Study of Dominating Singularity Behavior of Series Using Computer Based Approximation Techniques.

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Author's Declaration

I declare that the work done in this dissertation was followed in accordance with the regulations of Bangladesh University of Engineering and Technology, Dhaka. I also declare that this is an authentic record of my own work except where indicated by special reference in the text. No part of this text has been submitted for any other degree or diploma.

The dissertation has not been presented to any other University for examination either in Bangladesh or overseas.

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Abstract

Very few nonlinear problems can be solved exactly but it is sometimes possible to expand solution in powers of some parameters. In practice the presence of singularities prevents rapid convergence of the series, so it is necessary to seek an efficient approximate method. Our purpose is to analyze the dominating singularity behavior of a few problems and the performance of approximate methods.

The aim of this thesis is to analyze the approximate methods by applying them to various model problems. Firstly, we have applied them to some standard problems, whose singularities are known. Secondly we have analyzed numerically the critical behavior of the solutions of two non-linear differential equations.

Finally, we have studied the flow in a porous pipe with decelerating wall for analyzing the dominant singularity behavior of the flow. The series related to the Reynolds number R is developed by using algebraic programming language MAPLE. The series is then analyzed by various generalizations of the approximate methods. We observe that the convergence of both the series is limited by the dominating singularity located at $R = R_1 \approx 3.0724980042$ and surprisingly there is another turning point at $R_2 \approx 8.813114939$ .The result concluded that there is a reversal flow at the wall.

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Contents

Declaration
Abstract
Acknowledgements

CHAPTER		Page
1	**Introduction**	**1**
1.1	Overview of series	2
1.1.1	Perturbation series	3
1.2	Singularities	5
1.3	Elementary bifurcation theory	7
1.4	Continued fraction	10
1.5	Overview of the work	12
2	**Approximate Methods**	**13**
2.1	Pade'-Hermite approximants	14
2.2	Pade' approximants	15
2.3	Algebraic approximants	17
2.4	The Drazin-Tourigney approximants	19
2.5	Differential approximants	19
2.6	High-order differential approximants	20
2.7	Discussion	21
3	**Approximate Solution of Non-linear Systems**	**22**
3.1	Some test functions	23
3.2	Spontaneous singularities in the complex plane	24
3.3	Infinite number of spontaneous singularities	29
3.4	Flow in a porous pipe with decelerating wall	34
3.5	Discussion	41
4	**Conclusion**	**42**
4.1	Summary of the thesis	42
4.2	Future work	42
	References	**43**
	Appendix I	**45**
	Appendix II	**47**

List of Tables

		Page
Table2.2.1 :	The approximation of x_c by Pade' for the function in example 2.2.1	17
Table2.3.1 :	The approximation of x_c by AA for the function in example 2.3.1	18
Table2.5.1 :	The approximation of x_c by Differential Approximant for the function in example 2.5.1	20
Table-3.1.1 :	The number ρ_N of correct decimal figures in the approximation of x_c by various methods for the functions 1.-5.	24
Table-3.2.1 :	Estimates of the critical point $x_{c,N}$ and the corresponding exponent α_N by using various approximate methods for the differential equation $u' = 1 - xu^2, u(0) = 1$.	26
Table-3.2.2 :	Spontaneous singularities in the complex plane. Estimates of the critical point $x_{c,N}$ in the complex plane and the corresponding exponent α_N by using various approximate methods for the differential equation $u' = 1 - xu^2, u(0) = 1$.	27
Table-3.3 1 :	Estimates of the critical point $x_{c,N}$, $x_{c,N}$ and the corresponding exponent α_N by using various approximate methods for the differential equation $u' = u^2 + x$ with $u(0) = 0$.	33
Table-3.4.1 :	Estimates $x_{c,n}$ of $R_{1,N}^{(d)}$, $R_{2,N}^{(d)}$ and the critical exponent α_N of the corresponding exponent by HODA of centerline axial velocity. The last row shows the estimates obtained by using the D-T method [17].	38
Table-3.4.2 :	Estimates $x_{c,n}$ of $R_{1,d}$, $R_{2,d}$ and the critical exponent α_N of the corresponding exponent by HODA of skin friction. The last row shows the estimates obtained by using the D-T method [17].	38
Table-A.2	The coefficients of the series $F''(0) = \sum_{i=1}^{55} a_i R^i$:	45

List of figures

Page

Figure 1.3.1 : Bifurcation diagram of $F(x,u,\varepsilon)$ when $\varepsilon=0$. 9

Figure 1.3.2 . Bifurcation diagram of $F(x,u,\varepsilon)$ when $\varepsilon=0.01$. 9

Figure 1.3 3 : Bifurcation diagram of $F(x,u,\varepsilon)$ when $\varepsilon=0.1$ 10

Figure 3.2.1 : The tangent field indicate the solution $u(x)$ of $u'=1-xu^2$ which satisfies the initial condition $u(0)=1$. [Bender & Orszag, pp.149]. 28

Figure 3.2.2 : Approximate solution $u_N(x)$ for the Riccati equation $u'=1-xu^2$ with $u(0)=1$ with the Drazin-Tourigney approximant with d=12. 28

Figure 3.3.1 : Computer plot of $u(x)$ to $u'=u^2+x$, satisfying the initial condition $u(0)=0$. [Bender & Orszag, pp.150]. 29

Figure 3.4.1 : Schematic diagram of the problem. 35

Figure 3.4.2 : The bifurcation diagram in the $(F''(0),R)$ plane near the first and second turning points R_1 and R_2 for the problem using the Drazin-Tourigny method with $d=8$. Other curves are spurious. 39

Figure 3.4.3 : The bifurcation diagram in the $(F''(0),R)$ plane near the first and second turning points R_1 and R_2 for the problem using the Drazin-Tourigny method with $d=8$. Other curves are spurious. 39

Figure 3.4.4 : The bifurcation diagram in the (β,R) plane near the first and second turning points R_1 and R_2 for the problem using the Drazin-Tourigny method with $d=7$. Other curves are spurious. 40

Figure 3.4.5 : The bifurcation diagram in the (β,R) plane near the first and second turning points R_1 and R_2 for the problem using the Drazin-Tourigny method with $d=8$. Other curves are spurious. 40

Chapter 1

Introduction

It is an old maxim of mine that when you have excluded the impossible, what ever remains, however improbable, must be the truth

-Sherlock Holmes, The adventure of the Beryl Coronet

Sir Arthur ConanDoyle.

Very few nonlinear problems can be solved exactly but it is sometimes possible to expand solution in powers of some parameters. When the exact closed form solution of a problem is too complicated then one should try to ascertain the approximate nature of the solution. Approximation methods [1, 8, 11-13, 21-23] are the techniques for summing power series. A function is said to be approximant for a given series if its Taylor series expansion reproduces the first few terms of the series. The partial sum of a series is the simplest approximant, if the function has no singularities. For a rapidly convergent series such approximants can provide good approximation for the solution. In practice the presence of singularities prevents rapid convergence of the series. Therefore it is necessary to seek an efficient approximation method.

Khan [14] applied approximation methods to several fluid dynamical problems. Our purpose is to analyze the dominating singularity behavior of some standard problems and compare the performance of approximation methods numerically.

The remainder of this introductory chapter is as follows. Since we shall study the dominating singularity behavior of series by using approximation techniques, we begin with a brief review of series in §1.1. Then in §1.2 , we describe various types of singularities. We describe a brief review of elementary bifurcation theory in §1.3. We present the basic concept of continued fractions in §1.4 in order to connect with the approximation method in Chapter 2. Finally, in §1.5, we present a brief out line of the remaining thesis.

1.1 Overview of series

Consider a function $u(x)$ which can be represented by a power series

$$U(x)=\sum_{i=0}^{\infty}a_i x^i \text{ as } x\to 0. \tag{1.1.1}$$

The Nth partial sum is

$$U_N(x)=\sum_{i=0}^{N}a_i x^i . \tag{1.1.2}$$

If we can locate a point x_0, where the function $u(x)$ is analytic then it can be found in a power series -

$$U(x)=\sum_{i=0}^{\infty}a_i(x-x_0)^i \text{ as } x\to x_0 .$$

The series is said to be convergent if the sequence of the partial sums converges. When the series converges, the sum $U(x)$ can be approximated by the partial sum $U_N(x)$ and the error is defined by

$$e_N(x)=U(x)-U_N(x),$$

and the absolute error is defined by

$$e_N'(x)=\left|\frac{e_N(x)}{U(x)}\right| \quad \text{provided } U(x)\neq 0.$$

The number of accurate decimals for some particular value of x is given by

$$\rho_N = -\log_{10}\left|e_N'\right| .$$

We say that the error decays exponentially if there exists a particular constant σ such that $\sigma_N\to\sigma$ as $N\to\infty$, where

$$\sigma_N = -\frac{\ln\left|e_N\right|}{N}.$$

Sometimes the presence of singularity of the solution can delay the convergency of the series. So, we need to find the domain of convergence of the series. The series $U(x)$ converges for some x_c if it converges absolutely in the open disc

$$\{x : x < x_c\}$$

with centre at the origin. The largest such disc is called the disc of convergence and the radius, say R, of the disc is called the radius of convergence of the series of $U(x)$. If $u(x)$ is analytic at $x = 0$ then $R > 0$. If the series has a singularity at x_c such that $x_c = R$, then it diverges for $x \geq x_c$. Different methods such as ratio test, Domb Sykes plot etc. have been used to compute the radius of convergence by direct use of the coefficients of the series. We will apply various generalizations of the approximation methods to determine the singularity behavior of the series.

In applied mathematics, series are often obtained by expanding a solution in powers of some perturbation parameter. In the following subsection, we describe the basic literature on perturbation techniques [20][26]

1.1.1 Perturbation series

Perturbation theory is a collection of methods for the systematic analysis of the global behavior of solutions to nonlinear problems. Sometimes we solve nonlinear problems by expanding the solution in powers of one or several small perturbation parameters. The expansion may contain small or large parameters which appear naturally in the equations, or which may be artificially introduced Let us consider a problem of the form

$$f(u, x, \lambda) = 0 \tag{1.1.3}$$

where f may be an algebraic function or some non-linear differential operator, and λ is a parameter. It is seldom possible to solve the problem exactly, but there may exist some particular value of $x = x_0$ for which the solution is known. In this case, for $|x| << 1$, one can seek a series for u in powers of x such that

$$U(x) = \sum_{i=0}^{\infty} a_i(\lambda)(x - x_0)^i \text{ as } x \to x_0.$$

Then by substituting this into equation (1.1.3), expanding in powers of x and collecting the terms of $O(x^n)$, we can get the required coefficients of the perturbation series.

