Similarity Solutions for Unsteady Laminar Boundary Layer Flow Around a Vertical Heated Curvilinear Surface

A dissertation submitted in partial fulfillment of the requirements for the award of the degree

of
Master of Philosophy
in Mathematics

by

Md. Yeakub Ali
Roll No. 040209010F, Registration No. 0402410
Session: April-2002
Department of Mathematics
Bangladesh University of Engineering and Technology
Dhaka-1000

Supervised
by
Dr. Md. Zakerullah
Professor
Department of Mathematics, BUET, Dhaka

Department of Mathematics
Bangladesh University of Engineering and Technology
Dhaka-1000
April, 2005

The Thesis Entitled

Similarity Solutions for Unsteady Laminar Boundary Layer Flow Around a Vertical Heated Curvilinear Surface

Submitted by

MD.YEAKUB ALI

Registration No. 0402410, Roll No. 040209010F, Session: April-2002, a full time student of M. Phil (Mathematics) has been accepted as satisfactory in partial fulfillment for the degree of

Master of Philosophy in Mathematics

02/04/2005

Board of Examiners

1. Dr. Md. Zakerullah
   Professor
   Department of Mathematics
   BUET, Dhaka-1000

   Chairman
   (Supervisor)

2. **Head**
   Department of Mathematics
   BUET, Dhaka-1000

   Member
   (Ex-officio)

3. **Dr. Md. Abdul Maleque**
   Associate Professor
   Department of Mathematics
   BUET, Dhaka-1000.

   Member

4. **Dr. Gazi Md. Khalil**
   Professor
   Department of Naval Architecture and Marine Engineering
   BUET, Dhaka-1000.

   Member
   (External)

Dedications

This work is dedicated to

My parents

Contents

	Page No.
Declaration	v
Acknowledgements	vi
Abstract	vii
Nomenclature	viii
CHAPTER-1	
Introduction	1
CHAPTER-2	
Basic equations	7
Governing boundary layer equations	16
CHAPTER-3	
Transformations leading to similarity solution	18
CHAPTER-4	
Study of some possible similarity cases	28
CHAPETER-5	
Figures and Tables	63
Results and Discussion	84
CHAPETER-6	
Conclusion	86
References	87

Candidate's Declaration

None of the materials contained in this thesis is/will be submitted in support of any other degree or diploma at any other university or institution other than publications.

2nd April, 2005 (Md. Yeakub Ali)

Acknowledgement

The author expresses his indebtedness and deep sense of gratitude to his supervisor Dr. Md. Zakerullah, Professor of Mathematics, Bangladesh University of Engineering and Technology (BUET), Dhaka for his continuous guidance, valuable suggestions, constant encouragement and help all along during the period of this research work.

The author wishes to express his gratitude to Dr. Md. Abdul Maleque, Associate Professor of Mathematics, Bangladesh University of Engineering and Technology (BUET), Dhaka for his kind help in solving the equations numerically.

Thanks are due to the administration, Chittagong University of Engineering and Technology, Chittagong for granting him deputation and leave without pay for the period of his M.Phil study.

The author would like to thanks Chairman, University Grants commission of Bangladesh, Dhaka for offering him the UGC Fellowship.

The author would like to offer sincere thanks to his friends and colleagues specially all the teachers and staff of the department of mathematics, BUET, Dhaka for their all kinds of co-operation.

Finally, it will be an injustice if the author does not express his indebtedness, love and thanks to his wife Khandaker Shamema Akter and affection to his only daughter Zarin Tasnim.

(Md. Yeakub Ali)

Abstract

Similarity solution plays a vital role for the reduction of variables (dependents or independents) when the questions of partial differential equations arise in dealing with the boundary value problems. The present study deals with the possible similarity solutions of unsteady laminar boundary layer free convection flow around a vertical curvilinear surface. The free parameter method is introduced. Then the pertinent boundary layer partial differential equations under a suitable transformation are reduced to a set of ordinary differential equations. These simultaneous transformed ordinary differential equations with proper boundary conditions are solved numerically using Nachtsheim-Swigert iteration technique and Runge-Kutta shooting method. The results are compared with all other relevant works for special situations. The non-dimensional skin-friction factors $(=\tau_{w1}(0), \tau_{w2}(0))$ and heat transfer co-efficient $(=q_w(0))$ are displayed and shown graphically for some values of the controlling parameters by using the Software FORTRAN 77 and TECPLOT.

Nomenclature

a, b, c, d, R, k_1	constant
C_p	specific heat at constant pressure
F,S	dimensionless scaled stream function
f, s	dimensionless velocity
g	acceleration due to gravity
g_X	acceleration due to gravity along X–direction.
g_Y	acceleration due to gravity along Y–direction.
h_1, h_2, h_3	scale factor for curvilinear surface.
$Grxyt$,	modified Grashof number.
k	the coefficient of thermal diffusivity.
L_1,L_2	characteristic length.
m	temperature power/ exponent parameter.
P	pressure
R_F	Reynolds number
Pr	Prandtl number
Fr	Froude number
q_w	heat flux
T	temperature of the fluid.
T_∞	temperature of ambient fluid.
T_w	surface temperature.
u, v, w	velocity components in the boundary layer.
U_F, V_F	characteristic velocity generated by buoyancy effects.
x, y	co-ordinate along the edges of surface
z	co-ordinate normal to surface.

Greek letters

α, β	constant
β_T	the coefficient of volumetric expansion.
δ	boundary layer thickness
δ_T	thermal boundary layer thickness
θ	dimensionless temperature function
ψ, τ	mass flow components (stream function)
Φ	dissipation function
ϕ	similarity variable
ν	the kinemetic coefficient of viscosity.
ρ	the density of the ambient fluid
μ	coefficient of viscosity
κ	the coefficient of thermal diffusivity
τ_w	non dimensional skin friction
ξ, η, ζ	scaled co-ordinate defined in equations
γ	the square root of the boundary layer thickness

Chapter-1

Introduction

Free convection heat transfer occurs whenever a body is placed in a fluid at a higher or a lower temperature than that of the body. As a result of the temperature difference, the flows between the fluid and the body causes a change in the density of the fluid layers in the vicinity of the surface. The difference in density leads to downward flow of the heavier fluid and upward flow of the lighter one. If the motion of the fluid is caused solely by differences in density resulting from temperature gradients, without the aid of a pump or a fan, the associated heat transfer mechanism is called *natural* or *free convection*. The density difference gives rise to buoyancy effects due to which the flow is generated. A heated body cooling in ambient air generates such a flow in the region surrounding it. Similarly buoyant flow arising from heat rejection to the atmosphere and to other ambient media, circulations arising in heated rooms in the atmosphere and in bodies of water, causes thermal stratification of the medium. Many other such heat transfer process, in our natural environment as well as in many technological applications are included in the area of natural convection.

The concept of 'similarity' initially introduced by **Blasius** (1908) has become a useful tool now-a-days. On the basis of similarity transformations and finally the reduction of the set of partial differential equations to a set of ordinary differential equations have now reached stage of any great extent. A partial differential equation (PDE) with more than two independent variables becomes necessary to formulate the true picture of the happenings and reasoning of the problem. It is often difficult and even impossible to find the solution of PDE with the usual classical method. So applied mathematicians and engineers devote themselves to develop the ways and means for their solutions with simplifying assumptions. Similarity solution is one of the means, where the reduction of number of independent variables into one being done successfully. A vast literature of similarity solution has appeared in the arena of fluid mechanics, heat transfer, mass transfer etc. Different types of perturbation techniques are followed to solve the non-linear PDEs following primarily on local similarity solution.

The theoretical analysis and experiments carried out by **Schmidt** and **Bechmann** (1930) of the free convection boundary layer flow of air subject to the gravitational force about an isothermal, vertical flat plate constitute one of the earliest comprehensive studies of natural convection flow. **Eckert** and **Soehngen** (1948) verified and extended the experimental work of Schmidt. **Schuh** (1948) obtained the numerical solutions by computing velocity and temperature distributions for several Prandtl numbers. **Ostrach** (1953) studied aspects of natural convection heat transfer. He also analyzed laminar free convection flow and heat transfer about a flat plate parallel to the direction of the generating body force. **Yang** (1960) studied the unsteady laminar boundary layer equations for free convection on vertical plates and cylinder to establish necessary and sufficient conditions under which similarity solutions were possible. On the basis of these conditions, all possible cases were derived, for two-dimensional unsteady boundary layer flow. **Braun** et al. (1961) investigated free-convection similarity flows about two-dimensional and axisymetric bodies with closed lower ends. He found that families of bodies had similar velocity and temperature profiles along their entire extents. He also computed growth of the boundary-layer thickness and velocity along the surface as well as the heat transfer for a wide range of Prandtl numbers. **Stewart** (1971) derived boundary-layer solutions for free convection in laminar three-dimensional system by a temperature-dependent or composition-dependent density. He determined effects of rapid mass transfer for centrifugal forces and generalized the results of **Acrivos** (1960) for power-law fluids

Soundalgekar (1972) analyzed viscous dissipation effects on unsteady free convection flow past an infinite, vertical porous plate with constant suction. He derived the approximate solutions of the governing coupled, nonlinear equations for velocity and temperature field. He exhibited on graphs, the fluctuating parts of the velocity, the transient velocity profiles, the transient temperature profiles, the numerical values of amplitude and phase of the skin friction and heat transfer rate. **Johnson** and **Cheng** (1978) examined the necessary and sufficient condition under which similarity solutions exist for free convection boundary layers adjacent to flat plates in porous media. The solutions obtained in his work were more general than those appearing in the previous studies. Laminar free convection from vertical surfaces had been studied extensively by **Sparrow** and **Gregg** (1958) when the temperature of the surface was uniform and numerical solution were displayed also for Prandtl numbers in the range 0.1 to 100. **Suwono** (1980) applied the series of several variables to the solutions of the boundary-layer equations of free convection in laminar three dimensional system. He verified the numerical computation of the solution for the case of free convection over an inclined circular cylinder. The temperature profiles calculated from the first five terms of the series were compared with the experimental data. **Merkin** (1985) studied the similarity solutions for free convection on a vertical plate when the (non-dimensional) plate temperature is x^{λ} and when the (non-dimensional) surface heat flux was $-x^{\mu}$. He obtained solutions valid for $\lambda \geq 1$ and $\mu \geq 1$. He also discussed in both cases that the solutions became singular as $\lambda \to \lambda_0$ and as $\mu \to -1$ and the natures of these singularities. **Jafapur** and **Yovannovich** (1992) analyzed laminar free convection heat transfer from isothermal spheres by a analytical method. **Pop** and **Takhar** (1993) investigated the free convective flow over a non-isothermal two-dimensional body of arbitrary geometric configuration. It was shown that there were a certain family of body shape geometries and corresponding wall temperature distributions, which permited similarity solutions. He discussed in detail the effects of geometric shape parameter and Prandtl number on the velocity and temperature fields as well as on the heat transfer coefficient. Conjugate free convection on a vertical surface had been discussed in some detail by **Merkin** and **Pop** (1996).

Chaudhary et al. (1995) studied in detail the similarity solutions for free convection boundary layer flow over a permeable wall in fluid saturated porous medium. It was seen that the system depends on the two parameters m (the power law exponent) and γ (the dimensionless surface mass transfer rate). He also observed that the range of existence of solution depends on m and, for fluid, with drawl (γ <0) on γ as well, with a solution being possible for $m>m_0$ where $m_0 = -\frac{1}{2}$ for $\gamma \geq 0$ and for γ<0, m_0 decreasing monotonically from $-\frac{1}{2}$ at $\gamma = 0$ to -1 as $|\gamma| \to \infty$. **Jia** and **Gogose** (1996) studied numerically, steady state natural convection over a sphere. He obtained heat transfer and drag coefficients for a wide range of Grashof numbers ($10^1 \leq Gr \leq 10^8$) for Prandlt numbers 0.72 and 7.0. A plume with a mushroomshaped front forms above the sphere whose length and thickness decrease with the increasing value of *Gr*. At high *Gr* ($Gr \geq 10^7$ and Pr = 0.72), flow separation and an associated recirculation vortex exist in the wake of the sphere. The vortex size was found to increased with the increasing value of *Gr*. The local Nusselt number along the sphere surface first decreased, reached a minimum, and then increased steeply at the near of the sphere.

Ganapathy (1997) studied time dependent free convection motion and heat transfer in an infinite porous medium induced by a heated sphere. **Slaouti** et al.(1998) studied the unsteady free convection flow in the stagnation-point region of a three-dimensional body . He considered the case where there was an initial steady state that was perturbed by a step-change in the wall temperature. It was seen that the temperature and surface heat transfer were changed in a small interval of time. The surface heat transfer parameter increased with the increase of Prandtl number while the surface skin friction parameters decreased with the increase of Prandtl number. **Jayaraj** et al. (1999) discussed elaborately the analysis of thermophoresis in natural convection flow with variable fluid properties over a cold vertical plate. The effect of thermophoretic coefficient on wall concentration was also studied by him.

Hellums and **Churchill** (1962) presented a numerical solution of the coupled time dependent boundary layer equations governing transient natural convection flow over a semi-infinite vertical plate in air. **Goldstein** and **Brigges** (1964) and **Nanbu** (1971) studied the same problem analytically. **Elliot** (1970) analysed the problem of unsteady free convection boundary layer flow over two-dimensional and axisymmetric bodies for a step input in the surface temperature. **Williams** et al. (1987) studied the unsteady free convection flow over a vertical flat plate under the assumption of variations of the wall temperature with time and distance. They found possible semi-similar solutions for a variety of classes of wall temperature distributions. **Sattar** and **Alam** (1994) investigated the unsteady free convection flow of a viscous, incompressible and electrically conducting fluid past a moving infinite vertical porous plate taking into account the thermal diffusion effect. The unsteadiness in the flow field was introduced by time-dependent velocity of the moving plate. **Kumari** et al. (1996) studied the transient free convection flow over a continuous moving vertical sheet in an ambient fluid. The unsteadiness in the flow field was caused by the time-dependent velocity of the sheet. The constant temperature and the constant heat flux conditions were considered. **Kumari** and **Nath** (1984) considered the unsteady free convection flow in the stagnation-point region of a heated porous three-dimensional body where the unsteadiness in the flow field was caused by a time-dependent wall temperature. The semi-similar equation governing the flow were solved numerically.

Rees and **Pop** (1995) investigated the effects of large-scale surface non-uniformities on the boundary layer flow induced by a constant heat flux, vertically aligned, semi-infinite surface embedded in a porous medium. The analysis had been restricted to values of x that take O(1) values as $Ra \to \infty$. In this range of values of x, the boundary layer thickness is O($Ra^{-\frac{1}{3}}$), which was much smaller than the O(1) length scale associated with the waves of the surface. Three-dimensional, laminar, incompressible boundary layer similarity solutions were studied by **Hansen** and **Ohio** (1958). He found similarity requirements of the three-dimensional, laminar, boundary layer equations along with a general method of analysis. The problem of unsteady laminar mixed convection flow and heat transfer between two corotating disks with wall effects including both wall conduction and wall heat capacity were investigated numerically by **Monyan** and **Tzonglee** (1997). In his work, both the thermal boundary conditions of uniform heat flux (UHF) and uniform wall temperature (UWT) were considered. The Boussinesq approximation were used to characterize the centrifugal-buoyancy effects. He noticed the wall effects on the characteristics of fluid flow and thermal performance. The predicted results reveal that wall effects played a vital role in the unsteady mixed convection heat transfer, especially for the early transient period. Additionally, in the situation of buoyancy-opposing flow (Gr_{Ω}>0), the centrifugal buoyancy induced by the rotation had retarding effect on the skin friction coefficient and heat transfer rate. **Zakerullah** and **Maleque** (1998) studied theoretically three dimensional combined laminar boundary layer flows over inclined vertical orthogonal curvilinear surfaces. They described the detailed analyses of similarity requirements for an incompressible boundary layer fluid in order to reduce the governing partial differential equation into a set of ordinary differential equations. Different possible cases were exhibited in tabular form for ΔT-variations in addition to those of exterior velocity components tabulated by **Hansen** and **Ohio** (1958).

The present studies concerns with the systematic analyses in reducing the governing partial differential equations for unsteady free convective laminar incompressible flow over the vertical curvilinear surface into a set of ordinary differential equation. Finally similarity requirements is exhibited for ΔT, h_1, h_2, U_F and V_F variations. Numerical results are presented to predict flow characteristics for the different numerical values of the controlling parameters involved in the similarity transformation. Results are, therefore, compared with known results in literature.

Chapter-2

Basic Equations

Considering the flow direction along the ξ axis and η-axis and be defined in the surface over which the boundary layer is flowing. For simplicity $h_3(\xi,\eta)=1$ has been set such that ζ represents actual distance measured normal to the surface. The body force is taken as the gravitational force $\bar{g}(g_\xi(\xi,\eta),\, g_\eta(\xi,\eta),0)$ Thus the basic unsteady continuity, momentum and energy equations for a viscous and heat conducting fluid with variable fluid properties subject to the body force in curvilinear co-ordinates are,

continuity equation

$$\frac{D\rho}{Dt}+\rho\left\{\frac{\partial}{\partial\xi}(h_2u)+\frac{\partial}{\partial\eta}(h_1v)+\frac{\partial}{\partial\zeta}(h_1h_2w)\right\}=0 \qquad (2.1)$$

u-momentum equation

$$\rho\left[\frac{Du}{Dt}+\frac{uv}{h_1h_2}\frac{\partial h_1}{\partial\eta}+\frac{wu}{h_1}\frac{\partial h_1}{\partial\zeta}-\frac{v^2}{h_1h_2}\frac{\partial h_2}{\partial\xi}\right]=\frac{1}{h_1}\rho g_\xi-\frac{1}{h_1}\frac{\partial p}{\partial\xi}+\mu\nabla^2u \qquad (2.2)$$

v-momentum equation

$$\rho\left[\frac{Dv}{Dt}+\frac{uv}{h_1h_2}\frac{\partial h_2}{\partial\xi}+\frac{vw}{h_2}\frac{\partial h_2}{\partial\zeta}-\frac{u^2}{h_1h_2}\frac{\partial h_1}{\partial\eta}\right]=\frac{1}{h_2}\rho g_\eta-\frac{1}{h_2}\frac{\partial p}{\partial\eta}+\mu\nabla^2v \qquad (2.3)$$

w-momentum equation

$$\rho\left[\frac{Dw}{Dt}-\frac{u^2}{h_1}\frac{\partial h_1}{\partial\zeta}-\frac{v^2}{h_2}\frac{\partial h_2}{\partial\zeta}\right]=-\frac{\partial p}{\partial\zeta}+\mu\nabla^2w \qquad (2.4)$$

and **energy equation**

$$\rho C_p\frac{DT}{Dt}=k\nabla^2T+\mu\Phi \qquad (2.5)$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{h_1}\frac{\partial}{\partial \xi} + \frac{v}{h_2}\frac{\partial}{\partial \eta} + w\frac{\partial}{\partial \zeta}$ (2.6)

$$\nabla^2 = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \xi}\left(\frac{h_2}{h_1}\frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{h_1}{h_2}\frac{\partial}{\partial \eta}\right) + \frac{\partial}{\partial \zeta}\left(h_1 h_2 \frac{\partial}{\partial \zeta}\right)\right] \tag{2.7}$$

and Φ is the dissipation function.

The Boussinesq approximation

The study of unsteady free convective laminar boundary layer equations needs a discussion of the Boussinesq approximation. In this approximation density variations other than the variation in the body force term in the momentum equation are ignored. Thus the elimination of the first term ($= \frac{D\rho}{Dt}$) in the continuity equation will be found to lead to great simplifications in the boundary layer equations, particularly when the latter are expressed in terms of a stream function. Fluid property variations are ignored completely in this approximations and this factor, together with the removal of density variations in the convection terms, removes the requirement for the use of Howarth-Dorodnitsyn transformation.

Since the equation of the state plays an important role for a fluid, we consider this in general form as $\rho = \rho(T, p)$ (2.8)

One may write, $d\rho = -\rho\beta_T dT + \rho\overline{K}dp$ (2.9)

where $\beta_T = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$ is the volumetric expansion coefficient.

and $\overline{K} = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T$ is the isothermal compressibility coefficient.

In non-dimensional form the equation (2.9) may be written as

$$\frac{d\overline{\rho}}{\overline{\rho}} = -\beta_T \Delta T d\theta + \overline{K}p_0\left(\frac{\rho_0 U^2}{p_0}\right)d\overline{p} \tag{2.10}$$

In the case of slow motion one obtains for a gas $\overline{K}p_0 \cong 0(1)$, $\dfrac{\rho_0 U^2}{p_0}$ is proportional to $M^2 << 1$; but for a liquid $\dfrac{\rho_0 U^2}{p_0}$ is more significant than for a gas. However for a liquid $\overline{K}p_0 << 1$, hence the equation (2.10) becomes

$$\frac{d\overline{\rho}}{\overline{\rho}} = -\beta_T \Delta T d\theta$$

$$\Rightarrow \rho = \rho(T)$$

$$\Rightarrow d\rho = -\rho\beta_T dT$$

$$\Rightarrow \rho - \rho_r = -\rho_r \beta_T (T - T_r)$$

(Suffix 'r' represents the small changes from a reference condition as a first approximation.)

Hence $$\rho = \rho_r \{1 - \beta_T (T - T_r)\} \tag{2.11}$$

Similarly, we can write for another transport prooerty $\mu = \mu(T)$. It yields

$$\mu = \mu_r \{1 + a(T - T_r)\} \tag{2.12}$$

where $$a = \left(\frac{1}{\mu}\frac{\partial \mu}{\partial T}\right)_r$$

To the first order of small quantities equations (2.11-2.12) and similar equations for k and C_p provide

$$\rho \cong \rho_r,\quad \mu \cong \mu_r,\quad k \cong k_r,\quad C_p \cong C_{p_r}$$

In the present case, heating due to viscous dissipation is neglected and fluid is considered unsteady and incompressible. For simplicity, the fluid considered here is a Boussinesq one

Here *u*, *v*, *w* denote velocity components in the ξ, η and ζ directions, ρ is the density, t denotes time, μ is the dynamic viscosity coefficient, h_1, h_2 denote the scale factors in the ξ and η directions. p is the pressure, T is the temperature, C_p is the specific heat at constant pressure, k is the thermal conductivity of the fluid.

The boundary conditions to be imposed on the present problem may be determined as follows:

(i) The fluid must adhere to the surface (the no slip condition):

$$u(t,\xi,\eta,0) = v(t,\xi,\eta,0) = 0 \qquad (2.13)$$

(ii) The temperature of the fluid at the surface must be function of t, ξ, and η (non-isothermal surface).

$$T(t,\xi,\eta,0) = T_w(t,\xi,\eta) \qquad (2.14)$$

(iii) The fluid at large distances from the surface must remain undisturbed:

$$u(t,\xi,\eta,\infty) = v(t,\xi,\eta,\infty) = 0 \qquad (2.15)$$

(iv) The temperature at large distances from the surface must be equal to the undisturbed fluid temperature.

$$T(t,\xi,\eta,\infty) = T_\infty (=\text{constant}) \qquad (2.16)$$

The pressure gradients in the ξ and η directions result from the change is elevation up the curved surface.

Thus the hydrostatic conditions are

$$-\frac{1}{h_1}\frac{\partial P}{\partial \xi} = -\rho_\infty g_\xi$$

$$\text{and } -\frac{1}{h_2}\frac{\partial p}{\partial \eta} = -\rho_\infty g_\eta$$

For Boussinesq approximation, the continuity equation becomes

$$\frac{\partial}{\partial \xi}(h_2 u) + \frac{\partial}{\partial \eta}(h_1 v) + \frac{\partial}{\partial \zeta}(h_1 h_2 w) = 0 \qquad (2.17)$$

Thus the eliminations of pressure terms, the equations (2 2) and (2.3) become

$$\rho\left[\frac{Du}{Dt} + \frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta} + \frac{wv}{h_1}\frac{\partial h_1}{\partial \zeta} - \frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right] = \frac{1}{h_1}(\rho - \rho_\infty)g_\xi + \mu\nabla^2 u \qquad (2.18)$$

$$\rho\left[\frac{Dv}{Dt}+\frac{uv}{h_1h_2}\frac{\partial h_2}{\partial \xi}+\frac{vw}{h_2}\frac{\partial h_2}{\partial \zeta}--\frac{u^2}{h_1h_2}\frac{\partial h_1}{\partial \eta}\right]=\frac{1}{h_2}(\rho-\rho_\infty)g_\eta+\mu\nabla^2 v \tag{2.19}$$

For incompressible flow, introducing the Boussinesq approximation $\rho-\rho_\infty=-\rho\,\beta_T\Delta T\theta$ in equating (2.18 - 2.19). We get

$$\rho\left[\frac{Du}{Dt}+\frac{uv}{h_1h_2}\frac{\partial h_1}{\partial \eta}+\frac{wv}{h_1}\frac{\partial h_1}{\partial \zeta}-\frac{v^2}{h_1h_2}\frac{\partial h_2}{\partial \xi}\right]=-\frac{1}{h_1}\rho\beta_T\Delta T\theta g_\xi+\mu\nabla^2 u \tag{2.20}$$

$$\rho\left[\frac{Dv}{Dt}+\frac{uv}{h_1h_2}\frac{\partial h_2}{\partial \xi}+\frac{vw}{h_2}\frac{\partial h_2}{\partial \zeta}-\frac{u^2}{h_1h_2}\frac{\partial h_1}{\partial \eta}\right]=-\frac{1}{h_2}\rho\beta_T\Delta T\theta g_\eta+\mu\nabla^2 v \tag{2.21}$$

Before proceeding to obtain solutions of the equations (2.17), (2.20), (2.21), (2.4) and (2.5), it is proposed first to discover the dimensionless groups upon which the solution depend. We begin by introducing dimensionless quantities into the equation, referring all lengths to some characteristic length L along the surface, velocities with reference to some characteristic velocity U and t by $\frac{U}{L}$. The density will made dimensionless with respect to ρ_0, the pressure will be referred to $\rho_0 U^2$ and the temperature to the temperature difference between the wall and $T_\infty, (=T_w-T_\infty=\Delta T)$. The other transport properties other the fluid μ, k, C_p and the gravitational components g_ξ, g_η will be made dimensionless by μ_0, k_0, C_{p_0} and g respectively. We use suffix 0 to refer to some convenient constant reference conditions far from the surface.

Hence the substitutions are as follows:

$$\bar{\xi}=\frac{\varsigma}{L},\ \bar{\eta}=\frac{\eta}{L},\ \bar{\zeta}=\frac{\zeta}{L},\ \bar{t}=\frac{U}{L}t$$

$$\bar{u}=\frac{u}{U},\ \bar{v}=\frac{v}{U},\ \bar{w}=\frac{w}{U}$$

$$\bar{\rho}=\frac{\rho}{\rho_0},\ \bar{\mu}=\frac{\mu}{\mu_0},\ \bar{k}=\frac{k}{k_0}$$

$$\overline{p}=\frac{1}{\rho_0 U^2}P,\ \overline{g}_\xi=\frac{g_\xi}{g},\ \overline{g}_\eta=\frac{g_\eta}{g}$$

$$\overline{C_p}=\frac{C_p}{C_{p_0}},\ \theta=\frac{T-T_\infty}{T_w-T_\infty}=\frac{T-T_\infty}{\Delta T},\ \Delta T=T_w-T_\infty.$$

Here U is the maximum fluid velocity generated by buoyancy effect and defined by $U^2=g\beta_T\Delta TL$ where $\beta_T=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p$ is a volumetric expansion coefficient. For simplicity, h_1, h_2 are themselves dimensionless because they are different in individual co-ordinate system, L is some characteristic length.

