Free Convection Flow of Visco-Elastic Fluid Past an Infinite Vertical Porous Plate.

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Declaration

I do hereby declare this thesis has been composed by myself and it is a record of my own research work under the supervision of Mr. Md. Mustafa Kamal Chowdhury. It has not been accepted in any previous application for a higher degree. All source of information are specifically acknowledge by means of references. None of the materials contained in this thesis will be submitted in support of any other degree or diploma at any other university or institution other than publications.

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Abstract

This work provides a comprehensive theoretical analysis of a two-dimensional unsteady free convection flow of an incompressible, visco-elastic fluid past an infinite vertical porous plate. Solutions for the zero-order perturbation velocity profile, the first order perturbation velocity profile and temperature profile in closed form are obtained with the help of Laplace transform technique.

It covers the area of boundary layer flow of viscous, incompressible and electrically conducting fluid in the presence of strong magnetic field along a heated vertical flat plate. The ensuing boundary layer flows considered here are governed by a non-similar set of parabolic equations. Local non-similarity method is employed to investigate the solutions of boundary layer equations representing the flow and temperature fields. The numerical solutions are carried out for Prandtl's number , 0.1, 0.72, 1.0, 1.5 and 2.0 which are appropriate for different types of liquid metals and for different values of magnetic field parameter, M.

Finally, a problem on free convection boundary layer flow of visco-elastic incompressible and electrically conducting fluid past an infinite vertical porous plate along an isothermal vertical surface are studied in the presence of a transverse magnetic field. The results thus obtained have a graphical illustration for different valnes of the magnetic field parameter M, transpiration parameter a, Grashof number Gr, Visco-elasticity parameter s and the Prandtl number Pr.

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Introduction

Magnetofluid dynamics deals with the study of electrically conducting fluids in electric and magnetic fields. It unifies a framework between the electromagnetic and fluid dynamic theories to yield a description of the concurrent effects of magnetic field on the flow and the flow on the magnetic field. There are many natural phenomena and engineering problems susceptible to Magnetofluid dynamics analysis. It is useful in astrophysics because much of the universe is filled with widely spaced, charged particles and permeated by magnetic fields, and so the continuum assumption becomes applicable.

The natural convection boundary layer flow of an electrically conducting fluid up a hot vertical wall in the presence of strong magnetic field has been studied by **Sing and Cowling** (1963), **Sparrow** and **Cess** (1961), **Riley** (1964) and **Kuiken** (1970) because of its application in the nuclear engineering in connection with the cooling of reactors. Sometimes it becomes necessary to control the connective boundary-layer flows by injecting or withdrawing fluid through a porous heated boundary wall. Since, this can enhance heating(or cooling) the system and can help to delay the transition from the laminar flow. Most of the work on the effect of transpiration on free convection boundary layer. had been confined to cases where there is a prescribed wall temperature. The power law variations of the plate temperature and the transpiration velocity considered by **Eichhorn** (1960) are those for which similarity solntions exist.

Sparrow and **Cess** (1961) first considered the case of uniform transpiration velocity through an isothermal vertical wall. They looked at the problem of a nniform plate temperature and transpiration velocity. The problem was considered in more detail by **Merkin** (1972) who obtained asymptotic solution, valid at large distances from the leading edge. For both suction and blowing on general body shapes a similarity solution had been investigated by **Merkin** (1975).

A transformation of the equations for general blowing and wall temperature variations was studied by **Vedanayagam** et. el. (1980). **Clark** and **Riley** (1975, 1976) **Lin** and **Yu** (1988) studied the case of heated isothermal horizontal surface with transpiration in detail and recently by **Chaudhary** and **Merkin** (1993). Since the method proposed by **Cess** (1961) had a serious drawback in the results it provides us low accuracy at moderate values of x, **Sparrow** and **Yu** (1971) introduced a method known as local non-similarity method for the thermal boundary layer. This technique is locally autonomous. Since solutions of the non-similarity equations at any specified steamwise station can be obtained without first obtaining upstream solutions. This local non-similarity method has been used efficiently by **Chen** and **Sparrow** (1976), **Minkowycz** and **Sparrow** (1978), **Chen** (1988) and very recently by **Hossain** et.

The presence of roughness on the flat surface disturbs the flow and alters the heat transfer rate. Using a simple transformation on the governing equation for the flow considered here, non-similar boundary layer equations for a wavy surface are derived. The present study deals with the effects of the magnetic field parameter, M and the amplitude of the transpiration parameter a, on the velocity field, the temperature field and local Nusselt number on the flow characteristics.

A study of considerable importance in areas concerned with the energy generation and its utilization is that of heat transfer. The study of heat transfer has, over the past several years, been related to a wide variety of problems ,each with its own demands of precision and elaboration in the understanding of the particular processes of interest. Atmospheric, geophysical and environmental problems in connection with heat rejection, space research and manufacturing system require such type of studies.

In the diversity of studies related to heat transfer, considerable effort directed at connective mode, in which relative motion of the fluid provides an additional mechanism for the transfer of energy.

The connective mode of heat transfer is divided into two basic processes. If the motion of the fluid arises due to an external agent, such as the externally-imposed flow of a fluid stream over a heated object, the process is termed as forced convection. This type of fluid flow is caused in general by a fan, blower, the wind or the motion of the heated object itself. Such problems are very frequently encountered in technology, where the heat transfer to, or from a body is often due to imposed flow of a fluid at a temperature different from that of the body.

If, on the otherhand, no such externally induced flow is provided and the flow arises "naturally" due to the effect a density difference, resulting from a temperature in a body, the process is termed as natural or free convection. The density difference gives rise to buoyancy effects due to which the flow is generated. A heated body cooling in ambient air generates such a flow in the region surrounding it. Similarly buoyant flow arising from heat rejection to the atmosphere and to other ambient media, circulation's arising in heated rooms in the atmosphere, and in bodies of water causes thermal stratification of the medium. Many other such heat transfer processes, in our natural environment as well as in many technological applications, are included in the area of natural convection.

The flow of an incompressible viscous fluid past an impulsively started infinite horizontal plate, in its own plane was studied first by Stoke's. Because of its particular importance, it has been extended to bodies of different shapes by a number of researchers. Amongst them are **Illingworth** (1950), **Stewartson** (1951), **Hall** (1969) and **Elliot** (1969). **Illingworth** (1950) considered the flow of a compressible gas with variable viscosity near an impulsively started vertical plate and the problem was solved by the method of successive approximation. **Elliot** (1969) generalized **Illingworth's** (1950) problem by assuming a time-dependent velocity and temperature for the plate but neglected the viscous dissipation. However, in both papers, only mathematical results were derived and no physical situation was discussed. From the engineering point of view the physical aspects of such types of problems are important . In these two papers the flow past an impulsively started semi-infinite horizontal plate has been considered. **Stewartson** (1951) studied it by analytical methods whereas **Hall** (1969) discussed the same problem by finite-difference method.

Analytical studies on unsteady laminar free-convection problems have received much attention by many research workers such as **Surgawara** and **Michiyoshi** (1951 & 1952). **Siegel** (1958), **Gebhart** (1961), **Chung** and **Anderson** (1961), **Sparrow** and **Gregg** (1960) and **Yang** (1960) . Exact solutions were available only for the infinite vertical plate with Prandtl number of unity and under transient conditions of step change in either the surface temperature or the surface heat flux . These represent asymptotic solutions expected to be valid not only at large distances away from the leading edge, but also during a short time interval after the communication of the free convection flow along a finite plate. Exact asymptotic solution for the saue problem with arbitrary surface temperature or heat flux variations were obtained by **Menold and Yang** (1962) and **Soundalgeker** (1977) . In these studies, solutions for the coupled equations governing the flow field were obtained in exact form using the Laplace transform technique. Study of the flow of electrically couducting fluid in presence of magnetic field across a surface is also important from the technological point of view. Fluid flowing across a transverse magnetic field produces an electromagnetic force. The current and the magnetic field combine to produce force that resists the fluid's motion. The current also generates itself magnetic field which distorts the original magnetic field. An opposing or pumping force on the fluid can be produced by applying an electric field perpendicular to the magnetic field . The disturbance created in the magnetic field or the fluid can produce magneto-fluid dynamics waves in upstream and downstreamwake phenomena **Cramer and Pai** (1974). In natural convection boundary layer flow of a electrically conducting fluid up a vertical wall in presence of a strong cross-magnetic field has been studied by **Singh** and **Cowling** (1963). **Sparrow** and **Cess**(1961), **Riley** (1964), **Kuiken** (1970), **Wilks** (1976), **Hossain** and **Ahmed** (1990) and many others. All the above studies were confined to forced , free and combined forced and free convection flow of an electrically conducting fluid along a vertical surface in presence of a transverse strong magnetic field.

Compared with the steady state situation, there are relatively few solutions available for the transient flow of the electrically conducting fluid in presence of a magnetic field. **Rossow** (1958) investigated the problem of an infinite flat plate given impulsive motion in presence of transverse magnetic field. Later **Gupta** (1960) , **Singh** (1964) , **Pop** (1970) studied the transient free convection flow in presence of magnetic field. Recently **Hossain** and **Mandal** (1985) have investigated the free convection flow of an electrically conducting fluid past an accelerated vertical porous plate with time dependent suction for an arbitrary Prandtl number.

Stndies in the visco-elastic fluid , which exhibits both viscous and elastic properties such as bituminous, flour dough, napalm and similar jellies, polymers and polymer melts such as nylon and many polymer solutions, are of great interest from the technological point of view to many researchers. **Oldroyd** (1950) was the first to outline a method of formulating consecutive equations which would be valid for large deformation. The equation

$$\tau + \lambda_1 \frac{d\tau}{dt} = \mu\left(\frac{dv}{dt} + \lambda_2 \frac{d^2 v}{dt^2}\right)$$

was taken as the basis for his theory which gives a liner relation between the shear stress τ, the rate of shear $\frac{dv}{dt}$ and their time rates of change.

The expression contains 3 constants, a viscosity μ and " relaxation times" λ_1 and λ_2. The relaxation times obviously have the physical significance that if the motion is suddenly stopped, the shear stress decays as $\exp\left(-\frac{t}{\lambda_1}\right)$ and if the stress is removed the rate of strain decays as $\exp\left(-\frac{t}{\lambda_2}\right)$. The quantity μ is the Newtonian viscosity observed in the fluid at very low rates of shear. This equation reduces to the Newtonian fluid (λ_1-λ_2=0) and to Maxwell fluid (λ_2=0) as special case.

The boundary layer treatment for an idealized visco-elastic fluid was introduced by **Beard** and **Walters** (1964) . There has been a continued interest in the investigation of natural convection heat transfer of non-Newtonian flnid, which exhibit visco-elasticity. Recently **Rajagopal** (1980) and others investigated the heat transfer in the forced convection flow of a visco-elastic fluid of Walters model. Most recently **Dandpath** and **Gupta** (1989) have investigated the flow and heat transfer in an incompressible second order fluid caused by a stretching sheet with a view to examining the inflnence of visco-elasticity on the flow and heat transfer characteristics. The above work were confined to the study of steady forced convection flow. Less interest was shown in the problem of transient forced and free convection flow of a visco-elastic fluid. **Teipel** (1981) first studied the transient flow of non -Newtonian visco-elastic fluid for an inpnlsive motion of a flat plate.

The flow-along a harmonically oscillating flat plate of the visco-elastic flnid has been studied by **Rajagopal** (1983) and **Panda** (1979) et.al . It is now a well-known fact that magnetic field has stabilizing effect on the boundary layer growth.

With this understanding, **Singh** (1983) and **Singh** (1984) have investigated the effect of a transverse magnetic field of an electrically conducting visco-elastic fluid past an accelerated flat plate of infinite extension. The effect of elasticity on MHD flow of an elastico-viscous fluid past an accelerated plate has been investigated. and in , analysis of the Stokes problem for the MHD free convection flow of a visco-elastic fluid past an impulsively started vertical plate has been performed employing the Laplace transform technique for Prandtl number P_r=1. In this work. it is proposed to study the effects of free convection flow of an electrically conducting visco- elastic fluid past an infinite vertical porous plate in presence of transverse magnetic fluid . The plate is allowed to move in its own plane with a velocity $u=u_0f(t)$, where u_0 is constant and f(t) is a function of time only. In chapter 1, the energy and momentum equation of have been deduced by introducing the non- dimensional quantities into the boundary layer equations. In chapter 2 . the energy equation has been solved for different values of Prandtl number. The rate of heat transfer has beeu studied. In chapter 3, the problem of free convection boundary layer flow of a visco-elastic fluid along an infinite vertical plate in presence of magnetic field has been studied. Here both impulsive and uniformly accelerated motion of the plate have been considered . The results for the velocity fields are shown graphically. The skin friction factor is represented graphically and in tabular form for different values of the parameters P_r (prandtl number), M (Magnitic field parameter), Gr(Grashof number), α(Transpiration parameter) and S(visco-elastic parameter). Finally the flow of visco-elastic fluid past an infinite vertical porous plate has been studied and results are discussed in chapter 4. The velocity profiles for impulsive and uniformly accelerated starts of the plate are giveu graphically. The skin friction factor is represented graphically and in tabular form for different values of the parameters P_r ,S , α, τ and Gr.

Non-Newtonian fluids:

The Newtonian hypothesis has worked very well in explaining many physical phenomena in various branches of fluid dynamics. This allures us to remark that most of fluids at least in ordinary situations behave like Newtonian fluids. But in the recent years, especially with the introduction of polymers, it has been found that there are fluids which show a distinct deviation from Newtonian hypothesis. Such fluids are called non-Newtonian fluids.

There arises a strong feeling to develop new theories to explain the behavior of different types of non-Newtonian fluids. The non-Newtonian fluids are broadly classified into the following three categories:

(i) Viscous fluids,

(ii) Visco-plastic fluids and perfectly plastic materials, and

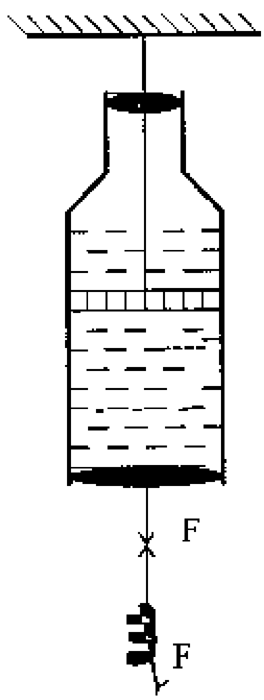
(iii) Visco-elastic fluids

The idea is that the stress tensor is isotropic when the fluid is at rest, is followed for all fluids whether it is a Newtonian fluid or a non-Newtonian fluid. Therefore we define $T_{ij} = -p\delta_{ij} + p_{ij}$ where p is the pressure , p_{ij} is zero when the flnid is at rest and T_{ij} is the stress tensor .When a non-Newtonian flnid is undergoing a general deformation then p_{ij} may not be a deviatric stress tensor and so its values is other than.

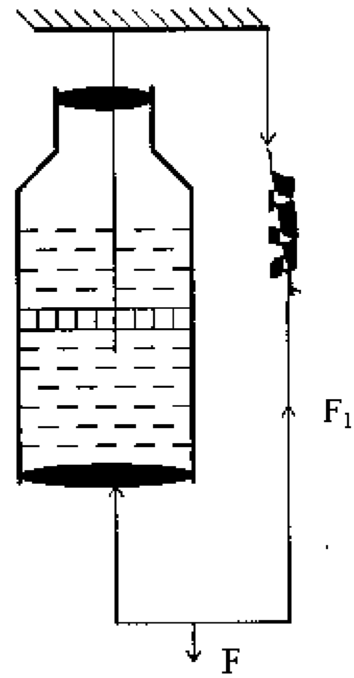
(i) Viscous fluids: The term "viscous" has come from the internal friction between the fluid layers. In Newtonian fluids the strain-rate depends linearly on the applied stress. Moreover , when the fluid is at rest the stress tensor is isotropic and thns the results in pressure only. So one naturally asks why not take the stress tensor as a general function of the strain-rate tensor but which is isotropic when the strain -rate is zero. Well a fluid in which the stress tensor p_{ij} is a given function of the strain-rate is called a purely viscous fluid, provided in the absence of strainrate the stress tensor is zero.

(ii) Visco-plastic fluids and perfectly plastic materials: One of the important observations in the viscous fluids is that if we apply a certain shearing stress on a fluid, however small it may be, it causes a continuous deformation in the fluid. But in many materials like paints , pastes, etc., we find that if we apply a shearing stress less than a certain qnantity, the materials does not move at all. But when this shearing stress exceeds a certain value the material starts moving and the strain-rate of the material depends upon the applied stress. Such materials are called plastics.

(iii) Visco-elastic fluids: In the elastic materials the stress depends on the strain only; that is , the stress is a certain function of the strain. Thus if we apply a certain stress on an elastic material, the material undergoes some deformation and when this stress is removed the material returns to its original position. So we can say that the elastic materials have memory (rather perfect memory), i.e., it is capable of recognizing its original shape. On other hands, the stress depends upon the rate of deformation and when the stress is removed the strain-rate becomes zero. But the deformation it has accumulated persists. That is , it forgets its original position. In other words , we can say that fluids have no memory. But there are some fluids like soap solution, polymers, which have some elastic properties, besides having usual fluid properties. Such fluids are called visco-elastic fluid. A visco-elastic model can easily be illustrated by a spring dashpot assembly (figs 1.1 & 1.2). Now if we take a spring and apply a certain force to it, we see that the spring extends by a certain amount which is proportional to the force applied to it. On the other hand , in a dashpot if we apply a certain force on the piston we find that the piston moves with a certain speed which is proportional to the force applied to it. We can say that the elastic properties of a material can be represented by a spring and the viscous



Fig(1.1)



Fig(1.2)

properties by a dashpot. Now we shall take two models containing a dashpot and a spring to illustrate visco-elastic models.

In general heat transfer estimates the rate at which heat is transferred across the system, where the boundaries are subjected to specific temperature differences and the temperature distribution of the system during the process. The physical processes involved in the generation and utilization of heat are of practical importance.

There are three basic processes of thermal energy transport :

(i) conduction, (ii) convection and (iii) radiation.