Example-1.1.1. Let us take the cubic polynomial

$$u^3 - (4+x)u + 2x = 0 \tag{1.1.4}$$

The perturbation series for (1.1.4) in powers of x may be taken in the form of

$$U(x) = \sum_{i=0}^{\infty} a_i x^i \tag{1.1.5}$$

for small x. For $x = 0$, the polynomial has three distinct roots, namely $a_0 = -2, 0, 2$. Which, when $a_0 = -2$, substituting the expansion

$$U(x) = -2 + a_1 x + a_2 x^2 + \dots$$

in (1.1.4) and equating the coefficients of x gives $a_1 = -\frac{1}{2}, a_2 = \frac{1}{8}$. Therefore, the perturbation series for $a_0 = -2$ is

$$U = -2 - \frac{1}{2}x + \frac{1}{8}x^2 + O(x^3).$$

In a similar process other series are $U = 0 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$ at $a_0 = 0$ and

$$U = 2 + 0.x + 0.x^2 + O(x^3) \text{ at } a_0 = 2.$$

Example-1.1.2. Consider the differential equation

$$u' = \frac{u^2}{1 - xu} \text{ with } u(0) = 1. \tag{1.1.6}$$

Let us consider the series

$$U(x) = \sum_{i=0}^{\infty} a_i x^i \quad \text{as } x \to 0 \tag{1.1.7}$$

of the solution of the above nonlinear initial value problem (1.1.6).

Equating the coefficients of x^0, we obtain $a_1 = a_0^2$. Equating the coefficients of x and x^2 etc, we obtain $a_2 = \frac{3}{2}a_0 a_1 = \frac{3}{2}a_0^3$ and $a_3 = \frac{1}{3}(4a_0 a_2 + 2a_1^2) = \frac{8}{3}a_0^4$ respectively. These values of a_1, a_2 and a_3 lead us to the recurrence relation

$$(i+1)a_{i+1} - \sum_{j=0}^{i-1}(i-j)a_j a_{i-j} = \sum_{j=0}^{i} a_j a_{i-j},$$

which implies,

$$a_i = \frac{(i+1)^{(i-1)}}{i!} a_0^{i+1} \quad \text{for } i = 1,2,3,\ldots$$

According to the initial condition, $a_0 = 1$. Since the coefficients a_i are functions of a_0, the coefficients will vary with the initial condition.

1.2 Singularities

Singularity of a function is a value of the independent variable or variables for which the function is undefined. Singularities are crucial points of a function, because the expansion of a function into a power series depends on the nature of singularities of the function. For the purpose of this thesis, we are interested to analyze those functions, which have several types of singularities. Practically, one of these singularities dominates the function. Therefore it is important to know about this singular point to analyze the local behavior of the function around this point.

The convergency of the sequence of partial sums depends crucially on the singularities of the function represented by the series. Several types of singularities may arise in physical (nonlinear) problems. The dominating behavior of the function $u(x)$ represented by a series may be written as

$$U(x) \sim A\left(1 - \frac{x}{x_c}\right)^{\alpha} \quad \text{as } x \to x_c \tag{1.2.1}$$

Where A is a constant and x_c is the critical point with the critical exponent α. If α is a negative integer then the singularity is a pole; otherwise if it is a nonnegative rational number then the singularity is a branch point. We can include the correction terms with the dominating part in (1.2.1) to estimate the degree of accuracy of the critical points. It can be as follows

$$U(x) \sim A\left(1 - \frac{x}{x_c}\right)^{\alpha}\left[1 + A_1\left(1 - \frac{x}{x_c}\right)^{\alpha_1} + A_2\left(1 - \frac{x}{x_c}\right)^{\alpha_2} + \ldots\right] \quad \text{as } x \to x_c \tag{1.2.2}$$

Where $0 < \alpha_1 < \alpha_2 < M$ and A_1, A_2, M are constants. $\alpha_i + \alpha \notin N$ for some i, then the correction terms are called confluent. Sometimes the correction terms can be logarithmic. e.g,

$$U(x) \sim A\left(1-\frac{x}{x_c}\right)^{\alpha}\left\{1+\ln\left|1-\frac{x}{x_c}\right|\right\} \text{ as } x \to x_c. \tag{1.2.3}$$

Sometimes the sign of the series coefficients indicate the location of the singularity. If the terms are of the same sign the dominant singular point lie on the positive x-axis. If the terms take alternately positive and negative signs then the singular point is on the negative x-axis.

Following are few examples with different types of singularities:

Example-1.2.1. (Singularities for single variable functions)

1) Singularities that are poles: $u(x)=(3-2x)^{-1}+\cos(x)$.

Here $u(x)$ is an algebraic function whose singularity is at $x_c=\frac{3}{2}$, the critical exponent α =-1, which makes the singularity a pole.

2) Algebraic singularities with different exponents:

$$u(x)=(3-2x)^{-1/2}+\left(2-\frac{x}{3}\right)^{-1/3}+\left(1-\frac{x}{2}\right)^{-1/4}.$$

Here $u(x)$ has several singular points. The singular points are at $x_c=\frac{3}{2},6,2$ and the critical exponents are $\alpha=-\frac{1}{2},-\frac{1}{3},-\frac{1}{4}$ respectively. In this example the singular points are branch points. Though there are a number of singularities for $u(x)$, only one of these singularities will dominate the local behavior of $u(x)$.

3) Logarithmic singularity:

$$u(x)=\ln\left(1+\frac{x}{3}\right)+\sin(x).$$

Here $u(x)$ has a logarithmic singularity at $x_c=-3$.

4) Essential singularity:

$$u(x)=\exp(3-2x)^{-1}.$$

Here $u(x)$ has an essential singularity at $x_c=\frac{3}{2}$ with critical exponent α = -1.

5) Algebraic dominant singularity with a secondary logarithmic behavior:

$$u(x)=\left(1-\frac{x}{2}\right)^{-1/2}+\ln\left(1-\frac{x}{3}\right).$$

The algebraic dominant singularity of $u(x)$ here is at $x_c=2$ with critical exponent $\alpha=-\frac{1}{2}$, which makes it a branch point. And a logarithmic singularity at $x_c=3$.

6) nth root singularity: $u(x)=\left(1-\frac{x}{2}\right)^{-1/n}+\exp(x)$.

Here $u(x)$ has a branch point with the critical exponent $\alpha=-\frac{1}{n}$ at $x_c=2$.

1.3 Elementary bifurcation theory

In this thesis we have investigated a nonlinear problem in fluid dynamics. Solutions of nonlinear problems often involve one or several parameters. As the parameter varies, so does the solution set. A bifurcation occurs where the solution of a nonlinear system alter their qualitative behavior while a parameter changes its value. In particular, bifurcation theory shows how the number of steady solutions of a system depends on parameters. Examples of bifurcation are: Simple turning points, in which two real solutions become complex conjugate solutions and pitchfork bifurcation, in which the number of real solutions changes discontinuously from one to three (or vice versa) We intend to introduce some basic concepts of bifurcation theory. Drazin [10] discussed the bifurcation theory in detail.

Consider a functional map $F:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. We seek for the solutions

$$u=U(x) \text{ of } F(x,u)=0. \tag{1.3.1}$$

Bifurcation diagrams can show the solutions. In these diagrams solution curves are drawn in the (x,u) plane. Let (x_0,u_0) be a solution of equation (1.3.1), i.e.

$$F(x_0,u_0)=0 \tag{1.3.2}$$

then, F can be expanded in a Taylor series about (x_0,u_0) and we can study the solution set in that neighborhood provided that F is smooth. Thus we obtain

$$0 = F(x,u)$$

$$= F(x_0,u_0) + (u-u_0)F_u(x_0,u_0) + (x-x_0)F_x(x_0,u_0) + \frac{1}{2}(u-u_0)^2 F_{uu}(x_0,u_0) + \ldots \qquad (1.3.3)$$

If, we assume that, $F_u(x_0,u_0) \neq 0$, then

$$u(x) = u_0 - (x-x_0)\frac{F_x(x_0,u_0)}{F_u(x_0,u_0)} + O(x-x_0), \text{ as } x \to x_0. \qquad (1.3.4)$$

This gives only one solution curve in the neighborhood of the point $(x_0.u_0)$ in the bifurcation diagram. However, if we replace (x_0,u_0) with (x_c,u_c), where

$$F(x_c,u_c) = 0, F_u(x_c,u_c) = 0, \qquad (1.3.5)$$

then the expansion (1.3.3) shows that there are at least two solution curves in the neighborhood of (x_c,u_c). The point (x_c,u_c) is called a bifurcation point.

Example-1.3.1. Let F be a function defined as

$$F(x,u,\varepsilon) = \frac{1}{2}u^2 - \frac{1}{3}x^3 + x - \frac{2}{3} + \varepsilon = 0 \qquad (1.3.6)$$

Where ε is some real parameter. When ε =0, Figure1.3.1, there arise a bifurcation point at $(1,0)$ and a turning point at $(-2,0)$.

When $0 < \varepsilon < 4/3$, there are two separate branches of the bifurcation curve, one an isola and the other unbounded. When, the value of ε increases, in the considered interval, these two branches move apart from each other. Figure 1.3.2 and Figure 1.3.3 shows this behavior of F for different values of ε.

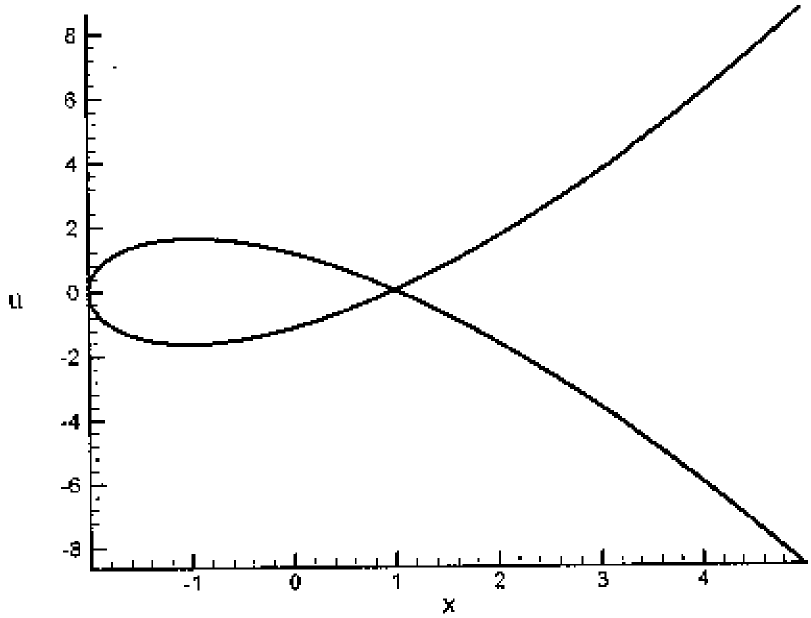

Figure1 3 1: Bifurcation diagram of $F(x,u,\varepsilon)$ in (x,u) plane when $\epsilon = 0$.

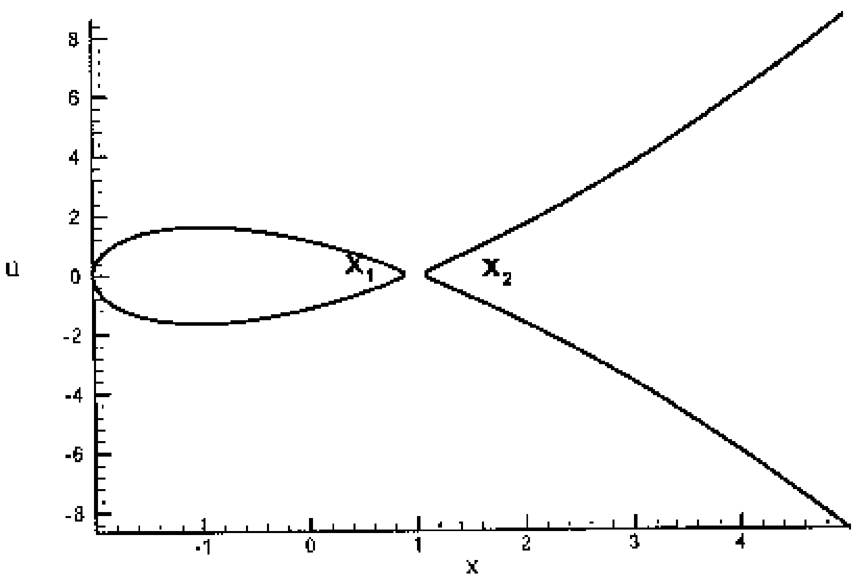

Figure 1.3.2: Bifurcation diagram of $F(x,u,\varepsilon)$ in (x,u) plane when $\varepsilon = 0.01$.