We obtain the following non-dimensional equations:

continuity equation

$$\frac{\partial}{\partial\overline{\xi}}(h_2\overline{u})+\frac{\partial}{\partial\overline{\eta}}(h_1\overline{v})+\frac{\partial}{\partial\overline{\zeta}}(h_1h_2\overline{w})=0 \tag{2.22}$$

u-momentum equation

$$\frac{\partial\overline{u}}{\partial\overline{t}}+\frac{\overline{u}}{h_1}\frac{\partial\overline{u}}{\partial\overline{\xi}}+\frac{\overline{v}}{h_2}\frac{\partial\overline{u}}{\partial\overline{\eta}}+\overline{w}\frac{\partial\overline{u}}{\partial\overline{\zeta}}+\frac{\overline{u}\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial\overline{\eta}}+\frac{\overline{w}\,\overline{u}}{h_1}\frac{\partial h_1}{\partial\overline{\zeta}}-\frac{\overline{v}^2}{h_1h_2}\frac{\partial h_2}{\partial\overline{\xi}}=-\frac{1}{h_1}\beta_T\Delta T\theta\frac{\overline{g}_\xi}{F_r}$$
$$+\frac{1}{R_F}\overline{\nu}\left[\frac{1}{h_1h_2}\left\{\frac{\partial}{\partial\overline{\xi}}\left(\frac{h_2}{h_1}\frac{\partial\overline{u}}{\partial\overline{\xi}}\right)+\frac{\partial}{\partial\overline{\eta}}\left(\frac{h_1}{h_2}\frac{\partial\overline{u}}{\partial\overline{\eta}}\right)+\frac{\partial}{\partial\overline{\zeta}}\left(h_1h_2\frac{\partial\overline{u}}{\partial\overline{\zeta}}\right)\right\}\right] \tag{2.23}$$

v-momentum equation

$$\frac{\partial\overline{v}}{\partial\overline{t}}+\frac{\overline{u}}{h_1}\frac{\partial\overline{v}}{\partial\overline{\xi}}+\frac{\overline{v}}{h_2}\frac{\partial\overline{v}}{\partial\overline{\eta}}+\overline{w}\frac{\partial\overline{v}}{\partial\overline{\zeta}}+\frac{\overline{u}\overline{v}}{h_1h_2}\frac{\partial h_2}{\partial\overline{\xi}}+\frac{\overline{w}\,\overline{v}}{h_2}\frac{\partial h_2}{\partial\overline{\zeta}}-\frac{\overline{u}^2}{h_1h_2}\frac{\partial h_1}{\partial\overline{\eta}}=-\frac{1}{h_2}\beta_T\Delta T\theta\frac{\overline{g}_\eta}{F_r}$$
$$+\frac{1}{R_F}\overline{\nu}\left[\frac{1}{h_1h_2}\left\{\frac{\partial}{\partial\overline{\xi}}\left(\frac{h_2}{h_1}\frac{\partial\overline{v}}{\partial\overline{\xi}}\right)+\frac{\partial}{\partial\overline{\eta}}\left(\frac{h_1}{h_2}\frac{\partial\overline{v}}{\partial\overline{\eta}}\right)+\frac{\partial}{\partial\overline{\zeta}}\left(h_1h_2\frac{\partial\overline{v}}{\partial\overline{\zeta}}\right)\right\}\right] \tag{2.24}$$

w-momentum equation

$$\frac{\partial \bar{w}}{\partial \bar{t}}+\frac{\bar{u}}{h_1}\frac{\partial \bar{w}}{\partial \bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial \bar{w}}{\partial \bar{\eta}}+\bar{w}\frac{\partial \bar{w}}{\partial \bar{\zeta}}-\frac{\bar{u}^2}{h_1}\frac{\partial h_1}{\partial \bar{\zeta}}-\frac{\bar{v}^2}{h_2}\frac{\partial h_2}{\partial \bar{\zeta}}-=-\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial \bar{\zeta}}$$

$$+\frac{1}{R_F}\bar{v}\left[\frac{1}{h_1h_2}\left\{\frac{\partial}{\partial \bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial \bar{w}}{\partial \bar{\xi}}\right)+\frac{\partial}{\partial \bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial \bar{w}}{\partial \bar{\eta}}\right)+\frac{\partial}{\partial \bar{\zeta}}\left(h_1h_2\frac{\partial \bar{w}}{\partial \bar{\zeta}}\right)\right\}\right] \tag{2.25}$$

and energy equation

$$\bar{\rho}\bar{C}_p\left[\left\{\frac{\partial \theta}{\partial \bar{t}}+\frac{\bar{u}}{h_1}\frac{\partial \theta}{\partial \bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial \theta}{\partial \bar{\eta}}+\bar{w}\frac{\partial \theta}{\partial \bar{\zeta}}\right\}+\theta\left\{\frac{\partial(\ln\Delta T)}{\partial \bar{t}}+\frac{\bar{u}}{h_1}\frac{\partial(\ln\Delta T)}{\partial \bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial(\ln\Delta T)}{\partial \bar{\eta}}+\bar{w}\frac{\partial(\ln\Delta T)}{\partial \bar{\zeta}}\right\}\right]$$

$$=\frac{\bar{k}}{P_rR_F}\frac{1}{h_1h_2}\left[\left\{\frac{\partial}{\partial \bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial \theta}{\partial \bar{\xi}}\right)+\frac{\partial}{\partial \bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial \theta}{\partial \bar{\eta}}\right)+\frac{\partial}{\partial \bar{\zeta}}\left(h_1h_2\frac{\partial \theta}{\partial \bar{\zeta}}\right)\right\}\right.$$

$$\left.+\theta\left\{\frac{\partial}{\partial \bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial(\ln\Delta T)}{\partial \bar{\xi}}\right)+\frac{\partial}{\partial \bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial(\ln\Delta T)}{\partial \bar{\eta}}\right)+\frac{\partial}{\partial \bar{\zeta}}\left(h_1h_2\frac{\partial(\ln\Delta T)}{\partial \bar{\zeta}}\right)\right\}\right] \tag{2.26}$$

where $R_F=\frac{UL}{\nu_0}$, The Reynolds number based on free convection fluid velocity.

$P_r=\frac{\mu_0 C_{p_0}}{k_0}$, The Prandtl number of the fluid.

$F_r=\frac{U^2}{gL}$, The Froude number.

The boundary conditions in dimensionless form are

$$\bar{u}(\bar{t},\bar{\xi},\bar{\eta},0)=\bar{v}(\bar{t},\bar{\xi},\bar{\eta},0)=0 \tag{2.27}$$

$$\bar{\theta}(\bar{t},\bar{\xi},\bar{\eta},0)=1 \tag{2.28}$$

$$\bar{u}(\bar{t},\bar{\xi},\bar{\eta},\infty)=\bar{v}(\bar{t},\bar{\xi},\bar{\eta},\infty)=0 \tag{2.29}$$

$$\bar{\theta}(\bar{t},\bar{\xi},\bar{\eta},\infty)=0 \tag{2.30}$$

If δ be the boundary layer thickness, then the dimensionless boundary layer thickness is

$$\bar{\delta}=\frac{\delta}{L}<<1 \text{ since } L>>1.$$

Order of the magnitude of each of the terms in equations (2.22 to2.26) are estimated, so that very small terms can be neglected Since,

$$\frac{\partial \overline{u}}{\partial \overline{t}} \sim 0(1),\ \frac{\partial \overline{u}}{\partial \overline{\xi}} \sim 0(1),\ \frac{\partial \overline{u}}{\partial \overline{\eta}} \sim 0(1),\ \frac{\partial \overline{v}}{\partial \overline{t}} \sim 0(1),\ \frac{\partial \overline{v}}{\partial \overline{\xi}} \sim 0(1),\ \frac{\partial \overline{v}}{\partial \overline{\eta}} \sim 0(1)$$

then also $\frac{\partial \overline{w}}{\partial \overline{\zeta}} \sim 0(1)$, since $\overline{\zeta}$ is of order $\overline{\delta}$ so that $w \sim 0(\overline{\delta})$.

and $\frac{\partial^2 \overline{u}}{\partial \overline{\xi}^2} \sim 0(1),\ \frac{\partial^2 \overline{v}}{\partial \overline{\xi}^2} \sim 0(1),\ \frac{\partial^2 \overline{u}}{\partial \overline{\eta}^2} \sim 0(1),\ \frac{\partial^2 \overline{u}}{\partial \overline{\zeta}^2} \sim 0\left(\frac{1}{\overline{\delta}^2}\right),\ \frac{\partial^2 \overline{v}}{\partial \overline{\zeta}^2} \sim 0\left(\frac{1}{\overline{\delta}^2}\right),$

$$\frac{\partial^2 \overline{w}}{\partial \zeta^2} \sim 0\left(\frac{1}{\overline{\delta}}\right),\ \frac{\partial \overline{w}}{\partial \overline{\zeta}} \sim 0(1),\ \frac{\partial \overline{w}}{\partial \overline{\xi}} \sim 0(\overline{\delta}),\ \frac{\partial \overline{w}}{\partial \overline{\eta}} \sim 0(\overline{\delta}),\ R_F \sim 0(1),\ F_r \sim 0(1)$$

Let δ_T be the thermal boundary layer thickness, the conduction term becomes of the same order of magnitude as the convectional term,, only if the thickness of the thermal boundary layer is order of

$$\left(\frac{\delta_T}{L}\right)^2 \sim \frac{1}{R_F P_r}$$

In view the previously obtained estimation for the thickness of the velocity boundary layer

$$\delta \sim \frac{1}{\sqrt{R_F}},\ \text{it is found that}\ \frac{\delta_T}{\delta} \sim \frac{1}{\sqrt{P_r}},$$

Assuming that h_1, h_2 and all their first derivatives is of $0(1)$.

Setting the order of magnitude in each terms of equations (2.22 - 2.26), one obtains,

continuity equation

$$\frac{\partial}{\partial \overline{\xi}}(h_2 \overline{u}) + \frac{\partial}{\partial \overline{\eta}}(h_1 \overline{v}) + \frac{\partial}{\partial \overline{\zeta}}(h_1 h_2 \overline{w}) = 0$$

(0) → (1) (1) (1)

u-momentum equation

$$\frac{\partial \overline{u}}{\partial \overline{t}} + \frac{\overline{u}}{h_1}\frac{\partial \overline{u}}{\partial \overline{\xi}} + \frac{\overline{v}}{h_2}\frac{\partial \overline{u}}{\partial \overline{\eta}} + \overline{w}\frac{\partial \overline{u}}{\partial \overline{\zeta}} + \frac{\overline{u}\overline{v}}{h_1 h_2}\frac{\partial h_1}{\partial \overline{\eta}} + \frac{\overline{w}\,\overline{u}}{h_1}\frac{\partial h_1}{\partial \overline{\zeta}} - \frac{\overline{v}^2}{h_1 h_2}\frac{\partial h_2}{\partial \overline{\xi}} = -\frac{1}{h_1}\beta_r \Delta T \theta \frac{\overline{g}_\xi}{F_r}$$

(0)→ (1) (1) (1) (1) (1) $(\overline{\delta})$ (1) (1)

$$+\frac{1}{R_F}\bar{\nu}\left[\frac{1}{h_1h_2}\left\{\frac{\partial}{\partial\bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial\bar{u}}{\partial\bar{\xi}}\right)+\frac{\partial}{\partial\bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial\bar{u}}{\partial\bar{\eta}}\right)+\frac{\partial}{\partial\bar{\zeta}}(h_1h_2)\frac{\partial\bar{u}}{\partial\bar{\zeta}}+h_1h_2\frac{\partial\bar{u}}{\partial\bar{\zeta}^2}\right\}\right]$$

$(\bar{\delta}^2)$ (1) (1) $\frac{1}{(\bar{\delta})}$ $\frac{1}{(\bar{\delta}^2)}$

v-momentum equation

$$\frac{\partial\bar{v}}{\partial\bar{t}}+\frac{\bar{u}}{h_1}\frac{\partial\bar{v}}{\partial\bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial\bar{v}}{\partial\bar{\eta}}+\bar{w}\frac{\partial\bar{v}}{\partial\bar{\zeta}}+\frac{\bar{u}\bar{v}}{h_1h_2}\frac{\partial h_2}{\partial\bar{\xi}}+\frac{\bar{v}\bar{w}}{h_2}\frac{\partial h_2}{\partial\bar{\zeta}}-\frac{\bar{u}^2}{h_1h_2}\frac{\partial h_1}{\partial\bar{\eta}}=-\frac{1}{h_2}\beta_r\Delta T\theta\frac{\bar{g}_{\bar{\eta}}}{F_r}$$

(0)→ (1) (1) (1) (1) (1) $(\bar{\delta})$ (1) (1)

$$+\frac{1}{R_F}\bar{\nu}\left[\frac{1}{h_1h_2}\left\{\frac{\partial}{\partial\bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial\bar{v}}{\partial\bar{\xi}}\right)+\frac{\partial}{\partial\bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial\bar{v}}{\partial\bar{\eta}}\right)+\frac{\partial}{\partial\bar{\zeta}}(h_1h_2)\frac{\partial\bar{v}}{\partial\bar{\zeta}}+h_1h_2\frac{\partial\bar{v}}{\partial\bar{\zeta}^2}\right\}\right]$$

$(\bar{\delta}^2)$ (1) (1) $\frac{1}{(\bar{\delta})}$ $\frac{1}{(\bar{\delta}^2)}$

w-momentum equation

$$\frac{\partial\bar{w}}{\partial\bar{t}}+\frac{\bar{u}}{h_1}\frac{\partial\bar{w}}{\partial\bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial\bar{w}}{\partial\bar{\eta}}+\bar{w}\frac{\partial\bar{w}}{\partial\bar{\zeta}}-\frac{\bar{u}^2}{h_1}\frac{\partial h_1}{\partial\bar{\zeta}}-\frac{\bar{v}^2}{h_2}\frac{\partial h_2}{\partial\bar{\zeta}}-=-\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial\bar{\zeta}}$$

(0)→ $(\bar{\delta})$ $(\bar{\delta})$ $(\bar{\delta})$ $(\bar{\delta})$ (1) (1) $\frac{1}{(\bar{\delta})}$

$$+\frac{1}{R_F}\bar{\nu}\left[\frac{1}{h_1h_2}\left\{\frac{\partial}{\partial\bar{\xi}}\left(\frac{h_2}{h_1}\frac{\partial\bar{w}}{\partial\bar{\xi}}\right)+\frac{\partial}{\partial\bar{\eta}}\left(\frac{h_1}{h_2}\frac{\partial\bar{w}}{\partial\bar{\eta}}\right)+\frac{\partial}{\partial\bar{\zeta}}(h_1h_2)\frac{\partial\bar{w}}{\partial\bar{\zeta}}+(h_1h_2)\frac{\partial^2\bar{w}}{\partial\bar{\zeta}^2}\right\}\right]$$

$(\bar{\delta}^2)$ $(\bar{\delta})$ $(\bar{\delta})$ (1) $\frac{1}{(\bar{\delta})}$

and **energy equation**

$$\bar{\rho}\bar{C}_p\left[\left\{\frac{\partial\theta}{\partial\bar{t}}+\frac{\bar{u}}{h_1}\frac{\partial\theta}{\partial\bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial\theta}{\partial\bar{\eta}}+\bar{w}\frac{\partial\theta}{\partial\bar{\zeta}}\right\}+\theta\left\{\frac{\partial(\ln\Delta T)}{\partial\bar{t}}+\frac{\bar{u}}{h_1}\frac{\partial(\ln\Delta T)}{\partial\bar{\xi}}+\frac{\bar{v}}{h_2}\frac{\partial(\ln\Delta T)}{\partial\bar{\eta}}+\bar{w}\frac{\partial(\ln\Delta T)}{\partial\bar{\zeta}}\right\}\right]$$

(0)→ (1) (1) (1) (1) (1) (1) (1) (1) (0)

$$= \frac{\bar{k}}{P_r R_r} \frac{1}{h_1 h_2} \left[\left\{ \frac{\partial}{\partial \bar{\xi}} \left(\frac{h_2}{h_1} \frac{\partial \theta}{\partial \bar{\xi}} \right) + \frac{\partial}{\partial \bar{\eta}} \left(\frac{h_1}{h_2} \frac{\partial \theta}{\partial \bar{\eta}} \right) + \frac{\partial}{\partial \bar{\zeta}} (h_1 h_2) \frac{\partial \theta}{\partial \bar{\zeta}} + h_1 h_2 \frac{\partial^2 \theta}{\partial \bar{\zeta}^2} \right\} \right.$$

$$(\bar{\delta}_r^2) \qquad (1) \qquad (1) \qquad \frac{1}{(\bar{\delta})} \qquad \frac{1}{(\bar{\delta}^2)}$$

$$\left. + \theta \left\{ \frac{\partial}{\partial \bar{\xi}} \left(\frac{h_2}{h_1} \frac{\partial (\ln \Delta T)}{\partial \bar{\xi}} \right) + \frac{\partial}{\partial \bar{\eta}} \left(\frac{h_1}{h_2} \frac{\partial (\ln \Delta T)}{\partial \bar{\eta}} \right) + \frac{\partial}{\partial \bar{\zeta}} \left(h_1 h_2 \frac{\partial (\ln \Delta T)}{\partial \bar{\zeta}} \right) \right\} \right]$$

$$(1) \qquad (1) \qquad (0)$$

Governing equations

The governing boundary layer equations of the flow field in general orthogonal curvilinear co-ordinates are

continuity equation

$$\frac{\partial}{\partial \xi}(h_2 u) + \frac{\partial}{\partial \eta}(h_1 u) + \frac{\partial}{\partial \zeta}(h_1 h_2 w) = 0 \qquad (2.31)$$

***u*-momentum equation**

$$\frac{Du}{Dt} + \frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta} - \frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi} = -\frac{1}{h_1}\beta_T \Delta T \theta g_\xi + \nu \frac{\partial^2 u}{\partial \zeta^2} \qquad (2.32)$$

***v*-momentum equation**

$$\frac{Du}{Dt} + \frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi} - \frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta} = -\frac{1}{h_2}\beta_T \Delta T \theta g_\eta + \nu \frac{\partial^2 v}{\partial \zeta^2} \qquad (2.33)$$

and energy equation

$$\frac{D\theta}{Dt} + \theta \left\{ \frac{\partial}{\partial t}(\ln \Delta T) + \frac{u}{h_1}\frac{\partial}{\partial \xi}(\ln \Delta T) + \frac{v}{h_2}\frac{\partial}{\partial \eta}(\ln \Delta T) \right\} = \frac{\nu}{\Pr}\frac{\partial^2 \theta}{\partial \zeta^2} \qquad (2.34)$$

where $\Pr = \frac{\mu C_p}{k}$ is the Prandtl number of the fluid.

The boundary conditions are

$$u(t,\xi,\eta,0) = v(t,\xi,\eta,0) = 0 \tag{2.35}$$

$$\theta(t,\xi,\eta,0) = 1 \tag{2.36}$$

$$u(t,\xi,\eta,\infty) = v(t,\xi,\eta,\infty) = 0 \tag{2.37}$$

$$\theta(t,\xi,\eta,\infty) = 0 \tag{2.38}$$

Chapter-3

Transformations leading to Similarity solution

Equations (2.31–2.34) are non-linear, simultaneous partial differential equations and the solutions of these equations are extremely difficult to obtain. Hence our aim is to reduce equations (2.31 – 2.34) to ordinary differential equations with the help of (2.31) which permits possible variations in $\Delta T, U_F, V_F, h_1$ and h_2 with respect to t ,ξ, and η.

Let us now change the variables t , ξ, η, and ζ to a new set of variables τ, X, Y and $\bar{\phi}$.

$$(t,\xi,\eta,\zeta)\rightarrow(\tau,X,Y,\bar{\phi})$$

by the set of following equations,

$$t=\tau, \xi=X,\ \eta=Y, and\ \bar{\phi}=\frac{\zeta}{\gamma(\tau,X,Y)} \tag{3.1}$$

$\gamma(\tau,X,Y)$ is considered primarily here to be proportional to the square root of the local beundary layer thickness.

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial \tau}-\frac{\bar{\phi}}{\gamma}\gamma_\tau\frac{\partial}{\partial \bar{\phi}} \tag{3.2}$$

$$\frac{\partial}{\partial \xi}=\frac{\partial}{\partial X}-\frac{\bar{\phi}}{\gamma}\gamma_X\frac{\partial}{\partial \bar{\phi}} \tag{3.3}$$

$$\frac{\partial}{\partial \eta}=\frac{\partial}{\partial Y}-\frac{\bar{\phi}}{\gamma}\gamma_Y\frac{\partial}{\partial \bar{\phi}} \tag{3.4}$$

$$\frac{\partial}{\partial \zeta}=\frac{1}{\gamma}\frac{\partial}{\partial \bar{\phi}} \tag{3.5}$$

$$\frac{\partial^2}{\partial \zeta^2}=\frac{1}{\gamma^2}\frac{\partial^2}{\partial \bar{\phi}^2} \tag{3.6}$$

Let two stream functions ψ and Φ be defined as the mass flow components within the boundary layer for the case of incompressible flow.

The following equations can be written

$$\psi_\zeta = h_2 u,\ \Phi_\zeta = h_1 v \tag{3.7}$$

$$-(\psi_\zeta + \Phi_\eta) = h_1 h_2 w \tag{3.8}$$

to satisfy the equation of continuity (2.31). Guided by the idea of similarity procedure of **Hansen** and **Ohio** (1958) are allowed to write

$$\int_0^{\bar{\phi}} \frac{u}{U_F(\tau,X,Y)} d\bar{\phi} = F(\tau,X,Y,\bar{\phi}) \quad \text{where}\, U_F^2 = -g_X \beta\Delta T L_1 \tag{3.9}$$

$$\int_0^{\bar{\phi}} \frac{v}{V_F(\tau,X,Y)} d\bar{\phi} = S(\tau,X,Y,\bar{\phi}) \quad \text{where}\, V_F^2 = -g_Y \beta\Delta T L_2 \tag{3.10}$$

where L_1 and L_2 are primarily considered to be some characteristic lengths.

Let us assume that the separation of variables are

$$\left.\begin{aligned} F(\tau,X,Y,\bar{\phi}) &= L(\tau,X,Y)\bar{F}(\bar{\phi}) \\ S(\tau,X,Y,\bar{\phi}) &= M(\tau,X,Y)\bar{S}(\bar{\phi}) \\ \theta(\tau,X,Y,\bar{\phi}) &= N(\tau,X,Y)\bar{\theta}(\bar{\phi}) \end{aligned}\right\} \tag{3.11}$$

where $\bar{F}, \bar{S}$ and $\bar{\theta}$ are the functions of single variable $\bar{\phi}$. From (3.9–3.10) and (3.11), it is found that

$$\left.\begin{aligned} u &= U_F L\bar{F}_{\bar{\phi}} \\ \text{and } v &= V_F M\bar{S}_{\bar{\phi}} \end{aligned}\right\} \tag{3.12}$$

Again, from (3.7), we have

$$\int_0^{\bar{\phi}} \frac{u}{U_F} d\bar{\phi} = \frac{1}{h_2 \gamma U_F} \left\{\psi(\tau,X,Y,\bar{\phi}) - \psi(\tau,X,Y,0)\right\} \tag{3.13}$$

$$\therefore \psi(\tau,X,Y,\bar{\phi}) = h_2 \gamma U_F L\bar{F}(\bar{\phi}) + \psi(\tau,X,Y,0) \tag{3.14}$$

Similarly,

$$\Phi(\tau,X,Y,\bar{\phi}) = h_1 \gamma V_F M\bar{S}(\bar{\phi}) + \Phi(\tau,X,Y,0) \tag{3.15}$$

and

$$\begin{aligned} h_1 h_2 w = &-(h_2 \gamma U_F L)_X \bar{F} + \bar{\phi}\gamma_X h_2 U_F L\bar{F}_{\bar{\phi}} - (h_1 \gamma V_F M)_Y \bar{S} \\ &+ \bar{\phi}\gamma_Y h_1 V_F M\bar{S}_{\bar{\phi}} + h_1 h_2 w_0(\tau,X,Y,0) \end{aligned} \tag{3.16}$$

where $w_0(\tau, X, Y, 0) = -\frac{1}{h_1 h_2}\{\psi_X(\tau, X, Y, 0) + \Phi_Y(\tau, X, Y, 0)\}$ (3.16a)

is the suction / injection velocity normal to the surface.

The convective operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{1}{h_1 h_2}\left[h_2 u \frac{\partial}{\partial \xi} + h_1 v \frac{\partial}{\partial \eta} + h_1 h_2 w \frac{\partial}{\partial \zeta}\right]$$

in terms of new set of varibles τ, X, Y and $\bar{\phi}$ may be derived as

$$\frac{D}{Dt} = \frac{\partial}{\partial \tau} + \frac{1}{h_1} U_F L \bar{F}_{\bar{\phi}} \frac{\partial}{\partial X} + \frac{1}{h_2} V_F M \bar{S}_{\bar{\phi}} \frac{\partial}{\partial Y} -$$

$$\frac{1}{\gamma}\left\{\frac{(h_2 \gamma U_F L)_X \bar{F}}{h_1 h_2} + \frac{(h_1 \gamma V_F M)_Y \bar{S}}{h_1 h_2} - w_0 + \bar{\phi}\gamma_\tau\right\}\frac{\partial}{\partial \phi} \tag{3.17}$$

[By using equations (3.2 - 3.5)].

In view of equation (3.17), equations (2.32), (2.33) and (2.34) become

u-momentum equation

$$\nu \bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{\gamma(h_2 \gamma U_F L)_X}{h_1 h_2} \bar{F}\bar{F}_{\bar{\phi}\bar{\phi}} + \frac{\gamma(h_1 \gamma V_F M)_Y}{h_1 h_2} \bar{S}\bar{F}_{\bar{\phi}\bar{\phi}} - (w_0\gamma - \bar{\phi}\gamma\gamma_\tau)\bar{F}_{\bar{\phi}\bar{\phi}}$$

$$-\frac{\gamma^2}{h_1}(U_F L)_X \bar{F}_{\bar{\phi}}^2 - \frac{\gamma^2}{h_2} V_F M \left\{\frac{(U_F L)_Y}{U_F L} + \frac{h_{1Y}}{h_1}\right\}\bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}} + \frac{\gamma^2}{h_1 h_2}\frac{(V_F)^2 M^2}{U_F L} h_{2X} \bar{S}_{\bar{\phi}}^2$$

$$-\frac{\gamma^2 (U_F L)_\tau}{U_F L}\bar{F}_{\bar{\phi}} - \frac{\gamma^2}{U_F L}\frac{1}{h_1}\beta_T \Delta T N \bar{\theta} g_X = 0 \tag{3.18}$$

v-momentum equation

$$\nu \bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{\gamma(h_1 \gamma V_F M)_Y}{h_1 h_2} \bar{S}\bar{S}_{\bar{\phi}\bar{\phi}} + \frac{\gamma(h_2 \gamma U_F L)_X}{h_1 h_2} \bar{F}\bar{S}_{\bar{\phi}\bar{\phi}} - (w_0\gamma - \bar{\phi}\gamma\gamma_\tau)\bar{S}_{\bar{\phi}\bar{\phi}}$$

$$-\frac{\gamma^2}{h_2}(V_F M)_Y \bar{S}_{\bar{\phi}}^2 - \frac{\gamma^2 U_F L}{h_1}\left\{\frac{(V_F M)_X}{V_F M} + \frac{h_{2X}}{h_2}\right\}\bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}} + \frac{\gamma^2}{h_1 h_2}\frac{(U_F L)^2 M^2}{V_F M} h_{1Y} \bar{F}_{\bar{\phi}}^2$$

$$-\frac{\gamma^2 (V_F M)_\tau}{V_F M}\bar{S}_{\bar{\phi}} - \frac{\gamma^2}{V_F M}\frac{1}{h_2}\beta_T \Delta T N \bar{\theta} g_Y = 0 \tag{3.19}$$

and **energy equation**

$$\frac{\nu}{\Pr}\bar{\theta}_{\bar{\phi}\bar{\phi}}+\frac{\gamma(h_2\gamma U_F L)_X}{h_1h_2}\bar{F}\bar{\theta}_{\bar{\phi}}+\frac{\gamma(h_1\gamma V_F M)_Y}{h_1h_2}\bar{S}\bar{\theta}_{\bar{\phi}}-(w_0\gamma-\bar{\phi}\gamma\gamma_\tau)\bar{\theta}_{\bar{\phi}}$$

$$-\frac{\gamma^2U_FL}{h_1}\{(\ln N)_X+(\ln\Delta T)_X\}\bar{F}_{\bar{\phi}}\bar{\theta}-\frac{\gamma^2V_FM}{h_2}\{(\ln N)_Y+(\ln\Delta T)_Y\}\bar{S}_{\bar{\phi}}\bar{\theta}$$

$$-\gamma^2\{(\ln N)_\tau+(\ln\Delta T)_\tau\}\bar{\theta}=0 \qquad (3.20)$$

The associated boundary conditions are

$$U_F(\tau,X,Y,0)=0=\bar{F}_{\bar{\phi}}(0)$$
$$V_F(\tau,X,Y,0)=0=\bar{S}_{\bar{\phi}}(0)$$
$$w(\tau,X,Y,0)=-w_0$$

where w_0 is considered to be the surface suction or injection velocity for the curvilinear surface. For the temperature function the boundary condition becomes

$$\theta(\tau,X,Y,0)=N(\tau,X,Y)\bar{\theta}(0)=1$$

$$\Rightarrow N(\tau,X,Y)=1 \text{ and } \bar{\theta}(0)=1$$

In order to satisfy the boundary conditions (2.35) and (2.37) without loss of generality we may put $L=M=1$.

