# THERMAL-DIFFUSION AND DIFFUSION-THERMO EFFECTS ON MAGNETOHYDRODYNAMICS HEAT AND MASS TRANSFER

by

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## MASTER OF PHILOSOPHY



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# THERMAL-DIFFUSION AND DIFFUSION-THERMO EFFECTS ON MAGNETOHYDRODYNAMICS HEAT AND MASS TRANSFER

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by

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The Thesis titled

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Roll No. 040209007P, Registration No. 0402407, Session-April, 2002 a part time student of M. Phil. (Mathematics) has been accepted as satisfactory in partial fulfillment for the degree of **MASTER OF PHILOSOPHY** in Mathematics on December 20, 2004.

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It is hereby declared that this thesis or any part of it has not been submitted elsewhere for the award of any degree or diploma.

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# Abstract

The thermal-diffusion and diffusion-thermo effects on Magnetohydrodynamics (MHD) heat and mass transfer boundary layer flow have been studied numerically. The governing equations are then made dimensionless by using usual similarity transformations. The dimensionless equations are solved numerically by applying Nachtsheim-Swigert shooting iteration technique along with Runge-Kutta sixth order integration method.

A two-dimensional steady MHD combined free-forced convection and mass transfer flow past a semi-infinite vertical plate is investigated. The effects of magnetic parameter $M$, thermal-diffusion parameter $Sr$ (Soret number) and diffusion-thermo parameter $Df$ (Dufour number) have been examined on the flow field of a hydrogen-air mixture as a non-chemical reacting fluid pair. The analysis has shown that the flow field is appreciably influenced by the thermal-diffusion and diffusion-thermo effects. In our study we have also found that the present results are in excellent agreement with Kafoussias (1990), when $M = Df = Sr = 0$.

Next, the above effects have been analyzed on unsteady MHD free convection and mass flow past an infinite vertical porous plate. Two cases are considered (I) Impulsively started plate (ISP) and (ii) Uniformly accelerated plate (UAP). Similarity equations of the momentum, energy and concentration equations are derived by introducing a time dependent length scale. The dimensionless velocity, temperature and concentration profiles are shown graphically for different values of the parameters entering into the problem. The numerical values of the local skin-friction, local Nusselt number and local Sherwood number are also presented in tabular form.

# Nomenclature

| | | |
|---|---|---|
| $B_0$ | : | applied magnetic field |
| $C$ | : | species concentration |
| $c_p$ | : | specific heat at constant pressure |
| $c_s$ | : | concentration susceptibility |
| $C_w$ | : | concentration at the plate |
| $C_\infty$ | : | species concentration at infinity |
| $Df$ | : | Dufour number |
| $D_m$ | : | mass diffusivity |
| $f$ | : | dimensionless stream function |
| $f_i, f_a$ | : | dimensionless longitudinal velocities |
| $g$ | : | acceleration due to gravity |
| $Gr$ | : | local temperature Grashof number |
| $Gm$ | : | local mass Grashof number |
| $k_T$ | : | thermal-diffusion ratio |
| $M$ | : | magnetic parameter |
| $Nu$ | : | Nusselt number |
| $Pr$ | : | Prandtl number |
| $Sc$ | : | Schmidt number |
| $Sr$ | : | Soret number |
| $Sh$ | : | Sherwood number |

| | | |
|---|---|---|
| $T$ | : | temperature of the flow field |
| $T_m$ | : | mean fluid temperature |
| $T_w$ | : | temperature at the plate |
| $T_\infty$ | : | temperature of the fluid at infinity |
| $U_0$ | : | constant plate velocity |
| $U_\infty$ | : | free stream velocity |
| $u$, $v$ | : | velocity components in the $x$ and $y$ directions, respectively |
| $x$, $y$ | : | Cartesian coordinates |

## Greek symbols

| | | |
|---|---|---|
| $\alpha$ | : | thermal diffusivity |
| $\beta$ | : | coefficient of thermal expansion |
| $\beta^*$ | : | coefficient of concentration expansion |
| $\sigma$ | : | electrical conductivity |
| $\rho$ | : | density of the fluid inside the boundary layer |
| $\nu$ | : | kinematic viscosity |
| $\eta$ | : | dimensionless similarity variable |
| $\delta$ | : | time dependent length scale |
| $\theta$ | : | dimensionless temperature |
| $\phi$ | : | dimensionless concentration |
| $\tau_x$ | : | skin-friction |

