Critical Behaviour of the Solution of Hydromagnetic Flows in Convergent-Divergent Channels

A dissertation submitted in partial fulfillment of the
requirements for the award of the degree
of

Master of Philosophy

in Mathematics

by

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Session: April 2005
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January 2009

The Thesis entitled ,

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Registration No: 0405409, Roll No: 040509009P, Session: April 2005, a part time student of M. Phil (Mathematics) has been accepted as satisfactory in partial fulfillment for the degree of

Master of Philosophy in Mathematics

on 10^{th} January 2009

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Author's Declaration

I declare that the work done in this dissertation was carried out in accordance with the regulations of Bangladesh University of Engineering and Technology, Dhaka. I also declare that this is an authentic record of my own work except where indicated by special reference in the text. No part of this text has been submitted for any other degree or diploma.

The dissertation has not been presented to any other University for examination either in Bangladesh or overseas.

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List of Symbols

$U(x)$	Power series
$U_N(x)$	N th partial sum of $U(x)$
$u_N(x)$	Approximant function using $U_N(x)$
$P_N(x)$	Polynomial
x_c	Singular point using $U_N(x)$
$x_{c,N}$	Critical point using $U_N(x)$
α_N	Critical exponent using $U_N(x)$
x'	Dimensional coordinate tangential to the flow boundary
y'	Dimensional coordinate normal to the flow boundary
U_0	Speed of the stream at infinity
a	Characteristic length
υ	Kinematic viscosity
t'	Time
t_c	Critical point in time using $U_N(x)$
r_c	Radius of convergence
θ_c	Angle with the positive real axis using $U_N(x)$
τ'	Shear stress
α	Semi-angle in Chapter 4
u	Radial velocity
v	Tangential velocity
Q	Volumetric flow rate
B_0	Electromagnetic induction
μ_e	Magnetic permeability
H_0	Magnetic intensity
σ_e	Conductivity of the fluid
ρ	Density of the fluid
α_c	Critical Channel angular width using $U_N(x)$ in Chapter 4
Re_c	Critical Reynolds number using $U_N(x)$
Ha	Hartmann number
G	Stream function
β_c	Critical exponent using $U_N(x)$ in Chapter 4

Acknowledgements

I have the privilege to express my deep respect, gratitude and sincere appreciation to my supervisor Professor Dr. Md. Abdul Hakim Khan, Department of Mathematics, Bangladesh University of Engineering and technology, Dhaka who has initiated me into the realm of mathematical research. Without his valuable guidance, constant encouragement and generous help it was difficult to complete this thesis. I am grateful to him to give me the opportunity to work with him as a research student and for every effort that he made to get me on the right track of the thesis.

I wish to express my gratitude to Professor Dr. Md. Mustafa Kamal Chowdhury and Professor Dr. Md. Abdul Maleque, previous Head and incumbent Head, Department of Mathematics, BUET for their encouragement and providing all necessary help in the department during the course of M.Phil degree.

I would like to extend my sincere thanks to all other respected teachers of this department for their valuable comment and inspiration and time to time guidance, the whole Mathematics Department for providing all necessary help during the course of M. Phil. degree.

I would like to record my gratefulness to Rifat Ara Rouf, Senior Lecturer, Independent University, Bangladesh for her help and encouragement during my work.

I should mention my heartfelt thanks to my wife, my son and my parents for their support, encouragement and patience in every step of my life.

Above all I want to express my gratefulness to the Almighty Allah for enabling me to complete the work successfully.

Abstract

In this thesis under the title "Critical behavior of the Solution of Hydromagnetic Flows in Convergent-Divergent Channels", two problems have been studied namely Hydromagnetic Flows in Convergent-Divergent Channels and the Laminar Unsteady Flow of a Viscous Fluid away from a Plane Stagnation Point, which belong to two different realms. Initially we have discussed some basic topics in order to study the problems and the approximation methods.

Firstly, we have studied the location and nature of dominant singularity in the complex plane for laminar unsteady flow of a viscous fluid at a plane stagnation point. The series expansion with 44 terms in time of the shear stress is investigated with High-order differential approximant to determine the poles in the complex plane using algebraic programming language MAPLE. The series-improvement techniques are employed to improve its convergence properties. It is observed that the performance of High-order differential approximant is better than that of Pade' approximant and Drazin-Tourigny approximant.

Finally, we have studied the two-dimensional, steady, nonlinear flow of an incompressible conducting viscous fluid in Convergent-Divergent Channels under the influence of an externally applied homogeneous magnetic field by means of $Hermite - Pade'$ approximation especially differential approximate method. We have obtained the series related to similarity parameters by using algebraic programming language MAPLE. The series is then analysed by approximate methods to show the dominating singularity behavior of the flow and the critical relationship among the parameters of the solution.

Chapter 1

Introduction

Observations of fluid flows in daily life show the various types motions that a fluid may undertake. When one turns the bath tap on slightly, the column of water that flows out does so in a smooth manner. However, on opening the tap further the flow becomes erratic and random-like. In modern times the theory of flow through Convergent-Divergent Channels have many applications in aerospace, chemical, civil, environmental, mechanical and bio-mechanical engineering as well as in understanding rivers and canals.

Very few nonlinear problems can be solved exactly but it is sometimes possible to expand solution in powers of some parameters. When the exact closed form solution of a problem is too complicated then one should try to ascertain the approximate nature of the solution.

Approximation methods [5,6,8,9,11,17,26] are the techniques for summing power series. A function is said to be approximant for a given series if its Taylor series expansion reproduces the first few terms of the series. The partial sum of a series is the simplest approximant, if the function has no singularities. For a rapidly convergent series such approximants can provide good approximation for the solution. In practice the presence of singularities prevents rapid convergence of the series. Therefore it is necessary to seek an efficient approximation method.

Khan [19] applied approximation methods to several fluid dynamical problems. Our purpose is to analyse the critical behavior of two standard fluid dynamical problems and compare the performance of approximation methods numerically and graphically

The remainder of this introductory chapter is as follows. Since we shall study the critical behavior of series by using approximation techniques, we begin with a brief review of series in §1.1. Then in §1.2 and §1.3, we describe perturbation series and various types of singularities. We describe a brief review of elementary bifurcation theory in §1.4. Finally, in §1.5, we present a brief outline of the remaining thesis.

1.1 Overview of Series

Consider a function $u(x)$ which can be represented by a power series

$$U(x)=\sum_{i=0}^{\infty}a_i x^i \text{ as } x\to 0. \tag{1.1}$$

The Nth partial sum is

$$U_N(x)=\sum_{i=0}^{N}a_i x^i. \tag{1.2}$$

If we can locate a point x_0, where the function $u(x)$ is analytic then it can be found in a power series $U(x)=\sum_{i=0}^{\infty}a_i(x-x_0)^i$ as $x \to x_0$.

The series is said to be convergent if the sequence of the partial sums converges When the series converges, the sum $U(x)$ can be approximated by the partial sum $U_N(x)$ and the error is defined by $e_N(x)=U(x)-U_N(x)$,

and the absolute error is defined by $e'_N(x)=\left|\dfrac{e_N(x)}{U(x)}\right|$ provided $U(x)\neq 0$.

The number of accurate decimals for some particular value of x is given by

$$\rho_N=-\log_{10}\left|e'_N\right|$$

We say that the error decays exponentially if there exists a particular constant σ such that $\sigma_N\to\sigma$ as $N\to\infty$, where

$$\sigma_N=-\frac{\ln\left|e_N\right|}{N}.$$

Sometimes the presence of singularity of the solution can delay the convergency of the series. So, we need to find the domain of convergence of the series. The series $U(x)$ converges for some x_c if it converges absolutely in the open disc $\{x : x < x_c\}$

with centre at the origin. The largest such disc is called the disc of convergence and the radius, say R, of the disc is called the radius of convergence of the series of $U(x)$. If $u(x)$ is analytic at $x=0$ then $R>0$. If the series has a singularity at x_c such that $x_c=R$, then it diverges for $x \ge x_c$. Different methods such as ratio test, Domb Sykes plot etc. have been used to compute the radius of convergence by direct use of the coefficients of the series. We will apply various generalizations of the approximation methods to determine the critical behavior of the series.

In applied mathematics, series are often obtained by expanding a solution in powers of some perturbation parameter. In the following subsection, we describe the basic literature on perturbation techniques [24][31].

1.2 Perturbation Series

Perturbation theory is a collection of methods for the systematic analysis of the global behavior of solutions to nonlinear problems. Sometimes we solve nonlinear problems by expanding the solution in powers of one or several small perturbation parameters. The expansion may contain small or large parameters which appear naturally in the equations, or which may be artificially introduced. Let us consider a problem of the form

$$f(u,x,\lambda)=0 \tag{1.3}$$

where f may be an algebraic function or some non-linear differential operator, and λ is a parameter. It is seldom possible to solve the problem exactly, but there may exist some particular value of $x=x_0$ for which the solution is known. In this case, for $|x| \ll 1$, one can seek a series for u in powers of x such that

$$U(x)=\sum_{i=0}^{\infty} a_i(\lambda)(x-x_0)^i \text{ as } x \to x_0 .$$

Then by substituting this into equation (1.3), expanding in powers of x and collecting the terms of $O(x^n)$, we can get the required coefficients of the perturbation series.

Example 1.1 Let us take the cubic polynomial

$$u^2-(3+\varepsilon)u+2\varepsilon=0. \tag{1.4}$$

The perturbation series for (1.4) in powers of ε may be taken in the form of

$$U(\varepsilon)=\sum_{i=0}^{\infty}b_i\varepsilon^i \tag{1.5}$$

for small ε. For $\varepsilon=0$, the polynomial has two distinct roots, namely $b_0=0,3$. Which, when $b_0=0$, substituting the expansion

$$U(\varepsilon)=0+b_1\varepsilon+b_2\varepsilon^2+\ldots$$

in (1.4) and equating the coefficients of ε gives $b_1=\frac{2}{3}, b_2=-\frac{2}{27}$. Therefore, the perturbation series for $b_0=0$ is

$$U=0+\frac{2}{3}\varepsilon-\frac{2}{27}\varepsilon^2+O(\varepsilon^3).$$

In a similar process other series is $U=3+\frac{1}{3}\varepsilon+\frac{2}{27}\varepsilon^2+O(\varepsilon^3)$ at $b_0=3$ and

1.3 Singularities

Singularity of a function is a value of the independent variable or variables for which the function is undefined. Singularities are crucial points of a function, because the expansion of a function into a power series depends on the nature of singularities of the function. For the purpose of this thesis, we are interested to analyze those functions, which have several types of singularities. Practically, one of these singularities dominates the function. Therefore it is important to know about this singular point to analyze the critical behavior of the function around this point.

The convergency of the sequence of partial sums depends crucially on the singularities of the function represented by the series. Several types of singularities may arise in physical (nonlinear) problems. The dominating behavior of the function $u(x)$ represented by a series may be written as

$$u(x)\sim A\left(1-\frac{x}{x_c}\right)^{\alpha} \text{ as } x\to x_c \tag{1.6}$$

Where A is a constant and x_c is the critical point with the critical exponent α. If α is a negative integer then the singularity is a pole; otherwise if it is a nonnegative rational number then the singularity is a branch point. We can include the correction terms with the dominating part in (1.6) to estimate the degree of accuracy of the critical points. It can be as follows

$$u(x) \sim A\left(1-\frac{x}{x_c}\right)^{\alpha}\left[1+A_1\left(1-\frac{x}{x_c}\right)^{\alpha_1}+A_2\left(1-\frac{x}{x_c}\right)^{\alpha_2}+\ldots\right] \text{ as } x \to x_c \tag{1.7}$$

Where $0<\alpha_1<\alpha_2<M$ and A_1, A_2, M are constants. $\alpha_i+\alpha \notin N$ for some i, then the correction terms are called confluent. Sometimes the correction terms can be logarithmic. e.g,

$$u(x) \sim A\left(1-\frac{x}{x_c}\right)^{\alpha}\left\{1+\ln\left|1-\frac{x}{x_c}\right|\right\} \text{as } x \to x_c. \tag{1.8}$$

Sometimes the sign of the series coefficients indicate the location of the singularity. If the terms are of the same sign the dominant singular point lie on the positive x-axis. If the terms take alternately positive and negative signs then the singular point is on the negative x-axis.