In various types of studies related to heat transfer or thermal energy transport, considerable effort has been directed at the connective mode, in which heat transfer processes take place with the motion of the fluid. As a consequence of this fluid motion, the heat transfer rate, as given by conduction is considerably altered.We are interested here to deal with the forecs of convection only.

The convective heat transfer is divided into two basic processes namely (a) forced convection and (b) natural or free convection.

Free convection: If there is no internally induced flow but the flow arises "naturally" due to the effect of a density difference, resulting from a temperature or concentration difference in a body force field, such as gravitational field, the process is termed as "natural" or "free" convection. The density difference causes buoyancy effects due to which the flow is generated.

Porous plates: By porous plates we mean that plates possess very fine holes distributed uniformly over the enter surface of the plates through which fluid can flow freely and continuously.

Plates with suction and injection : The plate from which the fluid enters the flow region is known as the plate with injection and the plate from which the fluid leaves the flow region is known as the plate with suction.

Chapter 1
BASIC EQUATIONS GOVERNING THE FLOW

An unsteady free convection flow of an electrically conducting viscous incompressible fluid past an infinite vertical porous plate with time dependent suction/injection has been considered. A magnetic field of uniform strength B_0 is applied transversely to the plate. The induced magnetic field is neglected as the magnetic Reynolds number of the flow is taken very small. We assume that all the fluid properties are constant and the influence of density variation with temperature is considered only the body force term. The flow is assumed to be in the x-direction which is along the vertical plate in the upward direction and y-axis is taken to be the surface of the plate. Initially the temperature of the plate and the fluid are same.

Continuity equation

$$\frac{\partial v}{\partial y} = 0 \tag{1.1}$$

Momentum equation

$$\frac{\partial u}{\partial t} - \lambda \frac{\partial u}{\partial y} = \beta g (T - T_\infty) + \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma_0 \beta_0^2 u}{\rho} - \kappa_0^* \frac{\partial^3 u}{\partial y^2 \partial t} \tag{1.2}$$

Energy equation

$$\frac{\partial T}{\partial t} - v \frac{\partial T}{\partial y} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial y^2} \tag{1.3}$$

Here u and v are the velocity components associated with the direction of increasing x and y co-ordinates. T is the temperature of the fluid in the boundary layer , g is the acceleration due to gravity , β is the volumetric coefficient of expansion , κ is the thermal conductivity, ρ the density of the fluid , σ_0 is the electric conductivity, ν is the kinematics coefficient of viscosity , C_P is the specific heat at constant pressure, T_∞ is the temperature of the ambient fluid , κ_0^* defined rotational viscosity coefficient and λ is the suction parameter.

The associated initial and boundary conditions are

$$t \le 0;\ u(y,t) = 0;\ T(y,t) = T_{\infty}, \qquad (1.4)$$
$$t > 0;\ u(0,t) = u_0 f(t);\ T(0,t) = T_W,$$
$$t > 0;\quad u(\infty,t) = 0;\quad T(\infty,t) = T_{\infty}.$$

Initially the temperature of the plate is the same as that of the fluid (no slip condition). At t>0 , the plate starts moving in its own plane with a velocity $u=u_0 f(t)$ where u_0 is constant and f(t) is a function of time . The plate temperature is instantaneously raised or lowered to T_w, which is therefore maintained constant in order to produce boundary effect. The heat due to viscous and joules dissipation are neglected in the energy equation because of small velocity usually encounters in free convection flow.

In order to non-dimensionalize the governing equations, we introduce the following non-dimensional variables:

$$W = \frac{u}{u_0},\quad \eta = \frac{u_0}{\nu} y,\quad \tau = \frac{u_0^2}{\nu} t,\quad \theta = \frac{T - T_{\infty}}{T_W - T\infty} \qquad (1.5)$$

with the help of these non-dimensional variables the momentum equation (1.2) take the following forms:

$$\frac{\partial u}{\partial t} = \frac{u_0^3}{\nu}\frac{\partial w}{\partial \tau}$$

$$\frac{\partial u}{\partial y} = \frac{u_0^2}{\nu}\frac{\partial w}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_0^2}{\nu^2}\frac{\partial^2 w}{\partial \eta^2}$$

$$\frac{\partial^3 u}{\partial y^2 \partial t} = \frac{u_0^5}{\nu^3}\frac{\partial^3 w}{\partial \eta^2 \partial \tau}$$

Put $T - T_\infty = \Delta T \quad , T - T_\infty = \theta \Delta T$

$$\frac{u_0^3}{\nu}\frac{\partial w}{\partial \tau} - \lambda \frac{u_0^2}{\nu}\frac{\partial w}{\partial \eta} = \beta g.\Delta T.\theta + \nu \frac{u_0^3}{\nu^2}\frac{\partial^2 w}{\partial \eta^2} - \frac{\sigma_0 \beta_0^2 u_0 w}{\rho} - \kappa_0^* \frac{u_0^5}{\nu^3}\frac{\partial^3 w}{\partial \eta^2 \partial \tau}$$

$$\frac{\partial w}{\partial \tau} - \frac{\lambda}{u_0}\frac{\partial w}{\partial \eta} = \frac{\beta g.\Delta T.\nu}{u_0^3}\theta + \frac{\partial^2 w}{\partial \eta^2} - \frac{\sigma_0 \beta_0 \nu}{\rho\, u_0^2} w - \kappa_0^* \frac{u_0^3}{\nu^2}\frac{\partial^3 w}{\partial \eta^2 \partial \tau}$$

$$\frac{\partial w}{\partial \tau} - \frac{\lambda}{u_0}\frac{\partial w}{\partial \eta} = \frac{\partial^2 w}{\partial \eta^2} - s\frac{\partial^3 w}{\partial \eta^2 \partial \tau} - Mw + G_r \theta \tag{1.6}$$

Energy equation (1.3) takes the form

$$\frac{\partial T}{\partial t} = \frac{\partial \theta}{\partial \tau}\frac{u_0^2}{\nu}\Delta T$$

$$\frac{\partial T}{\partial y} = \frac{u_0 \Delta T}{\nu}\frac{\partial \theta}{\partial \eta}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{u_0 \Delta T}{\nu}\frac{\partial \theta}{\partial \eta}\right)$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{u_0 \Delta T}{\nu}\frac{\partial}{\partial \eta}\left(\frac{\partial \theta}{\partial \eta}\right)\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 T}{\partial y^2}=\frac{u_0^2\Delta T}{\nu^2}\frac{\partial^2\theta}{\partial\eta^2}$$

$$\frac{u_0^2\Delta T}{\nu}\frac{\partial\theta}{\partial\tau}-\lambda\frac{u_0\Delta T}{\nu}\frac{\partial\theta}{\partial\eta}=\frac{K}{\rho C_p}\frac{u_0^2}{\nu^2}\frac{d^2\theta}{\partial\eta^2}$$

$$\frac{\partial\theta}{\partial\tau}-\frac{\lambda}{u_0}\frac{\partial\theta}{\partial\eta}=\frac{k}{\rho C_p}\frac{1}{\nu}\frac{\partial^2\theta}{\partial\eta^2}$$

$$\frac{\partial\theta}{\partial\tau}-\frac{\lambda}{u_0}\frac{\partial\theta}{\partial\eta}=\frac{k}{C_p\mu}\frac{\partial^2\theta}{\partial\eta^2}$$

where $\nu=\frac{\mu}{\rho}$

$$\frac{\partial\theta}{\partial\tau}-\frac{\lambda}{u_0}\frac{\partial\theta}{\partial\eta}=\frac{1}{p_r}\frac{\partial^2\theta}{\partial\eta^2} \tag{1.7}$$

where

$Gr=\frac{g\beta\Delta T}{u^3}$ (Grashof Number),

$P_r=\frac{\mu C_p}{K}$ (Prandtl Number),

$S=\kappa_0^{*}\left(\frac{u_0}{\nu}\right)^2$ (Visco- elasticity Parameter),

$M=\frac{\nu\sigma_0\beta_0^2}{u_0^2\rho}$ (Magnetic field of Parameter).

subject to the boundary conditions:

$$\begin{aligned} &\tau\le 0, \quad w(\eta,\tau)=\theta(\eta,\tau)=0 \\ &\tau>0, \quad w(0,\tau)=\tau^n, \quad \theta(0,\tau)=1 \\ &\qquad\quad w(\infty,\tau)=0, \quad \theta(\infty,\tau)=0 \end{aligned} \tag{1.8}$$

where n is an exponent.

CHAPTER 3

SOLUTION FOR THE VELOCITY DISTRIBUTION AND SKIN FRICTION FACTOR IN PRESENCE OF TRANSVERSE MAGNETIC FIELD

Here free convection flow of a visco-elastic fluid past an infinite vertical flat plate in presence of a transverse magnetic field has been discussed . The form of the boundary layer equations is invariant under the transformation, and the surface condition can therefore the applied on a transformed flat plate surface. The equations for the velocity field have been solved with the help of the Laplace transform technique. In the present chapter a problem on free convection boundary layer flow of a viscous incompressible and electrically condncting fluid along an isothermal vertical surface in the presence of a transverse magnetic field is proposed to be discussed. The results thus obtained are shown graphically for different values of the magnetic field parameter M, transpiration parameter(suction parameter) a, visco-elasticity parameter S, Grashof number Gr and the Prandtl's number Pr. Finally, effects of the above parameter on the zero-order perturbation velocity profile, first order perturbation velocity profile and temperature are also shown graphically.

3.1 Solution of momentum equation

Momentum equation is

$$\frac{\partial w}{\partial \tau} - a\frac{\partial w}{\partial \eta} = \frac{\partial^2 w}{\partial \eta^2} - S\frac{\partial^3 w}{\partial \eta^2 \partial \tau} - Mw + G_r \theta \tag{3.1}$$

the boundary conditions are,

$$\begin{aligned} &\tau \le 0, \quad w(\eta,\tau) = 0, \\ &\tau > 0, \quad w(0,\tau) = \tau^n, \\ &\tau > 0, \quad w(\infty,\tau) = 0. \end{aligned} \tag{3.2}$$

Equation (3.1) is a third order partial differential equation. For S=0, it reduces to equation governing the Newtonian fluid. Hence, the presence of elastic parameter increases the order of the governing equation from two to three. There are prescribed only two boundary conditions (3.2). Therefore it needs one boundary and

$$\overline{\theta}(0,\tau)=\frac{1}{q},\overline{\theta}(\infty,\tau)=0 \tag{2.6}$$

Where $\overline{\theta}$ is the transformed function of θ and q is the Laplace transformed independent variable.

The solution of equation (2.5) satisfying the boundary conditions (2.6) is obtained as

$$\overline{\theta}=\frac{e^{\frac{-\frac{\lambda P_r}{u_0}-\sqrt{\frac{\lambda^2 P_r^2}{u_0}+4P_r q}}{2}\eta}}{q} \tag{2.5b}$$

where $\frac{\lambda}{u_0}=a$

$$\overline{\theta}=\frac{e^{\frac{-aP_r-\sqrt{a^2P_r^2+4P_r q}}{2}\eta}}{q} \tag{2.5c}$$

$$\overline{\theta}=\frac{e^{-\sqrt{P_r}\left(\frac{a\sqrt{P_r}}{2}+\sqrt{\frac{a^2P_r+q}{4}}\right)\eta}}{q} \tag{2.5d}$$

$$\overline{\theta}=\frac{e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}}{q} \tag{2.7}$$

where $a\sqrt{P_r}=\alpha$

The temperature distribution θ is now obtained by taking the inverse Laplace transformation of equation (2.6). Since $\overline{\theta}$ has poles of different orders we apply Bromwich integral defined as

$$\theta = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} \bar{\theta}\, e^{St}\, dt$$

equation (2.7) to obtain

$$\theta(\eta,t) = \frac{1}{2\pi i}\int \frac{e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta+qt}}{q}\,dq \tag{2.7a}$$

where $\alpha^2 + q = k^2 \Rightarrow dq = 2k\,dk$

$$\theta(\eta,t) = \frac{1}{2\pi i}\int \frac{e^{-\sqrt{P_r}(\alpha+k)\eta+\left(k^2-\alpha^2\right)t}}{k^2-\alpha^2}\,2k\,dk \tag{2.7b}$$

$$\theta(\eta,t) = \frac{1}{2\pi i}\, e^{-\alpha\sqrt{P_r}\,\eta-\alpha^2 t}\int \frac{e^{-k\sqrt{P_r}\,\eta+k^2 t}}{k^2-\alpha^2}\,2k\,dk \tag{2.7c}$$

$$\theta(\eta,t) = \frac{1}{2\pi i}\, e^{-\alpha\sqrt{P_r}\,\eta-\alpha^2 t}\int\left(\frac{1}{k+\alpha}+\frac{1}{k-\alpha}\right) e^{-k\sqrt{P_r}\,\eta+k^2 t}\, dk \tag{2.7d}$$

$$\theta(\eta,t) = e^{-\sqrt{P_r}\,\eta-\alpha^2 t}\,\frac{1}{2}\left[e^{\alpha^2 t+\alpha\sqrt{P_r}\,\eta}\, erfc\left(\frac{\eta\sqrt{p_r}}{2\sqrt{t}}+\sqrt{\alpha\, t}\right)\right.$$
$$\left. + e^{\alpha^2 t-\alpha\sqrt{P_r}\,\eta}\, erfc\left(\frac{\eta\sqrt{p_r}}{2\sqrt{t}}-\sqrt{\alpha\, t}\right)\right] \tag{2.7e}$$

$$\theta(\eta,t) = \frac{1}{2} e^{-\alpha\sqrt{P_r}\,\eta}\left[e^{\alpha\sqrt{P_r}\,\eta}\, erfc\left(\frac{\eta\sqrt{P_r}}{2\sqrt{t}}+\sqrt{\alpha t}\right) + e^{-\alpha\sqrt{P_r}\,\eta}\, erfc\left(\frac{\eta\sqrt{P_r}}{2\sqrt{t}}-\sqrt{\alpha t}\right)\right] \tag{2.8}$$

For $\alpha=0$ i.e. $\lambda=0$ the solution in (2.8) is exactly the same given by Menold and Yang (1962).

For $P_r=1$, equation (2.8) turns into

$$\theta(\eta,t)=\frac{1}{2}e^{-\alpha\eta}\left[e^{\alpha\eta}erfc\left(\frac{\eta}{2\sqrt{t}}+\sqrt{\alpha t}\right)+e^{-\alpha\eta}erfc\left(\frac{\eta}{2\sqrt{t}}-\sqrt{\alpha t}\right)\right] \tag{2.9}$$

For equation (2.9) we get the rate of heat transfer defined by

$$q_w=-k\left.\frac{\partial t}{\partial y}\right|_{y=0}$$

$$q_w=-\frac{k\,u_0}{\nu}\Delta T\frac{\sqrt{P_r}}{\sqrt{(\pi T)}} \tag{2.10}$$

It follows from equation (2.10) that the rate of heat transfer is directly proportional to the square root of the Prandtl number P_r and inversely proportional to the square root of the time variable T.

CHAPTER 3

SOLUTION FOR THE VELOCITY DISTRIBUTION AND SKIN FRICTION FACTOR IN PRESENCE OF TRANSVERSE MAGNETIC FIELD

Here free convection flow of a visco-elastic fluid past an infinite vertical flat plate in presence of a transverse magnetic field has been discussed . The form of the boundary layer equations is invariant under the transformation, and the surface condition can therefore the applied on a transformed flat plate surface. The equations for the velocity field have been solved with the help of the Laplace transform technique. In the present chapter a problem on free convection boundary layer flow of a viscous incompressible and electrically conducting fluid along an isothermal vertical surface in the presence of a transverse magnetic field is proposed to be discussed. The results thus obtained are shown graphically for different values of the magnetic field parameter M. transpiration parameter(suction parameter) a, visco-elasticity parameter S, Grashof number Gr and the Prandtl's number Pr. Finally, effects of the above parameter ou the zero order perturbation velocity profile, first order perturbation velocity profile, heat transfer and temperature are also shown graphically.