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# Chapter-1



## General introduction and review of previous works

Magnetohydrodynamics (MHD) is that branch of continuum mechanics, which deals with the flow of electrically conducting fluids in electric and magnetic fields. Probably the largest advance towards an understanding of such phenomena comes from the field of Astrophysics. Originally, MHD included only the study of strictly incompressible fluid, but today the terminology is applied to studies of ionized gases as well. Other names have been suggested, such as magnetofluid-mechanics or magneto-aerodynamics, but original nomenclature has persisted.

Many natural phenomena and engineering problems are susceptible to MHD analysis. It is useful in Astrophysics. Geophysicists encounter MHD phenomena in the interactions of conducting fluids and magnetic fields that are presented in and around heavenly bodies. Engineers employ MHD principles in the design of heat exchangers, pumps and flowmeters, in space vehicle propulsion, control and re-entry, in creating novel power generating systems, and in developing confinement schemes for controlled fusion.

The most important application of MHD is in the generation of electrical power with the flow of an electrical conducting fluid through a transverse magnetic field. Recently, experiments with ionized gases have been performed with the hope of producing power on a large scale in stationary plants with large magnetic fields. Generation of MHD power on a smaller scale is of interest for space applications.

In the studies related to heat transfer, considerable effect has been directed towards the convective mode, in which the relative motion of the fluid provides an additional mechanism for the transfer of energy and of material, the latter being a more important consideration in cases where mass transfer, due to a concentration difference, occurs. Convection is possible only in the presence of a fluid medium. When a fluid flows inside a duct or over a solid body while temperatures of the fluid and the solid surface are different, heat transfer between the fluid and the solid surface takes place as a consequence of the motion of fluid relative to the surface; this mechanism of heat transfer is called convection. The convective mode of heat transfer is divided into two basic processes. If the fluid motion is artificially induced say with a pump or a fan that forces the fluid flow over the surface, the heat transfer is termed as forced convection. Such problems are very frequently encountered in technology where the heat transfer to or from a body is often due to an imposed flow of a fluid at a different temperature from that of the body. If the fluid motion is set up by buoyancy effects resulting from density difference caused by temperature difference in the fluid, the heat transfer is said to be free or natural convection. There are essentially three factors, which govern the natural convection processes, namely the body force, the temperature variation in the flow field and the fluid density variation with temperature. Free convection is the principal mode of heat transfer from pipes, transmission lines, refrigerating coils, hot radiators and many other practical situations in everyday life. But in many cases of practical interest, both processes are important and heat transfer is by mixed convection, in which neither mode is truly predominant. It is expected that for large Reynolds number (i.e. correspondingly large flow velocities) and small Grashof number, the influence of free convection on the heat transfer can be neglected. On the other hand, for large Grashof number and small Reynolds number the free convection should be the dominating factor. In nature we face such situation where forced and free convection is of comparable order, the phenomena may be termed as the mixed or combined convection flows. This type of combined convection flow is the main topics of this study with a special attention paid to local similarity solution for a vertical flat plate.

Model studies of the free and mixed convection flows have earned reputations because of their applications in geophysical, geothermal and nuclear engineering problems. Sparrow et al. (1959), were the first investigator, who dealt with the combined forced and free convective boundary layer flow about a vertical flat plate. Later Mori (1961) and Sparrow and Minkowycz (1962) studied mixed convection flow in the boundary layer of a micropolar fluid over a horizontal flat plate using a perturbation series in terms of the buoyancy parameter. Their investigations presented formulae for calculating the shear stress and heat transfer rate in the mixed flow regime.

Cheng (1977) investigated the combined free and forced convection boundary layer flow along an inclined surface embedded in porous media. It was found that when both the wall temperature distribution of the plate and the velocity parallel to the plate outside of the boundary layer vary according to the some power function of distance, i. e. $x^{\lambda}$ (where $\lambda$ = constant), then the similarity solutions exist.

