elements.

Just as for the linear element and with reference to figure 3.14, we now define the global shape functions in such a manner that one global shape function is related to each nodal point. The following definitions are made:

$$\begin{aligned}
 N_1 &= \begin{cases} N_1^1 & \text{for } x \text{ in element 1} \\ 0 & \text{otherwise} \end{cases} \\
 N_2 &= \begin{cases} N_2^1 & \text{for } x \text{ in element 1} \\ 0 & \text{otherwise} \end{cases} \\
 N_3 &= \begin{cases} N_3^1 & \text{for } x \text{ in element 1} \\ N_1^2 & \text{for } x \text{ in element 2} \\ 0 & \text{otherwise} \end{cases} \\
 N_4 &= \begin{cases} N_2^2 & \text{for } x \text{ in element 2} \\ 0 & \text{otherwise} \end{cases} \\
 N_5 &= \begin{cases} N_3^2 & \text{for } x \text{ in element 2} \\ 0 & \text{otherwise} \end{cases}
 \end{aligned} \tag{3.15}$$

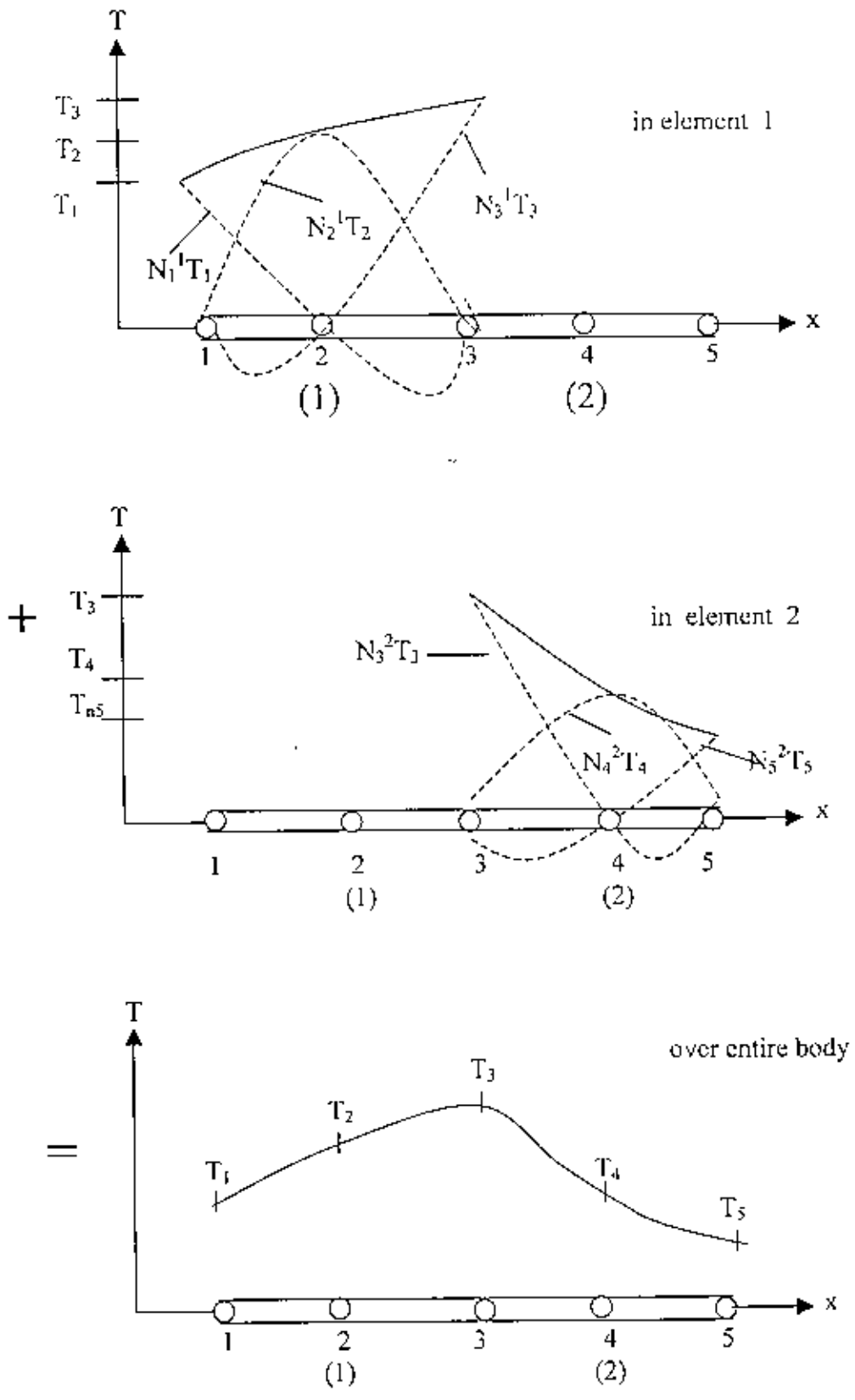
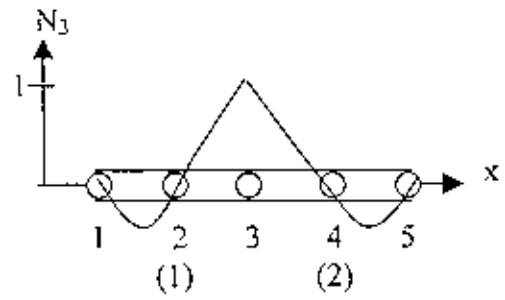
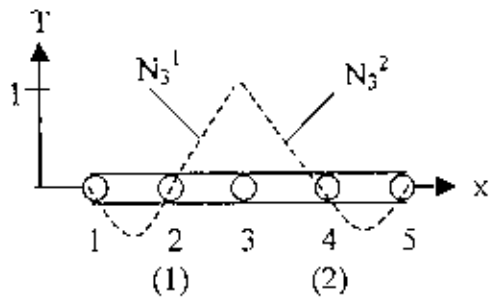
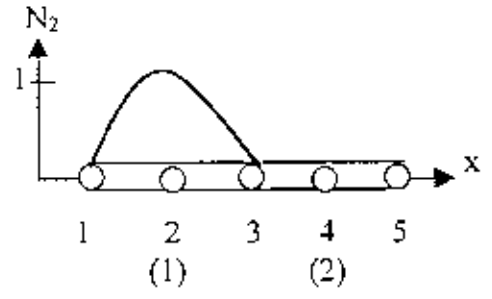
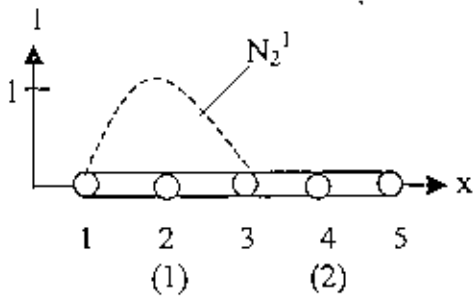
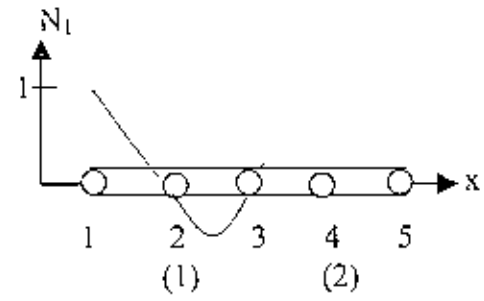
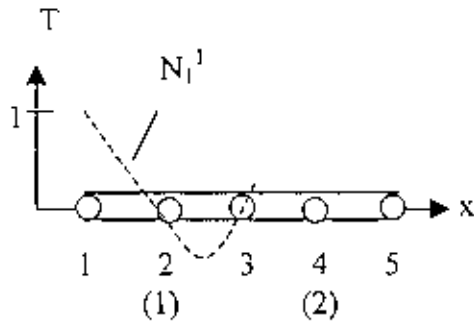


Fig. 3.14 Approximation of temperature over two quadratic elements

For the body shown in figure 3.14, these global shape functions take, as well as definitions (3.15), the form shown in figure 3.15.