3.1 Solution of momentum equation

Momentum equation is

$$\frac{\partial w}{\partial \tau} - a\frac{\partial w}{\partial \eta} = \frac{\partial^2 w}{\partial \eta^2} - S\frac{\partial^3 w}{\partial \eta^2 \partial \tau} - Mw + G_r\theta \tag{3.1}$$

the bouudary conditions are,

$$\begin{aligned} \tau \le 0, \quad & w(\eta,\tau) = 0, \\ \tau > 0, \quad & w(0,\tau) = \tau^n, \\ \tau > 0, \quad & w(\infty,\tau) = 0. \end{aligned} \tag{3.2}$$

Equation (3.1) is a third order partial differential equation. For S=0, it reduces to equation governing the Newtonian fluid. Hence, the presence of elastic parameter increases the order of the governing equation from two to three. There are prescribed only two boundary conditions (3.2). Therefore it needs one boundary condition more for a unique solution. Thus to overcome the difficulty we adopt the perturbation technique in which the elastic parameter S can be regarded as a small quantity. We therefore, follow the technique of Beard and Walters and assume the solution in the form $w=w_0+Sw_1$.

putting $w=w_0+Sw_1$ in equation (3.1)

$$\frac{\partial(w_0+Sw_1)}{\partial\tau}-a\frac{\partial(w_0+Sw_1)}{\partial\eta}=\frac{\partial^2(w_0+Sw_1)}{\partial\eta^2}-M(w_0+Sw_1)-S\frac{\partial^3(w_0+Sw_1)}{\partial\eta^2\partial\tau}+G_r\theta$$

equating the coefficient of S^0 and S, we obtain the following equations:

$$\frac{\partial w_0}{\partial\tau}-a\frac{\partial w_0}{\partial\eta}=\frac{\partial^2 w_0}{\partial\eta^2}-Mw_0+G_r\theta. \tag{3.3}$$

$$\frac{\partial w_1}{\partial\tau}-a\frac{\partial w_1}{\partial\eta}=\frac{\partial^2 w_1}{\partial\eta^2}-Mw_1-\frac{\partial^3 w_0}{\partial\eta^2\partial\tau}. \tag{3.4}$$

the boundary conditions (3.2) becomes

$$\begin{aligned}&\tau\le 0,\quad w_0(\eta,\tau)=0,\quad w_1(\eta,\tau)=0,\\ &\tau>0,\quad w_0(0,\tau)=\tau^n,\quad w_1(0,\tau)=0,\\ &\qquad w_0(\infty,\tau)=w_1(\infty,\tau)=0.\end{aligned} \tag{3.5}$$

Applying Laplace transform on (3.3), we get

$$\frac{\partial^2\overline{w}_0}{\partial\eta^2}+a\frac{\partial\overline{w}_0}{\partial\eta}-q\overline{w}_0-M\overline{w}_0+G_r\overline{\theta}=0.$$

$$\frac{\partial^2\overline{w}_0}{\partial\eta^2}+a\frac{\partial\overline{w}_0}{\partial\eta}-(q+M)\overline{w}_0=-G_r\overline{\theta}. \tag{3.6}$$

and the boundary conditions takes the from

$$\overline{w}_0(\eta,q)=0,\quad \overline{w}_0(0,q)=\frac{1}{q},\quad \overline{w}_0(\infty,q)=0 \quad . \tag{3.7}$$

$$(\overline{w}_0)_c = Ae^{\frac{-a+\sqrt{a^2+4(q+M)}}{2}\eta} + Be^{\frac{-a+\sqrt{a^2+4(q+M)}}{2}\eta}$$

where $D=\dfrac{d}{d\eta}$, $\alpha=\dfrac{a\sqrt{P_r}}{2}$ and $m=\dfrac{-a\pm\sqrt{a^2+4(q+M)}}{2}$

$$(\overline{w}_0)_p = \frac{1}{D^2+aD-(q+M)}(-G_r)\frac{e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}}{q}$$

$$=-\frac{G_r}{q}\frac{e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}}{\left\{\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\right\}^2+a\left\{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\right\}-(q+M)}$$

$$=-\frac{G_r}{P}\frac{e^{-\sqrt{P_r}\left(\alpha+\sqrt{a^2+q}\right)\eta}}{P_r\left(\alpha+\sqrt{\alpha^2+q}\right)^2-a\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)-(q+M)}$$

the solution of differential equation (3.6) is

$$\overline{w}_0 = Ae^{\frac{-a+\sqrt{a^2+4(q+M)}}{2}\eta} + Be^{\frac{-a+\sqrt{a^2+4(q+M)}}{2}\eta} - \frac{G_r}{q}\frac{e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}}{P_r\left(\alpha+\sqrt{\alpha^2+q}\right)^2-a\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)-(M+q)} \tag{3.8}$$

where $Q_r^2=\dfrac{M}{P_r-1}$

Using inverse Laplace transform on (3.8) we obtain

$$w_0 = L^{-1}\frac{e^{\frac{-a+\sqrt{a^2+4(q+M)}}{2}\eta}}{q} + \frac{G_r}{P_r-1}L^{-1}\frac{e^{\frac{-a+\sqrt{a^2+4(q+M)}}{2}\eta}}{q\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right]} \tag{3.9}$$

$$-\frac{G_r}{P_r-1}L^{-1}\frac{e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}}{q\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right]}$$

$$w_0(\eta,\tau) = I_0 + \frac{G_r}{P_r-1}I_1 - \frac{G_r}{P_r-1}I_2 \tag{3.10}$$

where

$$I_0 = L^{-1}\frac{e^{\frac{-a+\sqrt{a^2+4(q+M)}}{2}\eta}}{q} \tag{3.11}$$

$$I_1 = L^{-1}\frac{e^{\frac{-a+\sqrt{a^2+4(q+M)}}{2}\eta}}{q\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right]} \tag{3.12}$$

$$I_2 = L^{-1}\frac{e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}}{q\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right]} \tag{3.13}$$

The values of I_0 , I_1, and I_2 are obtained as

$$I_0=\frac{1}{2}e^{-\frac{1}{2}a\eta}\left[e^{\frac{\sqrt{a^2+4M}}{2}\eta}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\frac{\sqrt{\left(a^2+4M\right)\tau}}{2}\right)\right.$$

$$\left.+e^{-\frac{\sqrt{a^2+4M}}{2}\eta}erfc\left(\frac{\eta}{2\sqrt{\tau}}-\frac{\sqrt{\left(a^2+4M\right)\tau}}{2}\right)\right] \tag{3.14}$$

$$I_1=L^{-1}\frac{e^{\frac{-a+\sqrt{a^2+4(q+M)}}{2}\eta}}{q}\frac{1}{\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right]} \tag{3.15}$$

$$L^{-1}\frac{1}{\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right]}=\frac{e^{Q_r^2\tau}}{2Q_r}\left[\left(\alpha+Q_r\right)e^{2\alpha Q_r\tau}erfc\left\{\left(\alpha+Q_r\right)\sqrt{\tau}\right\}\right.$$

$$\left.-\left(\alpha-Q_r\right)e^{-2\alpha Q_r\tau}erfc\left\{\left(\alpha-Q_r\right)\sqrt{\tau}\right\}\right] \tag{3.16}$$

From (3.14) and (3.16) by convolution theorem

$$I_1=\int_0^{\tau}\frac{1}{2}e^{-\frac{a\eta}{2}}\left\{e^{\frac{\sqrt{a^2+4M}}{2}\eta}erfc\left(\frac{\eta}{2\sqrt{v}}+\frac{\sqrt{\left(a^2+4M\right)v}}{2}\right)\right.$$

$$\left.+e^{-\frac{\sqrt{a^2+4M}}{2}\eta}erfc\left(\frac{\eta}{2\sqrt{v}}-\frac{\sqrt{\left(a^2+4M\right)v}}{2}\right)\right\}\frac{e^{Q_r^2(\tau-v)}}{2Q_r}\left\{\left(\alpha+Q_r\right)e^{2\alpha Q_r(\tau-v)}\right. \tag{3.17}$$

$$\left.erfc\left(\left(\alpha+Q_r\right)\sqrt{\tau-v}\right)-\left(\alpha-Q_r\right)e^{-2\alpha Q_r(\tau-v)}erfc\left(\left(\alpha-Q_r\right)\sqrt{\tau-v}\right)\right\}dv$$

$$I_2 = L^{-1}\frac{e^{\frac{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}{2}}}{q\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right]}$$

$$\begin{aligned}
I_2 = {} & \frac{e^{-\sqrt{P_r}\alpha\eta}}{2Q_r^2\left(4\alpha^2-Q_r^2\right)}\left[Q_r^2 e^{-\sqrt{P_r}\alpha\eta}\, erfc\left(\frac{\sqrt{P_r}\eta}{2\sqrt{\tau}}-\alpha\sqrt{\tau}\right)\right. \\
& -\left(4\alpha^2-Q_r^2\right)e^{\sqrt{P_r}\alpha\eta}\, erfc\left(\frac{\sqrt{P_r}\eta}{2\sqrt{\tau}}+\alpha\sqrt{\tau}\right) \\
& +\left(\alpha+Q_r\right)\left(2\alpha-Q_r\right)e^{\left(Q_r^2\tau+2\alpha Q_r\tau+\sqrt{P_r}\alpha\eta+\sqrt{P_r}Q_r\eta\right)} erfc\left(\frac{\eta\sqrt{P_r}}{2\sqrt{\tau}}+\alpha\sqrt{\tau}+Q_r\sqrt{\tau}\right) \\
& \left.+\left(\alpha-Q_r\right)\left(2\alpha+Q_r\right)e^{\left(Q_r^2\tau-2\alpha Q_r\tau+\sqrt{P_r}\alpha\eta-\sqrt{P_r}Q_r\eta\right)} erfc\left(\frac{\eta\sqrt{P_r}}{2\sqrt{\tau}}+\alpha\sqrt{\tau}-Q_r\sqrt{\tau}\right)\right]
\end{aligned} \tag{3.18}$$

similarly applying Laplace transform on equation (3.4), we obtain

$$L\frac{\partial w_1}{\partial \tau} - aL\frac{\partial w_1}{\partial \eta} = L\frac{\partial^2 w_1}{\partial \eta^2} - LMw_1 - L\frac{\partial^3 w_0}{\partial \eta^2 \partial \tau}$$

$$q\overline{w}_1 - w_1(0,\eta) - a\frac{d\overline{w}_1}{d\eta} = \frac{d^2\overline{w}_1}{d\eta^2} - M\overline{w}_1 - \frac{d^2}{d\eta^2}\left[q\overline{w}_0 - w_0(0,\eta)\right]$$

where q is the Laplace transform parameter

$$\frac{d^2\overline{w}_1}{d\eta^2} + a\frac{d\overline{w}_1}{d\eta} - (M+q)\overline{w}_1 = q\frac{d^2\overline{w}_0}{d\eta^2} \tag{3.19}$$

substituting w_0 from (3.9) on (3.19) we get

$$\frac{d^2\overline{w}_1}{d\eta^2}+a\frac{d\overline{w}_1}{d\eta}-(M+q)\overline{w}_1$$

$$=q\frac{d^2}{d\eta^2}\left[\frac{1}{q}+\frac{G_r e^{-\frac{1}{2}\left(a+\sqrt{a^2+4(q+M)}\right)\eta}}{q(P_r-1)\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right]}-\frac{G_r}{q(P_r-1)}\frac{e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}}{\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right]}\right]$$

$$=\left[1+\frac{G_r}{(P_r-1)\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right]}\right]\left[-\frac{1}{2}\left\{a+\sqrt{a^2+4(q+M)}\right\}\right]^2 e^{-\frac{1}{2}\left\{a+\sqrt{a^2+4(q+M)}\right\}\eta}$$

$$-\frac{G_r}{(P_r-1)}\frac{P_r\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right]}e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}$$

$$\left(\overline{w}_1\right)_c=Ae^{-\frac{1}{2}\left(a+\sqrt{a^2+4(q+M)}\right)\eta}+Be^{\frac{1}{2}\left(-a+\sqrt{a^2+4(q+M)}\right)\eta}$$

$$\left(\overline{w}_1\right)_p=\left[1+\frac{G_r}{(P_r-1)\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right]}\right]\left[-\frac{a+\sqrt{a^2+4(q+M)}}{2}\right]^2$$

$$\frac{\eta e^{-\frac{\left\{a+\sqrt{a^2+4(q+M)}\right\}\eta}{2}}}{2\frac{a+\sqrt{a^2+4(q+M)}}{2}+a}-\frac{G_r}{P_r-1}\frac{P_r\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2}e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}$$

$$\frac{1}{\left[-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\right]^2+a\left[-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\right]-M-q}$$

$$=\frac{1}{4}\left[1+\frac{G_r}{(P_r-1)\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right]}\right]\frac{\left[a+\sqrt{a^2+4(q+M)}\right]^2}{-\sqrt{a^2+4(q+M)}}e^{-\frac{\left\{a+\sqrt{a^2+4(q+M)}\right\}\eta}{2}}$$

$$-\frac{G_r}{P_r-1}\frac{P_r\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2}\frac{e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}}{(P_r-1)\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right]}$$

$$\overline{w}_1=Ae^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}+Be^{\frac{-a+\sqrt{a^2+4(q+M)}}{2}\eta}-\frac{1}{4}\left[1+\frac{G_r}{(P_r-1)\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right\}}\right]$$

$$\frac{\left[a+\sqrt{a^2+4(q+M)}\right]^2}{\sqrt{a^2+4(q+M)}}\eta e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}-\frac{G_r}{(P_r-1)^2}\frac{P_r\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right\}^2}$$

$$e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta} \qquad (3.20)$$

applying boundary condition, $\overline{w}_1=0$ where $\eta\to\infty$. This gives B=0 . Again $\overline{w}_1=0$ where $\eta=0$.

$$0 = A - \frac{G_r P_r\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{(P_r-1)^2\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right\}^2}$$

$$A = \frac{G_r P_r\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{(P_r-1)^2\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right\}^2}$$

substituting A and B equation (3.20) we obtain

$$\overline{w}_1 = \frac{G_r P_r\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{(P_r-1)^2\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right\}^2} e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}$$

$$-\frac{\eta\left[a+\sqrt{a^2+4(q+M)}\right]^2}{4\sqrt{a^2+4(q+M)}}\left[1+\frac{G_r}{(P_r-1)\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right\}}\right] \quad (3.21)$$

$$e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta} - \frac{G_r}{(P_r-1)^2}\frac{P_r\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right\}^2} e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}$$

using inverse Laplace transform on (3.21) we obtain

$$w_1 = \frac{G_r P_r}{(P_r-1)^2} L^{-1} \frac{\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right\}^2} e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}$$

$$-\frac{\eta}{4} L^{-1} \frac{\left[a+\sqrt{a^2+4(q+M)}\right]^2}{\sqrt{a^2+4(q+M)}} \left[1+\frac{G_r}{(P_r-1)\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right\}}\right] e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}$$

$$-\frac{G_r P_r}{(P_r-1)^2} L^{-1} \frac{\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right\}^2} e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta} \tag{3.22}$$

$$\overline{w}_1 = \frac{G_r P_r}{(P_r-1)^2} J_1 - \frac{\eta}{4} J_2 - \frac{G_r P_r}{(P_r-1)^2} J_3 \tag{3.23}$$

where

$$J_1 = L^{-1} \frac{\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right]^2} e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta} \tag{3.24}$$

$$J_2 = L^{-1} \frac{\left[a+\sqrt{a^2+4(q+M)}\right]^2}{\sqrt{a^2+4(q+M)}} \left[1+\frac{G_r}{(P_r-1)\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right\}}\right] \tag{3.25}$$

$$e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}$$

$$J_3 = L^{-1}\frac{\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right\}^2} e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta} \tag{3.26}$$

Now

$$L^{-1}\frac{\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right]^2} = \frac{1}{2\pi i}\int \frac{e^{q\tau}\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right]^2} dq$$

putting

$\alpha^2 + q = k^2 \Rightarrow dq = 2k\,dq$

$q = k^2 - \alpha^2$

$$= \frac{1}{2\pi i}\int \frac{e^{\left(k^2-\alpha^2\right)\tau}(\alpha+k)^2}{\left[(\alpha+k)^2 - Q_r^2\right]^2} 2k\,dk$$

$$= \frac{1}{2\pi i}\int \frac{e^{\left(k^2-\alpha^2\right)\eta}(\alpha+k)^2}{(\alpha+k+Q_r)^2(\alpha+k-Q_r)^2} 2k\,dk \tag{3.27}$$

let

$$\frac{(\alpha+k)^2 k}{(\alpha+k+Q_r)^2(\alpha+k-Q_r)^2} = \frac{A}{(\alpha+k+Q_r)} + \frac{B}{(\alpha+k+Q_r)^2} + \frac{C}{(\alpha+k-Q_r)} + \frac{D}{(\alpha+k-Q_r)^2}$$

$$k(\alpha+k)^2 = A(\alpha+k-Q_r)^2(\alpha+k+Q_r)+B(\alpha+k-Q_r)^2$$
$$+C(\alpha+k-Q_r)(\alpha+k+Q_r)^2 + D(\alpha+k-Q_r)^2$$

where $k = Q_r - \alpha$

$$(Q_r-\alpha)(\alpha+Q_r-\alpha)^2 = D(\alpha+Q_r-\alpha+Q_r)^2$$

$$Q_r^2(Q_r-\alpha) = 4Q_r^2 D$$

$$D = \frac{Q_r-\alpha}{4} = -\frac{\alpha-Q_r}{4}$$

$$k = -Q_r - \alpha$$

$$(-\alpha-Q_r)(\alpha-\alpha-Q_r)^2 = B(\alpha-\alpha-Q_r-Q_r)^2$$

$$-(\alpha+Q_r)Q_r^2 = 4Q_r^2 B$$

$$B = -\frac{\alpha+Q_r}{4}$$

equating the coefficient of k^3

$$1 = A + C \Rightarrow A = 1 - C \quad \text{and} \quad k=0$$

$$0 = A(\alpha-Q_r)^2(\alpha+Q_r)+B(\alpha-Q_r)^2+C(\alpha-Q_r)(\alpha+Q_r)^2+D(\alpha+Q_r)^2$$

$$0 = A(\alpha-Q_r)^2(\alpha+Q_r)-\frac{(\alpha+Q_r)}{4}(\alpha-Q_r)^2+C(\alpha-Q_r)(\alpha+Q_r)^2-\frac{(\alpha-Q_r)}{4}(\alpha+Q_r)^2$$

$$0 = (\alpha-Q_r)^2(\alpha+Q_r)\left(A-\frac{1}{4}\right)+(\alpha-Q_r)(\alpha+Q_r)^2\left(c-\frac{1}{4}\right)$$

$$0 = (\alpha-Q_r)\left(A-\frac{1}{4}\right)+(\alpha+Q_r)\left(c-\frac{1}{4}\right)$$

$$0=(\alpha-Q_r)\left(1-C-\frac{1}{4}\right)+(\alpha+Q_r)\left(c-\frac{1}{4}\right)$$

$$0=-C(\alpha-Q_r)+\frac{3}{4}(\alpha-Q_r)+C(\alpha+Q_r)-\frac{1}{4}(\alpha+Q_r)$$

$$0=C(-\alpha+Q_r+\alpha+Q_r)-\frac{1}{4}(-3\alpha+3Q_r+\alpha+Q_r)$$

$$2Q_rC=\frac{1}{4}(4Q_r-2\alpha)$$

$$C=\frac{4Q_r-2\alpha}{4.2Q_r}=\frac{2Q_r-\alpha}{4Q_r}$$

$$A=1-\frac{-\alpha+2Q_r}{4Q_r}=\frac{4Q_r+\alpha-2Q_r}{4Q_r}=\frac{2Q_r+\alpha}{4Q_r}$$