Hassanien (1977) studied the combined forced and free convection in boundary layer flow of a micropolar fluid over a horizontal plate. In his work similarity solutions are obtained for the case of a wall temperature that is inversely proportional to the square root of the distance from the leading edge.

The problem of mixed convection boundary layer flow along a vertical surface in the absence of magnetic field has been discussed in some details by Merkin (1969), Wilks (1974). Hunt and Wilks (1980), Gryzagoridus (1975), Carey and Gebahart (1982), Raju et al. (1984) and Harris et al. (1999). Hunt and Wilks (1980) introduced a group of continuous transformations for computations of the boundary layer equations between similarity regimes for mixed convection flow. In the case of similarity regimes Hunt and Wilks (1980) recognized $\zeta$ ($=\frac{Gr}{\mathrm{Re}^2}$, where $Gr$ is the local Grashof number and Re is the local Reynolds number) as the governing parameter for the flow from vertical plate. Forced convection exists as a limit when $\zeta$ goes to zero which occurs at the leading edge, and the free convection limit can be reached at large values of $\zeta$. Perturbation solutions have been developed in both cases, since both the forced convection and free convection limits admit similarity solutions.

Extensive studies have also been carried out on MHD free and mixed convection flows by many researchers. Steady hydromagnetic free convection flow past heated vertical flat plate has been considered by Gupta (1961), Poots (1961), Osterle and Young (1961), Sparrow and Cess (1961), Lykoudis (1962), Cramer (1963) and Riley (1964). Gupta (1961) and Lykoudis(1962) have studied the similarity solutions by assuming that the magnetic field varies inversely as the fourth root of the height above the bottom edge of the plate. They have used the approximate momentum integral technique for solutions of basic equations. Their results are therefore of very limited applications, since for very low and very high Prandtl number the viscous and thermal boundary layers are unequal. Later Nanda and Mohanty (1970) employed the same method to solve the hydromagnetic free convection of high and low Prandtl numbers due to practical applications, since for liquid metals the Prandtl number is always small. Lykoudis (1962) has considered a third degree polynomial for the temperature distribution in the entire boundary layer, while Gupta (1961) has considered a third degree velocity profiles which contains a second order zero at the outer edge of the boundary layer. Sparrow and Cess (1961) have considered the case of a constant magnetic field, and have used a perturbation scheme, taking the non-magnetic case as the first approximation. Their results are applicable in the immediate neighborhood of the leading edge and for weak magnetic fields. Cramer (1963) studied the influence of magnetic field on the laminar free convection flow of liquid metals over a vertical flat plate and between two parallel plates. He obtained an analytical solution for liquid metals. Riley (1964) has considered a uniform magnetic field and has integrated the boundary layer equations over a single boundary layer thickness. Effects of transversely applied magnetic field on free convection of an electrically conducting fluid past a semi-infinite plate are studied by Cobble (1979),Wilks (1976) and Wilks and Hunt (1984). MHD mixed convection flow investigated by many researchers such as Yu (1965), Gardner and Lo (1975), Hossain and Ahmed (1990) and Al-Khawaja (1999). Yu (1965) showed the stabilizing effect on combined forced and free convection channel flows similar to the case of horizontal layer heated from below. Grardner and Lo (1975) investigated the laminar problem using a perturbation method, which produced some details of the secondary flow but his result, were limited to small values of the Hartman number. Hossain and Ahmed (1990) studied the combined forced and free convection of an electrically conducting fluid past a vertical flat plate at which the surface heat flux was uniform and a magnetic field was applied parallel to the direction normal to the plate. The equations governing the flow are solved numerically using the method of superposition for small buoyancy parameter $\zeta = \left( \frac{Gr}{\mathrm{Re}^{\frac{5}{2}}} \right)$, where $Gr$ is the Grashof number and $\mathrm{Re}$, the Reynolds number.

The unsteady mixed convection flow past an infinite vertical isothermal plate of an incompressible fluid is a physical situation, which is often experienced, in the industrial application. Some important contributions in this aspect have been given by Schneider (1979), Jha (1991) and Sattar et al. (1997). Jha (1991) investigated the MHD unsteady mixed convection flow through a porous medium. He, however, obtained the solutions by employing Laplace transform technique and taking the value of the Prandtl number to be equal to one. Latter Sattar et al. (1997) obtained an analytical solution of an unsteady flow through a porous medium taking a constant heat source and a variable suction velocity. Unlike the work of Jha (1991), the solutions have been obtained for different Prandtl number.

