Effect Of Hyperbolic Interpolation In The Finite Element Solution Of An Eigenvalue Problem

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by

Humaira Farzana
Student No. 100109011 P
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Department of Mathematics
BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY,
DHAKA-1000
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Submitted by

Humaira Farzana

Student No. 100109010 P, Session: October- 2001, a part time student of M. Phil. (Mathematics) has been accepted as satisfactory in partial fulfillment of the requirement for the degree of

Master of Philosophy in Mathematics

on 24 March, 2007.

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(Humaira Farzana)

Date:

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The Author

Date :

Abstract

Finite element method is an efficient method for solving ordinary and partial differential equations in both linear and nonlinear cases that arise in different branches of applied sciences such as heat transfer, fluid flow, solid mechanics, quantum mechanics. All kinds of problems such as initial and boundary value problems and eigenvalue problems are solved by using finite element method. In all these cases algebraic polynomial or Lagrange interpolation function is used to approximate the field variable.

In our present study we have replaced the Lagrange interpolation function by the Hyperbolic interpolations namely sine and tangent hyperbolic interpolation in solving an eigenvalue problem by finite element method. The result shows that eigenvalues obtained by using sine and tangent hyperbolic interpolation agree well with those of Lagrange interpolation.

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Chapter 1
INTRODUCTION

1.1 General

Finite element method is one of the best numerical methods for solving a wide variety of practical problems in the field of applied science and engineering. It has its superiority other methods because of its ability to solve problems concerning domain with irregular geometry and heterogeneous composition. Various problems such as Heat Transfer [1], Nonlinear [2], Transient [3] and Eigenvalue Problems [4–6] are solved by using finite element method. Because of the close similarity between the equations of eigenvalue and boundary value problems the steps involved in the construction of their finite element models are entirely analogous. Galerkin weighted residual procedure is used to deduce the finite element model of eigenvalue problem.

Differential eigenvalue problems are then reduced to algebraic eigenvalue problems by means of the finite element approximation. For most of the problems A and B will be symmetric matrices of order n and $\bar{X}$ is a column vector with n components called the eigenvector. If the physical problem is the free vibration analysis of a structure A will be the stiffness matrix B will be mass matrix and λ is the square of natrual frequency and $\bar{X}$ is the mode shape of the vibrating structure. The eigenvalue problem can be written as $(\mathbf{A}-\lambda\mathbf{B})\mathbf{X}=0$. There are various methods of solving algebraic eigenvalue problem such as Jacobi method, Power method, subspace iteration method etc. In solving our problem Jacobi method has been used.

In this method the real symmetric matrix A is reduced to a diagonal matrix by a series of orthogonal transformation S_1, S_2---------- in 2×2 subspaces. When the diagonalization is completed the eigenvalues are located on the diagonal and the orthogonal matrix of eigenvectors is obtained as the product of all orthogonal transformation.

Here will be considered the solution of the eigenvalue problem $-\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\psi}{dx}\right)-\frac{2}{x}\psi=\lambda\psi$ which was solved by Ramachan et al [6], in a

different manner. In general Lagrange interpolation and Hermite interpolations are used in solving the problem by finite element method. Lagrange interpolation is used in solving second order problem and Hermite is used in fourth order problem. Rammohan et al used Lagrange interpolation. It has been replaced by Hyperbolic interpolation and is solved the problem to find the eigenvalues. Numerical calculation will be performed by using FORTRAN programming language.

1.2 Objectives of the Work

The main object of this study is to find the effect in the solution of the eigenvalue problem [6] by finite element method if the hyperbolic interpolation is used instead of Lagrange interpolation. Rammohan et al [6] calculated the eigenvalues using linear element and Lagrange interpolation. We shall find the eigenvalues for quadratic and cubic element using Lagrange interpolation.

Eigenvalues will be calculated for the linear, quadratic element using hyperbolic interpolation instead of Lagrange interpolation and compare the results with those obtained by Lagrange interpolation to observe the effect of hyperbolic interpolation on the solution of eigenvalue problem.

1.3 Review of the Earlier Works

The conception of finite element method has come from the idea of removing the difficulties faced by weighted residual methods where the approximation function is derived from intuitive idea and the body is considered as a whole body. In the finite element method the body is discretized into sub-domains and the systematic function is derived for each sub-domain.

The use of piecewise-continuous functions defined over a sub-domain to approximate an unknown function can be found in the work of Courant [7], who used an assemblage of triangular elements and the principle of minimum total potential energy to study the St. Venant torsion problem. in 1943. Although certain key features of the finite element method can be found in the works of Hrenikoff [8] and Courant [7], its formal presentation is attributed to Argyris and Kelsey [9] and Turner, Clough, Martin and Topp [10]. The term 'finite element' was first used by Clough [11] in 1960, in his paper titled, "The Finite Element Method in Plane Stress Analysis." After its introduction it

has continually developed and improved. Since its inception the literature on finite element application has grown exponentially. While in early days the contributors have been almost engineers, today a large of them comes from the field of mathematics.

Various types of problems such as Hyperbolic Problems [12] solid mechanics [13] are solved by using finite element method. Initial value problems and boundary value problems having Dirichlet, Neumann and mixed boundary conditions are solved by this method.

Most boundary value problems have an associated eigenproblem. The two are closely related both in physical meaning and mathematical expression. Most boundary value problems have an associated eigenproblem. The two are closely related, both in physical meaning and mathematical expression.

There are presently several methods that are appropriate for the medium to large generalized eigenproblems that occur in finite element application. They take full advantage of the symmetric, positive definite, banded properties of the stiffness and mass matrices. One or more of the following methods will be found in most commercial finite element programs that solve eigenproblem [14].

1. Generalized Jacobi method
2. Householder method
3. Givens method
4. Lanczos method [15]
5. Subspace iteration method [16]

The first method is based on the classical Jacobi method [1846] for the standard eigenproblem. It was modified during 1960s to handle the generalized eigenproblem. The modification is referred to as the generelized Jacobi method. The algorithm (Converging for a wide range of problems) is one of the easiest to understand and code. The generalized Jacobi method calculates the entire eigensystem (i.e. all eigenvalues and eigenvector) and therefore by itself, would be appropriate only for small problem. In order to extend its usefulness to large problems, it is first necessary to reduce the large eigenproblems to a much smaller eigenproblem by eleminating the values of dependent variable associated with higher eigenvalues.

The resulting smaller problem models only the lowest eigenvalues and corresponding eigenvectors. An algorithm that performs such a reduction is sometimes called an eigenvalue economizer. The most widely used one at present is the Guyan reduction [17].

Different solution techniques to solve the eigenproblems are illustrated in details in [18] and [19].

Rammohan et al [6] solved the eigenvalue problem to calculate the energy levels of quantum mechanical system by finite element method. This method provided a convenient procedure for the calculation of energy eigenvalues of the quantum mechanical system. They investigated the levels of accuracy that can be attained in the method of finite elements using various approximations. They illustrated it by considering two examples which formed a convenient basis for describing the calculation technique. One of them is the radial equation for hydrogen atom for spherically symmetric states and the other is simple harmonic Oscillator in one dimension. These two illustrative examples provide guidelines in the calculation of energy eigenvalues of the hydrogen atom in an arbitrary spatially uniform magnetic field, a problem not solvable by analytical means.

Chapter 2

WEIGHTED-RESIDUAL METHODS

Weighted Residual Methods and Rayley-Ritz method play an important role for the mathematical development of finite element method. The weighted residual methods are discussed here.

Suppose a differential equation is to be solved is

$$Au = f \text{ in } \qquad (2.1)$$

where A is the operator

The function u is not only required to satisfy the equation, it is also required to satisfy some boundary conditions.

In the Weighted method the solution u is approximated as

$$u_n = \Sigma c_j \phi_j + \phi_0 \qquad (2.2)$$

ϕ_0 and ϕ_j are function of x and c_j's are parameters. ϕ_0 is required to satisfy all specified boundary conditions and ϕ_j are required to satisfy homogeneous form of all boundary conditions of the problem.

Substitution of the approximate solution u_n into the left-hand side of equation (2.1) gives a function $f_n = A(u_n)$ that, in general, is not equal to the specified function f. The difference $A(u_n) - f$, called the residual of the approximation is nonzero.

$$R = A(u_n) - f = A\left(\sum c_j \phi_j + \phi_0\right) - f \neq 0.$$

The residual R is a function of position as well as of the parameters c_js. In the Weighted residual method as the name suggests the parameters c_js are determined by requiring the residual R to vanish in the weighted integral sense.

$$\int_{\Omega} w_i(x) R(x, c_j) dx = 0 \qquad i = 1, 2, 3, \ldots\ldots\ldots, n \qquad (2.3)$$

where w_i's are the weight functions and Ω is a one-dimensional domain, which in general, are not the same as the approximation function ϕ_j. The set $\{w_i\}$ must be a

linearly independent set, otherwise, the equations provided by equation (2 3) will not be linearly independent and hence will not be solvable.

$$\int w(A(u_n) - f)dx = 0$$

$$\int w\left(A\left(\sum c_j\phi_j + \phi_0\right) - f\right)dx = 0$$

$$\int wA\sum c_j\phi_j dx = \int w(f - A(\phi_0))dx$$

$$\Sigma\left[\int wA(\phi_j)dx\right]c_j = \int w\left(f - A(\phi_0)\right)dx$$

Some of the weighted residual methods are

1. The Petrov-Galerkin Method

The weighted residual is known as Petrov-Galerkin method when $w_i \neq \phi_i$. When the operator A is linear, equation (2.3) can be simplified to the form

$$\sum_{j=1}^{n}\left[\int_{\Omega} w_i A(\phi_j)dx\right]c_j = \int_{\Omega} w_i[f - A(\phi_0)]dx \qquad (2.4)$$

$$\sum_{j=1}^{n} A_{ij}c_j = F_i \qquad i = 1,2,3\ldots\ldots,n$$

Note that the coefficient matrix **[A]** is not symmetric:

$$A_{ij} = \int_{\Omega} w_i A(\phi_j)dx \neq A_{ji}$$

2. The Galerkin Method

For the choice of weight function w equal to the approximate function ϕ_i, the Weighted-residual method is better known as Galerkin method

Now, $w = \phi_i$ (2.5)

i.e. weight function = trial function

$$\sum_{j=1}^{n}\left[\int_{\Omega}\phi_i A(\phi_j)dx\right]c_j = \int_{\Omega}\phi_i[f - A(\phi_0)]dx$$

$$\sum_{j=1}^{n} A_{ij}c_j = F_i \qquad i = 1,2,3\ldots\ldots,n$$

where

$$A_{ij} = \int_{\Omega}\phi_i A(\phi_j)dx, \qquad F_i = \int\phi_i[f - A(\phi_0)]dx$$

Once again we note that A_{ij} is not symmetric.

If the equation permits, and one wishes, the differentiation can be transferred from the solution u to the weight function $w = \phi_i$, and on thereby obtains the weak form to relax the continuity requirements on the approximation functions and include the specified natural boundary conditions of the problem.

With equation (2.5), equation (2.3) yields

$$\int_{\Omega} \phi_i(x) R(x, c_j) dx = 0; \qquad i = 1,2,\ldots\ldots,n \tag{2.6}$$

Obviously, the Galerkin method is quite powerful since it can be applied to physical problems that do not have an alternative variational formulation

3. The Least Square Method

In this method we determine the parameters c_j by minimizing the integral of the square of the residual.

$$\frac{\partial}{\partial c_i} \int_{\Omega} R^2 \left(x, c_j\right) dx \tag{2.7}$$

Choosing $\qquad w_i = \dfrac{\partial R}{\partial c_{j_i}}$

So the equation (2.3) takes the form

$$\int_{\Omega} \frac{\partial R}{\partial c_{j_i}} R\left(x, c_j\right) dx = 0 \,. \qquad j = 1, 2, 3, \ldots\ldots\ldots, n \tag{2.8}$$

In order to evaluate the choice, consider the following quantity I;

$$I = \int R^2(x, c_1, c_2, \ldots\ldots c_n) dx \tag{2.9}$$

Since we integrate w. r. to x, it follows that the quantity I depends on the parameters c_j's only, i.e. $I = I(c_1, c_2, \ldots\ldots, c_n)$. From equation (2.9) it is concluded that

$$\frac{\partial I}{\partial c_j} = 2 \int_{\Omega} R \, \frac{\partial R}{\partial c_j} dx \tag{2.10}$$

An evaluation of equation (2.9) shows that I is the square of the error, i.e. the residual, integrated over the region of interest. A comparison equation (2.8) and (2.10) reveals that

$$\frac{\partial I}{\partial c_j} = 0 \tag{2.11}$$

i.e. our choice of weight function given by equation (2.7) implies that I is stationary. As the residual R, and thus also the quantity I, can be made arbitrarily large, it is concluded that the stationary of I as expressed through equation (2.11) is a minimum. Consequently, the choice of weight function given by equation (2.7) implies that the square of the error is a minimum; hence the terminology of the least-squares method. The Least-square method always results in a symmetric in a symmetric coefficient matrix, which is clearly an advantage in numerical calculations.

$$\int_{\Omega} A(\phi_i)\left[A\left(\sum_{j=1}^{n} c_j\phi_j + \phi_0\right) - f\right]dx = 0$$

$$\sum_{j=1}^{n}\left[\int_{\Omega} A(\phi_i)\,A(\phi_j)dx\right]c_j = \int_{\Omega} A(\phi_i)[f - A(\phi_0)]dx$$

$$\sum_{j=1}^{n} A_{ij}c_j = F_i$$

Here, $$[(Au - f)] = R$$

$$A(\Sigma c_j\phi_j + \phi_0) - f = R$$

$$w_i = \frac{\partial R}{\partial c_i}$$

4. The Collocation Method

In the Collocation method, we seek an approximate solution in the form of equation (2.11) by requiring the residual in the equation to be identically zero at n selected points x_i in domain Ω.

i.e. $R(x_i, c_j) = 0,\ i = 1, 2, 3, \cdots\cdots, n$

$w = \psi_i = \delta(x - x_i) = Dirac\ delta\ function$

In this method the weight function w is chosen based on Dirac's delta function. $\delta(x - x_i)$. This function is defined as

$$\delta(x - x_i) = \begin{cases} \infty & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases} \qquad (2.12)$$

$$\int_{-\infty}^{\infty} \delta(x - x_i)dx = 1 \qquad (2.13)$$

where x_i is a given fixed value. Alternatively due to equation (2.12), it may be written equation (2.13) as

$$\int_{x_i}^{x_i^+} \delta(x-x_i)dx = 1 \tag{2.14}$$

where x_i^+ and x_i^- denote x-values slightly larger than and smaller x_i, respectively

In the point collocation method, the weight function w is chosen such that

$$w = [\delta(x-x_1)\delta(x-x_2)\cdots\cdots\delta(x-x_n)] \tag{2.15}$$

where $w = w_1, w_2, w_3, \ldots\ldots\ldots, w_n$ are known functions of x.

The fixed points $x_1, x_2, \ldots, x_n$ are chosen arbitrarily within the region $a \le x \le b$ and are called collocation points. In order to illustrate the consequence of this choice of weight function, it is evaluated

$\int_a^b w\,dx = 0$ which becomes

$$\int_a^b \delta(x-x_i)R dx = 0 \quad i = 1,2,\ldots,n \tag{2.16}$$

As Dirac's delta function is zero unless $x = x_i$ we have

$$\int_a^b \delta(x-x_i)R(x,c_j)dx = \int_{x_i}^{x_i^+} \delta(x-x_i)R(x,c_j)dx = R(x,c_j)\int_{x_i}^{x_i^+} \delta(x-x_i)dx = R(x,c_j) \tag{2.17}$$

That is, the point collocation forces the residual $R(x,c_j)$ to be zero at the collocation points.

Illustration

Let us illustrate the above methods by considering an example

Here will be shown the various weighted residual differ in how the weight function w is chosen. Obviously, the specific choice of the weight function influences the values of the parameters $c_1, c_2, \ldots\ldots, c_n$, which are to be determined.

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \qquad u(0) = 0\,, \qquad u'(1) = 1$$

The exact solution is given by

$$u(x) = \frac{2\cos(1-x) - \sin x}{\cos 1} + x^2 - 2$$

For a weighted residual method ϕ_0 and ϕ_i should satisfy the following conditions:

$\phi_0(0)=0$, $\phi_0'(1)=1$ (satisfy the actual bc.)

$\phi_i(0)=0$, $\phi_i(1)=0$ (satisfy the homogeneous form of actual bc.)