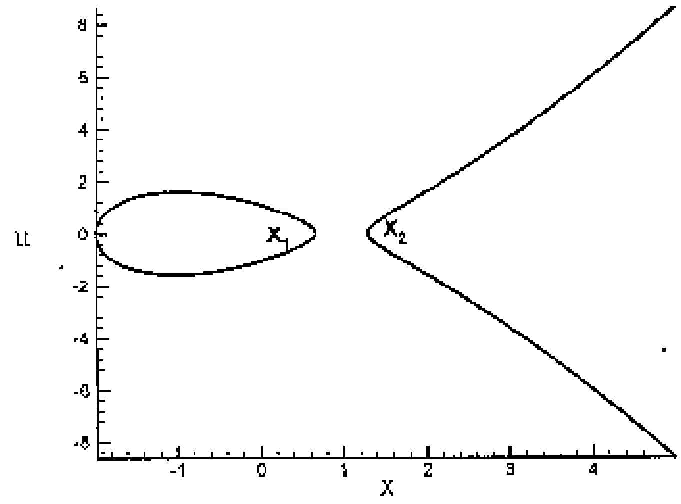

Figure 1.3.3: Bifurcation diagram of $F(x,u,\varepsilon)$ in (x,u) plane when $\varepsilon = 0.1$.

In Chapter 2, we will have an overview of approximation method. There are several variations on the Pade' method of summing power series. One such method consists of recasting the series into continued fraction instead of rational fraction form. This procedure closely resembles Pade' summation because here also only algebraic operations are required. In the next section we will discuss about continued fraction.

1.4 Continued Fraction

Continued fraction has a long history. For historical survey one can go through [5] and [16]. Continued fraction is very useful to analyze the dynamical systems, notably in connection with renormalization. We will discuss the basic concepts of continued fractions.

Let x be a rational or irrational number, then the simple continued fraction of x is

$$x = \alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cdots + \cfrac{1}{\alpha_{N-1} + \cfrac{1}{r_N}}}} = [\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{N-1}, r_N] \qquad (1.4.1)$$

Where, $\alpha_i = floor\left(\frac{1}{r_i}\right)$ (the integral part of $\left(\frac{1}{r_i}\right)$), for $0 \le i < N$. Here all α_i's are nonnegative integers and r_N is the Nth remainder. When $x \in Q$ (rational number), $r_N = 0$. If x is irrational number then the remainder can never vanish and we get the infinite continued fraction. ie. $x = [\alpha_0 . \alpha_1, \alpha_2, ...]$.

Example-1.4.1 Let $x = \frac{74}{23}$, then

$$\frac{74}{23} = 3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}} = [3,4,1,1,2]$$

Again, let $u(x) = e^x$, then

$$e^x = 1 + \cfrac{x}{1 + \cfrac{x}{-2 + \cfrac{x}{-3 + \cfrac{x}{2 + \cfrac{x}{5}}}}}.$$

Consider a function $u(x)$, which represents the power series

$$U(x) = \sum_{i=0}^{\infty} a_i x^i \quad \text{as} \quad x \to 0. \tag{1.4.2}$$

Let us now see how it can be expressed as a continued fraction. The Nth convergent of the series (1.4.2) is

$$U_N(x) = \sum_{i=0}^{N-1} a_i x^i . \tag{1.4.3}$$

In order to convert (1.4.3) into continued fraction, assume that all the inverse that we need to exist.

The continued fraction of (1.4.3) is

$$U_N(x) \sim a_0 + \cfrac{a_1 x}{1 + \cfrac{a_1^{(1)} x}{1 + \cfrac{a_1^{(2)} x}{1 + \ddots}}} \tag{1.4.4}$$

$$= \frac{a_0}{1+} \frac{a_1 x}{1+} \frac{a_1^{(1)} x}{1+} \frac{a_1^{(2)} x}{1+ \ldots}.$$

The convergent of (1.4.2) is rational function in the variable x.

In general, we obtain a rational approximant from (1.4.4) of the form

$$\frac{A_N(x)}{B_N(x)} = \frac{c_0 + c_1 x + c_2 x^2 + \ldots + c_N x^N}{d_0 + d_1 x + d_2 x^2 + \ldots + d_N x^N}, \qquad (1.4.5)$$

which matches with certain number of terms of the series (1.4.2).

In particular, the roots of the denominator $B_N(x)$ give the singularity of the series (1.4.2). When the series (1.4.2) represents a rational function, the remainder of (1.4.5) must eventually reduce to a constant, and the process (1.4.4) terminates after a finite number of iterations. Otherwise, it never terminates and we obtain the infinite continued fraction.

1.5 Overview of the work

This thesis is concerned with the study of computer based approximation techniques which are of Pade'-Hermite class. Many researchers have studied the application of these approximation techniques in fluid dynamical problems. For over the last quarter century many powerful approximants have been introduced for the approximation of the function by using its power series. Among them most of the methods are described for the series involving single independent variable and a few are derived for the power series involved with two or several independent variables. Many researchers hitherto have found remarkably more accurate results by using several approximation methods. The remainder of this thesis is as follows:

In Chapter 2, we have reviewed the Pade'-Hermite class of approximation techniques to determine the coefficients of the approximant. We have discussed several of these kind of approximants with some example. Then in Chapter 3, we have discussed the performance of these approximants on some model functions, on two model nonlinear differential equations and a fluid dynamical problem. Finally in chapter 4, we have summarized our work and give some ideas for future work.

Chapter-2

Approximation methods

Introduction

This thesis is concerned with the study of the application of computer based approximation techniques to reveal the local behavior of a perturbation series around its singular point. The approximation methods are widely used to approximate functions in many areas of applied mathematics

The mathematical model of physical phenomena usually results in nonlinear equations, which may be algebraic, ordinary differential, partial differential, integral or combination of these. A value of independent variable for which the function is undefined is known as a singularity of the function. Singularity plays an important role in many areas of applied science. Particularly in fluid dynamics, the presence of singularities may reflect some changes in the nature of the flow and their study is of great practical interest. Sometimes it is very difficult to find out the exact solution of physical problems. Particularly in statistical mechanics, there are a large number of problems for which the first few terms of the power series may be obtained exactly while the exact solution is unobtainable. The three dimensional Ising model is a good example. On the other hand, if the power series expansion of a nonlinear system is given, but their corresponding function is not known, then it becomes difficult to reproduce the function from the given power series. However, one can study their singularities by using the approximation methods. In order to study these problems many powerful techniques have been used to find the power series coefficients. At the same time, a variety of methods have been introduced for getting the required information about the singularities by using a finite number of series coefficients.

Brezinski [5] studied history of continued fraction and Pade′ approximants. Blanch [7] evaluated continued fractions numerically. Also the applications of continued fractions and their generalizations to problems in approximation theory have been studied by Khovanskii [15]. Baker and Graves-Morris[1] studied Pade′approximants and its properties. Algebraic and differential approximants [2] are some useful generalizations of Pade′ approximants. Khan [14] analyzed singularity behavior by summing power series. Khan [13] also introduced a new model of Differential approximant for single independent variable, called High-order differential approximant (HODA), for the summation of power series. The method is a special type of Pade′-Hermite class and it is one of the best methods of singularity analysis for the problems of single independent variable.

The reminder of this thesis paper is organized as follows:

We will study the Pade′-Hermite class of approximants and then the development of some approximants in this class such as Pade′, Algebraic and Differential Approximants. Drazin-Tourigney is one kind of Algebraic Approximant and High-order differential approximants is an extension of Differential Approximants.

2.1 Pade′-Hermite approximants

In 1893, Pade′ and Hermite introduced Pade′-Hermite class. All the one variable approximants that were used or discussed throughout this thesis paper belong to the Pade′-Hermite class. In its most general form, this class is concerned with the simultaneous approximation of several independent series. Firstly we describe the Pade′-Hermite class from it's point of view.

Let $d \in \mathrm{N}$ and let the $d+1$ power series

$$U_0(x), U_1(x), \dots, U_d(x)$$

are given. We say that the $(d+1)$ tuple of polynomials

$$P_N^{[0]}, P_N^{[1]}, \dots, P_N^{[d]}$$

where
$$\deg P_N^{[0]} + \deg P_N^{[1]} + \dots + \deg P_N^{[d]} + d = N, \tag{2.1.1}$$

is a Pade′-Hermite form of these series if

$$\sum_{i=0}^{d} P_N^{[i]}(x)U_i(x) = O(x^N) \text{ as } x \to 0. \qquad (2.1.2)$$

Here $U_0(x), U_1(x), \ldots, U_d(x)$ may be independent series or different form of a unique series. We need to find the polynomials $P_N^{[i]}$ that satisfy the equations (2.1.1) and (2.1.2). These polynomials are completely determined by their coefficients. So, the total number of unknowns in equation (2.1.2) is

$$\sum_{i=0}^{d} \deg P_N^{[i]} + d + 1 = N + 1. \qquad (2.1.3)$$

Expanding the left hand side of equation (2.1.2) in powers of x and equating the first N equations of the system equal to zero, we get a system of linear homogeneous equations. To calculate the coefficients of the Pade'-Hermite polynomials we require some sort of normalization, such as

$$P_N^{[i]}(0) = 1 \text{ for some } 0 \le i \le d. \qquad (2.1.4)$$

It is important to emphasize that the only input required for the calculation of the Pade'-Hermite polynomials are the first N coefficients of the series $U_0, \ldots, U_d$. The equation (2.1.3) simply ensures that the coefficient matrix associated with the system is square. One way to construct the Pade'-Hermite polynomials is to solve the system of linear equations by any standard method such as Gaussian elimination or Gauss-Jordan elimination.

2.2 Pade' approximants

Pade' approximant is a technique for summing power series that is widely used in applied mathematics [2]. Pade' approximant can be described from the Pade'-Hermite class in the following sense.