The phenomenon of natural convection heat and mass transfer is also carried on MHD flow by many investigators. Agrawal et al. (1977) have studied the combined buoyancy effects on the thermal and mass diffusion on MHD natural convection flows. The effects of mass transfer on free convective flow of an electrically conducting, viscous fluid past an infinite porous plate with constant suction and transversely applied magnetic field studied by Haldavneker and Soundalgeker (1977). Latter Soundalgeker et al. (1979) have made an exact analysis of the effects of mass transfer and the free convection current on the MHD Stokes (Rayleigh) problem for the flow of an electrically conducting, incompressible viscous fluid past an impulsively started vertical plate under the action of a transversely applied magnetic field. The effects of natural convection and mass transfer in a conducting fluid, when the fluid is subjected to a transverse magnetic field investigated by Georgantopolous et al. (1981). In the above discussion, the velocity and the skin friction are studied for the effects of Grashof number $Gr < 0$ of the plate by free convection current, $Gm$ (modified Grashof number), $Sc$ (Schmidt number) and $M$ (magnetic parameter). Georgantopolous and Nanousis (1980) have used the Laplace transform technique to solve the problem of the effects of the mass transfer on free convection flow of an electrically conducting viscous fluid past an impulsively started infinite vertical limiting surface on the presence of transverse magnetic field. Dimensionless velocity and skin-friction are obtained for the various values of $Sc$, $Pr$ and $M$ which are presented in graphical form. Raptis and Kafousias (1982) considered the free convection and mass transfer steady hydromagnetic flow of an electrically conducting viscous incompressible fluid through a porous medium, occupying a semi-infinite region of the space bounded by an infinite vertical and porous plate under the action of a transverse magnetic field. The solutions of velocity, temperature, concentration field and rate of heat transfer are obtained for the effects of different parameters. Further, Raptis and Trivanidis (1983) studied the unsteady case and solutions of the governing equations of the flow are obtained with the power series. An analytical study is performed to the effects of magnetic field on the free convection and mass transfer flow through porous medium by Jha and Prasad (1989). The solutions to the problem are obtained by Laplace transform technique. Later, the same method is employed by Jha et al. (1994) for the study of unsteady free convection and mass transfer flow past an exponentially accelerated infinite non-conducting vertical plate through a porous medium in the presence of uniform transverse magnetic field.

In all the above studies, the diffusion-thermo and thermal-diffusion terms were neglected from the energy and concentration equations respectively. When heat and mass transfer occur simultaneously in a moving fluid, the relations between the fluxes and the driving potentials are of more intricate nature. It has been found that an energy flux can be generated not only by temperature gradients but also by composition gradients as well. The energy flux caused by a composition gradient is called the Dufour or diffusion-thermo effect. On the other hand, mass fluxes can also be created by temperature gradients and this is the Soret or thermal-diffusion effect. In general, the thermal-diffusion and the diffusion thermo effects are of a smaller order of magnitude than the effects described by Fourier's or Fick's laws and are often neglected in heat and mass-transfer processes. There are, however, exceptions. The thermal-diffusion effect, for instance, has been utilized for isotope separation, and in mixture between gases with very light molecular weight ($H_2$, He) and of medium molecular weight ($N_2$, air) the diffusion-thermo effect was found to be of a magnitude such that it cannot be neglected (Eckert and Drake, 1972). In view of the importance of this diffusion-thermo effect, Jha and Singh (1990) presented an analytical study for free convection and mass transfer flow for infinite vertical plate moving impulsively in its own plane, taking into account the Soret effect. The Laplace transform technique was used to obtain the expressions for velocity and skin-friction. Kafoussias (1992) studied the MHD free convection and mass transfer flow, past an infinite vertical plate moving in its own plane, taken into account the thermal-diffusion effect when (i) the boundary surface is impulsively moving in its own plane (ISP) and (ii) it is uniformly accelerated (UAP). The problem was solved with the help of Laplace transform method and analytical expressions were given for the velocity field and for the skin-friction coefficients for the above mentioned cases. The effects of various dimensionless parameters entering into the problem were discussed. Later Kafoussias and Williams (1995) used a finite difference method to study the thermal-diffusion and diffusion-thermo effects on mixed (free-forced) convection and mass transfer flow with temperature dependent viscosity. The effects of the viscosity/temperature parameter $T_r$, the thermal-diffusion parameter $Sr$ (Soret number) and the diffusion-thermo parameter $Df$ (Dufour number) have been examined on the flow field of a hydrogen-air mixture as a non-chemical reacting fluid pair. Recently, Anghel et al. (2000) investigated the Dufour and Soret effects on free convection boundary layer over a vertical surface embedded in a porous medium using a double shooting method. Typical velocity, temperature and concentration profiles are presented for some values of buoyancy ratio parameter $N$, Lewis number $Le$, Dufour number $Df$ and Soret number $Sr$. Very recently, Adrian Postelnicu (2004) has studied the influence of a magnetic field on heat and mass transfer by natural convection from vertical surfaces in porous media considering Soret and Dufour effects. The problem is solved numerically using a finite difference method. Dimensionless velocity, temperature and concentration profiles are presented graphically for various values of the magnetic number $M$ and Lewis number $Le$, and for fixed values of the Dufour number $Df$, Soret number $Sr$ and buoyancy number $N$.