ELEMENT SHAPE FUNCTIONS

GLOBAL SHAPE FUNCTIONS



ELEMENT SHAPE FUNCTIONS

GLOBAL SHAPE FUNCTIONS

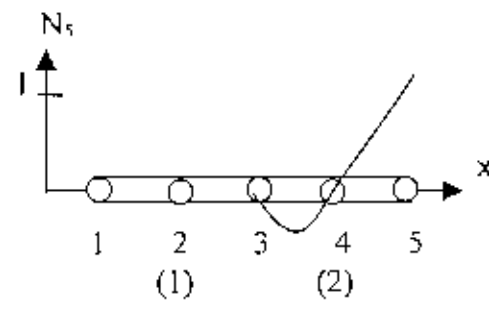
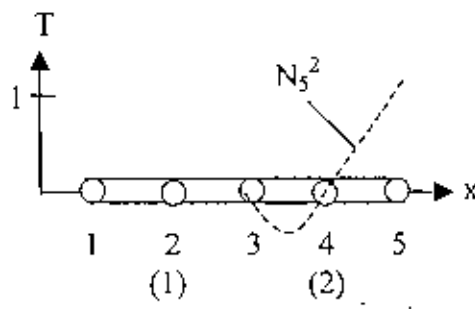
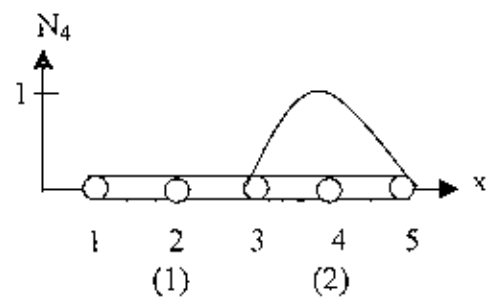
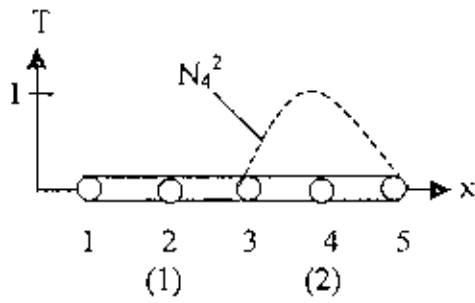
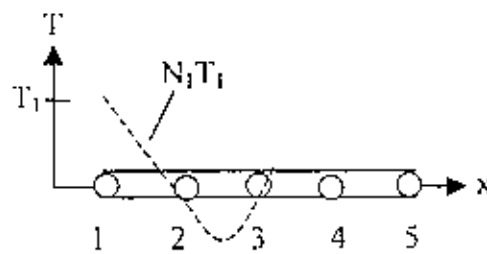


Fig. 3.15 Definition of global shape functions from element shape functions

It appears that the approximation over the entire body given in figure 3.14 can now be written as

$$T = N_1 T_1 + N_2 T_2 + N_3 T_3 + N_4 T_4 + N_5 T_5 \tag{3.16}$$

as illustrated in figure 3.16.





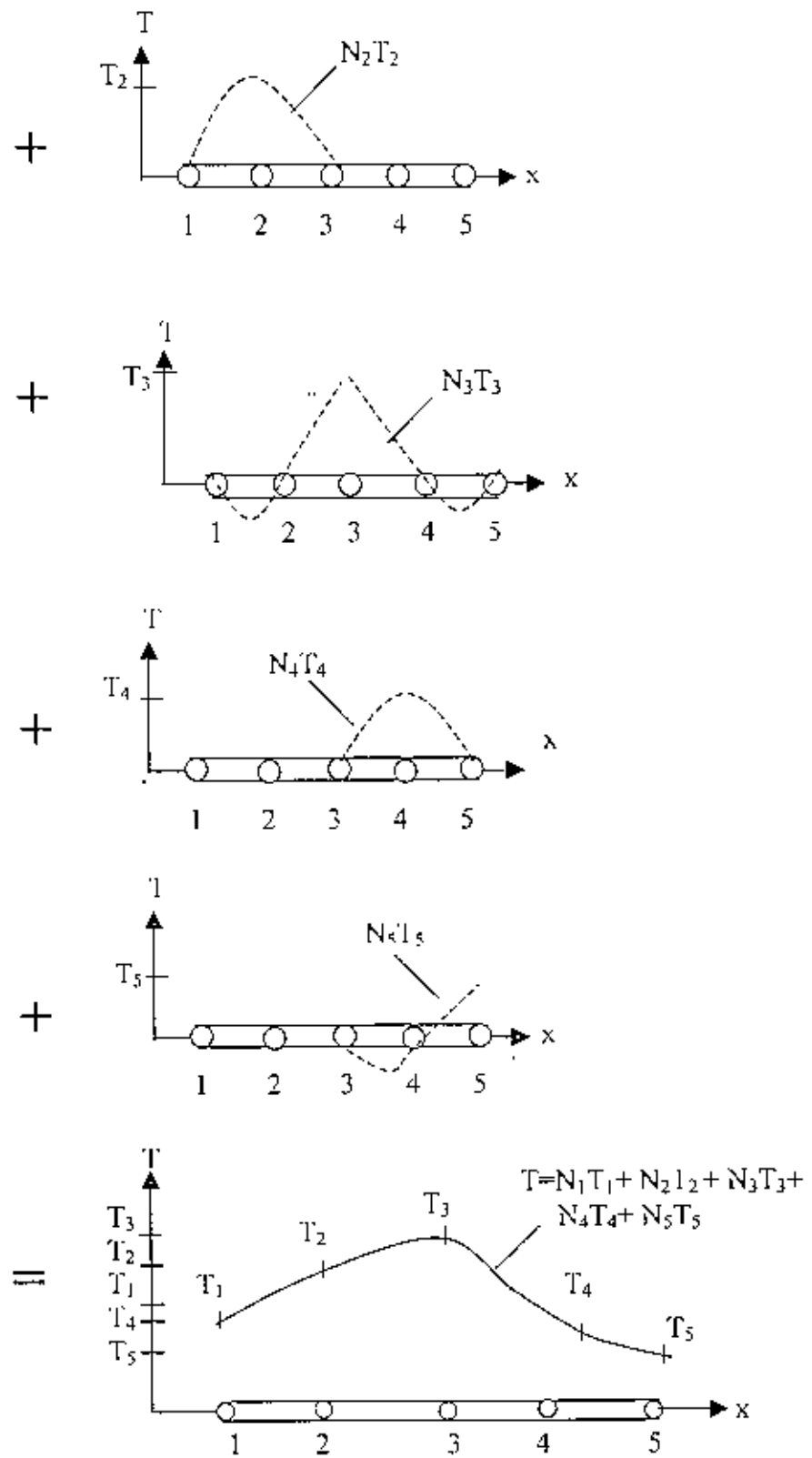


Fig. 3.16 Approximation of the temperature over entire body using global shape functions

### 3.1.4 CUBIC AND HIGHER ORDER ONE DIMENSIONAL ELEMENTS – LAGRANGE INTERPOLATION

Previously we established the linear and quadratic elements and we may proceed to consider the cubic element given by

$$T = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \quad (3.17)$$

In order to express the parameters  $\alpha_1, \dots, \alpha_4$  in terms of the temperatures at the nodal point, it is possible again to use the C-matrix concept given by equation (3.9), which requires that the inverse  $C^{-1}$  is established. For the linear element, this inverse is easily obtained, but for the quadratic element the calculations are rather cumbersome. This obstacle will be much more pronounced when an expression like (3.17) is considered, but it turns out that a general procedure exists by which the element shape functions can be written down directly.