In the Pade'-Hermite class, let $d = 1$ and the polynomials $P_N^{[0]}$ and $P_N^{[1]}$ satisfy equations (2.1.1) and(2.1.2). One can define an approximant $u_N(x)$ of the series $U(x)$ by

$$P_N^{[1]}U_N - P_N^{[0]} = 0, \qquad (2.2.1)$$

where

$$U_1 = U_N \text{ and } U_0 = -1.$$

Then we select the polynomials

$$P_N^{[0]}(x)=\sum_{i=0}^{n} b_i x^i \text{ and } P_N^{[1]}(x)=\sum_{i=0}^{m} c_i x^i . \tag{2.2.2}$$

Such that $n+m \le N$. the constants b_i's and c_i's are unknowns to be determined. So that,

$$U_N(x)P_N^{[1]}(x)-P_N^{[0]}(x)=O\left(x^{n+m+1}\right). \tag{2.2.3}$$

Equating the first $n+m$ equations of (2.2.3) equal to zero and the normalization condition in equation (2.1.4), we find the values of b_i's and c_i's. Then, the rational approximant known as Pade' approximant denoted as

$$u_N(x)=\frac{P_N^{[0]}(x)}{P_N^{[1]}(x)}, \tag{2.2.4}$$

help us to approximate the sum of the power series $U(x)$. And the zeroes of the polynomial $P_N^{[1]}(x)$ happens to be identical with the singular point (points) of $U(x)$. In order to evaluate the Pade' approximants for a given series numerically, we have used symbolic computation language such as MAPLE. The Pade' approximants have been used not only in tackling slowly convergent, divergent and asymptotic series but also to obtain singularity of a function from its series coefficients. The zeroes of the denominator $P_N^{[1]}$ give the singular point such as pole of the function $u(x)$ if it exists.

Example-2.2.1. Consider $u(x)=\dfrac{1}{1-2x}+e^x$, a function with a simple pole. After applying the normalization condition $c_0=1$, we obtain the polynomial coefficients $P_5^{[0]}$ and $P_5^{[1]}$ for deg $P_5^{[0]}=n$ and deg $P_5^{[1]}=m$. When $m=n=2$,

$$P_5^{[0]}=2-\frac{40}{17}x-\frac{17}{18}x^2 \text{ and } P_5^{[1]}=1-\frac{91}{34}x+\frac{791}{612}x^2.$$

When $m=n=3$,

$$P_7^{[0]}=2-\frac{27704}{13639}x-\frac{54457}{68195}x^2-\frac{2072}{13639}x^3$$

and

$$P_7^{[1]}=1-\frac{68621}{27278}x+\frac{153323}{136390}x^2-\frac{303409}{1636680}x^3.$$

The table below will show the convergence to the singular point of $u(x)$ on application of Pade' approximant.

Table2.2.1: The approximation of x_c by Pade' for the function in example 2.2.1

m, n	x_c
2,2	.4891882678
3,3	.5000370775

2.3 Algebraic approximants

Algebraic approximant is a special type of Pade'-Hermite approximants. In the Pade'-Hermite class we take

$$d \geq 1, U_0 = 1, U_1 = U, \ldots, U_d = U^d .$$

Let $U(x)$ represent power series of a function and $U_N(x)$ is the partial sum of that series. An algebraic approximant $u_N(x)$ of $U(x)$ can be defined as the solution of the equation:

$$P_N^{[0]}(x) + P_N^{[1]}(x)U_N(x) + P_N^{[2]}(x)U_N^2(x) + \ldots + P_N^{[d]}(x)U_N^d(x) = 0 \qquad (2.3.1)$$

Where d represent the degree of the partial sum $U_N(x)$. The algebraic approximant $u_N(x)$, is in general a multivalued function with d branches.

The solution of the equation (2.3.1) with $d \geq 1$ gives us the coefficients of the polynomials $P_N^{[\,]}(x)$. The discriminant of this equation approximates the singularity of $U(x)$.

Here,

$$\sum_{i=0}^{d} P_N^{[i]}(x)U_N^i(x) = O(x^N) \qquad (2.3.2)$$

And

$$\sum_{i=0}^{d} \deg P_N^{[i]} + d = N . \qquad (2.3.3)$$

And the total number of unknowns in (2.3.2) are

$$\sum_{i=0}^{d} \deg P_N^{[i]}(x) + d + 1 = N + 1 . \qquad (2.3.4)$$

In order to determine the coefficients of the polynomials $P_N^{[i]}$ one can set $P_N^{[0]}(0)=1$ for normalization. Through out this thesis we will indicate Algebraic approximants as AA.

Example-2.3.1 Consider

$$u(x)=(1-2x)^{1/3}+\cos x.$$

Let $d=2$ and $\deg P_8^{[0]}=\deg P_8^{[1]}=\deg P_8^{[2]}=2$ to apply the algebraic approximation method on the power series of the given function. After we set the normalization condition $P_8^{[0]}(0)=1$, we get the polynomials

$$P_8^{[0]}(x)=1+\frac{7118420073046 37}{477528552413468}x+\frac{34582752228787591}{51573083660654544}x^2,$$

$$P_8^{[1]}(x)=-\frac{3248460155432041}{955057104826936}+\frac{22334398779627}{477528552413468}x+\frac{1555136999112437}{128932709151636 36}x^2,$$

$$P_8^{[2]}=\frac{2770931603018573}{1910114209653872}+\frac{994609833893}{238764276206734}x+\frac{2429343756212869}{12893270915163636}x^2.$$

Here the discriminant gives us the singularity at $x_c=0.4825548636$. If we increase the degree of the polynomial coefficients it may give us a better approximation. So, again let $\deg P_{11}^{[0]}=\deg P_{11}^{[1]}=\deg P_{11}^{[2]}=3$ and $d=2$, following the same procedure we get the singularity at $x_c=0.5039567121$.

Again taking $d=2$ and $\deg P_9^{[0]}=\deg P_9^{[1]}=\deg P_9^{[2]}=4$ the singularity is calculated at $x_c=0.4989742074$. The table below will show the comparative results of the convergence of the algebraic approximation method to the singular point.

Table2.3.1: The approximation of x_c by AA for the function in example 2.3.1

$\deg P_N^i$	d	x_c
2	2	0.4825548636
3	2	0.5039567121
4	2	0.4989742074

Note that $d=3$ may be more accurate for this problem.

2.4 Drazin -Tourigney Approximants

Drazin and Tourigney in [8] implemented the idea $d = O(\sqrt{N})$ as $N \to \infty$. Their method is simply a particular kind of algebraic approximant, satisfying the equation (2.3.1). In this method they considered

$$\deg P_N^{[i]} = d - i \tag{2.4.1}$$

and

$$N = \frac{1}{2}\left(d^2 + 3d - 2\right). \tag{2.4.2}$$

Through out this thesis we will mention Drazin-Tourigney approximant as D-T method.

2.5 Differential approximants

By taking $d \geq 2, U_0 = 1, U_1 = U, U_2 = DU$ and $U_d = D^{d-1}U$, where $D \equiv \frac{d}{dx}$, a differential approximant $u_N(x)$ of the series $U(x)$ can be defined as the solution of the differential equation

$$P_N^{[0]} + P_N^{[1]}U_N + P_N^{[2]}DU_N + \ldots + P_N^{[d]}D^{d-1}U_N = 0. \tag{2.5.1}$$

Here (2.5.1) is homogeneous linear differential equation of order $(d-1)$ with polynomial coefficients. The singularities of $U(x)$ are located at the zeroes of the leading polynomial $P_N^{[d]}(x)$. Hence, the zeroes of $P_N^{[d]}(x)$ may provide approximations of the singularities of the function $u(x)$. Through out this thesis Differential appoximants are represented by DA.

Example 2.5.1 Consider

$$u(x) = \ln(1 - 2x).$$

Taking $d = 2$ for (2.5.1) and applying (2.1.1) and (2.1.2), we obtain $P_5^{[2]}(x) = \frac{1}{2} + 35x - 60x^2$ and the singular point at $x_c = 0.5972$. In a similar procedure taking $d=3$ gives us more accurate result, ie $x_c = 0.5000$. The table below shows a comparative result.

Table2.5.1: The approximation of x_c by Differential Approximant for the function in example 2.5.1

N	d	x_c
5	2	0.5972
9	3	0.5000
14	4	0.5000

2.6 High-order differential approximants

Khan [13] introduced an extension of differential approximant, which he mentioned as High-order differential approximant. When the function has a countable infinity of branches, then the fixed low-order differential approximants may not be useful. So, for these cases he considered d increase with N. It lead to a particular kind of differential approximant $u_N(x)$, satisfying equation (2.4.2). Here

$$N = \frac{1}{2}d(d+3) \text{ and } \deg P_N^{[i]} = i. \tag{2.6.1}$$

From (2.1.3) he deduced that there are

$$\frac{1}{2}\left(d^2 + 3d + 2\right)$$

unknown parameters in the definition of the Pade'-Hermite form. In order to determine those parameters, we use the N equations

$$P_N^{[0]}(x) + \sum_{i=1}^{d} P_N^{[i]}(x) D^{i-1} U_N(x) = O\left(x^N\right) \text{ as } x \to 0.$$

In addition one can normalize by setting $P_N^{[0]}(0) = 1$. Then there remains as many equations as unknowns. One of the roots, say $x_{c,N}$, of the coefficient of the highest derivative, i.e. $P_N^{[d]}\left(x_{c,N}\right) = 0$, gives an approximation of the dominant singularity x_c of the series U. If the singularity is of algebraic type, then the exponent α may be approximated by

$$\alpha_N = d - 2 - \frac{P_N^{[d-1]}(x_{c,N})}{DP_N^{[d]}(x_{c,N})}. \quad (2.6.2)$$

Through out this thesis High-order differential approximants are represented by HODA.

2.7 Discussion

Pade'-Hermite class is constructed over the technique of truncated continued fraction. It was discussed in equation (1.4.5) and the polynomial coefficients of Pade' were constructed by taking successive truncated continued fractions. In this chapter we had an overall study about the Pade'-Hermite class of approximation methods. Examples show the performance of Pade', AA and DA explicitly. We must mention that D-T method is an improved algebraic approximation technique. HODA is modified differential approximant. We notice that the performance of D-T method is better than that of Pade' when the singular point is a branch point. But performance of HODA is almost in every case convincing.

In the next chapter we will study some nonlinear differential equations with the application of Pade', D-T and HODA. And a fluid dynamical problem to reveal the behavior of the unknown solution with the results of the application of these techniques.

Chapter-3

Approximate Solution of Non-linear Systems

One cannot hope to obtain exact solutions to most nonlinear differential equations. There are only a limited number of systematic procedures for solving them, and these apply to a very restricted class of equations. Moreover, even when a closed-form solution is known, it may be so complicated that its qualitative properties are obscured. Thus for most nonlinear equations it is necessary to have reliable techniques to determine the approximate behavior of the solutions.

The solutions of differential equations encountered in practice are regular at almost every point; in the neighborhood of ordinary points Taylor series provide an adequate description of the solution. However, the distinguishing feamres of the solution are its singularities. Determining the location and nature of these singularities, without solving the differential equation, requires the techniques of local analysis.

A solution of a linear equation can only be singular at points where the coefficient functions are singular, and at no other points. But the solutions of nonlinear differential equations possess a richer spectrum of singular behaviors. Solutions of nonlinear equations, in addition to having fixed singularities, may also exhibit new kinds of singularities, which move around in the complex plane as the initial or boundary conditions vary. Such singularities are called Spontaneous or movable singularities.

Example 3.1 Consider the linear differential equation

$$u' + \frac{u}{x-1} = 0, \;\; u(0) = 1.$$

It has a singular point at $x=1$, so does the solution $u(x)=\frac{1}{1-x}$. For the differential equation it is a regular singular point, for the solution it is a pole. If we replace $u(0)=1$ with $u(0)=2$, the new solution $u(x)=\frac{2}{1-x}$ still has a pole at $x=1$.

