Convective flow along a horizontal circular cylinder and a vertical wavy surface with temperature dependent viscosity and thermal conductivity

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Abstract

In this dissertation, a steady two-dimensional natural convection flow along a horizontal circular cylinder and a vertical wavy surface with temperature dependent viscosity $\mu(T)$ and thermal conductivity $k(T)$ has been investigated. Using the appropriate transformations the basic equations are transformed to non-similar boundary-layer equations, which are solved numerically using a very efficient implicit finite-difference method together with Keller box scheme. Here we have focused our attention on the evolution of the surface shear stress in terms of local skin-friction, rate of heat transfer in terms of local Nusselt number, streamlines, isotherms, velocity distribution as well as viscosity distribution for a selection of parameter sets consisting of the viscosity-variation parameter ε, thermal conductivity-variation parameter γ, Prandtl number Pr and amplitude of the wavy surface.

CANDIDATE'S DECLARATION

I hereby declare that the work which is being presented in this dissertation entitled **"Convective flow along a horizontal circular cylinder and a vertical wavy surface with temperature dependent viscosity and thermal conductivity"** submitted in partial fulfillment of the requirements for the award of the degree of master of Philosophy in Mathematics in the Department of Mathematics, Bangladesh University of Engineering & Technology, Dhaka is an authentic record of my own work.

The matter presented in this dissertation has not been submitted elsewhere (University or Institution) for the award of any other degree.

Md. Mamun Molla

Date: 11 November 2003 (*Md. Mamun Molla*)

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Nomenclature

a	Radius of the circular cylinder
C_p	Specific heat at constant pressure
C_{fx}	Local skin-friction
f	Dimensionless stream function
g	Acceleration due to gravity
Gr	Grashof number
Gr_x	Local Grashof number
$k(T)$	Thermal conductivity
Nu_x	Local Nusselt number
Pr	Prandtl number
q_w	Heat flux at the surface
T	Temperature of the fluid in the boundary layer
T_∞	Temperature of the ambient fluid
T_w	Temperature at the surface
u,v	The dimensionless x and y- component of the velocity
$\hat{u},\hat{v}$	The dimensional $\hat{x}$ and $\hat{y}$ component of the velocity
x,y	Axis in the direction along and normal to of the surface

Greek symbols

β	Volumetric coefficient of thermal expansion
ψ	Stream function
τ_w	Shearing stress
ε	Viscosity-variation parameter
γ	Thermal conductivity-variation parameter
γ^*	Constant
η	Non-dimensional similarity variable
ρ	Density of the fluid
ν_∞	Reference kinematic viscosity
$\mu(T)$	Viscosity of the fluid
μ_∞	Dynamic viscosity of the ambient fluid
θ	Dimensionless temperature function
$\sigma(x)$	Surface profile function defined in (3.1)

Subscript

w Wall conditions

∞ Ambient temperature

x Differentiation with respect to x

Superscript

$'$ Differentiation with respect to η

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Chapter 1

Introduction:

Free convection from a horizontal circular cylinder to a surrounding fluid has received considerable interest not only due to its fundamental aspects but also due to its importance is many industrial applications. These applications include nuclear reactors, heat exchangers, hot wires, steam pipe and many others. Most of the studies on free convection from cylinder were focused on horizontal circular cylinders investigated for constant viscosity.

Sparrow and Lee [1], looked at the problem of vertical stream over a heated horizontal circular cylinder. They obtained a solution by expanding velocity and temperature profiles in powers of x, the co-ordinate measuring distance from the lowest point on the cylinder. The exact solution is still out of reach due to the non-linearity in the Navies-Stokes equations. It appears that Merkin [2,3], was the first who presented a complete solution of this problem using Blasius and Gortler series expansion method along with an integral method and a finite-difference scheme. Also the problem of free convection boundary layer flow on cylinder of elliptic cross-section was studied by Merkin [4]. Ingham [5], investigated the boundary layer flow on an isothermal horizontal cylinder. Hossain and Alim [6] have investigated natural convection-radiation interaction on boundary layer flow along a vertical thin cylinder. Hossain et al. [7], have studied radiation-conduction interaction on mixed convection from a horizontal circular cylinder. Recently, Nazar et al. [8], have considered the problem of natural convection flow from lower stagnation point to upper stagnation point of a horizontal circular cylinder immersed in a micro polar fluid.

Roughened surfaces are encountered in several heat transfer devices such as flat plate solar collectors and flat plate condensers in regenerators. Larger scale surface non-uniformities are encountered, for example, in cavity wall insulating systems and grain storage containers. The only papers to date that study the effects of such non-uniformities on the vertical convective boundary layer flow of a Newtonian fluid are those of Yao [9], and Moulic and Yao [10,11]. Hossain and Pop [12] investigated the magneto-hydrodynamic boundary layer flow and heat transfer from a continuous moving wavy surface while the problem of free convection flow from a wavy vertical surface in presence of a transverse magnetic field was studied by Alam et al [13]. On the other hand, Rees and Pop [14-16] investigated the free convection boundary layer induced by vertical and horizontal surface exhibiting small-amplitude waves embedded in a porous medium. Hossain and Rees [17] have investigated the combined effect of thermal and mass diffusion on the natural convection flow of a viscous incompressible fluid along a vertical wavy surface. The effect of waviness of the surface on the heat and mass flux in investigated in combination with the species concentration for a fluid having Prandtl number equal to 0.7. Natural convection heat and mass transfer near a vertical wavy surface with constant wall temperature and concentration in a porous medium was studied by Cheng [18]. Very recently, Molla et al, [19,20] have studied the problem of natural convection flow along a vertical wavy surface with uniform surface temperature and surface heat flux in presence of temperature dependent heat generation.

In all the above studies [1-20] the authors assumed that both the viscosity and thermal conductivity of the fluids are constant throughout the flow regime. But, these physical properties may be changed significantly with temperature. For instance, the viscosity of water decrease by about 240% when the temperature increases from 10^0C (μ = 0.0131 gm.cm^{-1}.s^{-1}) to 50^0C (μ = 0.00548 gm.cm^{-1}.s^{-1}). To predict accurately the flow behavior, it is necessary to take into account of viscosity. Gray and Kassery [21] and Mehta and Sood [22] found that the flow characteristics substantially changed when the effect of temperature dependent viscosity is considered. Hady et at [23] investigated the mixed convection boundary layer flow on a continuous flat plate with variable viscosity.

Kafoussius and Williams [24], and Kafoussius et al [25] have studied the effects of variable viscosity on the free and mixed convection flow from a vertical flat plate in the region near the leading edge. Recently, Hossain et al. [26,27] have considered the natural convection along a vertical wavy cone and surface placed in fluid having large Prandtl number with variable viscosity when the viscosity is inversely proportional to a linear function of temperature, a model that was proposed by Ling and Dybbs [28]. Besides these, it has been found that for liquid, such as organ gas, the viscosity varies with temperature in an approximately linear manner. From Kays [29], it has also been found that the thermal conductivity varies with temperature is an approximately linear manner in the range from 0^0 to 400^0 F.

A semi-empirical formula for the thermal conductivity was used by Arunachalam and Rajappa [30]. Assuming the viscosity and thermal conductivity of the fluid to be proportional to a linear function of temperature, two semi-empirical formulas were proposed by Charraudeau [31]. Following him Hossain et al [32] investigated the natural convection flow past a permeable wedge for the fluid having temperature dependent viscosity and thermal conductivity.

Since the natural convection flow along a horizontal circular cylinder with temperature dependent viscosity hasn't been considered yet, in Chapter 2, the attention has been given to a study of the natural convection flow of a viscous incompressible fluid having viscosity $\mu(T)$ depending on temperature from an isothermal horizontal circular cylinder. The surface temperature T_w of the cylinder is higher than that of the ambient fluid temperature T_∞ . In formulating the equations governing the flow the viscosity of the fluid has been assumed to be inversely proportional to a linear function of temperature, a semi-empirical formula for the viscosity had been used by Ling and Dybbs [28]. The governing partial differential equations are reduced to locally non-similar partial differential forms by adopting appropriate transformations. The transformed boundary layer equations are solved numerically using very efficient implicit finite-difference scheme together with Keller box technique [33,34]. Effect of viscosity-variation parameter ε , on the velocity and viscosity distribution of the fluid as well as on the local rate of heat transfer in terms of the Nusselt number and the local skin-friction are shown graphically for fluids having large Prandtl number ranging from 1.0 to 30.0.

In Chapter 3, we investigate the natural convection boundary layer flow along a vertical wavy surface maintained at a uniform surface temperature immersed in a fluid with a temperature dependent viscosity $\mu(T)$ and thermal conductivity $k(T)$ as [32, 35]. The boundary layer equations are solved by using implicit finite difference method [33,34]. We give our attention to the situation where the buoyancy forces assist the flow for various values of the viscosity-variation parameter ε and the thermal conductivity-variation parameter γ with the small Prandtl number Pr ranging from 0.01 to 1.0. From these results we can observe the different flow and heat transfer characteristics by varying the relevant parameters.

Chapter 2

Natural convection flow from an isothermal horizontal circular cylinder with temperature dependent viscosity

2.1 INTRODUCTION

Free convection flow from an isothermal horizontal circular cylinder immersed in a fluid with viscosity proportional to inverse linear function of temperature is studied. The governing boundary layer equations are transformed into a non-dimensional form and the resulting nonlinear system of partial differential equations are reduced to local non-similarity equations, which are solved numerically by very efficient implicit finite difference method together with Keller box scheme. Numerical results are presented by velocity and viscosity profiles of the fluid as well as heat transfer characteristics, namely the local heat transfer rate and the local skin-friction coefficients for a wide range of viscosity parameter ε (= 0.0, 0.5, 2.0, 3.0 and 4.0) and the Prandtle number Pr (= 1.0, 7.0, 10.0, 15.0, 20.0 and 30.0).

2.2. FORMULATION OF THE PROBLEM

A steady two-dimensional laminar free convective flow from a uniformly heated horizontal circular cylinder of radius a, which is immersed in a viscous incompressible fluid having temperature dependent viscosity. It is assumed that the surface temperature of the cylinder is T_w, where $T_w > T_\infty$. Here T_∞ is the ambient temperature of the fluid. The configuration considered is as shown in Fig. 2.1.

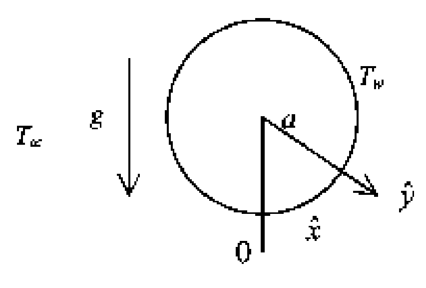

Fig. 2.1. Physical model and coordinate system.

The governing boundary layer equations of the flow are

$$\frac{\partial \hat{u}}{\partial \hat{x}}+\frac{\partial \hat{v}}{\partial \hat{y}}=0 \tag{2.1}$$

$$\rho\left(\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{u}}{\partial \hat{y}}\right)=\frac{\partial}{\partial \hat{y}}\left(\mu\frac{\partial \hat{u}}{\partial \hat{y}}\right)+\rho g\beta(T-T_\infty)\sin\left(\frac{\hat{x}}{a}\right) \tag{2.2}$$

$$\hat{u}\frac{\partial T}{\partial \hat{x}}+\hat{v}\frac{\partial T}{\partial \hat{y}}=\frac{1}{\rho c_p}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \tag{2.3}$$

The boundary conditions for equations (2.1) to (2.3) are

$$\hat{u}=\hat{v}=0, \quad T=T_w, \text{ at } \hat{y}=0 \tag{2.4a}$$

$$\hat{u}\to 0, \quad T\to T_\infty \quad \text{as} \quad \hat{y}\to\infty \tag{2.4b}$$

where $(\hat{u},\hat{v})$ are velocity components along the $(\hat{x},\hat{y})$axes, g is the acceleration due to gravity, ρ is the density, $\beta=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p$ is the coefficient of thermal expansion, $\mu\,(T)$ is the viscosity of the fluid depending on the fluid temperature T, k is the thermal conductivity, for this problem k is constant.