Therefore the aim of this dissertation is to study further Magnetohydrodynamics heat and mass transfer flow including thermal-diffusion and diffusion-thermo effects which have been of interest to the engineering community and to the investigators dealing with the problems in Geophysics and Astrophysics.

In Chapter 1 available informations regarding MHD heat and mass transfer flows along with various effects of physical parameters are discussed from both analytical and numerical point of view. In Chapter 2 we have considered a steady two-dimensional problem of MHD combined (free-forced) convection and mass transfer flow past a semi-infinite vertical plate taking into account the Dufour and Soret effects. In Chapter 3 an unsteady MHD free convection and mass transfer flow along an infinite vertical porous flat plate with previous Dufour and Soret effects is considered. The above two problems have been solved numerically using Nachtsheim-Swigert shooting iteration technique with sixth-order Runge-Kutta method. The effects of various parameters entering into the problems are discussed with the help of graphs and tables. In Chapter 4 we have presented overall conclusions of the models studied. Finally all references quoted in the text can be found at the end of the thesis.

# Chapter-2

## Steady MHD combined free-forced convection and mass transfer flow past a semi-infinite vertical plate

Model studies of the forced and free convection or mixed convection flows have many important applications in geophysical, geothermal and nuclear engineering problems. Some of the earlier and recent works are due to Sparrow et al. (1959), Wilks (1973), Chen et al. (1977) and Kafoussias et al. (1998). The effects of mass transfer on MHD free and forced convection flow have also been investigated by many researchers such as Daskalakis et al. (1989), Sattar and Alam (1995) and Rahman and Sattar (1999).

But in the above studies, the thermal-diffusion and diffusion-thermo effects were neglected on the basis that they are of a smaller order of magnitude than the effects described by Fourier's and Fick's laws. However, exceptions are observed therein. The thermal-diffusion effect, for instance, has been utilized for isotope separation, and in mixture between gases with very light molecular weight ($H_2$, He ) and of medium molecular weight ($N_2$ , air) the diffusion-thermo effect was found to be of order of considerable magnitude such that it cannot be ignored (Eckert and Drake, 1972). In view of the importance of this diffusion-thermo effect, Jha and Singh (1990) studied the free convection and mass transfer flow past an infinite vertical flat plate moving impulsively in its own plane, taking into account the Soret effects. Kafoussias (1992) studied the same problem in the case of MHD flow. Later Kafoussias and Williams (1995) studied thermal-diffusion and diffusion-thermo effects on mixed free-forced convective and mass transfer boundary layer flow with temperature dependent viscosity. Therefore, considering thermal-diffusion and diffusion-thermo effects we study the following steady two-dimensional MHD combined free-forced convection and mass transfer flow past a semi-infinite vertical flat plate.