We take n=2. Let the functions satisfying the above conditions be

$$\phi_0(x)=x, \quad \phi_1=x^2(x^2-2), \quad \phi_2=x(x^2-3)$$

the residual in this approximation is

$$R=-\left(0+c_i\sum_{i=1}^{n}\frac{d^2\phi_i}{dx^2}\right)-\left(\phi_0+\sum c_i\phi_i\right)+x^2$$

$$=-c_1(10x^2+x^4-4)-c_2(x^3+3x)+x^2-x$$

Petrov-Galerkin Method

Let the weight function be

$$w=\psi_1=e^x \qquad w=\psi_2=x^4$$

Then $\int_0^1 e^x R\,dx=0$ and $\int_0^1 x^4 R\,dx=0$

$$\frac{38711}{50000}c_1+\frac{17817}{5000}c_2+\frac{29}{100}=0 \qquad \frac{422}{315}c_1+\frac{5}{8}c_2+\frac{1}{42}=0$$

$$c_1=\frac{96406}{4289985}=0.0224724, \quad c_2=-\frac{369997}{4289985}=-0.0862468$$

The solution is $u_{PG}=\phi_0+c_1\phi_1+c_2\phi_2$

$$=1.258740\,x-0.0449448\,x^2-0.0862468\,x^3+0.0224724\,x^4$$

The Galerkin Method

The weight function in this case is

$$w=\phi_1=x^2(x^2-2) \qquad w=\phi_2=x(x^2-3)$$

$$\int_0^1 x^2(x^2-2)R\,dx=0, \qquad \int_0^1 x(x^2-3)R\,dx=0$$

$$\frac{914}{105}c_1+\frac{29}{24}c_2+\frac{8}{105}=0, \qquad \frac{121}{15}c_1-\frac{68}{35}c_2-\frac{13}{60}=0$$

$$c_1=\frac{3113}{1067116}=0.00291720 \qquad c_2=-\frac{22428}{266779}=-0.08406958$$

$$u_G=1.2522087x-0.0058344x^2-0.08406958x^3+0.0029172x^4$$

The Least Square Method

Taking $w = \psi_1 = \frac{\partial R}{\partial c_1}, \quad w = \psi_2 = \frac{\partial R}{\partial c_2}$

We have,

$$\int_0^1 -\left(10x^2 + x^4 - 4\right) R\, dx = 0 \qquad \int_0^1 -\left(x^3 + 3x\right) R\, dx = 0$$

$$\frac{161}{15}c_1 \frac{67}{24}c_2 - \frac{1}{7} = 0 \qquad \frac{67}{24}c_1 + \frac{152}{35}c_2 + \frac{17}{60} = 0$$

$$c_1 = \frac{141138}{3881993} = 0.036357 \qquad c_2 = -\frac{34399205}{388199302} = -0\ 08861223$$

$$u_{LS} = 1.2658366x - 0.072714x^2 - 0.0886122x^3 + 0.036357x^4$$

The Collocation Method

Choosing the points $x = \frac{1}{3}$ and $x = \frac{2}{3}$ as the collocation points, we evaluate the residuals at these points and set them equal to zero:

$$R\left(\frac{1}{3} = 0\right), \qquad \frac{233}{81}c_1 - \frac{28}{27}c_2 - \frac{2}{9} = 0$$

$$R\left(\frac{2}{3}\right) = 0; \qquad \frac{52}{81}c_1 + \frac{62}{27}c_2 + \frac{2}{9} = 0$$

$$c_1 = \frac{1836}{47736} = 0.0384615 \quad \text{and} \quad c_2 = -\frac{5130}{47736} = -0.107466$$

The solution is given by :

$$u_c = 1.322398x - 0.0769230x^2 - 0.107466x^3 + 0.0384615x^4$$

Comparison of the Weighted-residual and exact solutions of the boundary value problem $\frac{d^2u}{dx^2} - u + x^2 = 0, \quad u(0) = 0, \quad u'(1) = 1.$

Table 2.1: Comparison of the results of different Weighted-residual methods with exact solution

Solution, u(x)					
x	u_{exact}	u_{PG}	u_G	u_{LS}	u_C
0.0	0.0000	0.0000	0 0000	0.0000	0.0000
0.1	0.1262	0.1253	0.1248	0.1258	0.1314
0.2	0.2513	0.2493	0.2495	0.2496	0.2606
0.3	0.3742	0.3714	0.3729	0.3711	0.3872
0.4	0.4943	0.4914	0.4946	0.4900	0.5108
0.5	0.6112	0.6088	0.6143	0.6059	0.6309
0.6	0.7244	0.7233	0.7311	0.7189	0.7470
0.7	0.8340	0.8343	0.8455	0.8289	0.8603
0.8	0.9402	0.9433	0.9561	0.9357	0.9694
0.9	1.0433	1.0483	1.0628	1.0396	1.0747
1.0	1.1442	1.1500	1.1652	1.1408	1.1764

Difficulties in Weighted-residual Methods:

From the above discussion of different Weighted-residual method it is clear that one is to find an approximation function for the whole domain, that satisfies the boundary conditions. The difficultly is that there is no definite procedure of finding the approximation function. One is just to suppose it.

The finite element method overcomes the difficulty of the Weighted-residual methods by providing a systematic procedure for derivation of approximation function over sub-regions of the domain. The function is $u = \sum u_i, \varphi_i$ where u_1 are the nodal values of the function and ϕ_i are the Lagrange interpolation function

So finite element method is a piecewise application of Weighted-residual methods in which the approximation function is an algebraic polynomial that are obtained from interpolation theory.

Chapter 2

WEIGHTED-RESIDUAL METHODS

Weighted Residual Methods and Rayley-Ritz method play an important role for the mathematical development of finite element method. The weighted residual methods are discussed here.

Suppose a differential equation is to be solved is

$$Au = f \text{ in} \tag{2.1}$$

where A is the operator

The function u is not only required to satisfy the equation, it is also required to satisfy some boundary conditions.

In the Weighted method the solution u is approximated as

$$u_n = \Sigma c_j \phi_j + \phi_0 \tag{2.2}$$

ϕ_0 and ϕ_j are function of x and c_j's are parameters. ϕ_0 is required to satisfy all specified boundary conditions and ϕ_j are required to satisfy homogeneous form of all boundary conditions of the problem.

Substitution of the approximate solution u_n into the left-hand side of equation (2.1) gives a function $f_n = A(u_n)$ that, in general, is not equal to the specified function f The difference $A(u_n) - f$, called the residual of the approximation is nonzero.

$$R = A(u_n) - f = A\left(\sum c_j \phi_j + \phi_0\right) - f \neq 0.$$

The residual R is a function of position as well as of the parameters $c_j s$. In the Weighted residual method as the name suggests the parameters $c_j s$ are determined by requiring the residual R to vanish in the weighted integral sense.

$$\int_{\Omega} w_i(x) R(x, c_j) dx = 0 \qquad i = 1, 2, 3, \ldots\ldots, n \tag{2.3}$$

where w_i's are the weight functions and Ω is a one-dimensional domain, which in general, are not the same as the approximation function ϕ_i. The set $\{w_i\}$ must be a

$$y = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} y_3$$

$$= N_1 y_1 + N_2 y_2 + N_3 y_3$$

where $$N_1 = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$N_2 = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$N_3 = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

Shape functions are given in the following figure.

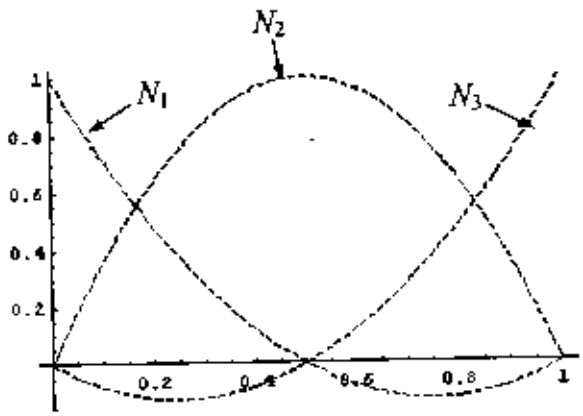


Fig. 3.2: Shape functions for quadratic element

For cubic element

$$y = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1 + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3 + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} y_4$$

$$= N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

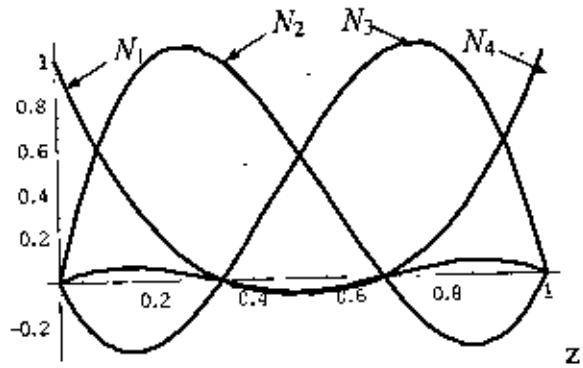


Fig. 3.3: Shape functions for cubic element

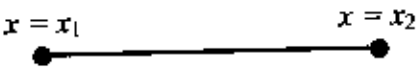


Fig. 3.4: Nodal numbers of the linear element

For the purpose of numerical integration using Gauss's quadrature formula and to convert to the limits of integration from $x = x_1$ to $x = x_2$ to the limits

$\xi = -1 \; to \; \xi = 1.$

Fig. 1 represents a linear element

Substitution is $x = \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2}\xi$

$N_1 = \frac{x_2 - x}{x_2 - x_1}$ and $N_2 = \frac{x - x_1}{x_2 - x_1}$

$$N_1 = \frac{x_2 - \left(\frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2}\xi\right)}{x_2 - x_1}$$

$$= \frac{1}{2}\frac{(x_2 - x_1)(1 - \xi)}{x_2 - x_1}$$

$$= \frac{1}{2}(1 - \xi) \quad \text{.....(3.1.1a)}$$

$$N_2 = \frac{\left(\frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2}\xi - x_1\right)}{x_2 - x_1}$$

$$= \frac{1}{2}\frac{(x_2 - x_1)(1 + \xi)}{x_2 - x_1}$$

$$= \frac{1}{2}(1 + \xi) \quad \text{.....(3.1.1b)}$$

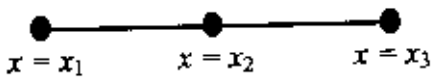


Fig. 3.5: Nodal numbers of the quadratic element.

Substituting $x = \frac{x_1 + x_3}{2} + \frac{x_3 - x_1}{2}\xi$ and $x_2 = \frac{x_1 + x_3}{2}$

$$N_1 = \frac{(x - x_2)(x - x_3)}{(x_1 - x_3)(x - x_3)}$$

$$N_1 = \frac{\left(\frac{x_1+x_3}{2}+\frac{x_3-x_1}{2}\xi-\frac{x_1+x_3}{2}\right)\left(\frac{x_1+x_3}{2}+\frac{x_3-x_1}{2}\xi-x_3\right)}{\left(x_1-\frac{x_1+x_3}{2}\right)(x_1-x_3)}$$

$$N_1 = \frac{\left(\frac{x_3-x_1}{2}\xi\right)\left\{\frac{x_3-x_1}{2}(\xi-1)\right\}}{\left\{\frac{1}{2}(x_1-x_3)\right\}(x_3-x_1)} = -\frac{1}{2}\xi(1-\xi) \qquad (3.1.2a)$$

Similarly,

$$N_2 = (1-\xi)(1+\xi) = 1-\xi^2 \qquad (3.1.2b)$$

$$N_3 = \frac{1}{2}\xi(1+\xi) \qquad (3.1.2c)$$

For cubic element

$$N_1 = -\frac{9}{10}(1-\xi)\left(\frac{1}{3}+\xi\right)\left(\frac{1}{3}-\xi\right) \qquad (3.1.3a)$$

$$N_2 = \frac{27}{16}(1+\xi)(1-\xi)\left(\frac{1}{3}-\xi\right) \qquad (3.1.3b)$$

$$N_3 = \frac{27}{16}(1+\xi)(1-\xi)\left(\frac{1}{3}+\xi\right) \qquad (3.1.3c)$$

$$N_4 = -\frac{9}{16}\left(\frac{1}{3}+\xi\right)\left(\frac{1}{3}-\xi\right)(1+\xi) \qquad (3.1.3d)$$

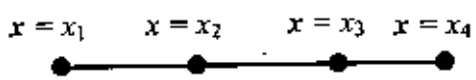


Fig. 3.6: Nodal numbers of the cubic element.

3.2 Hyperbolic Interpolation

Hyperbolic interpolation is

$$y = \frac{\sinh(x-x_2)\sinh(x-x_3)\cdots\cdots\sinh(x-x_n)}{\sinh(x_1-x_2)\sinh(x_1-x_3)\cdots\cdots\sinh(x_1-x_n)}y_1$$

$$+\frac{\sinh(x-x_1)\sinh(x-x_3)\cdots\cdots\sinh(x-x_n)}{\sinh(x_2-x_1)\sinh(x_2-x_3)}y_2+\cdots\cdots$$

$$+\frac{\sinh(x-x_1)\sinh(x-x_2)\cdots\cdots\sinh(x-x_{n-1})}{\sinh(x_n-x_1)\sinh(x_n-x_2)\cdots\cdots\sinh(x_n-x_{n-1})}y_n$$

Because hyperbolic interpolation has the property of shape function it has been selected it as a new shape function

For linear element

$$N_1 = \frac{\sinh(x_2 - x)}{\sinh(x_2 - x_1)}$$

$$N_2 = \frac{\sinh(x - x_1)}{\sinh(x_2 - x_1)}$$

Shape functions are given in the following figure.

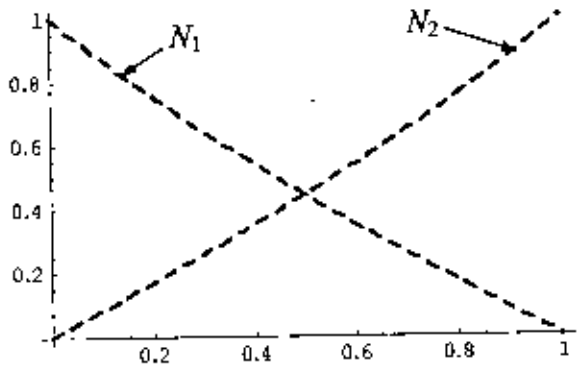


Fig. 3.7: Hyperbolic shape functions for linear element

Substituting $x = \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2}\xi$;

$$N_1 = \frac{\sinh\left\{\frac{1}{2}(x_2 - x_1)\right\}(1-\xi)}{\sinh(x_2 - x_1)}$$

Similarly, $N_2 = \frac{\sinh\left\{\frac{1}{2}(x_2 - x_1)\right\}(1-\xi)}{\sinh(x_2 - x_1)}$

Taking $x_1 = 0$, $x_2 = 1$

$$N_1 = \frac{\sinh\frac{1}{2}(1-\xi)}{\sinh(1)} \qquad (3.2.1a)$$

$$N_2 = \frac{\sinh\frac{1}{2}(1+\xi)}{\sinh(1)} \qquad (3.2.1b)$$

For Quadratic element

$$N_1 = \frac{\sinh(x - x_2)\sinh(x - x_3)}{\sinh(x_1 - x_2)\sinh(x_1 - x_3)}$$

$$N_2 = \frac{\sinh(x - x_1)\sinh(x - x_3)}{\sinh(x_2 - x_1)\sinh(x_2 - x_3)}$$

$$N_3 = \frac{\sinh(x - x_2)\sinh(x - x_1)}{\sinh(x_3 - x_1)(x_3 - x_2)}$$

Shape functions are given in the following figure.

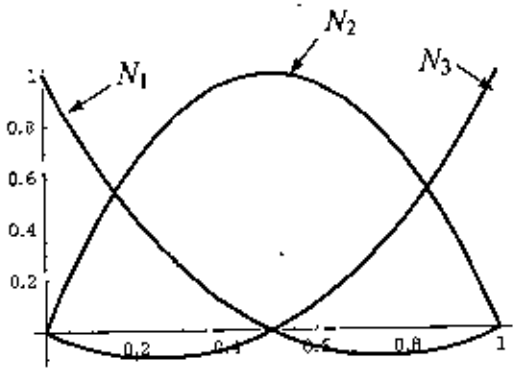


Fig. 3.8: Hyperbolic shape functions for quadratic element.

On substitution $x = \frac{x_1 + x_3}{2} + \frac{x_3 - x_1}{2}\xi$ and $x_2 = \frac{x_1 + x_3}{2}$ gives

$$N_1 = \frac{\sinh\left(\frac{x_1 + x_3}{2} + \frac{x_3 - x_1}{2}\xi - \frac{x_1 + x_3}{2}\right)\sinh\left(\frac{x_1 + x_3}{2} + \frac{x_3 - x_1}{2}\xi - x_3\right)}{\sinh\left(x_1 - \frac{x_1 + x_3}{2}\right)\sinh(x_1 - x_3)}$$

$$N_1 = \frac{\sinh\left(\frac{x_3 - x_1}{2}\xi\right)\sinh\left\{\frac{x_3 - x_1}{2}(\xi - 1)\right\}}{\sinh\left\{\frac{1}{2}(x_1 - x_3)\right\}\sinh(x_3 - x_1)}$$

Similarly, $N_2 = \frac{\sinh\left\{\left(\frac{x_3 - x_1}{2}(1 + \xi)\right)\right\}\sinh\left\{\frac{x_1 - x_3}{2}(1 - \xi)\right\}}{\sinh\left\{\frac{1}{2}(x_3 - x_1)\right\}\sinh\left\{\frac{1}{2}(x_1 - x_3)\right\}}$

$$N_3 = \frac{\sinh\left(\frac{x_3 - x_1}{2}\xi\right)\sinh\left\{\frac{x_3 - x_1}{2}(1 + \xi)\right\}}{\sinh(x_3 - x_1)\sinh\left\{\frac{1}{2}(x_3 - x_1)\right\}}$$

For $x_1 = 0,\ x_3 = 2$

$$N_1 = \frac{\sinh\xi\sinh(\xi - 1)}{\sinh(1)\sinh(2)} \qquad (3.2.2a)$$

$$N_2 = \frac{\sinh(1 + \xi)\sinh(\xi - 1)}{\sinh(1)\sinh(-2)} \qquad (3.2.2b)$$

$$N_3 = \frac{\sinh\xi \sinh(1+\xi)}{\sinh(1)\sinh(2)} \qquad (3.2.2c)$$

For cubic element

$$N_1 = \frac{\sinh(x-x_2)\sinh(x-x_3)\sinh(x-x_4)}{\sinh(x_1-x_2)\sinh(x_1-x_3)\sinh(x_1-x_4)}$$

$$N_2 = \frac{\sinh(x-x_1)\sinh(x-x_3)\sinh(x-x_4)}{\sinh(x_2-x_1)\sinh(x_2-x_3)\sinh(x_2-x_4)}$$

$$N_3 = \frac{\sinh(x-x_1)\sinh(x-x_2)\sinh(x-x_4)}{\sinh(x_3-x_4)\sinh(x_3-x_2)\sinh(x_3-x_4)}$$

$$N_4 = \frac{\sinh(x-x_1)\sinh(x-x_2)\sinh(x-x_3)}{\sinh(x_4-x_1)\sinh(x_4-x_2)\sinh(x_4-x_3)}$$

Shape functions are given in the following figure.

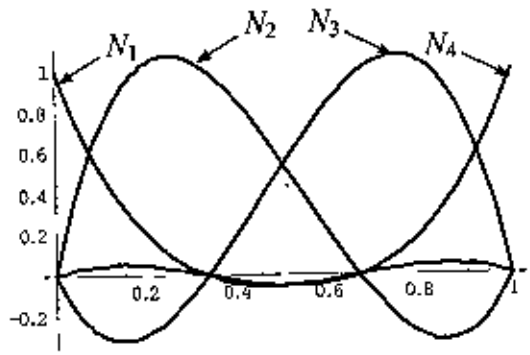


Fig. 3.9: Hyperbolic shape functions for cubic element.