Example 3.2 Let us consider the nonlinear differential equation

$$u'(x)=u^2,\ u(0)=1.$$

The solution is $u(x)=\frac{1}{1-x}$. Even though the equation is not singular at $x=1$, a pole spontaneously appeared. If we change the initial condition as $u(0)=2$, the solution will be changed to $u(x)=\frac{2}{1-2x}$, the pole has changed its position to $x=\frac{1}{2}$.

Bender and Orszag [2] discussed a nnmber of examples with local analysis. Without solving the equation they tried to locate the dominating singular points of this kind of nonlinear differential equations by the application of approximation method. And try to locate the dominating singular point with critical exponent which analyses the form of the singularity of this kind.

In this chapter we have examined few problems where approximation methods were applied to reveal the singularities and have compared our result with others.

Now we will see some test functions with different types of singularities, where we compare the performance of approximation methods described in Chapter 2.

3.1 Some Test Functions

Consider five test functions with different types of singularities:

1. Additive algebraic singularities with the same exponent:

$$u(x)=(1-x/2)^{-1/2}+2(1-x/3)^{-1/2}+3(1-x/4)^{-1/2}+4(1-x/5)^{-1/2}.$$

2. Additive algebraic singularities with different exponents:

$$u(x) = (1 - x/2)^{-1/2} + 2(1 - x/3)^{-1/3} + 3(1 - x/4)^{-1/4} + 4(1 - x/5)^{-1/5}.$$

3. Confluent algebraic/ logarithmic singularity:

$$u(x) = \exp(x)(1 - x/2)^{-1/2} + \ln(1 - x/2).$$

4. Algebraic dominant singularity with a secondary logarithmic singularity:

$$u(x) = \exp(x)(1 - x/2)^{-1/2} + \ln(1 - x/3).$$

5. Essential singularity:

$$u(x) = \exp\left[(1 - x/2)^{-1/2}\right].$$

The number, ρ_N of correct decimal figures in the approximation of x_c by various methods for the functions of the above examples is shown in the table below.

Here $$\rho_N = -\log_{10}\left|\frac{x_c - x_{c,N}}{x_c}\right|.$$

Table-3.1.1:The number ρ_N of correct decimal figures in the approximation of x_c by various methods for the functions 1.-5.

Example	N	Pade'	D-T	HODA
1.	26	2.42	2.13	Exact
2.	36	2.71	2.63	17.55
3.	20	2.10	2.08	4.31
4.	20	2.09	1.94	Exact.
5.	10	Exact	0.52	2.24

The results of approximating the dominating singularity x_c in each case by various methods of series analysis are shown in Table 3.1.1 where we have shown the number of correct decimal places. Here the value of N is rather small, and so one should be careful not to infer too much from the evidence. Nevertheless, it is interesting to note how badly the D-T method compares with the others. In most of the cases, we see that by using a small number of series coefficients, the High-order differential approximant produces the exact results The same is true for the critical exponent.

3.2 Spontaneous singularities in the complex plane

Consider the Riccati equation with the initial condition

$$u' = 1 - xu^2, u(0) = 1. \tag{3.2.1}$$

The solution of the Riccati equation becomes singular at a finite negative value of x. The presence of this singularity can be understood from the graph of the tangent field given in Figure 3.2.1 The tangent field indicate that the solution which satisfies the initial condition $u(0) = 1$ becomes large and negative for negative x. When u is sufficiently large and negative, 1 becomes negligible compared with $-xu^2$. The resulting approximate differential equation is

$$u' \sim -xu^2,\ u \to -\infty.$$

The solutions to $u' = -xu^2$, $u(x) = \left(\frac{x^2}{2} + C\right)^{-1}$, that are negative somewhere have $C < 0$, so they become infinite for some finite negative x.

To find the location of this singularity numerically, let $w(x) = 1/u(x), w(x)$ satisfies the differential equation $w' = x - w^2\left[w(0) = 1\right]$. Numerical integration of this differential equation gives a zero of w near $x = -2.12$. Thus, u becomes singular at $x \cong -2.12$.

From this result one might expect the Taylor series solution about $x = 0, u(x) = \sum_{k=0}^{\infty} b_k x^k$, to have a radius of convergence of *2.12*. However, a numerical evaluation of the Taylor coefficients b_k indicates that the true radius of convergence R is close to *1.228*:

$$R = \lim_{k \to \infty} \left|\frac{b_k}{b_{k+1}}\right| \cong 1.228. \tag{3.2.2}$$

R has this much smaller value because $u(x)$ also has complex spontaneous singularities. Further numerical integration shows that $w(x)$ has a zero in the complex plane at $x_0 \cong 0.313 + 1.188i$. This is the zero of $w(x)$ which is nearest to the origin in the complex-x plane. Its distance to the origin is $|x_0| \cong 1.228$. Therefore, it is this singularity and not the one at $x \cong -2.12$ that determines the radius of convergence of the Taylor series for $u(x)$.

Table-3.2 1 Estimates of the critical point $x_{c,N}$ and the corresponding exponent α_N by using various approximation methods for the differential equation $u' = 1 - xu^2, u(0) = 1$

N	m, n	Pade' $x_{c,N}$	N	d	HODA $x_{c,N}$	HODA α_N	N	d	D-T $x_{c,N}$
10	5,5	-2.144200829	20	5	-2.1122530506	-.9781950250	33	7	-2.103723308
12	6,6	-2.117890610	27	6	-2.1127172633	-.9959445675	42	8	-2.110655435
14	7,7	-2.111567345	35	7	-2.1127172633	-.9959445675	52	9	---------------
16	8,8	-2.112916314	44	8	-2.1128179244	-1.000003854	63	10	-2.083670213
18	9,9	-2.112826475	54	9	-2.1128178874	-.9999999979			
20	10,10	-2.112826475	65	10	-2.1128178874	-1.000000000			

We analyse the location and nature of the singularities by using various generalizations of the approximation methods. From the above analysis Bender and Orszag [2] indicates that the real singularities lies at $x_r \approx -2.112$, but using the ratio test the dominating singularities occurs in the complex plane at $x_c \approx 0.313 + 1.188i$.

On the other hand, the results in Table 3.2.2 indicate that it is possible, by using the High-order differential approximants (HODA), to obtain the radius of convergence- and the critical exponent α - to 18 digits of accuracy with $d = 10(N = 65)$. For comparison, the table also shows the results by using the Pade' and the Drazin-Tourigny (D-T) approximants. It is clear that the High-order differential approximants converges much faster.

Therefore the dominating singularity behavior of the solution is

$u(x) \sim A(x - x_c)^{\alpha}$ as $x \to x_c$,

where $x_c \approx .313409267155988995 \pm 1.1875279690275096i$ and

$\alpha \approx -1.000000000 + 0.879250\ 10^{-18} i$.

Table-3 2.2: Spontaneous singularities in the complex plane Estimates of the critical point $x_{c,N}$ in the complex plane and the corresponding exponent α_N by using various approximation methods for the differential equation $u' = 1 - xu^2, u(0) = 1$.

N	m,n	Pade	N	d	HODA	$\alpha_{d,N}$	N	d	D-T
6	3,3	.3009661 ± 1.1861691i	20	5	.313410540850118767 ± 1.1875314373780251i	-1.00024+.86065 10^{-5} i	33	7	.3135584 ± 1.1856023i
8	4,4	.3115770 ± 1.1878909i	27	6	.313409262641444066 ± 1.1875279736993554i	-1.00000-.6625838 10^{-6} i	42	8	.3134303 ± 1.1872631i
10	5,5	.3132811 ± 1.1875863i	35	7	.313409262641444066 ± 1.8752797369935546i	-1.00000-.6625838 10^{-6} i	52	9	.3134118 ± 1.1874953i
12	6,6	.3134103 ± 1.1875232i	44	8	.313409267155980864 ± 1.1875279690275096i	-.999999+.275891 10^{-13} i	63	10	.3134081 ± 1.1875265i
14	7,7	.3134092 ± 1.1875276i	54	9	.313409267155988995 ± 1.1875279690275096i	.999999-.684590 10^{-18} i	75	11	.3134090 ± 1.1875285i
16	8,8	.3134092 ± 1.1875279i	65	10	.313409267155988995 ± 1.1875279690275096i	-1.00000+.879250 10^{-18} i	88	12	.3134092 ± 1.1875279i
18	9,9	.3134092 ± 1.1875279i			------------------------------				------------------------------

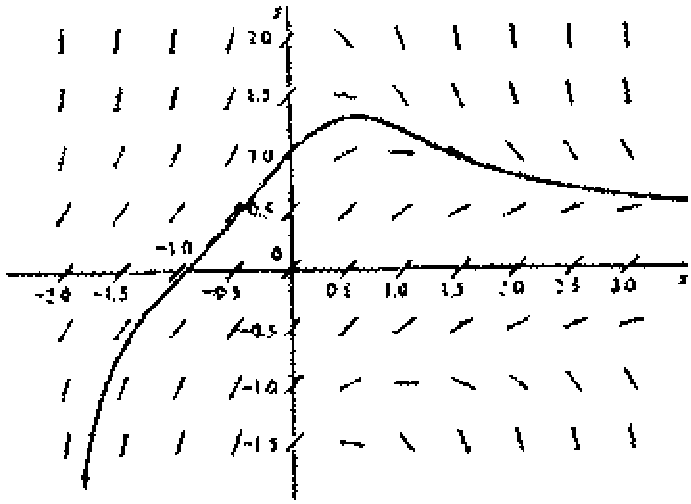

Figure 3.2.1 : The tangent field indicate the solution $u(x)$ of $u' = 1 - xu^2$ which satisfies the initial condition $u(0) = 1$ [Bender & Orszag [2], pp.149]

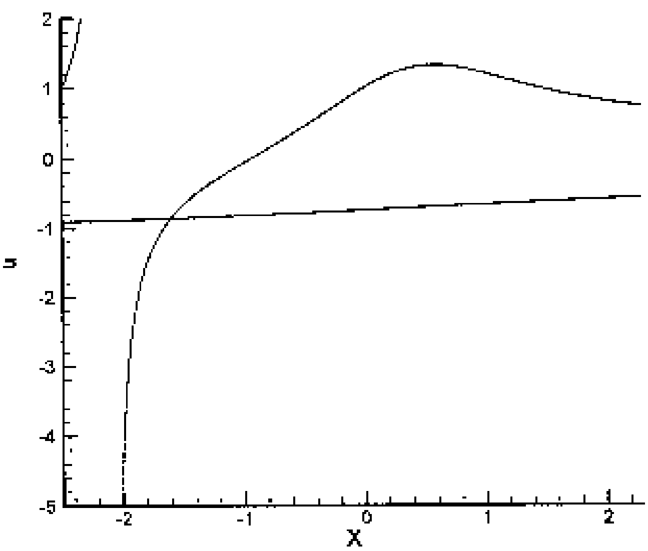

Figure 3.2.2: Approximate solution $U_N(x)$ for the Riccati equation $u' = 1 - xu^2$ with $u(0) = 1$ with the Drazin-Tourigney approximant with d =12

Table 3.2.1 shows the critical point and the corresponding critical exponent of the solution behavior in the real field. The approximate solution is also shown in Figure 3.2.2, which is comparable to Bender and Orszag [2] as in Figure 3.2.1.