The boundary conditions at large distance satisfy

$$U_FL\bar{F}_{\bar{\phi}}(\infty)=0\Rightarrow\bar{F}_{\bar{\phi}}(\infty)=0$$
$$V_FM\bar{S}_{\bar{\phi}}(\infty)=0\Rightarrow\bar{S}_{\bar{\phi}}(\infty)=0$$
$$\text{and }\bar{\theta}(\infty)=0$$

Then the two momentum equations (3.18) and (3.19) and the energy equation (3.20) take the following forms.

u-momentum equation

$$\nu\bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}}+\frac{\gamma(h_2\gamma U_F)_X}{h_1h_2}\bar{F}\bar{F}_{\bar{\phi}\bar{\phi}}+\frac{\gamma(h_1\gamma V_F)_Y}{h_1h_2}\bar{S}\bar{F}_{\bar{\phi}\bar{\phi}}-(w_0\gamma-\bar{\phi}\gamma\gamma_\tau)\bar{F}_{\bar{\phi}\bar{\phi}}$$

$$-\frac{\gamma^2}{h_1}(U_F)_X\bar{F}_{\bar{\phi}}^2-\frac{\gamma^2}{h_2}V_F\left\{\frac{(U_F)_Y}{U_F}+\frac{h_{1Y}}{h_1}\right\}\bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}}+\frac{\gamma^2}{h_1h_2}\frac{V_F^2}{U_F}h_{2X}\bar{S}_{\bar{\phi}}^2$$

$$-\frac{\gamma^2(U_F)_\tau}{U_F}\bar{F}_{\bar{\phi}}-\frac{\gamma^2}{U_F}\frac{1}{h_1}\beta_T\Delta T\theta g_X=0 \qquad (3.21)$$

v-momentum equation

$$\nu\bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}}+\frac{\gamma(h_1\gamma V_F)_Y}{h_1h_2}\overline{SS}_{\bar{\phi}\bar{\phi}}+\frac{\gamma(h_2\gamma U_F)_X}{h_1h_2}\overline{F}\overline{S}_{\bar{\phi}\bar{\phi}}-(w_0\gamma-\bar{\phi}\gamma\gamma_\tau)\bar{S}_{\bar{\phi}\bar{\phi}}$$

$$-\frac{\gamma^2}{h_2}(V_F)_Y\bar{S}^2_{\bar{\phi}}-\frac{\gamma^2U_F}{h_1}\left\{\frac{(V_F)_X}{V_F}+\frac{h_{2X}}{h_2}\right\}\bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}}+\frac{\gamma^2}{h_1h_2}\frac{(U_F)^2}{V_F}h_{1Y}\bar{F}^2_{\bar{\phi}}$$

$$-\frac{\gamma^2(V_F)_\tau}{V_F}\bar{S}_{\bar{\phi}}-\frac{\gamma^2}{V_Fh_2}\beta_T\Delta T\bar{\theta}g_Y=0 \qquad (3.22)$$

and **energy equation**

$$\frac{\nu}{\Pr}\bar{\theta}_{\bar{\phi}\bar{\phi}}+\frac{\gamma(h_2\gamma U_F)_X}{h_1h_2}\bar{F}\theta_{\bar{\phi}}+\frac{\gamma(h_1\gamma V_F)_Y}{h_1h_2}\bar{S}\bar{\theta}_{\bar{\phi}}-(w_0\gamma-\bar{\phi}\gamma\gamma_\tau)\bar{\theta}_{\bar{\phi}}$$

$$-\frac{\gamma^2U_F}{h_1}(\ln\Delta T)_X\bar{F}_{\bar{\phi}}\bar{\theta}-\frac{V_F\gamma^2}{h_2}(\ln\Delta T)_Y\bar{S}_{\bar{\phi}}\bar{\theta}-\gamma^2(\ln\Delta T)_\tau\bar{\theta}=0 \qquad (3.23)$$

The boundary conditions are

$$\left.\begin{aligned}&\bar{F}_{\bar{\phi}}(0)=\bar{S}_{\bar{\phi}}(0)=0\\&\bar{F}_{\bar{\phi}}(\infty)=\bar{S}_{\bar{\phi}}(\infty)=0\\&\bar{\theta}(0)=1,\ \theta(\infty)=0\end{aligned}\right\} \qquad (3.24)$$

The coefficients of $\overline{FF}_{\bar{\phi}\bar{\phi}}$ and $\overline{SS}_{\bar{\phi}\bar{\phi}}$ in (3.21) and (3.22) may be expressed as

$$\frac{\gamma(\gamma h_2U_F)_X}{h_1h_2}=\frac{1}{2}\left[\left(\frac{\gamma^2U_F}{h_1}\right)_X+\frac{\gamma^2(h_2U_F)_X}{h_1h_2}-\gamma^2U_Fh_2\left(\frac{1}{h_1h_2}\right)_X\right]$$

and

$$\frac{\gamma(\gamma h_1V_F)_X}{h_1h_2}=\frac{1}{2}\left[\left(\frac{\gamma^2V_F}{h_2}\right)_Y+\frac{\gamma^2(h_1V_F)_Y}{h_1h_2}-\gamma^2V_Fh_1\left(\frac{1}{h_1h_2}\right)_Y\right]$$

Thus momentum and energy equations become

$$\nu\bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}}+\frac{1}{2}(a_0+a_1-a_2)\overline{FF}_{\bar{\phi}\bar{\phi}}+\frac{1}{2}(a_3+a_4-a_5)\overline{SF}_{\bar{\phi}\bar{\phi}}-(a_6-\bar{\phi}a_7)\bar{F}_{\bar{\phi}\bar{\phi}}$$

$$-a_8\bar{F}^2_{\bar{\phi}}-(a_9+a_{10})\bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}}+a_{11}\bar{S}^2_{\bar{\phi}}-a_{12}\bar{F}_{\bar{\phi}}+a_{13}\bar{\theta}=0 \qquad (3.25)$$

$$\nu\bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}}+\frac{1}{2}(a_3+a_4-a_5)\overline{SS}_{\bar{\phi}\bar{\phi}}+\frac{1}{2}(a_0+a_1-a_2)\bar{S}_{\bar{\phi}\bar{\phi}}\bar{F}-(a_6-\bar{\phi}a_7)\bar{S}_{\bar{\phi}\bar{\phi}}$$

$$-a_{14}\bar{S}^2_{\bar{\phi}}-(a_{15}+a_{16})\bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}}+a_{17}\bar{F}^2_{\bar{\phi}}-a_{18}\bar{S}_{\bar{\phi}}+a_{19}\bar{\theta}=0 \qquad (3.26)$$

and
$$\frac{\nu}{\Pr}\overline{\theta}_{\bar\phi\bar\phi}+\frac{1}{2}(a_0+a_1-a_2)\overline{F\theta}_{\bar\phi}+\frac{1}{2}(a_3+a_4-a_5)\overline{S\theta}_{\bar\phi}-$$
$$-(a_6-\overline{\phi}a_7)\overline{\theta}_{\bar\phi}-(a_{20}\overline{F}_{\bar\phi}+a_{21}\overline{S}_{\bar\phi})\theta-a_{22}\overline{\theta}=0 \qquad (3.27)$$

where the constant a's and the differential equation involving the independent variable τ, X, and Y are given by the following differential equations:

$$\left(\frac{\gamma^2 U_F}{h_1}\right)_X=a_0, \qquad \frac{\gamma^2(h_2U_F)_X}{h_1h_2}=a_1 \qquad (3.28a,b)$$

$$\gamma^2U_Fh_2\left(\frac{1}{h_1h_2}\right)_X=a_2, \qquad \left(\frac{\gamma^2V_F}{h_2}\right)_Y=a_3 \qquad (3.29a,b)$$

$$\frac{\gamma^2(h_1V_F)_Y}{h_1h_2}=a_4, \qquad \gamma^2V_Fh_1\left(\frac{1}{h_1h_2}\right)_Y=a_5 \qquad (3.30a,b)$$

$$w_0\gamma=a_6\ , \qquad \gamma\gamma_\tau=a_7 \qquad (3.31a,b)$$

$$\frac{\gamma^2}{h_1}(U_F)_X=a_8, \qquad \frac{\gamma^2}{h_2}V_F\left\{\frac{(U_F)_Y}{U_F}\right\}=a_9 \qquad (3.32a,b)$$

$$\frac{\gamma^2}{h_2}V_F\frac{h_{1Y}}{h_1}=a_{10}, \qquad \frac{\gamma^2}{h_1h_2}\frac{(V_F)^2}{U_F}h_{2X}=a_{11} \qquad (3.33a,b)$$

$$\frac{\gamma^2(U_F)_\tau}{U_F}=a_{12}\ , \qquad \frac{\gamma^2}{h_1U_F}\beta_T\Delta Tg_X=a_{13} \qquad (3.34a,b)$$

$$\frac{\gamma^2}{h_2}(V_F)_Y=a_{14}, \qquad \frac{\gamma^2U_F}{h_1}\frac{(V_F)_X}{V_F}=a_{15} \qquad (3.35a,b)$$

$$\frac{\gamma^2U_F}{h_1}\frac{h_{2X}}{h_2}=a_{16}, \qquad \frac{\gamma^2}{h_1h_2}\frac{(U_F)^2}{V_F}h_{1Y}=a_{17} \qquad (3.36a,b)$$

$$\frac{\gamma^2(V_F)_\tau}{V_F}=a_{18}, \qquad \frac{\gamma^2}{h_2V_F}\beta_T\Delta Tg_Y=a_{19} \qquad (3.37a,b)$$

$$\frac{\gamma^2U_F}{h_1}(\ln\Delta T)_X=a_{20}, \qquad \frac{\gamma^2V_F}{h_2}(\ln\Delta T)_Y=a_{21} \qquad (3.38a,b)$$

$$\gamma^2(\ln\Delta T)_\tau=a_{22} \qquad (3.39)$$

Similar solutions for (3.25), (3.26) and (3.27) exist only when all the a's are finite and independent of τ , X and Y that is to say that all a's must be constants. Thus the boundary layer momentum equations and the energy equation will become non-linear ordinary differential equations. If $\Delta T(\tau,X,Y), h_1(\tau,X,Y), h_2(\tau,X,Y)$, $U_F(\tau,X,Y), V_F(\tau,X,Y)$ and $\gamma(\tau,X,Y)$ satisfy the equations (3.28-3.39).

To find $\Delta T(\tau,X,Y), h_1(\tau,X,Y), h_2(\tau,X,Y), U_F(\tau,X,Y), V_F(\tau,X,Y)$ and $\gamma(\tau,X,Y)$ in different situations.

We first ignore the suction or injection effects i.e. $a_6 = 0$.

From the expression for a's, we have

$$a_1 + a_2 = \gamma^2\left(\frac{U_F}{h_1}\right)_X \tag{3.40}$$

similarly,

$$a_4 + a_5 = \gamma^2\left(\frac{V_F}{h_2}\right)_Y \tag{3.41}$$

From (3.28a), we have
$$a_0 = \left(\frac{\gamma^2 U_F}{h_1}\right)_X$$

$$\Rightarrow\ 2\gamma\gamma_X = \frac{h_1}{U_F}(a_0 - a_1 - a_2) \tag{3.42}$$

Again from (3.29b) , we have
$$a_3 = \left(\frac{\gamma^2 V_F}{h_2}\right)_Y$$

$$\Rightarrow\ 2\gamma\gamma_Y = \frac{h_2}{V_F}(a_3 - a_4 - a_5) \tag{3.43}$$

By virtue of equation (3.28a), we get

$$\frac{\gamma^2 U_F}{h_1} = a_0 X + A(Y,\tau) \tag{3.44}$$

where $A(Y,\tau)$ is either constant or function of Y and τ. Differentiating (3.44) with respect to Y and in view of similarity requirements, One obtains

$$\frac{\partial A(Y,\tau)}{\partial Y} = \frac{h_2}{h_1}\frac{U_F}{V_F}(a_3 - a_4 - a_5 + a_9 - a_{10}) \tag{3.45}$$

Again, differentiating (3.44) with respect to τ

$$\frac{\partial A(Y,\tau)}{\partial \tau} = \frac{U_F}{h_1}(a_{12} + 2a_7 - a_{23}) \text{ [using constant a's]} \tag{3.46}$$

where
$$a_{23} = \frac{\gamma^2}{h_1} h_{1\tau} \tag{3.47}$$

Similarly, in view of equation (3.29b), we get

$$\frac{\gamma^2 V_F}{h_2} = a_3 Y + B(\tau, X) \tag{3.48}$$

where B is either constant or function of τ and X. Differentiating (3.48) with respect to X and τ respectively and in view similarity requirements. We obtain

$$\frac{\partial B(\tau,X)}{\partial X} = \left(\frac{\gamma^2 V_F}{h_2}\right)_X$$

$$\Rightarrow \frac{\partial B(\tau,X)}{\partial X} = \frac{h_1 V_F}{h_2 U_F}(a_0 - a_1 - a_2 - a_{11} + a_{15}) \tag{3.49}$$

and
$$\frac{\partial B(\tau,X)}{\partial \tau} = \left(\frac{\gamma^2 V_F}{h_2}\right)_\tau$$

$$\Rightarrow \frac{\partial B(\tau,X)}{\partial \tau} = \frac{V_F}{h_2}(2a_7 + a_{18} - a_{24}) \tag{3.50}$$

where
$$a_{24} = \frac{\gamma^2 h_{2\tau}}{h_2} \tag{3.51}$$

By virtue of (3.31b), we get $a_7 = \gamma\gamma_\tau$

$$\Rightarrow \gamma^2 = 2a_7\tau + C(X,Y) \tag{3.52}$$

where C is either constant or function of X and Y. Differentiating (3.52) with respect to X and Y respectively and in view of similarity requirments. We have

$$\frac{\partial C(X,Y)}{\partial X} = (\gamma^2)_X$$

$$\Rightarrow \frac{\partial C(X,Y)}{\partial X} = \frac{h_1}{U_F}(a_0 - a_1 - a_2) \tag{3.53}$$

and
$$\frac{\partial C(X,Y)}{\partial Y} = \left(\gamma^2\right)_Y$$

$$\Rightarrow \frac{\partial C(X,Y)}{\partial Y} = \frac{h_2}{V_P}(a_3 - a_4 - a_5) \tag{3.54}$$

Taking the product of (3.45), (3.46), (3.49), (3.50), (3.53) and (3.54), we get

$$\frac{\partial B(\tau,X)}{\partial \tau}\frac{\partial B(\tau,X)}{\partial X}\frac{\partial C(X,Y)}{\partial X}\frac{\partial C(X,Y)}{\partial Y}\frac{\partial A(Y,\tau)}{\partial Y}\frac{\partial A(Y,\tau)}{\partial \tau}$$

$$= (2a_7 + a_{18} - a_{24})(a_0 - a_1 - a_2 - a_{11} + a_{15})(a_0 - a_1 - a_2)(a_3 - a_4 - a_5)$$

$$(a_3 - a_4 - a_5 + a_9 - a_{10})(2a_7 + a_{12} - a_{23}) \tag{3.55}$$

The form of similarity solution, the scale factors $\Delta T(\tau,X,Y), h_1(\tau,X,Y), h_2(\tau,X,Y)$, $U_P(\tau,X,Y), V_P(\tau,X,Y)$ and $\gamma(\tau,X,Y)$ depend wholly on the equation (3.55). This situation leads to the following possibilities:

Case–1:
$$\frac{\partial B(\tau,X)}{\partial \tau} \neq 0, \frac{\partial B(\tau,X)}{\partial X} \neq 0 \frac{\partial C(X,Y)}{\partial X} \neq 0, \frac{\partial C(X,Y)}{\partial Y} \neq 0,$$
$$\frac{\partial A(Y,\tau)}{\partial Y} \neq 0, \frac{\partial A(Y,\tau)}{\partial \tau} \neq 0$$

Case–2:
$$\frac{\partial B(\tau,X)}{\partial \tau} \neq 0, \frac{\partial B(\tau,X)}{\partial X} \neq 0, \frac{\partial C(X,Y)}{\partial X} \neq 0, \frac{\partial C(X,Y)}{\partial Y} = 0,$$
$$\frac{\partial A(Y,\tau)}{\partial Y} = 0, \frac{\partial A(Y,\tau)}{\partial \tau} \neq 0$$

Case–3:
$$\frac{\partial B(\tau,X)}{\partial \tau} = 0, \frac{\partial B(\tau,X)}{\partial X} \neq 0, \frac{\partial C(X,Y)}{\partial X} \neq 0, \frac{\partial C(X,Y)}{\partial Y} = 0,$$
$$\frac{\partial A(Y,\tau)}{\partial Y} = 0, \frac{\partial A(Y,\tau)}{\partial \tau} = 0$$

Case–4:
$$\frac{\partial B(\tau,X)}{\partial \tau} \neq 0, \frac{\partial B(\tau,X)}{\partial X} = 0, \frac{\partial C(X,Y)}{\partial X} = 0, \frac{\partial C(X,Y)}{\partial Y} \neq 0,$$
$$\frac{\partial A(Y,\tau)}{\partial Y} \neq 0, \frac{\partial A(Y,\tau)}{\partial \tau} \neq 0$$

Case–5: $$\frac{\partial B(\tau,X)}{\partial\tau}=0,\frac{\partial B(\tau,X)}{\partial X}=0,\frac{\partial C(X,Y)}{\partial X}=0,\frac{\partial C(X,Y)}{\partial Y}\neq 0,$$
$$\frac{\partial A(Y,\tau)}{\partial Y}\neq 0,\frac{\partial A(Y,\tau)}{\partial\tau}=0$$

Case–6: $$\frac{\partial B(\tau,X)}{\partial\tau}=0,\frac{\partial B(\tau,X)}{\partial X}\neq 0,\frac{\partial C(X,Y)}{\partial X}\neq 0,\frac{\partial C(X,Y)}{\partial Y}\neq 0,$$
$$\frac{\partial A(Y,\tau)}{\partial Y}\neq 0,\frac{\partial A(Y,\tau)}{\partial\tau}=0$$

Case–7: $$\frac{\partial B(\tau,X)}{\partial\tau}\neq 0,\frac{\partial B(\tau,X)}{\partial X}=0,\frac{\partial C(X,Y)}{\partial X}=0,\frac{\partial C(X,Y)}{\partial Y}=0,$$
$$\frac{\partial A(Y,\tau)}{\partial Y}=0,\ \frac{\partial A(Y,\tau)}{\partial\tau}\neq 0$$

Case–8: $$\frac{\partial B(\tau,X)}{\partial\tau}=0,\frac{\partial B(\tau,X)}{\partial X}=0,\frac{\partial C(X,Y)}{\partial X}=0,\frac{\partial C(X,Y)}{\partial Y}=0,$$
$$\frac{\partial A(Y,\tau)}{\partial Y}=0,\frac{\partial A(Y,\tau)}{\partial\tau}=0$$

Chapter-4

Study of some possible similarity cases

Case-1: $\frac{\partial B(\tau,X)}{\partial \tau} \neq 0, \frac{\partial B(\tau,X)}{\partial X} \neq 0 \frac{\partial C(X,Y)}{\partial X} \neq 0, \frac{\partial C(X,Y)}{\partial Y} \neq 0,$

$$\frac{\partial A(Y,\tau)}{\partial Y} \neq 0, \frac{\partial A(Y,\tau)}{\partial \tau} \neq 0$$

$$\text{Let } h_1 = h_2$$

$$\text{Let, } \frac{\partial A(Y,\tau)}{\partial Y} = \text{constant}$$

$$\therefore \quad \frac{\partial A(Y,\tau)}{\partial Y} = \frac{U_F}{V_F}(a_3 - a_4 - a_5 + a_9 - a_{10})$$

$$= k_1 l_1 \tag{4.1}$$

where $k_1 = \frac{U_F}{V_F}$, $l_1 = a_3 - a_4 - a_5 + a_9 - a_{10}$

$$\frac{\partial B(\tau,X)}{\partial \tau} = \text{constant}$$

$$\therefore \quad \frac{\partial B(\tau,X)}{\partial \tau} = \frac{V_F}{h_1}(2a_7 + a_{18} - a_{24})$$

$$= k_2 l_2 \tag{4.2}$$

where $k_2 = \frac{V_F}{h_1}$, $l_2 = 2a_7 + a_{18} - a_{24}$

$$\frac{\partial C(X,Y)}{\partial X} = \text{constant}$$

$$\therefore \quad \frac{\partial C(X,Y)}{\partial X} = \frac{h_1}{U_F}(a_0 - a_1 - a_2)$$

$$= k_3 l_3 \tag{4.3}$$

where $k_3 = \frac{h_1}{U_F}$ and $l_3 = a_0 - a_1 - a_2$

$$\frac{\partial C(X,Y)}{\partial X} = \text{constant}$$

$$\therefore \quad \frac{\partial C(X,Y)}{\partial X} = \frac{h_1}{V_F}(a_3 - a_4 - a_5)$$

$$= \frac{1}{k_2} l_4 \tag{4.4}$$

where $l_4 = a_3 - a_4 - a_5$

$$\frac{\partial A(X,\tau)}{\partial \tau} = \text{constant}$$

$$\therefore \quad \frac{\partial A(Y,\tau)}{\partial \tau} = \frac{U_F}{h_1}(2a_7 + a_{12} - a_{23})$$

$$= \frac{1}{k_3} l_5 \tag{4.5}$$

where $l_5 = 2a_7 + a_{12} - a_{23}$

and
$$\frac{\partial B(\tau,X)}{\partial X} = \text{constant}$$

$$\therefore \quad \frac{\partial B(\tau,X)}{\partial X} = \frac{V_F}{U_F}(a_0 - a_1 - a_2 - a_{11} + a_{15})$$

$$= \frac{1}{k_1} l_6 \tag{4.6}$$

where $l_6 = a_0 - a_1 - a_2 - a_{11} + a_{15}$

Integrating equations(4.1), (4.2), (4.3), (4.4), (4.5) and (4.6), we get

$$\left.\begin{array}{l} A(Y,\tau) = k_1 l_1 Y + A_0(\tau) \\ B(\tau,X) = k_2 l_2 \tau + B_0(X) \\ C(X,Y) = k_3 l_3 X + C_0(Y) \\ C(X,Y) = \dfrac{l_4}{k_2} Y + C_0(X) \\ A(Y,\tau) = \dfrac{l_5}{k_3} \tau + A_0(Y) \\ B(\tau,X) = \dfrac{l_6}{k_1} X + B_0(\tau) \end{array}\right\} \tag{4.7}$$

Taking linear combination of (4.7), we obtain

$$\left.\begin{aligned} A(Y,\tau) &= k_1 l_1 Y + \frac{l_5}{k_3}\tau + A_0 \\ B(\tau,X) &= k_2 l_2 \tau + \frac{l_6}{k_1} X + B_0 \\ C(X,Y) &= k_3 l_3 X + \frac{l_4}{k_2} Y + C_0 \end{aligned}\right\} \tag{4.8}$$

Again integrating equations (3.28a), (3.29b), (3.31b) and in view of equation (4.8), we get

$$\frac{\gamma^2 U_F}{h_1} = a_0 X + k_1 l_1 Y + \frac{l_5}{k_3}\tau + A_0 \tag{4.9}$$

$$\frac{\gamma^2 V_F}{h_1} = \frac{l_6}{k_1} X + a_3 Y + k_2 l_2 \tau + B_0 \tag{4.10}$$

$$\text{and } \gamma^2 = k_3 l_3 X + \frac{l_4}{k_2} Y + 2a_7 \tau + C_0 \tag{4.11}$$

From equation (4.9)

$$\gamma^2 = a_0 k_3 X + k_1 k_3 l_1 Y + l_5 \tau + A_0 \tag{4.12}$$

From equation (4.10)

$$\gamma^2 = l_6 \frac{1}{k_1 k_2} X + a_3 \frac{1}{k_2} Y + l_2 \tau + B_0 \tag{4.13}$$

From equations (4.11), (4.12) and (4.13), We have to write

$$a_0 = l_3 = l_6,\ a_3 = l_4 = l_1,\ 2a_7 = l_5 = l_2$$

$$\Rightarrow a_1 = -a_2,\ a_{11} = a_{15},\ a_4 = -a_5,\ a_9 = a_{10},\ a_{12} = a_{23},\ a_{18} = a_{24}$$

Hence

$$\gamma^2 = a_0 k_3 X + a_3 k_1 k_3 Y + 2a_7 \tau + A_0 \tag{4.14}$$

From equation (4.9), we have

$$\frac{\gamma^2 U_F}{h_1} = \frac{a_0 k_3 X + k_1 k_3 l_1 Y + l_5 \tau + A_0 k_3}{k_3}$$

$$= \frac{a_0 k_3 X + k_1 k_3 a_3 Y + 2a_7 \tau + A_0 k_3}{k_3} \tag{4.15}$$

By virtue of equation (3.36a), we get $a_{16}=\frac{\gamma^2 U_F}{h_1}\frac{h_{1X}}{h_1}$, since $h_1=h_2$

$$\Rightarrow \frac{h_{1X}}{h_1}=\frac{h_1}{\gamma^2 U_F}a_{16}$$

$$=\frac{k_3 a_{16}}{a_0k_3X+k_1k_3a_3+2a_7\tau+A_0k_3}$$

$$\Rightarrow h_1=b_1(a_0k_3X+k_1k_3a_3Y+2a_7\tau+C_1)^{\frac{a_{16}}{a_0}}$$

where b_1 is constant of integration.

$$\therefore\ h_1=b_1(a_0k_3X+k_1k_3a_3Y+2a_7\tau+C_1)^m \quad \text{where } \frac{a_{16}}{a_0}=m.$$

$$\therefore\ h_2=h_1=b_1(a_0k_3X+k_1k_3a_3Y+2a_7\tau+C_1)^m \tag{4.16}$$

Now using (4.15) and (4.17) in (4.9), we get

$$\frac{\gamma^2 U_F}{h_1}=a_0X+k_1a_3Y+\frac{2a_7}{k_3}\tau+A_0$$

$$\Rightarrow U_F=\frac{b_1}{k_1}(a_0k_3X+k_1k_3a_3Y+2a_7\tau+C_1)^m \tag{4.17}$$

Similarly substituting (4.15) and (4.17) in (4.10), we get

$$V_F=b_1k_2(a_0k_3X+k_1k_3a_3Y+2a_7\tau+C_1)^m \tag{4.18}$$

Substituting, the values of γ^2, h_1, h_2, U_F and V_F, we get the values of a's i.e.

a_0, a_3 and a_7 are arbitrary

$a_1=2ma_0$ $\quad a_2=-2ma_0$ $\quad a_4=2ma_3$ $\quad a_5=-2ma_3$

$a_6=0$ $\quad a_8=ma_0$ $\quad a_9=ma_3$ $\quad a_{10}=ma_3$

$a_{11}=m\frac{a_0}{k_1^2}$ $\quad a_{12}=2ma_7$ $\quad a_{13}=\frac{k_3}{b_1^2}(a_0k_3X+a_3k_1k_3Y+2a_7\tau+C_1)^{1-2m}\beta_l\Delta Tg_X$

$a_{14}=ma_3$ $\quad a_{15}=ma_0$ $\quad a_{16}=ma_0$ $\quad a_{17}=ma_3k_1^2$ $\quad$ (4.19)

$a_{18}=2ma_7$ $\quad a_{19}=\frac{1}{b_1^2k_2}(a_0k_3X+a_3k_1k_3Y+2a_7\tau+C_1)^{1-2m}\beta_T\Delta Tg_Y$

$a_{20}=(2m-1)a_0$ $\quad a_{21}=(2m-1)a_3$ $\quad a_{22}=2(2m-1)a_7$

Hence the transform equations (3.25), (3.26) and (3.27) reduce to

$$\nu\overline{F}_{\bar\phi\bar\phi\bar\phi}+\left(\frac{4m+1}{2}\right)a_0\overline{F}\overline{F}_{\bar\phi\bar\phi}+\left(\frac{4m+1}{2}\right)a_3\overline{S}\overline{F}_{\bar\phi\bar\phi}+a_7\overline{\phi}\overline{F}_{\bar\phi\bar\phi}-ma_0\overline{F}_{\bar\phi}^2$$

$$-2ma_3\overline{F}_{\bar\phi}\overline{S}_{\bar\phi}+ma_0\frac{1}{k_1^2}\overline{S}_{\bar\phi}^2-2ma_7\overline{F}_{\bar\phi}+a_{13}\overline{\theta}=0 \tag{4.20}$$

$$\nu\overline{S}_{\bar\phi\bar\phi\bar\phi}+\left(\frac{4m+1}{2}\right)a_3\overline{S}\overline{S}_{\bar\phi\bar\phi}+\left(\frac{4m+1}{2}\right)a_0\overline{F}\overline{S}_{\bar\phi\bar\phi}+a_7\overline{\phi}\overline{S}_{\bar\phi\bar\phi}-ma_3\overline{S}_{\bar\phi}^2$$

$$-2ma_3\overline{F}_{\bar\phi}\overline{S}_{\bar\phi}+ma_3k_1^2\overline{F}_{\bar\phi}^2-2ma_7\overline{S}_{\bar\phi}+a_{19}\overline{\theta}=0 \tag{4.21}$$

and
$$\frac{\nu}{\Pr}\overline{\theta}_{\bar\phi\bar\phi}+\left(\frac{4m+1}{2}\right)a_0\overline{F}\overline{\theta}_{\bar\phi}+\left(\frac{4m+1}{2}\right)a_3\overline{S}\overline{\theta}_{\bar\phi}+a_7\overline{\phi}\overline{\theta}_{\bar\phi}-$$

$$(2m-1)(a_0\overline{F}_{\bar\phi}+a_3\overline{S}_{\bar\phi})\overline{\theta}-2(2m-1)a_7\theta=0 \tag{4.22}$$

In order to simplify the above type of equation we substitute

$$\overline{F}=\alpha f,\ \overline{S}=\alpha s,\ \overline{\phi}=\alpha\phi,\ \overline{\theta}=\theta$$

Thus the above equation changed to

$$f_{\phi\phi\phi}+\left(\frac{4m+1}{2}\right)\frac{a_0\alpha^2}{\nu}ff_{\phi\phi}+\left(\frac{4m+1}{2}\right)\frac{a_3\alpha^2}{\nu}sf_{\phi\phi}+\frac{a_7\alpha^2}{\nu}\phi f_{\phi\phi}-m\frac{a_0\alpha^2}{\nu}f_\phi^2$$

$$-2m\frac{a_3\alpha^2}{\nu}f_\phi s_\phi+m\frac{a_0\alpha^2}{\nu}\frac{1}{k_1^2}s_\phi^2-2m\frac{a_7\alpha^2}{\nu}f_\phi+a_{13}\frac{\alpha^2}{\nu}\theta=0 \tag{4.23}$$

$$s_{\phi\phi\phi}+\left(\frac{4m+1}{2}\right)\frac{a_3\alpha^2}{\nu}ss_{\phi\phi}+\left(\frac{4m+1}{2}\right)\frac{a_0\alpha^2}{\nu}fs_{\phi\phi}+\frac{a_7\alpha^2}{\nu}\phi s_{\phi\phi}-m\frac{a_3\alpha^2}{\nu}s_\phi^2$$

$$-2m\frac{a_0\alpha^2}{\nu}f_\phi s_\phi+m\frac{a_3\alpha^2}{\nu}k_1^2f_\phi^2-2m\frac{a_7\alpha^2}{\nu}s_\phi+a_{19}\frac{\alpha^2}{\nu}\theta=0 \tag{4.24}$$

and $$\Pr^{-1}\theta_{\phi\phi}+\left(\frac{4m+1}{2}\right)\frac{a_0\alpha^2}{\nu}f\theta_\phi+\left(\frac{4m+1}{2}\right)\frac{a_3\alpha^2}{\nu}s\theta_\phi+\frac{a_7\alpha^2}{\nu}\phi\theta_\phi$$

$$-(2m-1)\left\{\frac{a_0\alpha^2}{\nu}f_\phi+\frac{a_3\alpha^2}{\nu}s_\phi\right\}\theta-2(2m-1)\frac{a_7\alpha^2}{\nu}\theta=0 \tag{4.25}$$

Choosing $\left(\frac{4m+1}{2}\right)\frac{a_0\alpha^2}{\nu}=1$ and writing $\frac{a_3}{a_0}=c, \frac{a_7}{a_0}=d, \frac{2m}{4m+1}=\beta.$

Also for purely free convection we have to put

$$\frac{2}{4m+1}\frac{a_{13}}{a_0}=1 \text{ and } \frac{2}{4m+1}\frac{a_{19}}{a_0}=R \text{ (constant).}$$

Finally, we get the equations (4.23), (4.24) and (4.25).

$$f_{\phi\phi\phi}+(f+cs)f_{\phi\phi}+(2-4\beta)d\phi f_{\phi\phi}-\beta\{(f_\phi+2cs_\phi)f_\phi$$
$$+\frac{1}{k_1^2}s_\phi^2-2df_\phi\}+\theta=0 \quad (4.26)$$

$$s_{\phi\phi\phi}+(f+cs)s_{\phi\phi}+(2-4\beta)d\phi s_{\phi\phi}-\beta\{(2f_\phi+cs_\phi)s_\phi$$
$$+k_1^2cf_\phi^2-2ds_\phi\}+R\theta=0 \quad (4.27)$$

and $\Pr^{-1}\theta_{\phi\phi}+(f+cs)\theta_\phi+(2-4\beta)d\phi\theta_\phi-(6\beta-2)(f_\phi+cs_\phi)\theta$
$$-(12\beta-4)d\theta=0 \quad (4.28)$$

The boundary condition are

$$\left.\begin{array}{l} f(0)=f_\phi(0)=0, f_\phi(\infty)=0 \\ s(0)=s_\phi(0)=0, s_\phi(\infty)=0 \\ \theta(0)=1, \ \theta(\infty)=0 \end{array}\right\} \quad (4.29)$$

If U_P, V_P and h_1 be constant then c is proportional to the ratio of the change of local boundary layer thickness with respect to position on the both edges and d is proportional to the ratio of the change of local boundary layer thickness with respect to time and position. If characteristic length with respect to both the edges be same, the parameter R determine the $\left(\frac{3}{2}\right)$ rd root of the gravitational ratio develops.