On substitution

$$x = \frac{x_1+x_4}{2} + \frac{x_4-x_1}{2}\xi \text{ and } x_2 = \frac{2x_1+x_4}{3}, \quad x_3 = \frac{x_1+2x_4}{3}$$

$$N_1 = \frac{\sinh\left(\frac{x_1+x_4}{2} + \frac{x_4-x_1}{2}\xi - \frac{2x_1+x_4}{3}\right)\sinh\left(\frac{x_1+x_2}{2} + \frac{x_4-x_1}{2}\xi - \frac{x_1+2x_4}{3}\right)\sinh\left(\frac{x_1+x_4}{2} + \frac{x_4-x_1}{2} - x_4\right)}{\sinh\left(x_1 - \frac{2x_1+x_4}{3}\right)\sinh\left(x_1 - \frac{x_1+2x_4}{3}\right)\sinh(x_1-x_4)}$$

$$N_1 = \frac{\sinh\left(\frac{3x_1+3x_4-4x_1-2x_4}{}\right)\sinh\left(7x_4+3x_4-2x_4-4x_4+\frac{x_1-x_4}{2}\xi\right)\sinh\left(\frac{x_1-x_4}{2}+\frac{x_4-x_1}{2}\right)}{\sinh\left(\frac{x_1-x_4}{2}\right)\sinh\left\{\frac{2}{3}(x_1-x_4)\right\}\sinh(x_1-x_4)}$$

$$N_1 = \frac{\sinh\left(\frac{x_4-x_1}{2}\right)\left(\frac{1}{3}+\xi\right)\sinh\left\{\frac{x_4-x_1}{2}\left(-\frac{1}{3}+\xi\right)\right\}\sinh\left(\frac{x_4-x_1}{2}\right)(-1+\xi)}{\sinh\left(\frac{x_1-x_4}{3}\right)\sinh\frac{2}{3}(x_1-x_4)\sinh(x_1-x_4)}$$

$$N_2 = \left(\frac{\sinh\left\{\left(\frac{x_4-x_1}{2}\right)(1+\xi)\right\}\sinh\left\{\left(\frac{x_4-x_1}{6}\right)(\xi-1)\sinh\left\{\left(\frac{x_4-x_1}{2}\right)(\xi-1)\right\}\right\}}{\sinh\left(\frac{x_4-x_1}{3}\right)\sinh\left\{\left(\frac{x_1-x_4}{3}\right)\right\}\sinh\left\{\frac{2}{3}(x_1-x_4)\right\}}\right)$$

$$N_3 = \left(\frac{\sinh\left\{(x_4-x_1)(1+\xi)\right\}\sinh\left\{\left(\frac{x_4-x_1}{6}\right)(3\xi+1)\sinh\left\{\left(\frac{x_4-x_1}{2}\right)(\xi-1)\right\}\right\}}{\sinh\left\{\frac{2}{3}(x_4-x_1)\right\}\sinh\left\{\frac{x_4-x_1}{3}\right\}\sinh\left(\frac{x_1-x_4}{3}\right)}\right)$$

$$N_4 = \left(\frac{\sinh\left\{\left(\frac{x_4-x_1}{2}\right)(1+\xi)\right\}\sinh\left\{\left(\frac{x_4-x_1}{6}\right)(3\xi+1)\sinh\left\{\left(\frac{x_4-x_1}{6}\right)(3\xi-1)\right\}\right\}}{\sinh(x_4-x_1)\sinh\left\{\frac{2}{3}(x_4-x_1)\right\}\sinh\left\{\frac{x_4-x_1}{3}\right\}}\right)$$

For $x_1 = 0,\ x_4 = 3$

$$N_1 = \frac{\sinh\left\{\frac{1}{2}(3\xi+1)\right\}\sinh\left\{\frac{1}{2}(3\xi-1)\right\}\sinh\left\{\frac{3}{2}(\xi-1)\right\}}{\sinh(-1)\sinh(-2)\sinh(-3)} \qquad (3.2.3a)$$

$$N_2 = \frac{\sinh\left\{\frac{3}{2}(1+\xi)\right\}\sinh\left\{\frac{1}{2}(3\xi-1)\right\}\sinh\left\{\frac{3}{2}(\xi-1)\right\}}{\sinh(1)\sinh(-1)\sinh(-2)} \qquad (3.2.3b)$$

$$N_3 = \frac{\sinh\left\{\frac{3}{2}(1+\xi)\right\}\sinh\left\{\frac{1}{2}(3\xi+1)\right\}\sinh\left\{\frac{3}{2}(\xi-1)\right\}}{\sinh(2)\sinh(1)\sinh(-1)} \qquad (3.2.3c)$$

$$N_4 = \frac{\sinh\left\{\frac{3}{2}(1+\xi)\right\}\sinh\left\{\frac{1}{2}(3\xi+1)\right\}\sinh\left\{\frac{1}{2}(3\xi-1)\right\}}{\sinh(3)\sinh(2)\sinh(1)}. \qquad (3.2.3d)$$

Shape functions for both Lagrange and Hyperbolic interpolations are given together for comparison as follows:

Shape functions for linear element are shown in figure 3.10; where f1, f2 denote the hyperbolic shape functions and g1, g2 denote the shape functions for Lagrange interpolations respectively.

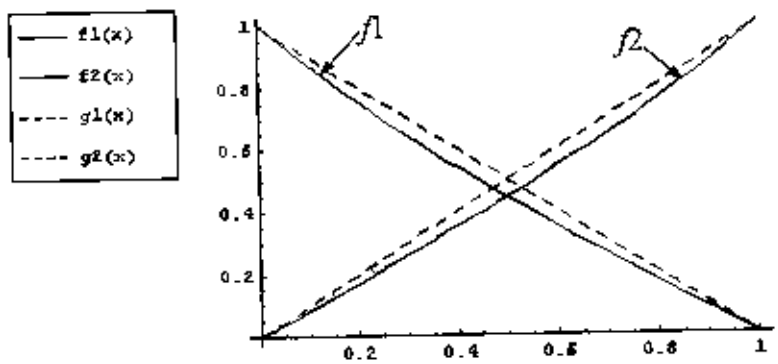


Fig. 3.10: Linear Lagrange and hyperbolic interpolation shape functions

Shape functions for quadratic element are shown in figure 3.11; where f1, f2, f3 and g1, g2, g3 are hyperbolic interpolation and Lagrange interpolation shape functions respectively.

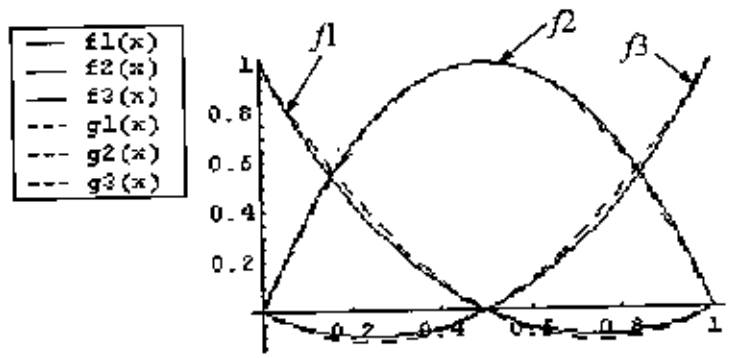


Fig. 3.11: Quadratic Lagrange and hyperbolic interpolation shape functions

Shape functions for cubic element are shown in figure 3.12 where h1, h2, h3, h4 and L1. L2, L3, L4 are hyperbolic interpolation and Lagrange interpolation shape functions respectively.

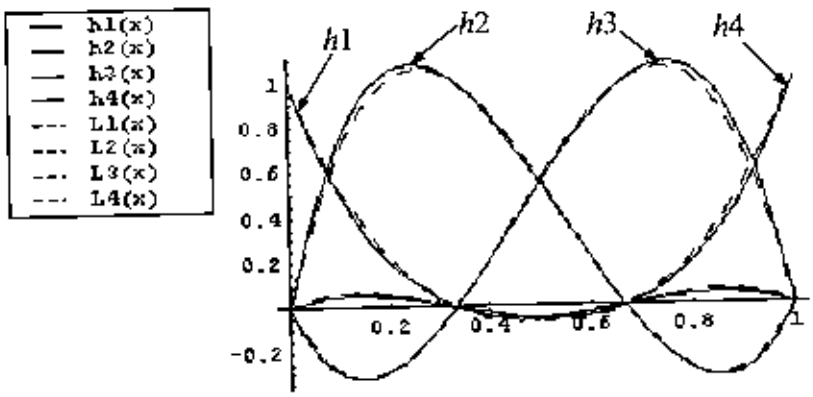


Fig. 3.12: Cubic Lagrange and hyperbolic interpolation shape functions

Chapter 4

EIGENVALUE PROBLEMS

4.1 Finite Element Model of an Eigenvalue Problem

Determination of the values of the parameter λ such that the equation

$$A(u) = \lambda B(u)$$

where A and B denote differential operators, has non trivial solutions u is called an eigenvalue problem. The values of λ are called eigenvalues and the associated functions u are called eigenfunctions. For example, the equations

$$-\frac{d^2u}{dx^2} = \lambda u \quad \text{with} \quad A = \frac{d^2}{dx^2} \qquad B = 1$$

which arises in connection with the axial oscillations of a bar or the transverse oscillations of a cable, constitutes an eigenvalue problem. Here λ denotes the square of the frequency of vibration ω.

In general the determination of the eigenvalues is of engineering as well as mathematical importance. In structural problems, the eigen values denote either natural frequencies or buckling loads. In quantum mechanics eigenvalues denote the energy levels. In fluid mechanics and heat transfer, eigenvalue problems arise in connection with the determination of the homogeneous parts of the solution. In these cases eigenvalues often denote amplitudes of the Fourier components making up the solution. Eigenvalues are also useful in determining the stability characteristics of temporal schemes.

In this section finite element models of eigenvalue problems will be developed.In view of the close similarity between the equations of eigenvalue and boundary value problems. The steps involved in the construction of the finite element models are entirely analogous. Differential eigenvalue problems are reduced to algebraic eigenvalue problems by means of the finite element approximation. The methods of solution of algebraic eigenvalue problems are then used to solve for the eigenvalues and eigenvectors.

Finite element model of eigen value problem using Galerkin, Weighted Residual method.Derivative of Finite element model for eigenvalue problem.is similar to those of initial and boundary value problems.

Consider an eigenvalue problem

$$-\frac{d}{dx}\left(a\frac{dU}{dx}\right)+\lambda U=0 \qquad (4.1.1)$$

Where a is function of x or constant and λ is the eigenvalue.

Over a typical element Ω^e; a finite element approximation of U in the form is

$$U=u_1N^e{}_1+u_2N^e{}_2+u_3N^e{}_3+\cdots\cdots\cdots\cdots+u_nN^e{}_n=\sum_{j=1}^{n}u_jN_j^e \qquad (4.1.2)$$

where e denotes the element number ; $u_1,u_2,\ldots u_n$ are the nodal values of the variable U in the node numbers 1,2, ...,n

To apply Weighted residual method multiply both sides of equation (4.1.1) by w and integrating over the element Ω^e which extends from $x=x_A$ to $x=x_B$

$$\int_{x_A}^{x_B}w\left(-\frac{d}{dx}\left(a\frac{dU}{dx}\right)+\lambda U\right)dx=0$$

$$-\left[wa\frac{dU}{dx}\right]_{x_A}^{x_B}+\int_{x_A}^{x_B}\frac{dw}{dx}a\frac{dU}{dx}dx+\lambda\int_{x_A}^{x_B}wU\,dx=0$$

$$\int\frac{dw}{dx}a\frac{dU}{dx}dx+w(x_A)\left(a\frac{dw}{dx}\right)_{x=x_A}-w(x_B)\left(a\frac{dU}{dx}\right)_{x=x_B}+\lambda\int_{x_A}^{x_B}wU\,dx=0$$

$$\int_{x_A}^{x_B}\frac{dw}{dx}a\frac{dU}{dx}dx+\lambda\int_{x_A}^{x_B}wU\,dx-w(x_A)Q_1-w(x_B)Q_n=0$$

writing , $Q_1=-\left(a\frac{dU}{dx}\right)_{x=x_A}$, $Q_n=\left(a\frac{dU}{dx}\right)_{x=x_B}$

Using Galerkin approach and putting $w=N_1^e,N_2^e,\ldots\ldots,N_n^e$

$$\int_{x_A}^{x_B}a\frac{dN_1^e}{dx}\left(\sum_{j=1}^{n}u_j^e\frac{dN_j^e}{dx}\right)+\lambda\int_{x_A}^{x_B}N_1^e\Sigma u_j^eN_j^e\,dx-\Sigma N_1^e\left(x_j^e\right)Q_j^e=0$$

$$\int_{x_A}^{x_B} a\frac{dN_2^e}{dx}\left(\sum_{j=1}^{n} u_j^e \frac{dN_j^e}{dx}\right) + \lambda\int_{x_A}^{x_B} N_2^e \Sigma u_j^e N_j^e dx - \Sigma N_2^e\left(x_j^e\right)Q_j^e = 0$$

$$\int_{x_A}^{x_B} a\frac{dN_n^e}{dx}\left(\sum_{j=1}^{n} u_j^e \frac{dN_j^e}{dx}\right) + \lambda\int_{x_A}^{x_B} N_n^e \Sigma u_j^e N_j^e dx - \Sigma N_n^e\left(x_j^e\right)Q_j^e = 0$$

The i-th algebraic equation can be written as

$$\sum_{j=1}^{n} K_{ij}^e u_j^e + \lambda\Sigma M_{ij}^e u_j^e - Q_i^e = 0 \qquad (i = 1,2,3,\ldots\ldots,n) \tag{4.1.3}$$

The interpolation property $N_i^e(x_j^e) = \delta_{ij} = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \end{cases}$ is used to write

$$\Sigma N_j^e\left(x_i^e\right)Q_j^e = Q_i^e$$

$$K_{ij} = \int_{x_A}^{x_B} a\frac{dN_i^e}{dx}\cdot\frac{dN_j^e}{dx}dx \tag{4.1.4}$$

$$M_{ij} = \int_{x_A}^{x_B} N_i^e N_j^e dx \tag{4.1.5}$$

Equations (4.1.3) can be expressed in terms of the co-efficients K_{ij}^e, u_i^e and Q_i^e as

$$K_{11}^e u_1^e + K_{12}^e u_2^e + \ldots\ldots + K_{1n}^e u_n^e + \left(M_{11}^e u_1^e + M_{12}^e u_2^e + \ldots\ldots + M_{1n}^e u_n^e\right) = Q_1^e$$

$$K_{21}^e u_1^e + K_{22}^e u_2^e + \ldots\ldots + K_{2n}^e u + \lambda\left(M_{21}^e u_1^e + M_{22}^e u_2^e + \ldots\ldots + M_{2n}^e u\right) = Q_2^e$$

$$K_{n1}^e u_1^e + K_{n2}^e u_2^e + \ldots\ldots + K_{nn}^e u_n^e + \lambda\left(M_{n1}^e u_1^e + M_{n2}^e u_2^e + \ldots\ldots + M_{nn}^e u_n^e\right) = Q_n^e$$

In matrix notation, the linear algebraic equations (4.13) can be written as

$$\left[K^e\right]\left\{u^e\right\} + \lambda[M^e]\left\{u^e\right\} = \left\{Q^e\right\}$$

In deriving the element equations, we isolated a typical element (the e-th) from the mesh and developed its finite element model. To solve the total problem, elements must be put back into their original positions. In doing this before discretization, the continuity of the primary variables and balance of the secondary variables at the connecting nodes between elements is imposed. Continuity of the primary variables refers here to the single valued nature of the solution; balance of secondary variables refers to the equilibrium of point sources at the junction of several elements. Thus imposing the following two conditions carries out the assembly of elements:

1. Continuity of primary variables at connecting nodes:

$$u_n^e = u_1^{e+1} \tag{4.1.6a}$$

the last nodal value of the element Ω^e is the same as the first nodal value of the adjacent element Ω^{e+1}.

2. Balance of secondary variables at connecting nodes:

$$Q_n^e + Q_1^{e+1} = \begin{cases} 0 & \text{if no external point source is applied} \\ Q_0 & \text{if an external point source of magnitude } Q_0 \text{ is applied} \end{cases} \tag{4.1.6b}$$

In writing equation (4.1.5), it is assumed that elements are connected in a sequence. The continuity of primary variables $u_2^e = u_1^{e+1}$ and balance of secondary variables $Q_2^e + Q_1^{e+1}$ for a mesh of linear elements is illustrated in fig. 4.1(a) and 4.1(b) .The balance of secondary variables can be interpretated as the continuity of $a\frac{du}{dx}$ (not $a\frac{dU^e}{dx}$) at the point common to elements Ω^e and Ω^{e+1} (when no change in $a\frac{du}{dx}$ is imposed externally):

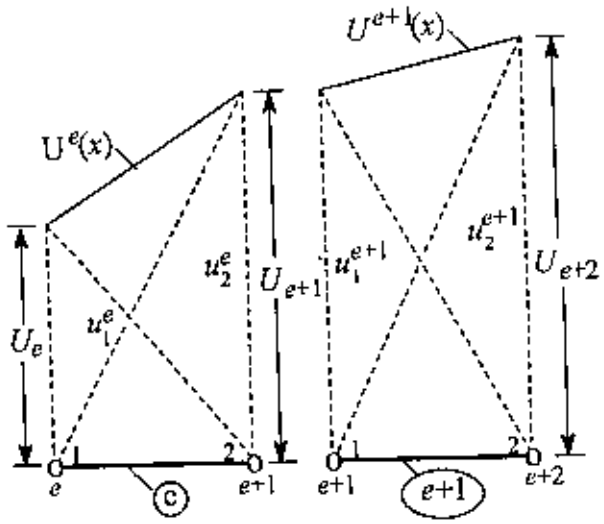


Fig. 4.1(a): Assembly of two linear Lagrange elements: continuity of the primary variables.

