A New Approach To Partial Differential Approximants

A dissertation submitted in partial fulfillment of the requirements for the award of the degree

of

Master of Philosophy
in Mathematics

by

Md. Mustafizur Rahman
Roll No. 100109005P, Registration No. 0110645
Session: October 2001
Department of Mathematics
Bangladesh University of Engineering and Technology
Dhaka-1000

Supervised
by

Dr. Md. Abdul Hakim Khan
Associate Professor
Department of Mathematics, BUET

Department of Mathematics
Bangladesh University of Engineering and Technology
Dhaka-1000
December 2004

The Thesis Entitled

A New Approach To Partial Differential Approximants

submitted by

MD.MUSTAFIZUR RAHMAN

Registration No. 0110645, Roll No. 100109005P, Session: October 2001, a part time student of M. Phil (Mathematics) has been accepted as satisfactory in partial fulfillment for the degree of

Master of Philosophy in Mathematics
on 29 December 2004.

Board of Examiners

1. MAHKhan

Dr. Md. Abdul Hakim Khan — **Chairman** (Supervisor)
Associate Professor
Department of Mathematics
BUET, Dhaka-1000

2. NFarhat

Dr.Nilufar Farhat Hossain — Member (Ex-officio)
Professor and Head
Department of Mathematics
BUET, Dhaka-1000

3. MZullah

Dr. Md. Zakerullah — Member
Professor
Department of Mathematics
BUET, Dhaka-1000

4. AHalder

Dr. Amal Krisna Halder — Member (External)
Professor
Department of Mathematics
University of Dhaka
Dhaka-1000.

Table of Contents

Abstract

The modelling of physical phenomena usually results to nonlinear problems whose solutions may have singularities Practically the locations of the singularities are important For many problems, a solution can be found as a series in powers of one or several independent variables In this thesis under the title "A New Approach To Partial Differential Approximants" we have analysed series in powers of two independent variables by High-order partial differential approximants We have developed the method using the concept of *Pade-Hermite* class. It consists of a high-order linear partial differential equation with polynomial coefficients that is satisfied approximately by the partial sum of the multivariable power series

We have also reviewed the different approximant methods for the summation of series in powers of one or more independent variables Our aim is to apply the new method to problems in physical field, particularly in fluid dynamics.

Candidate's Declaration

I hereby declare that the work, which is being presented in the thesis entitled "A New Approach To Partial Differential Approximant", submitted in partial fulfillment of the requirements for the award of the degree of Master of Philosophy in Mathematics, in the department of Mathematics, Bangladesh University of Engineering and Technology, Dhaka, is an authentic record of my own work.

The matter presented in this thesis has not been submitted by me for the award of any other degree in this or any other University.

Date: December 29, 2004

(Md. Mustafizur Rahman)

Acknowledgements

This thesis is prepared under the supervision of my honorable supervisor Dr Md. Abdul Hakim Khan, Associate Professor, Department of Mathematics, Bangladesh University of Engineering & Technology, Dhaka His generous help, guidance, constant encouragement and indefatigable assistance were available to me at all stages of my research work. I am highly grateful to him for his earnest feeling and help in matters concerning my research affairs

I express my deep regards to my respectable teacher, Dr. Nilufar Farhat Hossain, Professor & Head, Department of Mathematics, BUET for providing me advice and necessary research facilities

I also express my gratitude to my teachers Dr Md Zakerullah, Dr. Md. Mustafa Kamal Chowdhury, Mr. A K Hazra, Mr Md. Isa, Mr Md. Abdul Quddus Mean, Dr. Md. Elias, Dr Md Abdul Maleque, Mr Md Obayedullah, Department of Mathematics, Bangladesh University of Engineering & Technology, for their valuable comments and advice during my research work.

I indebted to my parents who have been constant source of encouragement and inspiration from my childhood. I am grateful to my wife and kids for their encouragement and moral support throughout the research work Indeed, they sacrificed a lot for my thesis

List of Tables

List of Figures

CHAPTER 1

INTRODUCTION

When you follow two separate chains of thought, Watson, you will find some point of intersection which should approximate the truth

Sherlock Holmes, *The Disappearance of Lady Farcis Carfaxt*
Sir Arthur Conan Doyle

This thesis is concerned with a new approach to Partial Differential Approximants along with review of some existing approximant methods. The approximant methods are widely used to approximate functions in many areas of applied mathematics.

The mathematical model of physical phenomena usually results in non-linear equations, which may be algebraic, ordinary differential, partial differential, integral or combination of these. The non-linear equations may contain one or several independent variables. The solutions of these non-linear systems are dominated by their singularities (if exist). A value of independent variable (or variables) for which the function is undefined is known as a singularity of the function. Singularity plays an important role in many areas of applied science. Particularly in fluid dynamics, the presence of singularities may reflect some changes in the nature of the flow and their study is of great practical interest. Sometimes it is very difficult to find out the exact solution of physical problems. Particularly in statistical mechanics, there are a large number of problems for which the first few terms of the power series may be obtained exactly while the exact solution is unobtainable. The three dimensional Ising model [18] is a good example. On the other hand, if the power series expansion of a non-linear system is given, but their corresponding function is not known, then it becomes difficult to reproduce the function from the given power series. However, one can study their singularities by some power series approximant methods. In order to study these problems many powerful techniques have been used to find the power series coefficients. At the same time a variety of methods have been introduced for getting the required information about the singularities by using a finite number of series coefficients.

Brezinski [5] studied history of continued fraction and *Padé* approximants. Blanch [6] evaluated continued fractions numerically. Also the applications of continued fractions and their generalizations to problems in approximation theory have been studied by Khovanskii [28]. Khan [24] analyzed singularity behavior by summing power series. Khan [25] also introduced Differential Approximant for single independent variable, where he developed a new form of ordinary differential approximant called High-order Differential Approximant (HODA), for the summation of power series. The method is a special type of *Padé-Hermite* class and it is one of the best methods of singularity analysis for the problems of single independent variable. Baker and Graves-Morris [1] studied multivariable *Padé* approximants and stated that the generalization of *Padé* approximants to more than one variable is as usual. In this regard multivariable algebraic approximants [26] are notable. Fisher and Styer [14] introduced partial differential approximants for multivariable power series. Styer [34] also investigated the invariance properties of partial differential approximants. Fisher and Kerr [15] studied multi-critical singularities by partial differential approximants. Recently Khan et al. [26] described a method for the summation of series in powers of several independent variables and its application in fluid dynamics.

The remainder of this introductory chapter is organized as follows:

Since the problems that we shall study in this thesis are nonlinear, we begin with a brief review of elementary bifurcation theory in §1.1. Then in §1.2 we also review some elementary facts about power series. In §1.3 we discuss various types of singularities with examples. We present the basic concept of continued fractions in §1.4. Finally in §1.5 we describe a brief out line of the remainder of the thesis.

1.1 Elementary bifurcation theory

In this thesis we have investigated an important nonlinear problem, which arises in fluid mechanics. Solution of nonlinear problems often involve one or several parameters. As a parameter varies, so does the solution set. A bifurcation occurs where the solutions of a nonlinear system change their qualitative character as a parameter changes. In particular, bifurcation theory [13] is about how the number of steady solutions of a system depends on parameters. Therefore, about the theory of bifurcation concerns with all nonlinear systems and thence has a great variety of applications. Examples of bifurcations are: Simple turning points, in which two real solutions becomes complex conjugate solutions and pitchfork bifurcation, in which the number of real solutions changes discontinuously from one to three (or vice versa). Our purpose in this section is to introduce some elementary concepts of bifurcation theory. Drazin has described bifurcation theory in detail in [13]. To illustrate the bifurcation points, first we consider the quadratic equation

$$x^2 - \alpha = 0 \tag{1.1.1}$$

The roots $\pm\sqrt{\alpha}$ are real for $\alpha > 0$ and are a complex conjugate pair for $\alpha < 0$. We say that there is a change in the character of the solutions at $\alpha = 0$, where there is a repeated root $x = 0$. If we confine our attention to real solutions, then there are two for $\alpha > 0$, one for $\alpha = 0$ and none for $\alpha < 0$. To illustrate the real solutions, we have sketched the parabola in the (α, x)-plane, shown in the Fig 1.1, which is called bifurcation diagram and (0,0) is called a bifurcation point.

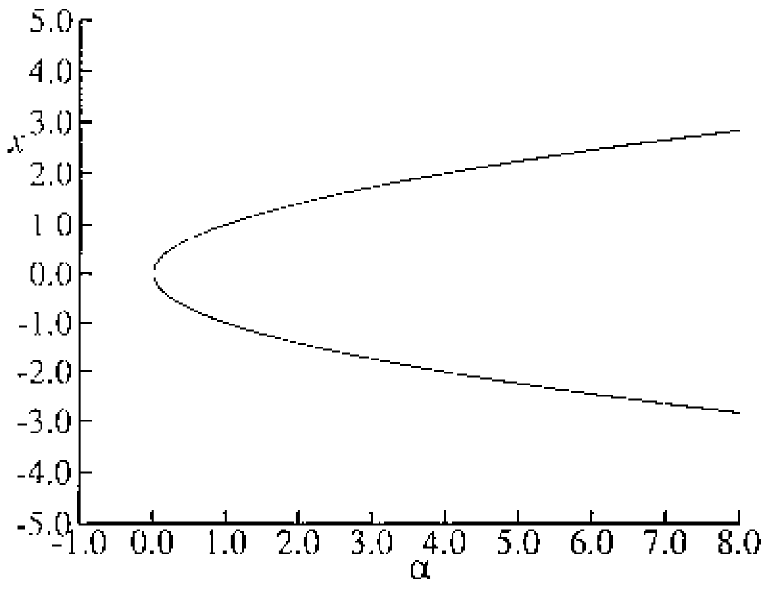

Figure1.1: Bifurcation diagram for (1.1.1) in the (α, x)-plane

In details, let us consider a functional map $f: \mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$, we seek solution $x = X(\alpha)$ of

$$f(\alpha, X) = 0 \tag{1.1.2}$$

The solutions can be visualized by means of a bifurcation diagram, in which the solution curves are drawn in the (α, x)-plane.

Consider (α_0, x_0) be a solution of (1.1.2)

$$\text{i.e. } f(\alpha_0, x_0) = 0 \tag{1.1.3}$$

Then, we may expand f in a Taylor series about (α_0, x_0) and so study the solution set in that neighborhood. We get

$$0 = f(\alpha, x) = f(\alpha_0, x_0) + (x - x_0) f_x(\alpha_0, x_0) + (\alpha - \alpha_0) f_\alpha(\alpha_0, x_0) + \frac{1}{2}(x - x_0)^2 f_{xx}(\alpha_0, x_0) + \cdots \tag{1.1.4}$$

If we take $f_x(\alpha_0, x_0) \neq 0$, then

$$x(\alpha) = x_0 - (\alpha - \alpha_0)\frac{f_\alpha(\alpha_0, x_0)}{f_x(\alpha_0, x_0)} + O(\alpha - \alpha_0) \text{ as } \alpha \to \alpha_0 \tag{1.1.5}$$

and we see that there is only one solution curve in the neighborhood of the point (α_0, x_0) in the bifurcation diagram.

However, if $(\alpha_0, x_0) = (\alpha_c, x_c)$ where

$$f(\alpha_c, x_c) = 0,\ f_x(\alpha_c, x_c) = 0 \tag{1.1.6}$$

then the expansion (1.1.4) shows that there are at least two solution curves in the neighborhood of (α_c, x_c). The point (α_c, x_c) is called a **bifurcation point**.

Example 1.1.1 Let f be given by

$$f(\alpha, x) = (x-2)\{(x-2)^2 + \alpha + 1\} - \varepsilon \tag{1.1.7}$$

where ε is some real parameter. By solving the equation (1.1.6), we see that the bifurcation point depends on ε. For $\varepsilon = 0$, the bifurcation occurs at $(\alpha_c, x_c) = (-1, 2)$, which terms as the singularity point and there are three solution branches that intersect at that point, hence the "pitchfork bifurcation" shown in Figure 1.2.