Following are few examples with different types of singularities:

Example 1.2 (Singularities for single variable functions)

1. Singularities that are poles: $u(x)=\frac{1}{2}(2-x)^{-1}+\sin(2x)$.

   Here $u(x)$ is an algebraic function whose singularity is at $x_c=2$, the critical exponent $\alpha=-1$, which makes the singularity a pole.

2. Algebraic singularities with different exponents:

   $$u(x)=\frac{1}{2}(2-x)^{-1/2}+\left(1-\frac{x}{3}\right)^{-1/3}+\left(1-\frac{x}{2}\right)^{-1/4}.$$

   Here $u(x)$ has several singular points. The singular points are at $x_c=2, 3, 2$ and the critical exponents are $\alpha=-\frac{1}{2},-\frac{1}{3},-\frac{1}{4}$ respectively. In this example the singular points are branch points. Though there are a number of singularities for

$u(x)$, only one of these singularities will dominate the local behavior of $u(x)$.

3. Logarithmic singularity:

$$u(x) = \ln\left(1+\frac{x}{5}\right)+\cos(x).$$

Here $u(x)$ has a logarithmic singularity at $x_c = -5$.

4. Essential singularity:

$$u(x) = \exp(1-2x)^{-2}.$$

Here $u(x)$ has an essential singularity at $x_c = \frac{1}{2}$ with critical exponent $\alpha = -2$.

5. Algebraic dominant singularity with a secondary logarithmic behavior:

$$u(x) = \left(1-\frac{x}{4}\right)^{-1/4} + \ln\left(1-\frac{x}{7}\right)$$

The algebraic dominant singularity of $u(x)$ here is at $x_c = 4$ with critical exponent $\alpha = -\frac{1}{4}$, which makes it a branch point. And a logarithmic singularity at $x_c = 7$.

6. nth root singularity. $u(x) = \left(1-\frac{x}{2}\right)^{-1/n} + \exp(x)$.

Here $u(x)$ has a branch point with the critical exponent $\alpha = -\frac{1}{n}$ at $x_c = 2$.

1.4 Elementary Bifurcation Theory

In this thesis we have investigated two nonlinear problem in fluid dynamics. Solutions of nonlinear problems often involve one or several parameters. As the parameter varies, so does the solution set. A bifurcation occurs where the solution of a nonlinear system alter their qualitative behavior while a parameter changes its value. In particular, bifurcation theory shows how the number of steady solutions of a system depends on parameters. Examples of bifurcation are: Simple turning points, in which two real solutions become complex conjugate solutions and pitchfork bifurcation, in which the number of real solutions changes discontinuously from one to three (or vice versa). We intend to introduce some basic concepts of bifurcation theory. Drazin [7] discussed the bifurcation theory in detail.

Consider a functional map $F:\Re\times\Re\rightarrow\Re$. We seek for the solutions

$$u=U(x)\text{ of }F(x,u)=0. \tag{1.9}$$

Bifurcation diagrams can show the solutions. In these diagrams solution curves are drawn in the (x,u) plane. Let (x_0,u_0) be a solution of equation (1.9), i.e.

$$F(x_0,u_0)=0 \tag{1.10}$$

then, F can be expanded in a Taylor series about (x_0,u_0) and we can study the solution set in that neighborhood provided that F is smooth. Thus we obtain

$$0=F(x,u)$$

$$=F(x_0,u_0)+(u-u_0)F_u(x_0,u_0)+(x-x_0)F_x(x_0,u_0)+\frac{1}{2}(u-u_0)^2F_{uu}(x_0,u_0)+\ldots \tag{1.11}$$

If, we assume that, $F_u(x_0,u_0)\neq 0$, then

$$u(x)=u_0-(x-x_0)\frac{F_x(x_0,u_0)}{F_u(x_0,u_0)}+O(x-x_0),\text{ as }x\rightarrow x_0. \tag{1.12}$$

This gives only one solution curve in the neighborhood of the point (x_0,u_0) in the bifurcation diagram. However, if we replace (x_0,u_0) with (x_c,u_c), where

$$F(x_c,u_c)=0, F_u(x_c,u_c)=0, \tag{1.13}$$

then the expansion (1.11) shows that there are at least two solution curves in the neighborhood of (x_c,u_c) The point (x_c,u_c) is called a bifurcation point.

Example 1.3 Let F be a function defined as

$$F(x,u,\varepsilon)=\frac{1}{4}u^2-\frac{1}{3}x^3+x-\frac{3}{4}+\varepsilon=0 \tag{1.14}$$

Where ε is some real parameter. When ε=0, Figure1.1, there arise a bifurcation point at $(1,0)$ and a turning point at $(-2,0)$.

When $0<\varepsilon<4/3$, there are two separate branches of the bifurcation curve, one an isola and the other unbounded. When, the value of ε increases, in the considered interval, these two branches move apart from each other. Figure 1.2 and Figure 1.3 shows this behavior of F for different values of ε.

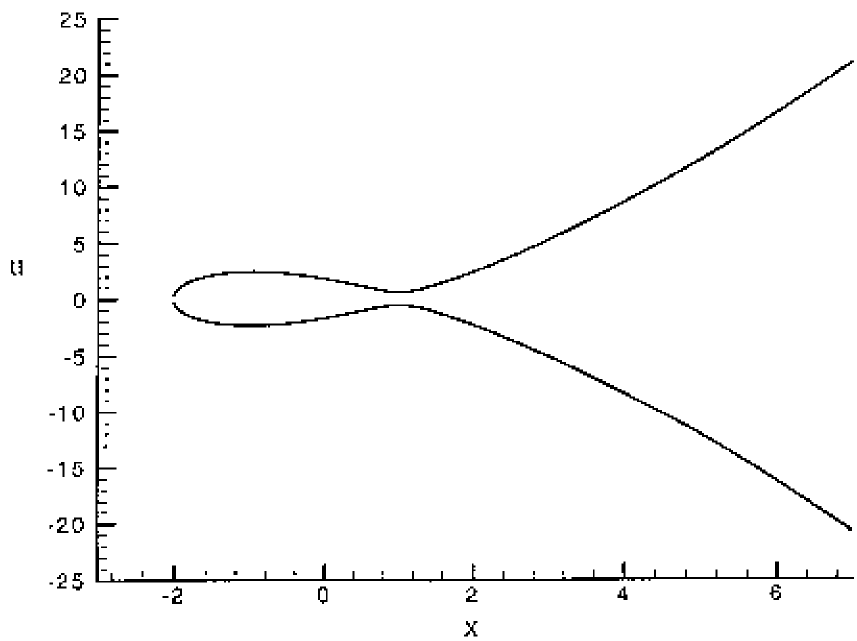


Figure 1.1: Bifurcation diagram of $F(x,u,\varepsilon)$ in (x,u) plane when $\varepsilon=0$.

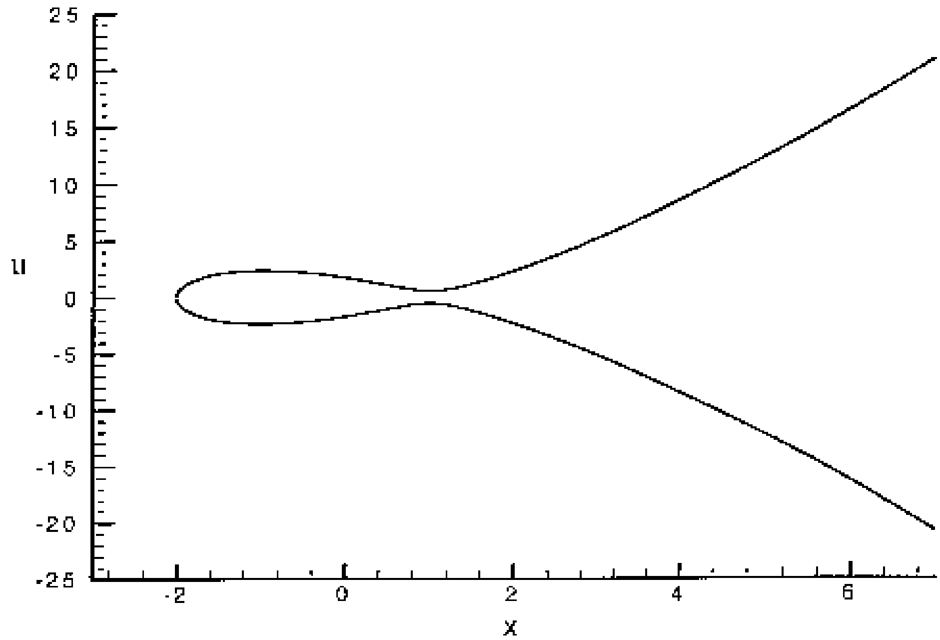


Figure 1. 2: Bifurcation diagram of $F(x,u,\varepsilon)$ in (x,u) plane when $\varepsilon = 0.01$.

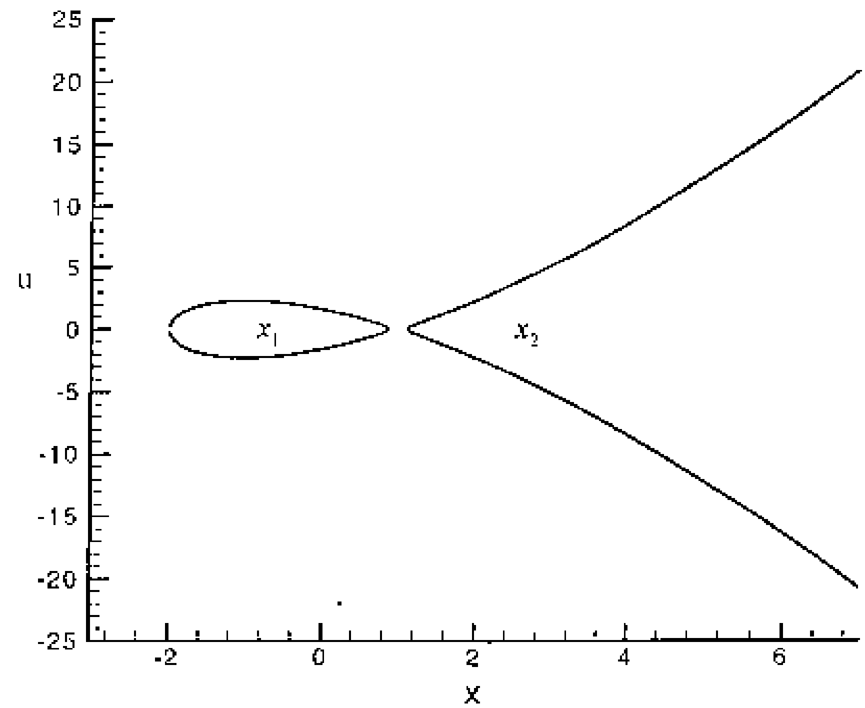


Figure 1. 3: Bifurcation diagram of $F(\mathbf{x},u,\varepsilon)$ in (x,u) plane when $\varepsilon = 0.1$.

1.5 Overview of the Work

This thesis is concerned with the study of computer based approximation techniques which are of *Hermite – Pade'* class. Many researchers have studied the application of these approximation techniques in fluid dynamical problems. For over the last quarter century many powerful approximants have been introduced for the approximation of the function by using its power series. Among them most of the methods are described for the series involving single independent variable and a few are derived for the power series involved with two or several independent variables. Many researchers hitherto have found remarkably more accurate results by using several approximation methods. The remainder of this thesis is as follows:

In Chapter 2, we have reviewed the *Hermite – Pade'* class of approximation techniques to determine the coefficients of the approximant. We have discussed several of these kind of approximants with some examples. Then in Chapter 3, we have discussed the comparative performance of these approximants to the Laminar Unsteady Flow of a Viscous Fluid away from a Plane Stagnation Point.

In chapter 4, we have studied the Critical Behaviour of the Hydromagnetic Flows in Convergent-Divergent Channels. Makinde [23] analysed the magnetic effect in the classical Jeffry-Hamel flow in Convergent-Divergent Channels. We extend the work by the comparison of our method with Makinde [23] and the bifurcation study for the effect of magnetic intensity and the critical relation among the parameters of the flow. Finally in chapter 5, we have summarized our work and give some ideas for future work.