3.3 Infinite number of spontaneous singularities

Consider

$$u' = u^2 + x. \tag{3.3.1}$$

Bender and Orszag [2] studied the leading behavior of the solution to the Riccati equation as $x \to +\infty$. The previous example shows that the solution to a nonlinear differential equation may exhibit several spontaneous singularities. We will see that the solution to the above nonlinear equation has an infinite number of singularities along the positive real axis! The Figure 3.4.1 given by Bender and Orszag [2] is a computer plot of solution to the equation satisfying the initial condition $u(0) = 0$. Note that the graph of $u(x)$ resembles that of the function $\tan x$.

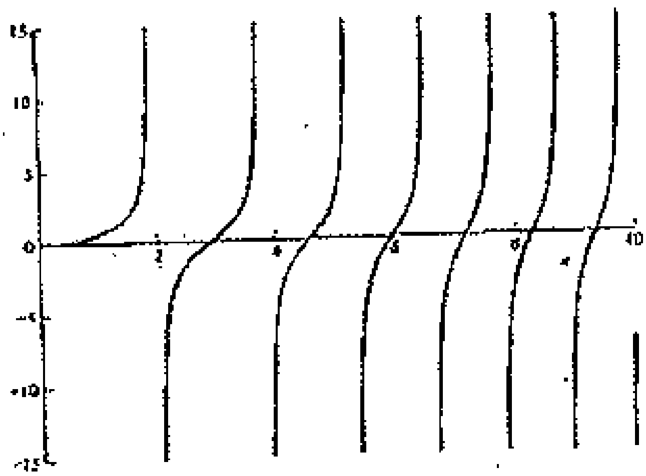

Figure 3.3.1: Computer plot of $u(x)$ to $u' = u^2 + x$, satisfying the initial condition $u(0) = 0$. [Bender & Orszag [2], pp.150]

The ultimate goal of our analysis is to construct a function which closely approximates $u(x)$ as $x \to 0$. However, we begin with a more modest investigation: let us try to determine the nature of the singularities of $u(x)$.

Can the singularities of $u(x)$ be poles? We know that in the neighborhood of a pole the leading behavior is given by $u(x) \sim \frac{A}{(x-a)^{\alpha}} (x \to a)$, where a is the location and α is the order of the pole. Substituting this asymptotic relation into the differential equation and comparing leading terms gives $A = -1, \alpha = 1$. Thus, solutions of the differential equation probably have simple poles. But to prove this conjecture we must show that in some neighborhood of $x = a$ there is a solution in the form of a (convergent) Laurent series

$$u(x) = -\frac{1}{x-a} + \sum_{k=0}^{\infty} b_k (x-a)^k . \qquad (3.3.2)$$

It is left as an exercise to compute the coefficients b_k directly from the differential equation and to verify that the series converges in a neighborhood of $x = a$. Unfortunately, this series expansion is only valid in a disk which does not contain any other singularity of u. It would be much more desirable to have a uniform description valid for large x which exhibits the multiple singularity structure of $u(x)$.

To obtain such an expression it is necessary to approximate the differential equation by one that has an analytical solution. However, in this case an approximation which reveals the nature of the nonlinear differential equation is not easy to find! It would certainly be nice if one could neglect x in favor of u^2 or u^2 in favor of x in the differential equation. Unfortunately, a glance at the figure 3.3.1 shows that as $x \to +\infty$, sometimes $u^2 > x$ and sometimes $x > u^2$; we need a more subtle approximation which is uniformly valid as $x \to +\infty$.

An ingenious trick is to substitute

$$u(x) = x^{1/2} v(x). \qquad (3.3.3)$$

The equation for $v(x)$ is then

$$v' = (1 + v^2) x^{1/2} - \frac{v}{2x} . \qquad (3.3.4)$$

Now the term $\frac{v}{2x}$ is uniformly negligible for large x because $v \le 1+v^2$ for all v and $x^{-1} << x^{1/2}, x \to +\infty$.

The resulting asymptotic differential equation $v' \sim (1+v^2)x^{1/2}, x \to +\infty$ is easily solved because it is separable:

$$u(x) = x^{1/2}v(x) = x^{1/2}\tan\phi(x). \tag{3.3.5}$$

Where $$\phi(x) \sim \frac{2}{3}x^{3/2},\ x \to +\infty.$$

This result suggests that for large x the solution of the Riccati equation has an infinite sequence of first order poles having an accumulation point at $x = \infty$.

The accuracy of this result may be tested in several ways. We could plot the function $\sqrt{x}\tan\left(\frac{2}{3}x^{3/2}\right)$ for large x and compare the result with the figure in Bender and Orszag[2] (see Figure: 3.3.1). However, a better test of this result is to compute $u(x)$ numerically and to plot $\frac{\tan^{-1}\left(\frac{u}{\sqrt{x}}\right)}{\frac{2}{3}x^{3/2}}$ and verify that this ratio approaches 1 as $x \to +\infty$.

We analyse the location and nature of the singularities by using various generalizations of the approximation methods. From the above analysis Bender and Orszag [2] indicates that the real singularities lies at $x_c \approx 1.98635, 3.82534, 5.29562$ etc.

On the other hand, the results in Table 3.3.1 indicate that it is possible, by using the High-order differential approximants (HODA), to obtain the radius of convergence- and the critical exponent α- to 18 digits of accuracy with $d = 10(N = 65)$. For comparison, the table also shows the results by using the Drazin-Tourigny (D-T) approximants. It is clear that the High-order differential approximants converges much faster.

Therefore the dominating singularity behavior of the solution is

$$u(x) \sim A_1(x - x_{c,1})^{\alpha_1} + A_2(x - x_{c,2})^{\alpha_2} + \cdots \text{ as } x \to 0,$$

where $x_{c,1} \approx 1.986352707$ and $x_{c,2} \approx 3.825339191$ with $\alpha_1 = \alpha_2 \approx -\frac{1}{3}$.

The table shows two consecutive singular points of the solution calculated by High-order differential approximant (HODA). Drazin Tourigney method (D-T) gives us the approximate value of the first singular point where the performance of Pade' was not satisfactory. Column five shows the critical exponents calculated as in equation (2.6.2). According to these values it confirms that the singularities are branch points with critical exponent $\alpha_1 = \alpha_2 \approx -\frac{1}{3}$.

Table-3.3.1: Estimates of the critical point $x_{c,N}$ and the corresponding exponent α_N by using two approximation methods for the differential equation

$$u' = u^2 + x \text{ with } u(0) = 0.$$

Bender-Orszag	d	N	HODA	$\alpha_{d,N}$	d	N	D-T
1.98635	3	9	1.9863527074551836589581741443	-0.33333333370031518	4	12	1.988851254
	4	14	1.9863527074304728141430784896	-0 333333333333333335	5	18	1.986350399
	5	20	1.9863527074304728134718183900	-0.33333333333333333	6	25	1.975527722
	6	27	1.9863527074304728134718183899	-0.33333333333333333	7	33	1.986377301
	7	35	1.9863527074304728134718183899	-0.33333333333333333	8	42	1.986352145
	8	44	1.9863527074304728134718183899	-.03333333333333333			
3.82534	3	9	3.8127999524726231235206077488	-0.28137697134512616			-----------
	4	14	3.8253554401122866541502681962	-0.33349723033033788			-------------
	5	20	3.8253391905355145769483591171	-0.33333332165213668			------------
	6	27	3.8253391911604671135914070386	-0.33333333333382770			------------
	7	35	3.8253391911604526481822632051	-0.33333333333333333			-------------
	8	44	3.8253391911604526481815598886 3	-0.33333333333333333			----------------
5.29562			------------------	-------------			-----------------

3.4 Flow in a porous pipe with decelerating wall

The flow in a pipe driven by suction or injection was first considered approximately fifty years ago. Berman [3] first considered the steady Navier-Stokes equations to a fourth order ordinary differential equation. Since then Brady [6], Zaturska and Banks [27] have considered various aspects of the flow, both steady and unsteady. Brady and Acrivos [4] analysed the flow in a pipe with accelerating wall.

The flow in a porous pipe with decelerating wall is important in physical point of view. Practically it is found that there is a range of Reynolds numbers for which there is no real solution to the steady similarity equation; this absence of solutions and the determination of any bifurcation has been the motivation for this study.

In this present problem we have considered the flow in a porous pipe with decelerating wall. We have studied the temporal stability of the flow by using the various generalization of approximation methods.

The steady axisymmetric flow of a viscous incompressible fluid driven along a pipe by the combined effect of the wall deceleration and suction is considered. This type of problem was first investigated by Berman [3] and subsequently by many authors, for example, Terrill and Thomas [24], Zaturska and Banks [27], Makinde [18].

Formulation of the problem:

We consider the steady axisymmetric flow of a viscous incompressible fluid driven along a porous pipe with decelerating wall. Let E be a parameter such that the axial velocity of the wall is Ez. It is assumed that $aE/V = O(1)$ and $V \neq 0(V > 0$ represents suction velocity and $V < 0$ represents the injection velocity).

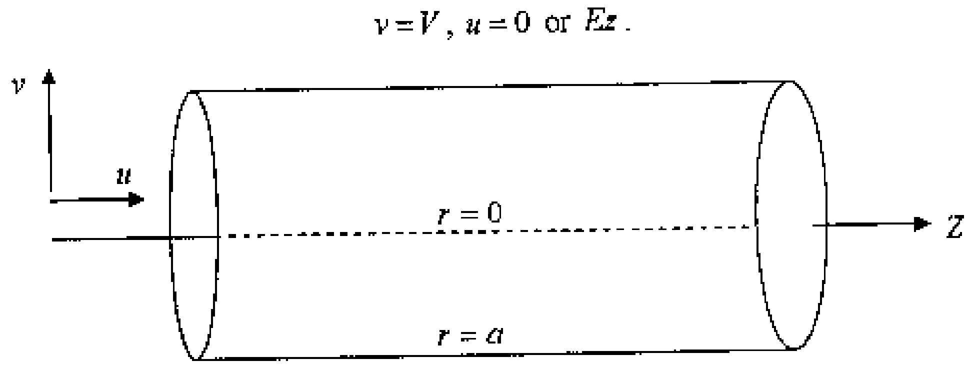

Figure 3.4.1: Schematic diagram of the problem.