Such pitchforks often arise as a result of some symmetry inherent in the problem. It is interesting to note that for non-zero values of ε, no matter how small, this pitchfork is replaced by a simple turning point, as shown in Figure 1.3. The existence of the parameter ε "breaks the symmetry" in the problem.

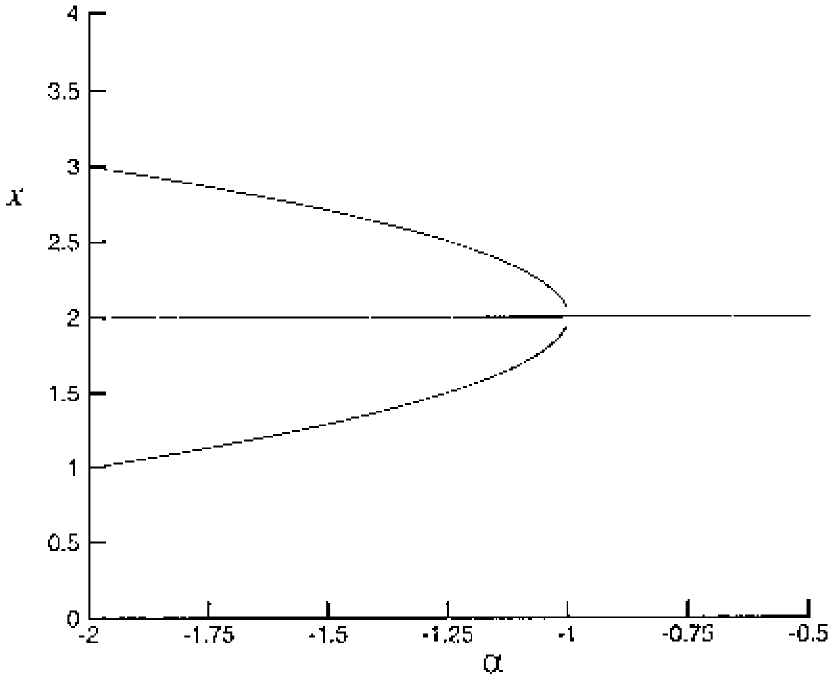

Figure1.2: Pitchfork bifurcation diagram for (1.1.7) in the (α, x)-plane ε =0.0

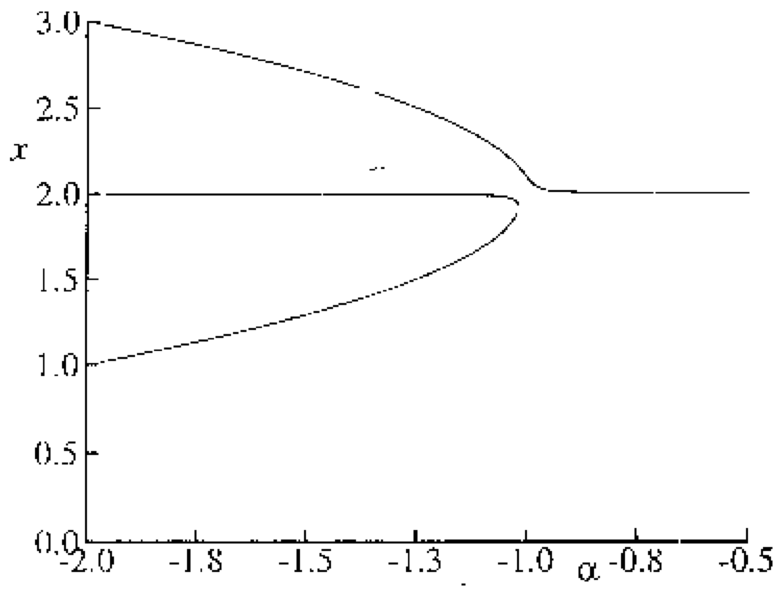

Figure1 3: Symmetry breaking into the bifurcation diagram for (1.1.7) in the (α, x)-plane when ε =0.01

1.2 Review of Power Series

The solution of nonlinear problem can be expressed as a series in powers of one or several independent variables The first few terms of the series expansion contain major information of the problem One can use this information to carry out further research on the problem

1.2.1 Single variable series

Consider a function $f(x)$ which can be represented by a power series

$$S(x)=\sum_{i=0}^{\infty} c_i x^i \qquad \text{as} \qquad x \to 0 \tag{1.2.8}$$

The Nth partial sum is

$$S_N(x)=\sum_{i=0}^{N-1} c_i x^i \tag{1.2.9}$$

The series converges if the sequence of partial sums converges. When the series converges, the series $S(x)$ can be approximated by the partial sum $S_N(x)$ and the error is defined by

$$e_N(x)=S(x)-S_N(x) \tag{1.2.10}$$

When $S(x)\neq 0$, the absolute relative error is defined by

$$e'_N(x)=\left|\frac{e_N(x)}{S(x)}\right| \tag{1.2.11}$$

1.2.2 Multivariable series

Consider a function $f(x,y)$ of two independent variables, which can be represented by a power series

$$S(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij} x^i y^j \qquad \text{as} \quad (x,y)\to(0,0) \tag{1.2.12}$$

The Nth partial sum is

$$S_N(x,y)=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}c_{i,j}x^i y^j \tag{1.2.13}$$

The series converges if the sequence of partial sums converges. When the series converges, the series $S(x,y)$ can be approximated by the partial sum and the error is defined by

$$e_N(x,y)=S(x,y)-S_N(x,y) \tag{1.2.14}$$

When $S(x,y)\neq 0$, the absolute relative error is defined by

$$e'_N(x,y)=\left|\frac{e_N(x,y)}{S(x,y)}\right| \tag{1.2.15}$$

In applied mathematics, power series are often obtained by expanding a function in powers of some perturbation parameters. In the following subsection, we describe the basic literature on perturbation techniques. See [4] & [31] for details.

1.2.3 Perturbation series

Perturbation theory is a collection of methods for the systematic analysis of the global behavior of solutions to nonlinear problems. Sometimes we solve nonlinear problems by expanding the solution in powers of one or several small perturbation parameters. The expansion may contain small or large parameters which appear naturally in the equations, or which may be artificially introduced. A perturbative solution is constructed about the perturbation parameter $\varepsilon=0$ as a series of powers of ε

$$s(x)=c_0(x)+\varepsilon c_1(x)+\varepsilon^2 c_2(x)+\cdots \tag{1.2.16}$$

where the coefficients c_n are independent of ε. This series is called a perturbation series. It is to note that the perturbation series for $s(x)$ is local in ε but that it is global in x. If ε is very small, we expect that $s(x)$ will be well approximated by only a few terms of the perturbation series.

Following are elementary examples to introduce the ideas of perturbation series.

Example1.2.2 Roots of a cubic polynomial

Consider the algebraic equation

$$x^3 - (4+\varepsilon)x + 2\varepsilon = 0 \qquad (1.2.17)$$

for small ε.

For $\varepsilon = 0$ it have the roots $x = -2, 0, 2$

Assume a perturbation series in powers of ε

$$x(\varepsilon) = \sum_{i=0}^{\infty} c_i \varepsilon^i \qquad (1.2.18)$$

When $x = -2$, by substituting the expansion

$$x = -2 + c_1\varepsilon + c_2\varepsilon^2 + \cdots$$

into (1.2.18) and equating the coefficients of ε, we get $c_1 = -\frac{1}{2}, c_2 = \frac{1}{8} \cdots$

Therefore, the perturbation series for $x = -2$ is

$$x_1 = -2 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots$$

The same procedure gives

$$x_2 = 0 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + O(\varepsilon^3), \text{ for } x = 0 \quad \text{and} \quad x_3 = 2 + 0.\varepsilon + 0.\varepsilon^2 + O(\varepsilon^3), \text{ for } x = 2$$

Example1.2.3 Consider the initial value problem

$$y'' = f(x)y,\ y(0) = 1,\ y'(0) = 1 \qquad (1.2.19)$$

First we introduce an ε in (1.2.19) such that the unperturbed problem is solvable

$$y'' = \varepsilon f(x)y,\ y(0) = 1,\ y'(0) = 1 \qquad (1.2.20)$$

Now assume a perturbation expansion for $y(x)$ in the following form

$$y(x) = \sum_{i=0}^{\infty} \varepsilon^i y_i(x) \qquad (1.2.21)$$

where $y(0) = 1,\ y'(0) = 1$ and $y_i(0) = 0,\ y_i'(0) = 0\ (i \geq 1)$

The zero-th order problem $y''(0) = 0$ is obtained by setting $\varepsilon = 0$ and the solution which satisfies the initial condition is $y_0 = 1 + x$

For the ith order problem $(i \geq 1)$, substituting (1.2.21) into (1.2.20) and equating the coefficient of ε^i $(i \geq 1)$ to zero, we get

$$y_i'' = y_{i-1} f(x), \quad y_i(0) = 0, \; y_i'(0) = 0 \tag{1.2.22}$$

The solution of (1.2.22) is

$$y_i = \int_0^x dt \int_0^t ds\, f(s) y_{i-1}(s), \; i \geq 1 \tag{1.2.23}$$

Equation (1.2.23) gives a simple iterative procedure for calculating successive terms in the perturbation series (1.2.21).

$$y(x) = 1 + x + \varepsilon \int_0^x dt \int_0^t ds (1+s) f(s) + \varepsilon^2 \int_0^x dt \int_0^t ds\, f(s) \int_0^s dv \int_0^v du (1+u) f(u) + \cdots \tag{1.2.24}$$

Putting $\varepsilon = 1$ yields the result.

1.3 Singularities

Singularity of a function is a value of the independent variable or variables for which the function is undefined. Singularities are crucial points of a function, because the expansion of a function into a power series depends on the nature of singularities of the function. For the purpose of this thesis, we are interested to analyze those functions, which have several types of singularities.

1.3.1 Singularities for single variable function

The rate of convergence of the sequence of partial sums depends crucially on the singularities of the function represented by the series. Several types of singularities may arise in physical (non-linear) problems. The dominating behavior of the function $f(x)$ represented by a series may be written as

$$f(x) \sim M\left(1 - \frac{x}{x_c}\right)^{\alpha} \quad \text{as } x \to x_c \tag{1.3.25}$$

where M is a constant and x_c is the critical point with the critical exponent α. If α is a negative integer then the singularity is a pole, otherwise if it is a nonnegative rational number then the singularity is a branch point. We can include the correction terms with the dominating part in (1.3.25) to estimate the degree of accuracy of the critical points. It may be

$$f(x) \sim M\left(1-\frac{x}{x_c}\right)^{\alpha}\left[1+M_1\left(1-\frac{x}{x_c}\right)^{\alpha_1}+M_2\left(1-\frac{x}{x_c}\right)^{\alpha_2}+\cdots\right] \qquad as\ x \to x_c \quad (1.3.26)$$

where $0<\alpha_1<\alpha_2<\Lambda$ and M_1, M_2, Λ are constants. $\alpha_i+\alpha \notin \mathbb{N}$ for some i, then the correction terms are called confluent. Sometimes the correction terms can be logarithmic. Such that $f(x) \sim M\left(1-\frac{x}{x_c}\right)^{\alpha}\left\{1+\ln\left|1-\frac{x}{x_c}\right|\right\} \quad as\ x \to x_c \quad (1.3.27)$

Sometimes the sign of the series coefficients may indicate the location of the singularity. If all terms are either positive or negative then the dominant singularity must be on the positive x-axis. If they alternate in sign then the dominant singularity is on the negative x-axis

1.3.2 Singularities for multivariable function

Several types of singularities may arise in physical problems that involve more than one independent variable. It is obvious that such functions might behave as

$$f(x,y) \sim \begin{cases} M\left((x,y)-(x_c,y_c)\right)^{\alpha} \\ M\ln\left(1+\dfrac{(x,y)}{(x_c,y_c)}\right) \end{cases} \qquad \text{as } (x,y)\to(x_c,y_c) \qquad (1.3.28)$$

near the critical points, where M is a constant, and (x_c, y_c) is the critical point with the critical exponent α. If α is negative integer then the singularity is a pole; otherwise if it is nonnegative rational number then it represents a branch point singularity.