Out of many forms of viscosity variation, which are available in the literature, we have considered only the following form proposed by Kafoussias and Williams [24] and Lings and Dybbs [28] for large Prandtl number

$$\mu=\frac{\mu_\infty}{1+\gamma_f(T-T_\infty)} \tag{2.5}$$

where $\gamma_f=\frac{1}{\mu_f}\left(\frac{\partial\mu}{\partial T}\right)_f$ is evaluated at the film temperature of the fluid $T_f=\frac{1}{2}(T_w-T_\infty)$ and $\mu_\infty=1.45\times10^{-6}T_\infty^{3/2}/(T_\infty+110.33)$ (Sutherland's law).

We now introduce the following non-dimensional variables:

$$x=\frac{\hat{x}}{a},\; y=Gr^{1/4}\left(\frac{\hat{y}}{a}\right),\; u=\frac{\rho a}{\mu_\infty}Gr^{-1/2}\hat{u}$$

$$v=\frac{\rho a}{\mu_\infty}Gr^{-1/4}\hat{v},\;\; \theta=\frac{T-T_\infty}{T_w-T_\infty},\quad Gr=\frac{g\beta(T_w-T_\infty)a^3}{\nu_\infty^2},\; \Pr=\frac{\nu_\infty}{\alpha} \tag{2.6a}$$

where $\nu_\infty(=\mu_\infty/\rho)$ is the reference kinematic viscosity, α is the thermal diffusivity, Gr is the Grashof number, Pr is the Prandtl number and θ is the non-dimensional temperature.

In order to non-dimensionalize the governing equations here we have shown some calculations:

$$\frac{\partial \hat{u}}{\partial \hat{x}} = \frac{\mu_\infty}{\rho a^2} Gr^{1/2} \frac{\partial u}{\partial x} \tag{2.6b}$$

$$\frac{\partial \hat{v}}{\partial \hat{y}} = \frac{\mu_\infty}{\rho a^2} Gr^{1/2} \frac{\partial v}{\partial y} \tag{2.6c}$$

$$\frac{\partial \hat{u}}{\partial \hat{y}} = \frac{\mu_\infty}{\rho a^2} Gr^{3/4} \frac{\partial u}{\partial y} \tag{2.6d}$$

$$\frac{\partial}{\partial \hat{y}}\left(\mu \frac{\partial \hat{u}}{\partial \hat{y}}\right) = \frac{\mu_\infty^2}{\rho a^3} Gr\left[\frac{1}{1+\varepsilon\theta}\frac{{}^2 u}{\partial y^2} - \frac{\varepsilon}{(1+\varepsilon\theta)^2}\frac{\partial \theta}{\partial y}\frac{\partial u}{\partial y}\right] \tag{2.6e}$$

$$\frac{\partial T}{\partial \hat{x}} = \frac{\Delta T}{a}\frac{\partial \theta}{\partial x} \tag{2.6f}$$

$$\frac{\partial T}{\partial \hat{y}} = \frac{\Delta T}{a} Gr^{1/4} \frac{\partial \theta}{\partial y} \tag{2.6g}$$

$$\frac{\partial^2 T}{\partial \hat{y}^2} = \frac{\Delta T}{a^2} Gr^{1/2} \frac{\partial^2 \theta}{\partial y^2} \tag{2.6h}$$

The equation (2.5) can be written as

$$\frac{\mu}{\mu_\infty} = \frac{1}{1+\varepsilon\theta} \tag{2.6i}$$

where $\varepsilon = \frac{1}{\mu_f}\left(\frac{\partial \mu}{\partial T}\right)_f (T_w - T_\infty)$ is the viscosity variation parameter

Substitution of (2.6) into equations (2.1)-(2.3) leads to the following non-dimensional equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.7}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{-\varepsilon}{(1+\varepsilon\theta)^2}\frac{\partial u}{\partial y}\frac{\partial \theta}{\partial y}+\frac{1}{1+\varepsilon\theta}\frac{\partial^2 u}{\partial y^2}+\theta\sin x \qquad (2.8)$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\Pr}\frac{\partial^2 \theta}{\partial y^2} \qquad (2.9)$$

With the boundary conditions (2.4) reduced to

$$u=v=0,\quad \theta=0 \text{ at } \quad x=0,\quad \text{for any} \quad y \qquad (2.10a)$$

$$u=v=0,\quad \theta=1,\quad \text{at} \quad y=0,\quad x>0 \qquad (2.10b)$$

$$u\to 0,\quad \theta\to 0,\quad as \quad y\to\infty,\quad x>0 \qquad (2.10c)$$

In equation (2.9) Pr is the Prandtl number defined in (2.11)

$$\Pr=\frac{\nu}{kC_p} \qquad (2.11)$$

To solve equations (2.7)-(2.9), subject to the boundary conditions (2.10), we assume the following variables

$$\psi=xf(x,y),\quad \theta=\theta(x,y) \qquad (2.12)$$

where ψ is the non-dimensional stream function defined in the usual way as

$$u=\frac{\partial \psi}{\partial y},\quad v=-\frac{\partial \psi}{\partial x} \qquad (2.13a)$$

Also we have found

$$u=x\frac{\partial f}{\partial y} \qquad (2.13b)$$

$$\frac{\partial u}{\partial y}=x\frac{\partial^2 f}{\partial y^2} \qquad (2.13c)$$

$$\frac{\partial^2 u}{\partial y^2}=x\frac{\partial^3 f}{\partial y^3} \qquad (2.13d)$$

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial y}+x\frac{\partial^2 f}{\partial x\partial y} \qquad (2.13e)$$

$$v=-\left[x\frac{\partial f}{\partial x}+f\right] \qquad (2.13f)$$

$$\frac{\partial \theta}{\partial x}=\frac{\partial \theta}{\partial x} \qquad (2.13g)$$

$$\frac{\partial \theta}{\partial y}=\frac{\partial \theta}{\partial y} \tag{2.13h}$$

$$\frac{\partial^2 \theta}{\partial y^2}=\frac{\partial^2 \theta}{\partial y^2} \tag{2.13I}$$

Substituting (2.13) into equations (2.8)-(2.9) we get, after some algebra the following transformed equations

$$\frac{1}{1+\varepsilon\theta}\frac{\partial^3 f}{\partial y^3}+f\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial f}{\partial y}\right)^2-\frac{\varepsilon}{(1+\varepsilon\theta)^2}\theta\frac{\partial^2 f}{\partial y^2}+\frac{\theta\sin x}{x} =x\left(\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}\right) \tag{2.14}$$

$$\frac{1}{\Pr}\frac{\partial^2 \theta}{\partial y^2}+f\frac{\partial \theta}{\partial y}=x\left(\frac{\partial f}{\partial y}\frac{\partial \theta}{\partial x}-\frac{\partial \theta}{\partial y}\frac{\partial f}{\partial x}\right) \tag{2.15}$$

Along with boundary conditions

$$f=\frac{\partial f}{\partial y}=0,\quad \theta=0 \text{ at } \quad x=0 \quad \text{any} \quad y \tag{2.16a}$$

$$f=\frac{\partial f}{\partial y}=0,\quad \theta=1,\quad \text{at} \quad y=0,\quad x>0 \tag{2.16b}$$

$$\frac{\partial f}{\partial y}\to 0,\quad \theta\to 0,\quad \text{as} \quad y\to\infty,\quad x>0 \tag{2.16c}$$

It can be seen that near the lower stagnation point of the cylinder i.e. for $x \approx 0$, equations (2.14)-(2.15) reduce to the following ordinary differential equations:

$$\frac{1}{1+\varepsilon\theta}f'''+ff''-f'^2-\frac{\varepsilon}{(1+\varepsilon\theta)^2}\theta' f''+\theta=0 \tag{2.17}$$

$$\frac{1}{\Pr}\theta''+f\theta'=0 \tag{2.18}$$

Subject to the boundary conditions

$$f(0)=f'(0)=0,\quad \theta(0)=1 \tag{2.19a}$$

$$f'\to 0,\quad \theta\to 0 \quad \text{as} \quad y\to\infty \tag{2.19b}$$

In practical applications, the physical quantities of principle interest are the rate of heat transfer and the skin-friction coefficients, which can be written, in non-dimensional form as

$$Nu_x = \frac{aGr_x^{-1/4}}{k(T_w - T_\infty)} q_w, \quad C_{f_x} = \frac{Gr_x^{-3/4} a^2}{\mu_\infty \nu_\infty} \tau_w \tag{2.20}$$

where $$q_w = -k\left(\frac{\partial T}{\partial \hat{y}}\right)_{\hat{y}=0}, \quad \tau_w = \left(\mu \frac{\partial \hat{u}}{\partial \hat{y}}\right)_{\hat{y}=0} \tag{2.21}$$

Using (2.6) into (2.12) and the boundary condition (2.16b), we get

$$Nu_x = -\frac{\partial \theta(x,0)}{\partial y} \tag{2.22}$$

$$C_{f_x} = \frac{x}{1+\varepsilon} \frac{\partial f(x,0)}{\partial y} \tag{2.23}$$

We also discuss the effect of the viscosity-variation parameter ε and the Prandtl number Pr on the velocity and viscosity distribution. The values of the velocity and viscosity distribution are calculated from the following relations:

$$u = \frac{\partial f}{\partial y}, \quad \frac{\mu}{\mu_\infty} = \frac{1}{1+\varepsilon\theta} \tag{2.24}$$

2.3. RESULTS AND DISCUSSION

Equations (2.14)-(2.15) subject to the boundary conditions (2.16) are solved numerically using a very efficient implicit finite finite-difference method together with Keller box, which is described by Cebeci and Bradshow [34]. The numerical solutions start at the lower stagnation point of the cylinder i.e. at $x \approx 0$, with initial profiles as given by equations (2.17)-(2.18) along with the boundary conditions (2.19) and proceed round the cylinder up to the upper stagnation point, $x \approx \pi$. Solutions are obtained for fluids having Prandtle number, Pr = 1.0, 7.0, 10.0, 15.0, 20.0, 30.0 and for a wide range of values of the variable viscosity parameter, $\varepsilon = 0.0, 0.5, 2.0, 3.0$ and 4.0.

Since values of $-\theta'(x,0)$ and $f''(x,0)$ are known from the solutions of the coupled equations (2.14) and (2.15), numerical values of the local heat transfer rate, Nu from (2.22) and the local skin-friction coefficients C_f from (2.23) are calculated for the surface of the cylinder from lower stagnation point to upper stagnation point. Numerical values of $-\theta'(x,0)$ and $f''(x,0)$ are depicted in Table 2.1, Table 2.2, Fig 2.3 and Fig. 2.5.

Numerical values of $-\theta'(x,0)$ and $f''(x,0)$ are depicted in Table 2.1, Table 2.2 respectively for Pr = 1.0, $\varepsilon = 0.0$, and the results of Merkin [2] and Nazar et at.[8] show excellent agreement among these three solutions.