## 2.1 Governing equations of the flow:

Consider a steady combined free-forced convection and mass transfer flow of an electrically conducting, non-magnetic fluid past a semi-infinite vertical flat plate under the influence of a transversely applied magnetic field. The flow is assumed to be in the $x$-direction, which is taken along the plate in the upward direction and $y$-axis is normal to it. A uniform magnetic field $B_0$ is taken to be acted along the $y$-axis. Initially it is assumed that the plate and the fluid are at the same temperature $T$ and the concentration level everywhere in the fluid is same. At time $t>0$, the plate temperature and concentration are instantly raised to $T_w(>T_\infty)$ and $C_w(>C_\infty)$, which are thereafter maintained constant, where $T$, $T_w$ and $T_\infty$ are the temperature of the fluid inside the thermal boundary layer, the plate temperature and the fluid temperature in the free stream, respectively, while $C$, $C_w$ and $C_\infty$ are the corresponding concentrations. The flow configuration and coordinate system is shown in figure 2.1.

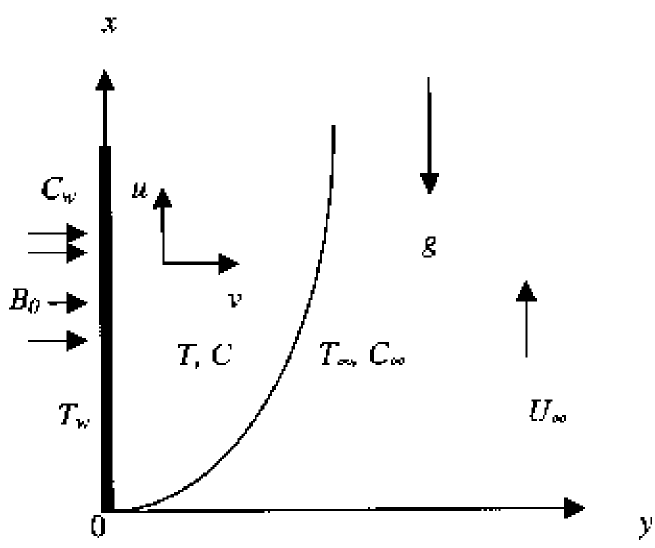


Fig. 2.1: Flow configuration and coordinate system.

The continuity, momentum, energy and concentration equations for steady, viscous, incompressible and electrically conducting flow are respectively given by [Cramer and Pai (1973)]

$$\nabla.\vec{q} = 0, \tag{2.1}$$

$$\rho(\vec{q}.\nabla)\vec{q} = -\nabla P + \mu\nabla^2\vec{q} + \vec{F} + \vec{J}\times\vec{B}, \tag{2.2}$$

$$(\vec{q}.\nabla)T = \alpha\nabla^2 T + \frac{D_m k_T}{c_s c_p}\nabla^2 C, \tag{2.3}$$

$$(\vec{q}.\nabla)C = D_m\nabla^2 C + \frac{D_m k_T}{T_m}\nabla^2 T, \tag{2.4}$$

Here $\vec{q} = \vec{q}(u,v)$ is the velocity vector, $\vec{F}$ is the body force per unit volume which is defined as $-\rho g$, the term $\vec{J}\times\vec{B}$ is the force on the fluid per unit volume produced by the interaction of current and magnetic field, $P$ is the pressure force, $\rho$ is the density of the fluid, $\mu$ is the viscosity, $g$ is the gravitational acceleration, $T$ is the temperature inside the boundary layer, $\alpha$ is the thermal diffusivity, $D_m$ is the mass diffusivity, $k_T$ is the thermal diffusion ratio, $c_s$ is the concentration susceptibility, $c_p$ is the specific heat at a constant pressure, $C$ is the concentration of the fluid inside the boundary layer, $T_m$ is the mean fluid temperature and $\nabla$ is the vector differential operator which is defined by

$$\nabla = \hat{i}_x\frac{\partial}{\partial x} + \hat{i}_y\frac{\partial}{\partial y}$$

where $\hat{i}_x$ and $\hat{i}_y$ are the unit vectors along $x$ and $y$ axes respectively.

When the external electric field is zero and the induced electric field is negligible, the current density $\vec{J}$ is related to the velocity by Ohm's law as follows

$$\vec{J} = \sigma(\vec{q}\times\vec{B}), \tag{2.5}$$

where $\sigma$ denotes the electrical conductivity of the fluid.