Following is a basic theorem that relates the asymptotic behavior of the power series coefficients to the form of the dominant singularity.

Theorem 1.3.1 (Darboux's in the case of single singularity)

Let the function $f(x)$ be analytic in the closed disc $|x| \le |x_c|$, apart from a branch cut for a single algebraic singularity at $x = x_c$, so that

$$f(x) = \left(1 - \frac{x}{x_c}\right)^{\alpha} P(x) + Q(x), \quad \text{for } \alpha \notin \mathbb{N} \tag{1.3.29}$$

where $P(x)$ and $Q(x)$ are analytic in a disc that includes the disc $|x| \le |x_c|$. Then the coefficients of the power series (1.2.8) satisfy the asymptotic relation

$$c_i \sim \sum_{k=0}^{N} \frac{(-1)^{k+i} P^{(k)}(x_c) 1 (i - \alpha - k) x_c^{k-i}}{k! i! \Gamma(-\alpha - k)} \qquad \text{as } i \to \infty, \tag{1.3.30}$$

for any N and some constants $P^{(k)}(x)$ independent of i.

Here are some artificial examples with different types of singularities.

Example1.3.4 Single variable functions

I. Singularities that are poles:

$$f(x) = (1 - 2x)^{-1} + \sin(1 - x).$$

II. Algebraic singularities with the same exponent:

$$f(x) = 2\left(1 - \frac{x}{3}\right)^{-1/2} + 3\left(1 - \frac{x}{4}\right)^{-1/2} + 4\left(1 - \frac{x}{5}\right)^{-1/2} + 5\left(1 - \frac{x}{6}\right)^{-1/2}.$$

III. Algebraic singularities with different exponents:

$$f(x) = 2\left(1 - \frac{x}{3}\right)^{-1/2} + 3\left(1 - \frac{x}{4}\right)^{-1/3} + 4\left(1 - \frac{x}{5}\right)^{-1/4} + 5\left(1 - \frac{x}{6}\right)^{-1/5}.$$

IV. Logarithmic singularity:

$$f(x) = \ln(1 + x) + \sin(x).$$

V. Essential singularity:

$$f(x) = \exp\left[2\left(1 - \frac{x}{3}\right)^{-1/2}\right].$$

VI. Algebraic dominant singularity with a secondary logarithmic behaviour:

$$f(x) = \exp(x)\left(1 - \frac{x}{3}\right)^{-1/2} + \ln\left(1 - \frac{x}{4}\right).$$

VII. Cube root singularity:

$$f(x) = (1 - x)^{1/3} + \exp(x)$$

Example1.3.5 Multivariable functions related to the :

I. Singularities that are poles:

$$f(x, y) = (1 - 2x + y)^{-1} + (1 - x + 2y)^{-2}.$$

II. Algebraic singularities with the same exponent.

$$f(x, y) = 2\left(1 - \frac{x}{3} + y\right)^{-1/2} + 3\left(1 - \frac{x}{4}\right)^{-1/2} + 4\left(1 - \frac{x}{5}\right)^{-1/2} + 5\left(1 - \frac{x}{6}\right)^{-1/2}.$$

III. Algebraic singularities with different exponents:

$$f(x, y) = 2\left(1 - \frac{x}{3} + y\right)^{-1/2} + 3\left(1 - \frac{x}{4}\right)^{-1/3} + 4\left(1 - \frac{x}{5}\right)^{-1/4} + 5\left(1 - \frac{x}{6}\right)^{-1/5}.$$

IV. Logarithmic singularity:

$$f(x, y) = \ln(1 + x - y) + \sin(x).$$

V. Essential singularity:

$$f(x, y) = \exp\left[2\left(1 - \frac{x}{3} + y\right)^{-1/2}\right].$$

VI. Algebraic dominant singularity with a secondary logarithmic singularity:

$$f(x, y) = \exp(x)\left(1 - \frac{x}{3} + y\right)^{-1/2} + \ln\left(1 - \frac{x}{4} + y\right).$$

VII. Cube root singularity:

$$f(x, y) = (1 - xy)^{1/3} + \exp(x + y)$$

To analyze the singularity behavior, it is very important to know about the continued fractions.

1.4 Continued Fractions

Continued fraction has a long history. For historical survey one can see [5] and [29]. Continued fraction is very useful to analyse the dynamical systems, notably in connection with renormalization. Here we present the basic concepts of continued fractions.

Let x be a rational number, then the simple continued fraction of x is

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{+ \cfrac{1}{a_{N-1} + \cfrac{1}{r_N}}}}} \tag{1.4.31}$$

where, for $0 \le i < N$, $a_i = floor\left(\frac{1}{r_i}\right)$ and $floor\left(\frac{1}{r_i}\right)$ denotes the integral part of $\left(\frac{1}{r_i}\right)$.

In this expression the a_i are positive integers and r_N is called the Nth remainder.

Example1.4.6 Let $x = \frac{95}{43}$.

Then

$$\frac{95}{43} = 2 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}}$$

$$= [2, 4, 1, 3, 2]$$

For every rational number, eventually the remainder must be equal to 0. On the other hand, if x is irrational, then the remainder can never vanish and we can get the infinite continued fraction

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \tag{1.4.32}$$

$$=[a_0, a_1, a_2, \cdots]$$

Example1.4.7 Let $x=\sqrt{3}$. Since $1 < x < 2$ then 1 is the greatest integer less than $\sqrt{3}$.

$$\sqrt{3} = 1+\left(\sqrt{3}-1\right)$$

Thus $$\sqrt{3} = 1+\cfrac{1}{1+\cfrac{1}{2+\left(\sqrt{3}-1\right)}}$$

Hence

$$\sqrt{3} = [1,1,2,1,2,1,2,\cdots]$$
$$= [1,\overline{1,2}]$$

By neglecting the Nth remainder in (1.4.31), we obtain a rational approximation x_N of x

$$x_N = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_{N-1}}}} \qquad (1.4.33)$$

x_N is called the Nth convergent of the continued fraction (1.4.32).

A power series may be manipulated into a form of continued fraction. It is just another way of writing fractions. It has some interesting connections with the approximate methods. Continued fractions can be simplified by cutting after a finite number of iterations. The result of the terminated continued fraction will give a true fraction, but it will be an approximation to the power series.

Consider a function $f(x)$, which represents the power series

$$S(x) = \sum_{i=0}^{n} c_i x^i \qquad \text{as} \qquad x \to 0. \qquad (1.4.34)$$

Let us now see how, it can be expressed as a continued fraction

The Nth convergent of the series (1.4.34) is

$$S_N(x)=\sum_{i=0}^{N-1} c_i x^i \quad (1.4.35)$$

In order to convert (1.4.35) into continued fraction, assume that all the inverse that we need exist.

The continued fraction of (1.4.35) is

$$S_N(x)\sim c_0+\cfrac{c_1 x}{1+\cfrac{c_1^{(1)}x}{1+\cfrac{c_1^{(2)}x}{1+\ddots}}} \quad (1.4.36)$$

$$=\frac{c_0}{1+}\;\frac{c_1 x}{1+}\;\frac{c_1^{(1)}x}{1+}\;\frac{c_1^{(2)}x}{1+\cdots}$$

The convergent of (1.4.34) is rational function in the variable x

In general, we can obtain a rational approximant from (1.4.36) of the form

$$\frac{P_N(x)}{Q_N(x)}=\frac{b_0+b_1x+b_2x^2+\cdots+b_Nx^N}{d_0+d_1x+d_2x^2+\cdots+d_Nx^N} \quad (1.4.37)$$

which matches certain number of terms of the series (1.4.34)

In particular, the roots of the denominator $Q_N(x)$ give the singularity of the series (1.4.34). When the series (1.4.34) represents a rational function, the remainder of (1.4.37) must eventually reduce to a constant, and the process (1.4.36) terminates after a finite number of iterations. Otherwise, it never terminates and we obtain the infinite continued fraction.

Example1.4.8 Consider the function

$$f(x)=\frac{1+x}{1-3x+2x^2} \quad (1.4.38)$$

The series expansion for the function (1.4.38) is

$$S(x)=1+4x+10x^2+22x^3+46x^4+94x^5+O(x^6)$$

and the continued fraction is

$$f(x)=1+\cfrac{x}{\cfrac{1}{4}+\cfrac{x}{\cfrac{-8}{5}+\cfrac{x}{\cfrac{-25}{12}+\cfrac{5}{3}x}}}.$$

1.5 Overview of the work

This thesis is concerned with the singularity analysis of power series arising in the solution of nonlinear system. For over the last quarter century many powerful approximants have been introduced for the approximation of function by using its power series. Among them most of the methods are described for the power series involving single independent variable and a few are derived for the power series involves with two or several independent variables. Many researchers hitherto have found remarkably more accurate results by using several approximant methods The remainder of this thesis is as follows.

In Chapter 2, we have reviewed some well-known approximant methods for the series in powers of one or several independent variables with examples. All these approximant methods are members of the *Padé-Hermite* class. The methods for single independent variable have been discussed at some length Then in Chapter 3, we have derived a new approach to partial differential approximant for the series in powers of two independent variables using the concept of *Padé-Hermite* class. Finally in Chapter 4 we have summarized our work and give some ideas for future work

CHAPTER 2

EXISTING APPROXIMANT METHODS

2.1 Introduction

Approximant methods are the techniques for summing power series. A function is said to be approximant for a given series if its Taylor series expansion reproduces the first few terms of the series. The partial sum of a series is the simplest approximant, which is very good approximant, if the function has no singularities. When the series converges rapidly, such approximants can provide good approximations for the series. In practice, however, the presence of singularities prevents rapid convergence of the series. It is then necessary to seek an efficient approximant method.

The convergent in the continued fraction expansion of a power series are rational approximants. In fact, it is a particular *Padé* approximants that have the property that the numerator and denominator are of the same degree. In general, such approximants are more accurate than the partial sum of the power series. See [1] and [4] for details.

In this chapter we describe some well-known approximant methods for the power series that have several types of singularities. The purpose of this chapter is to describe these approximant methods for constructing other often-powerful approximant methods. The advantages of these approximant methods are that they can be used, not only to approximate the rate of convergence of power series, but also to compute the location of its singularities.

The structure of this chapter is as follows: In §2.2 we review some well-known approximant methods for the series of single independent variable with some examples. In §2.3 we also describe some well-known approximant methods for the series that have two or several independent variables with some examples. Finally, we conclude with some remarks in §2.4.

2.2 Single Variable Approximant Methods

In this section, we describe a wide class *Padé-Hermite* approximants along with some single independent variable approximant methods. All the single independent variable approximants in this thesis belong to the *Pade-Hermite* class.

2.2.1 *Padé-Hermite* approximants

In 1893, *Padé* and *Hermite* introduced *Padé-Hermite* class. This class is related to the simultaneous approximations of several series and there is some advantage in first describing the *Padé -Hermite* class from that point of view.

Let $d \in \mathbb{N}$ and the $(d+1)$ power series

$$S_{[0]}(x),\ S_{[1]}(x),\ \cdots,\ S_{[d]}(x) \text{ be given.}$$

One can constructs the $(d+1)$-tuple of polynomials

$$\left[P_{[0]}(x),\ P_{[1]}(x),\cdots,P_{[d]}(x)\right]$$

Such that

$$\deg P_{[0]}(x)+\deg P_{[1]}(x)+\cdots+\deg P_{[d]}(x)+d=\mathrm{N} \tag{2.2.1}$$

and $$\sum_{i=0}^{d} P_{[i]}(x)S_i(x)=O\left(x^{\mathrm{N}}\right) \qquad \text{as } x\to 0 \tag{2.2.2}$$

Here $S_{[0]}(x),\ S_{[1]}(x),\ \cdots,\ S_{[d]}(x)$ may be independent series or different form of a unique series. However, since in this work we are interested to approximate a unique series $S(x)$, we shall take powers or derivatives of the partial sum $S_N(x)$ for other series.