Effect of viscosity variation parameter, ε (= 0.0, 0.5, 1.0, 2.0, 4.0) on the dimensionless velocity $f'(x,\eta)$ at $x = \pi/2$ for the fluid having Pr = 7.0 is shown in Fig. 2.2(a). The corresponding temperature dependent viscosity, μ/μ_∞ profiles are illustrated in Fig. 2.2(b). From Fig. 2.2(a) it can be observed that increase in the value of the viscosity-variation parameter ε leads to increase in the velocity profile. Increase in the value of the viscosity-variation parameter ε also leads to decrease in the viscosity at $x = \pi/2$ and these approach to unity at the outer edge of the boundary layer for every values of the viscosity-variation parameter.

Fig. 2.3(a) and 2.3(b) deal with the effect of viscosity-variation parameter ε (= 0.0, 0.5, 1.0, 2.0, 4.0) for Pr = 7.0 on the rate of heat transfer and the local skin-friction coefficient respectively. From Fig. 2.3(a) it is seen that the rate of heat, $-\theta'(x,0)$ increases monotonically with the increase of the viscosity-variation parameter ε. We also observe that the value of $-\theta'(x,0)$ reaches to some minimum values at $x = \pi$, the top surface of the cylinder for every values of ε. Such as for value of ε = 0.0, 0.5, 1.0, 2.0 and 4.0 the minimum values attained by the rate of heat transfer $-\theta'(x,0)$ are 0.2681, 0.2852, 0.2955, 0.3092 and 0.3312 respectively. From these we may conclude that these minimum values increase with the increase of the viscosity-variation parameter ε. Fig. 2.3(b) shows that for increasing values of ε, the local skin-friction coefficients decrease. The decreasing values of $f''(x,0)$ converge to individual finite values for every values of ε. We have also got for ε = 0.0, 0.5, 1.0, 2.0 and 4.0 the limiting values of $f''(x,0)$ are 0.1368, 0.1220, 0.1103, 0.0949 and 0.0813 respectively.

Fig. 2.4(a) depicts the velocity profile for different values of the Prandtl number Pr (= 7.0, 10.0, 15.0, 20.0, 30.0) while $x = \pi/2$ and the viscosity-variation parameter ε = 3.0. Corresponding distribution of the viscosity in the fluids is shown in Fig. 2.4(b). From Fig. 4(a) it can be seen that if the Prandtl number increases, the velocity of the fluid decreases. On the other hand, from Fig. 2.4(b) we observe that the viscosity of the fluid increases within the boundary layer region owing to increase the value of the Prandtl number. This viscosity profiles asymptote to the unit value near the edge of the boundary layer, since this where ambient conditions are recovered.

The effect of Prandtl number Pr, on the rate of heat transfer, $-\theta'(x,0)$ and the skin-friction, $f''(x,0)$ are illustrated in Fig. 2.5(a) and 2.5(b) respectively while ε = 3.0. From Fig. 2.5(a) it reveals that, increase in the value of the Prandtl number leads to increase the values of the rate of heat transfer. Opposite effects on the local skin-friction is observed due to increase the value of Prandtl number. We may also observe that the absolute maxima of the local skin-friction shifts near to the middle of the surface.

2.4. CONCLUSIONS

The effect of temperature-dependent viscosity on the natural convection boundary layer flow from an isothermal horizontal circular cylinder has been investigated theoretically. Numerical solutions of the equations governing the flow are obtained by using the very efficient implicit finite difference method together with Keller box scheme. From the present investigation, the following conclusions may be drawn:

1.The velocity distribution increases and the viscosity of the fluid decreases at the middle of the surface for increasing value of viscosity-variation parameter ε.

2. Increased value of the viscosity- variation parameter ε leads to increase in the local heat transfer rate and to decrease the local skin-friction.

3. It has been observed that the velocity distribution and skin-friction decrease as well as the viscosity distribution and the rate of heat transfer increase with the increase of Prandtl number Pr.

Table 2.1. Numerical values of $-\theta'(x, 0)$ for different values of curvature parameter x while Pr = 1.0 and $\varepsilon = 0.0$

	$-\theta'(x,0)$		
x	Merkin [2]	Nazar et al. [8]	Present
0.0	0.4214	0.4214	0.4241
$\pi/6$	0.4161	0.4161	0.4161
$\pi/3$	0.4007	0.4005	0.4005
$\pi/2$	0.3745	0.3741	0.3740
$2\pi/3$	0.3364	0.3355	0.3355
$5\pi/6$	0.2825	0.2811	0.2812
π	0.1945	0.1916	0.1917

Table 2.2. Numerical values of $f''(x,0)$ for different values of curvature parameter x while Pr = 1.0 and $\varepsilon = 0.0$

	$f''(x,0)$		
x	Merkin [2]	Nazar et al. [8]	Present
0.0	0.0000	0.0000	0.0000
$\pi/6$	0.4151	0.4148	0.4145
$\pi/3$	0.7558	0.7542	0.7539
$\pi/2$	0.9579	0.9545	0.9541
$2\pi/3$	0.9756	0.9698	0.9696
$5\pi/6$	0.7822	0.7740	0.7739
π	0.3391	0.3265	0.3264

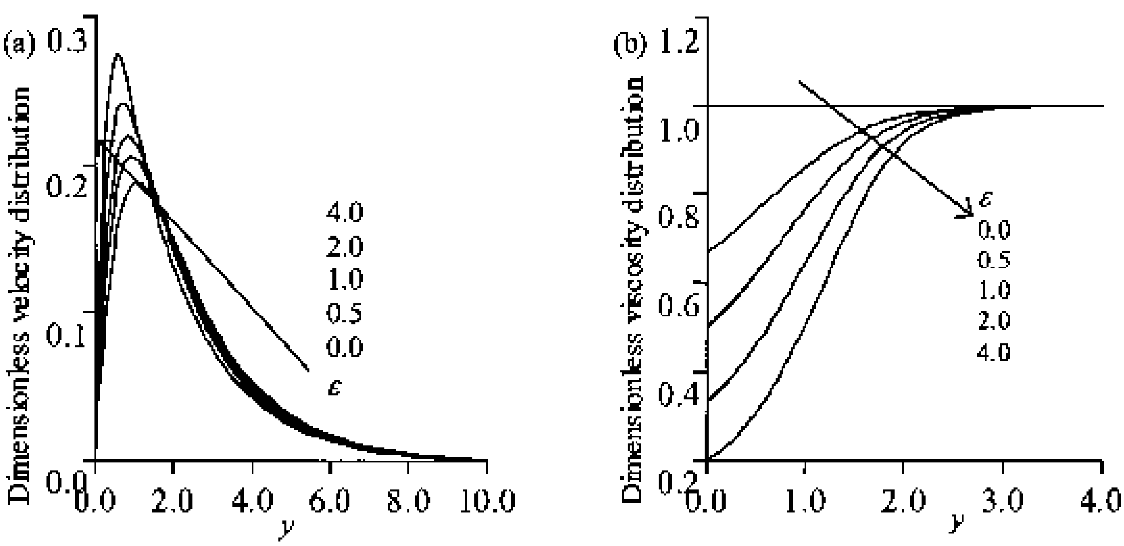

Fig. 2.2. (a) Velocity profiles (b) viscosity profiles for different ε while x = $\pi/2$ and Pr = 7.0

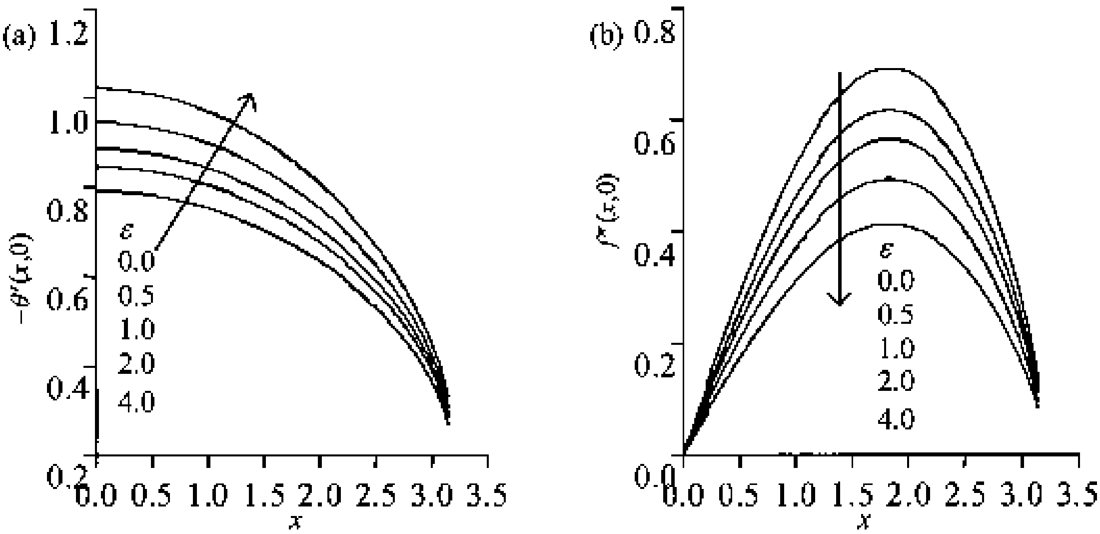

Fig. 2.3. (a) Rate of heat transfer (b) skin-friction coefficient for different ε while Pr = 7.0

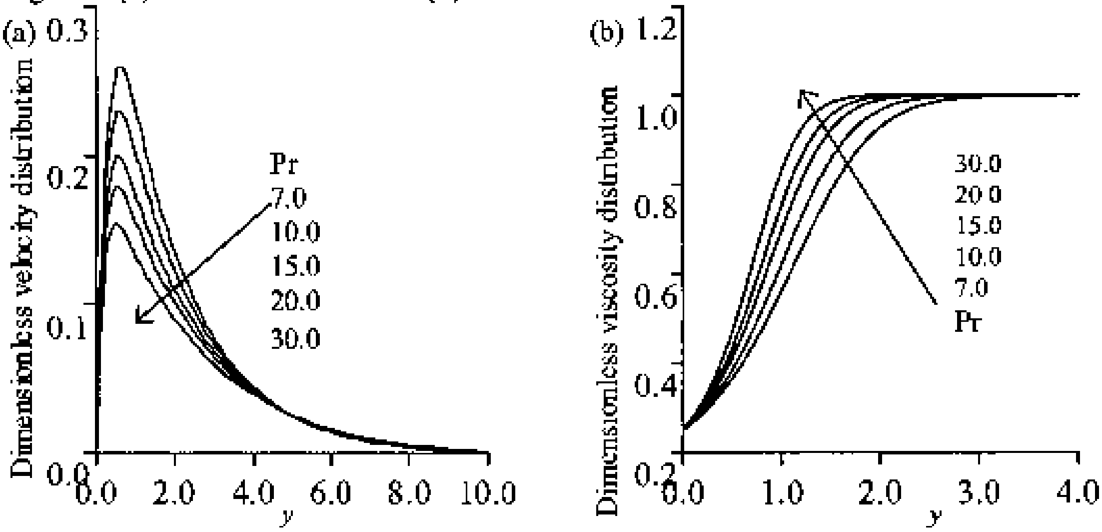

Fig. 2.4. (a) Velocity profiles (b) viscosity profiles for different Pr while $x = \pi/2$ and $\varepsilon = 3.0$