Chapter 2

Approximant Methods

Introduction

This thesis is based on the study of the application of computer based approximation techniques to reveal the local behavior of a perturbation series around its singular point and the critical relationship among the perturbation parameters.

The approximation methods are widely used to approximate functions in many areas of applied mathematics. Approximant methods are the techniques for summing power series. A function is said to be approximant for a given series if its Taylor series expansion reproduces the first few terms of the series.

Brezinski [1] studied history of continued fraction and Pade' approximants. Blanch [2] evaluated continued fractions numerically. Also the applications of continued fractions and their generalizations to problems in approximation theory have been studied by Khovanskii [20]. Baker and Graves-Morris [5] studied Pade'approximants and its properties. Algebraic and Differential approximants [4] are some useful generalizations of Pade' approximants. Khan [19] analysed singularity behavior by summing power series. Khan [17] also introduced a new model of Differential approximant for single independent variable, called High-order differential approximant (HODA), for the summation of power series. The method is a special type of $Hermite-Pade'$ class and it is one of the best methods of singularity analysis for the problems of single independent variable.

The reminder of this Chapter is organized as follows:

We study the $Hermite-Pade'$ class of approximants and then the development of some approximants in this class such as Algebraic and Differential Approximants Drazin-Tourigney is one kind of Algebraic approximant and High-order differential approximants and High-order partial differential approximants [26] is an extension of Differential Approximants.

2.1 Hermite-Pade' Approximants

In 1893, *Hermite* and *Pade'* introduced *Hermite – Pade'* class. The entire one variable approximants that were used or discussed throughout this thesis paper belong to the *Hermite – Pade'* class. In its most general form, this class is concerned with the simultaneous approximation of several independent series. Firstly we describe the *Hermite – Pade'* class from its point of view.

Let $d \in \mathrm{N}$ and let the $d+1$ power series $U_0(x), U_1(x), \ldots, U_d(x)$

are given. We say that the $(d+1)$ tuple of polynomials

$$P_N^{[0]}, P_N^{[1]}, \ldots, P_N^{[d]}$$

where $\deg P_N^{[0]} + \deg P_N^{[1]} + \ldots + \deg P_N^{[d]} + d = N,$ (2.1)

is a *Hermite – Pade'* form of these series if

$$\sum_{i=0}^{d} P_N^{[i]}(x) U_i(x) = O(x^N) \text{ as } x \to 0. \tag{2.2}$$

Here $U_0(x), U_1(x), \ldots, U_d(x)$ may be independent series or different form of a unique series. We need to find the polynomials $P_N^{[i]}$ that satisfy the equations (2.1) and (2.2). These polynomials are completely determined by their coefficients. So, the total number of unknowns in equation (2.2) is

$$\sum_{i=0}^{d} \deg P_N^{[i]} + d + 1 = N + 1\,. \tag{2.3}$$

Expanding the left hand side of equation (2.2) in powers of x and equating the first N equations of the system equal to zero, we get a system of linear homogeneous equations. . To calculate the coefficients of the *Hermite – Pade'* polynomials we require some sort of normalization, such as

$$P_N^{[i]}(0) = 1 \text{ for some } 0 \le i \le d\,. \tag{2.4}$$

It is important to emphasize that the only input required for the calculation of the *Hermite – Pade'* polynomials are the first N coefficients of the series $U_0, \ldots, U_d$. The equation (2.3) simply ensures that the coefficient matrix associated with the system is square. One way to construct the *Hermite – Pade'* polynomials is to solve the system of linear equations by any standard method such as Gaussian elimination or Gauss-Jordan elimination.

2.2 Pade′ Approximants

Pade′ approximant is a technique for summing power series that is widely used in applied mathematics [4]. Pade′ approximant can be described from the Pade′-Hermite class in the following sense.

In the Pade′-Hermite class, let $d = 1$ and the polynomials $P_N^{[0]}$ and $P_N^{[1]}$ satisfy equations (2.1) and(2.2). One can define an approximant $u_N(x)$ of the series $U(x)$ by

$$P_N^{[1]}U_N - P_N^{[0]} = 0, \tag{2.5}$$

where

$$U_1 = U_N \text{ and } U_0 = -1.$$

Then we select the polynomials

$$P_N^{[0]}(x) = \sum_{i=0}^{n} b_i x^i \text{ and } P_N^{[1]}(x) = \sum_{i=0}^{m} c_i x^i. \tag{2.6}$$

Such that $n + m \leq N$, the constants b_i's and c_i's are unknowns to be determined So that,

$$U_N(x)P_N^{[1]}(x) - P_N^{[0]}(x) = O\left(x^{n+m+1}\right). \tag{2.7}$$

Equating the first $n + m$ equations of (2.7) equal to zero and the normalization condition in equation (2.4), we find the values of b_i's and c_i's. Then, the rational approximant known as Pade′ approximant denoted as

$$u_N(x) = \frac{P_N^{[0]}(x)}{P_N^{[1]}(x)}. \tag{2.8}$$

help us to approximate the sum of the power series $U(x)$. And the zeroes of the polynomial $P_N^{[1]}(x)$ happens to be identical with the singular point (points) of $U(x)$. In order to evaluate the Pade′ approximants for a given series numerically, we have used symbolic computation language such as MAPLE. The Pade′ approximants have been used not only in tackling slowly convergent, divergent and asymptotic series but also to obtain singularity of a function from its series coefficients. The zeroes of the denominator $P_n^{[1]}$ give the singular point such as pole of the function $u(x)$ if it exists.

Example 2.1 Consider singularities $u(x)=\frac{1}{(1-3x)^2}+e^x$, a function with a simple pole

After applying the normalization condition $c_0=1$, we obtain the polynomial coefficients $P_5^{[0]}$ and $P_5^{[1]}$ for deg $P_5^{[0]}=n$ and deg $P_5^{[1]}=m$. When $m=n=2$,

$$P_5^{[0]}=1-\frac{939710307}{25638139}x+\frac{317480122}{25638139}x^2 \text{ and}$$

$$P_5^{[1]}=\frac{138234690}{25638139}-\frac{940857499}{25638139}x+\frac{3127666779}{51276278}x^2.$$

When m = n = 3

$$P_7^{[0]}=1-\frac{8453267322177}{19237554765697}x+\frac{113609517808274}{19237554765697}x^2+\frac{66275640000650}{19237554765697}x^3$$

and

$$P_7^{[1]}=\frac{10239385806720}{19237554765697}-\frac{68485409551560}{19237554765697}x+\frac{136474582726500}{19237554765697}x^2-\frac{75866508687612}{19237554765697}x^3$$

The table below shows the convergence to the singular point of $u(x)$ on application of Pade' approximant.

Table 2.1: The approximation of x_c by Pade' for the function in Example 2.1

m,n	x_c
2,2	.3466054085
3,3	$.2906911358-.1\times10^{-10}i$

2.3 Algebraic Approximants

Algebraic approximant is a special type of *Hermite – Pade'* approximants. In the *Hermite – Pade'* class we take

$$d \geq 1, U_0 = 1, U_1 = U, \ldots, U_d = U^d .$$

Let $U(x)$ represent power series of a function and $U_N(x)$ is the partial sum of that series. An Algebraic approximant $u_N(x)$ of $U(x)$ can be defined as the solution of the equation:

$$P_N^{[0]}(x) + P_N^{[1]}(x)U_N(x) + P_N^{[2]}(x)U_N^2(x) + \ldots + P_N^{[d]}(x)U_N^d(x) = 0 \qquad (2.9)$$

Where d represent the degree of the partial sum $U_N(x)$. The Algebraic approximant $u_N(x)$, is in general a multivalued function with d branches.

The solution of the equation (2.9) with $d \geq 1$ gives us the coefficients of the polynomials $P_N^{[i]}(x)$. The discriminant of this equation approximates the singularity of $U(x)$.

Here,
$$\sum_{i=0}^{d} P_N^{[i]}(x)U_N^i(x) = O(x^N) \qquad (2.10)$$

And
$$\sum_{i=0}^{d} \deg P_N^{[i]} + d = N . \qquad (2.11)$$

And the total number of unknowns in (2.10) are

$$\sum_{i=0}^{d} \deg P_N^{[i]}(x) + d + 1 = N + 1 . \qquad (2.12)$$

In order to determine the coefficients of the polynomials $P_N^{[i]}$ one can set $P_N^{[0]}(0) = 1$ for normalization.

Example 2.2 Consider

$$u(x) = (1 - 3x)^{1/4} + \sin x .$$

Let $d = 2$ and $\deg P_8^{[0]} = \deg P_8^{[1]} = \deg P_8^{[2]} = 2$ to apply the Algebraic approximation method on the power series of the given function. After we set the normalization condition $P_8^{[0]}(0) = 1$, we get the polynomials

$$P_8^{[0]}(x) = 1 + \frac{405051011917681450135678429 33}{228268061031195168003057902 63}x + \frac{295733197856436286200527739 4}{228268061031195168003057902 63}x^2$$

$$P_8^{[1]}(x) = \frac{666327426223641455451}{228268061031195168003} - \frac{864976431391571779 84}{120141084753260614 73}x + \frac{93700980909939642 96}{228268061031195168 0}x^2$$

$$P_8^{[2]} = -\frac{371472358962442535320}{228268061031195168003} + \frac{102929046962042066735}{228268061031195168003}x - \frac{496752609063763819700}{228268061031195168003}x^2$$

Here the discriminant gives us the singularity at $x_c = 0.4742842500$. If we increase the degree of the polynomial coefficients it may give us a better approximation. So, again let $\deg P_{11}^{[0]} = \deg P_{11}^{[1]} = \deg P_{11}^{[2]} = 3$ and $d = 2$, following the same procedure we get the singularity at $x_c = 0.2906496342$.

Again taking $d = 2$ and $\deg P_{14}^{[0]} = \deg P_{14}^{[1]} = \deg P_{14}^{[2]} = 4$ the singularity is calculated at $x_c = 0.3126797065$. The table below shows the comparative results of the convergence of the Algebraic approximation method to the singular point.

Table 2.2: The approximation of x_c by Algebraic approximants for the function in Example 2.2

deg $P_N^{[i]}$	d	x_c
2	2	0.4742842500
3	2	0.2906496342
4	2	0.3126797065

Note that $d = 3$ may be more accurate for this problem.

2.4 Drazin -Tourigney Approximants

Drazin and Tourigney in [6] implemented the idea $d = O(\sqrt{N})$ as $N \to \infty$. Their method is simply a particular kind of Algebraic approximant, satisfying the equation (2.9). In this method they considered

$$\deg P_N^{[i]} = d - i \tag{2.13}$$

and

$$N = \frac{1}{2}\left(d^2 + 3d - 2\right). \tag{2.14}$$

2.5 Differential Approximants

Differential approximants is an important member of the *Hermite – Pade'* class. It is obtained by taking

$$d \geq 2, \quad U_0 = 1, U_1 = U, U_2 = DU \text{ and } U_d = D^{d-1}U,$$

where $D \equiv \dfrac{d}{dx}$, a differential approximant $u_N(x)$ of the series $U(x)$ can be defined as the solution of the differential equation

$$P_N^{[0]} + P_N^{[1]}U_N + P_N^{[2]}DU_N + \ldots + P_N^{[d]}D^{d-1}U_N = 0. \tag{2.15}$$

Here (2.15) is homogeneous linear differential equation of order $(d-1)$ with polynomial coefficients. There are $(d-1)$ linearly independent solutions, but only one of them has the same first few Taylor coefficients as the given series $U(x)$. When $d > 2$, the usual method for solving such an equation is to construct a series solution

Differential approximants are used chiefly for series analysis. They are powerful tools for locating the singularities of a series and for identifying their nature.

The singularities of $U(x)$ are located at the zeroes of the leading polynomial $P_N^{[d]}(x)$. Hence, the zeroes of $P_N^{[d]}(x)$ may provide approximations of the singularities of the function $u(x)$.

Example 2.3 Consider $u(x)=\dfrac{e^x(1+\sin x)}{\sqrt{1-\frac{1}{3}x}}$.

Taking $d=4$ for (2.11) and applying (2.1) and (2.2), we obtain the singular point at $x_c=3.000563091$. In a similar procedure taking $d=5$ gives us more accurate result, i.e. $x_c=2.999999734$. The table below shows a comparative result.