Now attention is given on the problem of finding polynomials $P_{[i]}(x)$ that satisfy the equations (2.2.1) and (2.2.2). The Polynomials are completely determined by their coefficients. So the total number of unknowns in the equation (2.2.2) is

$$\sum_{i=0}^{d} \deg P_{[i]}(x)+d+1=\mathrm{N}+1$$

If we expand the left-hand side of the equation (2.2.2) in powers of x, we see that the equation (2.2.2) is equivalent to equating the first N terms in the expansion to zero. This gives a system of N linear equations for the unknown coefficients of the *Padé-Hermite* polynomials. In order to obtain non zero solutions of that system of linear equations we must normalize by setting

$$P_{[i]}(0)=1 \qquad \text{for some} \quad 0 \le i \le d \tag{2.2.3}$$

The equation (2.2.3) then simply ensures that the coefficient matrix associated with the system is square. One way to construct the *Padé-Hermite* polynomials is to solve the system of linear equations by any standard method such as Gaussian elimination or Gauss-Jordan elimination.

2.2.2 *Padé* approximants

Padé approximant is a technique for summing power series that is widely used in applied mathematics [4]. *Padé* approximant can be described from the *Padé-Hermite* class in the following sense.

In the *Padé-Hermite* class, let

$$d=1, \quad S_0=-1, \quad S_1=S \tag{2.2.4}$$

and the polynomials $P_{[0]}$ *and* $P_{[1]}$ satisfy (2.2.1) and (2.2.2). One can define an approximant $S_N(x)$ of the series $S(x)$ by

$$P_{[1]}S_N - P_{[0]} = 0 \tag{2.2.5}$$

We call the rational relation $S_N(x)$ is a *Padé* approximant of the power series $S(x)$.The Nth convergent of the continued fraction expansion of the power series $S(x)$is itself analogous to *Padé* approximant. Indeed, the *Padé* approximants are a particular type of rational fraction of two polynomials so that it would tend to a finite limit as N tends to infinity. Hence the *Padé* approximants to a power series is a sequence of rational functions (a rational function is a ratio of two polynomials) of the form

$$\frac{P_{[0]}}{P_{[1]}}=\frac{\sum_{i=0}^{l} a_i x^i}{\sum_{i=0}^{m} b_i x^i} \tag{2.2.6}$$

Without loss of generality we choose $b_0 = 1$. Also we can calculate the remaining $(l+m+1)$ coefficients $a_0, a_1, \cdots a_l, b_1, b_2, \cdots b_m$, so that the first $(l+m+1)$ terms in the Taylor series expansion of $\frac{P_{[a]}}{P_{[r]}}$ matches the first $(l+m+1)$ terms of the power series $\sum_{i=0}^{\infty} c_i x^i$. Suppose that $\sum_{i=0}^{\infty} c_i x^i$ is a power series representation of the function $f(x)$, then $\frac{P_{[a]}}{P_{[r]}} \to f(x)$ as $l, m \to \infty$, even if $\sum_{i=0}^{\infty} c_i x^i$ is a divergent series. Since Padé approximants involved only algebraic operations, they are more convenient for computational purposes. In fact, the general *Padé* approximant can be expressed as

$$\sum_{j=0}^{l+m} c_j x^j \sum_{k=0}^{m} b_k x^k - \sum_{n=0}^{l} a_n x^n = O\left(x^{l+m+1}\right). \tag{2.2.7}$$

In order to evaluate the *Padé* approximants for a given series numerically, we have used symbolic computation language such as MAPLE. The Padé approximants have been used not only in tackling slowly convergent, divergent and asymptotic series but also to obtain singularity of a function from its series coefficients. The zeroes of the denominator $P_{[1]}(x)$ give the singular point such as pole of the function $f(x)$, if exist.

Example2.2.1 Consider the function

$$f(x) = (1-2x)^{-2} + \ln(1-x)$$

After using the normalization condition $b_0 = 1$, we get the following numerator and denominator for the *Padé* approximants of the function,

For $\deg P_{[0]}(x) = \deg P_{[1]}(x) = 2$

$$P_{[0]}(x) = 1 + \frac{124}{7485}x + \frac{11471}{7485}x^2$$

and $$P_{[1]}(x) = 1 - \frac{9107}{2495}x + \frac{46151}{14970}x^2$$

For $\deg P_{[0]}(x) = \deg P_{[1]}(x) = 3$

$$P_{[0]}(x) = 1 - \frac{7292173}{11804610}x + \frac{38077294}{29511525}x^2 - \frac{95080207}{88534575}x^3$$

and $$P_{[1]}(x) = 1 - \frac{16858581}{3934870}x + \frac{50822853}{9837175}x^2 - \frac{47513951}{39348700}x^3.$$

For $\deg P_{[0]}(x) = \deg P_{[1]}(x) = 4$

$$P_{[0]}(x) = 1 - \frac{6696433085}{5846639613}x + \frac{41892911919}{27284318194}x^2 - \frac{215602875185}{122779431873}x^3 + \frac{1087472177237}{2455588637460}x^4$$

and $$P_{[1]}(x) = 1 - \frac{9378037222}{1948879871}x + \frac{100217195343}{13642159097}x^2 - \frac{7091162434}{1948879871}x^3 + \frac{7564234217}{19488798710}x^4$$

The following table 2.1 shows the convergence of the singularity point by *Padé* approximant:

Table 2.1: Convergence of singularity by *Padé* approximant for the function in the example 2.2.1.

d	x_c
2	0.4304862724
3	0.4818596590
4	0.4962878690

2.2.3 Algebraic approximant

Algebraic approximants is a special type of *Padé-Hermite* approximants.

In the *Padé-Hermite* class, we take

$d \geq 1,\ S_0 = 1, S_1 = S, \cdots S_d = S^d$.

Consider a function $f(x)$ represented by the power series $S(x)$ and $S_N(x)$ is the partial sum of that series.

Using *Padé-Hermite* polynomials defined by (2.2.1) and (2.2.2) an algebraic approximant $S_N(x)$ of $S(x)$ can be defined as the solution of the equation

$$P_{[0]} + P_{[1]} S_N + \cdots + P_{[d]} S_N{}^d = 0 \qquad (2.2.8)$$

Since the equation (2.2.8) is a polynomial of $S_N(x)$ in degree *d*, the algebraic approximant $S_N(x)$ is in general a multivalued function with d branches. At first this may appear to be an undesirable feature of the method, in that case we have the problem of identifying the particular branch that approximates $S(x)$. On the other hand, the series $S(x)$ is the expansion of a particular type of function $f(x)$ that is itself multivalued. For algebraic approximants, one uses the partial sum $S_N(x)$ to construct the $(d+1)$ polynomials

$$P_{[0]}(x), P_{[1]}(x), \cdots, P_{[d]}(x) \qquad (2.2.9)$$

such that

$$\sum_{i=0}^{d} P_{[i]}(x) S_N{}^i(x) = O(x^N) \qquad (2.2.10)$$

and $\quad \sum_{i=0}^{d} \deg P_{[i]} + d = N$

The total number of unknowns in the equation (2.2.10) is

$$\sum_{i=0}^{k} \deg P_{[i]} + d + 1 = N + 1.$$

In order to determine the coefficients of the polynomials (2.2.10), without loss of generality one can set $P_{[0]}(0) = 1$ for normalization. The discriminant of the equation (2.2.8) gives singularity of the function.

Example2.2.2 Consider the function

$$f(x) = (1 - 2x)^{\frac{1}{2}}$$

and take $\deg P_{[0]}(x) = \cdots = \deg P_{[d]}(x) = 2$

For $d = 2$, after using the normalization condition $P_{[0]}(0) = 1$, we get the polynomials

$$P_{[0]}(x) = 2x - 1$$

$$P_{[1]}(x) = 0$$

and $$P_{[2]}(x) = 1.$$

Here the discriminant $D(x) = P_{[1]}^2 - 4P_{[0]}P_{[2]}$, which gives the singularity $x_c = \frac{1}{2}$ for the above mentioned function.

Example2.2.3 Consider the function

$$f(x) = (1-2x)^{\frac{1}{3}} + \sin x$$

and take $\deg P_{[0]}(x) = \cdots = \deg P_{[d]}(x) = 2$

For $d = 2$, after using the normalization condition $P_{[0]}(0) = 1$, we get the polynomials

$$P_{[0]}(x) = -1 - \frac{25204}{47295}x - \frac{155377}{141885}x^2$$

$$P_{[1]}(x) = -\frac{2648}{1051} + \frac{83168}{47295}x + \frac{237764}{141885}x^2$$

and $$P_{[2]}(x) = \frac{1597}{1051} - \frac{57964}{47295}x - \frac{45532}{141885}x^2,$$

which gives the singularity x_c =0.4084607608 for the above mentioned function. It will be close to the actual singularity if increase d as well as the degree of $P_{[d]}(x)$.

2.2.4 Differential approximants

Differential approximants is an important member of the *Padé-Hermite* class. It is obtained by taking

$$d \geq 2, \quad S_0 = 1, \ S_1 = S, \ S_2 = DS, \cdots S_d = D^{d-1}S \qquad (2.2.11)$$

Where D is the differential operator

$$D = \frac{d}{dx}$$

Once the *Padé–Hermite* polynomials have been found, a differential approximant $S_N(x)$ of the series $S(x)$ can then be defined as the solution of the differential equation

$$P_{[0]} + P_{[1]}S_N + P_{[2]}DS_N + \cdots + P_{[d]}D^{d-1}S_N = 0 \tag{2.2.12}$$

Equation (2.2.12) is non-homogeneous linear differential equation of order *(d-1)* with polynomial coefficients. There are *(d-1)* linearly independent solutions, but only one of them has the same first few Taylor coefficients as the given series $S(x)$. When $d > 2$, the usual method for solving such an equation is to construct a series solution.

Differential approximants are used chiefly for series analysis. They are powerful tools for locating the singularities of a series and for identifying their nature [20]. It is not necessary to solve the differential equation (2.2.12) in order to find the singularities of $f(x)$. In practice, one usually finds that its only singularities are located at the zeros of the leading polynomials $P_{[d]}(x)$. Hence, the zeroes of $P_{[d]}(x)$ may provide approximations of the singularities of the function $f(x)$.

A less general form of the method of differential approximants was developed by Guttmann and Joyce [20] and Hunter and Baker [22] for series analysis. However, these studies considered only low-order differential approximants, where d is not related to N. When the function has countably infinite branches, then the low-order differential approximants may not be useful. It is to note that Sergeyev and Goodson [33] for algebraic approximants suggests that $d \propto \sqrt{N}$. Tourigny and Drazin [36] and Khan [25] had already implemented this idea for algebraic approximants and High-order Differential Approximants respectively. Khan [25] established the relation

$$\mathrm{N} = \frac{1}{2}d(d+3) \tag{2.2.13}$$

between the numbers d and N for the High-order Differential Approximant of $S_N(x)$ and considered

$$\deg P_{[k]} = k \quad . \tag{2.2.14}$$

From (2.2.14), he deduced that there are $\sum_{k=0}^{d}(k+1)=\frac{1}{2}(d+1)(d+2)$ unknowns by the definition of the *Padé -Hermite* class. In order to determine those unknowns, he used the N linear equations those satisfy the equation

$$P_{[0]}(x)+\sum_{i=1}^{d}P_{[i]}(x)D^{i-1}S_N(x)=O(x^N) \tag{2.2.15}$$

The normalizing condition

$$P_{[0]}(0)=1 \tag{2.2.16}$$

ensures that there are as many equations as unknowns. One of the roots, say x_c, of the coefficient polynomial of the highest derivative

i.e. $P_{[d]}(x_c)=0$

gives an approximation of the dominant singularity x_c of the function $f(x)$.

Example2.2.4 Consider the function

$$f(x)=(1-2x)^{-2}+\ln(1-x)$$

and take $\deg P_{[k]}=k$

For $d=2$, $N=5$ the leading polynomial

$$P_{[2]}(x)=\frac{15}{11}-\frac{108}{11}x+\frac{144}{11}x^2$$

gives the singularity $x_c=0.5019319169$ approximately for the function.