For $\beta=1$, $c=0$, $d=0$, $s=f$ and $R=1$, the equations (4.26 - 4.28) with the boundary conditions coincide with possible similarity solutions for laminar free convection on vertical plates, analysed by **Yang** (1960) which was also identical with similar solutions for free convection from a Non-isothermal vertical plate discussed by **Sparrow** and **Gregg** (1958)

We have, in this case, the similarity requirements are

$$h_1 = b_1(a_0k_3X + k_1k_3a_3Y + 2a_7\tau + c_1)^m$$

$$= b_1a_0^mk_3^m\left(X + \frac{k_1a_3}{a_0}Y + \frac{2a_7}{a_0k_3}\tau + \frac{c_1}{a_0k_3}\right)^m$$

$$= \alpha_1(\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})^m$$

$$\therefore\ h_1 \propto (\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})^m$$

where

$$\bar{x} = X + X_0$$

$$\bar{y} = Y + Y_0$$

$$\bar{t} = \tau + \tau_0$$

$$\alpha_1 = b_1a_0^mk_3^m$$

$$\bar{b} = \frac{k_1a_3}{a_0}$$

$$\bar{c} = \frac{2a_7}{a_0k_3} = \frac{2d}{k_3} = 2k_1k_2d$$

$$\frac{c_1}{a_0k_3} = X_0 + \frac{k_1a_3}{a_0}Y_0 + \frac{2a_7}{a_0k_3}\tau_0$$

$\therefore\ U_F = \alpha_2(\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})^m$ where $\alpha_2 = b_1a_0^mk_3^{m-1}$

$U_F^2 \propto g_X\beta_T\Delta T$ (characteristic length L_1), where $L_1 = (\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})$

$\therefore\ V_F = \alpha_3(\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})^m$ where $\alpha_3 = b_1a_0^mk_2k_3^m$

$V_F^2 \propto g_Y\beta_T\Delta T$ (characteristic length L_2), where $L_2 = (\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})$

$\therefore\ \Delta T = \alpha_4(\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})^{2m-1}$ where $\alpha_4 = \dfrac{a_0^{2m}b_1^2k_3^{2m-2}(4m+1)}{2\beta_Tg_X}$

$\Delta T \propto (\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})^{2m-1}$

$\gamma^2 = \alpha_5(\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})$ where $\alpha_5 = a_0k_3$

If $\bar{b} = \frac{k_1 a_3}{a_0^{\cdot}}$ is non-dimensional quantity, $k_1 = \frac{U_I}{V_F}$ be also non-dimensional. Since $k_1 k_2 k_3 = 1$, it implies that $k_2 k_3$ is non-dimensional. According to given definition in (4.2) k_2 is the scale of velocity and k_3 is too. Hence $k_2 k_3$ is dimensionless.

Here $\bar{c}$ is finally the scale of velocity and $\bar{c}\bar{t}$ is the additive length added to the normal characteristic length $\bar{x} + \bar{b}\bar{y}$ then formed new characteristic length.

The similarity variable ϕ is

$$\phi = \frac{z}{a\gamma} = \frac{z}{\sqrt{\frac{2\nu}{(4m+1)a_0}}\sqrt{a_0 k_3(\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})}}$$

$$= Gr_{\bar{x}\bar{y}\bar{t}}^{\frac{1}{4}} \frac{z}{\left(\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t}\right)}$$

where $$Gr_{\bar{x}\bar{y}\bar{t}}^{\frac{1}{4}} = \left\{ \frac{(4m+1)}{2h_1^2} \frac{g_X \beta_F \Delta T(\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})^3}{\nu^2} \right\}^{\frac{1}{4}}$$

is the modified Grashof number.

The velocity components are

$$u = U_F f_\phi(\phi)$$
$$v = V_F s_\phi(\phi)$$

and $$w = \frac{1}{h_1 h_2}\left\{-(h_2 \gamma U_F\right\}_X \bar{F} + \bar{\phi}\gamma_X h_2 U_F \bar{F}_{\bar{\phi}} - (h_1 \gamma V_F)_Y \bar{S} + \bar{\phi}\gamma_Y h_1 V_F \bar{S}_{\bar{\phi}}$$

$$= \sqrt{\frac{2\nu}{(4m+1)k_3}}(\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})^{-\frac{1}{2}}\left\{-\left(2m + \frac{1}{2}\right)(f + cs) + \frac{1}{2}\phi(f_\phi + cs_\phi)\right\}$$

Skin frictions are

$$\tau_{w1} = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0}$$

$$\frac{\tau_{w1}}{\frac{1}{2}\rho U_F^2} = (4m+1)\frac{Gr_{xyt}^{-\frac{1}{4}}}{h_1} f_{\phi\phi}(0)$$

$$\tau_{w2} = \mu\left(\frac{\partial v}{\partial z}\right)_{z=0}$$

$$\frac{\tau_{w2}}{\frac{1}{2}\rho V_F^2} = (4m+1)k_1 \frac{Gr_{xyt}^{-\frac{1}{4}}}{h_1} s_{\phi\phi}(0)$$

Heat flux,

$$q_w = -k\left(\frac{\partial T}{\partial z}\right)_{z=0}$$

$$= -k\Delta T\left\{\frac{(4m+1)a_0}{2\nu k_3}\right\}^{\frac{1}{2}} (\bar{x} + \bar{b}\bar{y} + \bar{c}\bar{t})^{-\frac{1}{2}} \theta_\phi(0).$$

Case-2:

$$\frac{\partial A(Y,\tau)}{\partial Y}=0,\quad \frac{\partial B(\tau,X)}{\partial \tau}\neq 0,\quad \frac{\partial C(X,Y)}{\partial X}\neq 0$$

$$\frac{\partial C(X,Y)}{\partial Y}=0,\quad \frac{\partial A(Y,\tau)}{\partial \tau}\neq 0,\quad \frac{\partial B(\tau,X)}{\partial X}\neq 0$$

Let $h_1=h_2$

Let, $\frac{\partial A(Y,\tau)}{\partial Y}\neq$ constant.

$$\Rightarrow \quad \frac{U_F}{V_F}=k_1\neq \text{const},\ a_3-a_4-a_5+a_9-a_{10}=0=l_1 \tag{4.30}$$

$$\frac{\partial B(\tau,X)}{\partial \tau}=\text{constant}$$

$$=\frac{V_F}{h_1}(2a_7+a_{18}-a_{24})$$

$$=k_2 l_2 \tag{4.31}$$

$$\frac{\partial C(X,Y)}{\partial X}=\text{constant}$$

$$=\frac{h_1}{U_F}(a_0-a_1-a_2)$$

$$=k_3 l_3 \tag{4.32}$$

$$\frac{\partial C(X,Y)}{\partial Y}\neq \text{constant}$$

$$\Rightarrow \quad \frac{h_1}{V_F}=\frac{1}{k_2}\neq \text{const},\ a_3-a_4-a_5=0=l_4 \tag{4.33}$$

$$\frac{\partial A(Y,\tau)}{\partial \tau}=\text{constant}$$

$$=\frac{U_F}{h_1}(2a_7+a_{12}-a_{21})$$

$$=\frac{1}{k_3}l_5 \tag{4.34}$$

and $$\frac{\partial B(\tau, X)}{\partial X} = \text{constant}$$

$$= \frac{V_F}{U_F}(a_0 - a_1 - a_2 - a_{11} + a_{15})$$

$$= \frac{1}{k_1} l_6 \qquad (4.35)$$

By virtue of equation (4.8), we obtain

$$\left.\begin{aligned} A(Y,\tau) &= \frac{l_5}{k_3}\tau + A_0 \\ B(\tau, X) &= k_2 l_2 + \frac{l_6}{k_1} X + B_0 \\ C(X,Y) &= k_3 l_3 X + C_0 \end{aligned}\right\} \qquad (4.36)$$

In view of equations (4.11), (4.12) and (4.13), we get

$$\gamma^2 = k_3 l_3 X + 2a_7\tau + C_0 \qquad (4.37)$$

$$\gamma^2 = a_0 k_3 X + l_5\tau + A_0 \qquad (4.38)$$

and $\gamma^2 = l_5 \frac{1}{k_1 k_2} X + a_3 \frac{1}{k_2} Y + l_2\tau + B_0$, setting $a_3 = 0$.

$$\therefore \quad \gamma^2 = l_6 \frac{1}{k_1 k_2} X + l_2\tau + B_0 \qquad (4.39)$$

Comparing the above three equations for γ^2, we have to write,

$$a_0 = l_3 = l_6, \quad 2a_7 = l_5 = l_2 \text{ and } A_0 = B_0 = C_0.$$

$$\Rightarrow \qquad a_1 = -a_2,\ a_{11} = a_{15},\ a_{12} = a_{23},\ a_{18} = a_{24}.$$

Hence γ^2 is found to be

$$\gamma^2 = a_0 k_3 X + 2a_7\tau + A_0 k_3 \qquad (4.40)$$

In view of equation (3.44) $$\frac{\gamma^2 U_F}{h_1} = a_0 X + A(Y,\tau)$$

$$= a_0 X + \frac{l_5}{k_3}\tau + A_0$$

$$= \frac{a_0 k_3 X + 2a_7\tau + A_0 k_3}{k_3} \tag{4.41}$$

By virtue of equation (3.36a), we get

$$a_{16} = \frac{\gamma^2 U_F}{h_1} \frac{h_1}{h_{1X}}$$

$$\Rightarrow \quad \frac{h_{1X}}{h_1} = \frac{k_3 a_{16}}{a_0 k_3 X + 2a_7\tau + A_0 k_3}$$

$$\therefore h_1 = b_1 (a_0 k_3 X + 2a_7\tau + A_0 k_3)^{\frac{a_{16}}{a_0}}$$

when b_1 is constant of integration

$$\therefore h_2 = h_1 = b_1 (a_0 k_3 X + 2a_7\tau + A_0 k_3)^m, \text{ where } m = \frac{a_{16}}{a_0} \tag{4 42}$$

Again, in view of equation (3.48), we get

$$\frac{\gamma^2 V_F}{h_1} = a_3 Y + B(X,\tau)$$

$$= k_2 l_2 \tau + \frac{l_6}{k_1} X + B_0$$

$$= \frac{a_0}{k_1} X + 2a_7 k_2 \tau + B_0$$

$$= \frac{a_0 k_3 X + 2a_7\tau + B_0 k_1 k_3}{k_1 k_3} \tag{4.43}$$

From (4 41), we have

$$\frac{\gamma^2 U_F}{h_1} = \frac{a_0 k_3 X + 2a_7\tau + A_0 k_3}{k_3}$$

$$U_F = \frac{b_1}{k_3} (a_0 k_3 X + 2a_7\tau + A_0 k_3)^m \tag{4.44}$$

From (4.43), we have

$$\frac{\gamma^2 V_F}{h_2} = \frac{a_0 k_3 X + 2a_7\tau + B_0 k_1 k_3}{k_1 k_3}$$

$$\therefore \; V_F = b_1 k_2 (a_0 k_3 X + 2a_7\tau + A_0 k_3)^m \qquad (4.45)$$

The similarity requirements furnish us with the relations between the constants (a's). The relations are, a_0, a_7 are arbitrary,

$a_1 = 2ma_0$, $\quad a_2 = -2ma_0$, $\quad a_3 = a_4 = a_5 = a_6 = 0$, $\quad a_8 = ma_0$

$a_9 = a_{10} = 0$, $\quad a_{11} = \dfrac{ma_0}{k_1^2}$, $\quad a_{12} = 2ma_7$

$a_{13} = \dfrac{k_3}{b_1^2}(a_0 k_3 X + 2a_7\tau + A_0 k_3)^{1-2m} \beta_T \Delta T g_X$, $\quad a_{14} = 0$, $\quad a_{15} = ma_0$

$a_{16} = ma_0$, $\quad a_{17} = 0$, $\quad a_{18} = 2ma_7$,

$a_{19} = \dfrac{1}{b_1^2 k_2}(a_0 k_3 X + 2a_7\tau + A_0 k_3)^{1-2m} \beta_T \Delta T g_Y$, $\quad a_{20} = (2m-1)a_0$

$a_{21} = 0$, $\quad a_{22} = 2(2m-1)a_7$.

Hence the general equations (3.25-3.26) reduce to

$$\nu \overline{F}_{\bar\phi\bar\phi\bar\phi} + \frac{4m+1}{2}\overline{F}\,\overline{F}_{\bar\phi\bar\phi} + a_7\bar\phi\,\overline{F}_{\bar\phi\bar\phi} - ma_0\overline{F}_{\bar\phi}^2 + \frac{ma_0}{k_1^2}\overline{S}_{\bar\phi}^2 - 2ma_7\overline{F}_{\bar\phi} + a_{13}\overline{\theta} = 0$$

$$\nu \overline{S}_{\bar\phi\bar\phi\bar\phi} + \frac{4m+1}{2}a_0\overline{S}_{\bar\phi\bar\phi}\overline{F} + a_7\bar\phi\overline{S}_{\bar\phi\bar\phi} - 2ma_0\overline{F}_{\bar\phi}\overline{S}_{\bar\phi} - 2ma_7\overline{S}_{\bar\phi} + a_{13}\overline{\theta} = 0$$

and $$\frac{\nu}{\Pr}\overline{\theta}_{\bar\phi\bar\phi} + \frac{4m+1}{2}a_0\overline{F}\,\overline{\theta}_{\bar\phi} + a_7\bar\phi\,\overline{\theta}_{\bar\phi} - (2m-1)a_0\overline{F}_{\bar\phi}\theta - 2(2m-1) + a_7\overline{\theta} = 0$$

Subject to boundary conditions

$$\overline{F}(0) = \overline{F}_{\bar\phi}(0) = 0,\; \overline{F}_{\bar\phi}(\infty) = 0$$

$$\overline{S}(0) = \overline{S}_{\bar\phi}(0) = 0,\; \overline{S}_{\bar\phi}(\infty) = 0$$

for the dimensionless stream function and

$$\overline{\theta}(0) = 1,\; \overline{\theta}(\infty) = 0$$

for the dimensionless temperature function.

Let us now substitute

$$\overline{F} = \alpha f,\ \overline{S} = \alpha s,\ \overline{\phi} = \alpha\phi,\ \overline{\theta} = \theta$$

in the above equations. Thus the above equations changed to

$$f_{\phi\phi\phi} + \left(\frac{4m+1}{2}\right)\frac{a_0\alpha^2}{\nu} ff_{\phi\phi} + \frac{a_7\alpha^2}{\nu}\phi f_{\phi\phi} - m\frac{a_0\alpha^2}{\nu} f_\phi^2 + \frac{m}{k_1^2}\frac{a_0\alpha^2}{\nu} s_\phi^2 - 2m\frac{a_7\alpha^2}{\nu} f_\phi + a_{13}\frac{\alpha^2}{\nu}\theta = 0$$

$$s_{\phi\phi\phi} + \left(\frac{4m+1}{2}\right)\frac{a_0\alpha^2}{\nu} fs_{\phi\phi} + \frac{a_7\alpha^2}{\nu}\phi s_{\phi\phi} - 2m\frac{a_0\alpha^2}{\nu} f_\phi s_\phi - 2m\frac{a_7\alpha^2}{\nu} s_\phi + a_{19}\frac{\alpha^2}{\nu}\theta = 0$$

and

$$\Pr^{-1}\theta_{\phi\phi} + \left(\frac{4m+1}{2}\right)\frac{a_0\alpha^2}{\nu} f\theta_\phi + \frac{a_7\alpha^2}{\nu}\phi\theta_\phi - (2m-1)\frac{a_0\alpha^2}{\nu} f_\phi\theta - 2(2m-1)\frac{a_7\alpha^2}{\nu}\theta = 0$$

Choosing $\left(\frac{4m+1}{2}\right)\frac{a\alpha^2}{\nu} = 1$ and writing $\frac{a_7}{a_0} = c,\ \frac{2m}{4m+1} = \beta$. Also $\frac{2}{4m+1}\frac{a_{13}}{a_0} = 1$ and

$\frac{2}{4m+1}\frac{a_{19}}{a_0} = R$ for purely free convection.

Now, $\frac{2}{4m+1}\frac{a_{13}}{a_0} = 1, \qquad a_{13} = \frac{(4m+1)a_0}{2}$

$$\Rightarrow\ \frac{k_3}{b_1^2}(a_0k_3X + 2a_7\tau + A_0k_3)^{1-2m}\beta_T\Delta T g_X = \frac{4m+1}{2}a_0$$

$$\therefore\ \Delta T = \frac{b_1^2(4m+1)a_0}{2k_3\beta_t g_X}(a_0k_3X + 2a_7\tau + A_0k_3)^{1-2m}$$

We have finally the following similarity equations.

$$f_{\phi\phi\phi} + ff_{\phi\phi} + (2-4\beta)c\phi f_{\phi\phi} - \beta\left(f_\phi^2 - \frac{1}{k_1^2}s_\phi^2 + 2cf_\phi\right) + \theta = 0 \tag{4.46}$$

$$s_{\phi\phi\phi} + fs_{\phi\phi} + (2-4\beta)c\phi s_{\phi\phi} - 2\beta(f_\phi s_\phi + s_\phi) + \theta = 0 \tag{4.47}$$

and $$\Pr^{-1}\theta_{\phi\phi} + f\theta_\phi + (2-4\beta)c\phi\theta_\phi - (6\beta-2)f_\phi\theta - (12\beta-4)c\theta = 0 \tag{4.48}$$

The boundary conditions are

$$\left.\begin{aligned} &f(0) = f_\phi(0) = 0,\quad f_\phi(\infty) = 0 \\ &s(0) = s_\phi(0) = 0,\quad s_\phi(\infty) = 0 \\ &\theta(0) = 1,\quad \theta(\infty) = 0 \end{aligned}\right\} \tag{4.49}$$

We have, in this case, the similarity requirements are

$$h_1 = b_1(a_0k_3X + 2a_7\tau + A_0k_3)^m$$

$$= b_1a_0^mk_3^m\left(X + \frac{2a_7}{a_0k_3}\tau + \frac{A_0}{a_0}\right)^m$$

$$= \alpha_1(\bar{x} + \bar{c}\bar{t})^m$$

$$h_1 \propto (\bar{x} + \bar{c}\bar{t})^m$$

where $\bar{x} = X + X_0$

$$\bar{t} = \tau + \tau_0$$

$$\bar{c} = \frac{2a_7}{a_0k_3}$$

$$\frac{A_0}{a_0} = X_0 + \frac{2a_7}{a_0k_3}\tau_0$$

$$\alpha_0 = b_1a_0^mk_3^m$$

$\therefore$ $U_F = \alpha_2(\bar{x} + \bar{c}\bar{t})^m$, where $\alpha_2 = b_1k_3^{m-1}a_0^m$

$U_F^2 \propto g_X\beta_T\Delta T$ (characteristic length L_1), where $L_1 = (\bar{x} + \bar{c}\bar{t})$

$V_F = \alpha_3(\bar{x} + \bar{c}\bar{t})^m$, where $\alpha_3 = b_1k_2k_3^ma_0^m$

$V_F^2 \propto g_Y\beta_T\Delta T$ (characteristic length L_2), where $L_2 = (\bar{x} + \bar{c}\bar{t})$

$\Delta T = \alpha_4(\bar{x} + \bar{c}\bar{t})^{2m-1}$, where $\alpha_4 = \dfrac{b_1^2(4m+1)a_0^{2m}}{2k_3^{2-2m}\beta_Tg_X}$

$$\Delta T \propto (\bar{x} + \bar{c}\bar{t})^{2m-1}.$$

$\therefore$ $\gamma^2 = \alpha_5(\bar{x} + \bar{c}\bar{t})$, where $\alpha_5 = a_0k_3$

The similarity variable ϕ is,

$$\phi = \frac{z}{\alpha\gamma} = \frac{z}{\sqrt{\dfrac{2\nu}{(4m+1)a_0}\sqrt{\alpha_5(\bar{x} + \bar{c}\bar{t})}}}$$

$= (Gr_{\bar{x}\bar{t}})^{\frac{1}{4}}\dfrac{z}{(\bar{x} + \bar{c}\bar{t})}$ where $Gr_{\bar{x}\bar{t}}^{\frac{1}{4}} = \left[\dfrac{4m+1}{2h_1^2}\dfrac{g_X\beta_T\Delta T(\bar{x} + \bar{c}\bar{t})^3}{\nu^2}\right]$ is the modified Grashof number.

The velocity components

$$u = U_F f_\phi(\phi) \text{ where } U_F^2 = -g_X \beta_T \Delta T L_1$$

$$v = V_F s_\phi(\phi) \text{ where } V_F^2 = -g_Y \beta_T \Delta T L_2$$

and $w = \frac{1}{h_1 h_2}\left[-(h_2 \gamma U_F)_X \overline{F} + \overline{\phi}\gamma_X h_2 U_F \overline{F}_{\overline{\phi}}\right]$

$$= \left\{\frac{2\nu}{k_3(4m+1)(\overline{x}+\overline{c}\overline{t})}\right\}^{\frac{1}{2}}\left[\frac{1}{2}\phi f_\phi - \left(2m+\frac{1}{2}\right)f\right]$$

Skin frictions are

$$\tau_{w1} = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0}$$

$$\frac{\tau_{w1}}{\frac{1}{2}\rho U_F^2} = (4m+1)\frac{Gr_{xt}^{-\frac{1}{4}}}{h_1} f_{\phi\phi}(0)$$

$$\tau_{w2} = \mu\left(\frac{\partial v}{\partial z}\right)_{z=0}$$

$$\frac{\tau_{w2}}{\frac{1}{2}\rho V_F^2} = (4m+1)k_1\frac{Gr_{xt}^{-\frac{1}{4}}}{h_1} s_{\phi\phi}(0)$$

Heat flux, $q_w = -k\left(\frac{\partial T}{\partial z}\right)_{z=0}$

$$= -k\Delta T\left(\frac{4m+1}{2\nu}\right)\left(\frac{a_0}{k_3}\right)^{\frac{1}{2}}(\overline{x}+\overline{c}\overline{t})^{-\frac{1}{2}}\theta_\phi(0).$$

Case-4:

$$\frac{\partial A(Y,\tau)}{\partial Y}\neq 0,\ \frac{\partial B(\tau,X)}{\partial \tau}\neq 0,\ \frac{\partial C(X,Y)}{\partial X}=0$$

$$\frac{\partial C(X,Y)}{\partial Y}\neq 0,\ \frac{\partial A(Y,\tau)}{\partial \tau}\neq 0,\ \frac{\partial B(\tau,X)}{\partial X}=0$$

Choosing $h_1 = h_2$,

$$\text{Let, } \frac{\partial A(Y,\tau)}{\partial Y} = \text{constant.}$$

$$=\frac{U_F}{V_F}(a_3 - a_4 - a_5 + a_9 - a_{10})$$

$$= k_1 l_1 \tag{4.50}$$

$$\frac{\partial B(\tau,X)}{\partial \tau} = \text{constant}$$

$$=\frac{V_F}{h_1}(2a_7 + a_{18} - a_{24})$$

$$= k_2 l_2 \tag{4.51}$$

$$\frac{\partial C(X,Y)}{\partial X} \neq \text{constant}$$

$$\Rightarrow \quad \frac{h_1}{U_F} = k_3 \neq \text{constant}, a_0 - a_1 - a_2 = 0 = l_3 \tag{4.52}$$

$$\frac{\partial C(X,Y)}{\partial Y} = \text{constant}$$

$$=\frac{h_1}{V_F}(a_3 - a_4 - a_5)$$

$$=\frac{1}{k_2} l_4 \tag{4.53}$$

$$\frac{\partial A(Y,\tau)}{\partial \tau} = \text{constant}$$

$$=\frac{U_F}{h_1}(2a_7 + a_{12} - a_{23})$$

$$=\frac{1}{k_3} l_5 \tag{4.54}$$

and $\quad \dfrac{\partial B(\tau, X)}{\partial X} \neq$ constant

$$\Rightarrow \frac{V_F}{U_F} = \frac{1}{k_1} \neq \text{constant}, \; a_0 - a_1 - a_2 - a_{11} + a_{15} = 0 = l_6 \tag{4.55}$$

By virtue of equation (4.8), one obtains

$$\left.\begin{aligned} A(Y,\tau) &= k_1 l_1 Y + \frac{l_5}{k_3}\tau + A_0 \\ B(\tau, X) &= k_2 l_2 \tau + B_0 \\ C(X,Y) &= \frac{l_4}{k_2} Y + C_0 \end{aligned}\right\} \tag{4.56}$$

In view of equations (4.11), (4.12) and (4.13), we get

$$\gamma^2 = \frac{l_4}{k_2} Y + 2a_7\tau + C_0 \tag{4.57}$$

$$\gamma^2 = a_0 k_3 X + k_1 k_3 l_1 Y + l_5 \tau + A_0$$

without loss of any generality. Setting $a_0 = 0$

$$\therefore \gamma^2 = k_1 k_3 l_1 Y + l_5 \tau + A_0 \tag{4.58}$$

$$\text{and} \quad \gamma^2 = a_3 \frac{1}{k_2} Y + l_2 \tau + B_0 \tag{4.59}$$

Comparing the above three equations (4.57- 4.59) for γ^2, we have to write,

$$a_3 = l_1 = l_4, \quad 2a_7 = l_5 = l_2$$

$$\Rightarrow \qquad a_9 = a_{10}, a_4 = -a_5, a_{12} = a_{23}, \; a_{18} = a_{24}.$$

Hence γ^2 is found to be

$$\gamma^2 = a_3 k_1 k_3 Y + 2a_7 \tau + A_0 k_3 \tag{4.60}$$

In view of equation (3.44), we have $\dfrac{\gamma^2 U_F}{h_1} = a_0 X + A(Y,\tau) \quad \because a_0 = 0$

$$= \frac{k_1 k_3 a_3 Y + 2a_7 \tau + A_0 k_3}{k_3} \text{ [By using (4.56)]} \tag{4.61}$$

Again, in view of equation (3.48), we have

$$\frac{\gamma^2 V_F}{h_1} = a_3 Y + B(X,\tau)$$

$$= a_3 Y + k_2 2a_7\tau + B_0 \text{ [By using (4.56)]}$$

$$= \frac{k_1 k_3 a_3 Y + 2a_7\tau + k_1 k_3 B_0}{k_1 k_3}$$

$$= \frac{k_1 k_3 a_3 Y + 2a_7\tau + k_3 A_0}{k_1 k_3} \tag{4.62}$$

By virtue of equation (3.33a), we get

$$\frac{h_{1Y}}{h_1'} = \frac{a_{10} h_1}{V_F Y^2}$$

$$= \frac{k_1 k_3 a_{10}}{k_1 k_3 a_3 Y + 2a_7\tau + k_3 A_0}$$

$$\therefore\ h_1 = b_1 (k_1 k_3 a_3 Y + 2a_7\tau + k_3 A_0)^{\frac{a_{10}}{a_3}}$$

where b_1 is the constant of integration.

$$\Rightarrow\ h_1 = h_2 = b_1 (k_1 k_3 a_3 Y + 2a_7\tau + k_3 A_0)^m \quad \text{where } m = \frac{a_{10}}{a_3} \tag{4.63}$$

From (4.61), we have

$$U_F = \frac{b_1}{k_3}(k_1 k_3 a_3 Y + 2a_7\tau + k_3 A_0)^m \tag{4.64}$$

Similarly, from (4.62), we get

$$V_F = \frac{b_1}{k_1 k_3}(k_1 k_3 a_3 Y + 2a_7\tau + k_3 A_0)^m \tag{4.65}$$

Substituting (4.60, 4 63-4 65) in the similarity requirements one may obtain the following relations between the constants (a's):

a_3, a_7 are arbitrary.