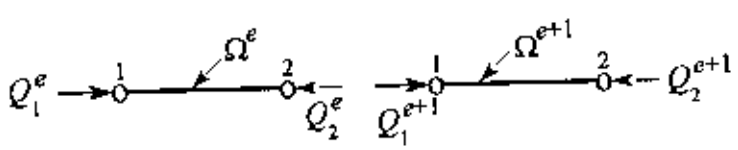


Fig. 4.1(b): Assembly of two linear Lagrange elements: balance of the secondary variables

$$\left(a\frac{du}{dx}\right)^{e} = \left(a\frac{du}{dx}\right)^{e+1}$$

$$\left(a\frac{du}{dx}\right)^{e} + \left(-a\frac{du}{dx}\right)^{e+1} = 0$$

$$Q_n^e + Q_1^{e+1} = 0$$

The interelements continuity of the primary variables is imposed by renaming the two variables u_n^e and u_1^{e+1} at $x = x_N$ as one and the same, namely the value of u at the global node N:

$$u_n^e = u_1^{e+1} = U_N$$

where N=(n-1)e +1 is the global node number corresponding to node n of the element Ω^e and node 1 of the element Ω^{e+1}. For example for a mesh of linear finite elements (n=2), thus obtained as

$$u_1^1 = U_1$$

$$u_2^1 = u_1^2 = U_2$$

$$u_2^2 = u_1^3 = U_3$$

$$\vdots$$

$$u_2^{E-1} = u_1^E = U_E$$

$$u_2^E = U_{E+1}$$

To enforce balance of the secondary variables Q_i^e it is clear that set $Q_n^e + Q_1^{e+1}$ can be set equal to zero or a specified value Q_0 only if there is such expressions in given equations. To obtain such expressions, the nth equation of the element Ω^e must be added

to the first equation of the element Ω^{e+1} ;that is it is added as

$$\sum_{j=1}^{n} K_{nj}^e u_j^e = Q_n^e$$

and

$$\sum_{j=1}^{n} K_{1j}^{e+1} u_j^{e+1} = Q_1^{e+1}$$

to give

$$\sum_{j=1}^{n} \left(K_{nj}^e u_j^e + K_{1j}^{e+1} u_j^{e+1} \right) = \left(Q_n^e + Q_1^{e+1}\right) = Q_0$$

[Here it is only written the first term of the two left hand side terms of the equation (4.1.3) to avoid the large calculation. This term will be taken in the final equation.]

This process reduces the number of equations from 2E to E+1. The first equation of the first element and the last equation of the last element will remain unchanged, except for renaming of the primary variables. The first term of the left hand side of equation (4.1.3) can be written in terms of the global nodal values as

$$\left(K^e_{n1}u^e_1 + K^e_{n2}u^e_2 + \cdots\cdots + K^e_{nn}u^e_n\right) + \left(K^{e+1}_{11}u^{e+1}_1 + K^{e+1}_{12}u^{e+1}_2 + \cdots\cdots + K^{e+1}_{1n}u^{e+1}_n\right)$$

$$= \left(K^e_{n1}U_N + K^e_{n2}U_{N+1} + \cdots\cdots + K^e_{nn}U_{N+n-1}\right) + \left(K^{e+1}_{11}U_{N+n-1} + K^{e+1}_{12}U_{N+n} + \cdots\cdots + K^{e+1}_{1n}U_{N+2n-2}\right)$$

$$= \left(K^e_{n1}U_N + K^e_{n2}U_{N+1} + \cdots\cdots + K^e_{n(n-1)}U_{N+n-2}\right) + \left(K^e_{nn} + K^{e+1}_{11}\right)U_{N+n-1} + K^{e+1}_{12}U_{N+n} + \cdots\cdots + K^{e+1}_{1n}U_{N+2n-2}$$

where N=(n-1)e+1

For a mesh of E linear elements (n=2), thus obtained

$$K^1_{11}U_1 + K^1_{12}u_2 = Q^1_1 \quad \text{(unchanged)}$$

$$K^1_{21}U_1 + \left(K^1_{22} + K^2_{11}\right)U_2 + K^2_{12}U_3 = Q^1_2 + Q^2_1$$

$$K^2_{21}U_2 + \left(K^2_{22} + K^3_{11}\right)U_3 + K^3_{12}U_4 = Q^2_2 + Q^3_1$$

$$\vdots$$

$$K^{E-1}_{21}U_{E-1} + \left(K^{E-1}_{22} + K^E_{11}\right)U_E + K^E_{12}U_{E+1} = Q^{E-1}_2 + Q^E_1$$

$$K^E_{21}U_E + K^E_{22}U_{E+1} = Q^E_2 \quad \text{(unchanged)}$$

These are called assembled equations. They contain the sum of the co- efficients and source term at nodes common to two elements. The numbering of the global equations corresponds to the numbering of the global primary degrees of freedom, U_1. This correspondence carries the symmetry of elements matrices to the global matrix. The above equations can be expressed as matrix equation

$$\begin{bmatrix} K^1_{11} & K^1_{12} & & & & \\ K^1_{21} & K^1_{22}+K^2_{11} & K^2_{12} & & & 0 \\ & K^2_{21} & K^2_{22}+K^3_{11} & & & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \\ 0 & & & K^{E-1}_{22}+K^E_{11} & K^E_{12} & \\ & & & K^E_{21} & K^E_{22} & \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_E \\ U_{E+1} \end{Bmatrix} = \begin{Bmatrix} Q^1_1 \\ Q^1_2 + Q^2_1 \\ Q^2_2 + Q^3_1 \\ \vdots \\ Q^{E-1}_2 + Q^E_1 \\ Q^E_2 \end{Bmatrix}$$

Thus from equation (4.1.3) the assembled matrix eigenvalue equation is

$$\mathbf{KU} + \lambda\mathbf{MU} = \mathbf{Q}.$$

4.2 Solution of Eigenvalue Problem Using Lagrange Interpolation

The eigenvalue problem solved by Rammohan et al [6] is

$$-\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\psi}{dx}\right)-\frac{2}{x}\psi=\lambda\psi$$

with domain having limit 0 to 20 taking 20 elements.

The sample eigenvalue problem discussed in the previous chapter was with constant coefficient. So the element stiffness equation was same for all elements. But for this case the coefficient is variable. So the element stiffness matrix will be different for different elements.

Multiplying the equation by x^2 is thus obtained

$$-\frac{d}{dx}\left(x^2\frac{d\psi}{dx}\right)-2x\psi=\lambda x^2\psi$$

Now multiply the equation by the weight function w and integrate from 0 to 20. An integration by parts leads to the equation.

$$\int_0^{20}\left(\ x^2\frac{dw}{dx}\cdot\frac{d\psi}{dx}-2xw\psi\ \right)dx=\int_0^{20}\lambda w\psi dx$$

where the boundary condition $\psi(20=0)$is applied.

For linear element, writing $\psi=N_1\psi_1+N_2\psi_2$ and considering the Galerkin approach $w=N_1$ and $w=N_2$, the element stiffness matrix can be written as

$$k_{ij}=\int_{x_A}^{x_B}\left(x^2\frac{dN_i}{dx}\cdot\frac{dN_j}{dx}-2xN_iN_j\right)dx \qquad \text{where } i,j=1,2$$

$$\text{with} \qquad x=\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi$$

$$dx=\frac{h}{2}d\xi \quad \text{and} \quad \frac{d\xi}{dx}=\frac{2}{h}$$

$x_B-x_A=h=$ length of an element

From equation (3.1.1) of chapter 3 the shape functions are

$$N_1=\frac{1}{2}(1-\xi), \qquad N_2=\frac{1}{2}(1+\xi)$$

$$k_{ij} = \int_{-1}^{1}\left\{\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi\right)^2 \frac{dN_i}{d\xi}\cdot\frac{d\xi}{dx}\cdot\frac{dN_j}{d\xi}\cdot\frac{d\xi}{dx} - 2\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi\right)N_iN_j\right\}\frac{h}{2}d\xi$$

$$= \int_{-1}^{1}\left\{\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi\right)^2 \frac{dN_i}{d\xi}\frac{2}{h}\frac{dN_j}{d\xi}\frac{2}{h} - \left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\right)N_iN_j\right\}\frac{h}{2}d\xi$$

Taking $x_A = 0, \quad x_B = 1, \quad h = 1$

$$k_{11} = \int_{-1}^{1}\left[\frac{1}{4}(1+\xi)^2(-1)(-1) - \frac{1}{2}(1+\xi)\frac{1}{2}(1-\xi)\frac{1}{2}(1+\xi)\right]\frac{1}{2}d\xi = 0.17$$

Similarly other elements of the matrix are obtained as

$k_{12} = k_{21} = -.500$

Thus for element 1 $\quad K^{(1)} = \begin{bmatrix} -.17 & -.500 \\ -.500 & -.17 \end{bmatrix}$

Taking $\quad x_A = 0, \; x_B = 2, \; h = 1$

For element 2 $\quad K^{(2)} = \begin{bmatrix} 1.50 & -2.8 \\ -2.8 & 1.17 \end{bmatrix}$

Also taking $\quad x_A = 19, \quad x_B = 20, \quad h = 1$

For element 20 $\quad K^{(20)} = \begin{bmatrix} 367.75 & -386.83 \\ -386.83 & 367.17 \end{bmatrix}$

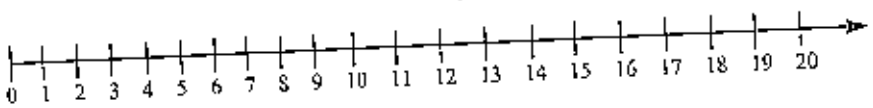


Fig. 4.2: Discretization of the body into 20 linear elements.

The elements of the mass matrix are

$$m_{ij} = \int_{x_A}^{x_B} x^2 N_i N_j dx$$

$$m_{ij} = \int_{-1}^{1}\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi\right)N_iN_j dx$$

$$m_{11} = \int_{-1}^{1}\frac{1}{4}(1+\xi)^2\frac{1}{2}(1-\xi)\frac{1}{2}d\xi = 0.0333$$

Similarly, $m_{12} = m_{21} = 0.05$, $\quad m_{22} = 0.2$

Thus for element 1 $M^{(1)} = \begin{bmatrix} .033 & .05 \\ .05 & 0.2 \end{bmatrix}$

Thus for element 2 $M^{(2)} = \begin{bmatrix} .533 & .383 \\ .383 & 1.033 \end{bmatrix}$

Thus for element 20 $M^{(20)} = \begin{bmatrix} 123.53 & 63.383 \\ 63.383 & 130.03 \end{bmatrix}$

The global stiffness matrix **K** and the global mass matrix **M** will be a 21×21 matrix.

The matrix eigenvalue equation to be solved is $\bar{K}\bar{\psi} = \lambda\bar{M}\bar{\psi}$ with

$$\psi = [\psi_1,\ \psi_2,\ \ldots\ \ldots\ \ldots,\ \psi_{19},\ \psi_{20},\ \psi_{21}]^t$$

and $\lambda = [\lambda_1, \lambda_2, \ldots\ \ldots\ \ldots, \lambda_{19}, \lambda_{20}]^t$

The equation is solved by Jacobi method using FORTRAN programming and the eigenvalues are obtained as follows.

Table-4.1: Eigenvalues obtained using 2-noded element

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
12.929	11.565	10.787	9.724	8.522	7.292	6.111	5.023	4.051	3.199

λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}	λ_{17}	λ_{18}	λ_{19}	λ_{20}
2.466	1.843	1.319	0.886	0.533	0.225	0.048	-0.093	-0.238	-0.942

This result is obtained by Rammohan et al [6]. Eigenvalues will now be found by using quadratic and cubic element.

For quadratic element writing, $\psi = N_1\psi_1 + N_2\psi_2 + N_3\psi_3$. The element stiffness matrix is obtained as

$$k_{ij} = \int_{x_1}^{x_n} \left(x^2 \frac{dN_i}{dx} \cdot \frac{dN_j}{dx} - 2xN_iN_j \right) dx \qquad \text{where } i, j = 1, 2, 3$$

From equation (3.2.2) of chapter 3 the shape functions are

$$N_1 = \frac{1}{2}\xi(\xi - 1),$$

$$N_2 = 1 - \xi^2$$

$$N_3 = \frac{1}{2}\xi(1 + \xi)$$

$$k_{ij} = \int_{-1}^{1}\left\{\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi\right)^2 \frac{dN_i}{d\xi}\frac{dN_j}{d\xi} - 2\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi\right)N_i N_j\right\}\frac{h}{2}d\xi$$

Taking $x_A = 0$, $x_B = 2$ and $h = 2$

$$k_{11} = \int_{-1}^{1}\left\{\left(\frac{1}{2} + \xi\right)^2\left(-\frac{1}{2} + \xi\right)\left(-\frac{1}{2} + \xi\right) - 2(1+\xi)\left(\frac{-1}{2}\xi(1-\xi)\right)\left(-\frac{1}{2}\xi(1-\xi)\right)\right\}d\xi = 0.77$$

Similarly the other elements of the matrix are obtained as

$k_{12} = k_{21} = -0.80$

$k_{13} = k_{31} = 0.53$

$k_{23} = k_{32} = -4.00$

$k_{22} = 2.13$

$k_{33} = 2.13$

Thus for element 1 $$K^{(1)} = \begin{bmatrix} 0.27 & 0.80 & 0.53 \\ -0.80 & 2.13 & -4.00 \\ 0.53 & -4.00 & 2.13 \end{bmatrix}$$

Taking $x_A = 2, \quad x_B = 4, \quad h = 2$

For element 2 $$K^{(2)} = \begin{bmatrix} 5.87 & -9.33 & 2.13 \\ -9.33 & 19.20 & -17.87 \\ 2.13 & -17.87 & 13.07 \end{bmatrix}$$

Taking $x_A = 18, \quad x_B = 20, \quad h = 2$

For element 10 $$K^{(10)} = \begin{bmatrix} 386.67 & -461.60 & 62.93 \\ -461.60 & 923.73 & -512.80 \\ 62.93 & -512.80 & 436.53 \end{bmatrix}$$

The mass matrix is, $$m_{ij} = \int_{-1}^{1}\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\right)^2 N_i N_j \frac{h}{2}d\xi$$

Taking $x_A = 0, \quad x_B = 2$ and $h = 2$

$$m_{11} = \int_{-1}^{1}(1+\xi)^2\left\{\frac{1}{2}\xi(\xi-1)\right\}\left\{\frac{1}{2}\xi(\xi-1)\right\}d\xi = 0.03$$

Similarly, $m_{12} = m_{21} = -0.05$

$m_{13} = m_{31} = -0.11$

$m_{23} = m_{32} = 0.48$

$$m_{22} = 1.17$$

$$m_{33} = 0.83$$

Thus for element 1 $$M^{(1)} = \begin{bmatrix} 0.03 & -0.05 & -0.11 \\ -0.05 & 1.17 & 0.48 \\ -0.11 & 0.48 & 0.83 \end{bmatrix}$$

For element 2 $$M^{(2)} = \begin{bmatrix} 1.36 & 0.48 & -0.64 \\ 0.48 & 9.71 & 2.08 \\ -0.64 & 2.08 & 3.76 \end{bmatrix}$$

For element 10 $$M^{(10)} = \begin{bmatrix} 88.83 & 43.15 & -24.11 \\ 43.15 & 385.17 & 53.28 \\ -24.11 & 53.28 & 104.03 \end{bmatrix}$$

The global stiffness matrix **K** and the global mass matrix **M** will be a 21×21 matrix.

The matrix eigenvalue equation to be solved is $\bar{K}\bar{\psi} = \lambda\bar{M}\bar{\psi}$ with

$$\psi = [\psi_1,\ \psi_2,\ \cdots\cdots\ \cdots\cdots\ \cdots\cdots,\ \psi_{19},\ \psi_{20},\ \psi_{21}]^T$$

and $$\lambda = [\lambda_1, \lambda_2, \cdots\cdots\ \cdots\cdots\ \cdots\cdots, \lambda_{19}, \lambda_{20}]^T$$

The equation is solved by Jacobi method using standard FORTRAN programming and the eigenvalues are obtained as follows.

Table-4.2: Eigenvalues obtained using 3-noded element

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
15.509	14.085	12.427	10.522	8.681	7.049	5.667	4.520	3.578	2.664

λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}	λ_{17}	λ_{18}	λ_{19}	λ_{20}
2183	1.634	1.179	0.798	0.483	0.229	0.035	0.099	-0.249	-0.990

For cubic element writing, $\psi = N_1\psi_1 + N_2\psi_2 + N_3\psi_3 + N_4\psi_4$. The element stiffness matrix is can be written as

$$k_{ij} = \int_{-1}^{1}\left\{\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi\right)^2 \frac{dN_i}{d\xi}\cdot\frac{2}{h}\cdot\frac{dN_j}{dx}\cdot\frac{2}{h} - 2\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi\right)N_i N_j\right\}\frac{h}{2}d\xi$$

From equation (3.1.3) of chapter 3 the shape functions are

$$N_1 = -\frac{9}{10}(1-\xi)\left(\frac{1}{3}+\xi\right)\left(\frac{1}{3}-\xi\right)$$

$$N_2 = \frac{27}{16}(1+\xi)(1-\xi)\left(\frac{1}{3}-\xi\right)$$

$$N_3 = \frac{27}{16}(1+\xi)(1-\xi)\left(\frac{1}{3}+\xi\right)$$

$$N_4 = -\frac{9}{16}\left(\frac{1}{3}+\xi\right)\left(\frac{1}{3}-\xi\right)(1+\xi)$$

Taking $x_A = 0, \quad x_B = 3$ and $h = 3$

$$k_{11} = \int_{-1}^{1}\left\{\frac{9}{64}(1+\xi)^2(27\xi^2-18\xi-1)^2 - \frac{27}{64}(9\xi^3-9\xi^2-\xi+1)^2\right\}\frac{3}{2}d\xi = .025$$

Thus for element 1 $K^{(1)} = \begin{bmatrix} 0.25 & -0.99 & 1.12 & -0.65 \\ -0.99 & 4.09 & -7.60 & 3.27 \\ 1.12 & -7.60 & 13.30 & -11.72 \\ -0.65 & 3.27 & -11.72 & 7.33 \end{bmatrix}$

For element 7 $K^{(7)} = \begin{bmatrix} 387.93 & -514.62 & 154.93 & -40.75 \\ -514.62 & 1192.88 & -893.87 & 177.66 \\ 154.93 & -893.87 & 13.40.95 & -643.64 \\ -40.75 & 177.66 & -643.64 & 497.73 \end{bmatrix}$

The mass matrix is, $m_{ij} = \int_{-1}^{1}\left(\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi\right)^2 N_i N_j \frac{h}{2}d\xi$

Taking $x_A = 0, \quad x_B = 3$ and $h = 3$

$$m_{11} = \int_{-1}^{1}\left\{\frac{81}{256}(1+\xi)^2(9\xi^3-9\xi^2-\xi+1)^2\right\}\frac{3}{2}d\xi = 0.03$$

For element 1 $M^{(1)} = \begin{bmatrix} .03 & -.06 & 0.12 & 0.10 \\ -.06 & 1.12 & -0.56 & -.50 \\ 0.12 & -0.56 & 4.50 & 1.19 \\ 0.10 & -.50 & 1.19 & 1.54 \end{bmatrix}$

For element 7 $M^{(7)} = \begin{bmatrix} 65.58 & 50.17 & -17.12 & 11.18 \\ 50.17 & 358.73 & -47.79 & -25.24 \\ -17.12 & -47.79 & 402.58 & 66.41 \\ 11.18 & -25.24 & 66.41 & 85.26 \end{bmatrix}$

The global stiffness matrix **K** and the global mass matrix **M** will be a 21×21 matrix.