By assuming a similarity form for the solution of the Navier-Stokes equation it is found, after non-dimentionalization, that the velocity components (u,v) increasing in the directions of (z,r), respectively, and vorticity ω of the flow may be expressed as (Makinde [19])

$$u=\frac{z}{r}\frac{dF}{dr}, v=-\frac{1}{r}F \text{ and } \omega=-zG.$$

And hence

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rG)\right]=R\left[\frac{G}{r}\frac{dF}{dr}-F\frac{d}{dr}\left(\frac{G}{r}\right)\right], G=\frac{d}{dr}\left(\frac{1}{r}\frac{dF}{dr}\right), \tag{3.4.1}$$

$$F=0, \frac{d}{dr}\left(\frac{1}{r}\frac{dF}{dr}\right)=0, \text{ on } r=0,$$

$$F=-1, \frac{dF}{dr}=-1, \text{ on } r=1.$$

Equations (3.4.1) with the boundary conditions govern the motion of an incompressible fluid through the porous pipe with decelerating wall An exact solution to this complicated nonlinear system of equations for $R \neq 0$ is not available, and so we resort to series analysis by approximation methods. When $R=0$, the equations (3.4 1) can be solved easily. The solution is a parabolic Poiseulli flow It is therefore natural to seek a power series in ascending powers series of R:

$$F(r)=F_0(r)+F_1(r)R+F_2(r)R^2+F_3(r)R^3+\cdots, \tag{3.4.2}$$

$$\left[\frac{1}{r}(rG_n)'\right]' = R\sum_{i=0}^{n-1}\left[G_i\left(\frac{F'_{n-i-1}}{r}\right)-F_i\left(\frac{G_{n-i-1}}{r}\right)'\right], G_n = \left(\frac{1}{r}F_n'\right)', \tag{3.4.3}$$

$$F_n = 0, \left(\frac{1}{r}F_n'\right)' = 0 \text{ on } r = 0.$$

$$F_0 = -1, F_{n+1} = 0, F_0' = -1, F'_{n+1} = 0, \text{ on } r = 1, n = 0,1,2,\ldots$$

Where the prime symbol denotes differentiation with respect to r.

In order to compute the series coefficients, let

$$F_i(r) = \sum_{j=1}^{2i+2} a_{0,2j} r^{2j}$$

and hence

$$F(r,R) = \sum_{i=0}^{\infty}\left(\sum_{j=1}^{2i+2} a_{0,2j} r^{2j}\right) R^i .$$

By substituting this to the equation (3.4.3), the recurrence relation for *F(r)* becomes

$$16j(j+1)^2(j+2)a_{2j+4}r^{2j-1}$$
$$= \sum_{j=1}^{k} 8\frac{(j+1)!}{(j-1)!}(k-j+1)a_{2j+2}a_{2k-2j-2}r^{2k-1} - \sum_{j=1}^{k} 8\frac{(k-j+3)!}{(k-j)!}a_{2j}a_{2k-2j+6}r^{2k+1}$$

We expand $F(r,R)$, $\beta = \left(\frac{F'}{r}\right)'$ at $r = 1$ and $F''(0)$ (that is, stream function, skin friction and centerline axial velocity parameter) in powers of the Reynolds number, to obtain

$$F(r) = -\frac{1}{2}r^2(3-r^2) - \frac{1}{144}r^2(7-r^2)(1-r^2)^2 R + \ldots,$$

$$\beta = 4 - \frac{1}{3}R - \frac{2}{27}R^2 - \frac{17}{1008}R^3 - \ldots. \tag{3.4.4}$$

and

$$F''(0) = -3 - \frac{7}{72}R - \frac{103}{4800}R^2 - \frac{760589}{152409600}R^3 - \ldots \text{ as } R \to 0. \tag{3.4.5}$$

These expansion yield a single solution of the equation (3.4.1), by taking

$$x := R \text{ and } U := F''(0) \quad or \quad \beta$$

in the notation of Chapter 2.

Using a symbolic algebra package such as Maple, the first 54 coefficients (see Appendix II for the coefficients of the series $F''(0)$) of the solution series were obtained. We observed that the sign of the coefficients are same and are monotonically decreasing in magnitude.

The convergence of the series may be limited by a singularity on the positive real axis (Van Dyke [25]). The graphical form of the D'Alembert's ratio test (Domb and Sykes [9]) together with Neville's extrapolation at $\frac{1}{n} = 0$ (that is, $n \to \infty$) reveal the radius of convergence $R = 3.07249$. Following the High-order differential approximant technique, we compute the first and second turning points R_1 and R_2 as $R \to 0$ on the secondary branch (Brady and Acrivos [4]). Our results show that $R_1 \approx 3.0724980042$, $R_2 \approx 8.813114939$.

We used the partial sums of the series to reconstruct the other solutions of the problem. The series has a real singularity at $R = R_1$ and this singularity corresponds to a turning point i.e. a value of R where the number of solutions changes abruptly. It seems to us that $R_{1,N}^{(d)}$ approximates R_1 very well as d increases. We computed a farther turning point at $R = R_2$. The bifurcation diagrams for the approximate solutions are shown in Figures 3.4.2-3.4.5. As can be seen in the figures, the method of Drazin & Tourigny succeeds in continuing the secondary singularity behavior beyond the circle of convergence of the series. The dominating singularity have the form

$$\beta \text{ or } F''(0) \sim A(R - R_1)^{\alpha} \quad as \quad R \to R_1$$

with $\alpha \approx \frac{1}{2}$. It is interesting to notice the absence of real solutions for $R_1 < R < R_2$, and that $\beta \to 0$ as $R \to 2.828847....$, that is, reversal of the flow at the wall will occur.

Table-3.4.1: Estimates $x_{c,n}$ of $R_{1,N}^{(d)}$, $R_{2,N}^{(d)}$ and the critical exponent α_N of the corresponding exponent by HODA of centerline axial velocity. The last row shows the estimates obtained by using the D-T method [17]

d	N	$R_{1,N}^{(d)}$	$\alpha_{1,N}$	$R_{2,N}^{(d)}$	$\alpha_{2,N}$
2	7	3.1748622903786039	0.7344695467871855866	-----------------	------------------
3	12	3.0768641905315211	0.5114909608937660009	-----------------	------------------
4	18	3.0725166917148683	0.4997196198569999405	-----------------	------------------
5	25	3.0724980065120133	0.4999999021203348536	8.8164063517596622	0.4954596938484847968
6	33	3.0724980042445946	0.5000000000982840711	8.8127995287711051	0.4991471015647478523
7	42	3.0724980042458199	0.4999999999999926717	8.8131460358460523	0.4997653746760795187
8	52	3.0724980042458197	0.500000000000000001	8.8131097616625635	0.4999534119259680143
9	**53**	**3.0724980042458197**		**8.813114939**	

Table-3.4.2: Estimates $x_{c,n}$ of $R_{1,d}$, $R_{2,d}$ and the critical exponent α_N of the corresponding exponent by HODA of skin friction. The last row shows the estimates obtained by using the D-T method [17].

d	N	$R_{1,N}^{(d)}$	$\alpha_{1,N}$	$R_{2,N}^{(d)}$	$\alpha_{2,N}$
2	7	3.169913601171531	0.1179160833837563	-----------------	------------------
3	12	3.078390933009397	0.4328071188188094	-----------------	------------------
4	18	3.072506610273949	0.4997190813662425	8.942239101069277	0.4060409189553583854
5	25	3.072498003441932	0.5000000089906489	8.441094619719760	2.7332819731928286414
6	33	3.072498004254962	0.4999999990682636	8.810061064410313	0.5076020616853571302
7	42	3.072498004245820	0.4999999999987227	8.813263038742537	0.4991927215986543196
8	52	3.072498004245819	0.4999999999999999	8.813083811310784	0.5003197469645002604
9	**53**	**3.0724980042458197**		**8.813114939**	

Table 3.4.1-3.4.2 shows that the accuracy of the approximations $R_{1,N}^{(d)}$ and $R_{2,N}^{(d)}$ increases very rapidly with the increase of d. It is remarkable that the secondary singularity also recovered from the information of a single series at the point of expansion.

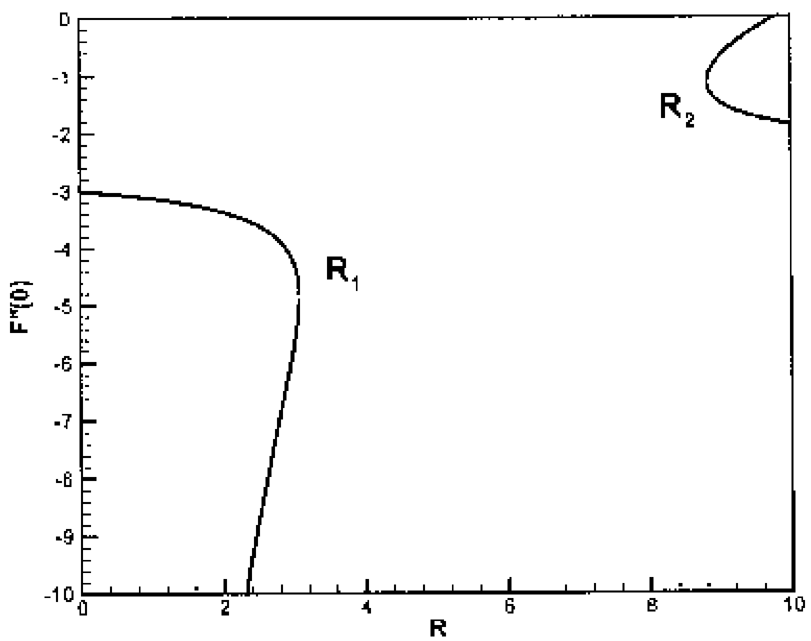

Figure 3.4.2: The bifurcation diagram in the $(F''(0), R)$ plane near the first and second turning points R_1 and R_2 for the problem using the Drazin-Tourigny method with $d = 8$. Other curves are spurious.

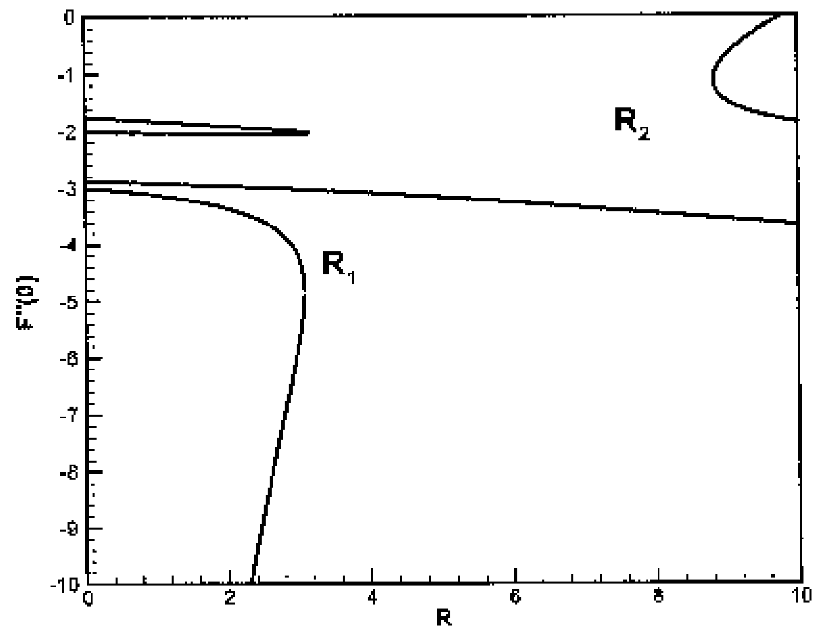

Figure 3.4.3: The bifurcation diagram in the $(F''(0), R)$ plane near the first and second turning points R_1 and R_2 for the problem using the Drazin-Tourigny method with $d = 9$. Other curves are spurious.