Table2.3: The approximation of x_c by Differential Approximant for the function in Example 2.3

N	d	x_c
15	3	2.991301923
21	4	3.000563091
28	5	2.999999734

2.6 High-Order Differential Approximants

Khan [17] introduced an extension of differential approximant, which he mentioned as High-order differential approximant. When the function has a countable infinity of branches, then the fixed low-order differential approximants may not be useful. So, for these cases he considered d increase with N. It lead to a particular kind of differential approximant $u_N(x)$, satisfying equation (2.14). Here

$$N=\frac{1}{2}d(d+3)\ \text{and}\ \deg P_N^{[i]}=i. \tag{2.16}$$

From (2.3) he deduced that there are

$$\frac{1}{2}\left(d^2+3d+2\right)$$

unknown parameters in the definition of the *Hermite – Pade'* form. In order to determine those parameters, we use the N equations

$$P_N^{[0]}(x)+\sum_{i=1}^{d}P_N^{[i]}(x)D^{i-1}U_N(x)=O\left(x^N\right)\ \text{as}\ x\to 0.$$

In addition one can normalize by setting $P_N^{[0]}(0)=1$. Then there remains as many equations as unknowns. One of the roots, say $x_{c,N}$, of the coefficient of the highest derivative, i.e. $P_N^{[d]}(x_{c,N})=0$, gives an approximation of the dominant singularity x_c of the series U. If the singularity is of-algebraic type, then the exponent α may be approximated by

$$\alpha_N = d-2-\frac{P_N^{[d-1]}(x_{c,N})}{DP_N^{[d]}(x_{c,N})}. \tag{2.17}$$

2.7 High-Order Partial Differential Approximants

Consider the function $f(x,y)$ of two independent variables, represented by its power series

$$U(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}x^i y^j \qquad (x,y)\to(0,0) \tag{2.18}$$

and the partial sum

$$U_N(x,y)=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} c_{ij}x^i y^j \tag{2.19}$$

By using that partial sum, we try to construct the following $(2d+1)$ polynomials

$$P_{[0,0]}, P_{[1,0]}, P_{[0,1]}, \ldots\ldots\ldots\ldots, P_{[d,0]}, P_{[0,d]} \tag{2.20}$$

in x and y such that

$$P_{[0,0]}U_N + P_{[1,0]}\frac{\partial U_N}{\partial x} + P_{[0,1]}\frac{\partial U_N}{\partial y} + \ldots\ldots + P_{[d,0]}\frac{\partial^d U_N}{\partial x^d} + P_{[0,d]}\frac{\partial^d U_N}{\partial y^d} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} e_{ij}x^i y^j \tag{2.21}$$

Where $\qquad e_{ij}=0 \quad \text{for } i+j<N=3d-1 \qquad (2.22)$

By equating the coefficients of the variables and their powers from (2.22), one can obtain a total of

$$N_e = \frac{3d(3d-1)}{2} \tag{2.23}$$

equations to determine the unknown coefficients of the polynomials in (2.21), we impose the normalization condition

$$P_{[0,0]}=1, \; or\, P_{[d,0]}=1 \quad or\, P_{[0,d]}=1 \quad \text{for } (x,y)=(0,0). \tag{2.24}$$

Thus the remaining unknowns

$$N_u = \frac{1}{3}d(d^2+6d+11)$$

must be found by the use of N_e equations.

It would be helpful to write the system of linear equations $e_{i,j}=0$ into the matrix form with the $N_u \times 1$ unknown matrix $\underline{x}$.

Thus the non-homogeneous system of N_e linear equations with N_u unknowns can be written in matrix form as

$$A\underline{x} = \underline{b}$$

where A is $N_e \times N_u$ matrix and $\underline{b}$ is the non-zero column matrix of order $N_e \times 1$. Thus system will be solvable if

$$N_e \leq N_u. \tag{2.25}$$

However, the system may be consistent or inconsistent. If the system is consistent , then the system can be solved by converting the augmented matrix $[A|\underline{b}]$ to row echelon or reduced row echelon form by using the Gaussian elimination or Gauss-Jordan elimination It is to note that, there will exist some free variables. Naturally the values of the free variables in the multivariable approximant methods can be chosen at random. For all the calculation reported in the remainder of this chapter, we have in fact set all the free variables to either zero or one. There is no particular reason to pick up these particular numbers. We might for instance seek a solution such that the polynomials in (2.20) have as few high-order terms as possible. Our experience suggests that the accuracy of the method does not depend critically on the particular choice made.

Once the polynomials (2.20) have been found, it is more practical to find the singular points by solving either of the polynomials coefficients of the highest derivatives

$P_{[d,0]}(x,y)=0$ or $P_{[0,d]}(x,y)=0$ or both simultaneously

2.8 Discussion

Hermite – Pade' class is constructed over the technique of truncated continued fraction. The polynomial coefficients were constructed by taking successive truncated continued fractions. In this chapter we had an overall study about the *Hermite – Pade'* class of approximation methods Examples show the performance of Algebraic approximant and Differential approximant explicitly. We must mention that Drazin -Tourigney method is an improved Algebraic approximation technique. High-order differential approximants is modified Differential approximant whose performance is almost in every case convincing. High-order partial differential approximants [9] is a multivariable differential approximants method is applied to determine critical relation between the solution parameter.

In Chapter 3 we study a nonlinear differential equation with the application of Drazin-Tourigney and High-order differential approximants and then the dominating singularity behaviour of the Laminar Unsteady Flow of a Viscous Fluid away from a Plane Stagnation Point as the application of these techniques. Finally in Chapter 4 we analyse the Critical Behaviour of the Hydromagnetic Flows in Convergent-Divergent Channels and the Critical relationship among the flow parameters.

Chapter 3

Application of Approximation Methods on Model Problems

3.1 Background

The solutions of differential equations encountered in practice are regular at almost every point; in the neighborhood of ordinary points. Taylor series provide an adequate description of the solution. However, the distinguishing features of the solution are its singularities. Determining the location and nature of these singularities, without solving the differential equation, requires the techniques of local analysis.

Bender and Orszag [4] discussed a number of examples with local analysis. Without solving the equation they tried to locate the dominating singular points of this kind of nonlinear differential equations by the application of approximation method. And try to locate the dominating singular point with critical exponent which analyses the form of the singularity of this kind.

In this chapter, we have studied the location and nature of dominant singularity in the complex plane for laminar unsteady flow of a viscous fluid at a plane stagnation point. The series expansion with 44 terms in time of the shear stress is investigated with High - order Differential Approximant to determine the poles in the complex plane using algebraic programming language MAPLE. The series-improvement techniques are employed to improve its convergence properties. It is observed that the performance of High-order differential approximant is better than that of Padeʹ approximant and Drazin-Tourigny approximant. We have also examined a problem where approximation methods were applied to reveal the singularities and have compared our result with others.

3.2 Behaviour of the first Painlev'e Transcendent as $x \to +\infty$

Consider the nonlinear differential equation

$$u'' = u^2 + x \tag{3.1}$$

This differential equation is the first of a set of six equations whose solutions are called the Painlev'e transcendent. These equations were discovered by Painlev'e in the course of classifying nonlinear differential equations. He considered all equations of the form

$$w'' = R(z,w)(w')^2 + S(z,w)w' + T(z,w)$$

having the properties (a) that R. S and T. are rational functions of w, but have arbitrary dependence on z and (b) that the solutions may have various kinds of fixed singularities (poles, branch points, essential singularities). but may not have any movable singularities expect for poles. There are 50 distinct types of equations having properties. Of course, 44 types are soluble in terms of elementary transcendent (sine, cosine, exponential), functions defined by linear second-order equations (Bessel functions, Legendre functions and so on)n , or elliptic functions. The remaining six equations define the six Painlev'e transcendent, one of which is (3.1).

Let us now return to the behaviour of the differential equation (3.1). This differential equation is similar in form to the first-order equation in [4] and its asymptotic properties are also similar in some respects [see equation (4.2.1), pp.150, Bender & Orszag [4]]. However, because this is a second-order equation, a more sophisticated analysis is required.

We begin by arguing that $u(x)$ exhibits movable singularities. Since the curvature of $u(x)$ is positive $(u'' > x > 0)$, it is likely that an arbitrary set of initial conditions will give rise to a solution which becomes singular at a finite value of x. To discover the leading behaviour of such a singularity, we substitute

$$u(x) \sim \frac{A}{(x-a)^b}, \quad x \to a.$$

into the differential equation (3.1). Comparing powers of $x \to a$ gives $A = 6$ and $b = 2$. This suggests that $u(x)$ has movable second –order poles.

However, this does not prove that the movable singularities are poles. To verify such a conjecture it is necessary to establish that a Laurent series solution of the form

$$u(x)=\frac{6}{(x-a)^2}+\sum_{n=-1}^{\infty}a_n(x-a)^n \qquad (3.2)$$

exists in the neighborhood of $x=a$.

Actually $u(x)$ has an infinite number of second-order poles along the positive real axis and not just one! The differential equation (3.1) has solved numerically, taking as initial conditions $u(0)=u'(0)=0$, and have plotted the result in Figure 3.1. Observe that there is a sequence of poles along the positive real axis.

The presence of the infinite number of second-order poles on the positive real axis can be understood from the graph of the tangent field given in Figure 3.1.

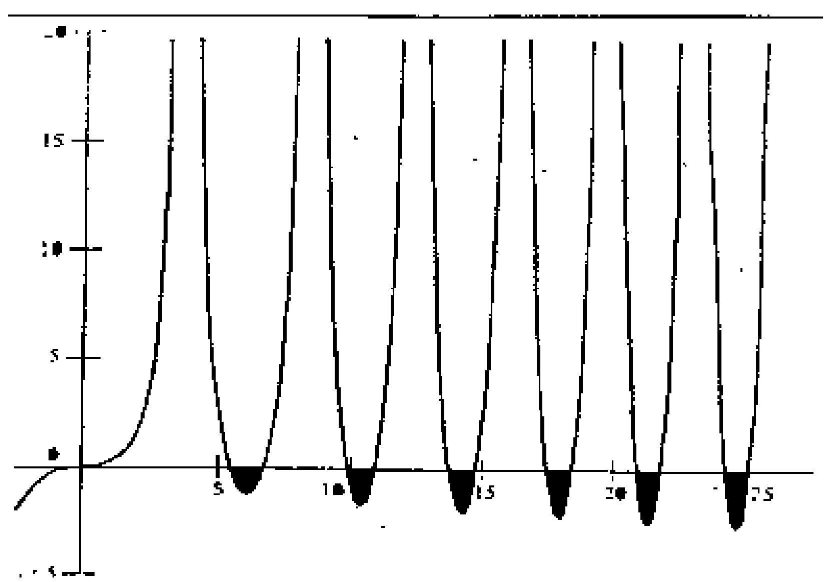


Figure 3.1: Computer plot of the solution to the initial-value problem $u''=u^2+x$ $[u(0)=u'(0)=0]$ has an infinite number of second-order poles on the positive real axis, Bender and Orszag [4]

Table 3.1: Estimates of the critical point $x_{c,N}$ and the corresponding exponent α_N by using High-order differential approximants[17] (HODA) and Drazin and Tourigney [6] (D-T) method for the differential equation $u'' = u^2 + x$ $[u(0) = u'(0) = 0]$.

N	d	HODA $x_{c,N}$	HODA α_N	N	d	D-T $x_{c,N}$
12	3	3.7428014612217258443	-1.999996754691065l			
18	4	3.7428015307707560902	-2.0000000000013l121			
25	5	3.742801530770741877	-2.000000000000000000			
33	6	3 7428015307707418775	-1.9999999999999999	25	6	3 742167390181541 4712
42	7	3 7428015307707418775	-1 99999999999999999	33	7	3.742626790401556 6872
52	8	3.7428015307707418778	-2 00000000000000002	42	8	3 740444794458224 5918
63	9	3.7428015307752701046	-2.00000000015497759	52	9	3.7074398382066125923
75	10	3.7428015307494165916	-1 99999999532936381			

We have analysed the location and nature of the singularities by using various generalizations of the approximation methods. From the above analysis Bender and Orszag [4] indicates that u(x) has an infinite number of second-order poles along the positive real axis and the nearest pole to the origin is $x_{c,N} \approx 3.7428$.