Equation (3.24) is obtain

$$L^{-1}\frac{\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right]^2}=\frac{1}{2\pi i}\int e^{\left(k^2-\alpha^2\right)\tau}\left[\frac{2Q_r+\alpha}{4Q_r}\frac{1}{k+\alpha+Q_r}-\frac{\alpha+Q_r}{4}\frac{1}{(k+\alpha+Q_r)^2}\right.$$
$$\left.-\frac{\alpha-2Q_r}{4Q_r}\frac{1}{k+\alpha-Q_r}-\frac{\alpha-Q_r}{4}\frac{1}{(k+\alpha-Q_r)^2}\right]dk$$

$$=\frac{2Q_r+\alpha}{4Q_r}\frac{e^{-\alpha^2\tau}}{\pi i}\int\frac{e^{k^2\tau}}{k+\alpha+Q_r}dk-\frac{\alpha+Q_r}{4}\frac{e^{-\alpha^2\tau}}{\pi i}\int\frac{e^{k^2\tau}}{(k+\alpha+Q_r)^2}dk$$
$$-\frac{\alpha-2Q_r}{4Q_r}\frac{e^{-\alpha^2\tau}}{\pi i}\int\frac{e^{k^2\tau}}{k+\alpha-Q_r}dk-\frac{\alpha-Q_r}{4}\frac{e^{-\alpha^2\tau}}{\pi i}\int\frac{e^{k^2\tau}}{(k+\alpha-Q_r)^2}dk$$

$$=\frac{2Q_r+\alpha}{4Q_r}e^{-\alpha^2\tau}e^{(\alpha+Q_r)^2\tau}erfc\left(\left(\alpha+Q_r\right)\sqrt{\tau}\right)+\frac{\alpha+Q_r}{4}e^{-\alpha^2\tau}\left[e^{(\alpha+Q_r)^2\tau}2\left(\alpha+Q_r\right)\tau\right.$$

$$\left.erfc\left(\left(\alpha+Q_r\right)\sqrt{\tau}\right)-2\sqrt{\frac{\tau}{\pi}}\right]-\frac{\alpha-2Q_r}{4Q_r}e^{-\alpha^2\tau}e^{(\alpha-Q_r)^2\tau}erfc\left(\left(\alpha-Q_r\right)\sqrt{\tau}\right)+\frac{\alpha-Q_r}{4}e^{-\alpha^2\tau}$$

$$\left[e^{(\alpha-Q_r)^2\tau}2\left(\alpha-Q_r\right)\tau\, erfc\left(\left(\alpha-Q_r\right)\sqrt{\tau}\right)-2\sqrt{\frac{\tau}{\pi}}\right]$$

$$=\frac{2Q_r+\alpha}{4\theta_r}e^{2\alpha Q_r\tau+Q_r^2\tau}erfc\left(\left(\alpha+Q_r\right)\sqrt{\tau}\right)-\frac{\alpha-2Q_r}{4Q_r}e^{-2\alpha Q_r\tau+Q_r^2\tau}erfc\left(\left(\alpha-Q_r\right)\sqrt{\tau}\right)$$

$$+\frac{\alpha+Q_r}{4}\left[e^{2\alpha Q_r\tau+Q_r^2\tau}2\left(\alpha+Q_r\right)\tau erfc\left(\left(\alpha+Q_r\right)\sqrt{\tau}\right)\right]+\frac{\alpha-Q_r}{4}\left[e^{-2\alpha Q_r\tau+Q_r^2\tau}\right. \tag{3.28}$$

$$\left.2\left(\alpha-Q_r\right)\tau\, erfc\left(\left(\alpha-Q_r\right)\sqrt{\tau}\right)\right]-\alpha\sqrt{\frac{\tau}{\pi}}e^{-\alpha^2\tau}$$

Again

$$L^{-1}e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}=\frac{1}{2\pi i}\int e^{q\tau-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}dq$$

$$=\frac{e^{-\frac{a\eta}{2}}}{2\pi i}\int e^{q\tau-\sqrt{\frac{a^2}{4}+q+M}\,\eta}dq$$

putting

$$\frac{a^2}{4}+q+M=k^2\Rightarrow dq=2kdk\quad and\ q=k^2-\left(\frac{a^2}{4}+M\right)$$

$$=\frac{e^{-\frac{a\eta}{2}}}{2\pi i}\int e^{k\tau^2-\left(\frac{a^2}{4}+M\right)\tau-k\eta}\,2k\,dk$$

$$=\frac{e^{-\frac{a\eta}{2}-\left(\frac{a^2}{4}+M\right)\tau}}{\pi i}\int k e^{k^2\tau-k\eta}dk$$

$$=\frac{1}{2}e^{-\frac{a\eta}{2}-\left(\frac{a^2}{4}+M\right)\tau}\frac{1}{\sqrt{\pi\tau^3}}\eta e^{-\frac{\eta^2}{4\tau}} \tag{3.29}$$

From (3.28) and (3.29) by convolution theorem

$$J_1=\int_0^\tau\left\{\frac{2Q_r+\alpha}{4Q_r}e^{\left(2\alpha Q_r+Q_r^2\right)v}erfc\left(\left(\alpha+Q_r\right)\sqrt{v}\right)-\frac{\alpha-2Q_r}{4Q_r}e^{\left(-2\alpha Q_r+Q_r^2\right)v}erfc\left(\left(\alpha-Q_r\right)\sqrt{v}\right)\right.$$
$$+\frac{\alpha+Q_r}{4}\left[e^{\left(2\alpha Q_r+Q_r^2\right)v}2\left(\alpha+Q_r\right)v\,erfc\left(\left(\alpha+Q_r\right)\sqrt{v}\right)\right]-\frac{\alpha-Q_r}{4}\left[e^{\left(-2\alpha Q_r+Q_r^2\right)v}2\left(\alpha-Q_r\right)v\right.$$
$$\left.\left.erfc\left(\left(\alpha-Q_r\right)\sqrt{v}\right)\right]-\alpha e^{-\alpha^2 v}\sqrt{\frac{v}{\pi}}\right\}\frac{1}{2}e^{-\frac{a\eta}{2}-\frac{a^2+4M}{4}(\tau-v)}\frac{\eta}{\sqrt{\pi}\sqrt{(\tau-v)^3}}e^{-\frac{\eta^2}{4(\tau-v)}}dv \tag{3.30}$$

$$J_2=L^{-1}\frac{\left\{a+\sqrt{a^2+4(q+M)}\right\}^2}{\sqrt{a^2+4(q+M)}}\left[1+\frac{G_r}{(P_r-1)\left\{\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2\right\}}\right]e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}$$

$$= L^{-1}\frac{\left\{a+\sqrt{a^2+4(q+M)}\right\}^2}{\sqrt{a^2+4(q+M)}}e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}+\frac{G_r}{P_r-1}L^{-1}\frac{\left\{a+\sqrt{a^2+4(q+M)}\right\}^2}{\sqrt{a^2+4(q+M)}}$$

$$\frac{1}{\left(a+\sqrt{a^2+q}\right)^2-Q_r^2}e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}$$

$$J_2=J_{21}+J_{22} \qquad (3.31)$$

where

$$J_{21}=L^{-1}\frac{\left\{a+\sqrt{a^2+4(q+M)}\right\}^2}{\sqrt{a^2+4(q+M)}}e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta} \qquad (3.32)$$

$$=\frac{1}{2\pi i}\int\frac{\left\{a+\sqrt{a^2+4(q+M)}\right\}^2}{\sqrt{a^2+4(q+M)}}e^{q\tau-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}dq$$

$$=\frac{1}{2\pi i}\int\frac{\left\{a+2\sqrt{\frac{a^2}{4}+q+M}\right\}^2}{2\sqrt{\frac{a^2}{4}+q+M}}e^{q\tau-\frac{a\eta}{2}-\sqrt{\frac{a^2}{4}+q+M}\;\eta}dq$$

Put $\frac{a^2}{4}+(q+M)=k^2 \Rightarrow q=k^2-\left(\frac{a^2}{4}+M\right)\Rightarrow dq=2kdk$

$$=\frac{1}{2\pi i}\int\frac{(a+2k)^2}{2k}e^{k^2\tau-\left(\frac{a^2}{4}+M\right)\tau-\frac{a\eta}{2}-k\,\eta}2k\,dk$$

$$= \frac{e^{-\frac{a\eta}{2}-\frac{a^2+4M}{4}\tau}}{2\pi i}\int a^2 e^{k^2\tau-k\eta}dk + \frac{e^{-\frac{a\eta}{2}-\frac{a^2+4M}{4}\tau}}{2\pi i}\int 4ake^{k^2\tau-k\eta}dk$$

$$+\frac{4e^{-\frac{a\eta}{2}-\frac{a^2+4M}{4}\tau}}{2\pi i}\int k^2 e^{k^2\tau-k\eta}dk$$

$$J_{21} = \frac{e^{-\frac{a\eta}{2}-\frac{a^2+4M}{4}\tau}}{2}\left[\frac{a^2}{\sqrt{\pi\tau}}e^{\frac{-\eta^2}{4\tau}} + \frac{4a}{2\sqrt{\pi\tau^3}}\eta e^{\frac{-\eta^2}{4\tau}} + \frac{4}{2\sqrt{\pi\tau^3}}\left(\frac{\eta^2}{2\tau}-1\right)e^{\frac{-\eta^2}{4\tau}}\right]$$

$$J_{21} = \frac{e^{-\frac{a\eta}{2}-\frac{a^2+4M}{4}\tau-\frac{\eta^2}{4\tau}}}{2\sqrt{\pi\tau}}\left[a^2+\frac{2a\eta}{\tau}+\frac{2}{\tau}\left(\frac{\eta^2}{2\tau}-1\right)\right] \qquad (3.33)$$

where

$$J_{22} = \frac{G_r}{P_r-1}L^{-1}\frac{\left\{a+\sqrt{a^2+4(q+M)}\right\}^2}{\sqrt{a^2+4(q+M)}}e^{-\frac{a+\sqrt{a^2+4(q+M)}}{2}\eta}\frac{1}{\left(\alpha+\sqrt{\alpha^2+q}\right)^2-Q_r^2} \qquad (3.34)$$

From (3.15) and (3.33) by convolution theorem,

$$J_{22} = \frac{G_r}{P_r-1}\int_0^\tau \frac{e^{-\frac{a\eta}{2}-\frac{a^2+4M}{4}v-\frac{\eta^2}{4v}}}{2\sqrt{\pi v}}\left\{a^2+\frac{2a\eta}{v}+\frac{2}{v}\left(\frac{\eta^2}{2v}-1\right)\right\}\frac{e^{Q_r^2(\tau-v)}}{2Q_r}\{(\alpha+Q_r)$$

$$e^{2\alpha Q_r^2(\tau-v)}erfc\left((\alpha+Q_r)\sqrt{\tau-v}\right)-(\alpha-Q_r)e^{-2\alpha Q_r^2(\tau-v)}erfc\left((\alpha-Q_r)\sqrt{\tau-v}\right)\Big\}$$

$$J_2 = J_{21}+J_{22}$$

Equation (3.26) is obtain,

$$J_3 = L^{-1} \frac{\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right]^2} e^{-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta}$$

$$= \frac{1}{2\pi i} \int \frac{\left(\alpha+\sqrt{\alpha^2+q}\right)^2}{\left[\left(\alpha+\sqrt{\alpha^2+q}\right)^2 - Q_r^2\right]^2} e^{q\tau-\sqrt{P_r}\left(\alpha+\sqrt{\alpha^2+q}\right)\eta} dq$$

where $\alpha^2 + q = k^2 \Rightarrow q = k^2 - \alpha^2 \Rightarrow dq = 2k\,dk$.

$$= \frac{1}{2\pi i} \int \frac{(\alpha+k)^2}{\left[(\alpha+k)^2 - Q_r^2\right]^2} e^{\left(k^2-\alpha^2\right)\tau-\sqrt{P_r}(\alpha+k)\eta} 2k.dk$$

$$= \frac{1}{2\pi i} e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta} \int \frac{2k(\alpha+k)^2 e^{k^2\tau-\sqrt{P_r}k\eta}}{(\alpha+k+Q_r)^2(\alpha+k-Q_r)^2} dk \qquad (3.35)$$

let

$$\frac{2k(\alpha+k)^2}{(\alpha+k+Q_r)^2(\alpha+k-Q_r)^2} = \frac{A}{(\alpha+k+Q_r)} + \frac{B}{(\alpha+k+Q_r)^2} + \frac{C}{(\alpha+k-Q_r)} + \frac{D}{(\alpha+k-Q_r)^2}$$

$$2k(\alpha+k)^2 = A(\alpha+k+Q_r)(\alpha+k-Q_r)^2 + B(\alpha+k-Q_r)^2$$
$$+C(\alpha+k-Q_r)(\alpha+k+Q_r)^2 + D(\alpha+k+Q_r)^2$$

where $k=-\alpha-Q_r$,

$$2(-\alpha-Q_r)(\alpha-\alpha-Q_r)^2=B(\alpha-\alpha-Q_r-Q_r)^2$$

$$-2(\alpha+Q_r)Q_r^2=4Q_r^2B\Rightarrow B=-\frac{\alpha+Q_r}{2}$$

where $k=-\alpha+Q_r$,

$$2(-\alpha+Q_r)(\alpha-\alpha+Q_r)^2=D(\alpha+Q_r-\alpha+Q_r)^2$$

$$-2(-\alpha+Q_r)Q_r^2=4Q_r^2D\Rightarrow D=-\frac{\alpha-Q_r}{2}$$

equating the coefficient of k^3,

$$2=A+C\Rightarrow A=2-C$$

when k=0,

$$0=A(\alpha+Q_r)(\alpha-Q_r)^2+B(\alpha-Q_r)^2+C(\alpha-Q_r)(\alpha+Q_r)^2+D(\alpha+Q_r)^2$$

$$0=A(\alpha+Q_r)(\alpha-Q_r)^2-\frac{(\alpha+Q_r)}{2}(\alpha-Q_r)^2+C(\alpha-Q_r)(\alpha+Q_r)^2-\frac{(\alpha-Q_r)}{2}(\alpha+Q_r)^2$$

$$0=A(\alpha-Q_r)-\frac{(\alpha-Q_r)}{2}+C(\alpha+Q_r)-\frac{(\alpha+Q_r)}{2}$$

$$0=(2-C)(\alpha-Q_r)-\frac{(\alpha-Q_r)}{2}+C(\alpha+Q_r)-\frac{(\alpha+Q_r)}{2}$$

$$-C(\alpha-Q_r)+2(\alpha-Q_r)-\frac{(\alpha-Q_r)}{2}+C(\alpha+Q_r)-\frac{(\alpha+Q_r)}{2}=0$$

$$C\left(-\alpha+Q_r+\alpha+Q_r\right)=-2\left(\alpha-Q_r\right)+\frac{\left(\alpha-Q_r\right)}{2}+\frac{\left(\alpha+Q_r\right)}{2}$$

$$2Q_rC=\frac{-4\alpha+4Q_r+\alpha-Q_r+\alpha+Q_r}{2}=\frac{-2\alpha+4Q_r}{2}=-\left(\alpha-2Q_r\right)$$

$$C=-\frac{\alpha-2Q_r}{2Q_r}$$

$$A=2+\frac{\alpha+2Q_r}{2Q_r}=\frac{4Q_r+\alpha-2Q_r}{2Q_r}=\frac{2Q_r+\alpha}{2Q_r}$$

$$J_3=A\frac{e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta}}{2\pi i}\int\frac{e^{k^2\tau-\sqrt{P_r}\eta k}}{k+\alpha+Q_r}dk+B\frac{e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta}}{2\pi i}\int\frac{e^{k^2\tau-\sqrt{P_r}\eta k}}{\left(k+\alpha+Q_r\right)^2}dq$$

$$+C\frac{e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta}}{2\pi i}\int\frac{e^{k^2\tau-\sqrt{P_r}\eta k}}{k+\alpha-Q_r}dq+D\frac{e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta}}{2\pi i}\int\frac{e^{k^2\tau-\sqrt{P_r}\eta k}}{\left(k+\alpha-Q_r\right)^2}dq$$

$$=\frac{A}{2}e^{\alpha^2\tau-\sqrt{P_r}\alpha\eta}e^{\left(\alpha+Q_r\right)^2+\left(\alpha+Q_r\right)\eta\sqrt{P_r}}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\left(\alpha+Q_r\right)\sqrt{\tau}\right)-B\frac{e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta}}{2}$$

$$\left[e^{\left(\alpha+Q_r\right)^2\tau+\left(\alpha+Q_r\right)\sqrt{P_r}\eta}\left\{2\left(\alpha+Q_r\right)\tau+\eta\sqrt{P_r}\right\}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\left(\alpha+Q_r\right)\sqrt{\tau}\right)-2\sqrt{\frac{\tau}{\pi}}e^{-\frac{\eta^2}{4\tau}}\right]$$

$$+\frac{C}{2}e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta}e^{\left(\alpha-Q_r\right)^2\tau+\left(\alpha-Q_r\right)\eta\sqrt{P_r}}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\left(\alpha-Q_r\right)\sqrt{\tau}\right)-\frac{D}{2}e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta}$$

$$\left[e^{\left(\alpha-Q_r\right)^2\tau+\left(\alpha-Q_r\right)\sqrt{P_r}\eta}\left\{2\left(\alpha-Q_r\right)\tau+\eta\sqrt{P_r}\right\}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\left(\alpha-Q_r\right)\sqrt{\tau}\right)-2\sqrt{\frac{\tau}{\pi}}e^{-\frac{\eta^2}{4\tau}}\right]$$

$$J_3=\frac{\alpha+2Q_r}{4Q_r}e^{-\alpha^2\tau+\sqrt{P_r}\alpha\eta+\alpha^2\tau+2\alpha\tau Q_r+Q_r^2\tau+\alpha\eta\sqrt{P_r}+Q_r\eta\sqrt{P_r}}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\alpha\sqrt{\tau}+Q_r\sqrt{\tau}\right)$$

$$+\frac{\alpha+Q_r}{4}e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta+\alpha^2\tau+2\alpha\tau Q_r+Q_r^2\tau+\alpha\eta\sqrt{P_r}+Q_r\eta\sqrt{P_r}}\left\{2(\alpha+Q_r)\tau+\eta\sqrt{P_r}\right\}$$

$$erfc\left(\frac{\eta}{2\sqrt{\tau}}+\alpha\sqrt{\tau}+Q_r\sqrt{\tau}\right)+\frac{\alpha+Q_r}{4}e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta}(-2)\sqrt{\frac{\tau}{\pi}}e^{-\frac{\eta^2}{4\tau}}-\frac{\alpha-2Q_r}{4Q_r}$$