2.3 Multivariable Approximant Methods

In this section we have reviewed some well-known approximant methods for the series in powers of two or more independent variables, which have been developed using the concept of *Padé–Hermite* class.

2.3.1 Multivariable *Padé* approximants

Many attempts have been made to generalize the concept of *Padé* approximants for multivariable functions. One can see [11], [12] and [40] for details. Here we have introduced the multivariable *Padé* approximants on the basis of *Padé–Hermite* class. Given a function $f(x,y)$ in the form of its Taylor series expansion at a certain point in the real plane is (for simplicity we use the Taylor series at the origin)

$$S(x,y)=\sum_{i=}^{\infty}\sum_{j=0}^{\infty}c_{ij}\,x^i\,y^j \qquad (x,y)\to(0,0) \tag{2.3.17}$$

The Nth partial sum of the series

$$S_N(x,y)=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}c_{ij}\,x^i\,y^j \tag{2.3.18}$$

For the formation of two variables rational approximants $S_N(x,y)$, we consider the polynomials.

$$P_{[0]}(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{n}p_{ij}x^i\,y^j \tag{2.3.19}$$

$$P_{[1]}(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{n}q_{ij}x^i\,y^j \tag{2.3.20}$$

such that

$$S_N(x,y)P_{[1]}(x,y)-P_{[0]}(x,y)=\sum_{i,j}e_{ij}\,x^i\,y^j \qquad i,j\in\mathbb{N} \tag{2.3.21}$$

where $\mathrm{N}=\deg P_{[0]}(x,y)+\deg P_{[1]}(x,y)$ (2.3.22)

and $e_{ij}=0$ for $i+j<\mathrm{N}$ (2.3.23)

The coefficients of the numerator $P_{[0]}(x,y)$ and the denominator $P_{[1]}(x,y)$ are determined from (2.3.23) by using the normalization condition $q_{00}=1$. The condition (2.3.23) then ensures that there are as many equations as unknowns. One can solve these equations by using symbolic programming language such as MAPLE. The zeros of the denominator $P_{[1]}(x,y)$ give the singularity of the function $f(x,y)$.

Example2.3.5 Consider the function

$$f(x,y)=(1-x+y)^{-2}$$

and take $\deg P_{[0]}(x,y)=\deg P_{[1]}(x,y)=2$

After using the normalization condition $q_{00}=1$, we get the numerator

$$P_{[0]}(x,y)=1$$

and the denominator

$$P_{[1]}(x,y)=1-2x+2y+x^2+y^2-2xy$$

The zeros of the denominator $P_{[1]}(x,y)$ give the singularity $(x_c,y_c)=(1,0)$ of the function $f(x,y)$.

2.3.2 Multivariable Algebraic approximants

Multivariable algebraic approximants have been developed by Khan [25]. Consider a function $f(x,y)$ of two independent variables, represented by the power series

$$S(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}\,x^i\,y^j \qquad (x,y)\to(0,0) \tag{2.3.24}$$

The Nth partial sum of the series (2.3.24) is

$$S_N(x,y)=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}c_{ij}x^iy^j \tag{2.3.25}$$

By using that partial sum, we are trying to construct $(d+1)$ polynomials

$$P_{[0]},P_{[1]},\cdots,P_{[d]} \tag{2.3.26}$$

in x and y such that

$$P_{[0]}+P_{[1]}S_N+\cdots+P_{[d]}S_N{}^d=\sum_{i,j}e_{ij}\,x^i\,y^j \qquad i,j\in\mathbb{N} \tag{2.3.27}$$

and $\quad e_{ij}=0 \quad \text{for } i+j<\mathrm{N}$ (2.3.28)

The equation (2.3.28) gives a total

$$N_e = \frac{1}{2}N(N+1) \tag{2.3.29}$$

equations to determine the unknown coefficients of the polynomials $P_{[i]}(x,y)$. One of these coefficients is specified by the normalization condition

$$\sum_{i=1}^{d} P_{[i]}(0,0)S_N{}^{i} = 1 \tag{2.3.30}$$

Thus, there remains

$$N_u = \sum_{i=0}^{d}\left[\frac{1}{2}\left(\deg P_{[i]}+1\right)\left(\deg P_{[i]}+2\right)\right]-1$$

unknown coefficients that must be found by use of the N_e linear equations .The equation (2.3.28) can be express in the matrix form as

$$AX = B \tag{2.3.31}$$

where A is a matrix of order $N_e \times N_u$ and the nonzero vector B of dimension N_e on the right hand side comes if we impose the condition (2.3.30).

This system will be solvable if

$$N_e \leq N_u \tag{2.3.32}$$

However, we must make it clear that, even with this condition, there is no guarantee that a solution will exist. In practical cases, a solution exists but it is not unique. By using algebraic- programming language such as MAPLE, it is straightforward to obtain the general solution of the system. But the general solution contains some free variables. It is, therefore, important to choose the value of the free variables. In this thesis we choose value of the free variables to zero or one. The discriminant of the equation

$$P_{[0]} + P_{[1]}S_N + \cdots + P_{[d]}S_N{}^{d} = 0 \tag{2.3.33}$$

will give singularities of the function $f(x,y)$.

Example2.3.6 Consider the function

$$f(x,y) = (1-2x+y)^{\frac{1}{2}}$$

and take $\deg P_{[0]}(x,y) = \cdots = \deg P_{[d]}(x,y) = 2$

For $d = 2$, the number of unknowns is $N_u = 17$. Therefore, we can take $N = 5$; so that the number of equations is $N_e = 15$. The coefficient polynomials are

$$P_{[0]} = 1+(-2+8c_1+48c_4-c_2+8c_5)x+(1-c_3-8c_4-4c_1)y+(-20c_1-108c_4-14c_5+2c_2)x$$
$$+(-c_3-5c_1-13c_4)y^2+(-c_2+7c_5+2c_3+20c_1+80c_4)xy$$

$$P_{[1]} = 16c_1+64c_4+(-24c_1-112c_4-8c_5)x+(12c_1+40c_4)y+(4c_1+32c_4+8c_5)x^2$$
$$+c_1y^2+(-16c_4-4c_5-4c_1)xy$$

$$P_{[2]} = -1-16c_1-64c_4+c_2x+c_3y+(-2c_5-4c_4)x^2+c_4y^2+c_5xy$$

The polynomials contains five free variables c_1, c_2, c_3, c_4 and c_5.

The particular polynomials can be obtained by setting 1 to the free variables

$$P_{[0]} = 1+61x-12y-140x^2-19y^2+108xy$$
$$P_{[1]} = 80-144x+52y+44x^2+y^2-24xy$$
$$P_{[2]} = -81+x+y-6x^2+y^2+xy$$

which gives the singularity (x_c, y_c) =(0.9934023773,1.0000000000) for the mentioned function.

In figure 2.1, we see how the critical line matches with the approximate critical values by using multivariable algebraic approximants.

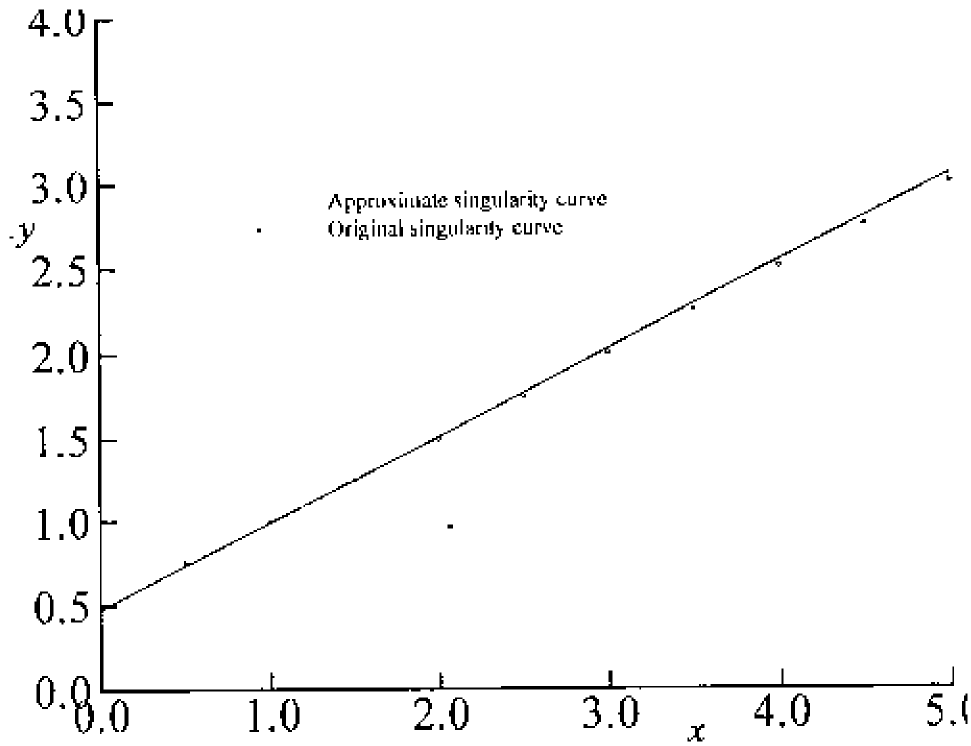

Figure 2.1. Approximate location of singularities of the example 2.3.6 obtained by using Multivariable Algebraic Approximants. The curve marked with square is the approximate singularity curve and the curve marked with diamond is the original singularity curve.

2.3.3 Fisher's approximants

Fisher [15] has suggested a new approximant method, derived from a first order homogeneous linear partial differential equation. For this he considered the function $f(x,y)$, which represent the power series

$$S(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{i,j}\,x^i y^j \qquad (x,y)\rightarrow(0,0) \tag{2.3.34}$$

The partial sum of this series is

$$S_N(x,y)=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}c_{i,j}\,x^i y^j \tag{2.3.35}$$

where its derivatives $\frac{\partial S_N(x,y)}{\partial x}$ and $\frac{\partial S_N(x,y)}{\partial y}$ exist. He considered the polynomials

$P_{[0,0]}(x,y)$, $P_{[1,0]}(x,y)$ *and* $P_{[0,1]}(x,y)$ which satisfy

$$P_{[0,0]}(x,y)S_N(x,y) = P_{[1,0]}(x,y)\frac{\partial S_N(x,y)}{\partial x} + P_{[0,1]}(x,y)\frac{\partial S_N(x,y)}{\partial y} + \text{ high-order terms.}$$

(2.3.36)

Using the partial sum $S_N(x,y)$, one can construct the polynomials

$P_{[0,0]}$, $P_{[1,0]}$ *and* $P_{[0,1]}$ in x and y such that

$$P_{[1,0]}(x,y)\frac{\partial S_N}{\partial x} + P_{[0,1]}(x,y)\frac{\partial S_N}{\partial y} - P_{[0,0]}(x,y)S_N = \sum_{i,j} e_{i,j}x^i y^j \qquad i,j \in \mathbb{N} \tag{2.3.37}$$

where $N = \deg P_{[0,0]} + \deg P_{[1,0]} + \deg P_{[0,1]}$

and $e_{i,j} = 0$ for $i+j < N$ (2.3.38)

The equation (2.3.38) defines a homogeneous linear system of equations for the coefficients of the polynomials $P_{[0,0]}(x,y)$, $P_{[1,0]}(x,y)$, *and* $P_{[0,1]}(x,y)$. One can evaluate the coefficients of these polynomials using any standard method such as Gaussian elimination or Gauss-Jordan elimination. Using these polynomials, the Fisher approximant of $f(x,y)$ is defined to be a solution of the partial differential equation

$$P_{[0,0]}(x,y)S_N(x,y) = P_{[1,0]}(x,y)\frac{\partial S_N(x,y)}{\partial x} + P_{[0,1]}(x,y)\frac{\partial S_N(x,y)}{\partial y} \tag{2.3.39}$$

The polynomials $P_{[1,0]}(x,y)$ *and* $P_{[0,1]}(x,y)$ will give singularities of the function $f(x,y)$.