To establish this general procedure, we shall first adopt a suitable numbering of the (local) nodal points belonging to an element. It is emphasized that this local numbering has no effect on the numbers of the global nodal points. With the nodal numbers of the linear element given in figure 3.17(a), equation (3.3) provides the element shape functions

$$N_1^e = -\frac{(x-x_2)}{(x_1-x_2)} ; \quad N_2^e = \frac{(x-x_1)}{(x_2-x_1)} \quad (3.18)$$

Likewise, with the nodal numbers of the quadratic element given in figure 3.17(b),

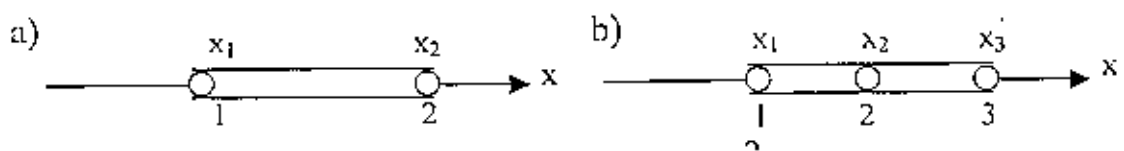


Fig. 3.17 (a) linear element ; (b) quadratic element

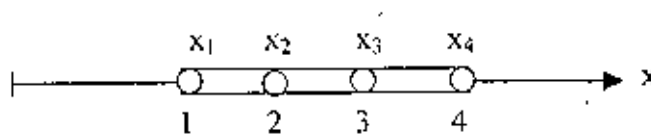


Fig. 3.18 Cubic element

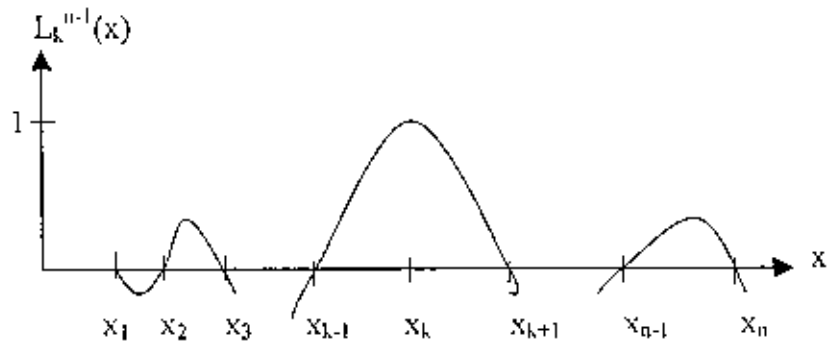


Fig. 3.19 Lagrange's interpolation function

equation (3.14) results in

$$\begin{aligned}
 N_1^e &= \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}; & N_2^e &= -\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \\
 N_3^e &= \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}
 \end{aligned}
 \quad (3.19)$$

Consider now the cubic element in figure 3.18. It is required that the element shape functions allow us to write

$$T = N_1^e T_1 + N_2^e T_2 + N_3^e T_3 + N_4^e T_4 \quad (3.20)$$

i.e. that the temperature is interpolated between its values at the nodal points, but in order to do so the element shape functions must possess the fundamental property given by equation (3.15). We also know that the element shape functions must be a polynomial and a polynomial fulfilling equation (3.15) is given directly by Lagrange's interpolation formula. In fact, for  $n$  given points this formula provides a polynomial of order  $n-1$  given by

$$l_k^{n-1}(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_1)(x_k-x_2)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}; \quad (3.21)$$

$k = 1, 2, \dots, n$

where it should be noted that the term  $(x_k - x_k)$  is not present in the denominator and, likewise, the term  $(x - x_k)$  is not present in the numerator.

It appears that  $l_k^{n-1}(x_k) = 1$  and  $l_k^{n-1}(x_i) = 0$  for  $k \neq i$ , as illustrated in figure 3.19. Moreover, these properties are in accordance with the fundamental property for element shape functions given by equation (3.15). Therefore, the element shape functions can be constructed directly by setting

$$N_k^e = l_k^{n-1}; \quad k = 1, 2, \dots, n \quad (3.22)$$

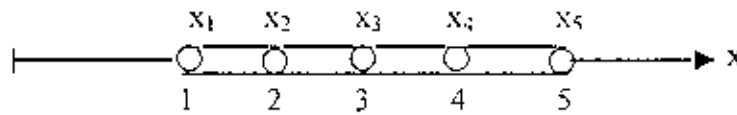


Fig. 3.20 Quartic element

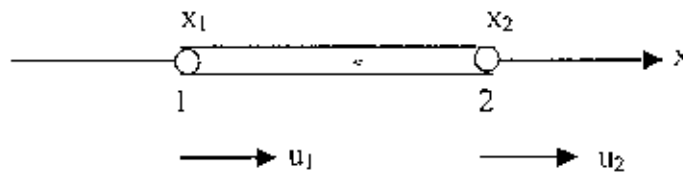


Fig. 3.21 Linear element applicable for axially loaded bar

To illustrate the use of Lagrange's interpolation function, we shall re-establish the element shape functions  $N_1^e$  and  $N_2^e$  for the linear element shown in figure 3.17(a). As the element comprises two nodal points, we have  $n=2$  and equation (3.22) then provides

$$N_k^e = l_k^l; \quad k = 1, 2$$

From equation (3.21) we get

$$l_1^l = \frac{x - x_2}{x_1 - x_2}; \quad l_2^l = \frac{x - x_1}{x_2 - x_1}$$

In a similar way, it can be seen that for the quadratic element, i.e.  $n=3$ , use of equation (3.22) yields the results given by equation (3.19). Therefore, the element shape functions of the cubic element shown in figure 3.18 are obtained directly from equation (3.22) by putting  $n = 4$ , whereas the element shape functions of the quadratic element given by figure 3.20 emerge for  $n = 5$ . This process may be continued for arbitrary  $n$ -values. As the element shape functions for all the one-dimensional elements considered may be derived using Lagrange's interpolation formula, these elements are termed Lagrange elements.

### 3.2 TRIGONOMETRIC INTERPOLATION

In mathematics, trigonometric interpolation is interpolation with trigonometric polynomials. Interpolation is the process of finding a function which goes through some given data points. When the function we desire to represent by an interpolation formula is known to be periodic, it is better to use trigonometric interpolation. Hermite's formula for interpolating periodic function is

$$y = \frac{\sin(x-x_1)\sin(x-x_2)\dots\sin(x-x_n)}{\sin(x_0-x_1)\sin(x_0-x_2)\dots\sin(x_0-x_n)}y_0 + \frac{\sin(x-x_0)\sin(x-x_2)\dots\sin(x-x_n)}{\sin(x_1-x_0)\sin(x_1-x_2)\dots\sin(x_1-x_n)}y_1$$

$$+ \dots + \frac{\sin(x-x_0)\sin(x-x_1)\dots\sin(x-x_{n-1})}{\sin(x_n-x_0)\sin(x_n-x_1)\dots\sin(x_n-x_{n-1})}y_n$$

It is evident also that  $y = y_0$  when  $x = x_0$ ,  $y = y_1$  when  $x = x_1$ , etc.

This formula of Hermite's for periodic functions corresponds to Lagrange's formula for non-periodic functions, and applies whether the given values of  $x$  are equidistant or not.