On the other hand, the results in Table 3.1 indicate that it is possible, by using the High-order differential approximants (HODA) to calculate the above pole and the critical exponent α to 19 and 15 digits of accuracy with $d = 7(N = 42)$. For comparison, the table also shows the results by the Drazin-Tourigny (D-T) approximants It is clear that the High-order differential approximants converges much faster and the value of α confirm that x_c is a pole. Therefore the dominating singularity behavior of the solution is

$$u(x) \sim A(x - x_c)^{\alpha} \text{ as } x \to x_c,$$

where $x_c \approx 3.7428015307707418775$ and $\alpha \approx -2$.

We now study the laminar unsteady flow of a viscous fluid at a plane stagnation point, where the singularity, for which the convergence of the series is limited, lie in the complex plane and is a pole. Various approximation techniques are applied to determine this singularity in the complex plane and its critical exponent.

3.3 The Laminar Unsteady Flow of a Viscous Fluid away from a Plane Stagnation Point

To model the laminar flow of an incompressible fluid with small viscosity away from a stagnation point, Hommel [13] took the approach to formulate a series expansion in time for the shear stress at the stagnation point. And then apply series improvement techniques with the hope of extrapolating to large times.

But in the present problem it was found that the singularity for which the convergence of the series is limited, lie in the complex plane and is a pole. This pole in the complex plane for the physical variable t can not be attached with a physical meaning. Hence, a finite number of terms for the series is used to determine the radius of convergence to banish the offending pole (or, more generally, singularity) to infinity with a linear fractional transformation, such as an Euler transformation.

The non-dimensional coordinates are defined as,

$$x=\frac{x'}{a},\quad y=y'\left(\frac{2U_0}{\nu a}\right)^{1/2},\qquad t=\frac{2U_0t'}{a},\tag{3.3}$$

Where x' is the dimensional coordinate tangential to the flow boundary measured away from the stagnation point, y' is the dimensional coordinate normal to the flow boundary. U_0 is the speed of the stream at infinity, a is the characteristic length, ν is the kinematic viscosity and t' is the time.

In a small region near the stagnation point, the potential flow corresponding to an impulsive start is described by the stream function,

$$\psi=-\left(2\nu aU_0\right)^{1/2}xy$$

Following Proudman & Johnson [25], this solution is enforced as the outer boundary condition for all time. Then the Navier-Stokes equations are solved exactly by setting,

$$\psi'=-\left(2\nu aU_0\right)^{1/2}xF\left(y,t\right)$$

To obtain the differential equation with initial and boundary conditions

$$\begin{aligned}&F_{yt}-F_{yyy}=\left(-1+F_y^2-FF_{yy}\right),\\&F\left(0,t\right)=F_y\left(0,t\right)=0,F_y\left(\infty,t\right)=1,\\&F_y\left(y,0\right)=1\left(y\neq0\right).\end{aligned}\tag{3.4}$$

To solve this differential equation Hommel set,

$$F = 2t^{1/2}\left\{f_1(\eta) + tf_2(\eta) + t^2 f_3(\eta) + \dots\right\}, \tag{3.5}$$

Where, $\eta = y/2t^{1/2}$. Substituting (3.4) the equations (3.5) are solved by employing the finite difference scheme. The main result for the dimensional shear stress at the boundary is

$$\tau'(x,y) = \rho\nu^{1/2}\left(\frac{U_0}{a}\right)^{3/2} x' \frac{1}{2t^{1/2}} \sum f_n''(0)t^{n-1}, \text{ where } \rho \text{ is the fluid density.}$$

3.3.1 Results and Discussion

Van Dyke [29] considers the sign pattern to obtain the location of the dominating singularity. For coefficients numbered 4,..., ,44 the sign pattern is $(+++---++---+++--)$ with the exceptions of coefficients numbers 33 and 41. Examining the sign pattern, there exists a complex conjugate pair of singularities forming angle with the real axis in the complex plane $\theta = \pm 67.5^{\circ}$.

Cauchy Root Test shows that the dominant singularity pair is located at a radius 3 approximately from the origin and at angles of $\pm\beta$ in the complex plane by choosing $\beta = 67.5^{\circ}$.

Calculations of Pade' $\left(\frac{22}{22}\right)$[4] propose that the dominant singularity is at t_c=1.201375433 ± 2.932027611i with radius r_c=3.168609923 and making angle θ_c = ± 67.61° with the positive t axis.

We have anlysed the coefficients of the series of shear stress τ' by algebraic and differential approximation method and represent the results of comparison by tabular and graphically.

Table 3.2: Coefficients of the series $f_n''(0)$ calculated by Hommel.

SL No	$f_n''(0)$	SL No	$f_n''(0)$
1	1.1283791671	23	-0.1976×10^{-10}
2	-1.60727816	24	-0.9133×10^{-11}
3	-0.2480917	25	-0.1530×10^{-12}
4	0.14290×10^{-1}	26	0.8296×10^{-12}
5	0.28692×10^{-1}	27	0.1985×10^{-12}
6	0.63774×10^{-2}	28	-0.3284×10^{-13}
7	-0.15147×10^{-2}	29	-0.2569×10^{-13}
8	-0.10750×10^{-2}	30	-0.2811×10^{-14}
9	-0.97361×10^{-4}	31	0.1717×10^{-14}
10	0.89268×10^{-4}	32	0.6626×10^{-15}
11	0.30662×10^{-4}	33	-0.262×10^{-17}
12	-0.18844×10^{-5}	34	-0.6351×10^{-16}
13	-0.34650×10^{-5}	35	-0.1522×10^{-16}
14	-0.61583×10^{-6}	36	0.2526×10^{-17}
15	0.19425×10^{-6}	37	0.2129×10^{-17}
16	0.10522×10^{-6}	38	0.2511×10^{-18}
17	0.58123×10^{-8}	39	-0.1575×10^{-18}
18	-0.8889×10^{-8}	40	-0.6075×10^{-19}
19	-0.26564×10^{-8}	41	0.2687×10^{-20}
20	0.24906×10^{-9}	42	0.6591×10^{-20}
21	0.3176×10^{-9}	43	0.1050×10^{-20}
22	0.4865×10^{-10}	44	-0.4285×10^{-21}

Table 3.3: Calculated values of singularity t_c, radius of the convergence r_c, angle with the positive real axis θ_c and the critical exponent α using Drazin-Tourigny method [6] and High-order differential approximants[17

D-T				
d, N	t_c	r_c	θ_c	
5.20	.831095912630 + 3.0883573518i	3.19822944	+ 74.93	
6,27	2.93037916321 + 3.41790041172141	4.50212897	+ 49.38	
7,35	1.250590891999 - 2 7191540106916i	2.99295441	- 65 30	
HODA				
d, N	t_c	r_c	θ_c	α
5, 25	1.098897417 ± 2.395072768i	2.63513739	± 65.47	-39.762218+22.1007012i
6,33	1.377750991 ± 3.017342269i	3.31700952	± 65.57	10 2145653+4 628815931
7,42	1.176206370 ± 2 759056686i	2.999309125	± 67.04	-2.070123085 +6.106652415i

The Table 3.3 shows the singularities calculated by Drazin-Tourigny method [6] and High-order differential approximants [17] by taking different number of coefficients from the Table 3.2. It is clear that Drazin-Tourigny method [6] can determine the dominating singularity but taking 42 terms High-order differential approximants [17] calculated the singularity more accurately and it is at $t_c = 1.176206370 \pm 2.759056686i$ and generating an angle $\theta_c = \pm 67.04°$ and also determine the type of the singularity which is a pole. The pole calculated by High-order differential approximants [17] agrees with the Cauchy Root test and Van Dyke sign pattern examination.

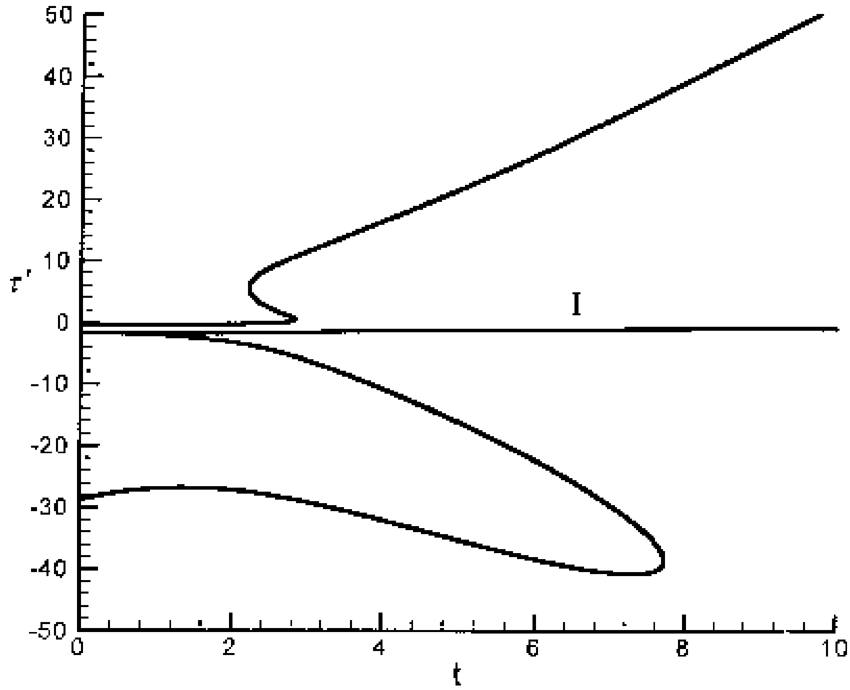


Figure 3.2: Approximate solution diagram (curve I) in the (t, τ') plane obtained by Drazin-Tourigny method [6] for $d = 6$. The other curves are spurious.

Figures 3.2 and 3.3 show that the shear stress τ' is negative which confirms the conjecture of Hommel [13]. Also numerically we have established that the singularities lie in the complex plane for which there is no physical change in figure in the real plane.

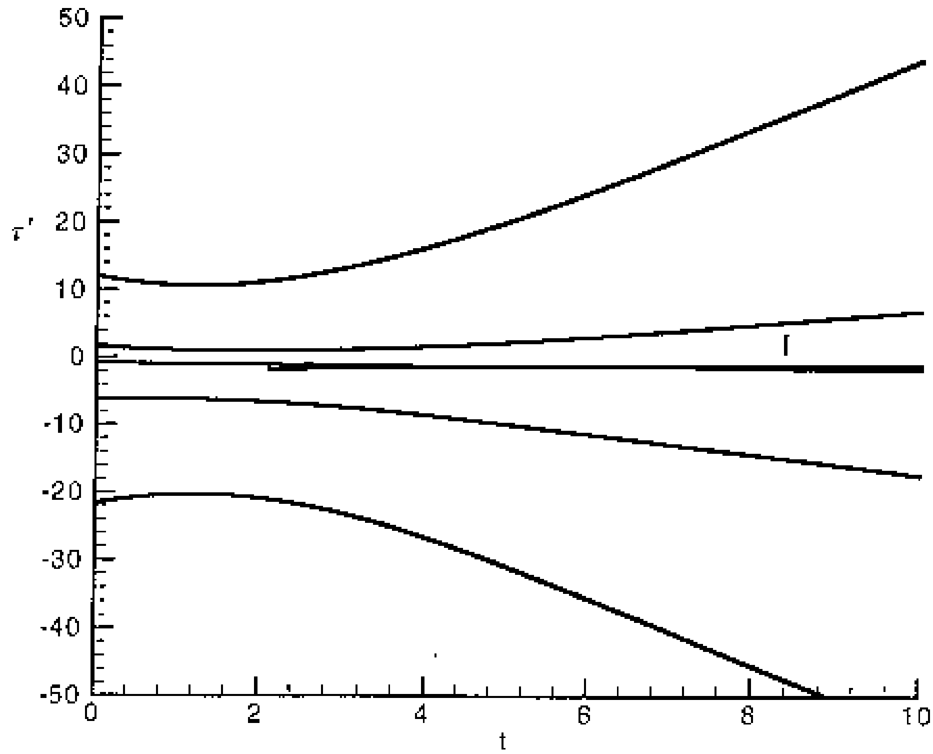


Figure 3.3: Approximate solution diagram (curve I) in the (t, τ') plane obtained by Drazin-Tourigny method [6] for $d = 7$. The other curves are spurious.