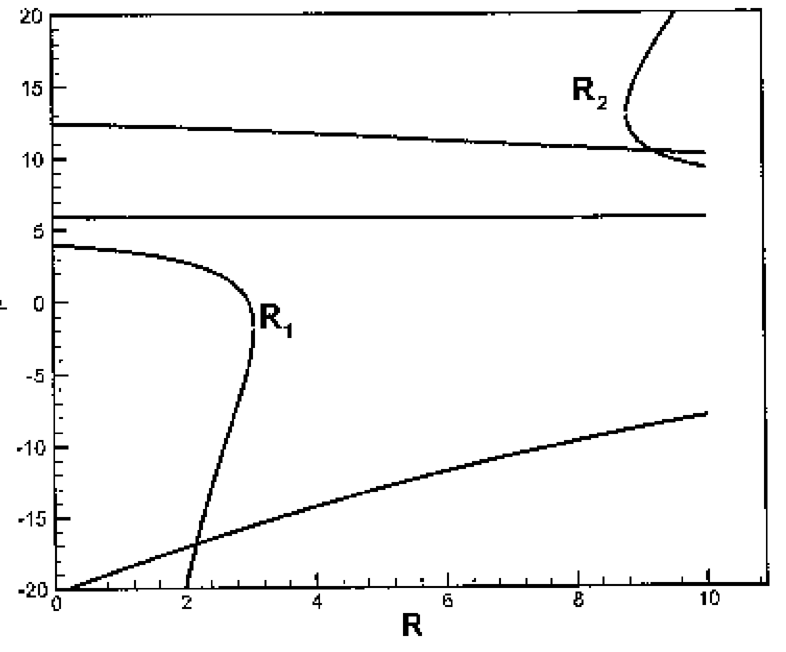

Figure 3.4.4: The bifurcation diagram in the (β, R) plane near the first and second turning points R_1 and R_2 for the problem using the Drazin-Tourigny method with $d = 7$. Other curves are spurious.

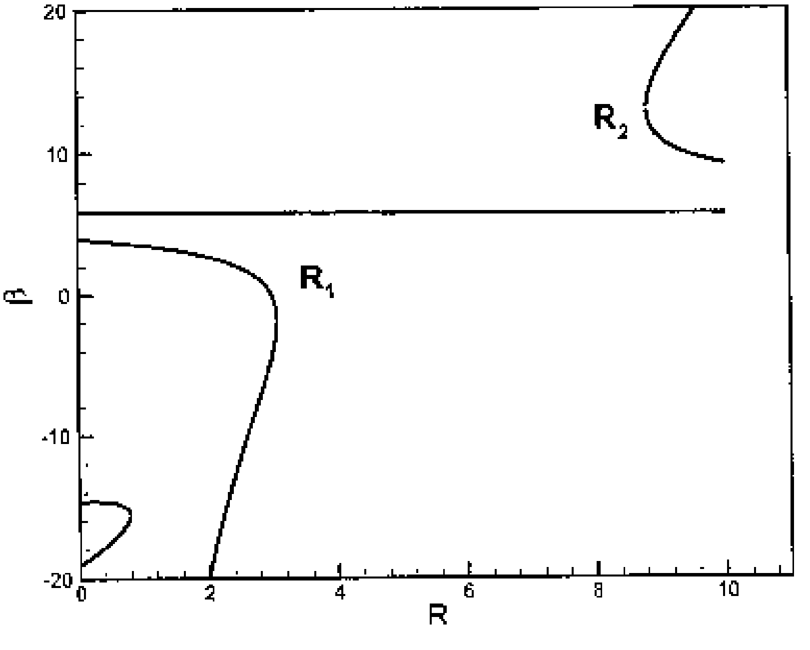

Figure 3.4..5: The bifurcation diagram in the (β, R) plane near the first and second turning points R_1 and R_2 for the problem using the Drazin-Tourigny method with $d = 8$. Other curves are spurious.

3.5 Discussion

In this chapter, we have applied Pade' approximants, Drazin-Tourigny approximants and High-order differential approximants to some model problems as well as a Fluid Dynamical problem. We have analyzed the approximate solution behavior of the problems by observing the dominating singularity of the problems.

We applied the approximation methods to series where the form of the singularity is not known with certainity such as the nonlinear differential equations with spontaneous singularity in the complex plane and infinite number of branch points. Generally, we have found that the High-order differential approximant method is very competitive, but for the approximate bifurcation diagram the Drazin-Tourigny approximant method is essential.

We have applied the High-order differential approximant to the series (3.4.3-3.4.4). The method produces very accurate approximations of the first bifurcation point R_1 and surprisingly a good estimate of the second turning point R_2. Not only the bifurcation point but also the critical exponent as shown in Table 3.4.1 and 3.4.2. The results are comparable with the results of Makinde [17] using the D-T method.

Chapter 4

Conclusion

In this final chapter, we discussed about the summary of the whole thesis. Finally, we sketch some ways in which this work may be exploited further.

4.1 Summary of the work

In this thesis, we studied by means of series summation techniques, the formation of singularities in the solutions of nonlinear problems. By expanding particular solutions in powers of some particular parameter, we obtained accurate numerical approximations of the singularity parameters.

In Chapter 2, we presented a general framework for the description of the Pade'-Hermite approximants with examples.

In Chapter 3, we investigated the dominating singularity behavior of some model problems as well as two nonlinear differential equations from Bender and Orszag's [2] analysis. We have determined spontaneous singularity in the complex plane very accurately in Section 3.2., but did not have enough series coefficients to find more branch points in Section 3.3.

4.2 Future work

In this section, we give some ideas to form the basis of future work:

1. Error analysis of High-order differential approximant.
2. Application of Approximation method to more physical models.
3. Application of Approximation method in other fields which include perturbation series and their performance in these fields.

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Appendix I

Program to compute the series coefficients of

$$F(r) = F_0(r) + F_1(r)R + F_2(r)R^2 + \dots$$

```
interface(quiet=true) :
N:=N;
F:=array(0..N);
F[0]:=expand(-(1/2)*(r^2)*(3-r^2));
G[0]:=expand(diff(((1/r)*diff(F[0],r)),r));

for n from 1 to N do
        a:=array(1..2*n+2);
        F[n]:=add(a[i]*r^(2*i),i=1..2*n+2);
        b[3]:=add(a[i],i=1..2*n+2)=0;
        b[4]:=add(2*i*a[i],i=1..2*n+2)=0;
        lb:=add(16*i*((i+1)^2)*(i+2)*a[i+2]*r^(2*i-1),i=1..2*n);
r1:=0;
r2:=0;

 for i from 0 to n-1 do
        g[i]:=expand(G[i]);
       for mm from 1 to nops(g[i]) do
           aa[mm]:=op(mm,g[i])
       od;
       f[n-i-1]:=expand(diff(F[n-i-1],r)/r);
        for nn from 1 to nops(f[n-i-1]) do
            bb[nn]:=op(nn,f[n-i-1])
         od;
   rh1:=0;
if (i=0) then
                for l from 1 to nops(f[n-i-1]) do
                       rh1:=rh1+g[i]*bb[l]
                od;
else
               for k from 1 to nops(g[i]) do
                    for l from 1 to nops(f[n-i-1]) do
                          rh1:=rh1+aa[k]*bb[l]
                     od;
               od;
fi;
   r1:=r1+rh1;
 od;

rhh1:=expand(r1);#print(1,rhh1);
 i:='i';mm:='mm';nn:='nn';k:='k';l:='l';


   for i from 0 to n-1 do
        g[n-i-1]:=expand(diff(G[n-i-1]/r,r));
```

```
        for mm from 1 to nops(g[n-i-1]) do
                aa[mm]:=op(mm,g[n-i-1]);#print(23,aa[mm]);
         od;
        f[i]:=expand(F[i]);
         for nn from 1 to nops(f[i]) do
                bb[nn]:=op(nn,f[i]);#print(25,bb[nn]);
          od;
  rh2:=0;
 if (n-i-1=0) then
          for ll from 1 to nops(f[i]) do
                rh2:=rh2+0*bb[ll]
           od;
 else
          for kk from 1 to nops(g[n-i-1]) do
                for ll from 1 to nops(f[i]) do
                        rh2:=rh2+aa[kk]*bb[ll]
                od;
 od;
 fi;
  r2:=r2+rh2;
 od;

rhh2:=expand(r2);#print(2,rhh2);
i:='i';mm:='mm';nn:='nn';kk:='kk';ll:='ll';
rh:=simplify(rhh1-rhh2);
leqn:=lh-rh;#print(n,leqn);j:='j';

for j from 1 to 2*n do
      eq[j]:=coeff(leqn,r,2*j-1)=0;
od:

sol:=solve({seq(eq[j],j=1..2*n),b[3],b[4]},{seq(a[i],i=1..2*n+2)});
assign(sol);

F[n]:=F[n]:
G[n]:=expand(add(4*i*(i+1)*a[i+1]*r^(2*i-1),i=1..2*n+1));

unassign(a);

save F, "makind.m";
print(n);

od:
```

Appendix II

The coefficients of the series $F''(0) = \sum_{i=1}^{55} a_i R^i$ **:**

Table A-2 contains the coefficients of $F''(0)$.

i	a_i	i	a_i	i	a_i	i	a_i
1	-3	15	$-.3310\ 10^{-8}$	29	$-.1850\ 10^{-15}$	43	$-.1535\ 10^{-22}$
2	$-9722\ 10^{-5}$	16	$-.9785\ 10^{-9}$	30	$-.5723\ 10^{-16}$	44	$-.4827\ 10^{-23}$
3	$-2145\ 10^{-5}$	17	$-.2909\ 10^{-9}$	31	$-.1773\ 10^{-16}$	45	$-.1519\ 10^{-23}$
4	$-4990\ 10^{-6}$	18	$-.8695\ 10^{-10}$	32	$-.5504\ 10^{-17}$	46	$-.4783\ 10^{-24}$
5	$-1216\ 10^{-6}$	19	$-.2610\ 10^{-10}$	33	$-.1710\ 10^{-17}$	47	$-.1507\ 10^{-24}$
6	$-3082\ 10^{-7}$	20	$-.7870\ 10^{-11}$	34	$-.5325\ 10^{-18}$	48	$-.4754\ 10^{-25}$
7	$-8062\ 10^{-8}$	21	$-.2381\ 10^{-11}$	35	$-.1659\ 10^{-18}$	49	$-.1500\ 10^{-25}$
8	$-2164\ 10^{-8}$	22	$-.7230\ 10^{-12}$	36	$-.5178\ 10^{-19}$	50	$-.4737\ 10^{-26}$
9	$-.5931\ 10^{-5}$	23	$-.2201\ 10^{-12}$	37	$-.1617\ 10^{-19}$	51	$-.1496\ 10^{-26}$
10	$-.1653\ 10^{-5}$	24	$-.6724\ 10^{-13}$	38	$-.5058\ 10^{-20}$	52	$-.4731\ 10^{-27}$
11	$-.4675\ 10^{-6}$	25	$-.2058\ 10^{-13}$	39	$-.1583\ 10^{-20}$	53	$-.1496\ 10^{-27}$
12	$-.1337\ 10^{-6}$	26	$-.6319\ 10^{-14}$	40	$-.4962\ 10^{-21}$	54	$-.4736\ 10^{-28}$
13	$-.3866\ 10^{-7}$	27	$-.1943\ 10^{-14}$	41	$-.1556\ 10^{-21}$	55	$-.1499\ 10^{-28}$
14	$-.1126\ 10^{-7}$	28	$-.5991\ 10^{-15}$	42	$-.4885\ 10^{-22}$		