The problem of trigonometric interpolation was first solved by Gauss, who derived several formulas similar to Hermite's. The formula usually called Gauss's formula differs from Hermite's only in having the factor  $\frac{1}{2}$  written in front of all the angles; thus,  $\sin \frac{1}{2}(x-x_0)$  etc.

## CHAPTER 4

### SOLUTION OF DIFFERENTIAL EQUATION BY FINITE ELEMENT METHOD

#### 4.1 INTRODUCTION

In the previous chapter we have derived the approximation function for the field variable (say temperature).

Now we are in a position to solve a differential equation using the finite element method. First we discuss the procedures involved in the solution of differential equation by FEM. We shall discuss here two methods.

1. Global finite element formulation i.e. finite element formulation of the whole body at a time and see how the stiffness matrix is related with each element of the body.
2. Element finite element formulation i.e. element formulation of each element separately and then assemble the element contribution to get the stiffness matrix for the whole body, which is used for the computer programming application.

#### 4.2 GLOBAL FE FORMULATION

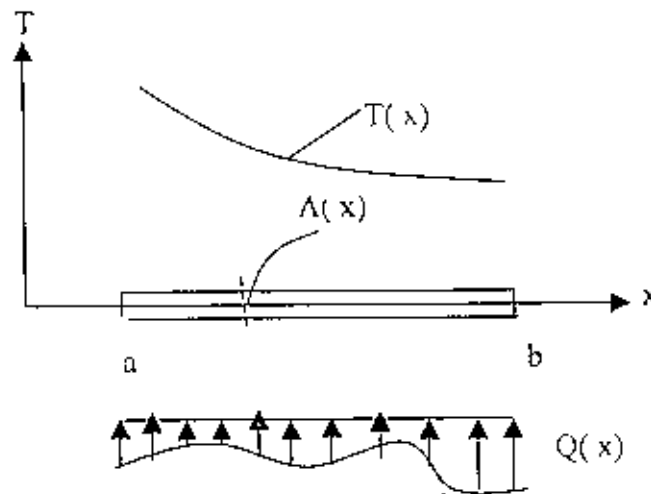


Fig. 4.1 Heat conduction in one-dimensional body

Consider one-dimensional heat flow in a body shown in figure 4.1. The differential equation is

$$\frac{d}{dx} \left( Ak \frac{dT}{dx} \right) + Q = 0 ; a \leq x \leq b \quad (4.1)$$

where it is recalled that  $A(x)$  is the cross-sectional area of the body,  $k(x)$  is the thermal conductivity and  $Q(x)$  is the heat supply per unit time and per unit length of the body.  $Q$  may be transferred to the body across its outer surface or created internally, for instance by electric heating. We shall leave the boundary conditions unspecified until later.

To obtain the finite element form of 4.1 we multiply by the arbitrary weight function  $v(x)$  and integrate over the region of interest, i.e.

$$\int_a^b v \left[ \frac{d}{dx} \left( Ak \frac{dT}{dx} \right) + Q \right] dx = 0$$

The first term is integrated by parts to provide

$$\int_a^b \frac{dv}{dx} Ak \frac{dT}{dx} dx = \left[ vAk \frac{dT}{dx} \right]_a^b + \int_a^b vQ dx \quad (4.2)$$

and using Fourier's law  $q = -k \frac{dT}{dx}$ , we get

$$\int_a^b \frac{dv}{dx} Ak \frac{dT}{dx} dx = - \left[ vAkq \right]_a^b + \int_a^b vQ dx \quad (4.3)$$

It appears that the integration by parts implies that the order of differentiation for  $T$  has decreased at the expense of the weight function  $v$  being differentiated.

The approximation for the temperature  $T$  is now introduced. Following the summary of chapter 3, the approximation over the entire region is in general written as

$$T = \mathbf{N} \mathbf{a} = N_1 T_1 + N_2 T_2 + \dots + N_n T_n \quad (4.4)$$

where  $\mathbf{N}$  is the global shape function matrix and  $\mathbf{a}$  contains the temperatures at the nodal points in the entire body. That is we have

$$\mathbf{N} = [N_1 \quad N_2 \quad \dots \quad N_n] ; \quad \mathbf{a} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} \quad (4.5)$$

where  $n$  denotes the number of nodal points for the entire body and  $N_i = N_i(x)$ . As  $\mathbf{a}$  does not depend on  $x$ , (4.4) implies that

$$\frac{dT}{dx} = \mathbf{B}\mathbf{a} \quad \text{where} \quad \mathbf{B} = \frac{d\mathbf{N}}{dx} \quad (4.6)$$

i.e. 
$$\mathbf{B} = \left[ \frac{dN_1}{dx} \quad \frac{dN_2}{dx} \quad \dots \quad \frac{dN_n}{dx} \right] \quad (4.7)$$

Inserting (4.6) in (4.3) we obtain

$$\left( \int_a^b \frac{dv}{dx} Ak\mathbf{B}dx \right) \mathbf{a} = -[v\Delta q]_a^b + \int_a^b vQdx \quad (4.8)$$

It is important that the approximation for  $dT/dx$  as given by (4.6) is not inserted into the boundary terms on the right-hand side of (4.8), which include the flux  $q = -k dT/dx$ . The reason is that the boundary conditions either specify the flux or the temperature itself at the boundary and there is no reason to make an approximation for matters that we know beforehand.

The last step is the choice of weight function  $v$ . In accordance with the Galerkin method, we choose the weight functions to be equal to the trial functions. In the present case the trial functions are the shape functions. Suppose

$$v = \mathbf{N}\mathbf{c} \quad (4.9)$$

since  $v$  is an arbitrary function, the matrix  $\mathbf{c}$  is arbitrary. Moreover as  $v$  is one number, we have  $v = v^T$ , i.e. can be written as

$$v = \mathbf{c}^T \mathbf{N}^T \quad (4.10)$$

which implies that  $dv/dx = \mathbf{c}^T \mathbf{B}^T$  where  $\mathbf{B}^T = d\mathbf{N}^T/dx$  (4.11)

Inserting (4.10) and (4.11) into (4.8) and using the fact that  $\mathbf{c}^T$  is independent of  $x$  results in

$$\mathbf{c}^T \left[ \left( \int_a^b \mathbf{B}^T Ak\mathbf{B}dx \right) \mathbf{a} + [\mathbf{N}^T \Delta q]_a^b - \int_a^b \mathbf{N}^T Qdx \right] = 0$$

As this expression should hold for arbitrary  $\mathbf{c}^T$  matrices, it is concluded that

$$\left( \int_a^b \mathbf{B}^T Ak\mathbf{B}dx \right) \mathbf{a} = -[\mathbf{N}^T \Delta q]_a^b + \int_a^b \mathbf{N}^T Qdx \quad (4.12)$$

which is the required FE formulation. Here, we have derived (4.12) by means of (4.8) and (4.9) and using the fact that  $v$  is arbitrary. Alternatively, in (4.8) we may directly choose  $n$  arbitrary weight functions and first choose  $v = N_1$ , then  $v = N_2$  and so on..