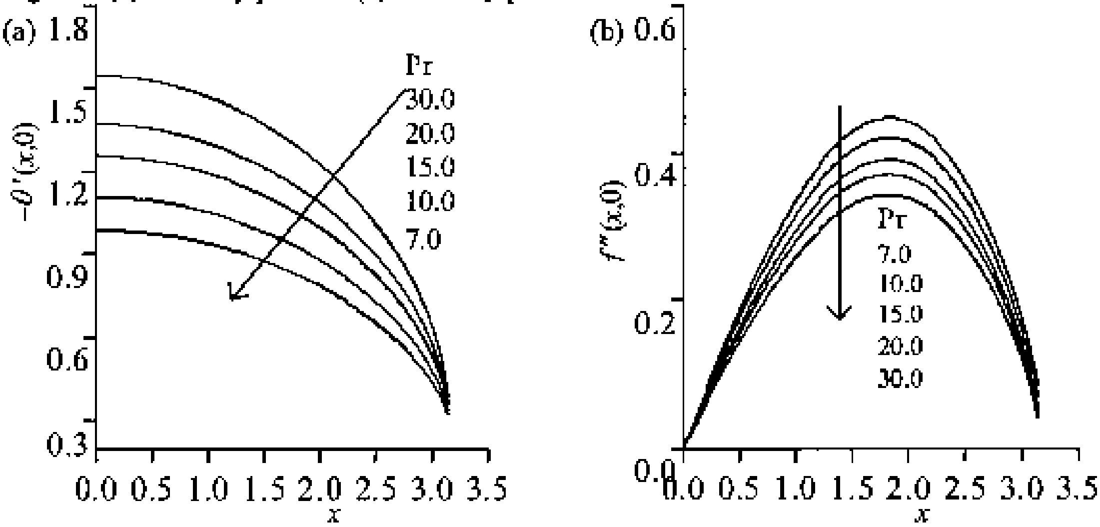

Fig. 2.5. (a) Rate of heat transfer (b) skin-friction coefficient for different Pr while $\varepsilon = 3.0$

Chapter 3

Natural convection flow with temperature dependent viscosity and thermal conductivity along a vertical wavy surface

3.1 INTRODUCTION:

The effect of variable viscosity and thermal conductivity on a steady two-dimensional natural convection flow of viscous incompressible fluid along a uniformly heated vertical wavy surface has been investigated. We consider the boundary layer regime having large Grashof number. Using the appropriate variables, the basic equations are transformed to non-similar boundary layer equations and then solved numerically employing the implicit finite difference method together with Keller-box scheme. Effects of the pertinent parameters, such as the viscosity parameter, thermal conductivity parameter, the amplitude of the waviness of the surface and Prandtl number on the rate of heat transfer in terms of the local Nusselt number (Nu_x), surface shear stress in terms of local skin-friction (C_{fx}), isotherms and the streamlines are discussed.

2. 2. FORMULATION OF THE PROBLEM

The boundary layer analysis outlined below allows $\hat{\sigma}(\hat{x})$ being arbitrary, but our detailed numerical work will assume that the surface exhibits sinusoidal deformations. The wavy surface may be described by

$$\hat{y}_w = \sigma(\hat{x}) = \alpha \sin\left(\frac{\pi\hat{x}}{L}\right) \tag{3.1}$$

where L is the characteristic length associated with the wavy surface.

The geometry of the wavy surface and the two-dimensional Cartesian coordinate system are shown in Fig. 3.1.

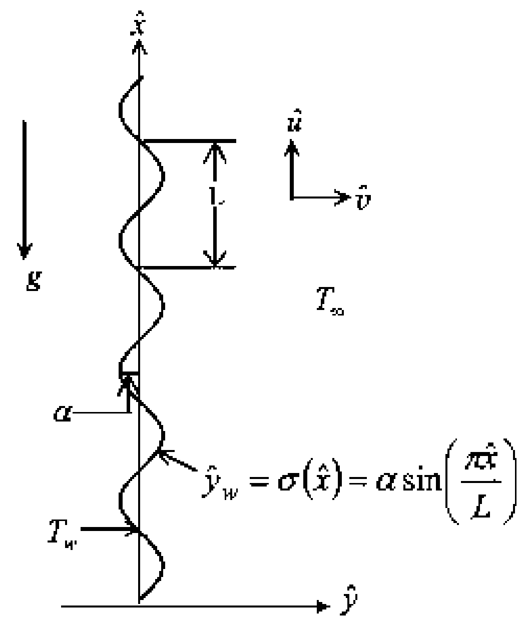

Fig. 3.1: Physical model and coordinate system

we consider the flow governed by the following equations:

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0 \tag{3.2}$$

$$\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{1}{\rho}\nabla.(\mu \nabla \hat{u}) + g\beta(T - T_\infty) \tag{3.3}$$

$$\hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{y}} + \frac{1}{\rho}\nabla.(\mu \nabla \hat{v}) \tag{3.4}$$

$$\hat{u}\frac{\partial T}{\partial \hat{x}} + \hat{v}\frac{\partial T}{\partial \hat{y}} = \frac{1}{\rho C_p}\nabla.(k \nabla T) \tag{3.5}$$

where $(\hat{x}, \hat{y})$ are the dimensional coordinates along and normal to the tangent of the surface and $(\hat{u}, \hat{v})$ are the velocity components parallel to $(\hat{x}, \hat{y})$, g is the acceleration due to gravity, $\hat{p}$ is the dimensionless pressure of the fluid, ρ is the density, μ (T) is the dynamic viscosity, $k(T)$ is the thermal conductivity of the fluid in the boundary layer region depending on the fluid temperature and C_p is the specific heat at constant pressure.

The boundary conditions for the present problem are

$$\hat{u}=0,\ \hat{v}=0,\ T=T_w \text{ at } \hat{y}=\hat{y}_w=\sigma(\hat{x})$$
$$\hat{u}=0,\ T=T_\infty,\quad \hat{p}=p_\infty \quad \text{as } \hat{y}\to\infty \tag{3.6}$$

where T_w is the surface temperature, T_∞ is the ambient temperature of the fluid.

There are very few forms of viscosity and thermal conductivity variations available in the literature. Among them we have considered that one which is appropriate for liquid introduced by Hossain et al. [32] and Gorla et al.[35] as follows:

$$\mu=\mu_\infty\left[1+\varepsilon^*(T-T_\infty)\right]$$
$$k=k_\infty\left[1+\gamma^*(T-T_\infty)\right] \tag{3.7}$$

where $\varepsilon^*=\frac{1}{\mu_f}\left(\frac{\partial\mu}{\partial T}\right)_f$ and $\gamma^*=\frac{1}{k_f}\left(\frac{\partial k}{\partial T}\right)_f$ are evaluated at the film temperature of the flow $T_f=½(T_w+T_\infty)$

Following Yao [9], we now introduce the following non-dimensional variables:

$$x=\frac{\hat{x}}{L},\quad y=\frac{\hat{y}-\sigma(\hat{x})}{L}Gr^{1/4},\quad u=\frac{\rho L}{\mu_\infty}Gr^{-1/2}\hat{u},\quad p=\frac{L^2}{\rho\nu_\infty^2}Gr^{-1}\hat{p},$$

$$v=\frac{\rho L}{\mu_\infty}Gr^{-1/4}\left(\hat{v}-\sigma_x\hat{u}\right),\theta=\frac{T-T_\infty}{T_w-T_\infty}, \tag{3.8a}$$
$$\sigma_x=\frac{d\hat{\sigma}}{d\hat{x}}=\frac{d\sigma}{dx},Gr=\frac{g\beta(T_w-T_\infty)}{\nu_\infty^2}L^3$$

where θ is the dimensionless temperature function and ν_∞ $(=\mu_\infty/\rho)$ is the kinematic viscosity. The (x,y) are not orthogonal, but a regular rectangular computational grid can be easily fitted in the transformed coordinates. It is also worthwhile to point out that (u,v) are the velocity components parallel to (x,y) which are not parallel to the wavy surface. Now we have shown some calculation for dimensionless of the above equations.

$$\frac{\partial\hat{u}}{\partial\hat{x}}=\frac{\mu_\infty}{\rho L^2}Gr^{1/2}\left(\frac{\partial u}{\partial x}-Gr^{1/4}\sigma_x\frac{\partial u}{\partial y}\right) \tag{3.8b}$$

$$\frac{\partial\hat{u}}{\partial\hat{y}}=\frac{\mu_\infty}{\rho L^2}Gr^{3/4}\frac{\partial u}{\partial y} \tag{3.8b}$$

$$\frac{\partial\hat{v}}{\partial\hat{x}}=\frac{\mu_\infty}{\rho L^2}\left[Gr^{1/4}\frac{\partial v}{\partial x}-Gr^{1/2}\sigma_x\frac{\partial v}{\partial y}+Gr^{1/2}\sigma_{xx}u+Gr^{1/2}\sigma_x\frac{\partial u}{\partial x}-Gr^{3/4}\sigma_x^2\frac{\partial u}{\partial y}\right] \tag{3.8c}$$

$$\frac{\partial \hat{v}}{\partial \hat{y}} = \frac{\mu_\infty}{\rho L^2}\left(Gr^{1/4}\frac{\partial v}{\partial y} + Gr^{1/2}\sigma_x \frac{\partial u}{\partial y}\right) \tag{3.8d}$$

$$\frac{\partial \hat{p}}{\partial \hat{x}} = \frac{\rho v_\infty^2}{L^3} Gr\left(\frac{\partial p}{\partial x} - Gr^{1/4}\sigma_x \frac{\partial p}{\partial y}\right) \tag{3.8e}$$

$$\frac{\partial \hat{p}}{\partial \hat{y}} = \frac{\rho v_\infty^2}{L^3} Gr \frac{\partial p}{\partial y} \tag{3.8f}$$

$$\frac{\partial T}{\partial \hat{x}} = \frac{\Delta T}{L}\left(\frac{\partial \theta}{\partial x} - Gr^{1/4}\sigma_x \frac{\partial \theta}{\partial y}\right) \tag{3.8g}$$

$$\frac{\partial T}{\partial \hat{y}} = \frac{Gr}{L}\Delta T \frac{\partial \theta}{\partial y} \tag{3.8h}$$

$$\begin{aligned} \nabla.(\mu\nabla\hat{u}) = \frac{\mu_\infty^2}{\rho L^3} Gr^{1/2}\Bigg[\varepsilon\Bigg(&\frac{\partial \theta}{\partial x}\frac{\partial u}{\partial x} - Gr^{1/4}\sigma_x \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial y} - Gr^{1/4}\sigma_x \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial x} \\ &+ Gr^{1/2}\sigma_x^2 \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial y}\Bigg) + (1+\varepsilon\theta)\left(\frac{\partial^2 u}{\partial x^2} - Gr^{1/4}\sigma_{xx}\frac{\partial u}{\partial y} + Gr^{1/2}\sigma_x \frac{\partial^2 u}{\partial y^2}\right)\Bigg] \\ &+ \frac{\mu_\infty^2}{\rho L^3} Gr\left[\varepsilon \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial y} + (1+\varepsilon\theta)\frac{\partial^2 u}{\partial y^2}\right] \end{aligned} \tag{3.8i}$$

$$\begin{aligned} \nabla.(\mu\nabla\hat{v}) = \frac{\mu_\infty^2}{\rho L^3}\Bigg[\varepsilon\Bigg(&Gr^{1/4}\frac{\partial \theta}{\partial x}\frac{\partial v}{\partial x} - Gr^{1/2}\sigma_x \frac{\partial \theta}{\partial x}\frac{\partial v}{\partial y} + Gr^{1/2}\sigma_{xx} u \frac{\partial \theta}{\partial x} \\ &+ Gr^{1/2}\sigma_x \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial x} - Gr^{3/4}\sigma_x^2 \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial y} - Gr^{1/2}\sigma_x \frac{\partial \theta}{\partial y}\frac{\partial v}{\partial x} \\ &+ Gr^{3/4}\sigma_x^2 \frac{\partial \theta}{\partial y}\frac{\partial v}{\partial y} - Gr^{3/4}\sigma_x\sigma_{xx} u \frac{\partial \theta}{\partial y} - Gr^{3/4}\sigma_x^2 \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial x} \\ &- Gr\sigma_x^3 \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial y}\Bigg) + \frac{\mu_\infty^2}{\rho L^3} Gr^{1/2}\Bigg[\varepsilon\left(Gr^{1/4}\frac{\partial \theta}{\partial y}\frac{\partial v}{\partial y} + Gr^{1/2}\sigma_x \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial y}\right) \\ &+ (1+\varepsilon\theta)\left(Gr^{1/4}\frac{\partial^2 v}{\partial y^2} + Gr^{1/2}\sigma_x \frac{\partial^2 u}{\partial y^2}\right)\Bigg]\Bigg] \end{aligned} \tag{3.8j}$$

$$\nabla.(\mu\nabla T)=\frac{\Delta T k_\infty}{\rho L^2}\left[\gamma\left(\left(\frac{\partial\theta}{\partial x}\right)^2-2Gr^{1/4}\sigma_x\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial y}+Gr^{1/2}\sigma_x^2\left(\frac{\partial\theta}{\partial y}\right)^2\right)\right.$$
$$\left.+(1+\gamma\theta)\left(\frac{\partial^2\theta}{\partial x^2}-Gr^{1/4}\sigma_{xx}\frac{\partial\theta}{\partial y}+Gr^{1/2}\sigma_x^2\frac{\partial^2\theta}{\partial y^2}\right)\right] \tag{8.3k}$$
$$+\frac{k_\infty\Delta T}{L}Gr^{1/2}\left[\gamma\left(\frac{\partial\theta}{\partial y}\right)^2+(1+\gamma\theta)\frac{\partial^2\theta}{\partial y^2}\right]$$