The matrix eigenvalue equation to be solved is $\vec{K}\vec{\psi} = \lambda\vec{M}\vec{\psi}$ with

$$\psi = [\psi_1, \ \psi_2, \ \ldots\ldots \ \ldots\ldots \ \ldots\ldots, \ \psi_{19}, \ \psi_{20}, \ \psi_{21}]'$$

and $\lambda = [\lambda_1, \lambda_2, \ldots\ldots \ \ldots\ldots \ \ldots\ldots, \lambda_{19}, \lambda_{20}]'$

The equation is solved by Jacobi method using FORTRAN programming and the eigenvalues are obtained as follows.

Table-4.3: Eigenvalues obtained using 4-noded element

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
21.295	18.987	16.012	12.985	10.383	8.354	5.897	4.896	4.058	32.87

λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}	λ_{17}	λ_{18}	λ_{19}	λ_{20}	λ_{21}
2.624	2.056	1.567	1.153	0.786	0.476	0.226	0.034	-0.100	-0.250	-0.998

Chapter 5
SOLUTION METHODS FOR EIGENPROBLEMS

5.1 Preliminaries to the Solution of Eigenproblems

The purpose of this chapter is to describe the actual solution procedures used to solve the eigenproblems of interest. Before presenting the algorithms, some important basic considerations for the solution of eigenproblems will be discussed in this chapter. The simplest problems encountered is the standard eigenproblem,

$$K\psi=\lambda\psi \tag{5.1}$$

Where **K** is the stiffness matrix of a single finite element or of an element assemblage. We recall that **K** has order n and half- bandwidth m (i.e., the total bandwidth is 2m+1) and **K** is a positive semidefinite or positive definite. Bandedness means that all elements beyond the bandwidth of the matrix are zero. Because **K** is symmetric, we can state this condition as

$$a_{ij}=0 \qquad \text{for} \quad j>i+m_K$$

where $2m_K+1$ is the bandwidth of **K**. If the half-bandwidth of a matrix is zero, we have nonzero elements only on the diagonal of the matrix. For example, the identity matrix is a diagonal matrix. There are n eigenvalues and corresponding eigenvectors satisfying equation (5.1). The i-th eigenpair is denoted as (λ_i,ψ_i), where the eigenvalues are ordered according to their magnitudes:

$$0\le\lambda_1\le\lambda_2\cdots\cdots\le\lambda_{n-1}\le\lambda_n \tag{5.2}$$

The solution for p-eigenpairs can be written

$$K\psi=\psi A \tag{5.3}$$

Where Ψ is an $n\times p$ matrix with its column equal to the p-eigenvalues and corresponding eigenvectors and **A** is a $p\times p$ diagonal matrix listing the corresponding eigenvalues. As an example equation equation(5.3) may represent the solution to the lowest p-eigenvalues and corresponding eigenvectors of K, in which case $\Psi=[\psi_1,\psi_2,\ldots\ldots,\psi_p]$ and A= diag(λ_i), $i=1,2,\ldots\ldots,p$. We recall that if **K** is positive definite, $\lambda_i>0$, $i=1,2,\ldots\ldots,n$, and if **K** is positive semidefinite, $\lambda_i\ge 0$,

$i = 1,2,\ldots\ldots,n$, where the numbers of zero eigenvalues is equal to the number of rigid modes in the systems.

A very frequently considered eigenvalue problem is the one to be solved in vibration mode superposition analysis. In this case we consider the generalized eigenvalue problem,

$$K\psi=\lambda M\psi \tag{5.4}$$

where **K** and **M** are, respectively, the stiffness matrix and mass matrix of the finite element assemblage. The eigenvalnes λ_i and eigenvectors ψ_i are the free vibration frequencies (radians/second) squared, ω_i^2, and corresponding mode shape vectors, respectively. The mass matrix may be banded, in which case its half-bandwidth m_M is equal to m_K, or **M** may be diagonal with $m_{ii} \geq 0$; i.e., some diagonal elements may possibly be zero. In general a banded mass matrix is positive semidefinite.

A variety of eigensystem solution methods have been developed and are reported in literature. Most of the techniques have been devised for rather general matrices. However, in finite element analysis we are concerned with the solution of the specific eigenvalue problems summarized above, in which each of the matrices have specific properties such as being banded, positive definite and so on. The eigensystem solution algorithm should take advantage of these properties in order to make a more economical solution possible.

5.2 Solution Methods for Eigenproblems

The solution methods that are considered here first can be subdivided into four groups, corresponding to which basic property is used as the basis of the solution algorithm.

The vector iteration methods make up the first group, in which the basic properties used is that

$$K\psi_i = \lambda_i M\psi_i$$

The transformation methods make up the second group, using and

$$\Psi^T K\Psi = A \tag{5.5}$$

$$\Psi^T M\Psi = I \tag{5.6}$$

where $\Psi = [\psi_1, \psi_2, \ldots\ldots, \psi_n]$ and $A = diag(\lambda_i)$, $i = 1,2,\ldots\ldots,n$. The solution methods of the third group are polynomial iteration techniques that operate on the fact that

$$p(\lambda_i) = 0$$

where $$p(\lambda) = \det(K - \lambda M)$$

The solution methods of fourth group employ the Sturm sequence property of the characteristic polynomials

$$p(\lambda) = \det(K - \lambda M)$$

$$p^{(r)}\left(\lambda^{(r)}\right) = \det\left(K^{(r)} - \lambda^{(r)} M^{(r)}\right); \quad r = 1, 2, \ldots\ldots, n-1$$

Where $p^{(r)}\left(\lambda^{(r)}\right)$ is the characteristic polynomial of the r-th associated constraint problem corresponding to $\mathbf{K}\psi=\lambda\mathbf{M}\psi$.

Before presenting the solution techniques of interest, a few basic additional points should be noted. It is important to realize that all solution methods must be iterative in nature because, basically, solving the eigenvalue problem $\mathbf{K}\psi=\lambda\mathbf{M}\psi$ is equivalent to calculating the roots of the polynomial $p(\lambda)$, which has order equal to the order of **K** and **M**. Since there are for the general case no explicit formulas available for the calculation of the roots of $p(\lambda)$ when the order of p is larger than 4, an iterative solution method has to be used. However, before iteration is started, we may choose to transform the matrices **K** and **M** into a form that allows a more economical solution of the required eigensystem.

Two general types of methods, namely transformation and iterative methods are available for solving eigenvalue problems. The transformation methods such as Jacobi, Givens and Householder schemes are preferable when all the eigenvalues and eigenvectors are required. The iterative methods such as the power method are preferable when few eigenvalues and eigenvectors are required.

The transformation method comprises a group of eigensystem solution procedures that employ the basic properties of eigenvectors in the matrix ψ.

To solve the generalized problem $\mathbf{K}\psi=\lambda\mathbf{M}\psi$, $\mathbf{M}\neq\mathbf{I}$, using standard Jacobi method, it would be necessary to first transform the problem into the standard form. However, this transformation can be dispensed with by using a generalized Jacobi solution method that operates directly on **K** and **M**. The algorithm proceeds as summarized in equation (5.1) to (5.6).

It is pointed out in section 5.2 that the transformation methods comprise a group of eigensystem solution procedures that employ the basic properties of the eigenvectors in the matrix Ψ,

$$\boldsymbol{\Psi}^T \mathbf{K}\boldsymbol{\Psi}=\mathbf{A} \tag{5.4.1}$$

$$\boldsymbol{\Psi}^T \mathbf{M}\boldsymbol{\Psi}=\mathbf{I} \tag{5.4.2}$$

Since the matrix Ψ, of order n×n, which diagonalizes **K** and **M** in the way given in equation (5.4.1) and (5.4.2) is unique, we can try to construct it by iteration. The basic scheme is to reduce **K** and **M** to diagonal form using successive pre- and post multiplication by matrices P_K^T and P_K, respectively, where $k = 1,2,\ldots\ldots$ Specifically, if it is defined $K_1 = K$ and $M_1 = M$, it can be formed

$$\left.\begin{aligned} K_2 &= P_1^T K_1 P_1 \\ K_3 &= P_2^T K_2 P_2 \\ &\vdots \\ K_{k+1} &= P_k^T K_k P_k \\ &\vdots \end{aligned}\right\} \tag{5.7}$$

Similarly,

$$\left.\begin{aligned} M_2 &= P_1^T M_1 P_1 \\ M_3 &= P_2^T M_2 P_2 \\ &\vdots \\ M_{k+1} &= P_k^T M_k P_k \\ &\vdots \end{aligned}\right\} \tag{5.8}$$

P_K are selected to bring K_k and M_k closer to diagonal form. Then for a proper procedure we apparently need to have

$$K_{k+1} \to A \quad \text{and} \quad M_{k+1} \to I \qquad \text{as } k \to \infty$$

in which case, with l being the last iteration

$$\Psi = P_1 P_2 \ldots\ldots P_l \tag{5.9}$$

In practice, it is not necessary that M_{k+1} converges to I and K_{k+1} to A, but they only need to converge to diagonal form. Namely if

$$K_{k+1} \to diag(K_r) \quad M_{k+1} \to diag(M_r) \text{ as } k \to \infty$$

$$A = diag\left(\frac{K_r^{(l+1)}}{M_r^{(l+1)}}\right) \tag{5.10}$$

$$\Psi = P_1 P_2 \ldots\ldots P_l diag\left(\frac{1}{\sqrt{M_r^{(l+1)}}}\right) \tag{5.11}$$

5.2.1 The Jacobi Method

The basic Jacobi solution method has been developed for the solution of standard eigenvalue problems (**M** being the identity matrix), and it will be considered in this section. The method was proposed over a century ago and has been used extensively. A major advantage of the procedure is its simplicity and stability. Since the eigenvector properties in equation (5.1) and (5.2) (with **M** = I) are applicable to all symmetric matrices **K** with no restriction on the eigenvalues, the Jacobi method can be used to calculate negative, zero, or positive eigenvalues.

Considering the standard eigenvalue problem

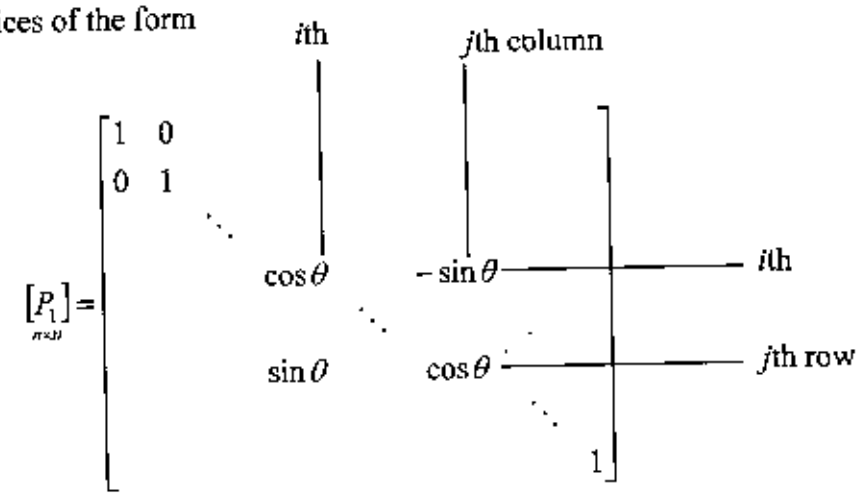
$$\mathbf{K}\psi=\lambda\psi \tag{5.12}$$

Where **[K]** is a symmetric matrix.

Method

The method is based on a theorem in linear algebra which states that a real symmetric matrix **[K]** has only real eigenvalues and that there exist a real orthogonal matrix **[P]** such that $[P]^T[K][P]$ is diagonal. The diagonal elements are the eigenvalues and the columns of the matrix **[P]** are the eigenvectors.

In the Jacobi method, the matrix [P] is obtained as a product of several "rotation" matrices of the form



where all elements other than those appearing in columns and rows i and j are identical with those of the identity matrix [I]. If the sine and cosine entries appear in positions (i,i), (i,j), (j,i) and (j,j), then the corresponding elements of $[P_1]^T[K][P_1]$ can be computed as

$$\underline{k}_{ii} = k_{ii}\cos^2\theta + 2k_{ij}\sin\theta\cos\theta + k_{jj}\sin^2\theta$$
$$\underline{k}_{ij} = \underline{k}_{ji} = (k_{jj} - k_{ii})\sin\theta\cos\theta + k_{ij}(\cos^2\theta - \sin^2\theta) \qquad (5.14)$$
$$\underline{k}_{jj} = k_{ii}\sin^2\theta - 2k_{ij}\sin\theta\cos\theta + k_{jj}\cos^2\theta$$

If θ is chosen as

$$\tan 2\theta = 2k_{ij}/(k_{ii} - k_{jj}), -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \qquad (5.15)$$

then, it makes $\underline{k}_{ij} = \underline{k}_{ji} = 0$. Thus each step of the Jacobi method reduces a pair of off-diagonal elements to zero. Unfortunately, in the next step, while the method reduces a new pair of zeroes, it introduces nonzero contributions to formerly zero positions. However, successive matrices introduce nonzero contributions to formerly zero positions.

$$[P_2]^T[P_1]^T[K][P_1][P_2],\ [P_3]^T[P_2]^T[P_1]^T[K][P_1][P_2][P_3],\ldots\ldots$$

converge to the required diagonal form and the desired matrix [P] (whose columns give the eigenvectors) would then be given by

$$[P] = [P_1][P_2][P_3]\ldots\ldots \qquad (5.16)$$

The minimum number of rotations required to bring **K** into a diagonal form is $n(n-1)/2$. A disadvantage of the Jacobi method is that the elements annihilated by a plane rotation may not necessarily remain zero during subsequent transformations.

5.2.2 The Generalized Jacobi Method

To solve the generalized problem $K\psi=\lambda M\psi$, $M\neq I$, using standard Jacobi method, it would be necessary to first transform the problem into the standard form. However, this transformation can be dispensed with by using a generalized Jacobi solution method that operates directly on **K** and **M**. The algorithm proceeds as summarized in equation (5.1) to (5.6) and is a natural extension of the standard Jacobi solution scheme; i.e., the generalized method reduces to the scheme presented for the problem $K\psi=\lambda\psi$ when **M** is an identity matrix.

Referring to the discussion in the previous section, in the generalized Jacobi iteration and using the following matrix P_k:

$$P_k = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & \alpha & & & \\ & & & \ddots & & 1 & \\ & & \gamma & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \tag{5.17}$$

where, the constants α and γ are selected in such a way as to reduce to zero simultaneously elements (i,j) in K_k and M_k. Therefore the values of α and γ are function of the elements $k_{ij}^{(k)}$, $k_{ii}^{(ki)}, k_{jj}^{(k)}$, $m_{ij}^{(k)}$, $m_{ii}^{(k)}$ and $m_{jj}^{(k)}$, where the superscript (k) indicates that the k-th iteration is considered. Performing the multiplications $P_k^T K_k P_k$ and $P_k^T M_k P_k$ and using the condition that $k_{ij}^{(k+1)}$ and $m_{ij}^{(k+1)}$ shall, we obtain the following two equations for α and γ:

$$\alpha k_{ii}^{(k)} + (1+\alpha\gamma) k_{ij}^{(k)} + \gamma k_{jj}^{(k)} = 0 \tag{5.18}$$

and

$$\alpha m_{ii}^{(k)} + (1+\alpha\gamma) m_{ij}^{(k)} \gamma m_{jj}^{(k)} = 0 \tag{5.19}$$

$$\frac{k_{ii}^{(k)}}{m_{ii}^{(k)}} = \frac{k_{jj}^{(k)}}{m_{jj}^{(k)}} = \frac{k_{ij}^{(k)}}{m_{ij}^{(k)}}$$

(i.e., the submatrices considered are scalar multiples, which may be regarded to be a trivial case), we use $\alpha = 0$ and $\gamma = -k_{ij}^{(k)} / k_{jj}^{(k)}$. In general, to solve for α and γ from equation (5.18) and (5.19), it can be defined

$$\left.\begin{aligned} \bar{k}_{ii}^{(k)} &= k_{ii}^{(k)} m_{ij}^{(k)} - m_{ii}^{(k)} k_{ij}^{(k)} \\ \bar{k}_{jj}^{(k)} &= k_{jj}^{(k)} m_{ij}^{(k)} - m_{jj}^{(k)} k_{ij}^{(k)} \\ \bar{k}^{(k)} &= k_{ii}^{(k)} m_{jj}^{(k)} - k_{jj}^{(k)} m_{ii}^{(k)} \end{aligned}\right\} \tag{5.20}$$

The relations for α and γ are used and have primarily been developed for the case of M being a positive full or banded mass matrix. In that case (and, in fact, also under less restrictive conditions), we have

$$\gamma = -\frac{\bar{k}_{ii}^{(k)}}{x}; \quad \alpha = -\frac{\bar{k}_{jj}^{(k)}}{x} \tag{5.21}$$

The value of x needed to obtain α and γ is then to be determined using

$$x = \frac{\bar{k}^{(k)}}{2} + sign\left(\bar{k}^{(k)}\right)\sqrt{\left(\frac{\bar{k}^{(k)}}{2}\right)^2 + \bar{k}_{\alpha}^{(k)}\bar{k}_{\beta}^{(k)}} \tag{5.22}$$

$$\left(\frac{\bar{k}^{(k)}}{2}\right)^2 + \bar{k}_{\alpha}^{(k)}\bar{k}_{\beta}^{(k)} > 0$$

and hence x is always nonzero. In addition, $\det P_k \neq 0$, which indeed is the necessary condition for the algorithm to work.