In order to write (4.12) in a more compact fashion, we define the following matrices:



$$\mathbf{K} = \left( \int_a^b \mathbf{B}^T A k \mathbf{B} dx \right)$$

$$\mathbf{f}_b = -[\mathbf{N}^T A q]_b \quad (4.13)$$

$$\mathbf{f}_l = \int_a^b \mathbf{N}^T Q dx$$

Referring to (4.7) it is obvious that  $\mathbf{K}$  is a square matrix with dimension  $n \times n$ ; it is called the stiffness matrix. Likewise, both  $\mathbf{f}_b$  and  $\mathbf{f}_l$  have the dimension  $n \times 1$ , and they are called the boundary vector and load vector, respectively, since  $\mathbf{f}_b$  refers to conditions at the boundary, whereas  $\mathbf{f}_l$  considers the effect of the 'loading'  $Q$ . With (4.13), (4.12) takes the form

$$\mathbf{K}\mathbf{a} = \mathbf{f}_b + \mathbf{f}_l \quad (4.14)$$

and defining the force vector  $\mathbf{f}$  by

$$\mathbf{f} = \mathbf{f}_b + \mathbf{f}_l \quad (4.15)$$

(4.14) can be written as

$$\mathbf{K}\mathbf{a} = \mathbf{f} \quad (4.16)$$

The temperatures at the nodal points given by  $\mathbf{a}$  are obtained by solving this system of linear equations. When  $\mathbf{a}$  is known, the temperature at an arbitrary point in the body is given by (4.4) and the temperature gradient at an arbitrary point in the body is given by (4.6). From this temperature gradient, the flux  $q$  at any location in the body is obtained from Fourier's law. Therefore, when  $\mathbf{a}$  has been determined from (4.16) all quantities of interest can be derived. Let us now evaluate the stiffness matrix  $\mathbf{K}$  and the force vector  $\mathbf{f}$  in more detail.

To evaluate the character of the stiffness matrix  $\mathbf{K}$  further we consider its components. From (4.13) and (4.7) it follows that

$$\begin{bmatrix} \int_a^b \frac{dN_1}{dx} Ak \frac{dN_1}{dx} dx & \int_a^b \frac{dN_1}{dx} Ak \frac{dN_2}{dx} dx & \dots & \int_a^b \frac{dN_1}{dx} Ak \frac{dN_n}{dx} dx \\ \int_a^b \frac{dN_2}{dx} Ak \frac{dN_1}{dx} dx & \int_a^b \frac{dN_2}{dx} Ak \frac{dN_2}{dx} dx & \dots & \int_a^b \frac{dN_2}{dx} Ak \frac{dN_n}{dx} dx \\ \dots & \dots & \dots & \dots \\ \int_a^b \frac{dN_n}{dx} Ak \frac{dN_1}{dx} dx & \int_a^b \frac{dN_n}{dx} Ak \frac{dN_2}{dx} dx & \dots & \int_a^b \frac{dN_n}{dx} Ak \frac{dN_n}{dx} dx \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \dots \\ T_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \dots \\ f_n \end{bmatrix}$$

using  $k_{ij} = \int_a^b \frac{dN_i}{dx} Ak \frac{dN_j}{dx} dx$ , we can write

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & \dots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \dots & \dots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \dots & \dots & k_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ k_{n1} & k_{n2} & k_{n3} & \dots & \dots & k_{nn} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \dots \\ T_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \dots \\ f_n \end{bmatrix}$$

The matrix  $\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & \dots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \dots & \dots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \dots & \dots & k_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ k_{n1} & k_{n2} & k_{n3} & \dots & \dots & k_{nn} \end{bmatrix}$  is called Global stiffness

matrix.

Let us now take a body composed of three linear elements and see the size and structure of the stiffness matrix of the entire body related to each of its elements.

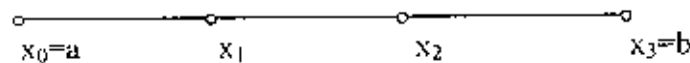


Fig. 4.2

For three elements having four nodes as shown in figure 4.2 the approximation function is

$$T = N_1 T_1 + N_2 T_2 + N_3 T_3 + N_4 T_4$$

and ultimately the stiffness matrix takes the form

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

### 4.3 ELEMENT FE FORMULATION

For one linear element approximation function is  $T^e = N_1^e T_1^e + N_2^e T_2^e$  where  $e$  is the element number. Exactly in the same way as before we get the stiffness matrix as  $\mathbf{K}^e \mathbf{a}^e = \mathbf{f}^e$  where

$$\mathbf{K}^e = \begin{bmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{bmatrix} \quad \mathbf{a}^e = \begin{bmatrix} T_1^e \\ T_2^e \end{bmatrix} \quad \mathbf{f}^e = \begin{bmatrix} f_1^e \\ f_2^e \end{bmatrix}$$

$$\text{where } k_{ij}^e = \int_{x_0}^{x_1} \frac{dN_i^e}{dx} A k \frac{dN_j^e}{dx} dx$$

For the element 1

$$\mathbf{K}^{(1)} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix}$$

For the element 2

$$\mathbf{K}^{(2)} = \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix}$$

For the element 3

$$\mathbf{K}^{(3)} = \begin{bmatrix} k_{11}^{(3)} & k_{12}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix}$$

Now we find the relation between the elements of Global stiffness matrix and element stiffness matrix.

We have

$$\begin{aligned}
 k_{ij} &= \int_a^b \frac{dN_i}{dx} Ak \frac{dN_j}{dx} dx \\
 k_{11} &= \int_a^b \frac{dN_1}{dx} Ak \frac{dN_1}{dx} dx \\
 &= \int_{x_u}^{x_1} \frac{dN_1^{(1)}}{dx} Ak \frac{dN_1^{(1)}}{dx} dx + \int_{x_1}^{x_2} \frac{d0}{dx} Ak \frac{d0}{dx} dx + \int_{x_2}^{x_3} \frac{d0}{dx} Ak \frac{d0}{dx} dx \\
 &= k_{11}^{(1)} \\
 k_{12} &= \int_a^b \frac{dN_1}{dx} Ak \frac{dN_2}{dx} dx \\
 &= \int_{x_u}^{x_1} \frac{dN_1^{(1)}}{dx} Ak \frac{dN_2^{(1)}}{dx} dx + \int_{x_1}^{x_2} \frac{d0}{dx} Ak \frac{dN_2^{(2)}}{dx} dx + \int_{x_2}^{x_3} \frac{d0}{dx} Ak \frac{d0}{dx} dx \\
 &\quad \text{[using equation (3.5)]} \\
 &= k_{12}^{(1)} \\
 k_{13} &= \int_a^b \frac{dN_1}{dx} Ak \frac{dN_3}{dx} dx \\
 &= \int_{x_u}^{x_1} \frac{dN_1^{(1)}}{dx} Ak \frac{d0}{dx} dx + \int_{x_1}^{x_2} \frac{d0}{dx} Ak \frac{dN_3^{(2)}}{dx} dx + \int_{x_2}^{x_3} \frac{d0}{dx} Ak \frac{dN_3^{(3)}}{dx} dx \\
 &= 0 \\
 k_{14} &= 0 \\
 k_{21} &= k_{12}^{(1)} \\
 \text{but } k_{22} &= \int_a^b \frac{dN_2}{dx} Ak \frac{dN_2}{dx} dx \\
 &= \int_{x_u}^{x_1} \frac{dN_2^{(1)}}{dx} Ak \frac{dN_2^{(1)}}{dx} dx + \int_{x_1}^{x_2} \frac{dN_2^{(2)}}{dx} Ak \frac{dN_2^{(2)}}{dx} dx + \int_{x_2}^{x_3} \frac{d0}{dx} Ak \frac{d0}{dx} dx \\
 &= k_{22}^{(1)} + k_{11}^{(2)} \\
 \\
 k_{23} &= k_{12}^{(2)}, k_{24} = 0, k_{31} = 0, k_{32} = k_{21}^{(2)} \\
 k_{33} &= k_{22}^{(2)} + k_{11}^{(3)}, k_{34} = k_{12}^{(3)}, k_{41} = 0, k_{42} = 0 \\
 k_{43} &= k_{21}^{(3)}, k_{44} = k_{22}^{(3)}
 \end{aligned}$$

Hence the global stiffness matrix becomes

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(1)} & k_{12}^{(1)} \\ 0 & 0 & k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix}$$

This is a square and symmetric matrix.