3.3.2 Conclusion

Table 3.3 shows the comparative study of the Drazin-Tourigny method [6] and High-order differential approximants [17] to approximate the dominating singularity using 44 terms and the figures show the physical location of the singularity of shear stress in the real plane for different values of d But High-order differential approximants [17] is able to calculate the dominating singular point using even lesser terms. And also it shows not only the singular points but also the critical exponent. But the only pole that it calculated by taking 42 terms agrees with the result of the Cauchy Root Test, and the assumption of Van Dyke according to the sign pattern. If more terms could be calculated and the value of d could be increased then we hope that the result will converge more accurately to the dominating singularity.

*Note: This problem has been presented in the **Bose Conference on Contemporary Physics** organized by Physics Department, University of Dhaka and The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy on 19 - 21 March, 2008.*

Chapter 4

Critical Behaviour of the Hydromagnetic Flows in Convergent-Divergent Channels

In this chapter we have studied the two-dimensional, steady, nonlinear flow of an incompressible conducting viscous fluid in Convergent-Divergent Channels under the influence of an externally applied homogeneous magnetic field by means of *Hermite – Pade'* approximation especially Differential approximate method. We have obtained the series related to similarity parameters by using algebraic programming language MAPLE. The series is then analysed by approximate methods to show the dominating singularity behavior of the flow and the critical relationship among the parameters of the solution.

4.1 Background

In modern times the theory of flow through Convergent-Divergent Channels have many applications in aerospace, chemical, civil, environmental, mechanical and bio-mechanical engineering as well as in understanding rivers and canals. The study of conducting viscous fluid flow through Convergent-Divergent Channels under the influence of an external magnetic field is not only fascinating theoretically, but it also finds application in mathematical modeling of several industrial and biological systems. The mathematical investigations of this type of problem were pioneered by Jeffery [16] and Hamel [14], which was the classical flow of ordinary fluid dynamics. Jeffery-Hamel flows are interesting models of boundary layers in Divergent Channel Fraenkel [10], Sobey and Drazin [28], Banks et al. [3] have studied extensively the problem in different ways. Makinde [23] investigated the Magneto Hydrodynamic (MHD) flows in Convergent-Divergent Channels. He extended the classical Jeffery-Hamel flows of ordinary fluid dynamics to MHD. Makinde [23] studied that in the MHD solution an external magnetic field acts as a control parameter for both Convergent and Divergent Channels flows Hence, beside the flow Reynolds number and the Channel angular width, at least an additional dimensionless parameter appears such as the Hartman number Ha. He obtained a perturbation series of twenty-four terms in powers of perturbation parameters Re, α and Ha and showed how the flows evolve and bifurcate as the flow parameters vary by using Algebraic approximate method [18].

4.2 Mathematical Formulation of the Problem

Consider the steady two-dimensional flow of an incompressible conducting viscous fluid from a source or sink at the intersection between two rigid plane walls under the influence of an externally applied homogeneous magnetic field as shown in Figure1. It is assumed that the fluid has small electrical conductivity and the electromagnetic force produced is very small. Let (r,θ) be polar coordinate with $r=0$ as the sink or source. Let α be the semi-angle and the domain of the flow be $-|\alpha| < \theta < |\alpha|$. Denote the velocity components in the radial and tangential direction by u and v respectively. The governing equations in terms of the vorticity (ω) and stream-function (ψ) can be written as [23]

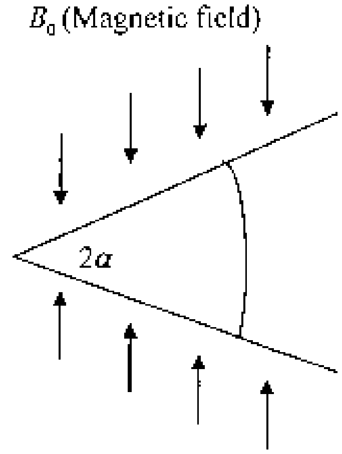


Figure 4.1: Convergent-Divergent Channels

$$\frac{1}{r}\frac{\partial(\psi,\omega)}{\partial(\theta,r)} = \upsilon\,\nabla^2\omega - \frac{\sigma_e B_0{}^2}{\rho r^2}\omega, \qquad \omega = -\nabla^2\psi\,, \tag{4.1}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{\partial}{r\partial r} + \dfrac{\partial^2}{r^2\partial\theta^2}$, with the boundary conditions

$$\psi = \frac{Q}{2}, \quad \frac{\partial\psi}{\partial\theta} = 0, \quad \text{at } \theta = \pm\alpha \tag{4.2}$$

Here $Q=\int_{-\alpha}^{\alpha} urd\theta$ is the volumetric flow rate, $B_0=(\mu_e H_0)$ the electromagnetic induction, μ_e the magnetic permeability, H_0 the intensity of magnetic field, σ_e the conductivity of the fluid, ρ the fluid density and υ is the kinematic viscosity coefficient..

For Jeffery-Hamel flow of conducting fluid, we assume a purely symmetric radial flow as described in [3], so that the tangential velocity $v=0$ and as a consequence of the mass conservation, we have the stream-function given by $\psi=\frac{QG(\theta)}{2}$. If we require $Q\geq 0$ then for $\alpha<0$ the flow is converging to a sink at $r=0$.

The dimensionless form of equations (1)-(2) is

$$\frac{d^4G}{d\eta^4}+2\mathrm{Re}\,\alpha\frac{dG}{d\eta}\frac{d^2G}{d\eta^2}+(4-Ha)\alpha^2\frac{d^2G}{d\eta^2}=0 \tag{4.3}$$

with

$$G=1,\ \frac{dG}{d\eta}=0,\ \text{at}\ \eta=\pm 1 \tag{4.4}$$

where $\eta=\frac{\theta}{\alpha}$ and $Ha=\sqrt{\frac{\sigma B_0{}^2}{\rho\upsilon}}$, $\mathrm{Re}=\frac{Q}{2\upsilon}$ are the Hartmann number and the flow Reynolds number respectively.

The problem defined by equation (4.3) is non-linear, for small Channel angular width, one can obtain a series of the form

$$G(\eta)=\sum_{i=0}^{\infty} G_i\alpha^i \tag{4.5}$$

We then find that $G(\eta)$ has a singularity at $\alpha=\alpha_c$ of the form

$$G(\eta)\sim C(\alpha_c-\alpha)^{\beta_c}$$

with the critical exponent β_c.

Substituting the above expressions (4.5) into (4.3) and collecting the coefficients of like powers of α and with the help of MAPLE, we have computed the first 18 terms for stream-function G in terms of α,Re,Ha. **[See Appendix I]**

The first few terms of the expansion of G are

$$G(\eta;\alpha,\mathrm{Re},\mathrm{Ha})=\frac{1}{2}\eta(3-\eta^2)-\frac{3}{280}\eta(\eta^2-5)(\eta-1)^2(\eta+1)^2\alpha\,\mathrm{Re}-\frac{1}{431200}\eta(98\eta^6-959\eta^4$$
$$+2472\eta^2-2875)(\eta-1)^2(\eta+1)^2\alpha^2\,\mathrm{Re}^2+\frac{1}{40}\eta(\eta-1)^2(\eta+1)^2(4-\mathrm{Ha})\alpha^2+ \tag{4.6}$$

Although the computational complexity increases rapidly, we managed to compute the first 75 terms for G in terms of single parameter α for $\mathrm{Re}=20$ and $Ha=0,1,2,3,4,5$. We also computed the first 75 terms for G in terms of single parameter Re at $\alpha=0.1$ for $Ha=4$. These series are then analyzed by Differential approximate methods [17, 26] to determine the critical behaviour of the flow and used the Algebraic approximate method [18] to show the bifurcation diagrams and the critical relationship among the parameters.

4.3 Results and Discussion

For the analysis, we make use of the series in powers of α, Re and Ha for the following functional form:

$$G'(\eta=0;\alpha,\mathrm{Re},Ha).$$

This quantity is proportional to the velocity of the flow along the centre line. By analyzing the series, we have calculated the location of the singularity with the critical exponents for different parameters. The results are obtained in terms of the critical Channel angular width α_c and critical Reynolds number Re_c for different values of magnetic parameter Ha and these are shown in tabular and graphically in figure.

Table 4.1: Estimates of critical angles α_c and corresponding exponent β_c at $Re = 20$ and $Ha = 0$ using High-order differential approximants [17].

d	N	α_c (single series)	α_c (multi-variable series)	β_c
2	7	.2749980919548527561596138	2749980919518527561596138	.10878684389230764954
3	12	2602059979752087905452916	.2602059979752087905452916	1 6592008206154575273
4	18	2679736610354297291067892	.2679736610354297291067892	4953933331461112855
5	25	.2679607925653470037213470		5000238205405836379
6	33	.2679608309216671706497074		5000002225349214361
7	42	2679608310828499910588531		.4999999999938879389
8	52	2679608310828466960601169		.4999999999999999565
9	63	2679608310828466960444344		.5000000000000000072
10	75	.2679608310828466960444352		5000000000000000002

Table 4.2: Estimates of critical angles α_c and corresponding exponent β_c at $Re = 20$ and $Ha = 1$ using High-order differential approximants [17].

d	N	α_c (single series)	α_c (multi-variable series)	β_c
2	7	27665112580188032650165569	276651125801880326501655691	7 429197021826671482
3	12	27176980615626427060569929	271769806156264270605699299	- 261785713430618567
4	18	.26925148848575837254505344	269251488485758372545053443	.4561906332084395651
5	25	26916241929088580028156064		.5000247289730165476
6	33	26916245977114395877256265		.4999999906897554027
7	42	.26916245976318174485150479		.4999999999573921934
8	52	.26916245976315907745130293		4999999999999998382
9	63	26916245976315907739346092		.5000000000000000050
10	75	26916245976315907739346096		4999999999999999923

Table 4.3: Estimates of critical angles α_c and corresponding exponent β_c at $Re = 20$ and $Ha = 2$ using High-order differential approximants [17].

d	N	α_c (single series)	α_c (multi-variable series)	β_c
2	7	.27823197900192240193407708	.27823197900192240193407708	5919848362542434934
3	12	.27379563943913053775199571	27379563943913053775199571	5448269070920006607
4	18	27037606595782002309656470	.27037606595782002309656470	5033015308595404481
5	25	27038644263339964600928911		.5000555023997966286
6	33	27038655906469678677030278		.4999999902052514669
7	42	.27038655905619460460995491		4999999999975450022
8	52	.27038655905619320425304318		4999999999999999820
9	63	.27038655905619320424824461		.4999999999999999990
10	75	.27038655905619320424824455		.5000000000000000029

Table 4.4: Estimates of critical angles α_c and corresponding exponent β_c at $Re = 20$ and $Ha = 3$ using High-order differential approximants [17].

d	N	α_c (single series)	α_c (multi-variable series)	β_c
2	7	.279762036762114265515350954	.279762036762114265515350954	5322217290273408261
3	12	.269213999914381062053064886	269213999914381062053064886	.4914784750977411087
4	18	.271630281415858951935860527	.271630281115858951935860527	.5011909765497291434
5	25	.271633911351507179960053773		.4999952474512750814
6	33	271633934182549283009691130		4999999973031274357
7	42	271633934180195495771995551		5000000000038244847
8	52	.271633934180197600883162104		5000000000000000062
9	63	271633934180197600883506790		.4999999999999999971
10	75	.271633934180197600883506774		.4999999999999999913

Table 4.5: Estimates of critical angles α_C and corresponding exponent β_c at $Re = 20$ and $Ha = 4$ using High-order differential approximants [17].

d	N	α_C (single series)	α_C (multi-variable series)	β_c
2	7	28125951132635432681080696	2812595113263543268108069639	533349118995105506
3	12	.27226274077853037159610100	.2722627407785303715961010067	.498228540512405286
4	18	27290283574099710569184306	2729028357409971056918430660	.500847691647703434
5	25	27290544292488388754260176		499994428166863026
6	33	.27290543430380172571596594		.500000001972260338
7	42	27290543430555951905919161		5000000000001361310
8	52	27290543430555959319476103		499999999999999997
9	63	27290543430555959319333598		.500000000000000115
10	75	27290543430555959319333598		.500000000000000100