$$e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta+\alpha^2\tau-2\alpha\tau Q_r+Q_r^2\tau+\alpha\eta\sqrt{P_r}-Q_r\eta\sqrt{P_r}}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\alpha\sqrt{\tau}-Q_r\sqrt{\tau}\right)$$

$$+\frac{\alpha-Q_r}{4}e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta+\alpha^2\tau-2\alpha\tau Q_r+Q_r^2\tau+\alpha\eta\sqrt{P_r}-Q_r\eta\sqrt{P_r}}\left\{2(\alpha-Q_r)\tau-\eta\sqrt{P_r}\right\}$$

$$erfc\left(\frac{\eta}{2\sqrt{\tau}}+\alpha\sqrt{\tau}-Q_r\sqrt{\tau}\right)+\frac{\alpha-Q_r}{4}e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta}(-2)\sqrt{\frac{\tau}{\pi}}e^{-\frac{\eta^2}{4\tau}}$$

$$=\left[\frac{\alpha+2Q_r}{4Q_r}+\frac{\alpha+Q_r}{4}\left(2\alpha\tau+2Q_r\tau+\eta\sqrt{P_r}\right)\right]e^{2\alpha\tau Q_r+Q_r\eta\sqrt{P_r}+Q_r^2\tau}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\alpha\sqrt{\tau}+Q_r\sqrt{\tau}\right)$$

$$+\left[-\frac{\alpha-2Q_r}{4Q_r}+\frac{\alpha-Q_r}{4}\left(2\alpha\tau-2Q_r\tau-\eta\sqrt{P_r}\right)\right]e^{-2\alpha\tau Q_r+Q_r\eta\sqrt{P_r}-Q_r^2\tau}$$

$$erfc\left(\frac{\eta}{2\sqrt{\tau}}+\alpha\sqrt{\tau}-Q_r\sqrt{\tau}\right)-\left[\frac{\alpha+Q_r}{2}+\frac{\alpha-Q_r}{2}\right]\sqrt{\frac{\tau}{\pi}}e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta-\frac{\eta^2}{4\tau}}$$

$$=\left[\frac{\alpha+2Q_r}{4Q_r}+\frac{\alpha+Q_r}{4}\left(2\alpha\tau+2Q_r\tau+\eta\sqrt{P_r}\right)\right]e^{2\alpha\tau Q_r+Q_r\eta\sqrt{P_r}+Q_r^2\tau}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\alpha\sqrt{\tau}+Q_r\sqrt{\tau}\right)$$

$$-\left[\frac{\alpha-2Q_r}{4Q_r}-\frac{\alpha-Q_r}{4}\left(2\alpha\tau-2Q_r\tau-\eta\sqrt{P_r}\right)\right]e^{-2\alpha\tau Q_r-Q_r\eta\sqrt{P_r}+Q_r^2\tau}$$

$$erfc\left(\frac{\eta}{2\sqrt{\tau}}+\alpha\sqrt{\tau}-Q_r\sqrt{\tau}\right)-\alpha\sqrt{\frac{\tau}{\pi}}e^{-\alpha^2\tau-\sqrt{P_r}\alpha\eta-\frac{\eta^2}{4\tau}} \quad (3.36)$$

substituting I_0, I_1 and I_2 , equation (3.9) we obtained

$$w_0 = \frac{e^{-\frac{a\eta}{2}}}{2}\left[e^{\frac{\sqrt{a^2+4M}}{2}\eta} erfc\left(\frac{\eta}{2\sqrt{\tau}} + \frac{\sqrt{\left(a^2+4M\right)\tau}}{2} \right) + e^{-\frac{\sqrt{a^2+4M}}{2}\eta} \right.$$

$$\left. erfc\left(\frac{\eta}{2\sqrt{\tau}} - \frac{\sqrt{\left(a^2+4M\right)\tau}}{2} \right) \right] + \frac{Ge^{-\frac{a\eta}{2}}}{4\left(P_r-1\right)Q_r}\int_0^{\tau}\left[e^{\frac{\sqrt{a^2+4M}}{2}\eta} erfc\left(\frac{\eta}{2\sqrt{v}} + \frac{\sqrt{\left(a^2+4M\right)v}}{2} \right) \right.$$

$$\left. + e^{-\frac{\sqrt{a^2+4M}}{2}\eta} erfc\left(\frac{\eta}{2\sqrt{\tau}} - \frac{\sqrt{\left(a^2+4M\right)}}{2}\eta \right) \right]\left[\left(\alpha+Q_r\right)e^{2\alpha Q_r(\tau-v)} erfc\left\{ \left(\alpha+Q_r\right)\sqrt{\tau-v} \right\} \right.$$

$$\left. - \left[\left(\alpha-Q_r\right)e^{-2\alpha Q_r(\tau-v)} erfc\left\{ \left(\alpha-Q_r\right)\sqrt{\tau-v} \right\} \right] e^{Q_r^2(\tau-v)} dv - \frac{Ge^{-\sqrt{P_r}\alpha\eta}}{2Q_r\left(P_r-1\right)\left(4\alpha^2-Q_r^2\right)} \right.$$

$$\left[Q_r^2 e^{-\sqrt{P_r}\alpha\eta} erfc\left(\frac{\sqrt{P_r}\eta}{2\sqrt{\tau}} - \alpha\sqrt{\tau} \right) - \left(4\alpha^2 - Q_r^2\right)e^{\sqrt{P_r}\alpha\eta} erfc\left(\frac{\sqrt{P_r}\eta}{2\sqrt{\tau}} + \alpha\sqrt{\tau} \right) \right] \quad (3.37)$$

substituting J_1, J_2 and J_3 equation (3.22) we obtain

$$w_1 = \frac{G_r P_r \eta e^{-\frac{a\eta}{2}}}{2(P_r-1)^2\sqrt{\pi}} \int_0^{\tau} \left[\left[\left\{ \frac{2Q_r+\alpha}{4Q_r} + \frac{(\alpha+Q_r)^2}{2} v \right\} e^{(2\alpha Q_r + Q_r^2)v} erfc\left(\alpha\sqrt{v} + Q_r\sqrt{v}\right) \right.\right.$$

$$\left. - \left\{ \frac{(\alpha-2Q_r)}{4Q_r} - \frac{(\alpha-Q_r)^2}{2} v \right\} e^{(-2\alpha Q_r + Q_r^2)v} erfc\left(\alpha\sqrt{v} - Q_r\sqrt{v}\right) - \alpha e^{-\alpha^2 v}\sqrt{\frac{v}{\pi}} \right]$$

$$\frac{e^{-\frac{(a^2+4M)}{4}(\tau-v) - \frac{\eta^2}{4(\tau-v)}}}{\sqrt{(\tau-v)^3}} dv - \frac{\eta}{8\sqrt{\pi\tau}} e^{-\frac{a\eta}{2} - \frac{a^2+4M}{4}\tau - \frac{\eta^2}{4\tau}} \left\{ a^2 + \frac{2a\eta}{\tau} + \frac{2}{\tau}\left(\frac{\eta^2}{2\tau} - 1\right) \right\}$$

$$- \frac{\eta G_r e^{-\frac{a\eta}{2}}}{2.8(P_r-1)\sqrt{\pi} Q_r} \int_0^{\tau} \left[\frac{e^{-\frac{a^2+4M}{4}v - \frac{\eta^2}{4v}}}{\sqrt{v}} \left\{ a^2 + \frac{2a\eta}{v} + \frac{2}{v}\left(\frac{\eta^2}{2v} - 1\right) \right\} \right] \left\{ (\alpha+Q_r) e^{2\alpha Q_r(\tau-v)} \right.$$

$$\left. \left. erfc\left(\alpha\sqrt{\tau-v} + Q_r\sqrt{\tau-v}\right) - (\alpha-Q_r) e^{-2\alpha Q_r(\tau-v)} erfc\left(\alpha\sqrt{\tau-v} - Q_r\sqrt{\tau-v}\right) \right\} e^{Q_r^2(\tau-v)} \right] dv$$

$$- \frac{G_r P_r}{(P_r-1)^2} \left[\left[\left\{ \frac{\alpha+2Q_r}{4Q_r} + \frac{(\alpha+Q_r)\left(2\alpha\tau + 2Q_r\tau + \sqrt{P_r}\eta\right)}{4} \right\} e^{2\alpha\tau Q_r + Q_r^2\tau + Q_r\eta\sqrt{P_r}} \right.\right.$$

$$erfc\left(\frac{\eta}{2\sqrt{\tau}} + \alpha\sqrt{\tau} + Q_r\sqrt{\tau}\right) - \left\{ \frac{\alpha-2Q_r}{4Q_r} - \frac{(\alpha-Q_r)\left(2\alpha\tau - 2Q_r\tau - \sqrt{P_r}\eta\right)}{4} \right\} e^{-2\alpha\tau Q_r + Q_r^2\tau - Q_r\eta\sqrt{P_r}}$$

$$\left. erfc\left(\frac{\eta}{2\sqrt{\tau}} + \alpha\sqrt{\tau} - Q_r\sqrt{\tau}\right) - \alpha\sqrt{\frac{\tau}{\pi}} e^{-\alpha^2\tau - \alpha\sqrt{P_r}\eta - \frac{\eta^2}{4\tau}} \right] \tag{3.38}$$

CHAPTER 4

Results and discussions

In the present investigation, free convection boundary layer flow of a visco-elastic fluid past an infinite vertical porous plate in the presence of a variable transverse magnetic field is solved numerically by the local non-similarity method.

Figures 1(a), 1(b) and 1(c) represent respectively the zero-order perturbation velocity profile, first order perturbation velocity profile and temperature profile against η for Prandtl nnmber Pr =0.1, 0.3, 0.5, 0.72 & 1.0 while Grashof number Gr=2.0, magnetic field parameter M=1.0, transpiration parameter a=0.5 and visco-elasticity parameters S=0.4.

From Figure 1(a) it can be concluded that the zero-order perturbation velocity profile decreases as the values of the Prandtl number Pr increases in the region $\eta \in (0.0, 25)$. Near the surface of the plate velocity profile increases, becomes maximnm and then decreases and finally takes asymptotic value.

From Figure 1(b) it is observed that near the surface of the plate the first order perturbation velocity profile decreases becomes minimum and then increases. We also observe that the velocity profile decreases as prandtl nnmber Pr increases.

From Figure 1(c) we see that the temperature profile is large near the surface of the plate and decreases away from the plate and finally takes asymptotic value. Here we also see that temperature profile decreases with the increases of the prandtl nnmber Pr.

Figures 2(a), 2(b) and 2(c) represent respectively the zero-order perturbation velocity profile, first order perturbation velocity profile and temperature profile against η for magnetic field parameter M =0.1, 0.2, 0.3, 0.4 & 0.5 while Grashof number Gr=2.0, Praudtl number Pr =0.72, trauspiration parameter a=0.5 and visco-elasticity parameters S=0.4.

From Figure 2(a) we observe that the zero-order perturbation velocity profile decreases owing to increase in the value of magnetic field parameter M. Near the surface of the plate velocity profile increases becomes maximum and then decreases and finally takes asymptotic values. From the same figure we may also conclude that an increase in the magnetic field parameter decreases the velocity profile more rapidly.

From Figure 2(b) we see that the magnetic field parameter M increases as first order perturbation velocity profile decreases. We also observed that near the surface of the plate the velocity profile decreases becomes minimum and then increases.

From Figure 2(c) we see that the temperature profile remains unchanged for different values of the Magnetic field parameter M. For $0.1 \le M \le 0.5$ the temperature profile becomes maximum at the surface of the plate then decreases away from the plate and finally takes asymptotic values at $\eta = 10.0$.

Figures 3(a), 3(b) and 3(c) represent respectively the zero-order perturbation velocity profile, first order perturbation velocity profile and temperature profile against η for Grashof number Gr=5.0, 4.0, 3.0, 2.0 & 1.0 while Prandtl number Pr =1.0, magnetic field parameter M=4.0, transpiration parameter a=0.8 and visco-elasticity parameters S=0.4.

From Figure 3(a) we see that the zero-order perturbation velocity profile increases with the increase of the Grashof number Gr. we also observe that near the surface of the plate the velocity profile increases becomes maximum and then decreases and finally takes the asymptotic value. It is also observed that the velocity profile moves away from the plate as the Grashof number Gr increases.

From Figure 3(b) it is observed that the fisrt order perturbation velocity profile increases as Grashof number increase. We also observe that for different values of Grashof number Gr near the surface of the plate velocity profile is maximum then decreases and finally takes asymptotic value.

From Figure 3(c) we may observe that the effect of Grashof number on temperature distribution is constant.

Figures 4(a), 4(b) and 4(c) represent respectively the zero-order perturbation velocity profile, first order perturbation velocity profile and temperature profile against η for transpiration parameter a =1.5. 2.0, 2.5, 3.0 & 3.5 while Prandtl number Pr =0.1, Grashof number Gr=2.0, magnetic field parameter M=1.0 and visco-elasticity parameters S=0.4.

From Figure 4(a) it can be concluded that the zero-order perturbation velocity profile decreases as the values of the transpiration parameter a increases in the region $\eta \in (0.0, 8.0)$. Near the surface of the plate velocity profile increases, becomes maximum and then decreases and finally takes asymptotic value.

From Figure 4(b) we see that the transpiration parameter a increases as the first order perturbation velocity profile decreases.

From Figure 4(c) we see that the temperature profile is large near the surface of the plate and decreases away from the plate and finally takes asymptotic value. Here we also see that temperature profile decreases with the increases of the transpiration parameter a.

Figures 5(a) and 5(b) represent respectively the zero-order perturbation velocity profile and first order perturbation velocity profile against η for visco-elasticity parameters S=0.0, 0.1, 0.5, 1.0 &2.0 while Prandtl number Pr =0.72, Grashof number Gr=2.0, magnetic field parameter M=1.0 and transpiration parameter a =0.5.

From Figure 5(a) it is observed that, the zero-order perturbation velocity profile decreases as the visco-elasticity parameters S increases. Near the surface of the plate velocity profile increases, becomes maximum and then decreases and finally takes asymptotic value.

From Figure 5(b) it is observed that near the surface of the plate the first order perturbation velocity profile decreases becomes minimnm and then increases. We also observe that the velocity profile decreases as visco-elasticity parameters S increases.

The ordinary differential equation (1.6) and (1.7) were solved numerically by local non-similarity technique. The calculations were carried out for several values of Prandtl numbers Pr (Table 1), Magnetic field parameter M, (Table 2) , Grashof numbers Gr (Table 3) and Transpiration parameter a (Table 4).

From Table 1, 2 and 4 we see that the zero-order perturbation velocity profile, first order perturbation velocity profile and temperature profile increases as the Prandtl number Pr, Magnetic field parameter M and Transpiration parameter a decreases. From Table 2 it is further observe that, the effect of Magnetic field parameter M on the temperature profile effect is negligible.

From Table 3 , it is observed that both the zero-order perturbation velocity profile and first order perturbation velocity profile increases with the increase of Grashof number Gr. From Table 3 it is further observe that, the effect of Grashof number Gr on the temperature profile effect is negligible.

CONCLUSIONS:

(i) The zero-order perturbation velocity profile, first order perturbation velocity profile velocity profile and temperature profile decrease with the increase of Prandtl numbers Pr.

(ii) Both the zero and first order perturbation velocity profiles decrease with the increase of the Magnetic field parameter *M*. But the effect of Magnetic field parameter *M* on the temperature profile is negligible.

(iii) Both the zero and first order perturbation velocity profiles increase with the increase of the Grashof number Gr. But the Grashof number Gr has no significant effect on the temperature profile.

(iv) The values of the zero-order perturbation velocity profile, first order perturbation velocity profile and temperature profile decrease with the increase of the transpiration parameter *a*.

(v) The values of the zero-order perturbation velocity profile and first order perturbation velocity profile for Newtonian fluid are greater than that for visco-elastic fluid.

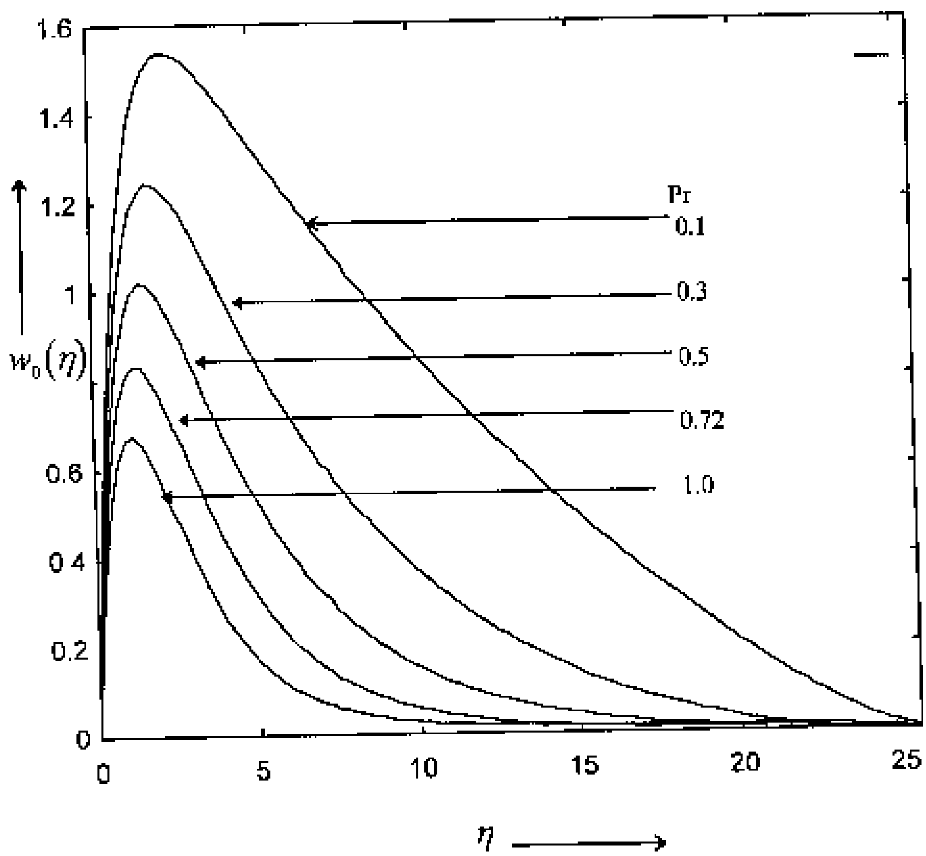


Figure 1(a): Zero- order perturbation velocity profile against η for different values of Pr while Gr=2.0, M=1.0, a=0.5 & S=0.4 for equation (3.37).