Introducing the above dimensionless dependent and independent variables into the equations (3.2)-(3.5) following dimensionless form of the governing equations are obtained after ignoring terms for large Gr, the Grashof number defined in (3.8a).

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3.9}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\sigma_x Gr^{1/4}\frac{\partial p}{\partial y}+(1+\sigma_x{}^2)(1+\varepsilon\theta)\frac{\partial^2 u}{\partial y^2}$$
$$+\varepsilon(1+\sigma_x{}^2)\frac{\partial\theta}{\partial y}\frac{\partial u}{\partial y}+\theta \tag{3.10}$$

$$\sigma_x\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+\sigma_{xx}u^2=-Gr^{1/4}\frac{\partial p}{\partial y}+\sigma_x(1+\sigma_x{}^2)(1+\varepsilon\theta)\frac{\partial^2 u}{\partial y^2}$$
$$+\varepsilon\sigma_x(1+\sigma_x{}^2)\frac{\partial\theta}{\partial y}\frac{\partial u}{\partial y} \tag{3.11}$$

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}=\frac{1}{\Pr}(1+\sigma_x{}^2)(1+\gamma\theta)\frac{\partial^2\theta}{\partial y^2}+\frac{1}{\Pr}(1+\sigma_x{}^2)\gamma\left(\frac{\partial\theta}{\partial y}\right)^2 \tag{3.12}$$

In the above equations, Pr is the Prandtl number, ε and γ are the viscosity and thermal conductivity variation parameter defined as respectively

$$\Pr=\frac{c_p\mu_\infty}{k_\infty},\quad \varepsilon=\left(\frac{1}{\mu}\frac{\partial\mu}{\partial T}\right)_f(T_w-T_\infty)\quad\text{and}\quad\gamma=\left(\frac{1}{k}\frac{\partial k}{\partial T}\right)_f(T_w-T_\infty) \tag{3.13}$$

It can easily be seen that the convection induced by the wavy surface is described by equations (3.9)-(3.12). We further notice that, equation (3.11) indicates that the pressure gradient along the y-direction is $O(Gr^{-1/4})$, which implies that lowest order pressure gradient along x-direction can be determined from the inviscid flow solution. For the present problem this pressure gradient is zero. Equation (3.11) further shows that $Gr^{-1/4}\partial p/\partial y$ is O(1) and is determined by the left-hand side of this equation. Thus, the elimination of $\partial p/\partial y$ from equation (3.10) and (3.11) leads to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\left(1+\sigma_x^{\,2}\right)\left(1+\varepsilon\theta\right)\frac{\partial^2 u}{\partial y^2}-\frac{\sigma_x\sigma_{xx}}{1+\sigma_x^{\,2}}u^2 \tag{3.14}$$
$$+\varepsilon\left(1+\sigma_x^{\,2}\right)\frac{\partial u}{\partial y}\frac{\partial \theta}{\partial y}+\frac{1}{1+\sigma_x^{\,2}}\theta$$

The corresponding boundary conditions for the present problem then turn into

$$u=v=0,\quad \theta=1 \quad \text{at} \quad y=0$$
$$u=0,\quad \theta=0,\quad p=0 \quad \text{as} \quad y\to\infty \tag{3.15}$$

Now we introduce the following transformations to reduce the governing equation to a convenient form:

$$\psi=x^{3/4}f(x,\eta),\quad \eta=x^{-1/4}y, \theta=\theta(x,\eta) \tag{3.16}$$

where η is the pseudo similarity variable and ψ is the stream-function that satisfies the equation (3.9) and is defined by

$$u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x} \tag{3.17a}$$

$$u=x^{1/2}f' \tag{3.17b}$$

$$v=-\frac{1}{4}x^{-1/4}\eta f'+x^{3/4}\frac{\partial f}{\partial x}+\frac{3}{4}x^{-1/4}f \tag{3.17c}$$

$$\frac{\partial u}{\partial x}=x^{1/2}\left[f''\left(-\frac{1}{4x}\eta\right)+\frac{\partial f'}{\partial x}\right]+\frac{1}{2}x^{-1/2}f' \tag{3.17d}$$

$$\frac{\partial u}{\partial y}=x^{1/4}f'' \tag{3.17e}$$

$$\frac{\partial^2 u}{\partial y^2}=f''' \tag{3.17f}$$

$$\frac{\partial\theta}{\partial x}=-\frac{1}{4x}\eta\theta'+\frac{\partial\theta}{\partial x} \tag{3.17g}$$

$$\frac{\partial \theta}{\partial y} = x^{-1/4}\theta' \tag{3.17h}$$

$$\frac{\partial^2 \theta}{\partial y^2} = x^{-1/2}\theta'' \tag{3.17i}$$

Introducing the transformations given in equation (3.16) and calculations (3.17) into the equations (3.14) and (3.12) we have,

$$\left(1+\sigma_x^{\,2}\right)\left(1+\varepsilon\theta\right)f''' + \frac{3}{4}ff'' - \left(\frac{1}{2} + \frac{x\sigma_x\sigma_{xx}}{1+\sigma_x^{\,2}}\right)f'^2 + \varepsilon\left(1+\sigma_x^{\,2}\right)\theta' f'' + \frac{1}{1+\sigma_x^2}\theta = x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) \tag{3.18}$$

$$\frac{1}{\Pr}\left(1+\sigma_x^{\,2}\right)\left(1+\gamma\theta\right)\theta'' + \frac{1}{\Pr}\left(1+\sigma_x^{\,2}\right)\gamma\theta'^2 + \frac{3}{4}f\theta' = x\left(f'\frac{\partial \theta}{\partial x} - \theta'\frac{\partial f}{\partial x}\right) \tag{3.19}$$

The boundary conditions (3.15) now take the following form:

$$\begin{aligned} &f(x,0) = f'(x,0) = 0,\ \theta(x,0) = 1 \\ &f'(x,\infty) = \theta(x,\infty) = 0 \end{aligned} \tag{3.20}$$

Solutions of local non-similar partial differential equations (3.18)-(3.19), subject to the boundary conditions (3.20), are obtained by using implicit finite difference method developed by Keller [33]. This method has extensively been used recently by Hossain et al. [12,13,17,19,20,26,27]. Since good description of this method and its application to boundary layer flow problems is given in the book by Cebeci and Bradshaw [32], the detail of this method has not been discussed in this chapter.

However, once we know the values of the function f and θ and their derivatives, it is important to calculate the values of the Nusselt number, Nu and the skin-friction coefficient, C_f from the following relations:

$$Nu = \frac{q_w L}{k_\infty\left(T_w - T_\infty\right)} \quad \text{and } C_f = \frac{2\tau_w}{\rho U_\infty^2} \tag{3.21}$$

where

$$q_w = -\left(k\hat{n}.\nabla T\right)_{y=0}$$

$$= -k_\infty(1+\gamma)\frac{\hat{i}f_x + \hat{j}f_y}{\sqrt{f_x^2 + f_y^2}}\left(\hat{i}\frac{\partial T}{\partial \hat{x}} + \hat{j}\frac{\partial T}{\partial \hat{y}}\right)_{y=0} \quad \text{where} \quad f(\hat{x},\hat{y}) = \hat{y} - \sigma(\hat{x}) \tag{3.22a}$$

$$= -k_\infty(1+\gamma)\frac{\Delta T Gr^{1/4}}{Lx^{1/4}}\sqrt{1+\sigma_x^2}\,\theta'(x,0) \quad \text{where} \quad \Delta T = (T_w - T_\infty)$$

$$\tau_w = \left(\mu\hat{n}.\nabla\hat{u}\right)_{y=0}$$

$$= \mu_\infty(1+\varepsilon)\frac{\hat{i}f_x + \hat{j}f_y}{\sqrt{f_x^2 + f_y^2}}\left(\hat{i}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{j}\frac{\partial \hat{u}}{\partial \hat{y}}\right)_{y=0} \tag{3.22b}$$

$$= \frac{\mu_\infty Gr^{3/4}}{\rho L^2}(1+\varepsilon)x^{1/4}\sqrt{1+\sigma_x^2}\,f''(x,0)$$

and $U_\infty = \mu_\infty Gr^{1/2}/\rho L$ (3.22c)

here $\hat{n}$ is the unit normal to the surface. Using the transformation (3.16) and (3.22) Nu and C_f take the following forms

$$Nu_x Gr_x^{-1/4} = -(1+\gamma)\sqrt{1+\sigma_x^2}\,\theta'(x,0) \tag{3.23}$$

$$C_{fx} Gr_x^{1/4}/2 = (1+\varepsilon)\sqrt{1+\sigma_x^2}\,f''(x,0) \tag{3.24}$$

3. 3. RESULTS AND DISCUSSION

Here we have investigated the problem of the natural convection flow of a viscous incompressible fluid with variable viscosity and thermal conductivity along a vertical wavy surface with uniform surface temperature. Solutions are obtained for the fluid having Prandtl number Pr = 0.01, 0.1, 0.7, 1.0, and for a wide range of values of the variable viscosity parameter ε = 0.0, 1.0, 2.0, 5.0 and the variable thermal conductivity parameter γ = 0.0, 1.0, 2.0, 5.0.

At first it should be mentioned that for constant viscosity and thermal conductivity (i.e. $\varepsilon = \gamma = 0$) of the fluid, we recover the problem qualitatively discussed by Yao [9] considering the form $\sigma(x) = \alpha \sin(2\pi x)$ for Pr = 1.0.

The effects of ε on the rate of heat transfer in terms of the local Nusselt number $Nu_x Gr_x^{-1/4}$ and the surface shear stress in terms of the local skin-friction, $C_{fx}\, Gr_x^{1/4}$ are give in Fig 3.2(a) and 3.2(b) respectively while $\alpha = 0.3$, $\gamma = 0.0$ and Pr = 0.7. Fig 3.2(a) indicates that increasing the values of the viscosity-variation parameter ε results in a decrease in the values of the rate of heat transfer and an increase in the surface shear stress along the wavy surface. Here we conclude that for high viscous fluid the skin-friction is large and the corresponding rate of heat transfer is slow. Increasing values of ε lead to increase the amplitude of the $Nu_x Gr_x^{-1/4}$ and $C_{fx}\, Gr_x^{1/4}$.