$a_0 = a_2 = a_2 = 0$, $\quad a_4 = 2ma_3$, $\quad a_5 = -2ma_3$, $\quad a_6 = a_8 = 0$

$a_{10} = ma_3$, $a_{11} = 0$, $\quad a_{12} = 2ma_7$

$a_{13} = \frac{k_3}{b_1^2}(k_1 k_3 a_3 Y + 2a_7\tau + k_3 A_0)^{1-2m} \beta_T \Delta T g_X$, $\quad a_{14} = ma_3$

$a_{15} = a_{16} = 0$, $\quad a_{17} = mk_1^2 a_3$, $\quad a_{18} = 2ma_7$,

$$a_{19} = \frac{k_1 k_3}{b_1^2}(k_1 k_3 a_3 Y + 2a_7 \tau + k_3 A_0)^{1-2m} \beta_T \Delta T g_Y , \qquad a_{20} = 0$$

$$a_{21} = (2m-1)a_3 , \qquad a_{22} = 2(2m-1)a_7 .$$

Furthermore, equations (3.25-3.27) reduce to

$$\nu \bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{4m+1}{2} a_3 \bar{S}\bar{F}_{\bar{\phi}\bar{\phi}} + a_7 \bar{\phi}\bar{F}_{\bar{\phi}\bar{\phi}} - 2ma_3 \bar{F}_{\bar{\phi}}\bar{S}_{\bar{\phi}} - 2ma_7 \bar{F}_{\bar{\phi}} + a_{13}\bar{\theta} = 0$$

$$\nu \bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + \frac{4m+1}{2} a_3 \bar{S}\bar{S}_{\bar{\phi}\bar{\phi}}\bar{F} + a_7 \overline{\phi S}_{\bar{\phi}\bar{\phi}} - ma_3 \bar{S}_{\bar{\phi}}^2 + mk_1^2 a_3 \bar{F}_{\bar{\phi}}^2 - 2ma_7 \bar{S}_{\bar{\phi}} + a_{19}\bar{\theta} = 0$$

$$\text{and } \frac{\nu}{\Pr}\bar{\theta}_{\bar{\phi}\bar{\phi}} + \frac{4m+1}{2} a_3 \bar{S}\bar{\theta}_{\bar{\phi}} + a_7 \bar{\phi}\bar{\theta}_{\bar{\phi}} - (2m-1)a_3 \bar{S}_{\bar{\phi}}\theta - 2(2m-1) + a_7 \bar{\theta} = 0$$

The boundary conditions are

$$\bar{F}(0) = \bar{F}_{\bar{\phi}}(0) = 0, \ \bar{F}_{\bar{\phi}}(\infty) = 0$$

$$\bar{S}(0) = \bar{S}_{\bar{\phi}}(0) = 0, \ \bar{S}_{\bar{\phi}}(\infty) = 0$$

$$\bar{\theta}(0) = 1, \ \bar{\theta}(\infty) = 0$$

As in previous cases, substituting $\bar{F} = \alpha f, \bar{S} = \alpha s, \bar{\phi} = \alpha\phi, \ \bar{\theta} = \theta$ choosing $\frac{4m+1}{2\nu} a_3 \alpha^2 = 1$ and writing $\frac{a_7}{a_3} = c, \ \frac{2m}{4m+1} = \beta$. Also for free convection we put $\frac{2}{4m+1}\frac{a_{13}}{a_3} = 1$ and $\frac{2}{4m+1}\frac{a_{19}}{a_3} = R$ (constant)

The above equations simplify to

$$f_{\phi\phi\phi} + sf_{\phi\phi} + (2-4\beta)c\phi f_{\phi\phi} - 2\beta(f_\phi s_\phi - cf_\phi) + \theta = 0 \tag{4.66}$$

$$s_{\phi\phi\phi} + ss_{\phi\phi} + (2-4\beta)c\phi s_{\phi\phi} - \beta(s_\phi^2 - k_1^2 f_\phi^2 - 2cs_\phi) + R\theta = 0 \tag{4.67}$$

$$\text{and } \Pr^{-1}\theta_{\phi\phi} + s\theta_\phi + (2-4\beta)c\phi\theta_\phi - (6\beta-2)f_\phi\theta - (12\beta-4)c\theta = 0 \tag{4.68}$$

The boundary conditions are

$$\left.\begin{aligned} &f(0) = f_\phi(0) = 0, \quad f_\phi(\infty) = 0 \\ &s(0) = s_\phi(0) = 0, \quad s_\phi(\infty) = 0 \\ &\theta(0) = 1, \quad \theta(\infty) = 0 \end{aligned}\right\} \tag{4.69}$$

In this case, the similarity requirements are

$$h_1 = b_1(k_1 k_3 a_3 Y + 2a_7\tau + k_3 A_0)^m$$

$$= b_1 k_1^m k_3^m a_3^m \left(Y + \frac{2a_7}{k_1 k_3 a_3}\tau + \frac{A_0}{k_1 a_3} \right)^m$$

$$= \alpha_1 (\bar{y} + \bar{c}\bar{t})^m$$

$$h_1 \propto (\bar{y} + \bar{c}\bar{t})^m$$

where $\bar{y} = Y + Y_0$

$$\bar{t} = \tau + \tau_0$$

$$\bar{c} = \frac{2a_7}{k_1 k_3 a_3}$$

$$\frac{A_0}{k_1 a_3} = Y_0 + \frac{2a_7}{k_1 k_3 a_3}\tau_0$$

$$\alpha_1 = b_1 k_1^m k_3^m a_3^m$$

$\therefore$ $U_F = \alpha_2 (\bar{y} + \bar{c}\bar{t})^m$, where $\alpha_2 = b_1 k_1^m k_3^{m-1} a_3^m$

$U_F^2 \propto g_X \beta_T \Delta T$ (characteristic length L_1), where $L_1 = (\bar{y} + \bar{c}\bar{t})$

$V_F = \alpha_3 (\bar{y} + \bar{c}\bar{t})^m$, where $\alpha_3 = b_1 k_1^{m-1} k_3^{m-1} a_3^m$

$V_F^2 \propto g_Y \beta_T \Delta T$ (characteristic length L_2), where $L_2 = (\bar{y} + \bar{c}\bar{t})$

$\Delta T = \alpha_4 (\bar{y} + \bar{c}\bar{t})^{2m-1}$, where $\alpha_4 = \dfrac{b_1^2 (4m+1) a_3^{2m}}{2k_1^{1-2m} k_3^{2-2m} \beta_T g_X}$

$\Delta T \propto (\bar{y} + \bar{c}\bar{t})^{2m-1}$.

$\gamma^2 = \alpha_5 (\bar{y} + \bar{c}\bar{t})$, where $\alpha_5 = k_1 k_3 a_3$

The similarity variable ϕ is,

$$\phi = \frac{z}{\alpha\gamma} \frac{z}{\sqrt{\dfrac{2\nu}{(4m+1)a_3}}\sqrt{\alpha_5 (\bar{y} + \bar{c}\bar{t})}}$$

$$= \left(Gr_{\bar{y}\bar{t}}\right)^{\frac{1}{4}} \frac{z}{(\bar{y} + \bar{c}\bar{t})}$$

where $\left(Gr_{\bar{y}\bar{t}}\right)^{\frac{1}{4}} = \left[\frac{(4m+1)}{2k_1h_1^2}\frac{g_T\beta_T\Delta T(\bar{y}+\bar{c}\bar{t})^3}{\nu^2}\right]^{\frac{1}{4}}$ is the modified Grashof number.

The velocity components are

$$u = U_F f_\phi(\phi), \qquad v = V_F s_\phi(\phi)$$

and $$w = \frac{1}{h_1h_2}\left[-(h_1\gamma V_F)_Y\bar{S} + \bar{\phi}\gamma_Y h_1 V_F \bar{S}_{\bar{\phi}}\right]$$

$$= \left\{\frac{2\nu k_1k_3}{(4m+1)(\bar{y}+\bar{c}\bar{t})}\right\}^{\frac{1}{2}}\left[\frac{1}{2}\phi s_\phi - \left(2m+\frac{1}{2}\right)s\right]$$

Skin frictions are

$$\tau_{w1} = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0}$$

$$\frac{\tau_{w1}}{\frac{1}{2}\rho U_F^2} = \frac{(4m+1)}{k_1}\frac{Gr_{yt}^{-\frac{1}{4}}}{h_1}f_{\phi\phi}(0)$$

$$\tau_{w2} = \mu\left(\frac{\partial v}{\partial z}\right)_{z=0}$$

$$\frac{\tau_{w2}}{\frac{1}{2}\rho V_F^2} = (4m+1)\frac{Gr_{yt}^{-\frac{1}{4}}}{h_1}s_{\phi\phi}(0)$$

Heat flux, $q_w = -k\left(\frac{\partial T}{\partial z}\right)_{z=0}$

$$= -k\Delta T\left(\frac{4m+1}{2\nu}\right)\left(\frac{a_3}{k_1k_3}\right)^{\frac{1}{2}}(\bar{y}+\bar{c}\bar{t})^{-\frac{1}{2}}\theta_\phi(0).$$

Case-7 :

$$\frac{\partial A(Y,\tau)}{\partial Y}=0,\ \frac{\partial B(\tau,X)}{\partial \tau}\neq 0,\ \frac{\partial C(X,Y)}{\partial X}=0,\ \frac{\partial C(X,Y)}{\partial Y}=0$$

$$\frac{\partial A(Y,\tau)}{\partial \tau}\neq 0,\ \frac{\partial B(\tau,X)}{\partial X}=0$$

Let $h_1=h_2$ and $a_3=a_0=0$.

Let $\frac{\partial A(Y,\tau)}{\partial Y}\neq$ constant

$$\Rightarrow \frac{h_2}{h_1}\frac{U_F}{V_F}(a_3-a_4-a_5+a_9-a_{10})\neq \text{constant.}$$

$$\therefore\ k_1=\frac{U_F}{V_F}\neq 0,\ l_1=a_3-a_4-a_5+a_9-a_{10}=0 \tag{4.70}$$

$$\frac{\partial B(\tau,X)}{\partial \tau}=\text{constant}$$

$$\Rightarrow \frac{\partial B(\tau,X)}{\partial \tau}=\frac{V_F}{h_1}(2a_7+a_{18}-a_{24})$$

$$=k_2 l_2 \tag{4.71}$$

$$\frac{\partial C(X,Y)}{\partial X}\neq \text{constant}$$

$$\Rightarrow \frac{h_1}{U_F}(a_0-a_1-a_2)\neq \text{constant}$$

$$\therefore\ k_3=\frac{h_1}{U_F}\neq 0,\ l_3=a_0-a_1-a_2=0 \tag{4.72}$$

$$\frac{\partial C(X,Y)}{\partial Y}\neq \text{constant}$$

$$\Rightarrow \frac{h_1}{V_F}(a_3-a_4-a_5)\neq \text{constant.}$$

$$\therefore\ \frac{1}{k_2}=\frac{h_1}{V_F}\neq 0,\ l_4=a_3-a_4-a_5=0 \tag{4.73}$$

$$\frac{\partial A(Y,\tau)}{\partial \tau}=\text{constant}$$

$$\Rightarrow \frac{\partial A(Y,\tau)}{\partial \tau} = \frac{U_F}{h_1}(2a_7 + a_{12} - a_{23})$$

$$= \frac{1}{k_3} l_5 \tag{4.74}$$

$$\frac{\partial B(\tau, X)}{\partial X} \neq \text{constant}$$

$$\Rightarrow \frac{V_F}{U_F}(a_0 - a_1 - a_2 + a_{11} - a_{15}) \neq \text{constant.}$$

$$\therefore \frac{1}{k_1} = \frac{V_F}{U_F} \neq 0,\ l_6 = a_0 - a_1 - a_2 - a_{11} + a_{15} = 0 \tag{4.75}$$

By virtue of equation (4.14), we have

$$\gamma^2 = 2a_7\tau + A_0 \tag{4.76}$$

(I) choosing $A_0 = 0$

$$\therefore \gamma^2 = 2a_7\tau \tag{4.77}$$

In view of equation (3.34a), we have

$$a_{12} = \frac{\gamma^2 (U_F)\tau}{U_F}$$

$$\Rightarrow \frac{(U_F)\tau}{U_F} = \frac{a_{12}}{2a_7\tau} \quad \text{[By using (4.77)]}$$

$$\therefore U_F = b_1(2a_7\tau)^m \tag{4.78}$$

where $m = \frac{a_{12}}{2a_7}$ and b_1 is the constant of integration

Similarly, from equation (3.37a), we get

$$a_{18} = \frac{\gamma^2 (V_F)\tau}{V_F}$$

$$\Rightarrow \frac{(V_F)\tau}{V_F} = \frac{a_{18}}{2a_7\tau}$$

$$\therefore V_F = b_2(2a_7\tau)^m \tag{4.79}$$

where $m = \frac{a_{18}}{2a_7}$ and b_2 is the constant of integration.

Also, by virtue of equation (3 47), we have

$$a_{23} = \frac{\gamma^2}{h_1} h_1 \tau$$

$$\Rightarrow \frac{h_1 \tau}{h_1} = \frac{a_{23}}{2a_7 \tau}$$

$$\therefore \; h_2 = h_1 = h_3 (2a_7 \tau)^m \tag{4.80}$$

where $m = \frac{a_{23}}{2a_7 \tau}$ and b_3 is the constant of integration.

With the help of the equations (4.77-4.80), the similarity requirement yield the following relations between the constants:

$$a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$$

a_7 is arbitrary.

$$a_8 = a_9 = a_{10} = a_{11} = 0$$

$$a_{12} = 2ma_7, \; a_{13} = \frac{1}{b_1 b_3 (2a_7 \tau)^{2m-1}} \beta_T \Delta T g_X$$

$$a_{14} = a_{15} = a_{16} = a_{17} = 0$$

$$a_{18} = 2ma_7, \quad a_{19} = \frac{1}{b_2 b_3 (2a_7 \tau)^{2m-1}} \beta_T \Delta T g_Y$$

$$a_{20} = a_{21} = 0, \; a_{22} = 2(2m-1)a_7$$

For this case the general equation (3.25-3.27) are therefore reduced to

$$\tau \bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}} + a_7 \bar{\phi} \bar{F}_{\bar{\phi}\bar{\phi}} - 2a_7 m \bar{F}_{\bar{\phi}} + a_{13} \bar{\theta} = 0$$

$$\tau \bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + a_7 \bar{\phi} \bar{S}_{\bar{\phi}\bar{\phi}} - 2a_7 m \bar{S}_{\bar{\phi}} + a_{19} \bar{\theta} = 0$$

$$\text{and } \frac{\tau}{\Pr} \bar{\theta}_{\bar{\phi}\bar{\phi}} + a_7 \bar{\phi} \bar{\theta}_{\bar{\phi}} - 2(m-1) a_7 \bar{\theta} = 0$$

The boundary conditions are

$$\left.\begin{array}{l} \bar{F}(0) = \bar{F}_{\bar{\phi}}(0) = 0, \; \bar{F}_{\bar{\phi}}(\infty) = 0 \\ \bar{S}(0) = \bar{S}_{\bar{\phi}}(0) = 0, \; \bar{S}_{\bar{\phi}}(\infty) = 0 \\ \bar{\theta}(0) = 1, \; \bar{\theta}(\infty) = 0 \end{array}\right\}$$

As in previous cases, substituting $\overline{F} = \alpha f, \overline{S} = \alpha s, \overline{\phi} = \alpha\phi, \overline{\theta} = \theta$ and choosing

$\frac{a_7\alpha^2}{\nu} = 1$ and later we have to put $\frac{a_{12}}{a_7} = 1$ and $\frac{a_{19}}{a_7} = R$ (Constant) for free convection.

Finally the above equations are reduced to

$$f_{\phi\phi\phi} + \phi f_{\phi\phi} - 2mf_\phi + \theta = 0 \tag{4.81}$$

$$s_{\phi\phi\phi} + \phi s_{\phi\phi} - 2ms_\phi + R\theta = 0 \tag{4.82}$$

$$\text{and } \Pr^{-1}\theta_{\phi\phi} + \phi\theta_\phi - 2(2m-1)\theta = 0 \tag{4.83}$$

The boundary conditions are

$$\left.\begin{aligned} &f(0) = f_\phi(0) = 0,\ f_\phi(\infty) = 0 \\ &s(0) = s_\phi(0) = 0,\ s_\phi(\infty) = 0 \\ &\theta(0) = 1,\ \theta(\infty) = 0 \end{aligned}\right\} \tag{4.84}$$

For $m = 1 + p$, p is a constant, s = f, $R = 1$, the equations (4 81- 4 84) with the boundary conditions coincide with unsteady free convection with uniform but unsteady surface temperature variations at large distance x, analysed by **Yang** (1960) We have, in this case, the similarity requirements are

$$h_1 = h_2 = b_3(2a_7\tau)^m$$

$$= b_3(2a_7)^m(\tau)^m$$

$$= \alpha_1(\bar{t})^m \text{ where } \alpha_1 = b_3(2a_7\tau)^m,\ \tau = \bar{t}.$$

$$h_1 \propto (\bar{t})^m$$

$$\therefore\ U_F = \alpha_2(\bar{t})^m, \text{ where } \alpha_2 = b_1(2a_7)^m$$

$U_F^2 \propto g_X\beta_T\Delta T$ (characteristic length L_1), where $L_1 = (\bar{t})^m$

$$\therefore\ V_F = \alpha_3(\bar{t})^m, \text{ where } \alpha_3 = b_2(2a_7)^m$$

$V_F^2 \propto g_Y\beta_T\Delta T$ (characteristic length L_2), where $L_2 = (\bar{t})^m$

$$\Delta T = \alpha_4(\bar{t})^{2m-1}, \text{ where } \alpha_4 = \frac{a_{13}b_1b_3(2a_7)^m}{\beta_T g_X}.$$

$$\Delta T \propto (\bar{t})^{2m-1}.$$

$$\gamma^2 = \alpha_5\bar{t}, \text{ where } \alpha_5 = 2a_7$$

The similarity variable ϕ is

$$\phi = \frac{z}{\alpha\gamma} = \frac{z}{\sqrt{\frac{\nu}{a_7}\sqrt{2a_7\bar{t}}}}$$

$$= \frac{z}{\sqrt{2\nu}(\bar{t})^{\frac{1}{2}}}.$$

The velocity components (u, v, w) are

$$u = U_F f_\phi(\phi)$$

$$v = V_{l_r} s_\phi(\phi)$$

and $w = 0$

Skin frictions are

$$\tau_{w1} = \frac{\mu a_2}{\sqrt{2\nu}}(\bar{t})^{m-\frac{1}{2}} f_{\phi\phi}(0)$$

$$\tau_{w2} = \frac{\mu a_3}{\sqrt{2\nu}}(\bar{t})^{m-\frac{1}{2}} s_{\phi\phi}(0)$$

Heat flux

$$q_w = \frac{-k a_4}{\sqrt{2\nu}}(\bar{t})^{2m-\frac{3}{2}} \theta_\phi(0)$$

(II) If we choose $a_7 = 0$, A_0 = arbitrary constant.

From (3.7.7), we have $\gamma^2 = A_0$ (4.85)

The equation (3.47) implies,

$$\frac{h_{1\tau}}{h_1} = \frac{a_{23}}{A_0}$$

$$\Rightarrow h_1 = b_1 e^{\frac{a_{23}}{A_0}\tau}$$

$$= b_1 e^{n\tau} \tag{4.86}$$

where b_1 is the constant of integration and $n = \frac{a_{23}}{A_0}$.

In view of equation (3.34a), we have

$$\frac{(U_F)_\tau}{U_F} = \frac{a_{12}}{A_0}$$

$$\Rightarrow \; U_F = b_2 e^{n\tau} \tag{4.87}$$

where b_2 is the constant of integration and $n = \dfrac{a_{12}}{A_0}$.

Again, from equation (3.37a), we get

$$\frac{(V_F)_\tau}{V_F} = \frac{a_{18}}{A_0}$$

$$\Rightarrow V_F = b_3 e^{n\tau}, \tag{4 88}$$

where b_3 is the constant of integration and $n = \dfrac{a_{18}}{A_0}$.

Therefore, the constants becomes

$$a_0 = a_1 = a_2 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = a_{11} = 0$$

$$a_{12} = nA_0, \qquad a_{13} = \frac{A_0}{b_1 b_2 e^{2n\tau}} \beta_T \Delta T g_X, \qquad a_{14} = a_{15} = a_{16} = a_{17} = 0, \quad a_{18} = nA_0.$$

$$a_{19} = \frac{A_0}{b_1 b_3 e^{2n\tau}} \beta_T \Delta T g_Y, \qquad a_{20} = a_{21} = 0 \qquad a_{22} = 2nA_0.$$

Thus the general equations (3.25-3.27) reduce to

$$\nu \overline{F}_{\overline{\phi\phi\phi}} - nA_0 \overline{F}_{\overline{\phi}} + a_{13} \overline{\theta} = 0$$

$$\nu \overline{S}_{\overline{\phi\phi\phi}} - nA_0 \overline{S}_{\overline{\phi}} + a_{19} \overline{\theta} = 0$$

$$\text{and } \frac{\nu}{\Pr} \overline{\theta}_{\overline{\phi\phi}} - nA_0 \overline{\theta} = 0$$

As before the above equations take the form

$$\left.\begin{array}{l} f_{\phi\phi\phi} - f_\phi + \theta = 0 \\ s_{\phi\phi\phi} - s_\phi + R\theta = 0 \\ \theta_{\phi\phi} - \Pr \theta = 0 \end{array}\right\} \tag{4.89}$$

with the boundary conditions

$$\left.\begin{aligned} &f(0) = f_\phi(0) = 0,\ f_\phi(\infty) = 0 \\ &s(0) = s_\phi(0) = 0,\ s_\phi(\infty) = 0 \\ &\theta(0) = 1,\ \theta(\infty) = 0 \end{aligned}\right\} \tag{4.90}$$

The analytical solutions of (4.89), are

$$f = \frac{1}{\Pr^{\frac{1}{2}}}\left[e^{-\Pr^{\frac{1}{2}}\phi} - e^{-\phi}\right], \qquad s = \frac{R}{\Pr^{\frac{1}{2}}}\left[e^{-\Pr^{\frac{1}{2}}\phi} - e^{-\phi}\right] \quad \text{and} \quad \theta = e^{-\Pr^{\frac{1}{2}}\phi}$$

The similarity requirements are

$$h_1 = b_1 e^{n\bar{t}}$$

$$h_1 \propto e^{n\bar{t}}$$

$$U_F = b_2 e^{n\bar{t}}$$

$U_F^2 \propto g_X \beta_T \Delta T$ (characteristic length L_1), where $L_1 = e^{n\bar{t}}$

$$V_F = b_3 e^{n\bar{t}}$$

$V_F^2 \propto g_T \beta_T \Delta T$ (characteristic length L_2), where $L_2 = e^{n\bar{t}}$

$$\Delta T = \alpha_1 e^{2n\bar{t}} \text{ where } \alpha_1 \frac{n}{b_1 b_2 \beta_T g_x}, \bar{t} = \tau$$

$$\Delta T \propto e^{2n\bar{t}}$$

$$\gamma^2 = A_0$$

The similarity variable ϕ is

$$\phi = \frac{z}{\alpha\gamma} = \frac{z}{\sqrt{\frac{\nu}{nA_0}}\sqrt{A_0}}$$

$$= \sqrt{\frac{n}{\nu}}z$$

The velocity components are

$$u = U_F f_\phi(\phi)$$

$$v = V_F s_\phi(\phi)$$

and $\quad w = 0.$

Skin friction are

$$\tau_{w1} = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0}$$

$$= \mu b_2 \sqrt{\frac{n}{\nu}} e^{nt} f_{\eta\phi}(0)$$

$$\tau_{w2} = \mu\left(\frac{\partial v}{\partial z}\right)_{z=0}$$

$$= \mu b_3 \sqrt{\frac{n}{\nu}} e^{nt} s_{\phi\phi}(0)$$

Heat flux

$$q_w = -k\left(\frac{\partial T}{\partial z}\right)_{z=0}$$

$$= -k\alpha_1 \sqrt{\frac{n}{\nu}} e^{2nt} \theta_\phi(0)$$

Case-8:

$$\frac{\partial A(Y,\tau)}{\partial Y}=0,\ \frac{\partial B(\tau,X)}{\partial \tau}=0,\ \frac{\partial C(X,Y)}{\partial X}=0$$

$$\frac{\partial C(X,Y)}{\partial Y}=0,\ \frac{\partial A(Y,\tau)}{\partial \tau}=0,\ \frac{\partial B(\tau,X)}{\partial X}=0$$

Choosing $h_1 = h_2 = 1$,

Let, $\dfrac{\partial A(Y,\tau)}{\partial Y} \neq$ constant

$\Rightarrow \dfrac{U_F}{V_F}(a_3 - a_4 - a_5 + a_9 - a_{10}) \neq$ constant

$$\Rightarrow k_1 = \frac{U_F}{V_F} \neq 0,\ l_1 = a_3 - a_4 - a_5 + a_9 - a_{10} = 0 \tag{4.91}$$

$\dfrac{\partial B(\tau,X)}{\partial \tau} \neq$ constant

$\Rightarrow V_F(2a_7 + a_{18} - a_{24}) \neq$ constant

$$\Rightarrow k_2 = V_F \neq 0,\ l_2 = 2a_7 + a_{18} - a_{24} = 0 \tag{4.92}$$

$\dfrac{\partial C(X,Y)}{\partial X} \neq$ constant

$\Rightarrow \dfrac{1}{U_F}(a_0 - a_1 - a_2) \neq$ constant

$$\Rightarrow k_3 = \frac{1}{U_F} \neq 0,\ l_3 = a_0 - a_1 - a_2 = 0 \tag{4.93}$$

$\dfrac{\partial C(X,Y)}{\partial Y} \neq$ constant

$\Rightarrow \dfrac{1}{V_F}(a_3 - a_4 - a_5) \neq$ constant

$$\Rightarrow \frac{1}{k_2} = \frac{1}{V_F} \neq 0,\ l_4 = a_3 - a_4 - a_5 = 0 \tag{4.94}$$

$\dfrac{\partial A(Y,\tau)}{\partial \tau} \neq$ constant

$\Rightarrow U_F(2a_7 + a_{12} - a_{23}) \neq$ constant

$$\Rightarrow \frac{1}{k_2} = U_F \neq 0,\ l_5 = 2a_7 + a_{12} - a_{23} = 0 \tag{4.95}$$

$$\frac{\partial B(\tau, X)}{\partial X} \neq \text{constant}$$

$$\Rightarrow \frac{V_F}{U_F}(a_0 - a_1 - a_2 - a_{11} + a_{15}) \neq \text{constant},$$

$$\Rightarrow \frac{1}{k_1} = \frac{V_F}{U_F} \neq 0,\ l_6 = a_0 - a_1 - a_2 - a_{11} + a_{15} = 0 \tag{4.96}$$

The equations (4.91 – 4.96) implies,

$$a_9 = a_{10},\ \ 2a_7 = -a_{12},\ \ a_{11} = a_{15},\ \ 2a_7 = -a_{18},\ \ a_0 = a_1 + a_2,\ \ a_3 = a_4 + a_5 \tag{4.97}$$

By virtue of equation (4.9), we get

$$\gamma^2 U_F = a_0 X + A_0 \tag{4.98}$$

[By using (4 91) and (4.95)]

From (4.10), we have

$$\gamma^2 V_F = a_3 Y + B_0 \tag{4.100}$$

[By using (4.92) and (4.96)]

In view of equation (4.11), we get

$$\therefore \gamma^2 = 2a_7\tau + C_0 \tag{4.101}$$

[By using (4.93) and (4.94)]

By virtue of equations (4.98-4.100), U_F and V_F are found to be

$$U_F = \frac{a_0 X + A_0}{2a_7\tau + V_0} \tag{4.102}$$

and

$$V_F = \frac{a_3 Y + B_0}{2a_7\tau + C_0} \tag{4.103}$$

Substituting (4 101-4.103) in the similarity requirements one may obtain the following relations between the constant (a's).

a_0, a_3 and a_7 are arbitrary

$$a_1 = a_0,\ \ a_2 = 0,\ \ a_4 = a_3,\ \ a_5 = a_6 = 0,\ \ a_8 = a_0,\ \ a_9 = a_{10} = a_{11} = 0,\ \ a_{12} = -2a_7$$

$$a_{13} = \frac{(2a_7\tau + C_0)^2}{a_0 X + A_0}\beta_T \Delta T g_X,\ \ a_{14} = a_3,\ \ a_{15} = a_{16} = a_{17} = 0,\ \ a_{18} = -2a_7,$$

$$a_{19} = \frac{(2a_7\tau + C_0)^2}{(a_3 Y + B_0)}\beta_T \Delta T g_Y,\ \ a_{20} = a_0,\ \ a_{21} = a_3,\ \ a_{22} = -4a_7.$$

Thus the general equation (3.25-3.27) take the forms for this case:

$$\nu \bar{F}_{\bar{\phi}\bar{\phi}\bar{\phi}} + a_0 \bar{F}\,\bar{F}_{\bar{\phi}\bar{\phi}} + a_3 \bar{S}\,\bar{F}_{\bar{\phi}\bar{\phi}} + a_7 \bar{\phi}\,\bar{F}_{\bar{\phi}\bar{\phi}} - a_0 \bar{F}_{\bar{\phi}}^2 + 2a_7 \bar{F}_{\bar{\phi}} + a_{13}\bar{\theta} = 0$$

$$\nu \bar{S}_{\bar{\phi}\bar{\phi}\bar{\phi}} + a_3 \bar{S}\,\bar{S}_{\bar{\phi}\bar{\phi}} + a_0 \bar{S}_{\bar{\phi}\bar{\phi}} \bar{F} + a_7 \bar{\phi}\,\bar{S}_{\bar{\phi}\bar{\phi}} - a_3 \bar{S}_{\bar{\phi}}^2 + 2a_7 \bar{S}_{\bar{\phi}} + a_{19}\bar{\theta} = 0$$

and $$\frac{\nu}{\Pr}\bar{\theta}_{\bar{\phi}\bar{\phi}} + a_0 \bar{F}\bar{\theta}_{\bar{\phi}} + a_3 \bar{S}\bar{\theta}_{\bar{\phi}} + a_7 \bar{\phi}\bar{\theta}_{\bar{\phi}} - (a_0 \bar{F}_{\bar{\phi}} + a_3 \bar{S}_{\bar{\phi}})\theta + 4a_7 \bar{\theta} = 0$$

The boundary conditions are

$$\bar{F}(0) = \bar{F}_{\bar{\phi}}(0) = 0,\ \bar{F}_{\bar{\phi}}(\infty) = 0$$

$$\bar{S}(0) = \bar{S}_{\bar{\phi}}(0) = 0,\ \bar{S}_{\bar{\phi}}(\infty) = 0$$

$$\bar{\theta}(0) = 1,\ \bar{\theta}(\infty) = 0$$

As in previous cases, substituting $\bar{F} = \alpha f,\ \bar{S} = \alpha s,\ \bar{\theta} = \theta,\ \bar{\phi} = \alpha\phi$ choosing $\frac{a_0\alpha^2}{\nu} = 1$ and later writing $\frac{a_3}{a_0} = c,\ \frac{a_7}{a_0} = d$. Also we put $\frac{a_{13}}{a_0} = 1$ and $\frac{a_{19}}{a_0} = R$ (constant) for free convection.