Now we shall show how the Global stiffness matrix can be assembled directly from the element stiffness matrix. The body is shown in the figure 3.3.

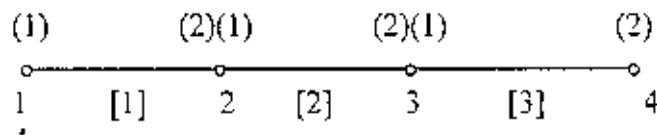


Fig. 4.3

[i] are the element number

(i) are node number of the element

i are the global node number of the body

element [1] is related with global node no. 1 and 2

element [2] is related with global node no. 2 and 3

element [3] is related with global node no. 3 and 4

For the element 1

$$\begin{matrix} 1 & 2 \\ 1 \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix} \end{matrix}$$

For the element 2

$$\begin{matrix} 2 & 3 \\ 2 \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \end{matrix}$$

For the element 3

$$\begin{matrix} 3 & 4 \\ 3 \begin{bmatrix} k_{11}^{(3)} & k_{12}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \end{matrix}$$

From the above we can form the global stiffness matrix from element stiffness matrix as follows :

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix}$$

Using the same procedure the right hand side of the global stiffness equation can also be formed from the right hand side of the element stiffness equation.

The right hand side of the element stiffness equation is

$$\mathbf{f}^e = \begin{bmatrix} f_1^e \\ f_2^e \end{bmatrix}$$

For element 1  $\begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{bmatrix}$   $\begin{matrix} 1 \\ 2 \end{matrix}$

For element 2  $\begin{bmatrix} f_1^{(2)} \\ f_2^{(2)} \end{bmatrix}$   $\begin{matrix} 2 \\ 3 \end{matrix}$

For element 3  $\begin{bmatrix} f_1^{(3)} \\ f_2^{(3)} \end{bmatrix}$   $\begin{matrix} 3 \\ 4 \end{matrix}$

Right hand side vector of the global stiffness matrix is

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} \end{bmatrix}$$

Thus the equation  $\mathbf{Ka}=\mathbf{f}$  becomes

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} \end{bmatrix} \quad (4.18)$$

After using proper boundary condition of the differential equation, the equation is solved for  $T_1, T_2, T_3, T_4$  by using any equation solver such as Gauss elimination method.

#### 4.4 DIFFERENT STEPS OF SOLVING A DIFFERENTIAL EQUATION BY FEM

The followings are the different steps of solving an equation by finite element method.

- (i) Discretization (or representation) of the given domain into a collection of pre selected finite elements.
- (ii) Derivation of approximation functions of  $U = \sum_{i=1}^n u_i \psi_i$  which is the interpolation function.
- (iii) Derivation of element (stiffness) equations for all typical element of the mesh.
- (iv) Assembly of element equations to obtain the equation of whole problem.
- (v) Imposition of the boundary condition of the problem.
- (vi) Solution of the assembled equations .
- (vii) Post processing the results.

## CHAPTER 5

### EIGENVALUE PROBLEM AND ITS SOLUTION BY FEM

#### 5.1 INTRODUCTION

The problem of determining the values of the constant  $\lambda$  such that

$$\begin{aligned} -\frac{d}{dx}\left(a\frac{du}{dx}\right) &= \lambda u \quad \text{for } 0 < x < 1 \\ u(0) &= d_0, \quad \left(a\frac{du}{dx}\right)_{x=1} = 0 \end{aligned} \quad (5.1)$$

is called the eigenvalue problem. The values of  $\lambda$  for which equation (5.1) can be satisfied are called eigenvalues, and the associated functions  $u$  are called eigenfunctions.

Here we shall solve a sample eigenvalue problem by finite element method showing different steps :

#### 5.2 SOLUTION OF AN EIGENVALUE PROBLEM BY FEM

The eigenvalue problem is

$$\frac{d^2U}{dx^2} + \lambda U = 0 \quad U(0) = 0, \quad U(1) = 0 \quad (5.2)$$

We first consider a typical element as shown in the figure 5.1 and derive the weighted integral form of the DE.

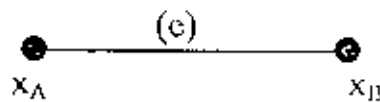


Fig. 5.1

Multiply the equation by the weight function  $w$  and integrating on the element extended from  $x_A$  to  $x_B$



$$\begin{aligned}
0 &= \int_{x_A}^{x_B} w \left( -\frac{d^2 U}{dx^2} - \lambda U \right) dx \\
&= \left[ -w \frac{dw}{dx} \right]_{x_A}^{x_B} + \int_{x_A}^{x_B} \left( \frac{dw}{dx} \frac{dU}{dx} - \lambda U \right) dx \\
&= \int_{x_A}^{x_B} \left( \frac{dw}{dx} \frac{dU}{dx} - \lambda U \right) dx - Q_1 w(x_A) - Q_n w(x_B)
\end{aligned}$$

The weighted integral form of equation (5.2) is

$$0 = \int_{x_A}^{x_B} \left( \frac{dw}{dx} \frac{dU}{dx} - \lambda w U \right) dx - Q_1 w(x_A) - Q_n w(x_B) \quad (5.3)$$

where  $Q_1$  and  $Q_n$  are the usual secondary variables ( $Q_i^e = 0, i \neq 1, n$ )

$$Q_1 = -\left( \frac{dU}{dx} \right)_{x_A} \quad Q_n = \left( \frac{dU}{dx} \right)_{x_B}$$

**Step-1:** Discretization of the given domain into a collection of finite elements.

The domain is discretized into two linear elements as shown in the figure 5.2.

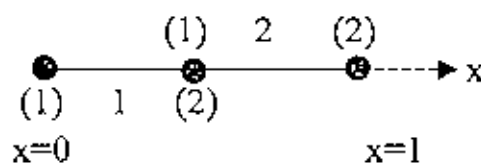


Fig. 5.2

**Step-2 :** Derivation of approximation function .

In this case it is considered that the elements are linear. So the approximation function is of the form ,

$$\begin{aligned}
U &= u_1^e \psi_1 + u_2^e \psi_2 \\
&= \frac{x_B - x}{x_B - x_A} u_1^e + \frac{x - x_A}{x_B - x_A} u_2^e \\
\therefore \psi_1 &= \frac{x_B - x}{x_B - x_A}, \quad \psi_2 = \frac{x - x_A}{x_B - x_A}
\end{aligned}$$

$\psi_1$  and  $\psi_2$  are called the shape functions.