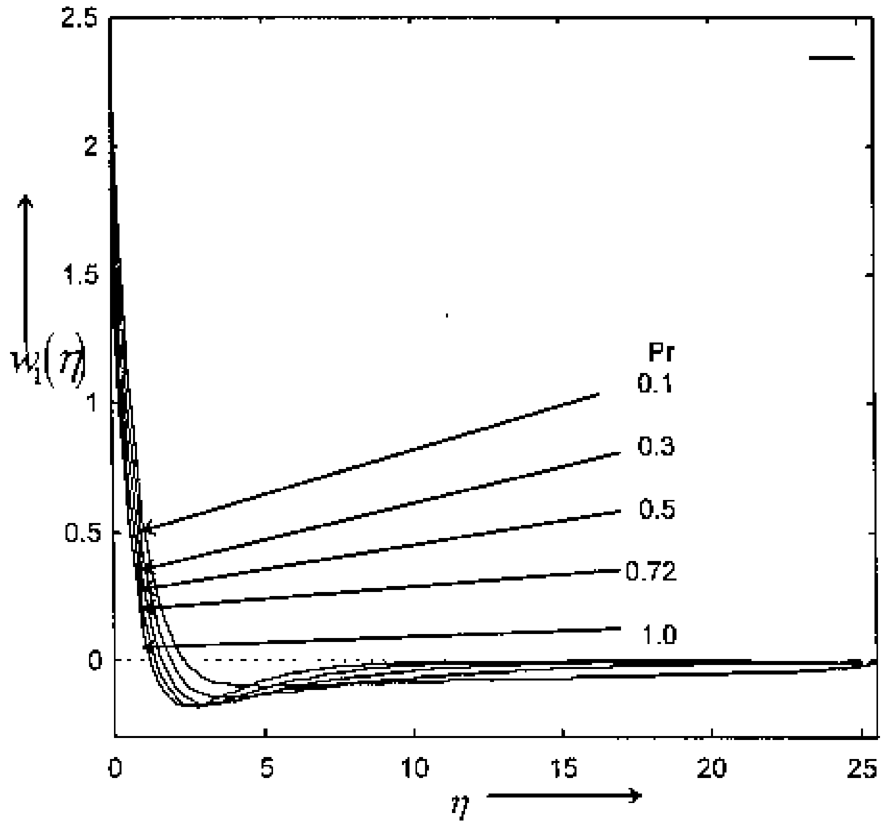


Figure 1(b): First order perturbation velocity profile for different values of Pr against η while Gr=2.0 , M=1.0, a=0.5 & S=0.4 for equation (3.38).

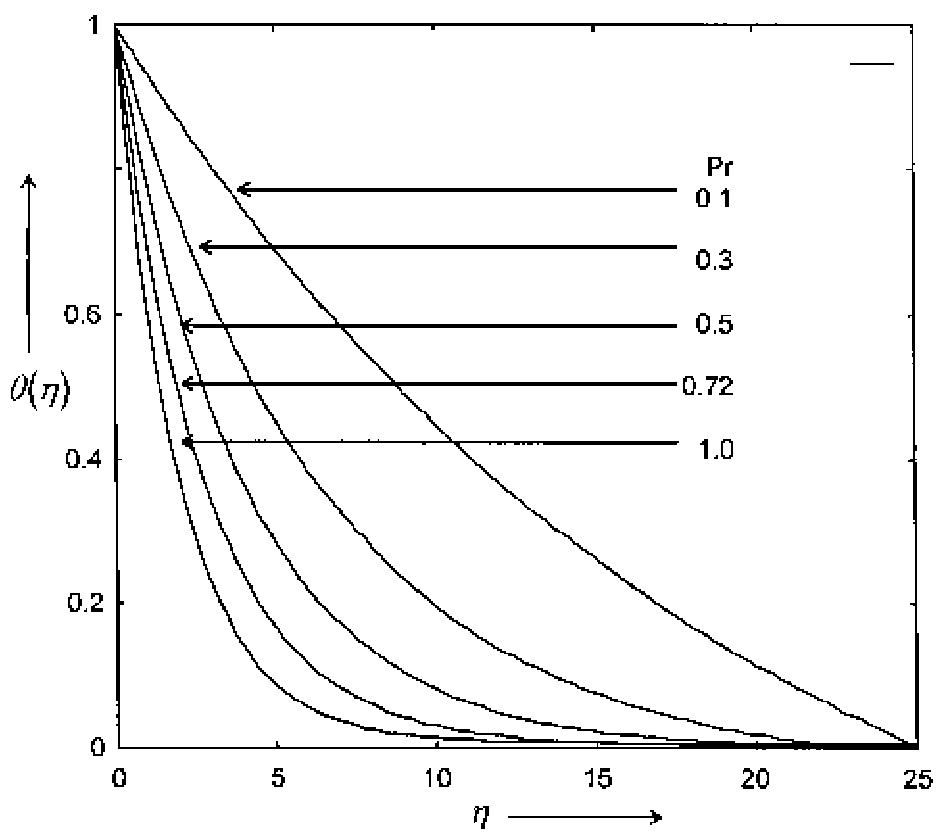


Figure 1(c): Temperature profile for different Pr agaainst η while Gr=2.0, M=1.0, a=0.5 & S=0.4 for equations arc (2.8).

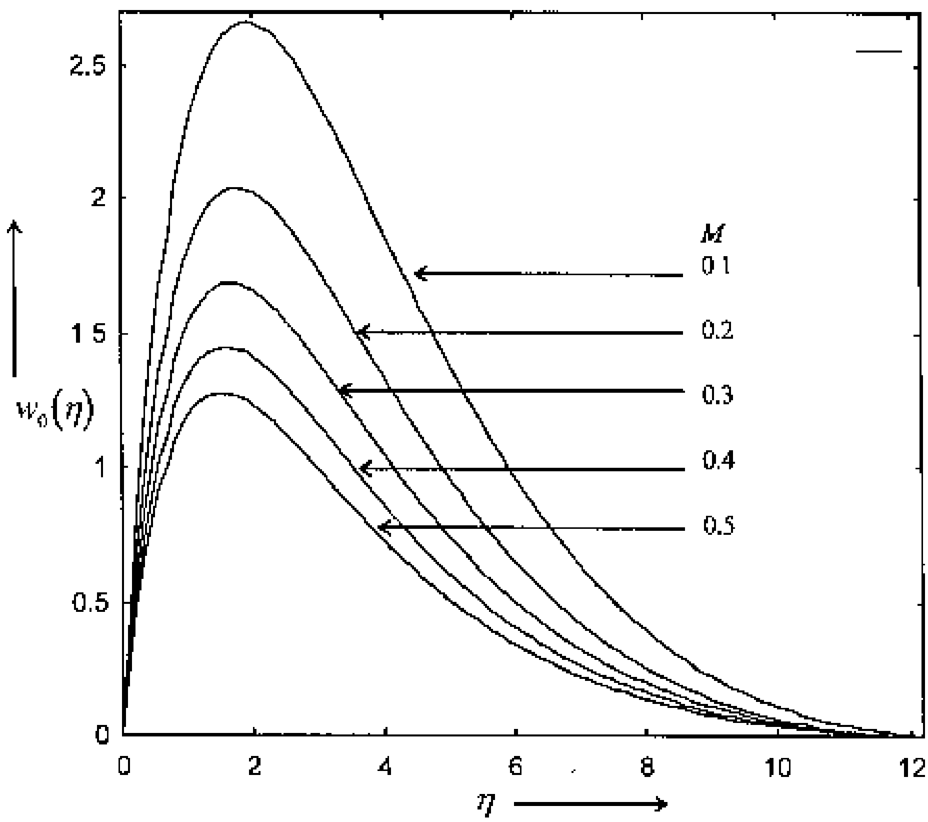


Figure 2(a): Zero-order perturbation velocity profile against η for different M while Pr =0.72 , Gr=2.0, a=0.5 & S=0.4 for equation (3.37).

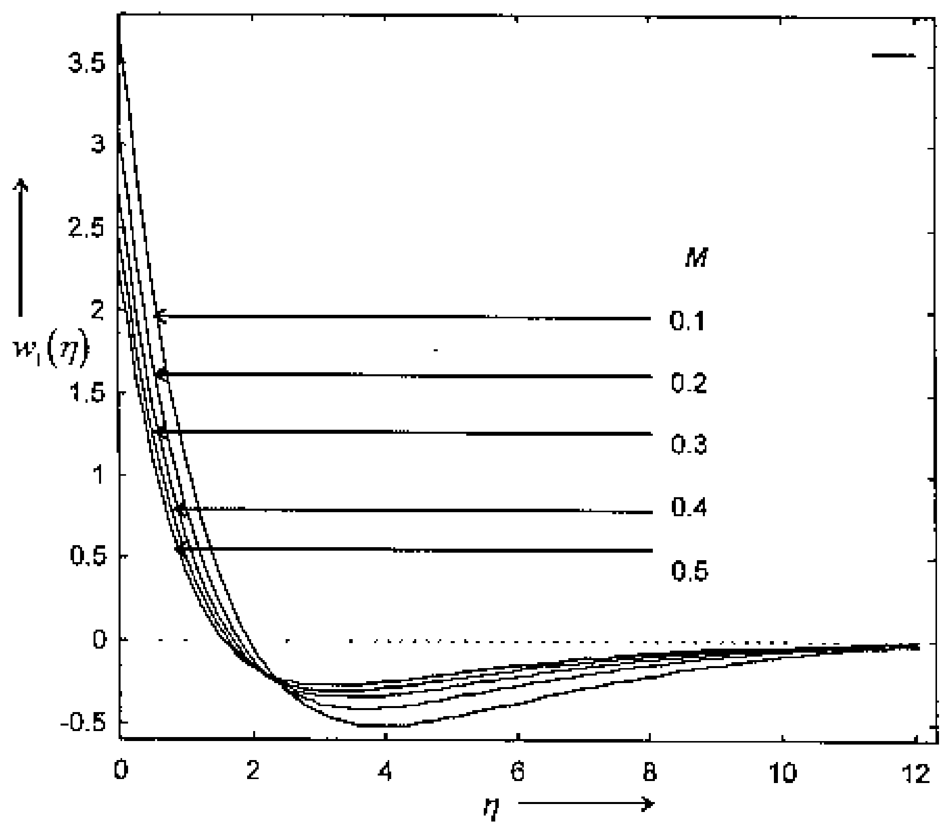


Figure 2(b): First order perturbation velocity profile for different M against η while Pr =0.72 , Gr=2.0, a=0.5 & S=0.4 for equation (3.38).

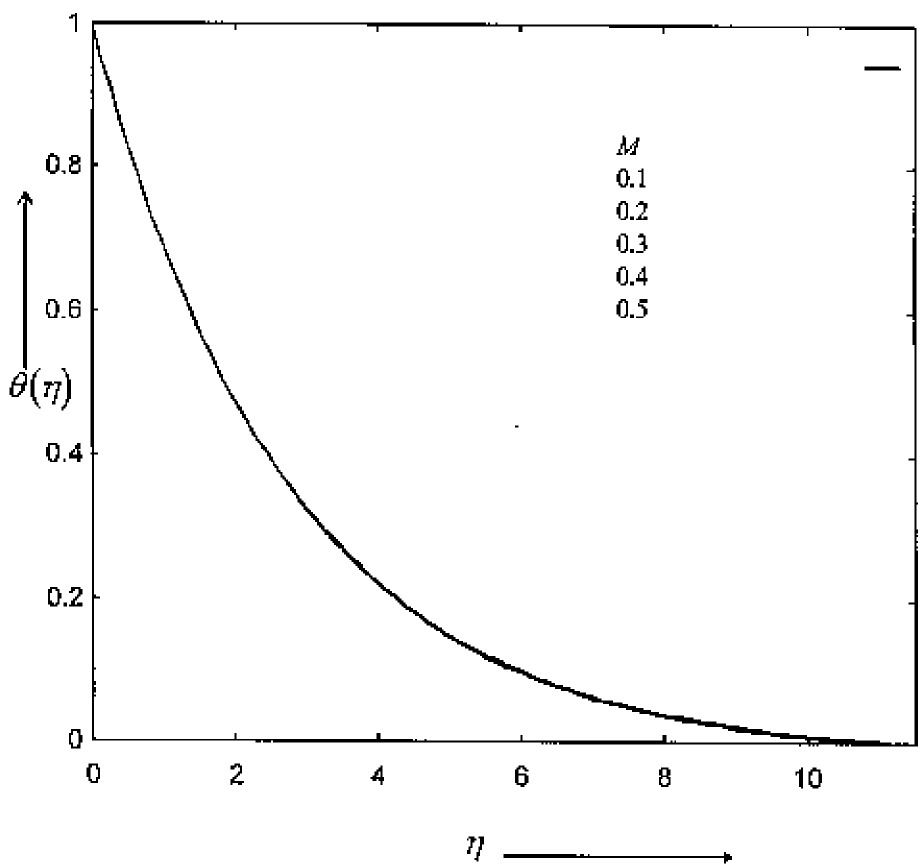


Figure 2(c): Temperature profile against η for different M while Pr =0.72 , Gr=2.0, a=0.5 & S=0.4 for equation (2.8).

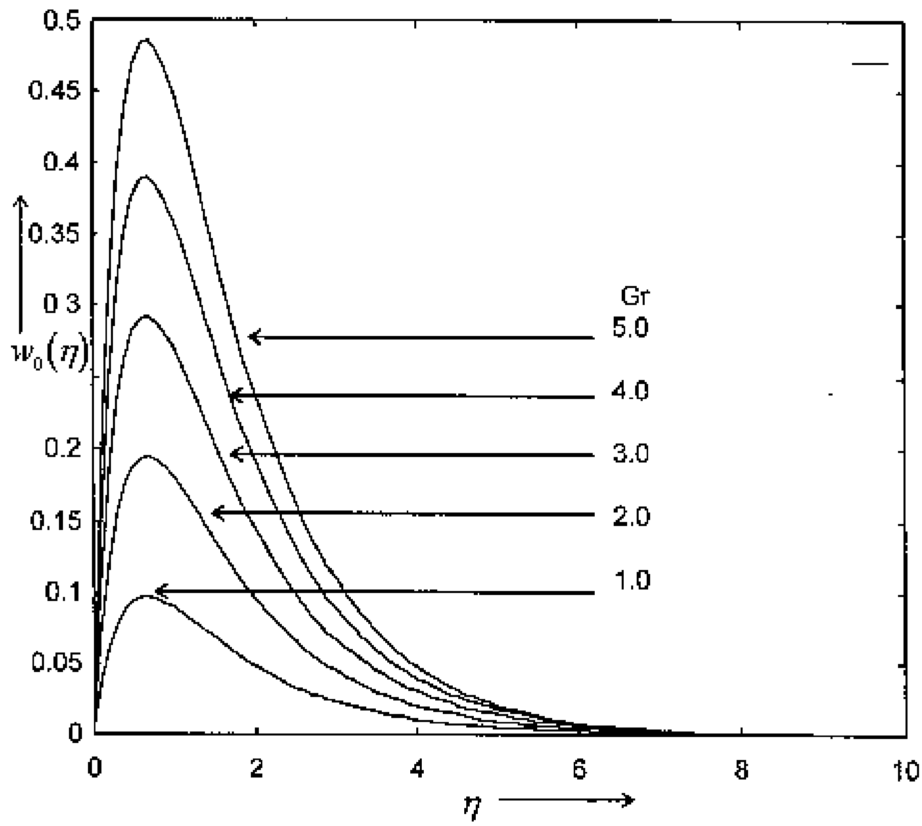


Figure 3(a): Zero-order perturbation velocity profile against η for diffrent values of Gr while Pr=1.0, M=4.0 , a=0.8 & S=0.4 for equation (3.37).

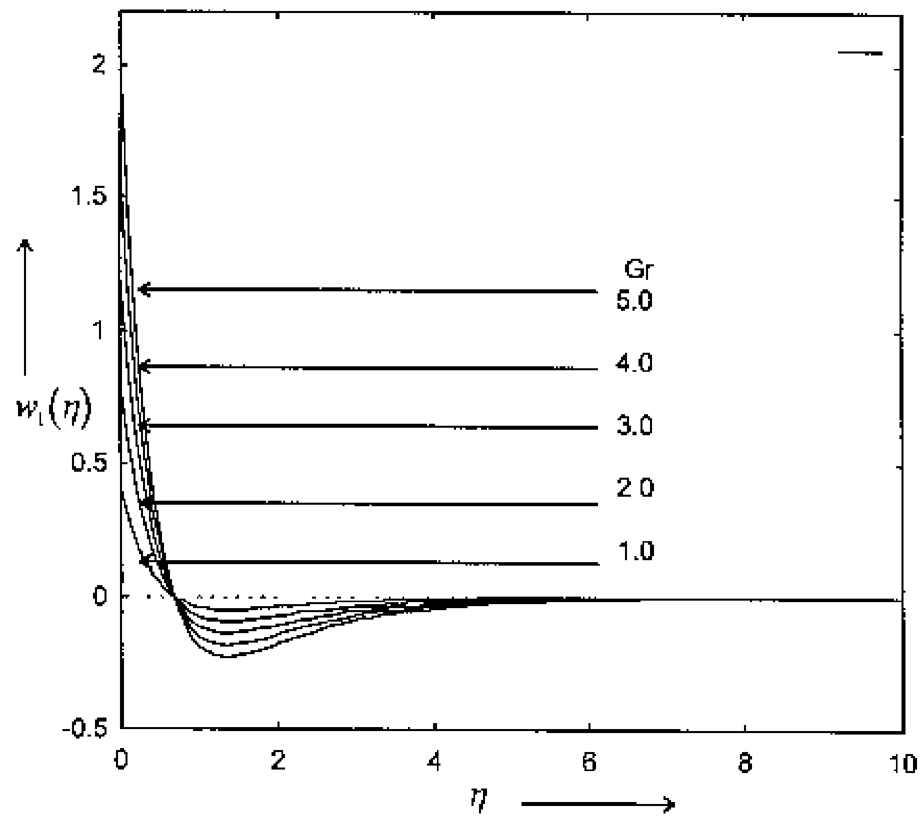


Figure 3(b): First order perturbation velocity profile for different Gr against η while Pr=1.0, M=4.0 , a=0.8 & S=0.4 for equation (3.38).