The above equation with their attached boundary conditions are simplified to

$$f_{\phi\phi\phi} + (f + cs)f_{\phi\phi} + d\phi f_{\phi\phi} - f_\phi^2 + 2df_\phi + \theta = 0 \qquad (4.104)$$

$$s_{\phi\phi\phi} + (f + cs)s_{\phi\phi} + d\phi s_{\phi\phi} - cs_\phi^2 + 2ds_\phi + R\theta = 0 \qquad (4.105)$$

and $$\Pr^{-1}\theta_{\phi\phi} + (f + cs)\theta_\phi + d\phi\theta_\phi - (f_\phi + cs_\phi)\theta + 4d\theta = 0 \qquad (4.106)$$

The boundary conditions are

$$\left.\begin{aligned} &f(0) = f_\phi(0) = 0, \quad f_\phi(\infty) = 0 \\ &s(0) = s_\phi(0) = 0, \quad s_\phi(\infty) = 0 \\ &\theta(0) = 1, \quad \theta(\infty) = 0 \end{aligned}\right\} \qquad (4.107)$$

The similarity requirements are

$$h_1 = h_2 = 1$$

$$U_F = \frac{a_0 X + A_0}{2a_7\tau + C_0}$$

$$= \alpha_1 \frac{\bar{x}}{\bar{t}}, \qquad \text{where } \bar{x} = X + X_0,\quad \bar{t} = \tau + \tau_0,\ \alpha_1 = \frac{a_0}{2a_7}$$

$U_F^2 \propto g_X \beta_T \Delta T$ (characteristic length L_1), where $L_1 = \dfrac{\bar{x}}{\bar{t}}$

$$V_F = \frac{a_3 Y + B_0}{2a_7 \tau + C_0}$$

$$= \alpha_2 \frac{\bar{y}}{\bar{t}} \ , \qquad \text{where } \alpha_2 = \frac{a_3}{2a_7}, \quad \bar{y} = Y + Y_0$$

$V_F^2 \propto g_Y \beta_T \Delta T$ (characteristic length L_2), where $L_2 = \dfrac{\bar{y}}{\bar{t}}$

$$\Delta T = \alpha_3 \frac{\bar{x}}{\bar{t}^2}, \qquad \text{where } \alpha_3 = \frac{a_0^2}{4a_7^2 g_X \beta_T}$$

$$\Delta T \propto \frac{\bar{x}}{\bar{t}^2}.$$

and $\Delta T = \alpha_4 \dfrac{\bar{y}}{\bar{t}^2}$, $\qquad$ where $\alpha_4 = \dfrac{a_0 a_3}{4a_7^2 \beta_T g_Y}$

$$\Delta T \propto \frac{\bar{y}}{\bar{t}^2}.$$

$\therefore \ \gamma^2 = \alpha_5 (\bar{t})$, where $\alpha_5 = 2a_7$

The similarity variable ϕ is,

$$\phi = \frac{z}{\alpha\gamma} = \frac{z}{\sqrt{\dfrac{\nu}{a_0}}\sqrt{2a_7(\bar{t})}}$$

$$= \frac{\sqrt{a_0}\, z}{\sqrt{2\nu a_0\, a_7(\bar{t})}}$$

The velocity components are

$$u = U_F f_\phi(\phi) \ , \qquad v = V_F s_\phi(\phi)$$

and $w = \dfrac{1}{h_1 h_2}\left\{-(h_2 \gamma U_F)_X \bar{F} - (h_1 \gamma V_F)_Y \bar{S}\right\}$

$$= -\sqrt{\frac{\nu}{2a_0 a_7 \bar{t}}}(a_0 f + a_3 s)$$

Skin frictions are

$$\tau_{w1} = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0}$$

$$= \mu\sqrt{\frac{a_0^3}{8a_7^3\nu}}\frac{\bar{x}}{(\bar{t})^{\frac{3}{2}}}f_{\eta\eta}(0)$$

$$\tau_{w2} = \mu\left(\frac{\partial v}{\partial z}\right)_{z=0}$$

$$= \mu\sqrt{\frac{a_0 a_3^2}{8a_7^3\nu}}\frac{\bar{y}}{(\bar{t})^{\frac{3}{2}}}s_{\eta\eta}(0)$$

Heat flux, $q_w = -k\left(\frac{\partial T}{\partial z}\right)_{z=0}$

$$= -k\alpha_3\sqrt{\frac{a_0}{2a_7\nu}}\frac{\bar{x}}{(\bar{t})^{\frac{5}{2}}}\theta_\phi(0)$$

and $q_w = -k\left(\frac{\partial T}{\partial z}\right)_{z=0}$

$$= -k\alpha_4\sqrt{\frac{a_0}{2a_7\nu}}\frac{\bar{y}}{(\bar{t})^{\frac{5}{2}}}\theta_\phi(0).$$

Chapter-5

Figures and Tables

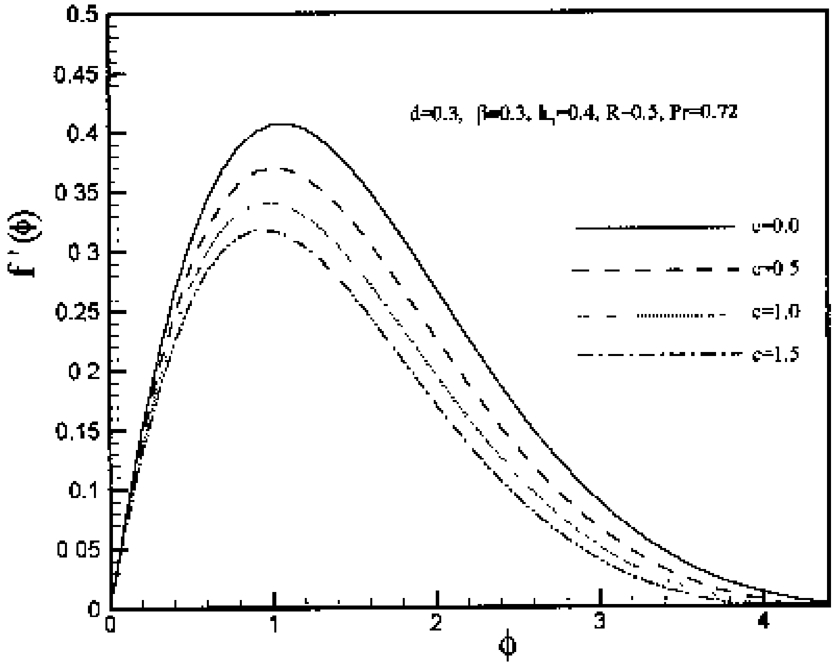

Figure 1(a). Variation of the dimensionless velocity f' along the u-direction with the similarity variable ϕ for different values of c [Based on equation(4.26-4.29)].

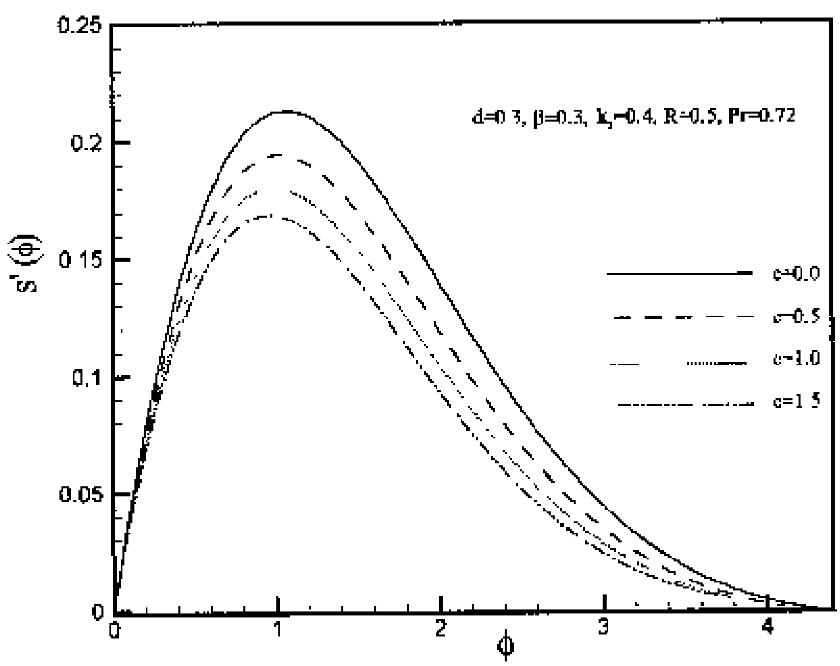

Figure 1(b): Variation of the dimensionless velocity s' along the v-direction with the similarity variable ϕ for different values of c [Based on equation(4.26-4.29)]

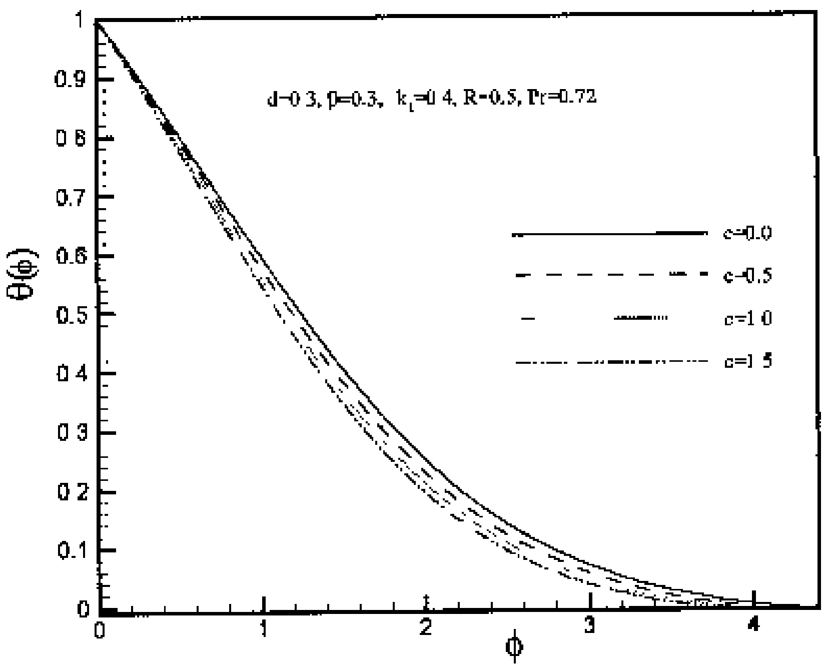

Figure 1(c): Variation of the dimensionless temperature θ with the similarity variable ϕ for different values of c [Based on equation(4 26-4.29)].

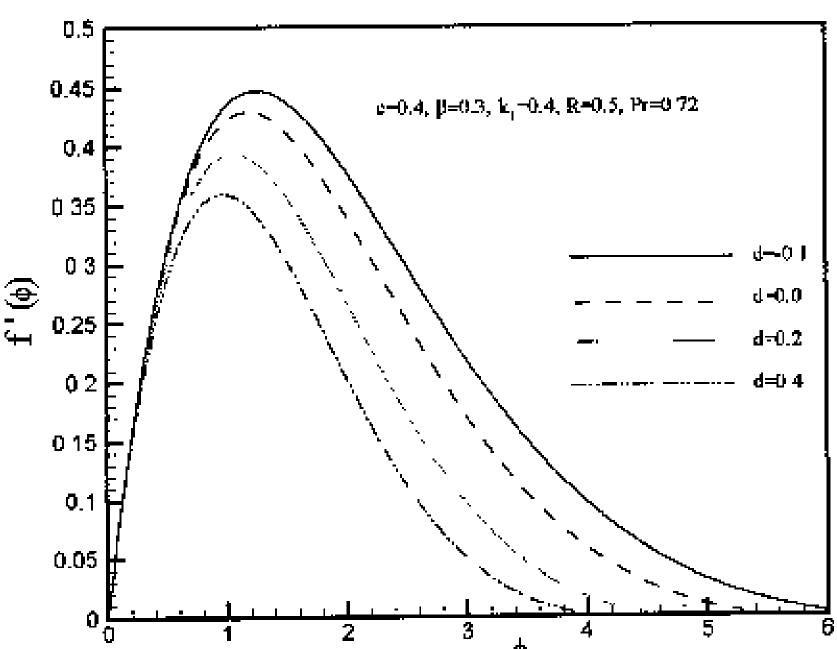

Figure 2(a): Variation of the dimensionless velocity f' along the u-direction with the similarity variable ϕ for different values of d [Based on equation(4.26-4.29)].

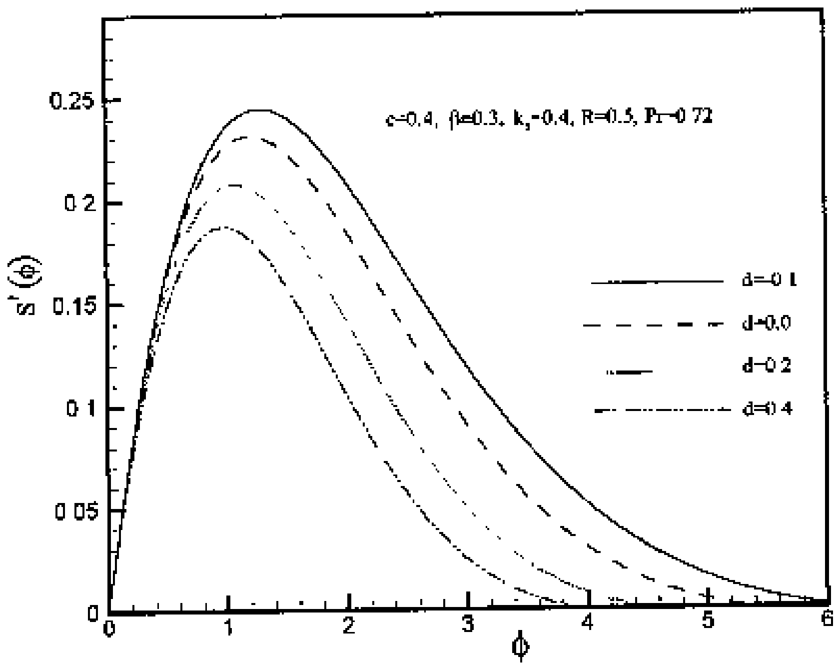

Figure 2(b): Variation of the dimensionless velocity s' along the y-direction with the similarity variable ϕ for different values of d [Based on equation(4.26-4.29)].

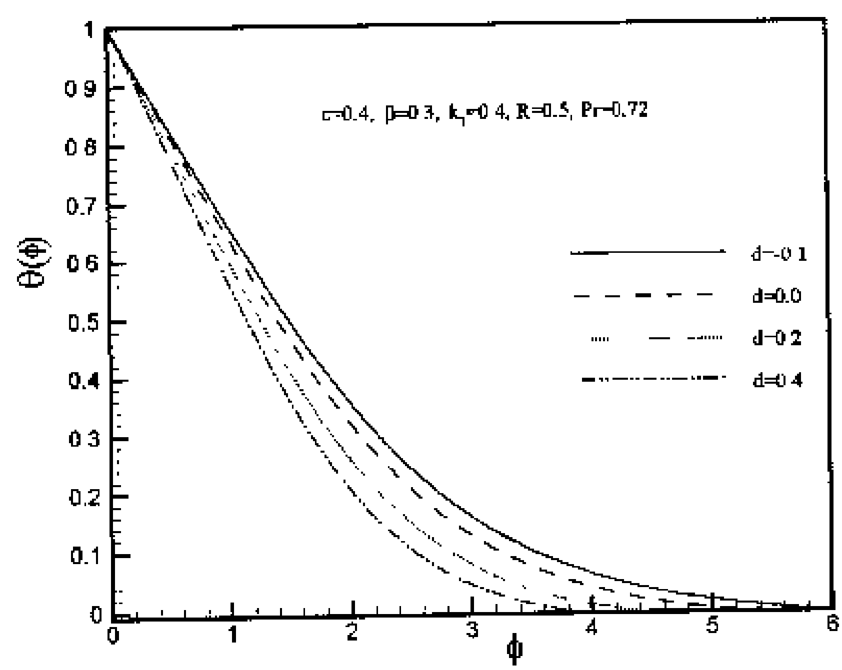

Figure 2(c). Variation of the dimensionless temperature θ with the similarity variable ϕ for different values of d [Based on equation(4.26-4.29)].

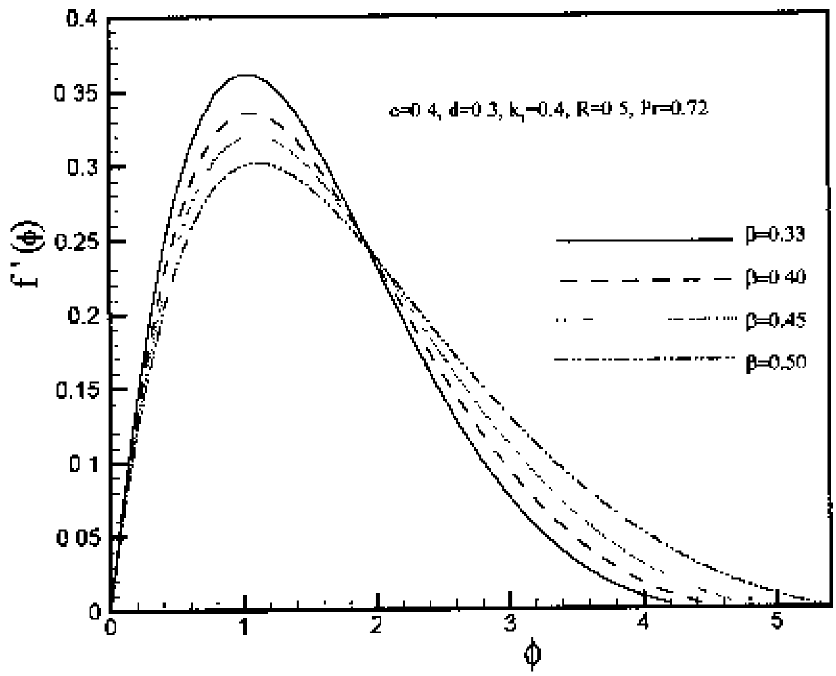

Figure 3(a): Variation of the dimensionless velocity f' along the u-direction with the similarity variable ϕ for different values of β [Based on equation(4.26-4.29)].

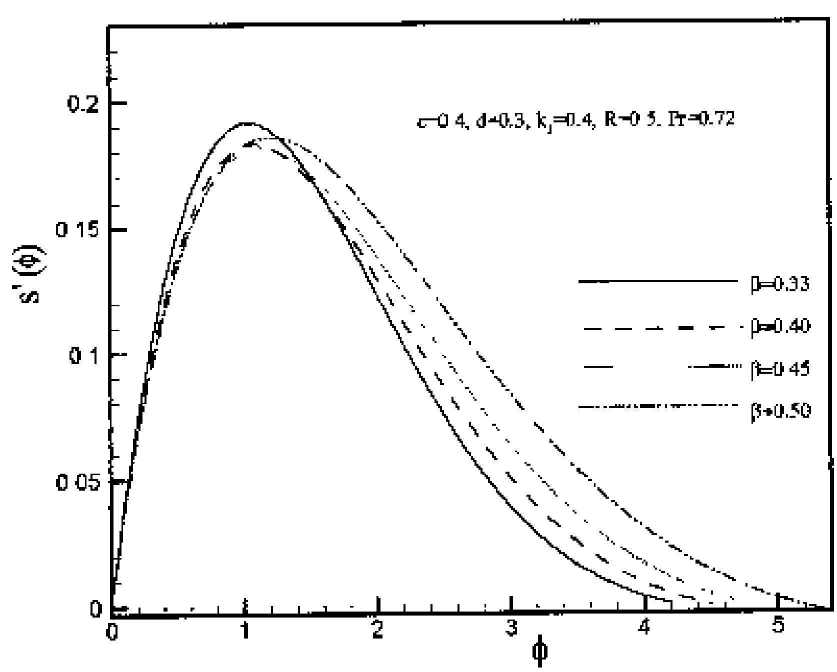

Figure 3(b): Variation of the dimensionless velocity s' along the v-direction with the similarity variable ϕ for different values of β [Based on equation(4.26-4 29)].

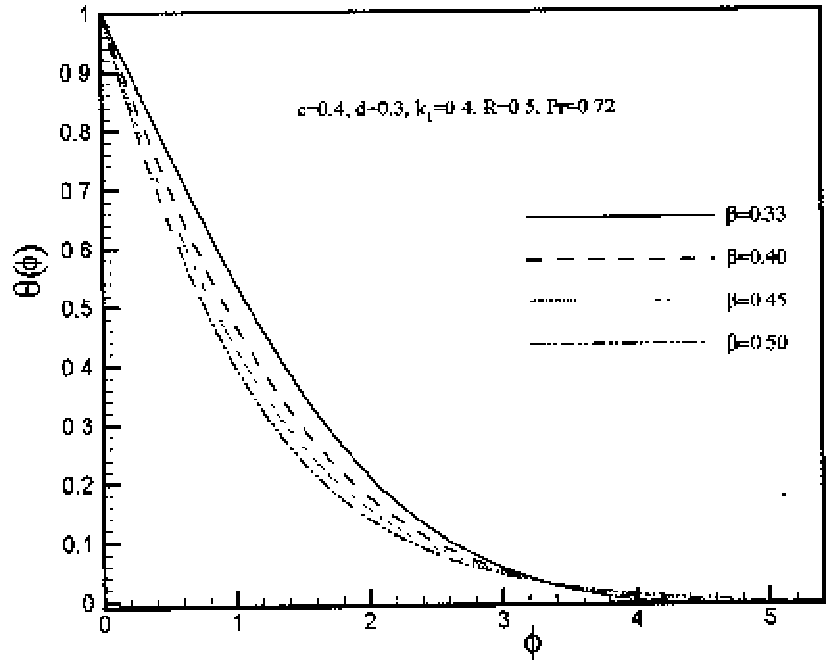

Figure 3(c): Variation of the dimensionless temperature θ with the similarity variable ϕ for different values of β [Based on equation(4.26-4.29)].

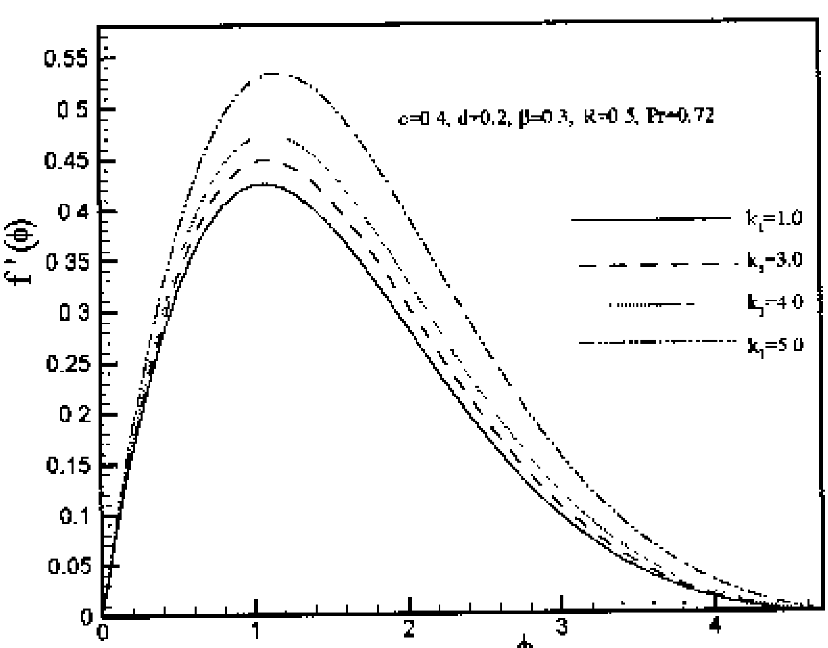

Figure 4(a). Variation of the dimensionless velocity f' along the u-direction with the similarity variable ϕ for different values of k_1 [Based on equation(4.26-4.29)].

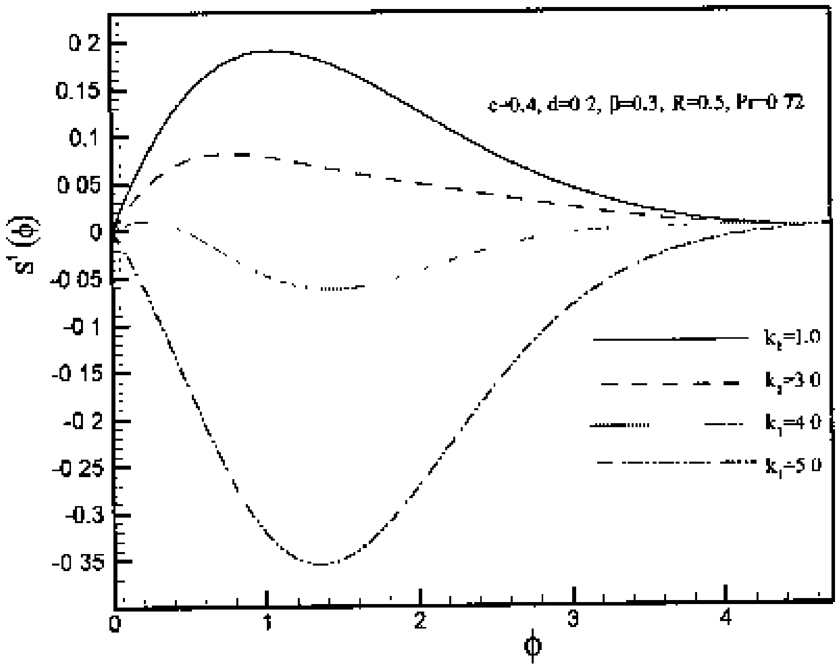

Figure 4(b): Variation of the dimensionless velocity s' along the y-direction with the similarity variable ϕ for different values of k_1 [Based on equation(4.26-4.29)].

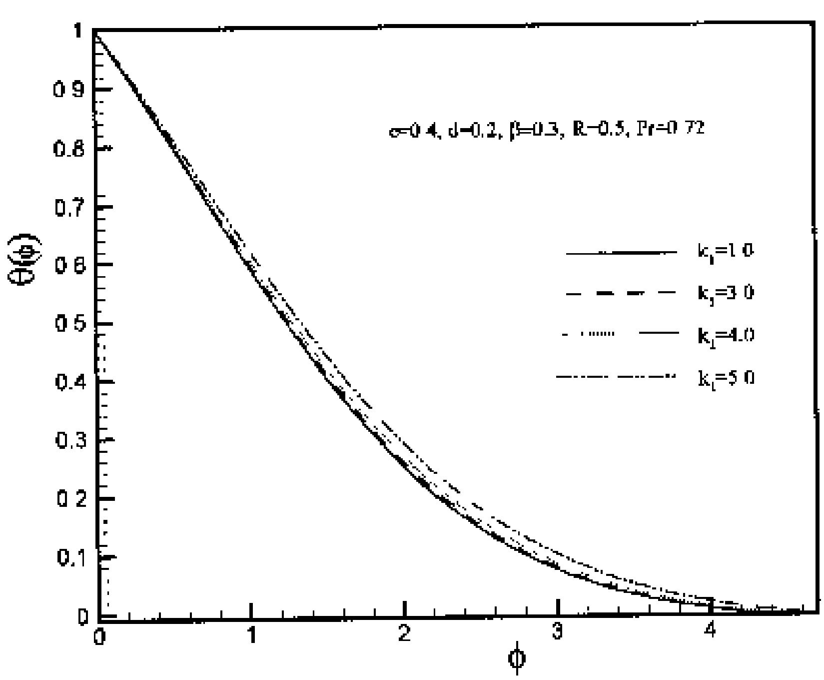

Figure 4(c): Variation of the dimensionless temperature θ with the similarity variable ϕ for different values of k_1 [Based on equation(4.26-4.29)].

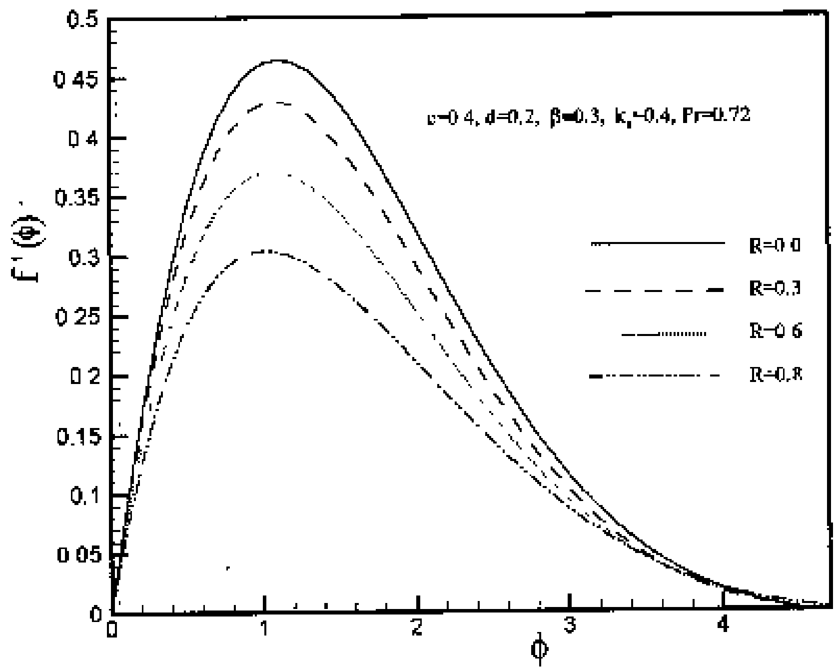

Figure 5(a): Variation of the dimensionless velocity f' along the u-direction with the similarity variable ϕ for different values of R [Based on equation(4.26-4.29)].