**Step-3 :** Derivation of element (stiffness) equation .

Substitution of the finite element approximation into equation (5.3) and consideration of Galerkin method in which the weight function is the shape function gives us the finite element model of the eigenvalue equation.

$$\begin{aligned}
\begin{bmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} - \lambda \begin{bmatrix} M_{11}^e & M_{12}^e \\ M_{21}^e & M_{22}^e \end{bmatrix} \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} &= \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \\
\left[ K^e \right] \left\{ u^e \right\} - \lambda \left[ M^e \right] \left\{ u^e \right\} &= \left\{ Q^e \right\}
\end{aligned} \tag{5.4}$$

where

$$k_{ij}^e = \int_{x_A}^{x_B} \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} dx \quad \text{and} \quad M_{ij}^e = \int_{x_A}^{x_B} \psi_i^e \psi_j^e dx \quad \text{with } i, j = 1, 2$$

Considering  $x_B - x_A = h_e$  we get

$$\begin{aligned}
k_{11} &= \frac{1}{h_e} & k_{12} &= -\frac{1}{h_e} & M_{11} &= \frac{h_e}{3} & M_{12} &= \frac{h_e}{6} \\
k_{21} &= -\frac{1}{h_e} & k_{22} &= \frac{1}{h_e} & M_{21} &= \frac{h_e}{6} & M_{22} &= \frac{h_e}{3}
\end{aligned}$$

Then the equation (5.4) becomes

$$\left( \frac{1}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\lambda h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

For the element 1, we have

$$2 \left( \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\lambda}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \end{Bmatrix}$$

For the element 2, we have

$$2 \left( \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\lambda}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} Q_1^{(2)} \\ Q_2^{(2)} \end{Bmatrix}$$

$$\text{here } u_1^{(1)} = U_1, \quad u_2^{(1)} = u_1^{(2)} = U_2, \quad u_2^{(2)} = U_3$$

**Step-4 :** Assembly of the element equation to obtain the equation of the whole problem.

For a mesh of two linear elements the assembled equations are

$$2 \left( \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\lambda}{12} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^t \\ 0 \\ Q_2^t \end{Bmatrix}$$

**Step-5 :** Imposition of boundary condition on the problem.

The boundary conditions  $U(0) = 0$  and  $U(1) = 0$  require  $U_1 = U_3 = 0$ .  
Therefore

$$2 \left( \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\lambda}{12} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} 0 \\ U_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} Q_1^t \\ 0 \\ Q_2^t \end{Bmatrix}$$

**Step-6 :** Solution of the assembled equation.

The eigenvalue problem reduces to the single equation.

$$\left( 4 - \frac{4}{12} \lambda \right) U_2 = 0 \quad \text{or } \lambda_1 = 12.0, \quad U_2 \neq 0$$

**Step-7 :** The corresponding eigenfunction amplitude is  $U_2=1.0$  (or any nonzero constant). So for the linear elements ( $h = 0.5$ ) the eigenfunctions are as follows :

$$U(x) = \begin{cases} U_1\psi_1^1 + U_2\psi_2^1 = \frac{x}{h} & \text{for } 0 \leq x \leq 0.5, \text{ element 1} \\ U_2\psi_1^2 + U_3\psi_2^2 = \frac{1-x}{h} & \text{for } 0.5 \leq x \leq 1.0, \text{ element 2} \end{cases}$$

The exact eigenvalues are  $\lambda_n=(n\pi)^2$  and  $\lambda_1=\pi^2=9.8696$ . The exact eigenfunctions are  $U_n(x)=\sin n\pi x$  and  $U_1=\sin \pi x$ .

## CHAPTER 6

### SOLUTION OF EIGENVALUE PROBLEM BY USING TRIGONOMETRIC INTERPOLATION

Here sine interpolation is used in place of algebraic interpolation. Shape functions used in this case are

$$N_1 = \frac{\sin(x_B - x)}{\sin(x_B - x_A)} \quad N_2 = \frac{\sin(x - x_A)}{\sin(x_B - x_A)}$$

Using the substitution  $x = \frac{x_A + x_B}{2} + \frac{x_B - x_A}{2} \xi$  shape function becomes

$$N_1 = \frac{\sin\left(\frac{x_B - x_A}{2}(1 - \xi)\right)}{\sin(x_B - x_A)} \quad N_2 = \frac{\sin\left(\frac{x_B - x_A}{2}(1 + \xi)\right)}{\sin(x_B - x_A)}$$

$$k_{ij} = \int_{-1}^1 \left[ \left( \frac{x_A + x_B}{2} + \frac{x_B - x_A}{2} \xi \right)^2 \frac{dN_i}{d\xi} \frac{d\xi}{dx} \frac{dN_j}{d\xi} \frac{d\xi}{dx} - 2 \left( \frac{x_A + x_B}{2} + \frac{x_B - x_A}{2} \xi \right) N_i N_j \right] \frac{h}{2} d\xi$$

For the first element  $x_A = 0, x_B = 1$

$$k_{11} = \int_{-1}^1 \left[ \frac{1}{4} (1 + \xi)^2 \frac{\cos \frac{1}{2}(1 - \xi) \left(-\frac{1}{2}\right)}{\sin(1)} \frac{\cos \frac{1}{2}(1 - \xi) \left(-\frac{1}{2}\right)}{\sin(1)} - 2 \cdot \frac{1}{2} (1 + \xi) \frac{\sin \frac{1}{2}(1 - \xi)}{\sin(1)} \frac{\sin \frac{1}{2}(1 - \xi)}{\sin(1)} \right] \frac{1}{2} d\xi$$

= .252

Similarly  $k_{12} = k_{21} = -.606, \quad k_{22} = -.303$

So for element 1

$$K^{(1)} = \begin{bmatrix} .252 & -.606 \\ -.606 & -.303 \end{bmatrix}$$

For element 2

$$K^{(2)} = \begin{bmatrix} 2.703 & -3.431 \\ -3.431 & 1.133 \end{bmatrix}$$

For element 20

$$K^{(20)} = \begin{bmatrix} 456.55 & -456.39 \\ -456.39 & 448.70 \end{bmatrix}$$

$$\text{Also } M_{ij} = \int_{-1}^1 \left( \frac{x_A + x_B}{2} + \frac{x_B - x_A}{2} \xi \right)^2 \psi_i \psi_j \frac{h}{2} d\xi$$

For the first element  $x_A = 0, x_B = 1$

$$M_{11} = \int_{-1}^1 \frac{1}{4} (1 + \xi)^2 \frac{1}{2} (1 - \xi) \frac{1}{2} (1 - \xi) \frac{1}{2} d\xi = .055$$

Similarly,  $M_{12} = M_{21} = 0.076, M_{22} = 0.247$

Thus for element 1

$$M^{(1)} = \begin{bmatrix} .055 & .076 \\ .076 & .247 \end{bmatrix}$$

For element 2

$$M^{(2)} = \begin{bmatrix} .774 & .603 \\ .603 & 1.349 \end{bmatrix}$$

For element 20

$$M^{(20)} = \begin{bmatrix} 169.46 & 100.20 \\ 100.20 & 176.95 \end{bmatrix}$$

The global stiffness matrix  $K_1$  and the global mass matrix  $M_1$  will be a  $21 \times 21$  matrix.

The matrix eigenvalue equation to be solved is  $K_1 \psi = \lambda M_1 \psi$

The equation is solved by using computer programming to find the eigenvalues as shown in the chapter 7.

Similarly we shall find eigenvalues using tangent interpolation

$$N_1 = \frac{\tan(x_B - x)}{\tan(x_B - x_A)} \quad N_2 = \frac{\tan(x - x_A)}{\tan(x_B - x_A)}$$

## CHAPTER 7

### RESULTS, DISCUSSION AND CONCLUSION

Ram-mohan et al calculated the eigenvalues using Lagrange interpolation for the domain 0 to 20 taking 20 elements having length 1 for each element.

We have calculated the eigenvalues using sine interpolation for the same domain having same number of elements. The results are shown below in the tabular form.

Table 7.1

Eigenvalues obtained by using Lagrange and sine interpolation for the domain 0 to 20 taking 20 elements having length 1 for each element.