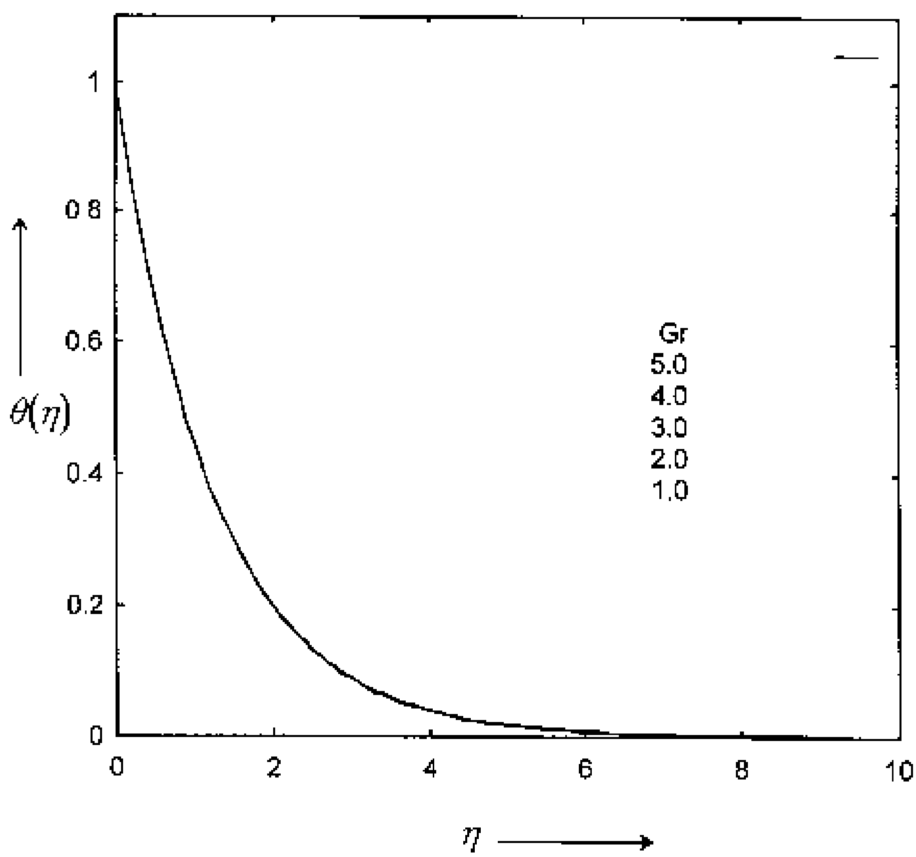


Figure 3(c): Temperature profile against η for different Gr while Pr=1.0, M=4.0 , a=0.8 & S=0.4 for equations (2.8).

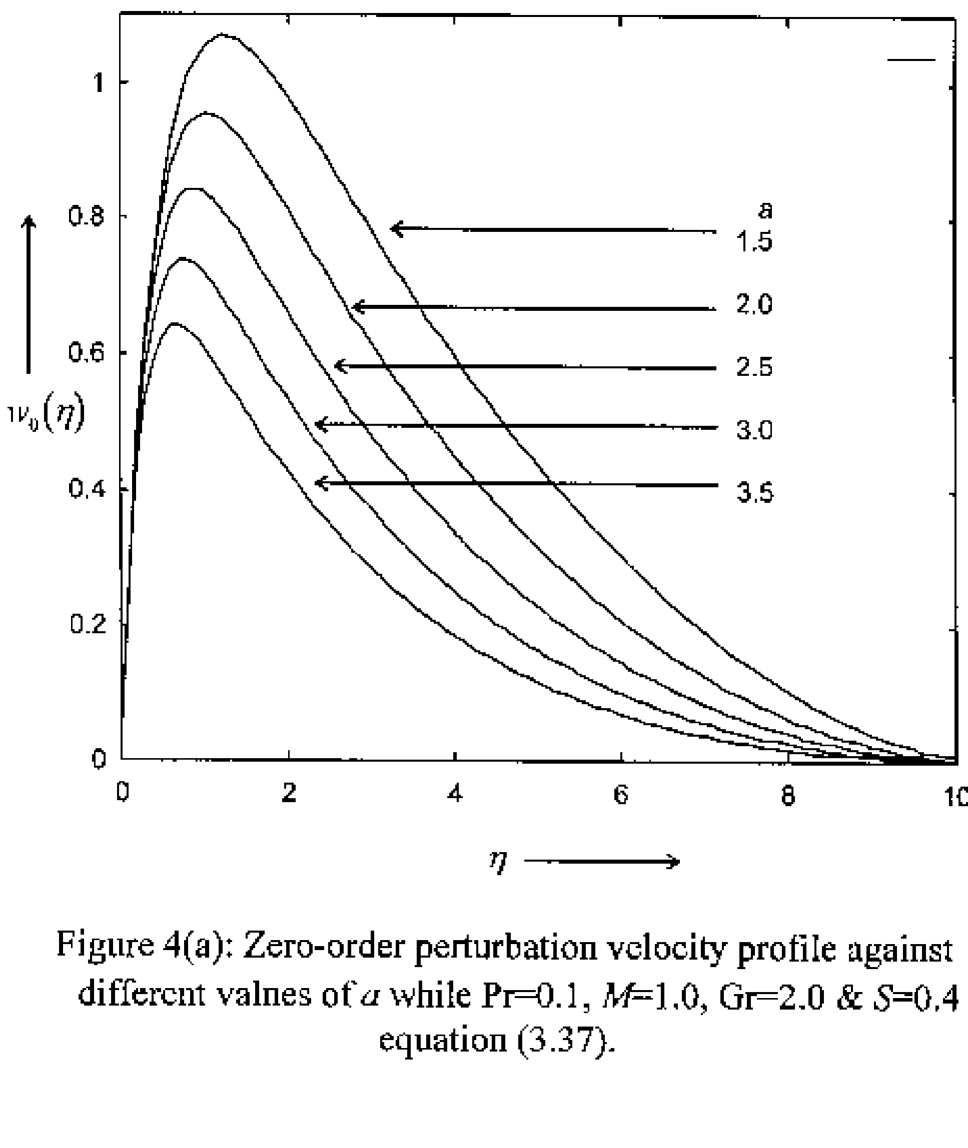


Figure 4(a): Zero-order perturbation velocity profile against η for different valnes of a while Pr=0.1, M=1.0, Gr=2.0 & S=0.4 for equation (3.37).

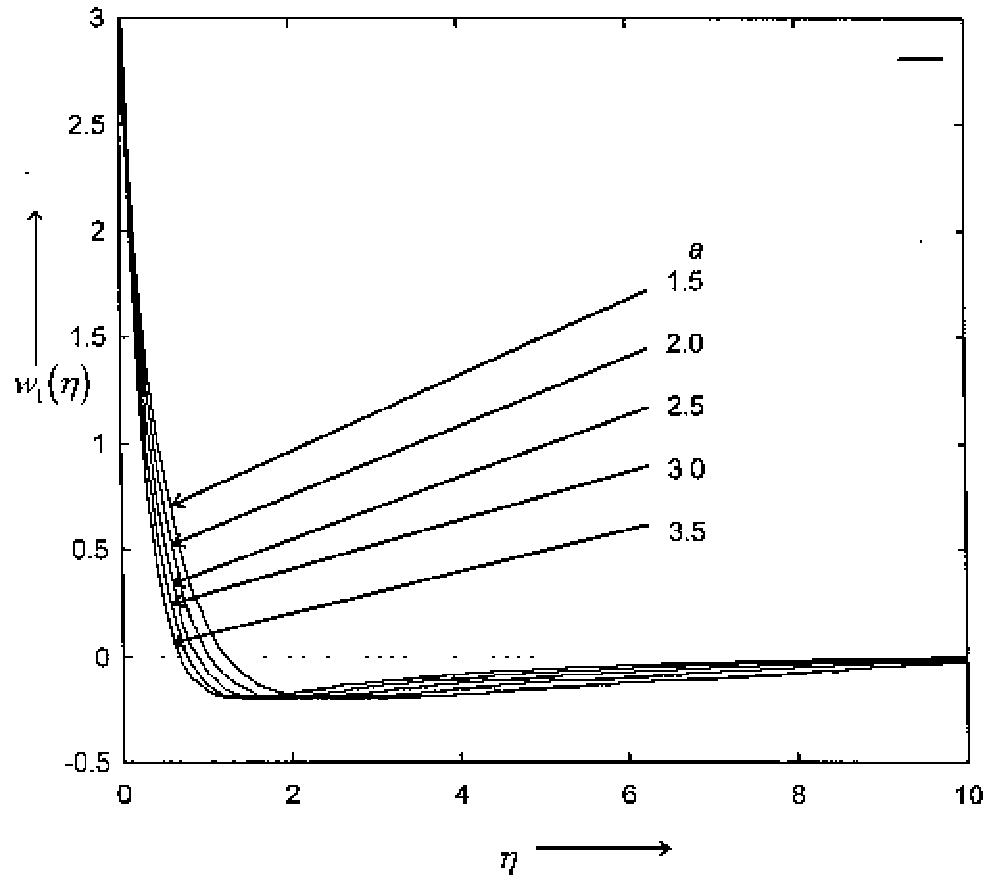


Figure 4(b): First order perturbation velocity profile for different values of a against η while Pr=0.1, M=1.0, Gr=2.0 & S=0.4 for equation (3.38).

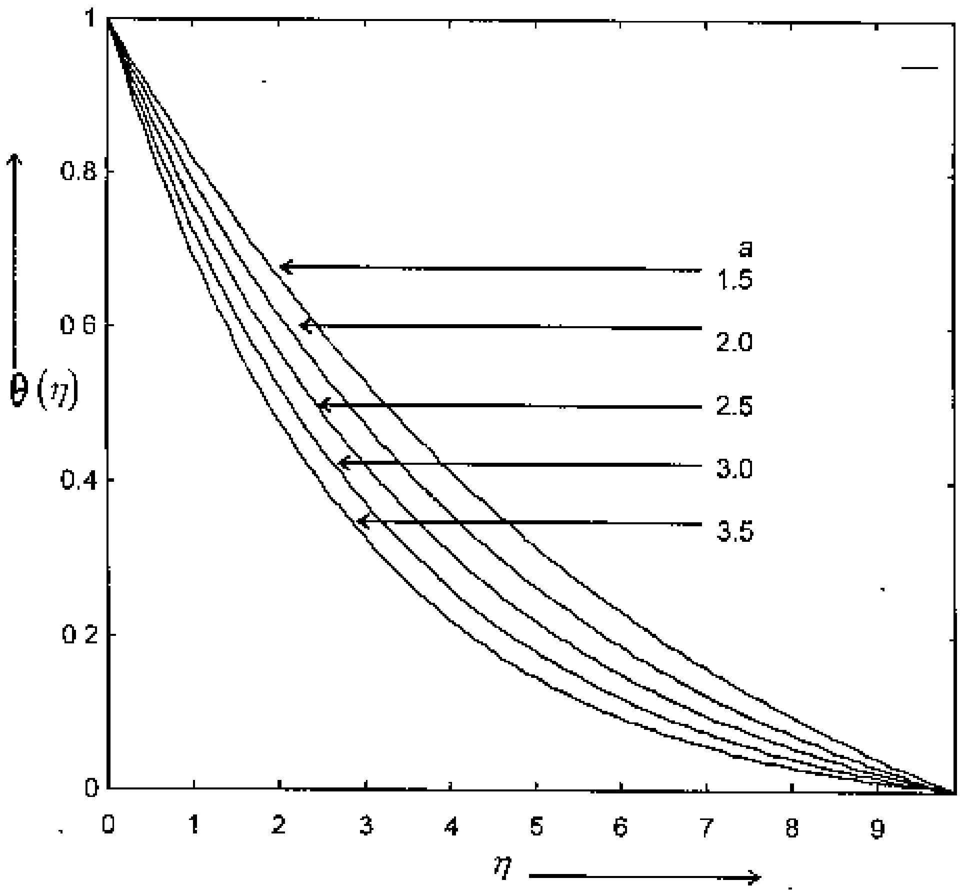


Figure (4c): Temperature profile against η for different values of a while Pr=0.1, M=1.0, Gr=2.0 & S=0.4 for equation (2.8).

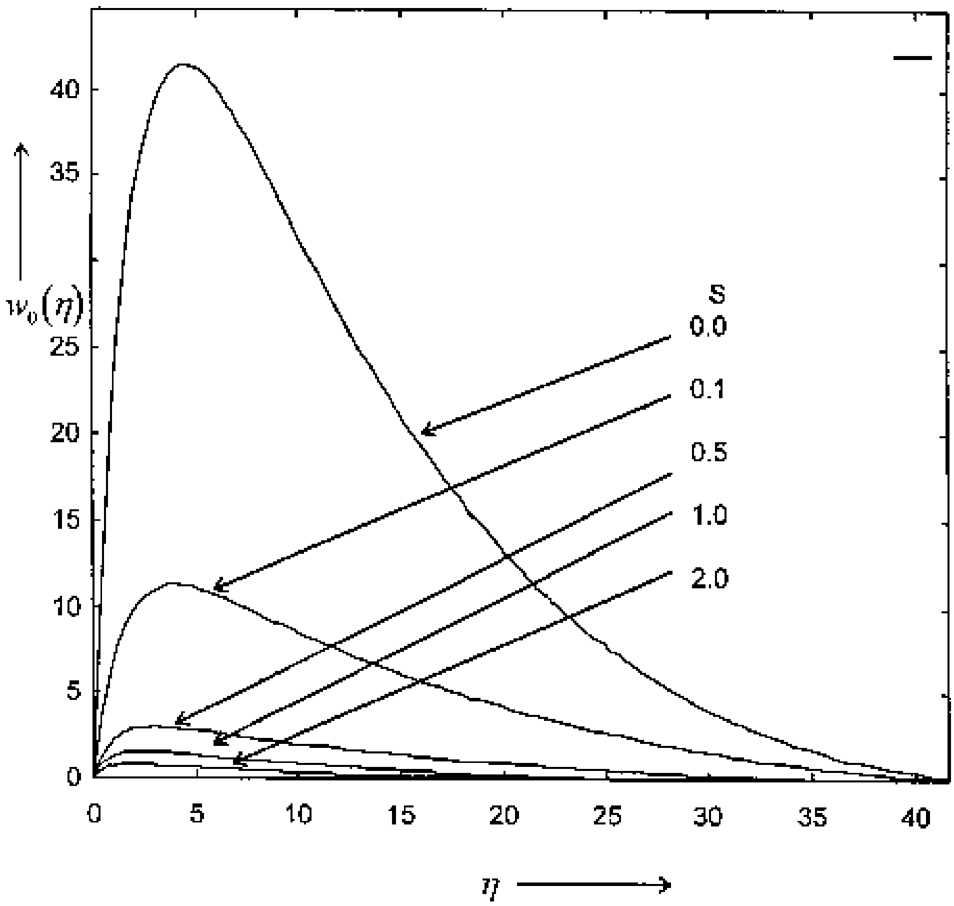


Figure 5(a): Zero-order perturbation velocity profile against for different S while Pr =0.72 , Gr=2, M=1.0 & a=0.5for equation (3.37).

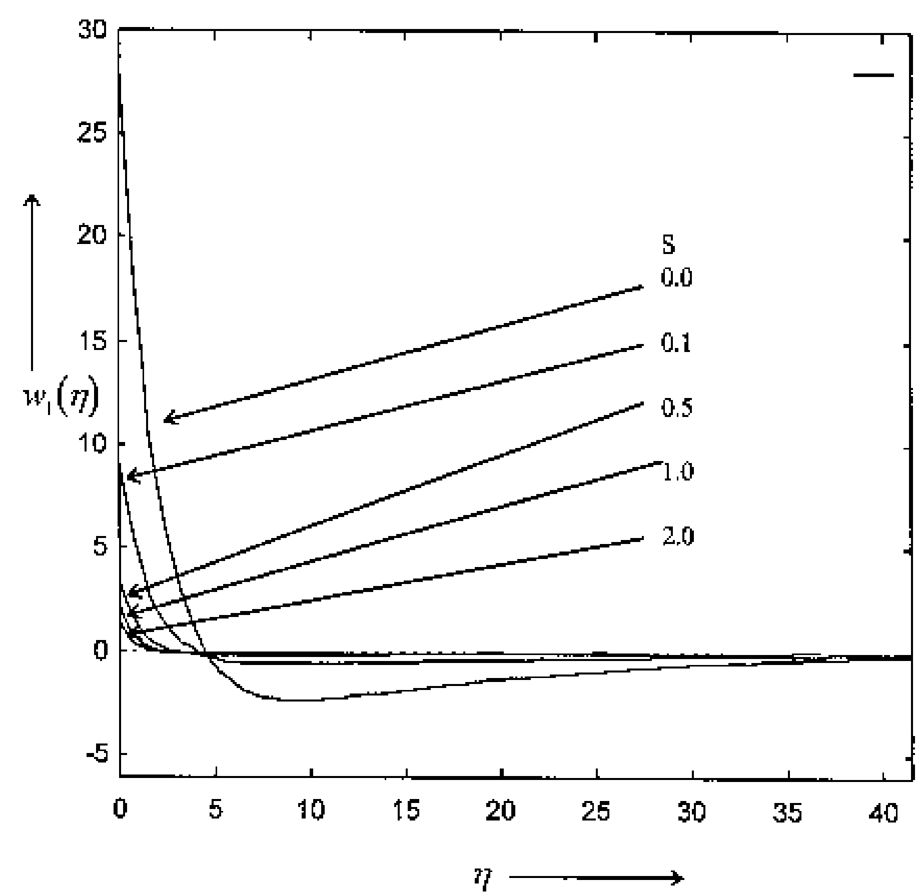


Figure 5(b): First order perturbation velocity profile for different S against η while Pr =0.72 , Gr=2, M=1.0 & a=0.5 for equation (3.38).