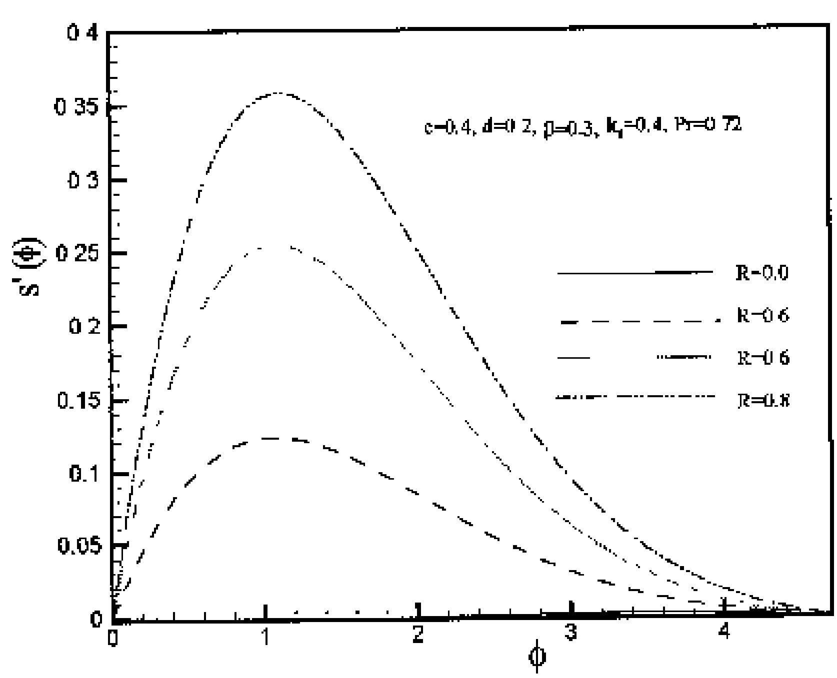

Figure 5(b): Variation of the dimensionless velocity s' along the v-direction with the similarity variable ϕ for different values of R [Based on equation(4.26-4.29)].

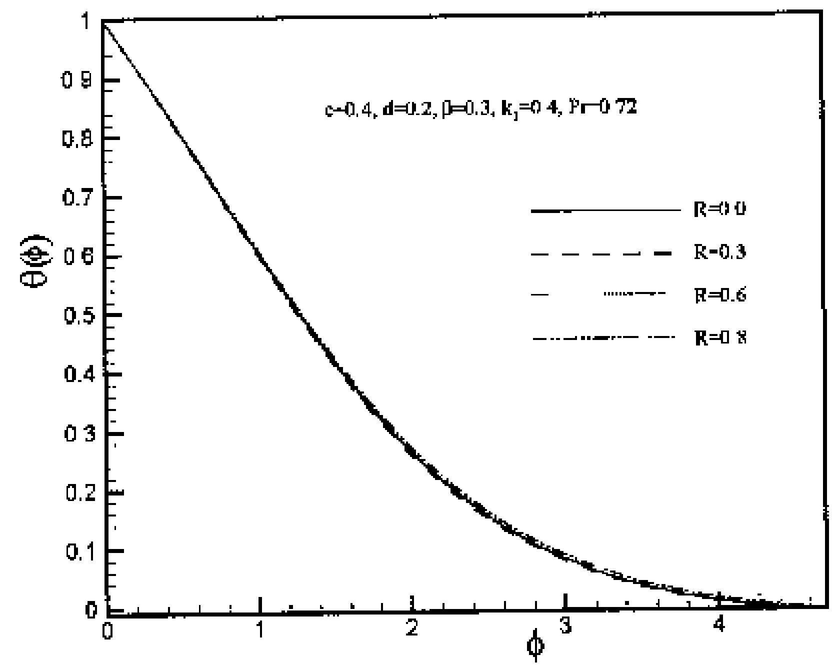

Figure 5(c): Variation of the dimensionless temperature θ with the similarity variable ϕ for different values of R [Based on equation(4.26-4.29)].

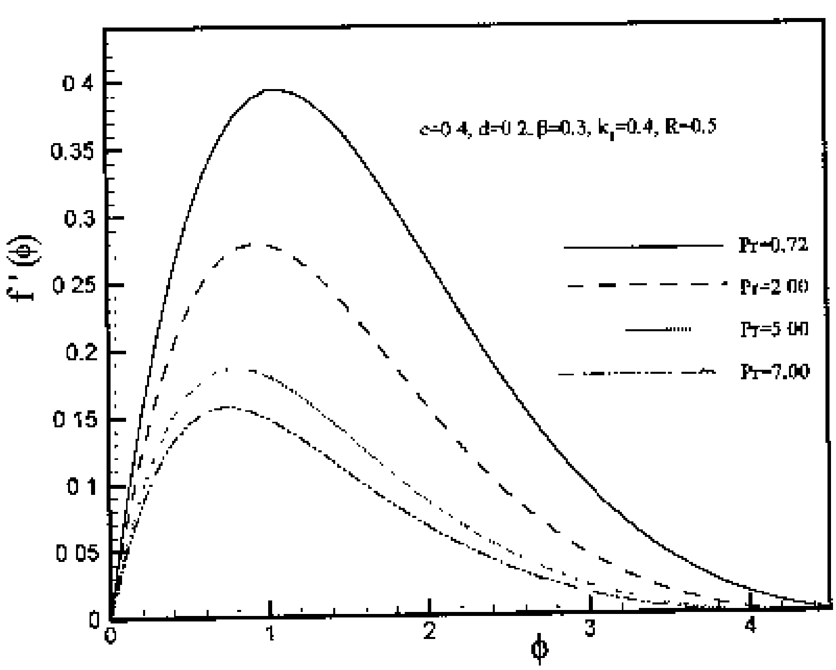

Figure 6(a): Variation of the dimensionless velocity f' along the u-direction with the similarity variable ϕ for different values of Pr [Based on equation(4.26-4.29)].

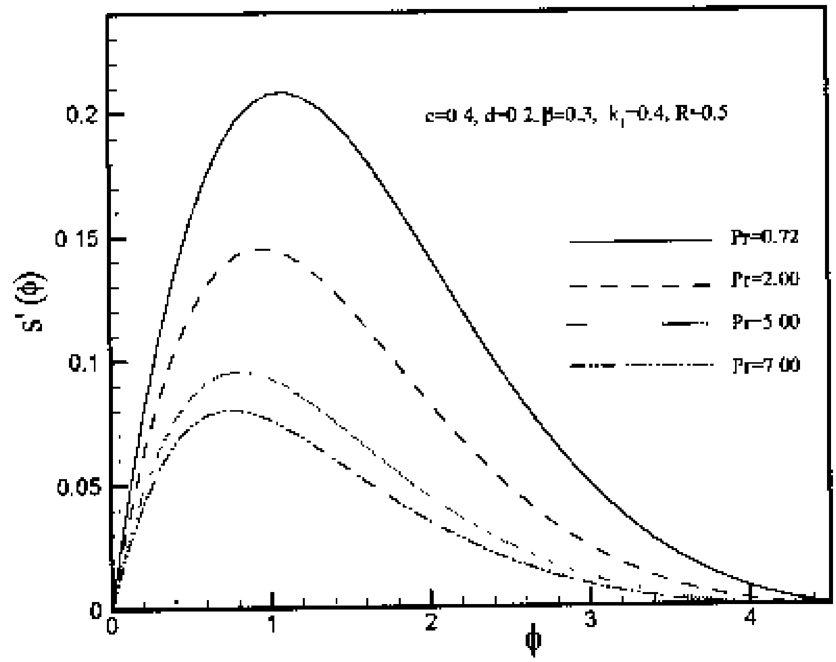

Figure 6(b): Variation of the dimensionless velocity s' along the y-direction with the similarity variable ϕ for different values of Pr [Based on equation(4.26-4.29)].

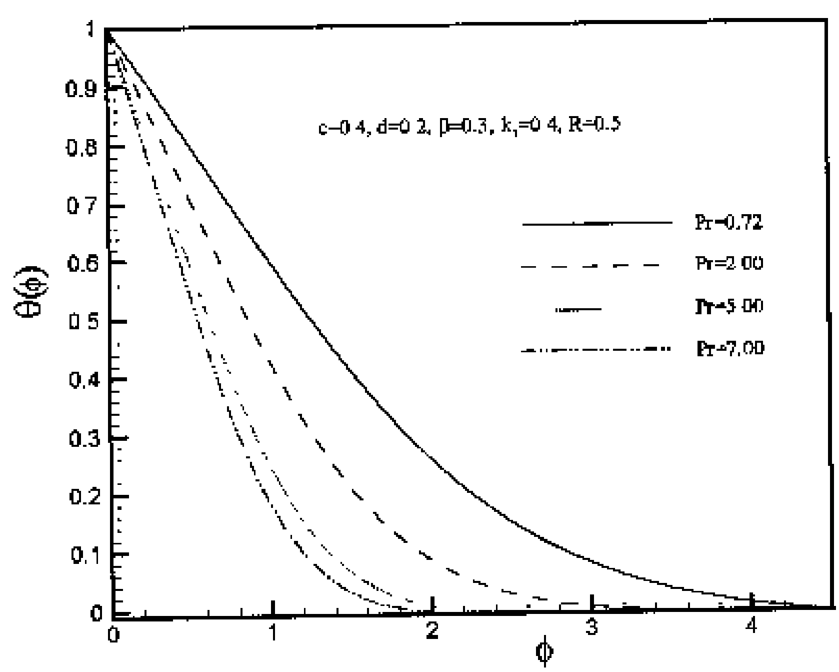

Figure 6(c): Variation of the dimensionless temperature θ with the similarity variable ϕ for different values of Pr [Based on equation(4.26-4.29)].

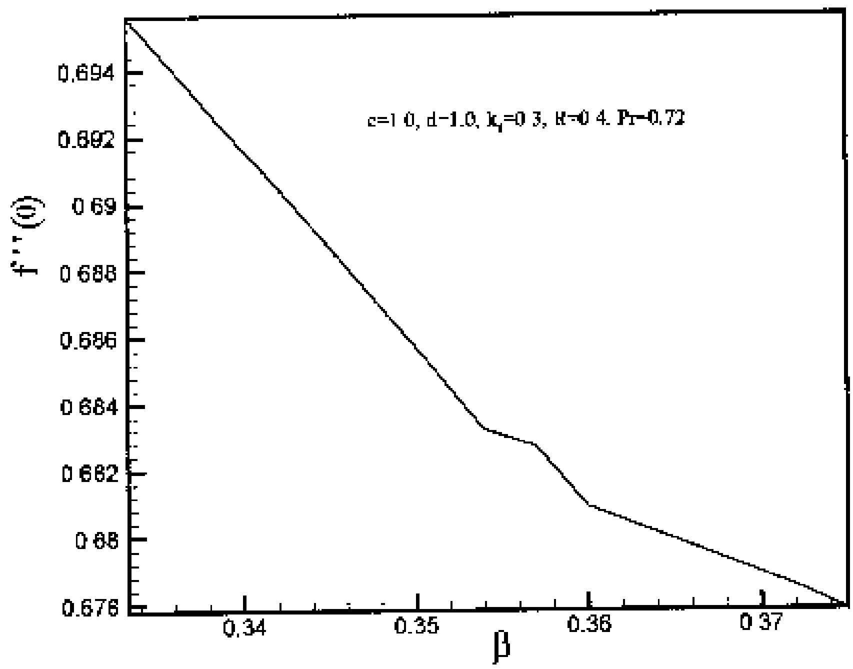

Figure 7(a): Variation of the dimensionless skin friction factor f'' along the u-direction with β [Based on equation(4.26-4.29)]

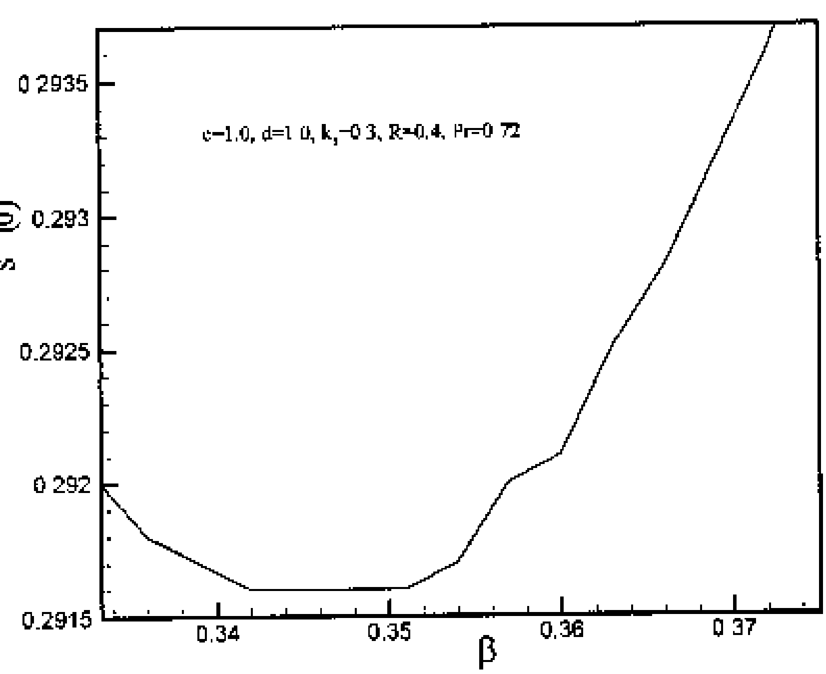

Figure 7(b): Variation of the dimensionless skin friction factor s'' along the v-direction with β [Based on equation(4.26-4.29)]

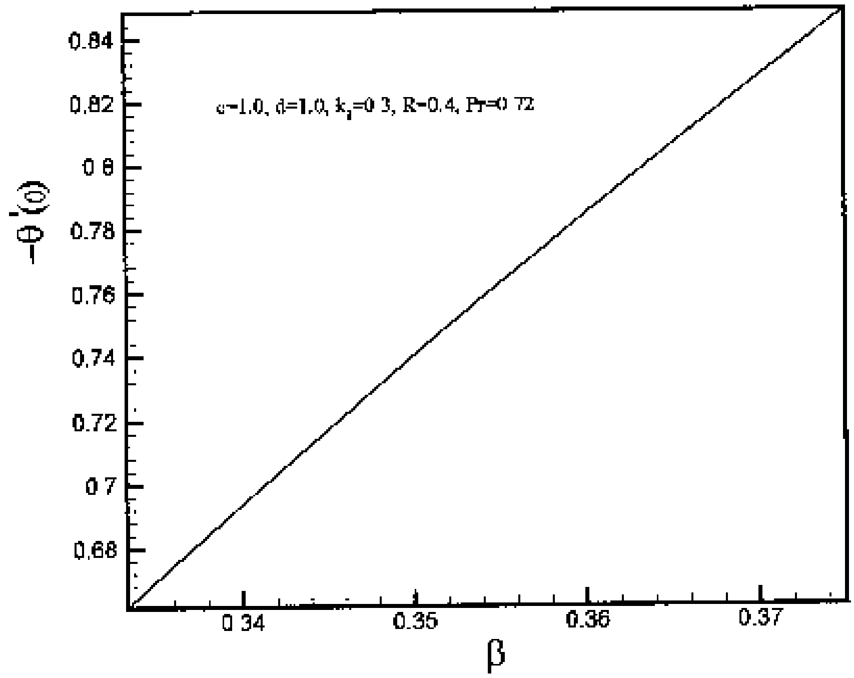

Figure 7(c). Variation of the dimensionless heat transfer factor $-\theta'$ with β [Based on equation(4.26-4.29)].

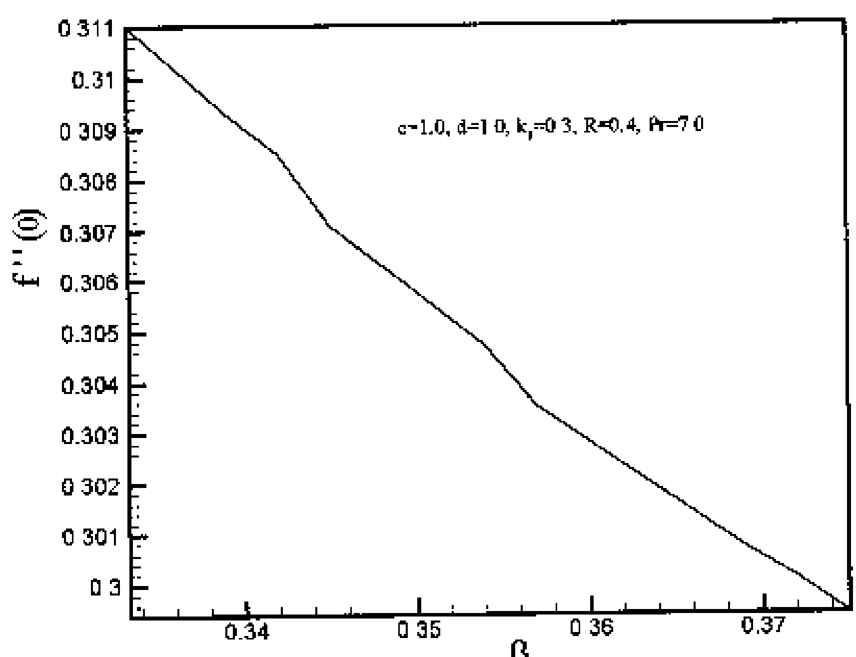

Figure 8(a): Variation of the dimensionless skin friction factor f'' along the u-direction with β [Based on equation(4.26-4.29)].

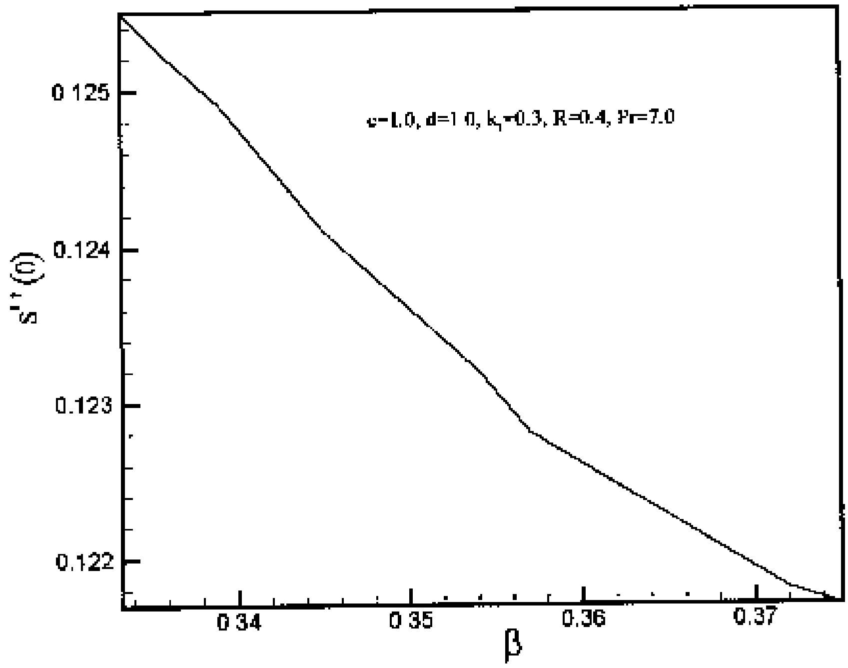

Figure 8(b): Variation of the dimensionless skin friction factor s'' along the y-direction with β [Based on equation(4.26-4.29)].

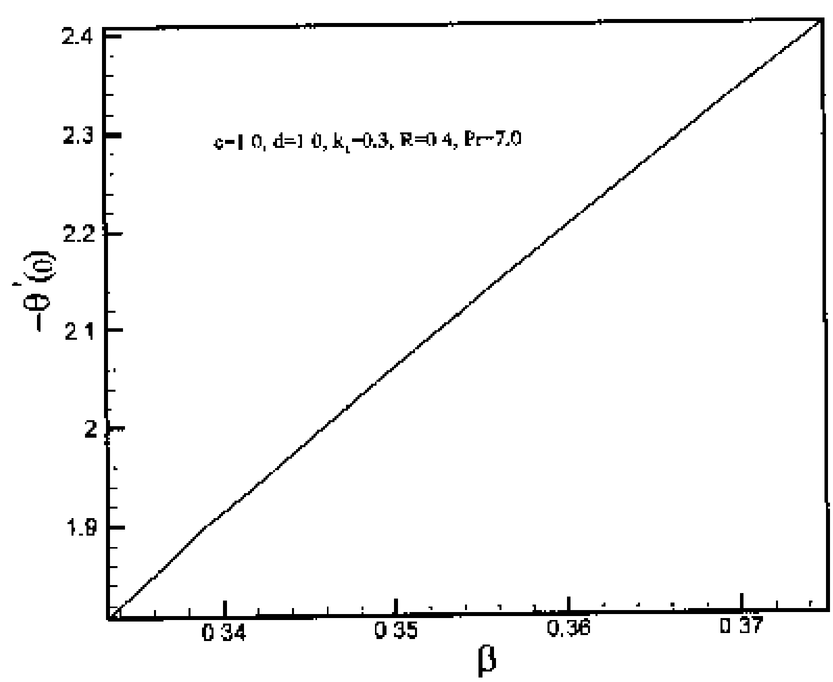

Figure 8(c): Variation of the dimensionless heat transfer factor $-\theta'$ with β [Based on equation(4.26-4.29)].

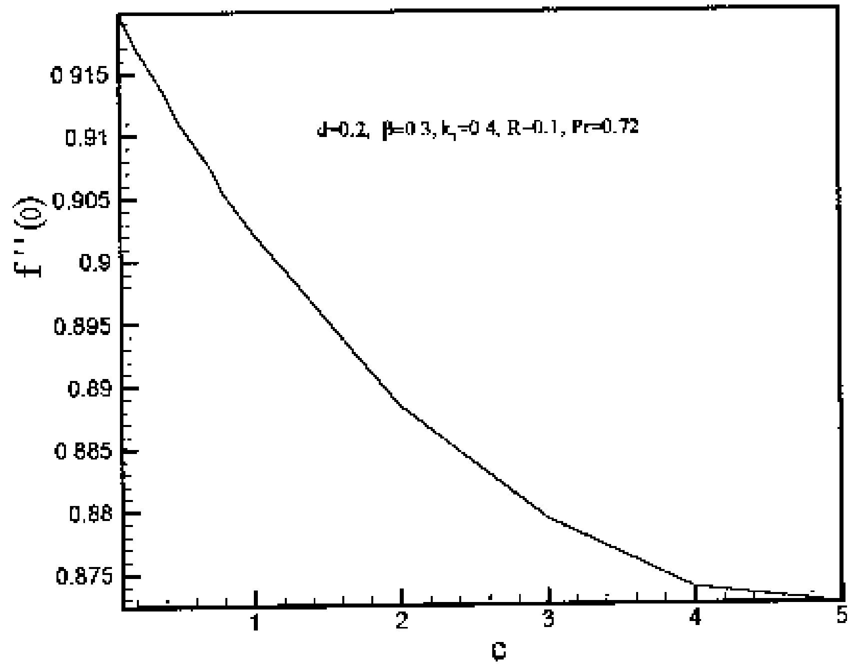

Figure 9(a): Variation of the dimensionless skin friction factor f'' along the u-direction with c [Based on equation(4.26–4.29)].

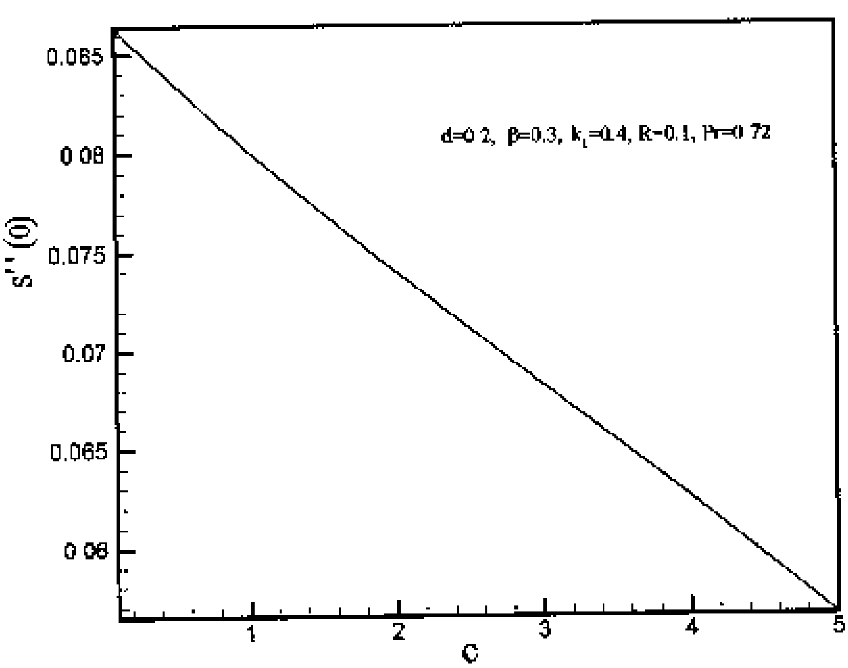

Figure 9(b): Variation of the dimensionless skin friction factor s'' along the v-direction with c [Based on equation(4.26–4.29)].

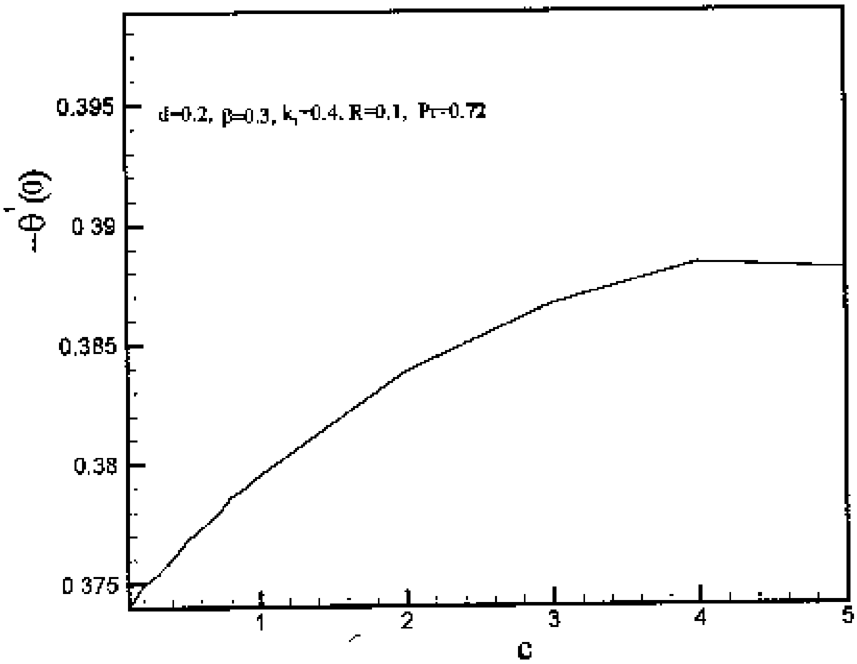

Figure 9(c): Variation of the dimensionless heat transfer factor $-\theta'$ with c [Based on equation(4.26-4.29)].

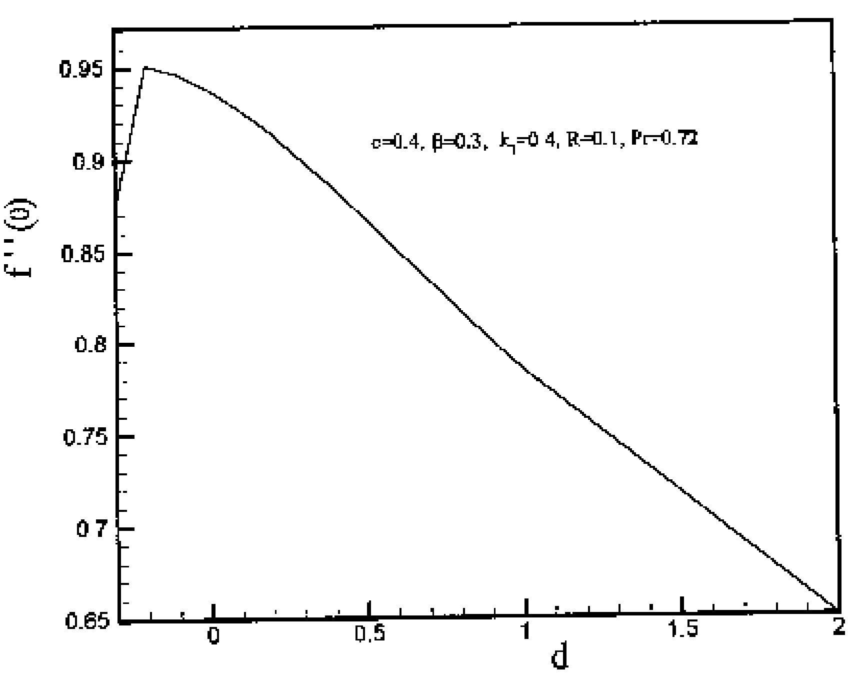

Figure 10(a): Variation of the dimensionless skin friction factor f'' along the u-direction with d [Based on equation(4.26-4.29)].