Table 4.6: Estimates of critical angles α_C and corresponding exponent β_c at $Re = 20$ and $Ha = 5$ using High-order differential approximants [17].

d	N	α_C (single series)	α_c (multi-variable series)	β_c
2	7	282740455470542194962772877	28274045547054219496277	.5316268843464719954
3	12	.273920667962368822833379792	.27392066796236882283337	.4997345430256748736
4	18	274199503848608338641744612	.27419950384860833864171	.5007915959995632061
5	25	.274201956842852361389389378		.4999993169363610355
6	33	274201955830691574449370731		.5000000014203295731
7	42	.274201955831973233791395193		4999999999998953730
8	52	274201955831973177295547888		5000000000000000020
9	63	274201955831973177293986707		.4999999999999999943
10	75	.274201955831973177293986712		4999999999999999865

Table 4.7: Estimates of critical Reynolds numbers Re_c, and corresponding exponent β_c at $\alpha = 0.1$ and $Ha = 4$ using High-order differential approximants [17].

d	N	Re_c	β_c
2	7	56.251902265270865362161392 79	72098188580721 79797
3	12	54 4525481557060743192202 0134	.48410683617161 14107
4	18	54.5805671481994211383686132 1	.49992965706807 48621
5	25	54 5810885849767775085203 5270	.49999442816686 30267
6	33	54.5810868607603451431931 8997	.50000000197226 03390
7	42	54.5810868611119038118383 2271	.50000000000013 61302
8	52	54.5810868611119186389522 0716	.19999999999999 99939
9	63	54.5810868611119186386671 9641	.49999999999999 99998
10	75	54 5810868611119186386671 9680	.49999999999999 99953

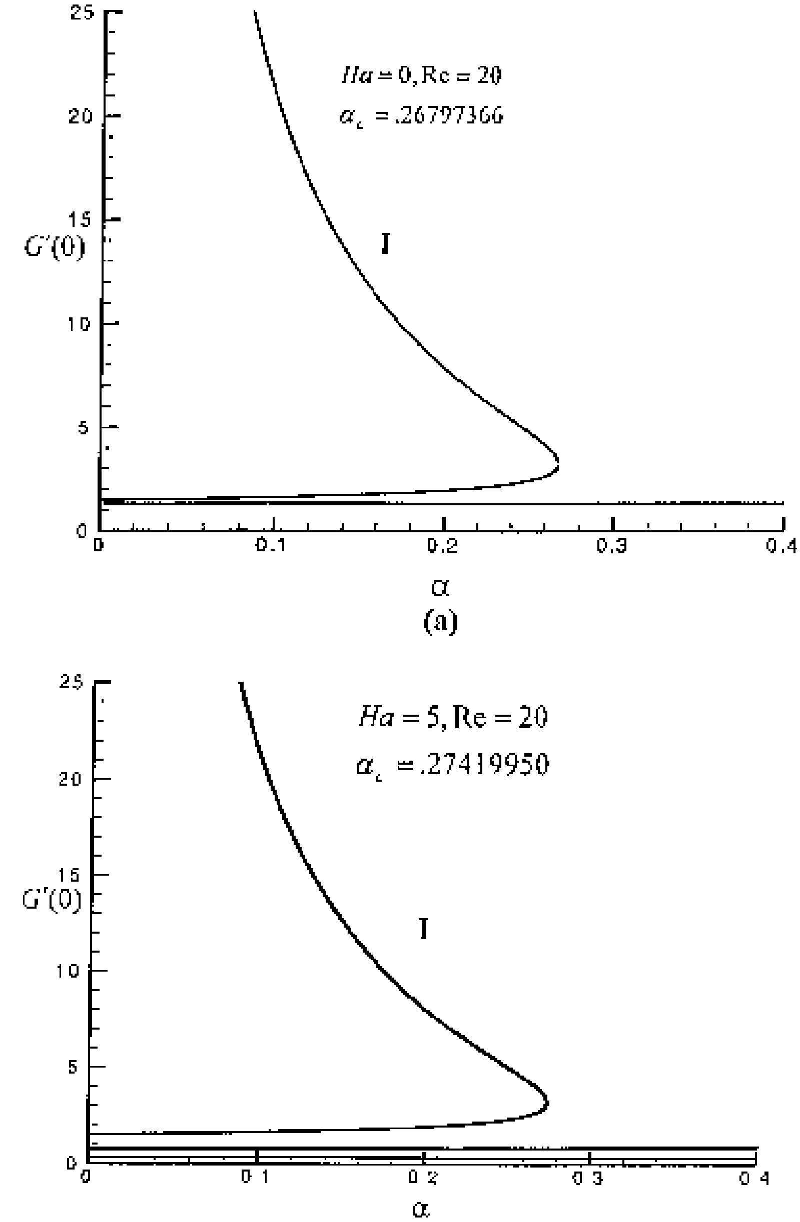
Table 4.8: Comparisons of critical angles α_c and corresponding critical exponent β_c at $Re = 20$ using High-order differential approximants [17]. The result is comparable with the result of Makinde [23].

HODA	Ha	0	1	2	3	4	5
$d = 4$ $N = 18$	α_c	.26797366	.26925148	.27037606	.27163028	.27290283	.27419950
	β_c	.49539333	.45619063	.50330153	.50119097	.50084769	50079159
Makinde	Ha	0	1	2	3	4	5
	α_c	0.267960	0.269162	0.272906	0 279878	0 290431	0 307406
	β_c	0.50000	0.50000	0.50000	0.50000	0 50000	0.50000

Table 4.9: Comparisons of critical Reynolds number Re_C and corresponding exponent β_c at $\alpha = 0.1$ using High-order differential approximants [17] and Makinde [23].

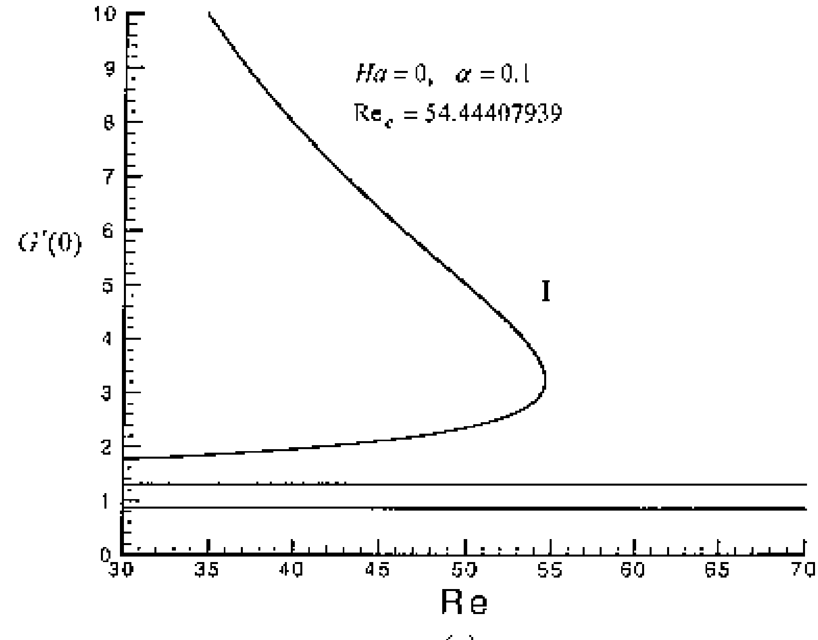
HODA	Ha	0	1	2	3	4	5
$d=4$ $N=18$	Re_C	54.44407939	54.47805874	54.51340970	54 54702585	54 58135150	54.61676356
	β_c	.4991155356	.4991155356	.4991155356	.4997069458	.5001231984	.4985944495
Makinde	Ha	0	1	2	3	4	5
	Re_C	54.4389	54.47179	54.58087	54.66510	55.22071	55 52727
	β_c	0 50000	0.50000	0 50000	0.50000	0.50000	0.50000

Figure 4.2(a) and 4.2(b) displays the results that as the magnetic parameter Ha increases, then the bifurcation point α_C changes form .26797366($Ha = 0$) to .27419950 ($Ha = 5$). Figure 4.3(a) and 4.3(b) shows that as the magnetic parameter Ha increases, the bifurcation point Re_c changes form 54.44407939 ($Ha = 0$) to 54.61676356 ($Ha = 5$). Therefore it is observed that the effect of magnetic intensity changes the solution behaviour of the problem.

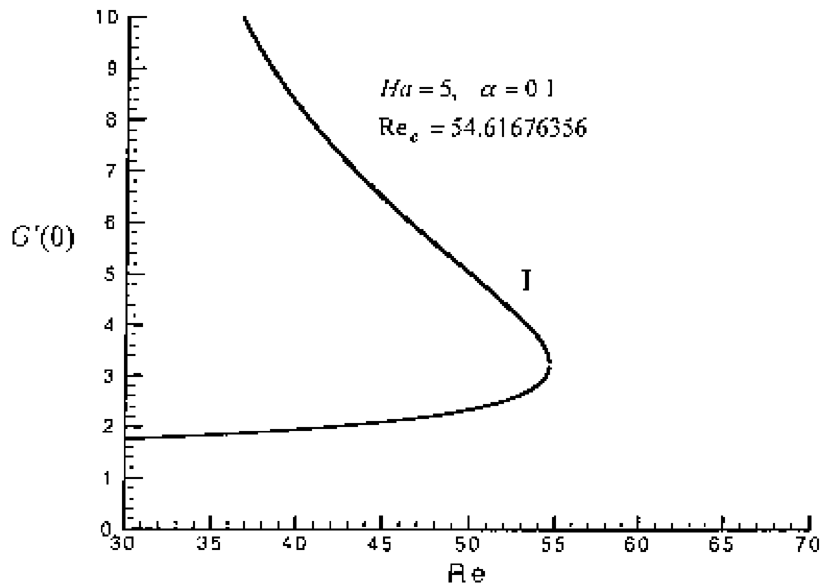


(b)

Figure 4.2: Approximate bifurcation diagram (curve I) of α_c in the $(\alpha,\ G'(0))$ Plane (a) with $Ha = 0$ and (b) with $Ha = 5$ obtained by Drazin-Tourigny method [6] for $d = 8$. The other curves are spurious.



(a)



(b)

Figure 4.3: Approximate bifurcation diagram (curve I) of Re_c in the $(\mathrm{Re},\ G'(0))$ Plane (a) with $Ha = 0$ and (b) with $Ha = 5$ obtained by Drazin-Tourigny method [6] for $d = 4$. The other curves are spurious.

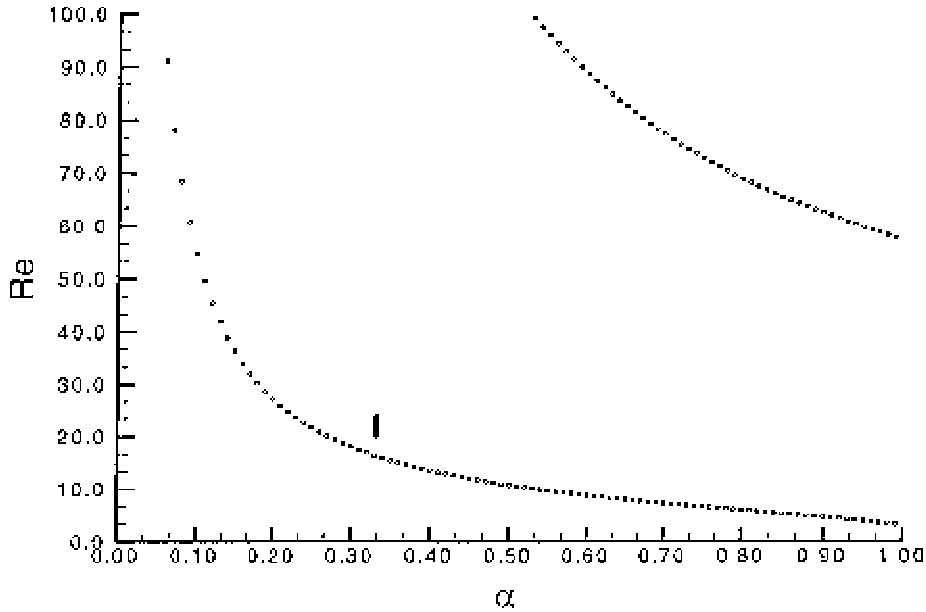


Figure 4.4: Critical α -Re relationship (curve I) using High-order partial Differential approximants [26] with $d = 6$ The other curve is spurious.