The generalized Jacobi solution procedure has been used a great deal in the subspace iteration method and when a consistent mass idealization is employed. Assume that M is a diagonal mass matrix, M ≠ I and $m_{ii} > 0$, in which case we employ in equation (5.19)

$$\bar{k}_{\alpha}^{(k)} = -m_{\alpha}^{(k)}k_{\alpha\beta}^{(k)}; \qquad \bar{k}_{\beta}^{(k)} = -m_{\beta}^{(k)}k_{\alpha\beta}^{(k)} \tag{5.23}$$

And otherwise equations (5.16) to (5.21) are used as before. However, if **M= I,** the relation in

equation (5.18) yields $\alpha = -\gamma$, and it is recognized that P_k in equation (5.18) is multiple of the rotation matrix defined equation (5.12) In addition, it should be mentioned that the solution procedure can be adapted to solve the problem Kψ = λψ when M is a diagonal matrix with some

zero diagonal elements.

The complete solntion process is analogous to the Jacobi iteration in the solution of the problem Kψ = λψ. The differences are that now a mass coupling factor $\left[\left(m_{\alpha\beta}^{(k)}\right)^2 / m_{\alpha}^{(k)}m_{\beta}^{(k)}\right]^{1/2}$ must also be calculated, unless M is a diagonal, and the transformation is applied to K_k and M_k.

Convergence is measured by comparing successive eigenvalue approximations and by testing if all off-diagonal elements are small enough; i.e., with l being the last iteration, convergence has been achieved if

$$\frac{\left|\lambda_i^{(l+1)} - \lambda_i^{(l)}\right|}{\lambda_i^{(l+1)}} \leq 10^{-S}; \quad i = 1,2,\ldots\ldots,n \tag{5.24}$$

where $\lambda_i^{(l)} = \frac{k_{ii}^{(l)}}{m_{ii}^{(l)}}; \qquad \lambda_i^{(l+1)} = \frac{k_{ii}^{(l+1)}}{m_{ii}^{(l+1)}}$ (5.25)

and $$\left[\frac{\left(k_{ij}^{(l+1)}\right)^2}{k_{ii}^{(l+1)}k_{jj}^{(l+1)}}\right]^{1/2} \le 10^{-s}; \qquad \left[\frac{\left(m_{ij}^{(l+1)}\right)^2}{m_{ii}^{(l+1)}m_{jj}^{(l+1)}}\right]^{1/2} \le 10^{-s} \quad \text{all } i,j;\ i<j \tag{5.26}$$

The above discussion of the generalized Jacobi solution method has already indicated in some way the advantage of the solution technique. First the transformation of the generalized eigenproblem to the standard form is avoided. This is particularly advantageous

(1) when the matrices are ill-conditioned, and (2) when the off-diagonal elements in **K** and **M** are already small or, equivalently when there are only a few non-zero off-diagonal elements. In the first case the direct solution of $\mathbf{K}\psi = \lambda\mathbf{M}\psi$ avoids the solution of a standard eigenproblem of a matrix with a very large and very small elements. In the second case the eigenproblem is already nearly solved, because the zeroing of small or only a few off-diagonal elements in **K** and **M** will not result in a large change in the diagonal elements of the matrices, the ratios of which are the eigenvalues. In addition fast convergence can be expected when the off-diagonal elements are small.

5.2.3 Givens Method for Symmetric Matrices

It is to be noted that in the Jacobi's method, the elements which were annihilated by plane rotation may not remain zero during subsequent rotations. Givens proposed an algorithm using plane rotations which preserves the zeroes in the off-diagonal element, once they are created. Let A be a real symmetric matrix.

The Givens method uses the following steps:

(a) Reduce **A** to a tridiagonal form **B**, using plane rotations,

(b) form a Sturm sequence for the characteristic equation of **B**, study the changes in sign in the sequences and find the eigenvalues, of **B**, which are also the eigenvalues of A,

(c) find the eigenvectors of **B** and then the eigenvectors of **A**

The reduction to a tridiagonal form is achieved by using the orthogonal transformations as in the Jacobi method. However, in this case to start with the subspace containing the elements $a_{22}, a_{23}, a_{32}, a_{33}$. Performing the plane rotation $S_1^{-1}AS_1$, using the orthogonal matrix

$$S_1^* = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{5.27}$$

Let the new matrix obtained be denoted by $A' = \left|a'_{ij}\right|$.

The angle θ is now obtained by putting $a'_{13} = a'_{31} = 0$ and not by putting $a'_{23} = 0 = a'_{32}$ as in Jacobi method. We find

$$a'_{13} = -a_{12}\sin\theta + a_{13}\cos\theta = 0$$

$$\text{or} \quad \tan\theta = a_{13}/a_{12}. \tag{5.28}$$

With this value of θ and performing the plane rotation, which produces zeroes in the (3,1) and (1,3) positions. Then, to perform rotation in (2,4) subspace and to put $a'_{14} = a'_{41} = 0$. This would not affect zeroes that have been obtained earlier. Proceeding in this manner, to put $a'_{15} = a'_{51} = 0$ etc. by performing rotations (2,5),.....,(2,n) subspaces. Then, we pass on to the elements $a'_{24}, a'_{25}, \ldots\ldots, a'_{2n}$ and make them zero by performing rotations in (3,4),.....,(3,n) subspaces. Finally, we produce the matrix

$$B = \begin{bmatrix} b_1 & c_1 & & & 0 & \\ c_1 & b_2 & c_2 & & & \\ & c_2 & b_3 & c_3 & & \\ & & \ddots & & & \\ & & & & & \\ & 0 & & c_{n-2} & b_{n-1} & c_{n-1} \\ & & & & c_{n-1} & b_n \end{bmatrix} \tag{5.29}$$

The number of plane rotations required to bring a matrix of order n to its tridiagonal form is $(n-1)(n-2)/2$. It is already known that **A** and **B** have the same eigenvalues. If $c_i \neq 0$, $i = 1,2\ldots\ldots n-1$, then the eigenvalues are distinct. Now, define

$$f_n = |\lambda I - B|$$

$$= \begin{vmatrix} \lambda - b_1 & -c_1 & & & 0 & \\ -c_1 & \lambda - b_2 & -c_2 & & & \\ & 0 & & & & \\ & & & -c_{n-2} & \lambda - b_{n-1} & -c_{n-1} \\ & & & & -c_{n-1} & \lambda - b \end{vmatrix}$$

Expanding by minors, the sequence $\{f_n\}$ satisfies

$$f_0 = 1 \quad f_1 = \lambda - b_1$$

$$f_r = (\lambda - b_r) f_{r-1} - c_{r-1}^2 f_{r-2}; \quad 2 \le r \le n. \tag{5.30}$$

Note that f_n

$f_n = 0$ is the characteristic equation.

If any $c_i = 0$, then the system degenerates as

$$B = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

Then, $f_n = |\lambda I - B|$ = (characteristic equation of P)×(characteristic equation of Q)

If none of the $c_1, c_2, \ldots\ldots, c_{n-1}$ vanish, then $\{f_n\}$ is a Sturm sequence. That is, if $V(x)$ denotes the number of changes in sign in the sequence for a given number x, then the number of zeroes of in (a,b) is $|V(a) - V(b)|$ (provided a or b is not a zero of f_n). In this way, one can approximately compute the eigenvalues and by repeated bisections, one can improve these estimates.

The eigenvectors of **B** are then found. If these are determined, then the eigenvectors of **A** can be determined, since it is known that if v and u are the eigenvectors of **B** and **A** respectively, then $\mathbf{u} = \mathbf{S}v$; where $\mathbf{S} = \mathbf{S}_1\mathbf{S}_2\ldots\ldots\mathbf{S}_j$ is the product of orthogonal matrices used in the plane rotations. Neglect a particular equation (say i-th) and then solve the remaining equations.This solution usually satisfies the equation that has been left. Then, v is the eigenvector determined from these solutions and by putting a zero in the i-th position. An advantage of the Givens method is that it takes only a finite number of plane rotations $((n-1)(n-2)/2)$ to reduce **A** to its tridiagonal form.

5.2.4 Householder-QR-Inverse Iteration Solution

Another most important transformation solution technique is the Housecholder-QR-inverse iteration (HQRI) method, although this method is restricted to the solution of the standard eigenproblem. Therefore, if the generalized eigenproblem $K\psi=\lambda M\psi$ is

considered, it must first be transformed into the standard form before the HQRI solution technique can be used. This transformation is effective in only some cases. The name "HQRI solution method" stands for the following three solution steps:

1. Householder transformations are employed to reduce the matrix **K** to a tridiagonal form.
2. QR iteration yields all eigenvalues.
3. Using inverse iteration the required eigenvectors of the tridiagonal matrix are calculated. The vectors are transformed to obtain the eigenvectors of **K**.

A basic difference from the Jacobi solution method is that the matrix is first transformed without iteration into a tridiagonal form. The matrix can then be used effectively in the QR iterative solution, in which all eigenvalues are calculated. Finally only those eigenvectors that are actually requested are evaluated.

5.2.4.1 The Householder Reduction

The Householder reduction to tridiagonal form involves $n-2$ transformations of the form of equation (5.7); i.e., using $\bar{K}_1 = \bar{K}$, it is calculated

$$K_{k+1} = P_k^t K_k P_k; \quad k = 1, 2, \ldots\ldots, n-2 \tag{5.31}$$

where P_k are Householder transformation matrices.

$$P_k = 1 - \theta w_k w_k^t \tag{5.32}$$

$$\theta = \frac{2}{w_k^t w_k} \tag{5.33}$$

To show how the vector w_i, that defines the matrix P_k is calculated, considering the case $k = 1$, which is typical. First, partitioning K_1, P_1, and w_1 into submatrices as follows:

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & \overline{P}_1 \end{bmatrix}; \qquad w_1 = \begin{bmatrix} 0 \\ \overline{w}_1 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} k_{11} & k_1^T \\ k_1 & K_{11} \end{bmatrix} \tag{5.34}$$

where K_{11}. $\overline{P}_1$ and $\overline{w}_1$ are of order $n-1$. In general case of step k, the corresponding matrices of order $n-k$. Performing the multiplications in (5.31), to obtain, using the notation of (5.34)

$$K_2 = \begin{bmatrix} k_{11} & k_1^T \overline{P}_1 \\ \overline{P}_1^T k_1 & \overline{P}_1^T K_{11} \overline{P}_1 \end{bmatrix} \tag{5.35}$$

The condition is now that the first column and row of K_2 to be n the form

$$K_2 = \begin{bmatrix} k_{11} & \times & 0 & \cdots & 0 \\ \times & & & & \\ 0 & & & & \\ \vdots & & \bar{K}_2 & & \\ 0 & & & & \end{bmatrix} \tag{5.36}$$

where $\times$ indicates a nonzero value and

$$\bar{K}_2 = \overline{P}_1^T K_{11} \overline{P}_1 \tag{5.37}$$

The form of $\bar{K}_2$ in equation (5.36) is achieved by realizing that $\bar{P}_1$ is a reflection matrix. Therefore, it can used $\overline{P}_1$ to reflect the vector k_1 of $\bar{K}_1$ in equation (5.34) into a vector that has only its first component nonzero. Since the length of the new vector must be the length of k_1, determining $\overline{w}_1$ from the condition

$$\left(I - \theta \overline{w}_1 \overline{w}_1^T\right) k_1 = \pm \|k_1\|_2 e_1 \tag{5.38}$$

where e_1 is a unit vector of dimension $n-1$; i.e., $e_1^T = [1 \;\; 0 \;\; 0 \;\; \ldots\ldots 0]$, and the + or − sign can be selected to obtain the best numerical stability. Noting that it is only needed to solve for a multiple of $\overline{w}_1$ (i.e., the direction of the vector normal to the plane of reflection is important, we obtain from equation (5.39) as a suitable value for $\overline{w}_1$,

$$\overline{w}_1 = k_1 + sign(k_{21}) \|k_1\|_2 e_1 \tag{5.39}$$

where k_{21} is element (2,1) of $\bar{K}_1$. With $\bar{w}_1$ defined in equation (5.39), the first Householder transformation, $k = 1$ in equation (5.33) can be carried out, in the next step, $k = 2$, it can be considered the matrix $\bar{K}_2$ in equation (5.36) in the same way as it is considered $\bar{K}_1$ in equation (5.34) to (5.39) because the reduction of the first column and row of $\bar{K}_2$ does not affect the first column and row in $\bar{K}_2$. Thus, the general algorithm for the transformation of $\bar{K}$ into tridiagonal form is established.

A disadvantage of the housholder transformation is that the matrix bandwidth is increased in the unreduced part of $\bar{K}_{k+1}$. Hence in the reduction, essentially no advantage can be taken of the bandedness of **K.**

5.2.4.2 The QR Iteration

In the HQRI solution procedure, the QR iteration is required could be applied to the tridiagonal matrix obtained by the Householder transformation of **K**. However, it should be realized that the **QR** iteration could be applied to the original matrix **K** as well, and that the transformation of **K** into tridiagonal form prior to the iteration is merely carried out to improve the efficiency of the solution. In the following it is therefore cousidered how the iteration is applied to a general symmetric matrix **K**.
The name "**QR** iteration" derives from the notation used in the algorithm. Namely the basic steps in the iteration is to decompose **K** in the form

$$K = QR \tag{5.40}$$

Where **Q** is an orthogonal and **R** is an upper triangular matrix, it is then formed

$$RQ = Q^T KQ \tag{5.41}$$

Therefore, by calculating RQ, it is in fact carried out a transformation of the form of equation (5.6).

The factorization in equation (5.39) could be obtained by applying the Gram-Schmidt process to the columns of **K**. In practice, it is more effective to reduce **K** into upper triangular form using Jacobi rotation matrices; i.e., evaluating

$$P_{n,n-1}^T \cdots\cdots P_{3,1}^T P_{2,1}^T K = R \tag{5.42}$$

where the rotation matrix $P_{j,i}^T$ is selected to zero element(j,i). Using equation (5.41), corresponding to equation (5.39) and (5.40). Using the notation $K_1 = K$, to form

$$K_k = Q_k R_k \tag{5.43}$$

$$K_{k+1} = R_k Q_k \tag{5.44}$$

Where, then, disregarding that eigenvalues and eigenvectors may not be in the usual order,

$$K_{k+1} \to A \quad \text{and} \quad Q_1, \cdots, Q_{k-1} Q_k \to \Psi \quad \text{as } k \to \infty$$

5.2.5 The Lanczos Transformation

The basic step of the Lanczos method transform, in theory, our generalized eigenproblem $K\psi = \lambda M\psi$, into a standard form with a tridiagonal coefficient matrix. Let us summarize the steps of transformation.

Pick a starting vector x and calculate

$$x_1 = \frac{x}{\gamma}; \quad \gamma = \left(x^t Mx\right)^{1/2} \tag{5.45}$$

Let $\beta_0 = 0$; for $i = 1,2,\ldots\ldots,n$ then calculate for $i = 1,2,\ldots\ldots,n$,

$$K\bar{x}_i = Mx_i \tag{5.46}$$

$$\alpha_i = \bar{x}_i^T Mx_i \tag{5.47}$$

and if $i \neq n$, $\qquad \bar{x}_i = \bar{x}_i - \alpha_i x_i - \beta_{i-1} x_{i-1}$ (5.48)

$$\beta_i = \left(\bar{x}_i^t M\bar{x}_i\right)^{1/2} \tag{5.49}$$

$$x_{i+1} = \frac{\bar{x}_i}{\beta_i} \tag{5.50}$$

Theoretically, the vectors x_i, $i = 1,2,\ldots\ldots,n$, generated using equation (5.45) to (5.50) are M- orthonormal

$$x_i^T Mx_j = \delta_{ij} \tag{5.51}$$

And the matrix $\qquad X_n = [x_1, x_2, \ldots\ldots, x_n]$ (5.52)

satisfies the relationship $X_n^T (MK^{-1}M) X_n = T_n$ (5.53)

Satisfies the relationship $X_n^T (MK^{-1}M) X_n = T_n$ (5.54)

Where $$T_n = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & & & & \\ & & & \alpha_{n-1} & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{bmatrix} \quad (5.55)$$

We can now relate the eigenvalues and vectors of T_n to those of the problem $\mathbf{K}\psi = \lambda\mathbf{M}\psi$, which can be written in the form

$$MK^{-1}\psi = \frac{1}{\lambda} M\psi \quad (5.56)$$

using the transformation

$$\psi = X_n \tilde{\psi} \quad (5.57)$$

and equation (5.51) and (5.53), we obtain from eqnation (5.57)

$$T_n \tilde{\psi} = \frac{1}{\lambda} \tilde{\psi} \quad (5.58)$$

Hence, the eigenvalues of T_n are the reciprocals of the eigenvalues of $\mathbf{K}\psi = \lambda\mathbf{M}\psi$, the eigenvectors of the two problems are related as in equation (5.56).

5.2.6 Subspace Iteration Method

Another iterative methodwhich can be used to find the lowest eigenvalues and the associated eigenvectors of the general eigenvalue problem, $[A]\vec{X} = \lambda [B]\vec{X}$, is the subspace iteration method. This method s very effective in finding the first few eigenvalnes and the corresponding eigenvectors of large eigenvalue problems whose stiffness ([A]) and mass ([B]) matrices have large bandwidths. The varions steps of this method are given below briefly.

Algorithm

Step 1: Start with q initial iteration vectors $\vec{X}_1, \vec{X}_2, \ldots\ldots, \vec{X}_q$, $q > p$, where p is the number of eigenvalues and eigenvectors to be calculated. Bath Wilson suggested a

value of $q = \min(2p, p+8)$ or good convergence. Define the initial modal matrix $[X_0]$ as

$$[X_0] = [\bar{X}_1, \bar{X}_2, \ldots\ldots, \bar{X}_q] \tag{5.59}$$

and set the iteration number as $k = 0$.

Step 2: Use the following subspace iteration procedure to generate an improve modal matrix $[\bar{X}_{k+1}]$.