Eigenvalues	Lagrange	Sine
$\lambda_1$	12.928	12.605
$\lambda_2$	11.565	11.936
$\lambda_3$	10.786	10.924
$\lambda_4$	9.724	9.654
$\lambda_5$	8.522	8.295
$\lambda_6$	7.292	6.967
$\lambda_7$	6.110	5.740
$\lambda_8$	5.022	4.656
$\lambda_9$	4.050	3.715
$\lambda_{10}$	3.199	2.913
$\lambda_{11}$	2.462	2.390
$\lambda_{12}$	1.842	1.676
$\lambda_{13}$	1.319	1.212
$\lambda_{14}$	0.8855	0.833
$\lambda_{15}$	0.5333	0.528
$\lambda_{16}$	0.2550	0.290
$\lambda_{17}$	0.0475	0.113
$\lambda_{18}$	-0.0929	-0.0064
$\lambda_{19}$	-0.2381	-0.1292
$\lambda_{20}$	-0.9417	-0.6538

Table 7.2

Eigenvalues obtained by using Lagrange and sine interpolation for the domain 0 to 10 taking 20 elements having length 0.5 for each element.

Eigenvalues	Lagrange	Sine
$\lambda_1$	57.483	56.408
$\lambda_2$	46.768	47.124
$\lambda_3$	43.905	44.062
$\lambda_4$	39.802	39.763
$\lambda_5$	35.072	34.869
$\lambda_6$	30.187	29.874
$\lambda_7$	25.471	25.100
$\lambda_8$	21.113	20.721
$\lambda_9$	17.206	16.838
$\lambda_{10}$	13.777	13.446
$\lambda_{11}$	10.815	10.533
$\lambda_{12}$	8.287	8.060
$\lambda_{13}$	6.155	5.982
$\lambda_{14}$	4.380	4.258
$\lambda_{15}$	2.923	2.848
$\lambda_{16}$	1.755	1.718
$\lambda_{17}$	0.851	0.8455
$\lambda_{18}$	0.196	0.2130
$\lambda_{19}$	-0.220	-0.1903
$\lambda_{20}$	-0.981	-0.8904



Table 7.3

Eigenvalues obtained by using Lagrange and sine interpolation for the domain 0 to 5 taking 20 elements having length 0.25 for each element.

Eigenvalues	Lagrange	Sine
$\lambda_1$	243.971	243.196
$\lambda_2$	188.434	188.097
$\lambda_3$	177.243	177.080
$\lambda_4$	160.941	160.971
$\lambda_5$	142.041	142.234
$\lambda_6$	122.491	122.797
$\lambda_7$	103.606	103.976
$\lambda_8$	86.160	86.550
$\lambda_9$	70.522	70.900
$\lambda_{10}$	56.799	57.144
$\lambda_{11}$	44.944	45.243
$\lambda_{12}$	34.826	35.074
$\lambda_{13}$	26.288	26.483
$\lambda_{14}$	19.165	19.311
$\lambda_{15}$	13.308	13.409
$\lambda_{16}$	8.585	8.647
$\lambda_{17}$	4.891	4.921
$\lambda_{18}$	2.144	2.151
$\lambda_{19}$	0.2985	0.2906
$\lambda_{20}$	-0.9623	-0.9876

Table 7.4

Eigenvalues obtained by using Lagrange and tangent interpolation for the domain 0 to 20 taking 20 elements having length 1 for each element.

Eigenvalues	Lagrange	Tangent
$\lambda_1$	12.928	16.055
$\lambda_2$	11.565	12.305
$\lambda_3$	10.786	11.824
$\lambda_4$	9.724	11.109
$\lambda_5$	8.522	10.217
$\lambda_6$	7.292	9.216
$\lambda_7$	6.110	8.165
$\lambda_8$	5.022	7.117
$\lambda_9$	4.050	6.107
$\lambda_{10}$	3.199	5.162
$\lambda_{11}$	2.462	4.298
$\lambda_{12}$	1.842	3.526
$\lambda_{13}$	1.319	2.848
$\lambda_{14}$	0.8855	2.266
$\lambda_{15}$	0.5333	1.780
$\lambda_{16}$	0.2550	1.390
$\lambda_{17}$	0.0475	1.094
$\lambda_{18}$	-0.0929	0.8961
$\lambda_{19}$	-0.2381	0.6656
$\lambda_{20}$	-0.9417	-1.334

**Table 7.5**

Eigenvalues obtained by using Lagrange and tangent interpolation for the domain 0 to 10 taking 20 elements having length 0.5 for each element.

Eigenvalues	Lagrange	Tangent
$\lambda_1$	57.483	59.868
$\lambda_2$	46.768	46.253
$\lambda_3$	43.905	43.729
$\lambda_4$	39.802	40.007
$\lambda_5$	35.072	35.602
$\lambda_6$	30.187	30.948
$\lambda_7$	25.471	26.362
$\lambda_8$	21.113	22.046
$\lambda_9$	17.206	18.114
$\lambda_{10}$	13.777	14.614
$\lambda_{11}$	10.815	11.552
$\lambda_{12}$	8.287	8.913
$\lambda_{13}$	6.155	6.668
$\lambda_{14}$	4.380	4.784
$\lambda_{15}$	2.923	3.230
$\lambda_{16}$	1.755	1.978
$\lambda_{17}$	0.851	1.007
$\lambda_{18}$	0.196	0.3061
$\lambda_{19}$	-0.220	-0.1393
$\lambda_{20}$	-0.981	-1.066

**Table 7.6**

Eigenvalues obtained by using Lagrange and tangent interpolation for the domain 0 to 5 taking 20 elements having length 0.25 for each element.

Eigenvalues	Lagrange	Tangent
$\lambda_1$	243.971	245.751
$\lambda_2$	188.434	187.462
$\lambda_3$	177.243	176.785
$\lambda_4$	160.941	161.058
$\lambda_5$	142.041	142.646
$\lambda_6$	122.491	123.439
$\lambda_7$	103.606	104.748
$\lambda_8$	86.160	87.365
$\lambda_9$	70.522	71.694
$\lambda_{10}$	56.799	57.873
$\lambda_{11}$	44.944	45.880
$\lambda_{12}$	34.826	35.607
$\lambda_{13}$	26.288	26.911
$\lambda_{14}$	19.165	19.638
$\lambda_{15}$	13.308	13.644
$\lambda_{16}$	8.585	8.805
$\lambda_{17}$	4.891	5.015
$\lambda_{18}$	2.144	2.196
$\lambda_{19}$	0.2985	0.3064
$\lambda_{20}$	-0.9623	-1.009

The eigenvalue problem has been solved for different trigonometric interpolation such as sine and tangent. These are calculated for different regions such as 0 to 20, 0 to 10, and 0 to 5 taking different length of elements 1, 0.5 and 0.25 respectively but having number of elements 20 in each case.

Rammohan et al [14] calculated the eigenvalues using Lagrange interpolation which is shown in the second column of table 7.1.

The tables 7.1, 7.2, and 7.3 shows eigenvalues obtained by Lagrange and sine interpolation.

It is seen that there is a good relation between the results obtained by two different interpolations. The eigenvalues obtained by Rammohan et al [14] shows a good agreement with that obtained by using sine interpolation for the same length and same number of elements. We also calculated two other results to show the behaviour in the case of smaller lengths of the elements, that is shown in table 7.2, and 7.3. It is noticeable that in all the cases eigenvalues obtained from sine interpolation shows lower results than those of Lagrange interpolation.

Similarly table 7.4, 7.5 and 7.6 show the eigenvalues obtained by tangent interpolation. In this case it is seen that eigenvalues obtained by tangent interpolation shows the higher values than those of Lagrange interpolation.

The eigenvalues calculated by using different trigonometric interpolation function shows that there is a good agreement of the Lagrange interpolation with the sine and tangent interpolation although sine interpolation shows the lower results and tangent interpolation shows the higher results than those of Lagrange interpolation.

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