Example2.3.7 Consider the function

$$f(x,y) = \ln(1-2x+y)$$

and take $\deg P_{[0,0]} = \deg P_{[1,0]} = \deg P_{[0,1]} = 1$

After using normalization condition, we get the following polynomials

$$P_{[1,0]} = 1 + c_1 x + c_2 y$$

and $$P_{[0,1]} = 2 + 2c_1 x + 2c_2 y$$

The polynomials contain two free variables c_1 *and* c_2 .

The particular polynomials are obtained by taking 1 to the free variables.

$$P_{[1,0]} = 1 + x + y$$

and $$P_{[0,1]} = 2 + 2x + 2y$$

which gives the singularity (x_c, y_c)=(0,-1) for the above mentioned function.

2.4 Conclusion

In this chapter, we have reviewed different approximant methods for the series of power one or several independent variables. There are some drawbacks in using to approximate singularities. Their success depends on the availability of a sufficient number of coefficients of the series $S(x)$. There are many possible sources of different errors. For example, the series coefficients may be known only approximately. Or accuracy may be lost in computing the *Padé–Hermite* polynomials and in solving the system of linear equations for the approximant. Particularly, in multivariable case the solution is not unique. There exist some free variables. It is, therefore, important to choose the value of the free variables. By using algebraic programming language such as MAPLE, one can also control the effect of round-off errors.

When the existing approximant methods are unable to give satisfactory answer, one can look for new, better approximant methods. Particularly, for a function which contains complicated term the existing methods fail to give satisfactory answer. The error analysis of approximant methods is certainly not easy. So the development of new methods, for which there is much scope, must be guided by numerical experimentation. In Chapter 3, we develop a new approximant method that is the differential analogue of the *Padé–Hermite* class. We also compare efficiency of the method numerically with the methods such as Multivariable *Padé* approximants (MPA), Multivariable Algebraic approximants (MAA), Fisher's approximants (FA) and High-order Differential approximants (HODA).

CHAPTER 3

A NEW APPROXIMANT METHOD

3.1 Introduction

Numerical approximation by power series expansion of a function is frequently used in many areas of science. Hitherto, we have studied several series in powers of a single independent variable. But many problems in applied mathematics may be related to the series in powers of two or more independent variables. So it is desirable to derive a new method to approximate such multivariable series efficiently. Generalizations of the Padé method to power series of several independent variables have been proposed by Cuyt [11], [12] and Guillaume [19]. Cuyt [11] studied multivariable *Padé* approximants by using abstract polynomials and she showed that the classical *Padé* approximant is a special case of the multivariate theory and many interesting properties of classical *Padé* approximants remain valid such as covariance properties of the *Padé*-table. Cuyt [12] compared and discussed many of the results to make it clear that simple properties or requirements, such as the uniqueness of the *Padé* approximant and consequently its consistency can play a crucial role in the development of the multivariate theory. Guillaume [19] introduced a new class of multivariable *Padé* approximants called nested *Padé* method, when dealing with two independent variables x and y his approach consists in applying the *Padé* approximation with respect to y to the coefficients of the *Padé* approximation with respect to x.

An efficient approximant method for a power series with two or several independent variables is a new approach to partial differential approximant, which we call the High-order Partial Differential Approximant (HPDA). We present in this chapter the new approximant method in some sense a "differential analogue" of the *Padé-Hermite* class. Even though the method can be described notationally as dth order partial differential equations with n independent variables, but in this chapter we describe the method for two independent variables. Our aim is to construct polynomials in x and y that can be used as coefficients in a partial differential equation for the approximant method.

The chapter is organized as follows: In § 3.2 we describe some basic ideas in order to clarify our new method and then we give a precise description of our method in §3.3. Some simple applications are given in §3.4. An application to a problem of fluid dynamics is discussed in §3.5. Finally, the results and discussion with conclusion are given in §3.6 and §3.7 respectively.

3.2 Basic ideas

Although our new approach, in this chapter involve only two independent variables (for simplicity), we see that it is in fact notationally simpler to describe the method in the context of an arbitrary number, say n, of variables. In order to describe the method for series in powers of n variables, we introduce the following notation. We shall use $\underline{x}$ and $\underline{k}$ to denote points in n-dimensional coordinates system. Thus, for instance, $\underline{x} \in \mathbb{C}^n$ and $\underline{k} \in \mathbb{N}^n$ denote the points $\underline{x} = (x_1, x_2, \cdots, x_n)$ and $\underline{k} = (k_1, k_2, \cdots, k_n)$ respectively, with $x_j \in \mathbb{C}$ and $k_j \in \mathbb{N}$ for $1 \le j \le n$. Then we shall write

$$\left|\underline{k}\right| = k_1 + k_2 + \cdots + k_n .$$

Where $\mathbb{C}^n$ and $\mathbb{N}^n$ denote the Euclidean n-space and $\mathbb{N}$ denotes the set of natural number (including zero). Further, since $\underline{k} \in \mathbb{N}^n$, we can consider

$$D^{\underline{k}} = \frac{\partial^{|\underline{k}|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \qquad \text{and } x^{\underline{k}} = x_1^{k_1} \cdot x_2^{k_2} \cdots x_n^{k_n}$$

Now consider $S\left(\underline{x}\right) = \sum_{\underline{i}} c_{\underline{i}} \underline{x}^{\underline{i}}$, $\underline{i} \in \mathbb{N}^n$ be a series in powers of the n independent variables $x_1, x_2, \cdots, x_n$.

The partial sum of length N

$$S_N\left(\underline{x}\right) = \sum_{|\underline{i}|<N} c_{\underline{i}} \underline{x}^{\underline{i}}$$

is obtained from the series by excluding all the terms of order N and higher.

A dth order partial differential approximant for $S(\underline{x})$ can be constructed as follows: First we seek polynomials $P_{[\underline{0}]}$ and $P_{[\underline{k}]}$ in $\underline{x}$ such that

$$P_{[\underline{0}]} + \sum_{|\underline{k}| \le d} P_{[\underline{k}]} D^{\underline{k}} S(\underline{x}) = \sum_{\underline{j}} e_{\underline{j}} \underline{x}^{\underline{j}}, \quad \underline{j} \in \mathbb{N}^n \tag{3.2.1}$$

$$\text{where } e_{\underline{j}} = 0 \quad \text{if } |\underline{j}| < N \tag{3.2.2}$$

Then, if such polynomials can be found, we define a partial differential approximant $S_N(\underline{x})$ as a solution of the dth order partial differential equation

$$P_{[\underline{0}]} + \sum_{|\underline{k}| \le d} P_{[\underline{k}]} D^{\underline{k}} S_N(\underline{x}) = 0 \tag{3.2.3}$$

When $n = 1$, we recover the differential approximant of §2.2.4. For $n = 2$, $d=1$ and $P_{[\underline{0}]} = 0$, the equation (3.2.3) reduces to the first order homogeneous linear partial differential equation

$$P_{[0,0]} S_N + P_{[1,0]} \frac{\partial S_N}{\partial x} + P_{[0,1]} \frac{\partial S_N}{\partial y} = 0 \tag{3.2.4}$$

considered by Fisher.

For $n = 2$, $d \geq 2$ and $P_{[\underline{0}]} = \text{constant}$, the equation (3.2.1) becomes higher order non-homogeneous linear partial differential equation. In this chapter we have considered new form of the partial differential equation (3.2.3), where the mixed derivative terms have been ignored.

3.3 Description of the method

Consider the function $f(x,y)$ of two independent variables, represented by its power series

$$S(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{i,j} x^i y^j \qquad (x,y) \to (0,0) \tag{3.3.5}$$

and the partial sum

$$S_N(x,y) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} c_{i,j} x^i y^j \tag{3.3.6}$$

By using that partial sum, we try to construct the following $(2d+1)$ polynomials

$$P_{[0,0]}, P_{[1,0]}, P_{[0,1]} \cdots, P_{[d,0]}, P_{[0,d]} \tag{3.3.7}$$

in x and y such that

$$P_{[0,0]} S_N + P_{[1,0]} \frac{\partial S_N}{\partial x} + P_{[0,1]} \frac{\partial S_N}{\partial y} + \cdots + P_{[d,0]} \frac{\partial^d S_N}{\partial x^d} + P_{[0,d]} \frac{\partial^d S_N}{\partial y^d} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} e_{i,j} x^i y^j \tag{3.3.8}$$

where $$e_{i,j} = 0 \qquad \text{for } i + j < N = 3d - 1 \tag{3.3.9}$$

By equating the coefficients of the variables and their powers from (3.3.9), one can obtain a total of

$$N_e = \frac{3d(3d-1)}{2} \tag{3.3.10}$$

equations to determine the unknown coefficients of the polynomials in (3.3.7). We impose the normalization condition

$$P_{[0,0]} = 1, \ or \ P_{[d,0]} = 1, \ or \ \ P_{[0,d]} = 1 \quad \text{for } (x,y) = (0,0) \ . \tag{3.3.11}$$

Thus the remaining unknowns

$$N_u = \frac{1}{3} d\left(d^2 + 6d + 11\right) \tag{3.3.12}$$

must be found by the use of N_e equations.

It would be helpful to write the system of linear equations $e_{i,j} = 0$ into the matrix form with the $N_u \times 1$ unknown matrix $\underline{x}$.

Thus the non-homogeneous system of N_e linear equations with N_u unknowns can be written in matrix form as

$$A\underline{x} = \underline{b} \tag{3.3.13}$$

where A is $N_e \times N_u$ matrix and $\underline{b}$ is the non-zero column matrix of order $N_e \times 1$. This system will be solvable if

$$N_e \le N_u . \tag{3.3.14}$$

However, the system may be consistent or inconsistent. If the system is consistent, then the system can be solved by converting the augmented matrix $\left[A\middle|\underline{b}\right]$ to row echelon or reduced row echelon form by using the Gaussian elimination or Gauss-Jordan elimination. It is to note that, there will exist some free variables. Naturally the values of the free variables in the multivariable approximant methods can be chosen at random. For all the calculations reported in the remainder of this chapter, we have in fact set all the free variables to either zero or one. There is no particular reason to pick up these particular numbers. We might for instance seek a solution such that the polynomials in (3.3.7) have as few high-order terms as possible. Our experience suggests that the accuracy of the method does not depend critically on the particular choice made.

Once the polynomials (3.3.7) have been found, it is more practical to find the singular points by solving either of the polynomials coefficients of the highest derivatives

$$P_{[d,0]}(x,y)=0 \text{ or } P_{[0,d]}(x,y)=0 \text{ or both simultaneously.} \quad (3.3.15)$$

Note that Programming MAPLE codes are in appendix.

As an example, let us consider the following function

$$f(x,y)=(1-2x+y)^{\frac{1}{2}}+\ln(1+x-2y)+\sin xy \quad (3.3.16)$$

Here the actual singularities for the dominating part of $f(x,y)$ lying on the line $1-2x+y=0$. However, the High-order Partial Differential Approximants approach the actual singularities quite smoothly as shown in Table 3.1 and Figure 3.1.