η	$w_0(\eta)$	$w_1(\eta)$	$\theta(\eta)$
	Pr=0.1		
2.0	1.53014	0.06204	-0.06319
4.0	1.42866	-0.09750	-0.05717
6.0	1.22614	-0.10008	-0.05173
8.0	1.03435	-0.09147	-0.04681
10.0	0.86015	-0.08284	-0.04236
12.0	0.70248	-0.07496	-0.03832
14.0	0.55981	-0.06783	-0.03468
16.0	0.43073	-0.06136	-0.03138
18.0	0.31399	-0.05546	-0.02839
20.0	0.20866	-0.04988	-0.02569
22.0	0.11487	-0.04361	-0.02324
	Pr=0.3		
2.0	1.24277	-0.03496	-0.11401
4.0	0.99934	-0.14647	-0.08446
6.0	0.73212	-0.11782	-0.06257
8.0	0.52746	-0.08800	-0.04635
10.0	0.37532	-0.06525	-0.03434
12.0	0.26258	-0.04834	-0.02544
14.0	0.17905	-0.03581	-0.01885
16.0	0.11718	-0.02652	-0.01396
18.0	0.07138	-0.01961	-0.01034
20.0	0.03764	-0.01433	-0.00766
22.0	0.01371	-0.00952	-0.00568
	Pr=0.5		
2.0	0.99227	-0.10394	-0.15097
4.0	0.67487	-0.15845	-0.09157
6.0	0.41363	-0.10347	-0.05554
8.0	0.24986	-0.06333	-0.03369
10.0	0.15013	-0.03845	-0.02043
12.0	0.08960	-0.02333	-0.01239
14.0	0.05289	-0.01415	-0.00752
16.0	0.03062	-0.00858	-0.00456
18.0	0.01710	-0.00521	-0.00277
20.0	0.00887	-0.00319	-0.00168
22.0	0.00381	-0.00198	-0.00102

Table 1: Numerical values of zero-order perturbation velocity profile, first order perturbation velocity profile and temparature profile against η for Pr=0.1, 0.3, & 0.5 while M=2.0, a=0.5, Gr=2.0, and S=0.4.

η	$w_0(\eta)$	$w_1(\eta)$	$\theta(\eta)$
	M=0.1		
1.0	2.24025	1.05843	-0.25527
2.0	2.66159	-0.04661	-0.17810
3.0	2.38025	-0.43892	-0.12426
4.0	1.88692	-0.51516	-0.08669
5.0	1.39093	-0.46513	-0.06048
6.0	0.96918	-0.37595	-0.04220
7.0	0.63944	-0.28481	-0.02944
8.0	0.39554	-0.20540	-0.02054
9.0	0.22365	-0.14083.	-0.01433
10.0	0.10933	-0.08991	-0.01000
11.0	0.04016	-0.05003	-0.00698
12.0	0.00653	-0.01840	-0.00487
	M=0.2		
1.0	1.79187	0.76767	-0.25477
2.0	2.04447	-0.11121	-0.17774
3.0	1.76843	-0.37878	-0.12401
4.0	1.36551	-0.40401	-0.08652
5.0	0.98717	-0.34603	-0.06036
6.0	0.67905	-0.26997	-0.04211
7.0	0.44509	-0.19982	-0.02050
9.0	0.15630	-0.09732	-0.01430
10.0	0.07708	-0.06260	-0.00998
11.0	0.02862	-0.03536	-0.00696
12.0	0.00474	-0.01301	-0.00486
	M=0.3		
1.0	1.51592	0.59477	-0.25440
2.0	1.67419	-0.13978	-0.17749
3.0	1.41137	-0.33393	-0.12383
4.0	1.06905	-0.33352	-0.08639
5.0	0.76277	-0.27515	-0.06027
6.0	0.52085	-0.20945	-0.04205
7.0	0.34078	-0.15266	-0.02934
8.0	0.21147	-0.10793	-0.02047
9.0	0.12133	-0.07392	-0.01428
10.0	0.06090	-0.04807	-0.00996
11.0	0.02333	-0.02778	-0.00695
12.0	0.00427	-0.01069	-0.00485

Table 2: Numerical values of Zero-order perturbation velocity profile, first order perturbation velocity profile and temperature profile against η for M=0.1,0.2 & 0.3 while P_r=.72, a=0.5, Gr=2.0 and S=0.4.

η	$w_0(\eta)$	$w_1(\eta)$	$\theta(\eta)$
	Gr=1.0		
1.0	0.09017	-0.03724	-0.35659
2.0	0.04822	-0.03553	-0.16023
3.0	0.02234	-0.01760	-0.07199
4.0	0.01010	-0.00805	-0.03235
5.0	0.00455	-0.00363	-0.01454
6.0	0.00205	-0.00163	-0.00653
7.0	0.00092	-0.00073	-0.00293
8.0	0.00042	-0.00033	-0.00132
9.0	0.00019	-0.00015	-0.00059
10.0	0.00010	-0.00006	-0.00027
	Gr=2.0		
1.0	0.18025	-0.07454	-0.35669
2.0	0.09633	-0.07108	-0.16027
3.0	0.04456	-0.03522	-0.07202
4.0	0.02007	-0.01611	-0.03236
5.0	0.00896	-0.00726	-0.01454
6.0	0.00395	-0.00327	-0.00653
7.0	0.00171	-0.00147	-0.00294
8.0	0.00070	-0.00066	-0.00132
9.0	0.00025	-0.00028	-0.00059
10.0	0.00009	-0.00005	-0.00027
	Gr=3.0		
1.0	0.27028	-0.11187	-0.35678
2.0	0.14435	-0.10665	-0.16031
3.0	0.06667	-0.05284	-0.07203
4.0	0.02993	-0.02417	-0.03237
5.0	0.01326	-0.01090	-0.01454
6.0	0.00575	-0.00490	-0.00653
7.0	0.00238	-0.00220	-0.00294
8.0	0.00087	-0.00098	-0.00132
9.0	0.00021	-0.00041	-0.00059
10.0	0.00000	0.00000	-0.00027

Table 3:Numerical values of zero-order perturbation velocity profile, first order perturbation velocity profile and temperature profile against η for Gr=1.0, 2.0 & 3.0 while P_r=1.0, M=4.0,and S=0.4.

η	$w_0(\eta)$	$w_1(\eta)$	$\theta(\eta)$
	a=1.5		
1.0	1.05616	0.13891	-0.16656
2.0	0.97922	-0.18127	-0.14336
3.0	0.78335	-0.19603	-0.12339
4.0	0.59814	-0.17296	-0.10621
5.0	0.43803	-0.14746	-0.09141
6.0	0.30269	-0.12344	-0.07868
7.0	0.19084	-0.10029	-0.06772
8.0	0.10236	-0.07636	-0.05829
9.0	0.03920	-0.04912	-0.05017
10.0	0.00652	-0.01459	-0.04318
	a=2.0		
1.0	0.95404	0.01416	-0.18974
2.0	0.81250	-0.20095	-0.15535
3.0	0.61700	-0.18263	-0.12719
4.0	0.45094	-0.14976	-0.10413
5.0	0.31616	-0.12053	-0.08526
6.0	0.20853	-0.09529	-0.06980
7.0	0.12465	-0.07283	-0.05715
8.0	0.06242	-0.05172	-0.04679
9.0	0.02132	-0.03028	-0.03831
10.0	0.00266	-0.00642	-0.03136
	a=2.5		
1.0	0.83690	-0.07554	-0.21198
2.0	0.66241	-0.19861	-0.16509
3.0	0.48090	-0.16186	-0.12857
4.0	0.33798	-0.12519	-0.10013
5.0	0.22812	-0.09562	-0.07798
6.0	0.14485	-0.07174	-0.06073
7.0	0.08326	-0.05199	-0.04730
8.0	0.03997	-0.03495	-0.03684
9.0	0.01290	-0.01931	-0.02869
10.0	0.00132	-0.00376	-0.02234

Table 4: Numerical values of zero-order perturbation velocity profile, first order perturbation velocity profile and temperature profile against η for a=1.5, 2.0 & 2.5 while P_r=0.1, M=1.0, Gr=2.0 and S=0.4.

Nomenclature

a	transpiration parameter
C_p	specific heat at constant pressure
f,s	dimensionless stream functions
g	acceleration due to gravity
Gr	average Grashof number
K	constant
k	the coefficient of thermal diffusivity
M	magnetic field of prameter
n	power/exponent
N_u	Nusselt number
P	pressure
P_r	Prandtl number
Q	Non-dimensional heat transfer.
S	visco-elasticity parameter
T	temperature of fluid
T_∞	temperature of ambient fluid
T_W	Surface temperature
u,v,w	velocity components in the boundary layer
x,y	coordinates along the edges of surface
z	coordinate normal to the surface

Greek letters

α	constant
β	coefficient of thermal expansion
ρ	density of the fluid
ν	kinematics coefficent viscosity
λ	suction parameter
σ_0	electric conductivity
δ	boundary layer thickness
θ	dimensionless temperature function
Ψ,ϕ	stream functions
ϕ	dissipation function
φ	similarity variable
ν	the kinematic coefficient of viscosity
ρ	the density of ambient fluid
μ	coefficient of viscosity
κ	the coefficient of thermal diffusivity
τ_w	nondimensional skin friction
ξ,η,ζ	scaled coordinate defined in equations
γ	the square root of the local boundary layer thickness

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APPENDIX

Derivation of I_0, I_1, I_2.

$$I_n = L^{-1}\left[\frac{\sqrt{n+1}}{q^{n+1}} e^{-\sqrt{q+M}\,\eta}\right] = \int_0^t (t-z)^n \frac{\eta}{2\sqrt{\pi z^3}} e^{-\frac{\eta^2}{4z}-MZ} dZ \qquad A(1)$$

Applying convolution Theorem.

Convolution Theorem: If $L^{-1}\{f(s)\} = F(t)$

and $L^{-1}\{g(s)\} = G(t)$ then

$$L^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)du = F^{*}G \qquad A(2)$$

Where $F^{*}G$ is the convolution of F and G and the above Theorem is called the Convolution Theorem.

For n=0 equation A(1) becomes

$$I_0 = \int_0^\tau \frac{\eta}{2\sqrt{\pi Z^3}} e^{-\frac{\eta^2}{4Z}-MZ} dZ$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\tau \left[e^{-\left(\frac{\eta}{2\sqrt{Z}}+\sqrt{MZ}\right)^2+\sqrt{M}\eta}\left(-\frac{\eta}{4Z^{3/2}}+\frac{1}{2}\sqrt{\frac{M}{Z}}\right) + e^{-\left(\frac{\eta}{2\sqrt{Z}}-\sqrt{MZ}\right)^2-\sqrt{M}\eta}\left(-\frac{\eta}{4Z^{3/2}}-\frac{1}{2}\sqrt{\frac{M}{Z}}\right)\right] dZ$$

$$I_0 = \frac{1}{2} e^{-\frac{1}{2}a\eta} \left[e^{\frac{\sqrt{a^2+4M}}{2}\eta} erfc\left(\frac{\eta}{2\sqrt{\tau}} + \frac{\sqrt{\left(a^2+4M\right)\tau}}{2} \right) + e^{-\frac{\sqrt{a^2+4M}}{2}\eta} erfc\left(\frac{\eta}{2\sqrt{\tau}} - \frac{\sqrt{\left(a^2+4M\right)\tau}}{2} \right) \right] \qquad A(3)$$

$$\text{Now } I_{n+1} = I_n \frac{\delta I_n}{\delta M}, \qquad n>0$$

$$I_1 = I_0 \frac{1}{\left[\left(\alpha + \sqrt{\alpha^2 + q} \right)^2 - Q_r^2 \right]},$$

$$I_1 = \int_0^{\tau} \frac{1}{2} e^{-\frac{a\eta}{2}} \left\{ e^{\frac{\sqrt{a^2+4M}}{2}\eta} erfc\left(\frac{\eta}{2\sqrt{v}} + \frac{\sqrt{\left(a^2+4M\right)v}}{2} \right) + e^{-\frac{\sqrt{a^2+4M}}{2}\eta} erfc\left(\frac{\eta}{2\sqrt{v}} - \frac{\sqrt{\left(a^2+4M\right)v}}{2} \right) \right\} \frac{e^{Q_r^2(\tau-v)}}{2Q_r} \left\{ (\alpha+Q_r) e^{2\alpha Q_r(\tau-v)} \right. \qquad A(4)$$

$$\left. erfc\left((\alpha+Q_r)\sqrt{\tau-v} \right) - (\alpha-Q_r) e^{-2\alpha Q_r(\tau-v)} erfc\left((\alpha-Q_r)\sqrt{\tau-v} \right) \right\} dv$$

$$I_2 = \tau I_1 + \frac{\delta I_1}{\delta M}$$

$$I_2 = \frac{e^{-\sqrt{P_r}\alpha\eta}}{2Q_r^2\left(4\alpha^2 - Q_r^2\right)}\left[Q_r^2 e^{-\sqrt{P_r}\alpha\eta} erfc\left(\frac{\sqrt{P_r}\eta}{2\sqrt{\tau}} - \alpha\sqrt{\tau}\right)\right.$$

$$-\left(4\alpha^2 - Q_r^2\right)e^{\sqrt{P_r}\alpha\eta} erfc\left(\frac{\sqrt{P_r}\eta}{2\sqrt{\tau}} + \alpha\sqrt{\tau}\right) + \left(\alpha + Q_r\right)\left(2\alpha - Q_r\right)e^{\left(Q_r^2\tau + 2\alpha Q_r\tau + \sqrt{P_r}\alpha\eta + \sqrt{p_r}Q_r\eta\right)}$$

$$erfc\left(\frac{\eta\sqrt{P_r}}{2\sqrt{\tau}} + \alpha\sqrt{\tau} + Q_r\sqrt{\tau}\right) + \left(\alpha - Q_r\right)\left(2\alpha + Q_r\right)e^{\left(Q_r^2\tau - 2\alpha Q_r\tau + \sqrt{P_r}\alpha\eta - \sqrt{p_r}Q_r\eta\right)} \qquad A(5)$$

$$\left.erfc\left(\frac{\eta\sqrt{P_r}}{2\sqrt{\tau}} + \alpha\sqrt{\tau} - Q_r\sqrt{\tau}\right)\right]$$

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Applying convolntion Theorem.

Convolution Theorem: If $L^{-1}\{f(s)\} = F(t)$

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$$I_0 = \frac{1}{2} e^{-\frac{1}{2}a\eta} \left[e^{\frac{\sqrt{a^2+4M}}{2}\eta} erfc\left(\frac{\eta}{2\sqrt{\tau}} + \frac{\sqrt{(a^2+4M)\tau}}{2} \right) \right.$$

$$\left. + e^{-\frac{\sqrt{a^2+4M}}{2}\eta} erfc\left(\frac{\eta}{2\sqrt{\tau}} - \frac{\sqrt{(a^2+4M)\tau}}{2} \right) \right] \qquad \text{A(3)}$$

$$\text{Now } I_{n+1} = I_n \frac{\delta I_n}{\delta M}, \qquad n>0$$

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$$I_1 = \int_0^{\tau} \frac{1}{2} e^{-\frac{a\eta}{2}} \left\{ e^{\frac{\sqrt{a^2+4M}}{2}\eta} erfc\left(\frac{\eta}{2\sqrt{v}} + \frac{\sqrt{(a^2+4M)v}}{2} \right) \right.$$

$$\left. + e^{-\frac{\sqrt{a^2+4M}}{2}\eta} erfc\left(\frac{\eta}{2\sqrt{v}} - \frac{\sqrt{(a^2+4M)v}}{2} \right) \right\} \frac{e^{Q_r^2(\tau-v)}}{2Q_r} \left\{ (\alpha + Q_r) e^{2\alpha Q_r(\tau-v)} \right. \qquad \text{A(4)}$$

$$\left. erfc\left((\alpha + Q_r)\sqrt{\tau - v} \right) - (\alpha - Q_r) e^{-2\alpha Q_r(\tau-v)} erfc\left((\alpha - Q_r)\sqrt{\tau - v} \right) \right\} dv$$

$$I_2 = \tau I_1 + \frac{\delta I_1}{\delta M}$$

$$
\begin{aligned}
J_2 = \frac{e^{-\sqrt{P_r}\alpha\eta}}{2Q_r^2\left(4\alpha^2 - Q_r^2\right)}\Bigg[& Q_r^2 e^{-\sqrt{P_r}\alpha\eta} erfc\left(\frac{\sqrt{P_r}\,\eta}{2\sqrt{\tau}} - \alpha\sqrt{\tau}\right) \\
& -\left(4\alpha^2 - Q_r^2\right) e^{\sqrt{P_r}\alpha\eta} erfc\left(\frac{\sqrt{P_r}\,\eta}{2\sqrt{\tau}} + \alpha\sqrt{\tau}\right) + \left(\alpha + Q_r\right)\left(2\alpha - Q_r\right) e^{\left(Q_r^2\tau + 2\alpha Q_r\tau + \sqrt{P_r}\alpha\eta + \sqrt{P_r}Q_r\eta\right)} \\
& erfc\left(\frac{\eta\sqrt{P_r}}{2\sqrt{\tau}} + \alpha\sqrt{\tau} + Q_r\sqrt{\tau}\right) + \left(\alpha - Q_r\right)\left(2\alpha + Q_r\right) e^{\left(Q_r^2\tau - 2\alpha Q_r\tau + \sqrt{P_r}\alpha\eta - \sqrt{P_r}Q_r\eta\right)} \\
& erfc\left(\frac{\eta\sqrt{P_r}}{2\sqrt{\tau}} + \alpha\sqrt{\tau} - Q_r\sqrt{\tau}\right)\Bigg]
\end{aligned}
\qquad A(5)
$$