(a) Find $[\bar{X}_{k+1}]$ from the relation

$$[A][\bar{X}_{k+1}] = [B][\bar{X}_k] \tag{5.60}$$

(b) Compute $[A_{k+1}] = [\bar{X}_{k+1}]^T [A][\bar{X}_{k+1}]$ and (5.61)

$$[B_{k+1}] = [\bar{X}_{k+1}]^T [B][\bar{X}_{k+1}] \tag{5.62}$$

(c) Solve for the eigenvalues and eigenvectors of the reduced system

$$[A_{k+1}][Q_{k+1}] = [B_{k+1}][Q_{k+1}][A_{k+1}] \tag{5.63}$$

and obtain $[A_{k+1}]$ and $[Q_{k+1}]$.

(d) Find an improved approximation to the eigenvectors of the original system as

$$[X_{k+1}] = [\bar{X}_{k+1}][Q_{k+1}]. \tag{5.64}$$

Note:

(1) It is assumed that the iteration vectors converging to the exact eigenvectors $\bar{X}_1^{(exact)}, \bar{X}_2^{(exact)} \ldots$, are stored as the first, second,..., columns of the matrix $[X_{k+1}]$.

(2) It is assumed that the vectors in $[X_0]$ are not orthogonal to one of the required eigenvectors.

Step 3: If $\lambda_i^{(k)}$ and $\lambda_i^{(k+1)}$ denote the approximations to the i-th eigenvalues in the iterations $k-1$ and k respectively, we assume convergence of the process whenever the following criteria are satisfied:

$$\left| \frac{\lambda_i^{(k+1)} - \lambda_i^{(k)}}{\lambda_i^{(k+1)}} \right| \le \varepsilon \qquad i = 1,2,\ldots\ldots,p \tag{5.65}$$

where $\varepsilon = 10^{-6}$. It is to be noted that although the iteration is performed with q vectors $(q > p)$, the convergence is measured only on the approximations predicted for the p smallest eigenvalues.

Chapter 6

SOLUTION OF EIGENVALUE PROBLEM BY USING HYPERBOLIC INTERPOLATION

6.1 Element Stiffness Matrix Formulation

Rammohan et al [6] solved the eigenvalue problem $-\frac{d}{dx}\left(x^2\frac{d\psi}{dx}\right)-2x\psi=\lambda x^2\psi$ using Lagrange interpolation. Sine hyperbolic interpolation will be used in place of Lagrange interpolation. From equation 3.2.1 of chapter 3,the shape functions used in this case are

$$N_1=\frac{\sinh\frac{1}{2}(1-\xi)}{\sinh(1)} \qquad N_2=\frac{\sinh\frac{1}{2}(1+\xi)}{\sinh(1)}$$

Now

$$k_{ij}=\int_{x_A}^{x_B}\left(x^2\frac{dN_i}{dx}\cdot\frac{dN_j}{dx}-2xN_iN_j\right)dx$$

$$k_{ij}=\int_{-1}^{1}\left\{\left(\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi\right)^2\frac{dN_i}{d\xi}\cdot\frac{d\xi}{dx}\cdot\frac{dN_j}{dx}\cdot\frac{d\xi}{dx}-2\left(\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi\right)N_iN_j\right\}\frac{h}{2}d\xi$$

$$=\int_{-1}^{1}\left\{\left(\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi\right)^2\frac{dN_i}{d\xi}\cdot\frac{2}{h}\cdot\frac{dN_j}{dx}\cdot\frac{2}{h}-2\left(\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi\right)N_iN_j\right\}\frac{h}{2}d\xi$$

Taking $x_A=0$, $x_B=1$ and $h=1$

$$k_{11}=\int_{-1}^{1}\left[\frac{1}{4}(1+\xi)^2\frac{\cosh\frac{1}{2}(1-\xi)\left(-\frac{1}{2}\right)\cosh\frac{1}{2}(1-\xi)\left(-\frac{1}{2}\right)}{\sinh(1)\cdot\sinh(1)}-2\cdot\frac{1}{2}(1+\xi)\frac{\sinh\frac{1}{2}(1-\xi)\sinh\frac{1}{2}(1-\xi)}{\sinh(1)\cdot\sinh(1)}\right]$$

$$=0.11$$

Similarly $k_{12}=k_{21}=-0.42$, $k_{22}=-0.05$

So for element 1 the stiffness matrix is $K^{(1)}=\begin{bmatrix}0.11 & -0.42\\ -0.42 & -0.05\end{bmatrix}$

For element 2 $K^{(2)}=\begin{bmatrix}1.17 & -2.35\\ -2.35 & 1.30\end{bmatrix}$

Also for element 20 $K^{(20)}=\begin{bmatrix}319.97 & -326.20\\ -326.20 & 325.39\end{bmatrix}$

6.1.1 Assembly of Element Stiffness to Form Global Matrix

For one linear element approximation function is, $\psi^e = N_1\Psi_1^e + N_2\Psi_2^e$, where e is the element number. Substitution of the finite element approximation and using Galerkin method in which the weight function is equal to the shape function gives us the finite element model of the eigenvalue equation.

$$\begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} \end{bmatrix} \begin{bmatrix} \Psi_1^e \\ \Psi_2^e \end{bmatrix} = \lambda \begin{bmatrix} m_{11}^{(e)} & m_{12}^{(e)} \\ m_{21}^{(e)} & m_{22}^{(e)} \end{bmatrix} \begin{bmatrix} \Psi_1^e \\ \Psi_2^e \end{bmatrix}$$

The stiffness matrix is obtained as $K^e\psi^e = \lambda M^e\psi^e$

Stiffness matrix for the element 1

$$k^{(1)} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix}$$

Stiffness matrix for element 2

$$k^{(2)} = \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix}$$

where $K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$ is the global stiffness matrix.

Now the relation between the elements of global stiffness matrix and element stiffness matrix has been shown.

$$k_{11} = 0.11, \quad k_{12} = k_{12}^{(1)} = -0.42, \quad k_{13} = 0$$

$$k_{21} = k_{21}^{(1)} = -0.42\ , \ k_{22} = k_{22}^{(1)} + k_{11}^{(2)} = -0.05 + 1.17 = 1.12\ , \quad k_{23} = k_{12}^{(2)} = -2.35$$

$$k_{31} = 0 \qquad k_{11} = k_{11}^{(1)} = 0.11\ k_{32} = k_{21}^{(2)} = -2.35 \qquad k_{33} = k_{22}^{(2)} = 1.30$$

Now it has been shown that how the global stiffness matrix can be assembled directly from the element stiffness matrix. The body is shown in the figure 6.1

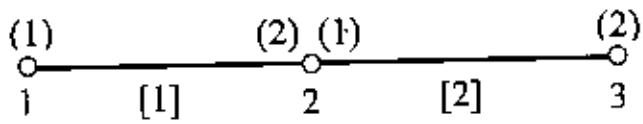


Figure 6.1: Local and global nodal point numbering

$[i]$ are the element number

(i) are the node number of the element

i are the global node number of the element

Element [1] is related with global node no. 1 and 2.

Element [2] is related with the global node no. 2 and 3.

For the element 1

$$k^{(1)} = \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{bmatrix} 1 & 2 \\ k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix}$$

For element 2

$$k^{(2)} = \begin{matrix} \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 2 & 3 \\ k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix}$$

Hence the global stiffness matrix will be a 3×3 matrix.

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} 0.11 & -0.42 & 0 \\ -0.42 & 1.12 & -2.35 \\ 0 & -2.35 & 1.30 \end{bmatrix}$$

This is a square symmetric matrix.

6.2 Formulation of Element Mass Matrix

$$m_{ij} = \int_{x_A}^{x_B} x^2 N_i N_j dx$$

$$m_{ij} = \int_{-1}^{1} \left\{ \left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2} \xi \right)^2 N_i N_j \right\} \frac{h}{2} d\xi$$

Taking $x_A = 0$, $x_B = 1$ and $h = 1$

$$m_{11} = 0.02$$

Similarly, $m_{12} = m_{21} = 0.03$ $\quad m_{22} = 0.17$

Thus for element 1 $\quad M^{(1)} = \begin{bmatrix} 0.02 & 0.03 \\ 0.03 & 0.17 \end{bmatrix}$

For element 2 $M^{(2)} = \begin{bmatrix} 0.39 & 0.25 \\ 0.25 & 0.83 \end{bmatrix}$.

For element 20 $M^{(20)} = \begin{bmatrix} 94.99 & 42.00 \\ 42.00 & 100.72 \end{bmatrix}$

Now, the element mass matrix is

6.2.1 Assembly of Element Mass Matrix

For the mass matrix M^e element stiffness matrices are given below:

For the element 1

$$M^{(1)} = \begin{matrix} 1 & 2 \end{matrix} \begin{bmatrix} m_{11}^{(1)} & m_{12}^{(1)} \\ m_{21}^{(1)} & m_{22}^{(1)} \end{bmatrix}$$

$$M^{(2)} = \begin{matrix} 2 & 3 \end{matrix} \begin{bmatrix} m_{11}^{(2)} & m_{12}^{(2)} \\ m_{21}^{(2)} & m_{22}^{(2)} \end{bmatrix}$$

$m_{11} = m_{11}^{(1)} = 0.02$, $m_{12} = m_{12}^{(1)} = 0.03$, $k_{13} = 0$

$m_{21} = m_{21}^{(1)} = 0.03$, $m_{22} = m_{22}^{(1)} + m_{11}^{(2)} = 0.17 + 0.39 = 0.56$, $m_{23} = m_{12}^{(2)} = -2.8$

$m_{31} = 0$, $m_{32} = m_{21}^{(2)} = 0.25$, $m_{33} = m_{22}^{(2)} = 0.83$

Hence the global mass matrix is

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} m_{11}^{(1)} & m_{12}^{(1)} & 0 \\ m_{21}^{(1)} & m_{22}^{(1)} + m_{11}^{(2)} & m_{12}^{(2)} \\ 0 & m_{21}^{(2)} & m_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} 0.02 & 0.03 & 0 \\ 0.03 & 0.56 & 0.25 \\ 0 & 0.25 & 0.83 \end{bmatrix}$$

Assemble of two linear elements has been shown below.

The element stiffness equation $K^{(1)}\psi = \lambda M^{(1)}\psi$ for element 1 is

$$\begin{bmatrix} 0.11 & -0.42 \\ -0.42 & -0.05 \end{bmatrix} \begin{bmatrix} \Psi_1^{(1)} \\ \Psi_2^{(1)} \end{bmatrix} = \lambda \begin{bmatrix} 0.02 & 0.03 \\ 0.03 & 0.17 \end{bmatrix} \begin{bmatrix} \Psi_1^{(1)} \\ \Psi_2^{(1)} \end{bmatrix}$$

Similarly, the element stiffness equation $K^{(2)}\psi = \lambda M^{(2)}\psi$ for element 2 is

$$\begin{bmatrix} 1.17 & -2.35 \\ -2.35 & 1.30 \end{bmatrix} \begin{bmatrix} \Psi_1^{(2)} \\ \Psi_2^{(2)} \end{bmatrix} = \lambda \begin{bmatrix} 0.39 & 0.25 \\ 0.25 & 0.83 \end{bmatrix} \begin{bmatrix} \Psi_1^{(2)} \\ \Psi_2^{(2)} \end{bmatrix}$$

Here $\Psi_1^{(1)} = \psi_1$, $\Psi_2^{(1)} = \Psi_1^{(2)} = \psi_2$, $\Psi_2^{(2)} = \psi_3$

The global stiffness equation $\mathbf{K\psi = \lambda M\psi}$ for two linear elements

$$\begin{bmatrix} 0.11 & -0.42 & 0 \\ -0.42 & 1.12 & -2.35 \\ 0 & -2.35 & 1.30 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \lambda \begin{bmatrix} 0.02 & 0.03 & 0 \\ 0.03 & 0.56 & 0.25 \\ 0 & 0.25 & 0.83 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$

Now the boundary conditions are imposed on the problem. Here $\psi(1) = 0$ requires $\psi_3 = 0$.

On substitution the above condition the global stiffness equation takes the form

$$\begin{bmatrix} 0.11 & -0.42 & 0 \\ -0.42 & 1.12 & -2.35 \\ 0 & -2.35 & 1.30 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 0.02 & 0.03 & 0 \\ 0.03 & 0.56 & 0.25 \\ 0 & 0.25 & 0.83 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ 0 \end{bmatrix}$$

The above assembled equations are solved by Gauss elimination method or by any other solver

Therefore, the eigenvalue problem reduces to two equations

$$0.11\psi_1 - 0.42\psi_2 = \lambda(0.02\psi_1 + 0.03\psi_2)$$

and $$-0.42\psi_1 + 1.12\psi_2 = \lambda(0.03\psi_1 + 0.56\psi_2)$$

Then solving the above two equations the values of λ are obtained as

$$\lambda_1 = -0.466$$

$$\lambda_2 = 11.068$$

The global stiffness matrix K_1 and M_1 for 20 elements will be a 21×21 matrix. The matrix eigenvalue equation to be solved by Jacobian method is

$$\bar{K}_1\bar{\psi} = \lambda\bar{M}_1\bar{\psi}$$

The equation is solved by using computer programming to find the eigenvalues which are given the following table

Table- 6.1 Eigenvalues obtained using 2-noded element

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
13.701	11.350	10.727	9.832	8.771	7.643	6.519	5.452	4.470	3589

λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}	λ_{17}	λ_{18}	λ_{19}	λ_{20}
2.813	2.141	1.568	1.087	0.693	0.379	0.143	-0.016	-0.185	-1.142

6.3 Stiffness and Mass Matrix for Quadratic Element

For quadratic element, writing

$$\psi = N_1\psi_1 + N_2\psi_2 + N_3\psi_3$$

The shape function in this case are given by equation (3.2.2) in chapter 3

$$N_1 = \frac{\sinh\xi\sinh(\xi-1)}{\sinh(-1)\sinh(-2)}$$

$$N_2 = \frac{\sinh(1+\xi)\sinh(\xi-1)}{\sinh(1)\sinh(-1)}$$

$$N_3 = \frac{\sinh\xi\sinh(1+\xi)}{\sinh(2)\sinh(1)}$$

Thus the element stiffness is obtained as

$$k_{ij} = \int_{x_A}^{x_B}\left(x^2\frac{dN_i}{dx}\cdot\frac{dN_j}{dx} - 2xN_iN_j\right)dx$$

$$k_{ij} = \int_{-1}^{1}\left\{\left(\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi\right)^2\frac{dN_i}{d\xi}\cdot\frac{d\xi}{dx}\cdot\frac{dN_j}{dx}\cdot\frac{d\xi}{dx} - 2\left(\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi\right)N_iN_j\right\}\frac{h}{2}d\xi$$

$$= \int_{-1}^{1}\left\{\left(\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi\right)^2\frac{dN_i}{d\xi}\,2/h\,\frac{dN_j}{dx}\,2/h - 2\left(\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi\right)N_iN_j\right\}\frac{h}{2}d\xi$$

Taking $x_A = 0$, $x_B = 2$ and $h = 2$

$$k_{11} = \int_{-1}^{1}\left\{(1+\xi)^2\left[\frac{\sinh\xi\cosh(\xi-1)}{\sinh(-1)\sinh(-2)}+\frac{\cosh\xi\sinh(\xi-1)}{\sinh(-1)\sinh(-2)}\right]^2\right.$$

$$\left.-2(1+\xi)\frac{\sinh\xi\sinh(\xi-1)\sinh\xi\sinh(\xi-1)}{\sinh(-1)\sinh(-2)\sinh(-1)\sinh(-2)}\right\}\frac{h}{2}d\xi = 0.16$$

Thus for element 1 $\quad K^{(1)} = \begin{bmatrix} 0.16 & -0.66 & 0.36 \\ 0.66 & 2.33 & -4.49 \\ 0.36 & -4.49 & 2.10 \end{bmatrix}$

For element 2 $\quad K^{(2)} = \begin{bmatrix} 6.75 & -9.88 & 1.81 \\ -9.88 & 2.212 & 20.70 \\ 1.81 & 20.70 & 16.88 \end{bmatrix}$

For element 10 $\quad K^{(10)} = \begin{bmatrix} 450.45 & -523.14 & 61.94 \\ -523.14 & 1059.42 & -589.90 \\ 61.94 & 589.90 & 516.32 \end{bmatrix}$

6.3.1 Assemble of Element Stiffness Matrix to Form Global Matrix

Now the relation between the elements of global stiffness matrix and element stiffness matrix by taking two quadratic elements has been shown below

$$k_{11} = k_{11}^{(1)} = 1.66, \quad k_{12} = k_{12}^{(1)} = 0.66, \quad k_{13} = k_{13}^{(1)} = 0.36, \quad k_{14} = 0, \quad k_{15} = 0$$

$$k_{21} = k_{21}^{(1)} = 0.66, \quad k_{22} = k_{22}^{(1)} = 2.33, \quad k_{23} = k_{23}^{(1)} = -4.49, \quad k_{24} = 0, \quad k_{25} = 0$$

$$k_{31} = k_{31}^{(1)} = 0.36, \quad k_{32} = k_{32}^{(1)} = -4.49, \quad k_{33} = k_{33}^{(1)} + k_{11}^{(2)} = 2.10 + 6.75 = 8.85$$

$$k_{34} = k_{12}^{(2)}, \quad k_{35} = k_{13}^{(2)}, \quad k_{41} = 0, \quad k_{42} = 0,$$

$$k_{43} = k_{21}^{(2)} = -9.88, \quad k_{44} = k_{22}^{(2)} = 2.212, \quad k_{45} = k_{23}^{(2)} = 20.70, \quad k_{51} = 0, \quad k_{52} = 0,$$

$$k_{53} = k_{31}^{(2)} = 1.81, \quad k_{54} = k_{32}^{(2)} = 20.70, \quad k_{55} = k_{33}^{(2)} = 16.88$$

Now it has been shown how the global stiffness matrix can be assembled directly from the element stiffness matrix. The body is shown in the figure 6.2

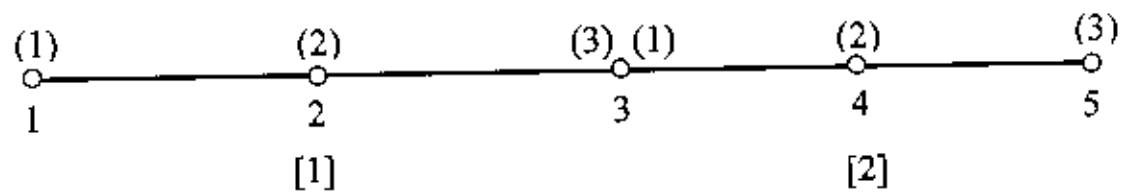


Figure 6.2: Local and global nodal point numbering

$[i]$ are the element number

(i) are the node number of the element

i are the global node number of the element

Element [1] is related with global node no. 1, 2 and 3.