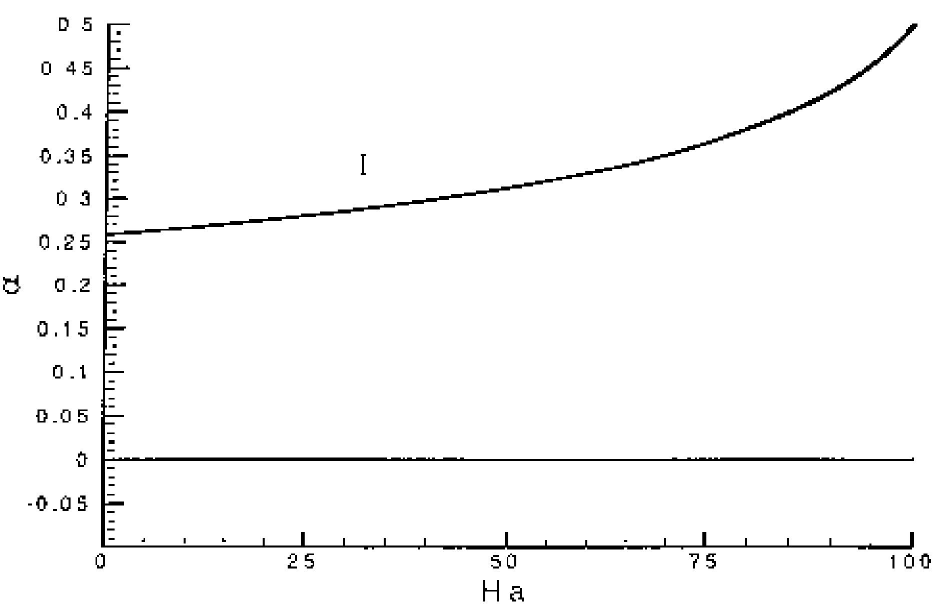


Figure 4.5: Critical $Ha - \alpha$ relationship (curve I) using High-order partial Differential approximants [26] with $d = 6$. The other curve is spurious.

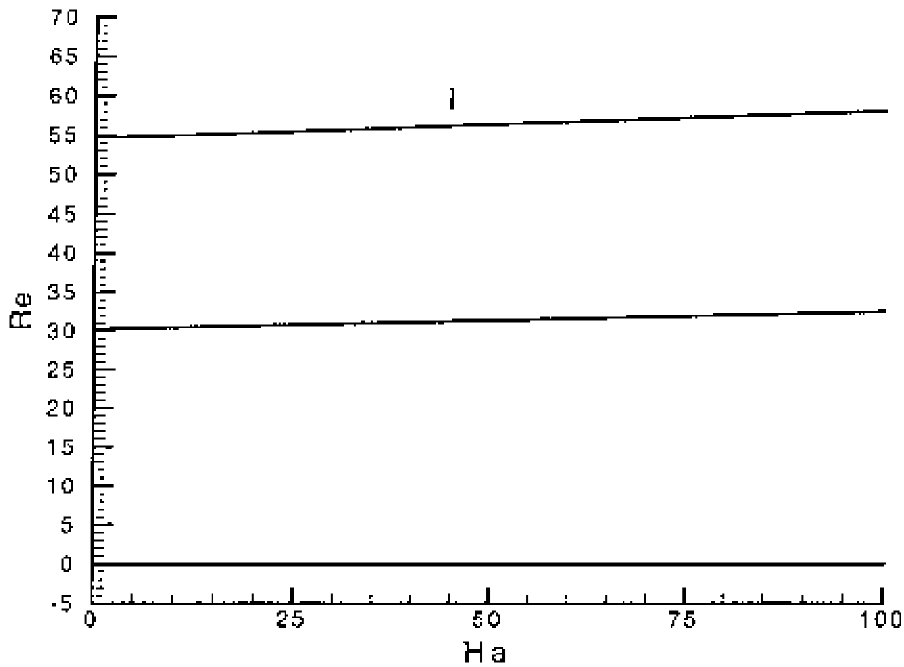


Figure 4.6: Critical $Ha-\mathrm{Re}$ relationship (curve I) using High-order partial Differential approximants [26] with $d = 6$. The other curves are spurious.

The High-order Partial Differential Approximant [26] is applied to the series $G'(0;\alpha,\mathrm{Re},Ha)$ in order to determine the critical relationship of $\alpha_c, \mathrm{Re}_c, Ha_c$. In Figure 4.4, it is observe that as Re_c increases then α_c decreases. Figure 4.5 and 4. 6 shows that α_c and Re_c changes significantly with the changes of magnetic parameter Ha.

4.4 Discussion

In this Chapter, we have studied the Critical Behaviour of the solution of Hydromagnetic flows in Convergent-Divergent Channels. A number of interesting features have been brought to attention, the foremost of which are different including the critical relationship among the parameters of the flow.

From the present investigation the following conclusions can be drawn:

- α_C and Re_C increases uniformly as Ha increases which are comparable with the results of Makinde.

- As the magnetic parameter Ha increases, then the bifurcation point α_C changes form .26797366($Ha = 0$) to .27419950 ($Ha = 5$). Also if the magnetic parameter Ha increases, the bifurcation point Re_c changes form 54.44407939 ($Ha = 0$) to 54.61676356 ($Ha = 5$). Therefore it is observed that the effect of magnetic intensity changes the solution behaviour of the problem.
- From the critical relationship it is observe that as Re_c increases then α_c decreases and α_c and Re_c changes significantly with the changes of magnetic parameter Ha.

4.5 Conclusion

We have used power series to study the singularity behaviour of the nonlinear problem of Hydromagnetic Jeffery-Hamel flow by using series summation technique. Our results confirm the conjecture of Makinde [23] with a little difference. Remarkably, we have shown the critical relationship among the parameters of the flow.

*Note: This problem has been presented in the 4^{th} **BSME-ASME International Conference on Thermal Engineering** on 27-29 December 2008 held in LGED HQ, Agargaon, Dhaka and IUT, Gazipur*

Chapter 5

Conclusion

5.1 Conclusion

We have described the Approximants method which is applied to analyse two fluid dynamical problems. The Critical Behaviour of the Laminar Unsteady Flow of a Viscous Fluid away from a Plane Stagnation Point and the Hydromagnetic Flows in Convergent – Divergent Channels have been studied using various types of Approximants methods. The critical relationships among the flow parameters have also been analysed.

By analysing the critical behaviour of the first Painlev'e transcendent we observed that High-order Differential approximant is able to determine more significantly that

$x_c \approx 3.742801530770741877$5 is a pole of order 2, which is the nearest pole to the origin.

The comparative study of the Drazin-Tourigny method [6] and High-order differential approximants [17] to approximate the dominating singularity behaviour of Laminar Unsteady Flow of a Viscous Fluid away from a Plane Stagnation Point using 44 terms is presented. High-order differential approximants [17] is able to calculate the dominating singular point using even lesser terms and agrees with the results of the Cauchy Root test and the assumption of Van Dyke [29].

Analysis of the Hydromagnetic Flows in Convergent – Divergent Channels show that the critical Channel angular width and Critical Reynolds number changes uniformly due to the effect of magnetic parameter Hartmann number. Critical relationships represent the significant variation due to the effect of magnetic intensity.

We try to provide a basis for guidance about what method of summing power series should be chosen for many problems in fluid dynamics to show the critical behaviour of the flow.

5.2 Future Work

There are some ideas to form the basis of future work:

- Critical Behaviour of the solution of Hydromagnetic Flows in Convergent-Divergent Channels has-been analysed using High-order differential approximant and Drazin-Tourigny approximant method. Therefore, further research in this regard could be carried out (i) by deriving series in more terms (ii) by using the series in terms of parameter Ha .
- Application of Approximation methods in other fields that include perturbation series.

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Appendix I

Program to compute the coefficients of multi-variable series (in Chapter 4)

$$G(\eta;\alpha,\mathrm{Re},Ha)=\frac{1}{2}\eta(3-\eta^2)-\frac{3}{280}\eta(\eta^2-5)(\eta-1)^2(\eta+1)^2\alpha\,\mathrm{Re}-\frac{1}{431200}\eta(98\eta^6-959\eta^4+2472\eta^2-2875)(\eta-1)^2(\eta+1)^2\alpha^2\,\mathrm{Re}^2+\frac{1}{40}\eta(\eta-1)^2(\eta+1)^2(4-Ha)\alpha^2+\ldots$$

```
#the parameter A= alpha*R and B= (4-H)*alpha^2:
interface(quiet=true):
N:=25:
#digits:=150:
G:=array(0..N,0..N):
h:=array(0..N,0..N):
p:=array(0..N,0..N):
for m from 0 to N do
for n from 0 to N do
if (m=0 and n=0) then
G[0,0]:=(1/2)*eta*(3-eta^2):
h[0,0]:=diff(G[0,0],eta):
p[0,0]:=diff(G[0,0],eta,eta):
else
i:='i':
#c.(m+n):=array(1..(4*(m+n)+7)):
c:=array(1..(4*(m+n)+7)):
G[m,n]:=sum(c[i]*eta^(i-1),i=1..(4*(m+n)+7)):
#G[m,n]:=sum(c.(m+n)[i]*eta^(i-1),i=1..(4*(m+n)+7)):
gg:=diff(G[m,n],eta):
bc[1]:=subs(eta=1,G[m,n])=0:
bc[2]:=subs(eta=-1,G[m,n])=0:
bc[3]:=subs(eta=1,gg)=0:
bc[4]:=subs(eta=-1,gg)=0:
l1:=diff(G[m,n],eta,eta,eta,eta);
j:='j':
```

```
l1:=diff(G[m,n],eta,eta,eta,eta):
j:='j':
l2:=0:
#for l from 0 to n do
#for k from 0 to m-1 do
#l2:=l2+h[k,l]*p[m-k-1,n-l]:
#od:
#od:
#l2:=2*l2:
l2:=2*sum(sum(h[k,l]*p[m-k-1,n-l],l=0..n),k=0..m-1):
if (n=0) then
l3:=0:
else
#l3:=p[m,n-1]):
l3:=p[m,n-1]:
fi:
FF:=expand(l1+l2+l3):
#print([m,n],FF);
for i from 0 to (4*(m+n)+2) do
beq[i]:=coeff(FF,eta,i)=0:
od:
aa:= array (1..(4*(m+n)+7),1..(4*(m+n)+7) ):
for i from 1 to 4 do
for j from 1 to 4*(m+n)+7 do
aa[i,j]:=coeff(lhs(bc[i]),c[j]):
#aa[i,j]:=coeff(lhs(bc[i]),c.(m+n)[j]):
od:
od:
for i from 5 to 4*(m+n)+7 do
for j from 1 to 4*(m+n)+7 do
aa[i,j]:=coeff(lhs(beq[i-5]),c[j]):
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```
#aa[i,j]:=coeff(lhs(beq[i-5]),c.(m+n)[j]):
od:
od:
b:=array(1..(4*(m+n)+7)):
for i from 1 to 4 do
b[i]:=0:
od;
i:='i':
j:='j':
for i from 5 to 4*(m+n)+7 do
b[i]:=-(lhs(beq[i-5])-(sum(aa[i,j]*c[j],j=1..4*(m+n)+7))):
#b[i]:=-(lhs(beq[i-5])-(sum(aa[i,j]*c.(m+n)[j],j=1..4*(m+n)+7))):
od:
with(linalg):
c:=array(1..(4*(m+n)+7)):
c:=linsolve(aa,b):
#c.(m+n):=array(1..(4*(m+n)+7)):
#c.(m+n):=linsolve(aa,b):
h[m,n]:=diff(G[m,n],eta):
p[m,n]:=diff(G[m,n],eta,eta):
G[m,n]:=factor((G[m,n])):
print([m,n]);
save G, "C://Sir/jef_sar25.m":
fi:
od:
od:
#done
```