Table3.1: Estimates of x_c ($y_c = 0$) *by* the High-order Partial Differential Approximants for the function (3.3.16)

d	N	N_c	N_u	x_c
2	5	15	18	0.6201633594
3	8	36	38	0.4879182224
4	11	66	68	0.4962775196
5	14	105	110	0.4815743527
6	17	153	166	0.5033611125

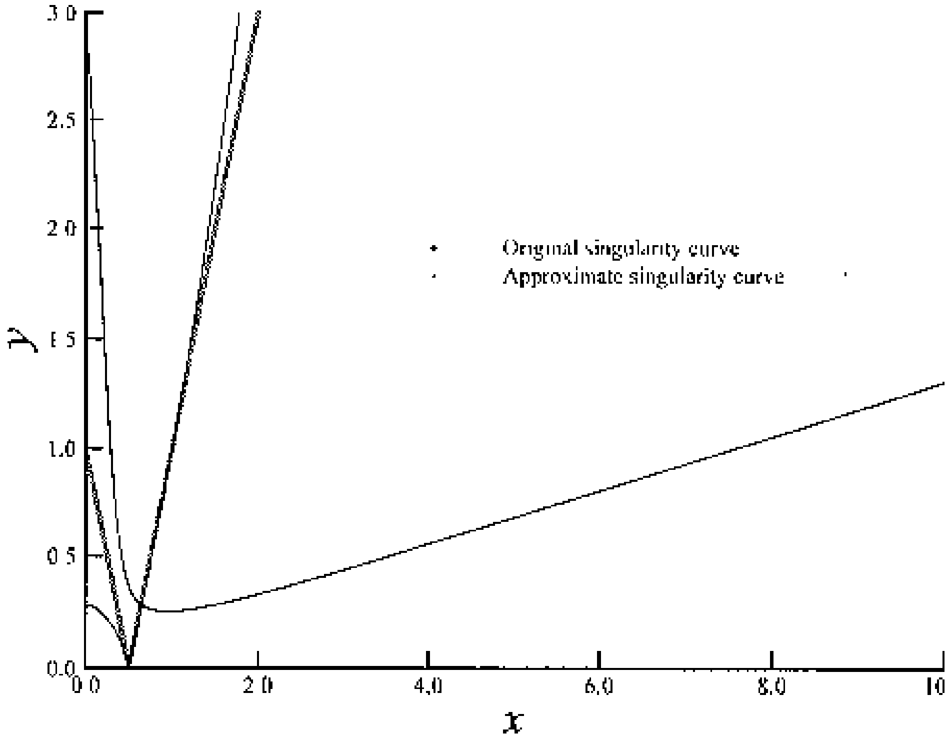

Figure 3.1. Approximate location of singularities of the function (3.3.16) by using High-order Partial Differential Approximants. The curve marked with square is the approximate singularity curve and the curve marked with diamond is the original singularity curve.

3.4 Some simple applications

The asymptotic error analysis for the new approximate method is very complicated. Instead, parallel to other existing methods, we apply it to some examples for which we can gain some insight into the effectiveness of the method.

Example3.4.1 We consider some test functions that have several types of singularities without secondary behavior.

I. Singularities are poles.

$$f(x,y)=(1-2x+y)^{-2}$$

II. Cube root branch point singularities

$$f(x,y)=(1-2x+y)^{\frac{1}{3}}$$

III. Logarithmic singularities

$$f(x,y)=\ln(1-2x+y)$$

IV. Essential singularities

$$f(x,y)=e^{\frac{1}{1-2x+y}}$$

The results of approximating the singularity in each case by various methods of series analysis are shown in table 3.2. where we considered fixed $y_c = -1$. Here the values of N is rather small but approximately same for all cases. It is interesting to note that the High-order Partial Differential Approximant produces sometimes the exact results.

Table 3.2:Estimates of x_c ($y_c = -1$) by various approximant methods for the functions in the example 3.4.1

Functions	HPDA	HODA	MPA	FA	MAA
I	exact	0.06666667	exact	exact	exact
II	exact	0.02994012	0.30892924	exact	-0.02122462
III	exact	0.01673453	0.25000000	exact	_
IV	exact	0.02133610	-0.16031689	exact	0.11470842

Example3.4.2 We consider here some test functions that have several types singularities with several types of secondary behavior.

I. Dominant singularities are poles and the remainder term has logarithmic singularities

$$f(x,y)=(1-2x+y)^{-2}+\ln(1-x+2y)+\sin xy$$

II. Dominant singularities are cubic and the remainder term has logarithmic singularities

$$f(x,y)=(1-2x+y)^{\frac{1}{3}}+\ln(1-x+2y)+e^{x+y}$$

III. Dominant singularities are logarithmic and the remainder term has no singularity

$$f(x,y)=\ln(1-2x+y)+\sin xy+e^{x+y}$$

IV. Dominant singularities are essential and the remainder term has logarithmic singularity

$$f(x,y) = e^{\frac{1}{1-2x+y}} + \sin(x+y) + \ln(1-x+2y)$$

Table 3.3: Estimates of x_c ($y_c = 0$) by various approximant methods for the functions in the example 3.4.2

Functions	HPDA	HODA	MPA	FA	MAA
I	0.49273522	0.50000000	0.17786075	0.202764977	0.10811102
II	0.60000000	0.50610591	0.67289720	1.618272003	0.35331233
III	0.50515411	0.50000000	0.63245553	1.683060109	0.80127134
IV	0.56512441	0.49481271	0.37500564	0.096310587	0.26242321

Comparable results of approximating the dominating x_c with $y_c = 0$ in each case by various methods of series analysis are shown in Table 3.3. For relatively same sizes of N, it is interesting to note how badly the Fisher's approximants and Multivariable algebraic approximants compares with the others. Most of the cases High-order Partial Differential Approximant produces very good results.

3.5 Application to symmetric Jeffery Hamel flows

We consider here the well-known problem named after Jeffery (1915) and Hamel (1916) for the steady two-dimensional flow of an incompressible viscous fluid from a source or sink at the intersection between two rigid plane walls. Jeffery-Hamel solutions are particular similarity solutions of the Navier-Stokes equations and are found by solving an ordinary differential equation. Fraenkel [16] described all these solutions in terms of elliptic functions. Sobey and Drazin [35] studied their bifurcations theoretically and experimentally. They showed that the symmetric solution, which is stable for low Reynolds numbers, undergoes pitchfork and Hopf bifurcations as the Reynolds number increases.

Let (r,θ) be polar coordinates, with $r = 0$ as the sink or source. Let α be the semi-angle and let the domain of the flow be $-|\alpha| < \theta < |\alpha|$, u and v be the velocity components in the radial and tangential directions respectively, ν be the kinematic viscosity and p be the pressure. The flow behavior can be expressed in terms of Navier-Stokes equations [16,35].

Further, we assume a symmetric radial flow, so that $v = 0$. Then the volumetric flow rate through the channel is

$$Q = \int_{-\alpha}^{\alpha} u\, r\, d\theta \tag{3.5.17}$$

If we require $Q \geq 0$ then for $\alpha < 0$ the flow is converging to a sink at $r = 0$.

Let $\psi = \psi(r,\theta)$ be the stream function. Then

$$\frac{\partial\psi}{\partial\theta} = u\, r, \qquad \frac{\partial\psi}{\partial r} = 0.$$

A Reynolds number Re for the flow can be defined by

$$Re = \frac{Q}{\nu}$$

Expressing in terms of the dimensionless variables

$$y = \frac{\theta}{\alpha} \text{ and } G(y; Re, \alpha) = \frac{\psi(\theta)}{Q},$$

the corresponding Navier-Stokes equations [16, 35] can be reduced to the ordinary differential equation

$$G'''' + 4\alpha^2 G'' + 2\alpha\, Re\, G'\, G'' = 0 \tag{3.5.18}$$

with the boundary conditions

$$G = \pm 1, \quad G' = 0 \text{ at } y = \pm 1 \tag{3.5.19}$$

The coefficients of the series for G in powers of Re and α can be computed by using MAPLE. The first few coefficients are

$$G(y;\alpha, Re) = \frac{1}{2} y\left(3 - y^2\right) - \frac{3}{280} y\left(y^2 - 5\right)(y-1)^2 (y+1)^2\, Re\,\alpha + \frac{1}{10} y (y-1)^2 (y+1)^2 \alpha^2$$
$$+ \frac{1}{1400} y\left(5y^4 - 22y^2 + 33\right)(y-1)^2 (y+1)^2 \alpha^3\, Re + \cdots \tag{3.5.20}$$

To investigate this Jeffery-Hamel flow by approximant methods, we take

$$(x, y) = (\alpha, Re)$$

and, the series

$$S(\alpha, Re) = G'(0, \alpha, Re)$$

We have used two approximate methods to compare the results with Fraenkel's asymptotic behavior. The High-order Differential Approximation and the High-Order Partial Differential Approximation are applied to the expression $S(\alpha, Re)$ in order to determine the critical behavior of α_c and Re_c. Some estimates are shown in Table 3.4. Since the flow really depends on the two parameters α and Re, we apply the High-order Partial Differential Approximants to the series $S(\alpha, Re)$ to calculate the critical α_c and Re_c relationship by considering the highest derivative polynomial coefficients of High-order Partial Differential Approximants. From Figure 3.2 we see that the result by our method agrees very well with the Fraenkel's asymptotic result, namely

$$Re_c \sim \frac{5.461}{\alpha_c} \quad \text{as } \alpha_c \to 0 \tag{3.5.21}$$

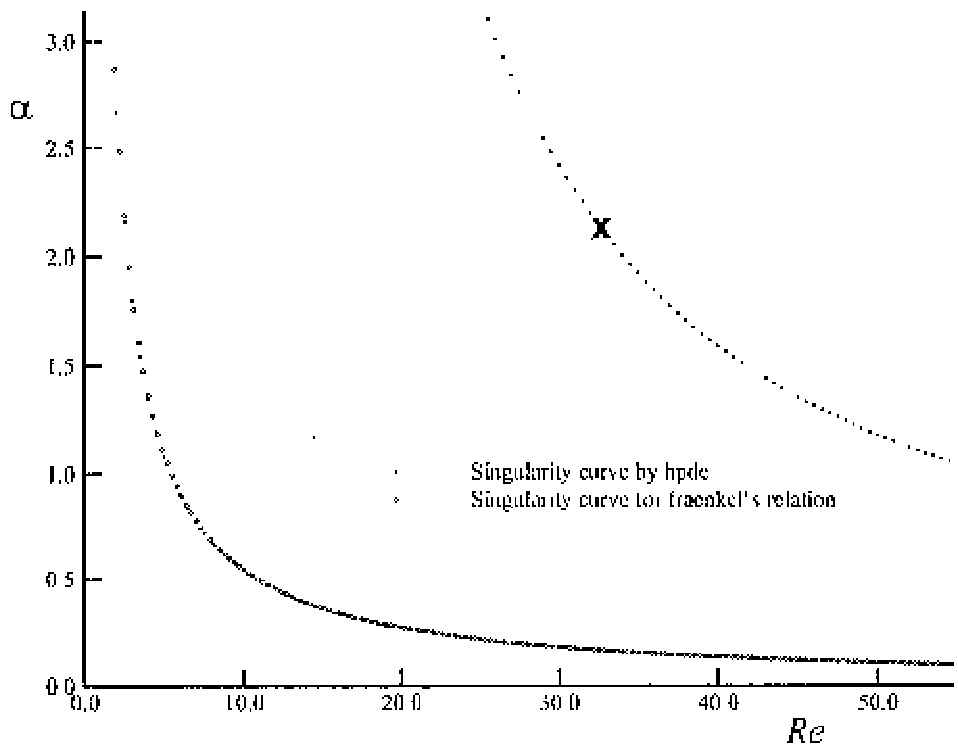

Figure3.2: The critical Re-α relationship (curve marked with square) for a symmetric flow by using High-order Partial Differential Approximants with $d = 6$. The curve marked with diamond is the Fraenkel's asymptotic result. The other curve (x) is spurious.

Table 3.4:Estimates of Re_c by the High-order Partial Differential Approximants (HPDA) and the High-order Differential Approximant (HODA) for the Jeffery-Hamel problem ($\alpha_c = 0.1$)

HPDA			HODA	
d	N	Re_c	N	Re_c
3	8	65.420143407049822707	9	54.580567148199421138
4	11	54.101790119138738101	14	54.580567148199421138
5	14	54.575430270369712868	20	54.581088584976777508
6	17	54.619749160234696701	27	54.581086860760345143

3.6 Results and discussion

For analysis, we make use of the series in powers of two independent variables. By analyzing the series, we have calculated the value of the singularities into the real field. Table 3.1 shows the results obtained by using the High-order partial differential approximant method. From these, we deduce that the method is well for the higher values of d, when the function contain complicated remainder terms. Also in the Figure 3.1 we see that sometimes the approximate singularity curve obtained by using the High-order partial differential approximant method coincides with the original singularity curve. The Table 3.2 and 3.3 show the results obtained by using the High-order partial differential approximant method and the other methods such as High-order Differential approximants (HODA), Multivariable *Padé* approximant (MPA) Fisher's approximants (FA) and Multivariable Algebraic approximants (MAA). The accuracy of the results obtained by using our method is very satisfactory in many cases. Figure 3.2 and Table 3.4 show singularity curve and singular points respectively for symmetric Jeffery-Hamel flow. The curve obtained by our new method matches very closely with that of Fraenkel's asymptotic result.