Fig. 3.3(a) and 3.3(b) show the axial distribution of the local Nusselt Number $Nu_x Gr_x^{-1/4}$ and the skin-friction respectively for different values of the thermal conductivity parameter γ while $\varepsilon = 0$, $\alpha = 0.3$ and Pr = 0.7. It is clearly seen that the local Nusselt number $Nu_x Gr_x^{-1/4}$ and the skin-friction $C_{fx}\, Gr_x^{1/4}$ lead to increase with the increase of the thermal conductivity parameter . Also at the downstream region the amplitude of the local Nusselt number and the skin-friction reduce gradually.

In Table 3.1 we entered the values of $f''(x,0)$ and $-\theta'(x,0)$ showing the variation of Pr while $\varepsilon = \gamma = 5.0$ and $\alpha = 0.3$. From which we can conclude that if the value of Pr increases the values of $f''(x,0)$ decreases and $-\theta'(x,0)$ increases. It is also seen that for liquid metal ($Pr = 0.01$) the rate of heat transfer is slow and the surface shear stress is higher than that of air.

Fig.3.4 and 3.5 illustrate the effect of the temperature dependent viscosity-variation parameter ε and the thermal conductivity-variation parameter γ, on the development of streamlines and isotherms respectively which are plotted for $Pr = 0.7$ and $\alpha = 0.3$ where $\Delta\psi = 1.5$ and $\Delta\theta = 0.1$. We find that for $\varepsilon = \gamma = 0$ the value of $\psi_{max} = 16.5$, for $\varepsilon = 5.0$ and $\gamma = 0$, $\psi_{max} = 15.0$, for $\varepsilon = \gamma = 5.0$ $\psi_{max} = 27.0$ and $\psi_{max} = 24.0$ where $\varepsilon = \gamma = 5.0$. From Fig.3.4(a), it is seen that the only effect of viscosity parameter ε, the flow rate in the boundary layer decreases slights. Besides only the effect of the thermal conductivity-variation parameter γ, the flow flux increases significantly. Simultaneously, the effect of viscosity parameter ε and the thermal conductivity parameter γ, the flow rate becomes higher comparison with the case of constant viscosity and thermal conductivity of the fluid. From Fig.3.5 we observe that owing to the effect of ε and γ, the thermal state of the fluid increases, causing the thermal boundary layer increase

The effect of variation of the surface roughness on the streamlines and the isotherms for the values of α equal to 0.0, 0.1, 0.2 and 0.3 are depicted by the Fig.3.6 and 3.7 respectively while $\varepsilon = \gamma = 5.0$, $Pr = 0.7$ $\Delta\psi = 1.5$ and $\Delta\theta = 0.1$. We observe that as the values of α increases the maximum values of ψ increases slightly. So we conclude that for much roughness of the surface with the effect of temperature dependent viscosity and thermal conductivity the velocity of the fluid flow increases in the boundary layer. Fig.3.7 shows that for increasing values of α, the thermal boundary layer thickness becomes larger.

Fig.3.8 and 3.9 show the effect of Prandtl number Pr (= 0.01, 0.1, 0.7 and 1.0) on the formation of streamlines and isotherms respectively while $\varepsilon = \gamma = 5.0$, $\alpha = 0.2$, $\Delta\psi = 1.5$ and $\Delta\theta = 0.1$. It can be seen that for Pr equal to 1.0, 0.7, 0.1 and 0.01 the maximum values of ψ, that is, ψ_{max} are 19.0, 22.5, 52.5 and 63.5 respectively. Also we observe that for small Pr with effect of temperature dependent viscosity and thermal conductivity the momentum and thermal boundary layer thickness increase.

4.CONCLUSIONS

The effect of viscosity and thermal conductivity variation parameter for small Prandtl number Pr (=0.01, 0.1, 0.7 and 1.0) on the natural convection boundary-layer flow along a uniformly heated vertical wavy snrface has been studied numerically. New variables transform to complex geometry into a simple shape were used a very efficient implicit-finite-difference together with Keller box scheme to solve the non-similar boundary layer equations. From the present investigations we may drawn the following conclusions:

1. The rate of heat transfer decreases and the surface shear stress increases for increasing values of the viscosity-variation parameter. It also leads to enhance the amplitude of $Nu_x Gr_x^{-1/4}$ and $C_{fx}\, Gr_x^{1/4}$.

2. The rate of heat transfer and the surface shear stress increase whet the values of the thermal conducvity-variation parameter increased. At this stage the amplitude of $Nu_x Gr_x^{-1/4}$ and $C_{fx}\, Gr_x^{1/4}$ reduce slowly.

3. With the effect of viscosity and thermal conductivity variation parameter the rate of fluid flow and the temperature distribution increase in the boundary layer significantly.

4. For small Prandtl number Pr = 0.01 (liquid metal) in presence of viscosity and thermal conductivity variation parameter the velocity and temperature distribution highly increase in the boundary layer, consequently the surface rate of heat transfer decreases.

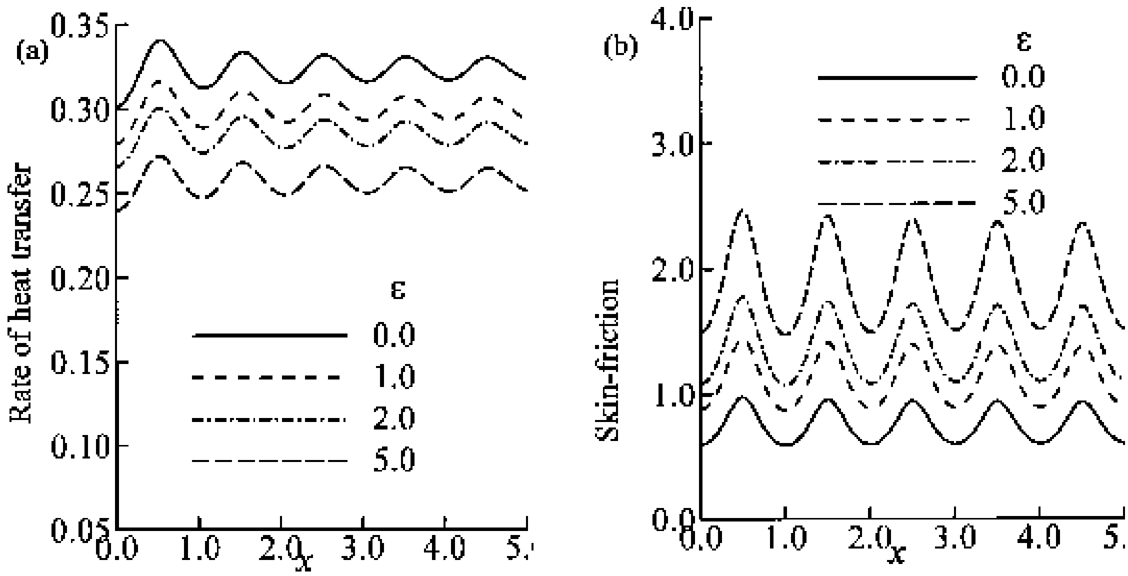

Fig. 3.2: (a) Rate of heat transfer (b) skin-friction for different values of the viscosity parameter ε while $\alpha = 0.3$, $\gamma = 0.0$ and Pr = 0.7

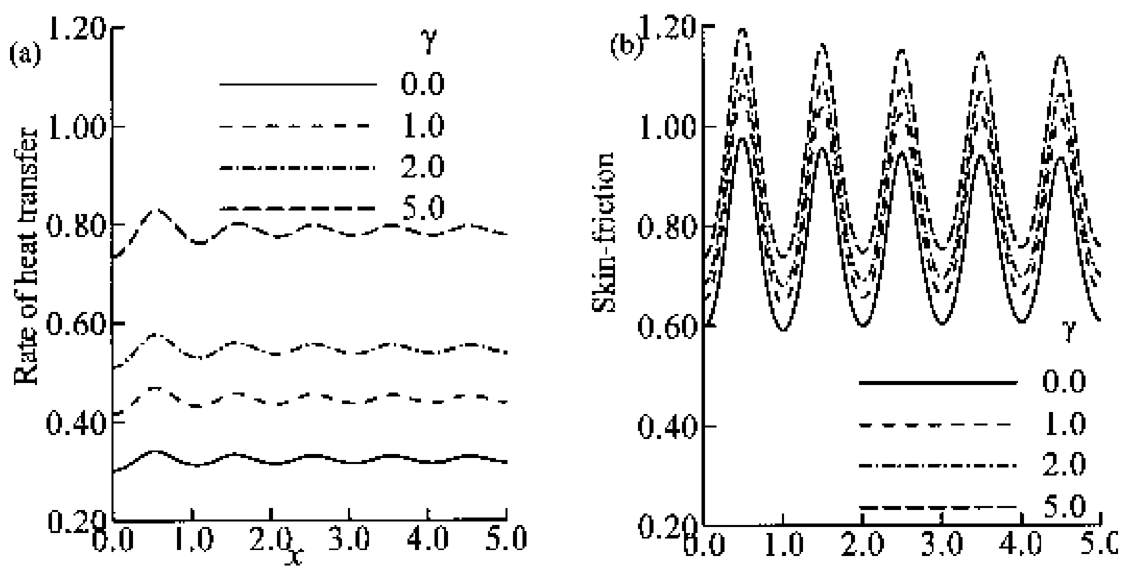

Fig. 3.3: (a) Rate of heat transfer (b) skin-friction for different values of the thermal conductivity parameter γ while $\alpha = 0.3$, $\varepsilon = 0.0$ and Pr = 0.7

Table 3.1: The values of $-\theta'(x,0)$ and $f''(x,0)$ while $\varepsilon = \gamma = 5.0$ for different values of Prandtl number Pr.

	Pr =0.01		Pr =0.1		Pr =1.0	
x	$-\theta'(x,0)$	$f''(x,0)$	$-\theta'(x,0)$	$f''(x,0)$	$-\theta'(x,0)$	$f''(x,0)$
0.0	0.29833	1.87630	0.34339	1.85304	0.66993	1.57560
0.5	0.22702	3.12810	0.34157	3.08450	0.76081	2.59743
1.0	0.29736	2.01861	0.32918	1.94338	0.69103	1.55365
1.5	0.23064	3.10188	0.34567	3.04521	0.74648	2.54459
2.0	0.29575	2.02056	0.32618	1.95484	0.70030	1.57907
2.5	0.23571	3.06400	0.34581	3.00626	0.73950	2.50812
3.0	0.29321	2.03080	0.32868	1.96862	0.70403	1.59470
3.5	0.24120	3.04872	0.34406	2.98605	0.73683	2.48817
4.0	0.28993	2.03486	0.33101	1.97599	0.70564	1.60365
4.5	0.24672	3.03863	0.34256	2.97270	0.73532	2.47543
5.0	0.28620	2.03792	0.33271	1.98138	0.70667	1.60974