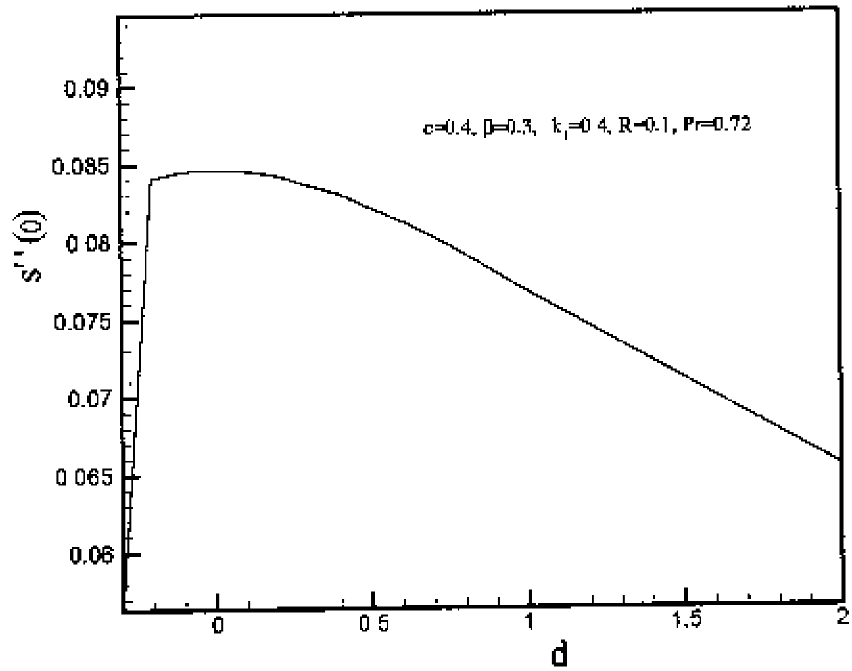

Figure 10(b): Variation of the dimensionless skin friction factor s'' along the y-direction with d [Based on equation(4.26-4.29)].

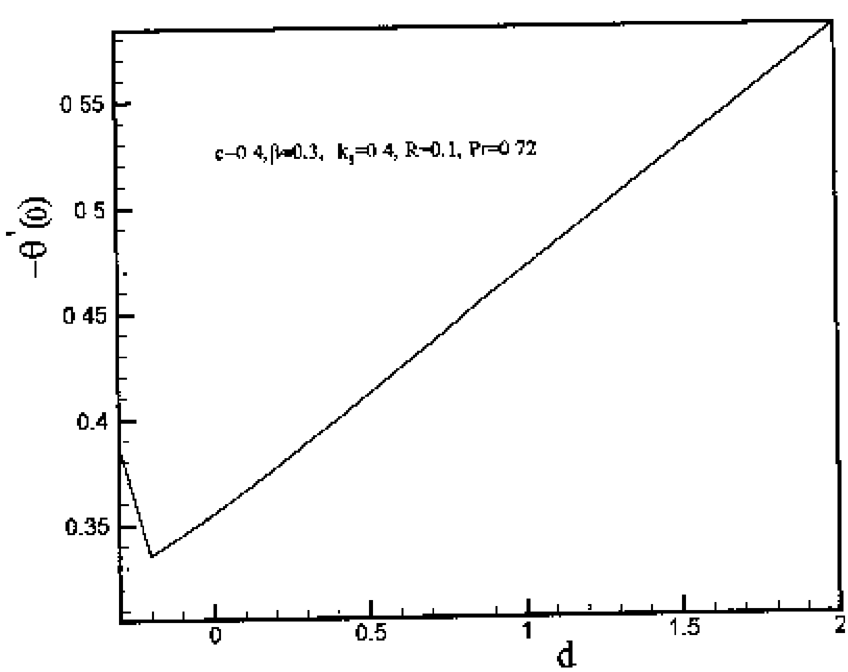

Figure 10(c): Variation of the dimensionless heat transfer factor $-\theta'$ with d [Based on equation(4.26-4.29)]

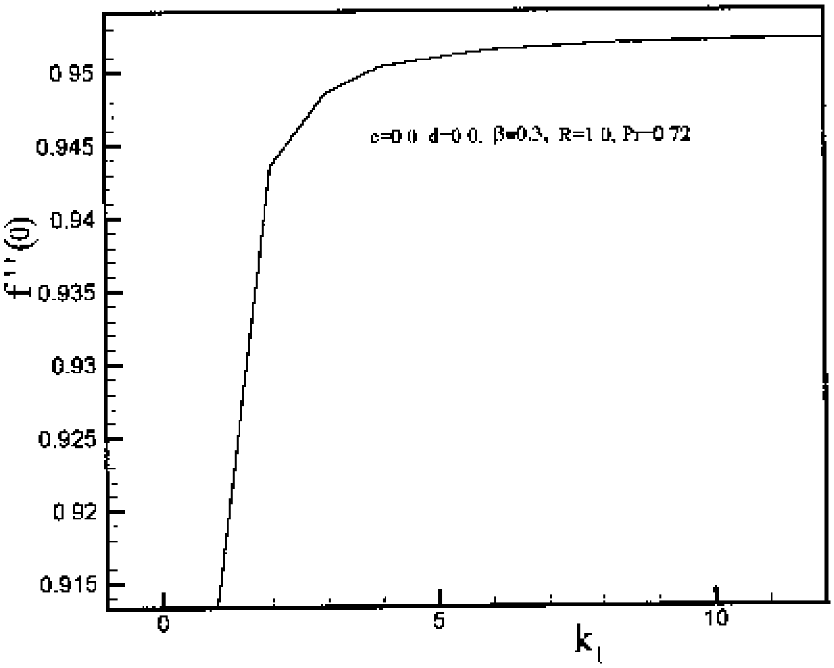

Figure 11(a): Variation of the dimensionless skin friction factor f'' along the u-direction with k_1 [Based on equation(4.26-4.29)].

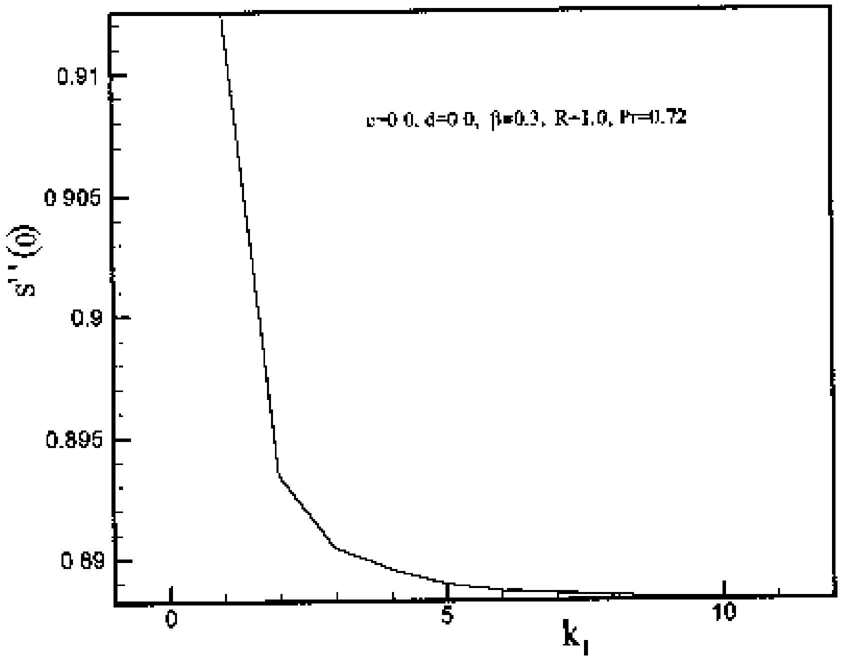

Figure 11(b): Variation of the dimensionless skin friction factor s'' along the v-direction with k_1 [Based on equation(4.26-4.29)]

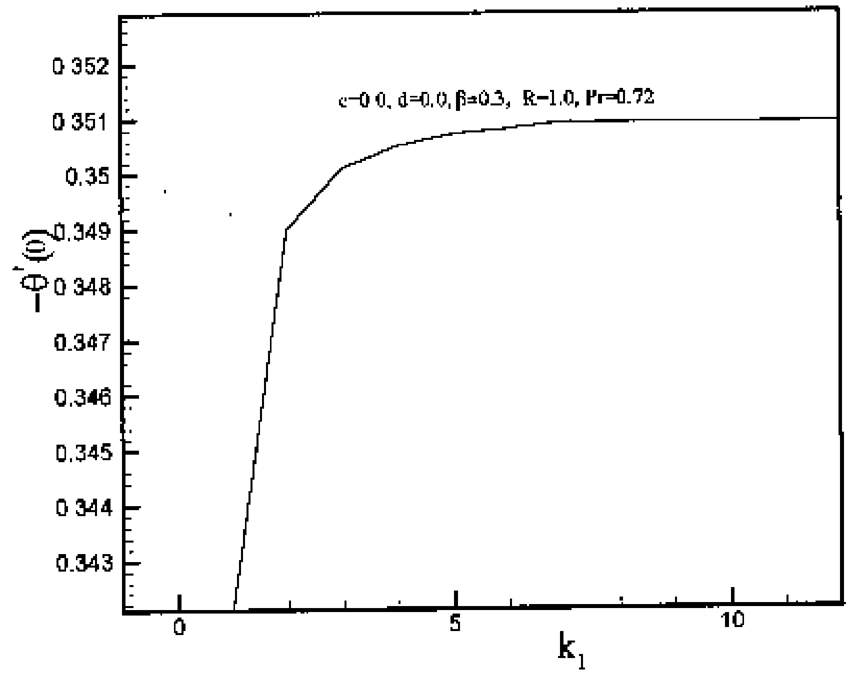

Figure 11(c): Variation of the dimensionless heat transfer factor $-\theta'$ with k_1 [Based on equation(4 26-4.29)].

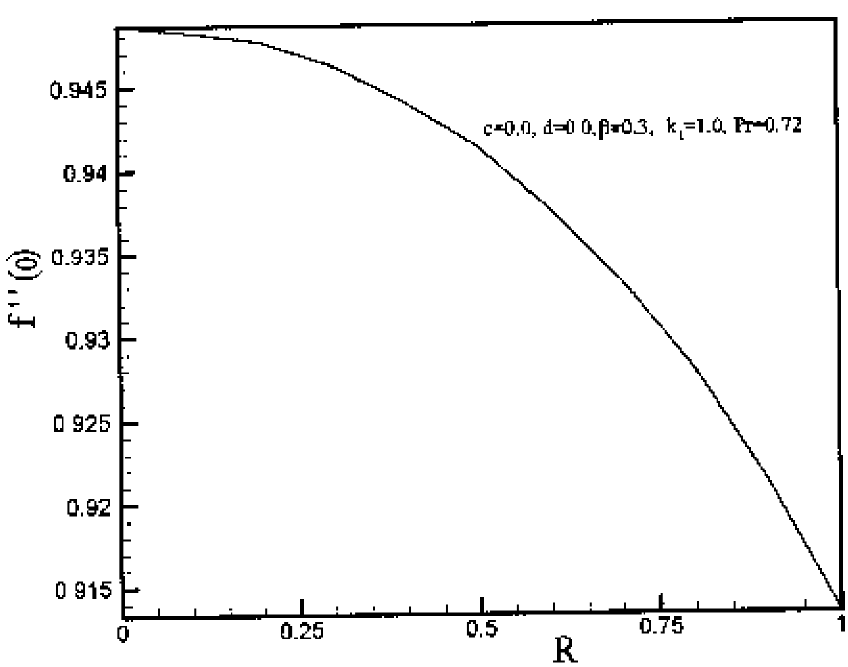

Figure 12(a): Variation of the dimensionless skin friction factor f'' along the u-direction with R [Based on equation(4 26-4.29)].

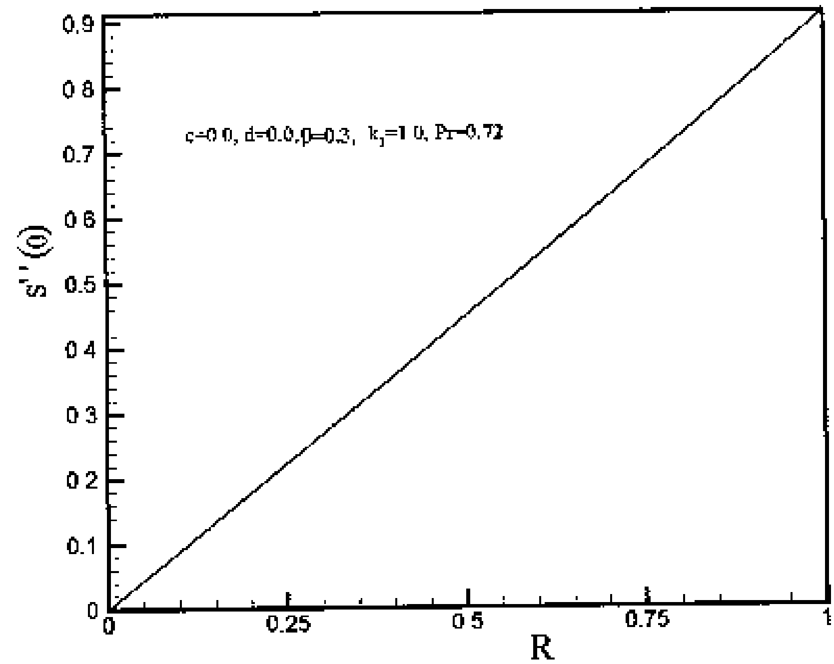

Figure 12(b): Variation of the dimensionless skin friction factor s'' along the y-direction with R [Based on equation(4.26-4.29)].

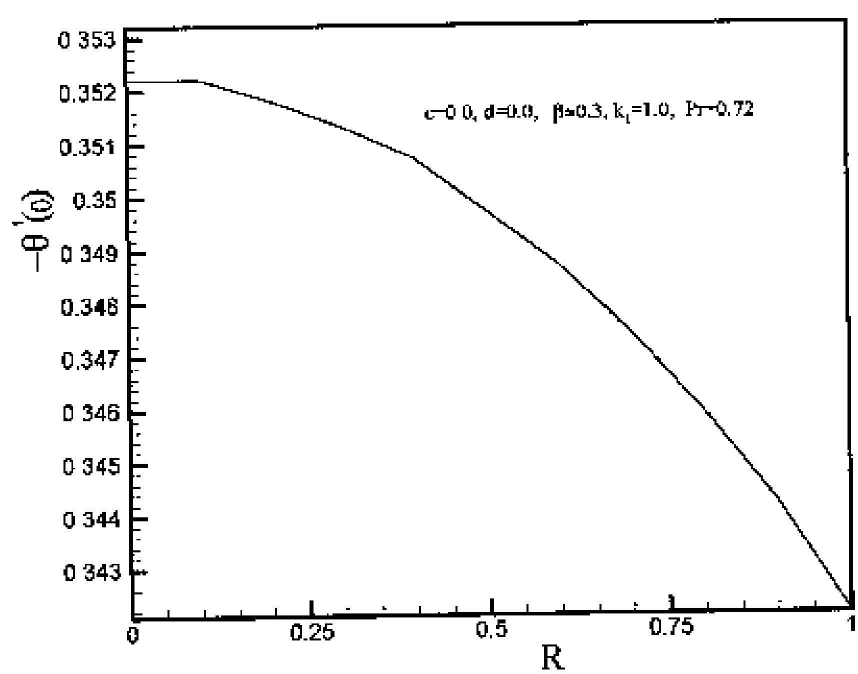

Figure 12(c): Variation of the dimensionless heat transfer factor $-\theta'$ with R [Based on equation(4.26-4.29)].

Table – (1)

β	$f''(0)$	$s''(0)$	$-\theta'(0)$
0.3333	0.6956	0.2920	0.6613
0.3360	0.6940	0.2918	0.6744
0.3390	0.6922	0.2917	0.6887
0.3420	0.6905	0.2916	0.7029
0.3450	0.6887	0.2916	0.7169
0.3480	0.6869	0.2916	0.7307
0.3510	0.6851	0.2916	0.7445
0.3540	0.6832	0.2917	0.7580
0.3570	0.6827	0.2920	0.7710
0.3600	0.6809	0.2921	0.7845
0.3630	0.6803	0.2925	0.7975
0.3660	0.6797	0.2928	0.8103
0.3690	0.6791	0.2932	0.8231
0.3720	0.6785	0.2936	0.8358
0.3750	0.6778	0.2941	0.8484

c=1.0, d=1.0, k_1=0.3, R=0.4, Pr=0.72

Table – (2)

β	$f''(0)$	$s''(0)$	$-\theta'(0)$
0.3333	0.3110	0.1255	1.8072
0.3560	0.3102	0.1252	1.8489
0.3390	0.3093	0.1249	1.8984
0.3420	0.3085	0.1245	1.9402
0.3450	0.3071	0.1241	1.9852
0.3480	0.3063	0.1238	2.0295
0.3510	0.3055	0.1235	2.0734
0.3540	0.3047	0.1232	2.1167
0.3570	0.3035	0.1228	2.1597
0.3600	0.3028	0.1226	2.2021
0.3630	0.3021	0.1224	2.2441
0.3660	0.3014	0.1222	2.2856
0.3690	0.3007	0.1220	2.3267
0.3720	0.3001	0.1218	2.3675
0.3450	0.2994	0.1217	2.4076

c=1.0, d=1.0, k_1=0.3, R=0.4, Pr=7.00

Table – (3)

c	$f''(0)$	$s''(0)$	$-\theta'(0)$
0.1000	0.9198	0.0864	0.3740
0.2000	0.9173	0.0856	0.3749
0.3000	0.9154	0.0849	0.3754
0.4000	0.9135	0.0842	0.3760
0.5000	0.9111	0.0835	0.3768
0.6000	0.9094	0.0828	0.3773
0.7000	0.9077	0.0821	0.3778
0.8000	0.9054	0.0814	0.3786
0.9000	0.9038	0.0808	0.3790
1.000	0.9022	0.0801	0.3795
2.000	0.8884	0.0740	0.3838
3.000	0.8793	0.0682	0.3866
4.000	0.8737	0.0625	0.3883
5.000	0.8724	0.0565	0.3887

d=0.2, β=0.3, k_1 = 0.4, R = 0.1, Pr=0.72

Table – (4)

d	$f''(0)$	$s''(0)$	$-\theta'(0)$
-0.3000	0.8724	0.0563	0.3887
-0.2000	0.9508	0.0841	0.3352
-0.1000	0.9458	0.0845	0.3445
0.0000	0.9374	0.0846	0.3545
0.1000	0.9263	0.0845	0.3651
0.2000	0.9135	0.0842	0.3760
0.3000	0.8984	0.0836	0.3875
0.4000	0.8834	0.0830	0.3988
0.5000	0.8671	0.0821	0.4105
0.6000	0.8501	0.0812	0.4225
0.7000	0.8338	0.0802	0.4341
0.8000	0.8166	0.0791	0.4463
0.9000	0.7999	0.0779	0.4583
1.0000	0.7841	0.0767	0.4700
2.0000	0.6493	0.0654	0.5839

c=0.4, β=0.3, k_1=0.4, R=0.1, Pr=0.72

Table – (5)

R	$f''(0)$	$s''(0)$	$-\theta'(0)$
0.0000	0.9486	0.0000	0.3522
0 1000	0.9482	0.0885	0.3522
0 2000	0.9476	0.1773	0.3518
0.3000	0.9462	0.2663	0.3513
0.4000	0.9440	0.3557	0.3507
0.5000	0.9414	0 4459	0.3497
0.6000	0.9376	0.5366	0.3487
0.7000	0.9332	0 6285	0.3474
0.8000	0.9279	0 7216	0.3459
0.9000	0.9212	0.8161	0 3442
1.0000	0.9133	0 9125	0.3421

c=0.0, d=0.0, β=0.3, k_1=1.0, Pr=0.72

Table – (6)

k_1	$f''(0)$	$s''(0)$	$-\theta'(0)$
-1.0000	0.9133	0.9125	0.3421
1.0000	0.9133	0 9125	0.3421
2.0000	0.9435	0.8934	0 3490
3.0000	0.9485	0.8904	0.3501
4.0000	0.9503	0.8895	0.3505
5 0000	0.9509	0 8889	0 3507
6 0000	0.9514	0.8886	0.3508
7.0000	0.9516	0.8885	0.3509
8.0000	0.9518	0.8884	0.3509
9.0000	0.9519	0.8883	0.3509
10.0000	0.9520	0.8883	0.3509
11.0000	0.9521	0.8882	0.3509
12.0000	0.9521	0.8882	0.3509

c=0.0, d=0 0, β = 0.3, R=1.0, Pr=0.72

Results and Discussion

In the present investigation, possible similarity solutions of unsteady laminar boundary layer free convection flow around a vertical curvilinear surface is solved numerically by Nachtsheim-Swigert iteration technique. The calculations were carried out for several values of parameter β (Table 1 and Table 2) for Pr = 0 72 and Pr = 7.0. For constant all temperature $\beta = \frac{1}{3}$ and constant heat flux $\beta = \frac{3}{8}$. We display numerical values of $f''(0), s''(0), -\theta'(0)$ in the range $.3333 \leq \beta \leq .3750$ in tabular form. The values for $f''(0)$, $s''(0)$, $-\theta'(0)$ may be obtained at the rectangular body surface ($\phi = 0$) which are required in evaluating the skin-frictions and heat transfer co-efficients.

Figures 1(a, b) and 1(c) represent respectively dimensionless velocity and temperature profiles for $d = 0.3$, $\beta = 0.3$, $k_1 = 0.4$, $R = 0.5$, Pr = 0.72 with several values of c. The velocity profiles vary as usual with the parameter c. From figure 1(a,b) it can be concluded that the velocity profile decreases as the values of the parameter c(0.0-1 5) increases Near the surface velocity profile increases, becomes maximum and then decreases and finally takes asymptotic values From figure 1(c) we observe that the temperature profile is large near the surface and decreases away from the surface and finally takes asymptotic value. Here we also see that temperature profile decreases with the increases of the parameter c

Figures 2(a) and 2(c) represent respectively dimensionless velocity and temperature profiles for c = 0.4, $\beta = 0.3$, $k_1 = 0.4$ R = 0 5 and Pr = 0.72 From Figure 2(a,b) we observe that the velocity profile decreases owing to increase in the value of the parameter d. Near the surface velocity profile becomes maximum and then decreases and finally takes asymptotic values.

From figure 5(c) we see that the temperature profile remains unchanged for different values of the parameter R For $0.0 \leq R \leq 0.8$ the temperature profile becomes maximum at the surface of the plate then decreases away from the plate and finally takes asymptotic value at ϕ = 4.2. From 5(b) it is observed that the velocity profile increases as the value parameter R increases.

From figure 9(a) and 9(b) we observe that along u-direction skin friction gradually decreases with the increasing of parameter c, the skin friction more decreases with the increasing of c. The fig. 9(c) asserts that the parameter c increases the heat transfer rate highly with its increasing value. With one of the parameter $R \to 0, V_F \to 0,$ the equation (3.26-3.29) may be well compared with 2-dimenssional equations of renowned authors.

Chapter-6

Conclusion

An analysis is made of three dimensional unsteady laminar boundary layer equations for free convection flow around a curvilinear surface, in order to establish necessary and sufficient conditions under which similarity solutions are possible. On the basis of these conditions, out of eight possible cases five cases have been studied here. The remaining three steady possible cases were derived by **Khan** (1998). An additional parameter d, which is the ratio of boundary layer thickness due to variations with respect to position and time is established here. For this situation; the possible variation in ΔT and the scale factors h_1 and h_2 are found in the similarity solution for the momentum equations and energy equation. Hence with the positive real value of d, the flow parameters like skin friction coefficients $\left(=\tau_{w1}(0), \tau_{w2}(0)\right)$ are found to decrease while heat transfer coefficient $\left(=q_w(0)\right)$ increases. Such effects are quite remarkable in presence of other parameter $k_1\left(=\dfrac{U_F}{V_F}\right)$ in the momentum equations, but the energy equation is of free from this parameter. Further investigation are necessary to draw the overall remarks conclusively.

References

Acrivos, A. 1960 A theoretical analysis of laminar natural convection heat transfer to non-Newtonian fluids. A. I. Ch. E. Jl 6, 584-590.

Blasius, H. 1908 Grenzschichten in Flussigkeiten mit kleiner Reibung. Z. Math. Physik, Bd. 56, 1-37. Engl. Translation in NACA-TM-1256.

Braun, W. H., Ostrach, S. and Heighway, J. E. 1961 Free-convection similarity flows about two-dimensional and axisymmetric bodies with closed lower ends. Int. J. Heat Mass Transfer Vol. 2, 121–135.

Chaudhary, M.A., Merkin, J. H. and Pop, I 1995 Similarity solutions in free convection boundary layer flows adjacent to vertical permeable surfaces in porous media. Eur J. Mech., B/Fluids, Vol. 14, No. 2, 217–237.

Eckert, E. R. G., and Soehngen, E. E 1948 Studies on heat transfer in laminar free convection with Zehnden-Mech interferometir. Tech. Rept 5747, ATI 44580, Air Materiel Command (Dayton, Ohio)

Elliot, L. 1970 Free convection on a two-dimensional or axisymmetric body Quarterly Journal of Mechanics and Applied Mathematics 23 153 – 62.

Ganapathy, R. 1997 Time-dependent free convection motion and heat transfer in an infinite porous medium induced by a heated sphere. Int. J. Heat Mass Transfer. Vol. 40, No. 7, 1551–1557.

Goldstein, R. J., and Brigges, D G 1964 Transient free convection about vertical plates and circular cylinders. Journal of heat Transfer 86, 490–500.

Hansen, A. G., and Ohio, C. 1958 Possible Similarity solutions of the Laminar, Incompressible, Boundary-Layer Equation. Trans. ASME 80, 1553.

Hellums, J D., and Churchill, S W. 1962 Transient and steady state, free and natural convection, Numerical solutions AIChE Journal 8, 690–5.

Jafarpur, K., and Yovannovich, M. M	1992	Laminar free convective heat transfer from isothermal spheres: a new analytical method Int J. Heat Mass Transfer. Vol. 35, No. 9, 2195–2201
Jayaraj, S., Dinesh, K. K., and Pillai, K. L	1999	Thermophoresis in natural convection with variable properties. Heat and Mass Transfer 34, 469–475.
Jai, H., and Gogose, G.	1996	Laminar natural convection heat transfer from isothermal spheres. Int. J. Heat Mass Transfer, Vol. 39, No. 8, 1603–1615.
Johnson, C H., and Cheng, P.	1978	Possible similarity solutions for free convection boundary layers adjacent to flat plates in porous media. Int. J. Heat Mass Transfer, Vol. 21, 709–718.
Khan, M. A. H.	1998	Free convective laminar boundary layer flows about an inclined vertical curved surface. M. Phil Thesis., Math , BUET
Kumari, M., Slaouti, A., Nakamura, S., Takhar, H. S., and Nath, G.	1996	Unsteady free convection flow over a continuous moving vertical surface. Acta Mechanica 116, 75–82.
Kumari, M., and Nath, G.	1984	Unsteady free convection boundary layer flow at a three-dimensional stagnation point. Engineering Transactions 32, 3–12.
Merkin, J H.	1985	A note on the similarity solutions for free convection on a vertical plate Journal of Engineering Mathematics 19, 189–201.
Merkin, J. H., and Pop, I.	1996	Conjugate free convection on a vertical surface. Int J. Heat Mass Transfer, Vol. 39, No. 7, 1527–1534.
Nanbu, K.	1971	Limit of pure conduction for unsteady free convection on a vertical plate Int. J. Heat Mass Tranfer, Vol. 14, No 4, 1531.

Ostrach, S. 1953 New aspects of natural convection heat transfer. Trans ASME 75, 1287.

Pop, I., and Takhar, H. S. 1993 Free Convection from a Curved Surface. ZAMM. Z. angew, Math. Mech. 73, 6, 534–539.

Rees, D. A. S., and Pop, I. 1995 Free Convection Induced by a Vertical Wavy Surface With Uniform Heat Flux in a Porous Medium. Transactions of the ASME, Vol. 117, 547–550

Sattar, M. A., and Alam, M. M. 1994 Thermal diffusion as well as transpiration effects on MHD free convection and mass transfer flow past an accelerated vertical porous plate. Indian Journal of Pure and Applied Mathematics 25, 679–88

Schmidt, E., and Bechmann, W. 1930 Das Temperature-und Geschwidigkeitsfeld vor einer Wärme abgebenden senkrechter platte bei natürlicher konvection. Tech. Mech u. Termodynamik, Bd. 1, (10), 341–349.

Schuh, H. 1948 Boundary Layer of temperature. Gt. Brit. Ministry of Air Production Volkenrode Repts. and Transls, 1007.

Slaouti, A., Takhar, H. S., and Nath, G. 1998 Unsteady free convection flow in the stagnation piont region of a three-dimensional body. Int. J. Heat Mass Transfer, 41, 3397–3408.

Soundalgekar, V. M. 1972 Viscous dissipation effects on unsteady free convective flow past an infinite, vertical porous plate with constant suction. Int. J. Heat Mass Transfer, Vol. 15, 1253–1261.

Sparrow, E. M. and Gregg, J. L. 1958 Similar solutions for free convection from a non-isothermal vertical plate. Trans. ASME, Vol. 80, 379.

Stewart, W. E. 1971 Asymptotic calculation of free convection in Laminar Three-dimensional systems. Int. J. Heat Mass Transfer, Vol. 14, 1013–1031

Suwono, A.	1980	Laminar free convection Boundary Layer in three-dimensional systems. Int. J. Heat Mass Transfer, Vol. 23, 53–61.
Williams, J. C., Mulligan, J. C., and Rhyne, T. B.	1987	Semi-similar solutions for unsteady free-convection boundary layer flow on a vertical that plate. Journal Fluid Mechanics 175, 309–32.
Yang, K. T.	1960	Possible similarity solutions for laminar free convection on vertical plates and cylinders J. Appl. Mech. 27, 230.
Zakerullah, M., and Maleque, K. A	1998	Similarity requirements for combined forced and free convective laminar boundary layer flow over vertical curvilinear surfaces. Journal of Bangladesh Academy of Sciences, Vol. 22, No. 1, 79–90.