3.7 Conclusion

In this chapter, we have proposed a new form of partial differential approximants to series in powers of two independent variables, which we name as High-order Partial Differential Approximants. This High-order Partial Differential Approximant is such that the order of the partial differential equation increases with N. From first order partial differential equation with two independent variables one can reproduces the Fisher's approximants. The method can be extended to three or more independent variables. One of the highest derivatives polynomial coefficients is a powerful tool to reveal the critical relationship between the independent variables.

We have applied this method and the multivariable approximant methods to a number of interesting test functions that contain different types of singularities. Our method gives better results than those obtained by other methods.

We have applied the new method to series where the form of the singularity is not known with certainty, such as the problem of Jeffery-Hamel flow. We have compared the results with the High-order Differential Approximant in tabular form and with the Fraenkel's asymptotic result graphically. We have found that the new method is very efficient. However, we have not yet developed a theory that would explain its strengths and limitations. So we may rely on intelligent numerical investigations.

CHAPTER 4

CONCLUSIONS

4.1 Discussion

In this thesis, we have studied several techniques for summation of series and introduced a new approach to partial differential approximants. The aim of this work was to approximate singularity behavior of nonlinear problems. The solution of nonlinear problems may be expanded into series in powers of one or more independent variables. Numerical approximation by power series expansion is frequently used, but the question of efficiency of such approximations are crucial

In Chapter1 we have presented elementary bifurcation theory, power series of one or several independent variables, singularities of one and multivariable power series and continued fraction of a function involved single independent variable.

In Chapter2, we have reviewed *Padé-Hermite* class as well as the several approximant methods. These methods would be helpful in describing a new method.

In Chapter3, we have developed a new method, which we call High-order Partial Differential Approximants The novel feature of this method is that the order of the partial differential equation increases infinitely with the number of series coefficients used. In many cases this method is more powerful to approximate singularity behavior than the other existing methods discussed in Chapter 2. For example, our new method gives more accurate results for the functions that contain complicated terms than those obtained by other methods. We also applied our new method to symmetric Jeffery-Hamel flows. We have shown this method match very nicely with the Fraenkel's asymptotic result and very much competitive with the High-order Differential Approximant.

4.2 Future work

In this thesis we have developed High-order Partial Differential Approximant where we considered the general form of homogeneous linear partial differential equation excluding the terms related to mixed derivative. Further research can be carried out on this field by taking the general form of partial differential approximants (3.2.3) as well as the asymptotic behavior of the error of proposed method.

Appendix

```
#program for the d-th order Homogeneous PDE.
#Series must be of v=v(lambda,mu)=Order (M+1).
d:=d;
N:=3*d-1;                          # Highest order of lambda and mu.
Ne:=add(r+1,r=0..N-1);
M:=(1/3)*d*((d^2)+6*d+11);         # Total number of unknown (Nu=M)
c:=array(1..M);
x:=array(1..Ne);
###########################################################
p:=0;
l:=0;
for m from 1 to d do
          u[p]:=diff(v,lambda$(m)):
#print(p,u[p]);
          p:=p+1;
          u[p]:=diff(v,mu$(m)):
          l:=l+1;
#print(p,u[p]);
          p:=p+1;
od:
###########################################################
p:=0;
l:=1;
for m from 1 to d do
for lm from 1 to 2 do
          A[p]:=0;
 for n from 0 to m do
 for k from 0 to n do
          A[p]:= A[p] + c[l] *lambda^(m-n)*mu^(n-k) :
          l:=l+1;
 od:
 od;
#print(p,A[p]);
          p:=p+1:
od:
od:
###########################################################
f:=v+add(A[i]*u[i],i=0..2*d-1);
#print(f);
f:=expand(f);
###########################################################
i:='i';
for i from 0 to N-1 do
          q[i]:=coeff(f,lambda,i);
```

```
#print(q[i]);
od:
################################################################
i:='i':
j:='j':
for i from 0 to N-1 do
   for j from 0 to (N-1-i) do
         eq[i,j]:= coeff(q[i],mu,j):
#print([i,j],eq[i,j]);
   od:
od:
################################################################
k:=1:
i:='i':
j:='j':
for i from 0 to N-1 do
   for j from 0 to (N-1-i) do
         x[k]:= eq[i,j]:
         #x[k]:= eq[i,j]+eq[j,i]:
#print(x[k]).
#print(k,x[k]);
          k:=k+1:
   od.
od:
aa:=array(1..Ne,1..M).
i:='i':
j:='j':
for i from 1 to Ne do
   for j from 1 to M do
         aa[i,j]:= coeff(x[i],c[j]):
    od:
od:
################################################################
i:='i':
j:='j':
b:=array(1..Ne);
for i from 1 to Ne do
         b[i]:=-(x[i]-add(aa[i,j]*c[j],j=1..M)):
od:
################################################################
with(linalg):
c:=linsolve(aa,b)
g:=f:
#print(g);
#Optional Check the method
for i from 1 to 500 do
```

```
        t[i] =1 .
od:
print(AA[2*d-2]=A[2*d-2]);
print(AA[2*d-1]=A[2*d-1]),
###################################################################
Ff:=F(lambda,mu):
p:=0:
l:=0.
for m from 1 to d do
        u[p]:=diff(Ff,lambda$(m)):
#print(p,u[p]);
        p:=p+1:
        u[p]:=diff(Ff,mu$(m)):
        l:=l+1:
#print(p,u[p]);
        p:=p+1:
od:
###################################################################
###################################################################
hpde:=Ff+add(A[i]*u[i],i=0..2*d-1)
#print(hpde);
######################
crtpt:=fnormal(fsolve ({A[2*d-2],A[2*d-1]},{lambda,mu},complex)),
```

REFERENCES

[1] Baker, G. A. Jr and Graves-Morris, P., Padé Approximants, Second Edition, University Press, 402-414, 1996.

[2] Beardon, A. F. On the location of poles of *Pade'* approximants, J. Math. Anal. Appl. 21. 469-74, 1968

[3] Brak, B. and Guttmann, A. J. Algebraic approximants a new method of series analysis. J Phys A Math. Gen . 23: L1331-L1337,1990.

[4] Bender, C. and Orszag, S. A.: Advanced Mathematical Methods for Scientist and Engineers. McGraw-Hill, New York, 1978

[5] Brezinski, C.: History of Continued Fraction and Padé Approximants. Springer, Berlin, 1990.

[6] Blanch, G.: The numerical evaluation of continued fractions, SIAM Rev. 6, 383-421,1964.

[7] Cheney, E. W.: Introduction to approximation Theory, McGraw-Hill, New York, 1966.

[8] Chisholm, J. S. R.: Rational approximants defined from double power series. Math. Compute., 27: 841-848, 1973

[9] Common, A. K. and Graves-Morris, P.: Some properties of Chisholm approximants. J. Inst. Math. Appl., 13.229-232, 1974.

[10] Cuyt, A. Driver, K. Tan, J. and Verdonk, B.: Exploring multivariate Padé approximants for multiple hyper-geometric series. Adv. Compute. Math , 10:29-49, 1999.

[11] Cuyt, A. M. Annie.: Multivarite Padé-Approximants. Journal of Mathematical Analysis and Applications 96, 283-293, 1983.

[12] Cuyt. Annie . How well can the concept of Pade approximant be generalized to the multivariate case? Journal of Computational and Applied Mathematics, 105, 25-50, 1999

[13] Drazin, P. G.. Non-linear Systems. Cambridge University Press, 1992.

[14] Fisher, M.E and Styer, D F · Partial Differential Approximants for Multivariable power series I, Definition and Faithfulness, Proc. Roy Soc. A 384, 259-287,1982.

[15] Fisher, M. E. and Kerr, R. M.· Partial differential approximants for multi-critical singularities. Phys. Rev. Lett., 39, 667-70, 1997.

[16] Fraenkel, L E. Laminar flow in symmetrical channels with slightly curved walls I: on the Jeffery-Hamel solutions for the flow between plane walls, Proc. Roy. Soc. London A 267, 119-138, 1962.

[17] Guttmann, A. J.· Asymptotic analysis of power series expansions. Academic Press, 1989.

[18] Gratenhaus, S. McCullough, W. S.. Higher order corrections for the quadratic Ising lattice susceptibility. Phys. Rev B, 38: 11689-11703, 1988.

[19] Guillaume, P. Nested multivariate Padé approximants. J. Comp. Appl. Math , 82: 149-158, 1997.

[20] Guttmann A. J. and Joyce G. S.: On a new method of series analysis in lattice statistics. J Phys. A Gen Phys., 5: L81-L84, 1972.

[21] Hairer, E. and Wanner G. Analysis by its History. Springer, New York, 1995.

[22] Hunter, D. L. and Baker G. A.· Methods of series analysis III: Integral approximant methods. Phys. Rev. B, 19: 3808-3821, 1979.

[23] Hunter, C. and Guerrieri, B.: Deducing the properties of singularities of functions from their Taylor series coefficients SIAM J. Appl Math., 39:248-263, 1980.

[24] Khan, M. A. H : Singularity analysis by summing power series. Ph.D. Thesis, University of Bristol, 2001.

[25] Khan M. A. H : High-order differential approximants, Journal of Computational and Applied Mathematics 149, 457-468, 2002.

[26] Khan, M A H., Drazin, P. G and Tourigny, Y : The summation of series in several variables and its application in fluid dynamics, Fluid Dynamics Research, 33,191-205, 2003.

[27] Khan, M. A H., Tourigny, Y and Drazin, P G : A case study of methods of series summation· Kelvin-Helmholtz instability of finite amplitude. Journal of computational Physics 187, 212-229, 2003.

[28] Khovanskii, A. N. The Application of Continued Fractions and their Generalizations to Problems in Approximation Theory. P. Noordhoff N. V., Groningen, 1963.

[29] Lorentzen. L., and Waadeland, H.: Continued Fractions with Applications, Nort-Holland, Amsterdam, 1992.

[30] McCabe, J.H.: A continued fraction expansion, with a truncation estimate, for Dawson's integral, Math. Comp 28, 811-16, 1974.

[31] Neyfeh A. H : Perturbation Methods. John Wiley & sons, Inc., 2000.

[32] Sergeyev, A V.: A recursive algorithm for Padé-Hermite approximations. U.S.S.R. Compute Math. Math. Phys., 26.17-22, 1986.

[33] Sergeyev A V and Goodson, D. Z : Summation of asymptotic expansions of multiple valued functions using algebraic approximants: Application to anharmonic oscillators. J. Phys. A: Math. Gen., 31:4301-4317, 1998.

[34] Styer, D. F. Partial Differential Approximants for Multivariable power series III; Enumeration of Invariance Properties, Proc. Roy. Soc. Lond. A 390, 321-39, 1983.

[35] Sobey, I.J., Drazin, P.G.. Bifurcation of two dimensional channel flows. J. Fluid Mech. 171, 263-287,1986.

[36] Tourigny, Y. and Drazin, P. G.. The asymptotic behavior of algebraic approximant, Proc. Roy. Soc. London A 456,1117-1137,2000.

[37] Tourigny, Y. and Drazin, P. G. The dynamics of Padé approximation, Nonlinearity, 15(3), 787-806, 2002.

[38] Tourigny, Y. and Drazin, P G : Numerical study of bifurcation by analytic continution of a function defined by a power series. SIAM J. Appl. Math , 56: 1-18,1996.

[39] Van Dyke, M. Perturbation Methods Fluid Mechanics Parabolic Press , 2nd edition 1975.

[40] Watson, P. J S. Two variable rational approximants-a new method , J. Phys. A 7, L 167-70, 1974.