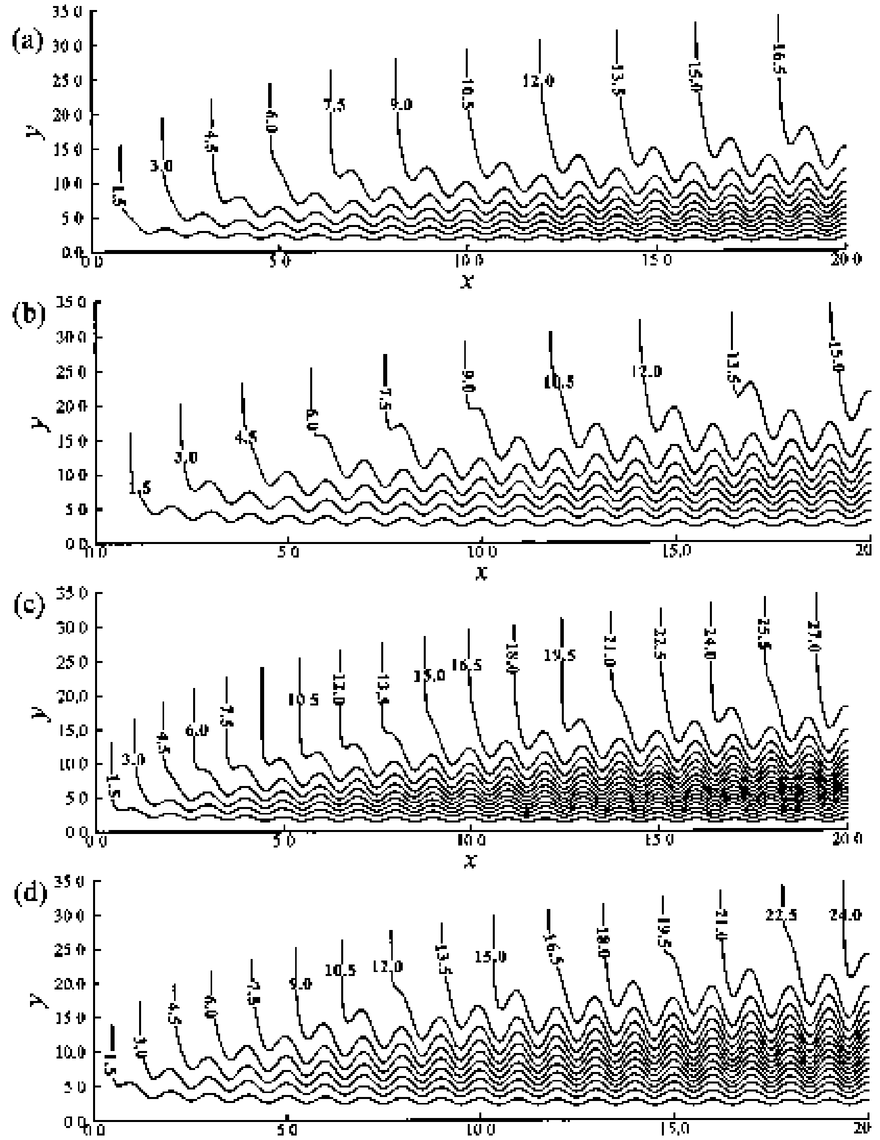

Fig. 3.4: Streamlines (a) $\varepsilon = 0.0$, $\gamma = 0.0$ (b) $\varepsilon = 5.0$, $\gamma = 0.0$ (c) $\varepsilon = 0.0$, $\gamma = 5.0$ (d) $\varepsilon = 5.0$, $\gamma = 5.0$ while $\alpha = 0.3$ and $\Pr = 0.7$ where $\Delta\psi = 1.5$

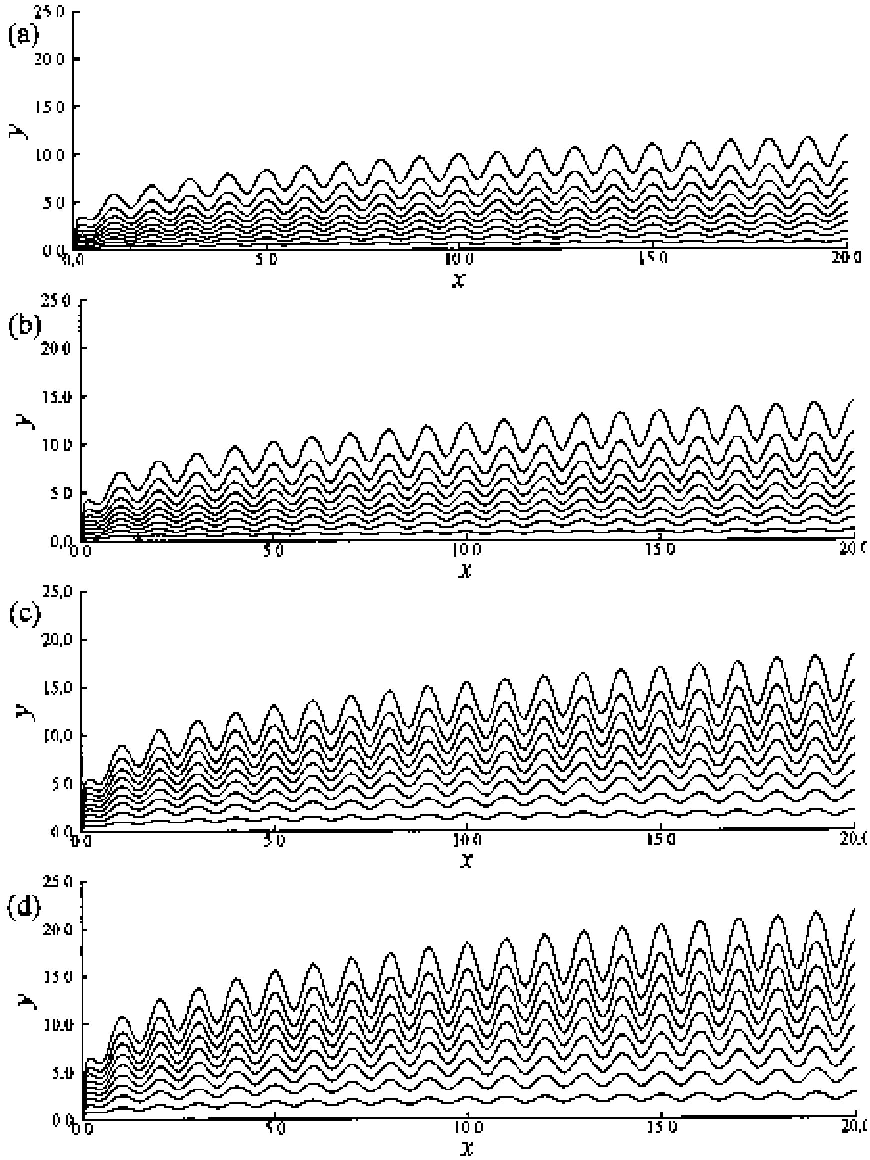

Fig. 3.5: Isotherms (a) $\varepsilon = 0.0$, $\gamma = 0.0$ (b) $\varepsilon = 5.0$, $\gamma = 0.0$ (c) $\varepsilon = 0.0$, $\gamma = 5.0$ (d) $\varepsilon = 5.0$, $\gamma = 5.0$ while $\alpha = 0.3$ and Pr = 0.7 where $\Delta\theta = 0.5$

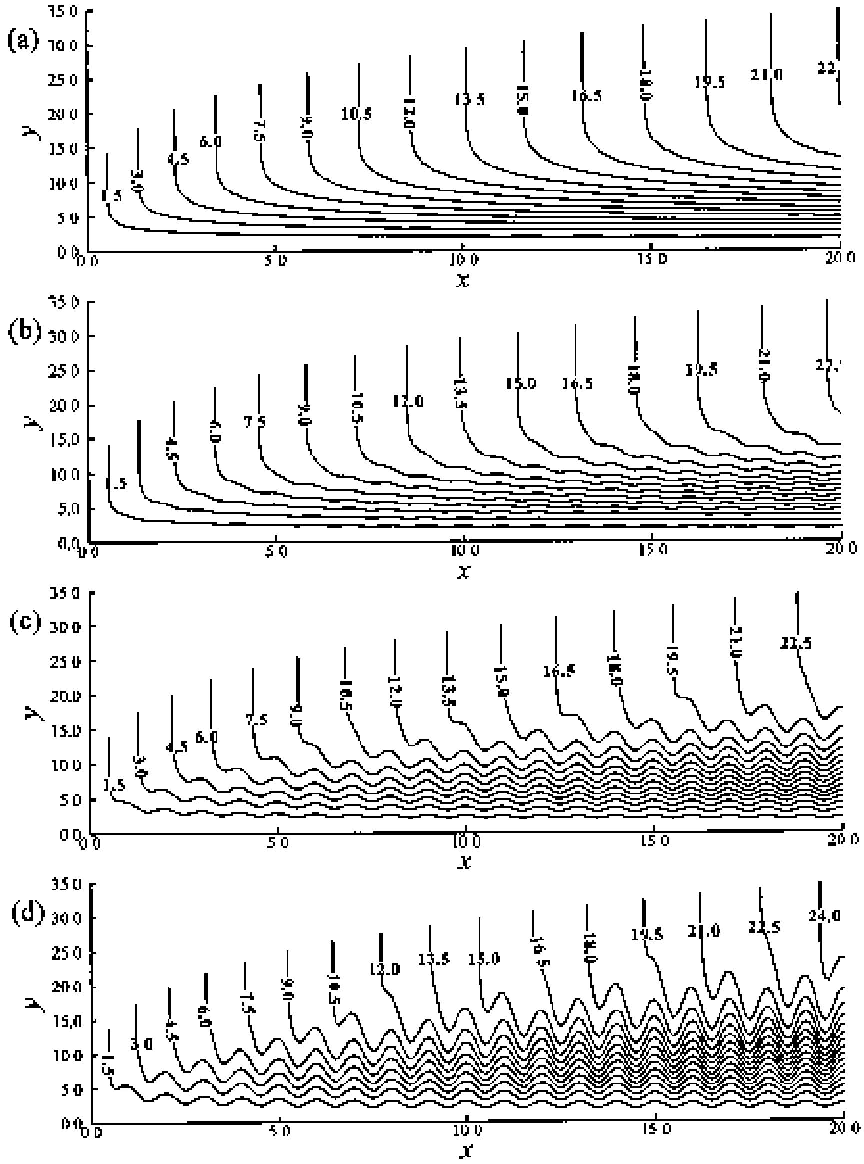

Fig. 3.6: Streamlines (a) $\alpha = 0.0$ (b) $\alpha = 0.1$ (c) $\alpha = 0.2$ (d) $\alpha = 0.3$ while $\varepsilon = 5.0$, $\gamma = 5.0$ and Pr = 0.7 where $\Delta\psi = 1.5$