Element [2] is related with the global node no. 3, 4, and 5.

Hence the global stiffness for 2 elements will be a 5 ×5 matrix.

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{bmatrix} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(2)} & k_{13}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{23}^{(1)} & 0 & 0 \\ k_{31}^{(1)} & k_{32}^{(1)} & k_{33}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(2)} \\ 0 & 0 & k_{21}^{(2)} & k_{22}^{(2)} & k_{23}^{(2)} \\ 0 & 0 & k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.16 & -0.66 & 0.36 & 0 & 0 \\ 0.66 & 2.33 & -4.49 & 0 & 0 \\ 0.36 & -4.49 & 8.85 & -9.88 & 1.81 \\ 0 & 0 & -9.88 & 2.212 & 20.70 \\ 0 & 0 & 1.81 & 20.70 & 16.88 \end{bmatrix}$$

This is a square symmetric matrix.

Similarly, global mass matrix can be obtained as

$$m_{ij} = \int_{x_A}^{x_B} x^2 N_i N_j dx$$

From $$m_{ij} = \int_{x_A}^{x_B} x^2 N_i N_j dx = \int_{-1}^{1} \left\{ \left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2} \xi \right)^2 N_i N_j \right\} \frac{h}{2} d\xi$$

we get $m_{11} = 0.03$

Thus for element 1 $$M^{(1)} = \begin{bmatrix} 0.03 & 0.00 & -0.03 \\ 0.00 & 1.25 & 0.33 \\ -0.03 & 0.33 & 0.58 \end{bmatrix}$$

For element 2 $$M^{(2)} = \begin{bmatrix} 1.03 & 0.68 & -0.25 \\ 0.68 & 10.52 & 1.67 \\ -0.25 & 1.67 & 2.69 \end{bmatrix}$$

For element 10 $$M^{(10)} = \begin{bmatrix} 88.84 & 43.12 & -24.10 \\ 43.12 & 385.22 & 53.26 \\ -24.10 & 53.26 & 104.04 \end{bmatrix}$$

Similarly, the assembled mass matrix for the first two will become:

$$M = \begin{bmatrix} m_{11}^{(1)} & m_{12}^{(2)} & m_{13}^{(3)} & 0 & 0 \\ m_{21}^{(1)} & m_{22}^{(1)} & m_{23}^{(1)} & 0 & 0 \\ m_{31}^{(1)} & m_{32}^{(1)} & m_{33}^{(1)} + m_{11}^{(2)} & m_{12}^{(2)} & m_{13}^{(2)} \\ 0 & 0 & m_{21}^{(2)} & m_{22}^{(2)} & m_{23}^{(2)} \\ 0 & 0 & m_{31}^{(2)} & m_{32}^{(2)} & m_{33}^{(2)} \end{bmatrix} = \begin{bmatrix} m_{11}^{(1)} & m_{12}^{(2)} & m_{13}^{(3)} & 0 & 0 \\ m_{21}^{(1)} & m_{22}^{(1)} & m_{23}^{(1)} & 0 & 0 \\ m_{31}^{(1)} & m_{32}^{(1)} & m_{33}^{(1)} + m_{11}^{(2)} & m_{12}^{(2)} & m_{13}^{(2)} \\ 0 & 0 & m_{21}^{(2)} & m_{22}^{(2)} & m_{23}^{(2)} \\ 0 & 0 & m_{31}^{(2)} & m_{32}^{(2)} & m_{33}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.03 & 0.00 & -0.03 & 0 & 0 \\ 0.00 & 1.25 & 0.33 & 0 & 0 \\ 0.03 & 0.33 & 1.06 & 0.68 & -0.25 \\ 0 & 0 & 0.68 & 10.52 & 1.67 \\ 0 & 0 & -0.25 & 1.67 & 2.69 \end{bmatrix}$$

The global stiffness matrix K_1 and M_1 will be a 21×21 matrix. The matrix eigenvalue equation to be solved by Jacobian method is

$$\vec{K}_1 \vec{\psi} = \lambda \vec{M}_1 \vec{\psi}$$

The equation is solved by using standard computer programming to find the eigenvalues which are given in the following table

Table- 6.2: Eigenvalues obtained using 3-noded element

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
18.711	17.709	16.083	14.903	12.021	10.059	8.314	6.844	5.721	4.093

λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}	λ_{17}	λ_{18}	λ_{19}	λ_{20}
2.440	1.972	1.468	1.013	0.629	0.320	0.083	-0.079	-0.232	-0.979

6.4 Formulation of Element Stiffness and Mass Matrix Using Tangent Hyperbolic Interpolation

(a) Stiffness and Mass Matrix for Linear Element

Tangent hyperbolic interpolatiou has been also used to calculate the eigenvalues. The shape functions in this case for linear element are

$$N_1 = \frac{\tanh\frac{1}{2}(1-\xi)}{\tanh(1)}$$

$$N_2 = \frac{\tanh\frac{1}{2}(1+\xi)}{\tanh(1)}$$

$$k_{11} = \int_{-1}^{1}\left[\frac{1}{4}(1+\xi)^2\frac{\sec h^2\frac{1}{2}(1-\xi)\left(-\frac{1}{2}\right)\sec h^2\frac{1}{2}(1-\xi)\left(-\frac{1}{2}\right)}{\tanh h(1)\cdot\tanh(1)} - 2\cdot\frac{1}{2}(1+\xi)\frac{\tanh\frac{1}{2}(1-\xi)\tanh\frac{1}{2}(1-\xi)}{\tanh(1)\cdot\tanh(1)}\right]\frac{1}{2}d\xi$$

$$= 0.30$$

$$k_{12} = k_{21} = -0.66 \qquad k_{22} = -0.36$$

So for elements 1 $\quad K^{(1)} = \begin{bmatrix} 0.30 & -0.66 \\ -0.66 & -0.36 \end{bmatrix}$

For element 2 $\quad K^{(2)} = \begin{bmatrix} 2.43 & -3.72 \\ -3.72 & 1.17 \end{bmatrix}$

Also for element 20 $\quad K^{(20)} = \begin{bmatrix} 512.43 & -484.28 \\ -484.28 & 500.52 \end{bmatrix}$

$$m_{ij} = \int_{x_A}^{x_B} x^2 N_i N_j dx$$

$$m_{11} = \int_{-1}^{1}(1+\xi)^2\left\{\frac{\tanh\frac{1}{2}(1-\xi)}{\tanh h(1)}\cdot\frac{\tanh\frac{1}{2}(1+\xi)}{\tanh(1)}\right\}\frac{1}{2}d\xi = 0.07$$

Thus for element 1 $M^{(1)} = \begin{bmatrix} 0.07 & 0.09 \\ 0.09 & 0.27 \end{bmatrix}$

For element 2 $M^{(2)} = \begin{bmatrix} 0.93 & 0.75 \\ 0.75 & 1.53 \end{bmatrix}$

For element 20 $M^{(20)} = \begin{bmatrix} 197.48 & 125.48 \\ 125.48 & 205.27 \end{bmatrix}$

The global stiffness matrix K_1 and M_1 will be a 21×21 matrix. The matrix eigenvalue equation to be solved by Jacobian method is

$$\bar{K}_1\bar{\psi} = \lambda\bar{M}_1\bar{\psi}$$

The equation is solved by using computer programming to find the eigenvalues which are given the following table

Table – 6.3: Eigenvalues obtained using 2-noded element

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
12.946	12.289	11.126	9.713	8.246	6.853	5.600	4.511	3.585	2.808

λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}	λ_{17}	λ_{18}	λ_{19}	λ_{20}
2.163	1.632	1.197	0.844	0.563	0.644	0.182	0.073	-0.042	-0.509

(b) Stiffness and Mass Matrix for Quadratic Element

writing, $\psi = N_1\psi_1 + N_2\psi_2 + N_3\psi_3$

The shape functions in this case are

$$N_1 = \frac{\tanh\xi\tanh(\xi-1)}{\tanh(-1)\tanh(-2)}$$

$$N_2 = \frac{\tanh(1+\xi)\tanh(\xi-1)}{\tanh(1)\tanh(-1)}$$

$$N_3 = \frac{\tanh\xi\tanh(1+\xi)}{\tanh(2)\tanh(1)}$$

We get the element stiffness as

$$k_{ij} = \int_{x_A}^{x_B}\left(x^2\frac{dN_i}{dx}\cdot\frac{dN_j}{dx} - 2xN_iN_j\right)dx$$

$$k_{ij} = \int_{-1}^{1}\left\{\left(\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi\right)^2\frac{dN_i}{d\xi}\cdot\frac{d\xi}{dx}\cdot\frac{dN_j}{dx}\cdot\frac{d\xi}{dx} - 2\left(\frac{x_A+x_B}{2}+\frac{x_B-x_A}{2}\xi\right)N_iN_j\right\}\frac{h}{2}d\xi$$

$$k_{ij} = \int_{-1}^{1}\left\{\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi\right)^2 \frac{dN_i}{d\xi}\frac{2}{h}\frac{dN_j}{dx}\frac{2}{h} - 2\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi\right)N_i N_j\right\}\frac{h}{2}d\xi$$

Taking $x_A = 0$, $x_B = 2$ Using and $h = 2$

$$k_{11} = \int_{-1}^{1}\left\{(1+\xi)^2\left[\frac{\tanh\xi\sec h^2(\xi-1)}{\tanh(-1)\tanh(-2)} + \frac{\sec h^2\xi\tanh(\xi-1)}{\tanh(-1)\tanh(-2)}\right]^2 - 2(1+\xi)\frac{\tanh\xi\tanh(\xi-1)\tanh\xi\tanh(\xi-1)}{\tanh(-1)\tanh(-2)\tanh(-1)\tanh(-2)}\right\}\frac{h}{2}d\xi = 1.26$$

Thus for element 1 $$K^{(1)} = \begin{bmatrix} 1.26 & 0.81 & 0.62 \\ 0.81 & 0.60 & 1.89 \\ 0.62 & 1.89 & 1.50 \end{bmatrix}$$

For element 10 $$K^{(10)} = \begin{bmatrix} 499.01 & -403.64 & -65.45 \\ -403.64 & 841.06 & 282.79 \\ -65.45 & 282.79 & 527.79 \end{bmatrix}$$

And the mass matrices are

From $$m_{ij} = \int_{x_A}^{x_B} x^2 N_i N_j dx = \int_{-1}^{1}\left\{\left(\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi\right)^2 N_i N_j\right\}\frac{h}{2}d\xi$$

Thus for element 1 $$M^{(1)} = \begin{bmatrix} 0.15 & -0.35 & -0.39 \\ -0.35 & 1.39 & 0.86 \\ -0.39 & 0.86 & 1.50 \end{bmatrix}$$

For element 10 $$M^{(10)} = \begin{bmatrix} 142.83 & 37.54 & -75.87 \\ 37.54 & 368.71 & 60.45 \\ -75.87 & 60.45 & 167.96 \end{bmatrix}$$

The global stiffness matrix K_1 and M_1 will be a 21×21 matrix. The matrix eigenvalue equation to be solved by Jacobian method is, $\bar{\bar{K}}_1\bar{\psi} = \lambda\bar{M}_1\bar{\psi}$

The equation is solved by using standard computer programming to find the eigenvalues which are given the following table

Table 6.4: Eigenvalues obtained using 3-noded element

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
33.248	7.689	6.642	5.606	4.821	4.312	4.016	3.833	3.563	2.723

λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}	λ_{17}	λ_{18}	λ_{19}	λ_{20}
1.733	1.636	1.483	1.306	1.113	0.935	0.797	0.694	0.503	-0.115

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6.5 Concluding Remarks:

From table 6.1 it has been observed that in case of linear element the sinc hyperbolic interpolation the eigenvalues shows discrepancy in case of higher eigenvalues than that of the lower values. But from table 6.2, it is seen that when quadratic elements are used then the sine hyperbolic interpolation gives the result where the eigenvalues are highly differ with each other than that of the values obtained by using linear sine hyperbolic interpolation functions in table 6.1.

From table 6.3 it has been observed that in case of linear tangent hyperbolic interpolation results shows less discrepancy than that of the eigenvalues obtained by using linear sinc hyperbolic interpolation. But on the other hand, it is seen from table 6.4 that the eigenvalues obtained from quadratic tangent hyperbolic interpolation are highly differ by their values in case of higher eigenvalues than the lower values. It is also noticed that in case of quadratic sine hyperbolic interpolation the results shows less discrepancy than those obtained by using quadratic tangent hyperbolic interpolation.

Chapter 7

RESULTS DISCUSSION AND CONCLUSION

Rammohan et al calculated the eigenvalue using Lagrange interpolation for the domain 0 to 20 taking 20 elements having length 1 for each element.

Eigenvalues are here calculated using quadratic Lagrange element for the same domain. Eigenvalues here are also calculated taking linear and quadratic element using sine hyperbolic and tangent hyperbolic interpolation. The results are shown below in the tabular form for comparison with the Lagrange interpolation.

Table 7.1: Eigenvalues obtained by using Lagrange and sine hyperbolic interpolation for linear element taking 20 elements having length 1 for each element.

Eigenvalues	Linear Lagrange	Linear sin*h*
λ_1	12.929	13.701
λ_2	11.565	11.350
λ_3	10.787	10.727
λ_4	9.724	9.832
λ_5	8.522	8.771
λ_6	7.292	7.643
λ_7	6.111	6.519
λ_8	5.023	5.452
λ_9	4.051	4.470
λ_{10}	3.199	3.589
λ_{11}	2.466	2.813
λ_{12}	1.843	2.141
λ_{13}	1.319	1.568
λ_{14}	0.886	1.087
λ_{15}	0.533	0.693
λ_{16}	0.225	0.379
λ_{17}	0.048	0.143
λ_{18}	-0.093	-0.016
λ_{19}	-0.238	-0.185
λ_{20}	-0.942	-1.142

Table 7.2: Eigenvalues obtained by using Lagrange and sine hyperbolic interpolation for quadratic element taking 10 elements having length 2 for each element.

Eigenvalues	Quadratic Lagrange	Quadratic sin*h*
λ_1	15.509	18.711
λ_2	14.085	17.709
λ_3	12.427	16.083
λ_4	10.522	14.093
λ_5	8.681	12.021
λ_6	7.049	10.059
λ_7	5.667	8.314
λ_8	4.520	6.844
λ_9	3.578	5.721
λ_{10}	2.664	4.093
λ_{11}	2.183	2.440
λ_{12}	1.634	1.972
λ_{13}	1.179	1.468
λ_{14}	0.798	1.013
λ_{15}	0.483	0.629
λ_{16}	0.229	0.320
λ_{17}	0.035	0.083
λ_{18}	-0.099	-0.079
λ_{19}	-0.249	-0.232
λ_{20}	-0.990	-0.979

Table 7.3: Eigenvalues obtained by using Lagrange and tangent hyperbolic interpolation for linear element taking 20 elements having length 1 for each element.

Eigenvalues	Linear Lagrange	Linear tan*h*
λ_1	12.929	12.946
λ_2	11.565	12.289
λ_3	10.787	11.126
λ_4	9.724	9.713
λ_5	8.522	8.246
λ_6	7.292	6.853
λ_7	6.111	5.600
λ_8	5.023	4.511
λ_9	4.051	3.585
λ_{10}	3.199	2.808
λ_{11}	2.466	2.163
λ_{12}	1.843	1.632
λ_{13}	1.319	1.197
λ_{14}	0.886	0.844
λ_{15}	0.533	0.563
λ_{16}	0.225	0.344
λ_{17}	0.048	0.182
λ_{18}	-0.093	0.073
λ_{19}	-0.238	-0.042
λ_{20}	-0.942	-0.509

Table 7.4: Eigenvalues obtained by using Lagrange and tangent hyperbolic interpolation for quadratic element taking 10 elements having length 2 for each element.

Eigenvalues	Quadratic Lagrange	Quadratric tan*h*
λ_1	15.509	17.713
λ_2	14.085	12.604
λ_3	12.427	10.763
λ_4	10.522	8.875
λ_5	8.681	7.285
λ_6	7.049	6.010
λ_7	5.667	4.985
λ_8	4.520	4.150
λ_9	3.578	3.462
λ_{10}	2.664	2.891
λ_{11}	2.183	2.296
λ_{12}	1.634	1.821
λ_{13}	1.179	1.438
λ_{14}	0.798	1.094
λ_{15}	0.483	0.496
λ_{16}	0.229	0.935
λ_{17}	0.035	0.233
λ_{18}	-0.099	0.020
λ_{19}	-0.249	-0.138
λ_{20}	-0.990	-0.513

The eigenvalue problem has been solved for different hyperbolic interpolation such as sine and Tangent. These are calculated for same region taking different length of elements 1 and 2 for linear and quadratic elements having number of elements 20 and 10 respectively. So that number of nodes remain same and number of eigenvalues remain same.

Table-7.1 shows that in case of linear element the Lagrange and sine hyperbolic interpolation the eigenvalues shows a good agreement.

Table-7.2 shows that when quadratic elements are used for the Lagrange and sine hyperbolic interpolation the result shows discrepancy in case of higher eigenvalues than the lower ones. It is also seen that for quadratic sine hyperbolic interpolation the agreement is good in case of lower eigenvalues than those of higher eigenvalues.

Table-7.3 it is seen that in case of Lagrange interpolation used for linear element, the result shows better agreement for the higher eigenvalues than the lower values.

Table-7.4 shows that for quadratic tangent interpolation results shows discrepancy in case of higher eigenvalues than the lower ones and the agreement is better in the later case. It is also noticeable that eigenvalues obtained by quadratic sine hyperbolic interpolation show better covergency in case of lower eigenvalues than those of lower eigenvalues obtained from quadratic tangent hyperbolic interpolation.

From the discussion it is clear that the effect of hyperbolic interpolation in the solution of eigenvalue problem is not good in case of qnadratic tangent hyperbolic compared to the linear sine hyperbolic, quadratic sine hyperbolic and linear tangent hyperbolic interpolation functions.

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