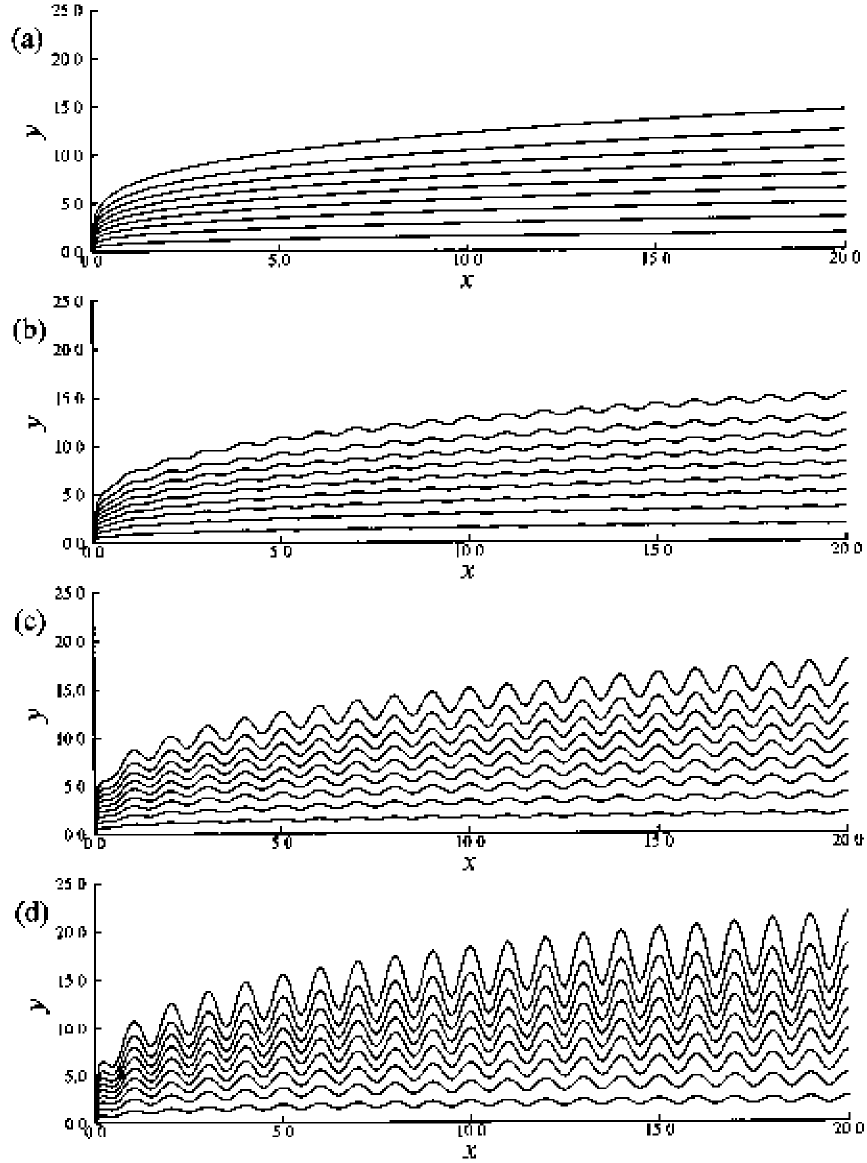

Fig. 3.7: Isotherms (a) $\alpha = 0.0$ (b) $\alpha = 0.1$ (c) $\alpha = 0.2$ (d) $\alpha = 0.3$ while $\varepsilon = 5.0$, $\gamma = 5.0$ and Pr = 0.7 where $\Delta\theta = 0.5$

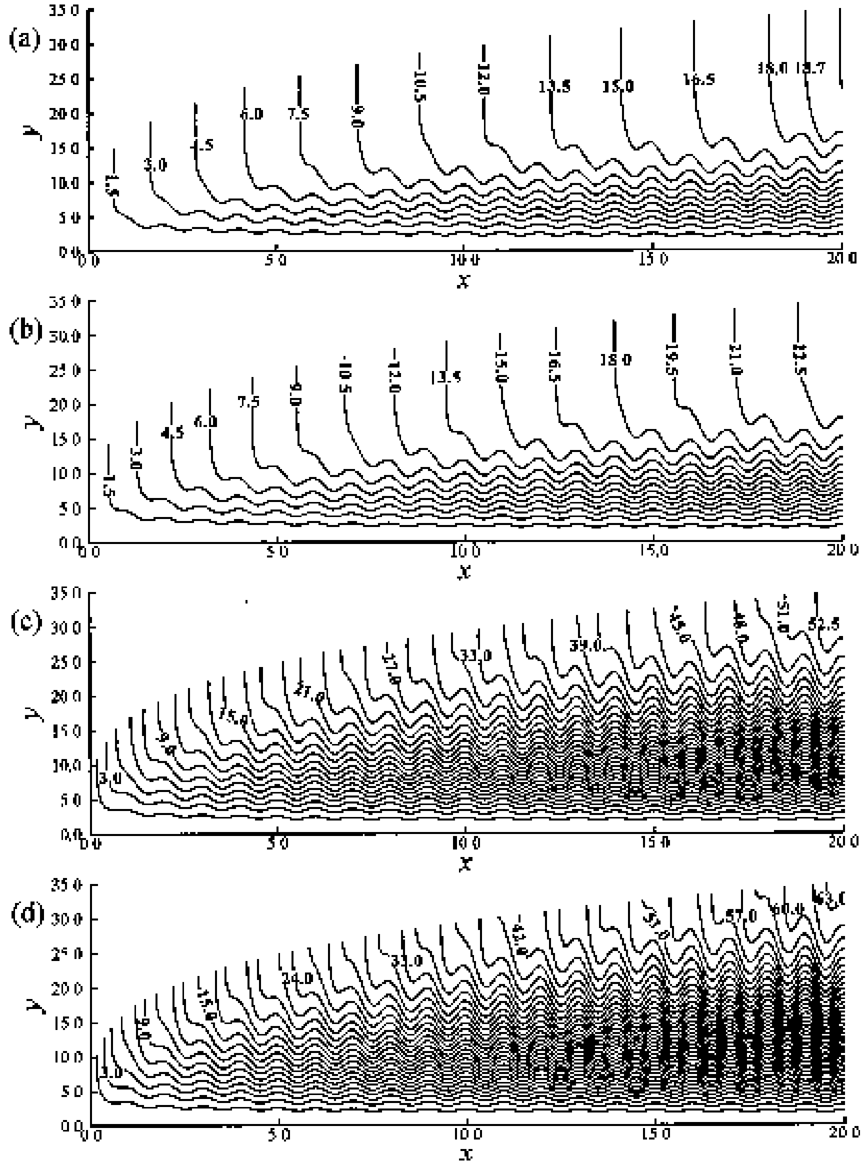

Fig. 3.8: Streamlines (a) Pr = 1.0 (b) Pr = 0.7 (c) Pr = 0.1 (d) Pr = 0.01 while $\varepsilon = \gamma = 5.0$ and $\alpha = 0.2$ where $\Delta\psi = 1.5$

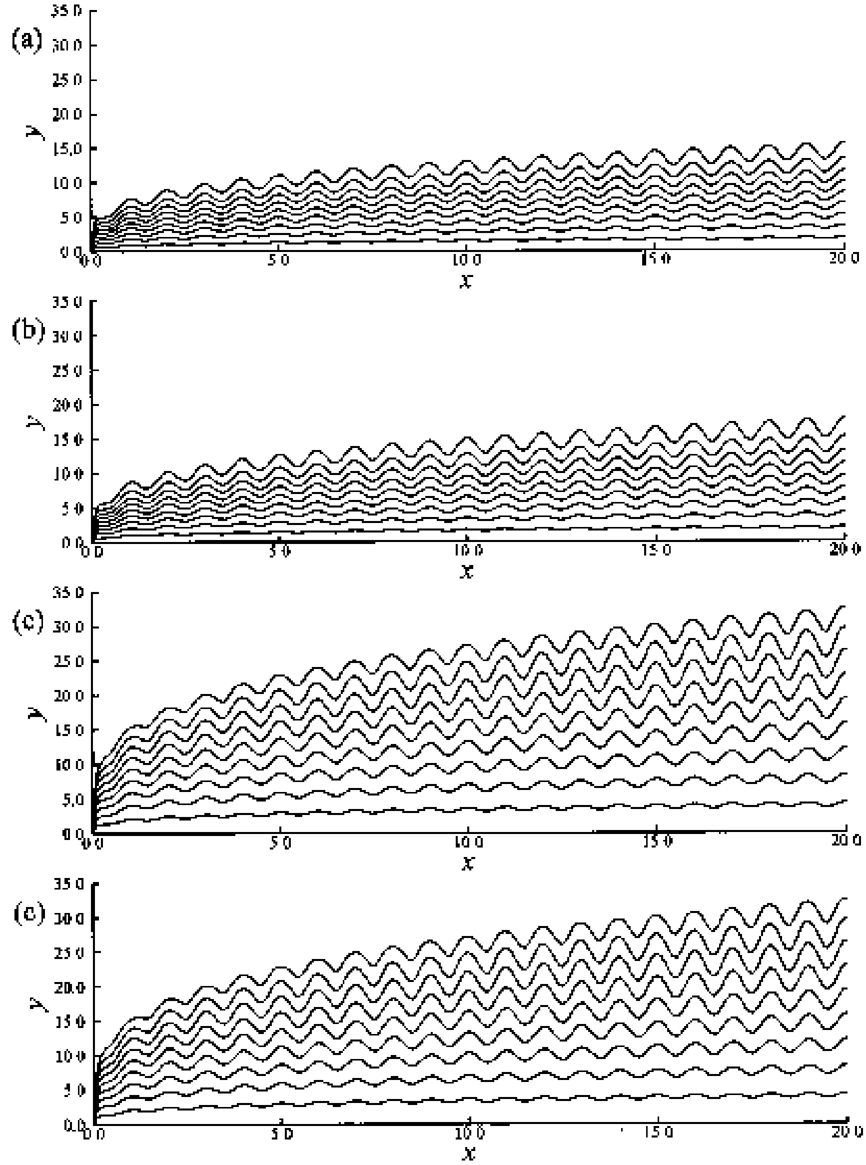

Fig. 3.9: Isotherms (a) Pr = 1.0 (b) Pr = 0.7 (c) Pr = 0.1 (d) Pr = 0.01 while $\varepsilon = 5.0$, $\gamma = 5.0$ and $\alpha = 0.2$ where $\Delta\theta = 0.5$

Chapter 4

CONCLUSIONS

In this dissertation, we have investigated the effect of temperature dependent viscosity $\mu(T)$ and thermal conductivity $k(T)$ on steady two-dimensional natural convection flow (i) from an isothermal horizontal circular cylinder (ii) along a vertical wavy surface with uniform surface temperature T_w. Using the appropriate transformations the basic equations are transformed to non-similar boundary-layer equations, which has been solved numerically using a very efficient implicit finite-difference method known as Keller box scheme. Here we have focused our attention on the evolution of the surface shear stress in terms of local skin-friction, rate of heat transfer in terms of local Nusselt number, streamlines, isotherms, velocity distribution, temperature distribution, viscosity distribution as well as thermal conductivity distribution for a selection of parameter sets consisting of the viscosity parameter, thermal conductivity parameter and amplitude of the wavy surface.

From Chapter 2:

The effect of temperature-dependent viscosity for large Pr on the natural convection boundary layer flow from an isothermal horizontal circular cylinder has been investigated theoretically. From the present investigation, the following conclusions may be drawn:

1.The velocity distribution increases and the viscosity of the fluid decrease at the middle of the surface for increasing value of viscosity-variation parameter ε.

2. Increased value of the viscosity- variation parameter ε leads to increase in the local heat transfer rate and to decrease the local skin-friction.

3. It has been observed that the velocity distribution and skin-friction decrease as well as the viscosity distribution and the rate of heat transfer increase with the increase of Prandtl number Pr.

From Chapter 3:

The effect of viscosity and thermal conductivity variation parameter for small Prandtl number Pr (=0.01, 0.1, 0.7 and 1.0) on the natural convection boundary-layer flow along a uniformly heated vertical wavy surface has been studied numerically. From the present investigation we may drawn the following conclusions:

1. The surface rate of heat transfer decreases and the surface shear stress increases for increasing values of the viscosity-variation parameter. It also leads to enhance the amplitude of $Nu_x Gr_x^{-1/4}$ and $C_{fx}\, Gr_x^{1/4}$.

2. The rate of heat transfer and the surface shear stress increase whet the values of the thermal conducvity-variation parameter increased. At this stage the amplitude of $Nu_x Gr_x^{-1/4}$ and $C_{fx}\, Gr_x^{1/4}$ reduce slowly.

3. With the effect of viscosity and thermal conductivity variation parameter the rate of fluid flow and the temperature distribution increase in the boundary layer significantly.

4. For small Prandtl number Pr = 0.01 (liquid metal) in presence of viscosity and thermal conductivity variation parameter the velocity and temperature distribution highly increase in the boundary layer, consequently the surface rate of heat transfer decreases.

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