Again, for small magnetic Reynolds number, the induced magnetic field is negligible in comparison with the applied magnetic field. So we can write

$$\vec{B} = \hat{i}_x B_0 . \tag{2.6}$$

Hence the retarding force per unit volume $\vec{J} \times \vec{B}$ acting along the $x$-axis takes the form

$$\vec{J} \times \vec{B} = -\sigma B_0^2 u . \tag{2.7}$$

Under the Boussinesq approximation, the density $\rho$ in the buoyancy term in equation (2.2) is considered to vary with temperature and concentration whereas the density appearing elsewhere in these equations is considered constant. We can then write

$$\rho = \rho_\infty [1 - \beta(T - T_\infty) - \beta^*(C - C_\infty)] \tag{2.8}$$

where $\rho_\infty$ is the density outside the boundary layer, $\beta$ is the coefficient of thermal expansion and $\beta^*$ is the coefficient of expansion with concentration which are defined by

$$-\frac{1}{\rho_\infty}\left(\frac{\partial \rho}{\partial T}\right)_P = \beta \tag{2.9}$$

and

$$-\frac{1}{\rho_\infty}\left(\frac{\partial \rho}{\partial C}\right)_P = \beta^* . \tag{2.10}$$

Using the above relations into equations (2.1) – (2.4), the steady laminar two-dimensional combined free-forced convection and mass transfer flow of a viscous incompressible and electrically conducting fluid with constant viscosity past a semi-infinite vertical impermeable flat plate in presence of a uniformly distributed transverse magnetic field take the following form:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.11}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}+g\beta(T-T_\infty)+g\beta^*(C-C_\infty)-\frac{\sigma B_0^2 u}{\rho}, \tag{2.12}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{D_m k_T}{c_s c_p}\frac{\partial^2 C}{\partial y^2}, \tag{2.13}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_m\frac{\partial^2 C}{\partial y^2}+\frac{D_m k_t}{T_m}\frac{\partial^2 T}{\partial y^2}, \tag{2.14}$$

where the variables and related quantities are defined in the Nomenclature.

The appropriate boundary conditions for the above problem are as follows:

$$u=0, v=0, T=T_w, C=C_w \text{ at } y=0, \tag{2.15a}$$

$$u=U_\infty, T=T_\infty, C=C_\infty \quad \text{as } y\to\infty. \tag{2.15b}$$

The last term on the right-hand side of the energy equation (2.13) and concentration equation (2.14) signifies the Dufour or diffusion-thermo effect and the Soret or thermal-diffusion effect, respectively.

We observe that the equations (2.11)-(2.14) together with the boundary conditions (2.15) are non-linear partial differential equations. In the following section non-dimensionalisation and similarity analysis of the problem are discussed in details.

## 2.2 Similarity analysis

To obtain similarity solutions we introduce the following similarity transformations:

$$\left.\begin{aligned} \eta &= y\sqrt{\frac{U_\infty}{\upsilon x}}, \\ u &= U_\infty f'(\eta), \\ \theta(\eta) &= \frac{T-T_\infty}{T_w-T_\infty}, \\ \phi(\eta) &= \frac{C-C_\infty}{C_w-C_\infty}. \end{aligned}\right\} \tag{2.16}$$

From the continuity equation (2.11), we have

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \tag{2.17}$$

Integrating both sides of (2.17) with respect to $y$, we get

$$v = -\frac{1}{2}\sqrt{\frac{U_\infty \upsilon}{x}}\left[f(\eta) - \eta f'(\eta)\right] \tag{2.18}$$

Then substituting the relations (2.16) and (2.18) into equations (2.12)-(2.14), we get the following local similarity equations:

$$f''' + \frac{1}{2}ff'' + g_s\theta + g_c\phi - Mf' = 0, \tag{2.19}$$

$$\theta'' + \frac{1}{2}\Pr f\theta' + \Pr Df\phi'' = 0, \tag{2.20}$$

$$\phi'' + \frac{1}{2}Scf\phi' + SrSc\theta'' = 0. \tag{2.21}$$

The transformed boundary conditions are:

$$f = 0, f' = 0, \theta = 1, \phi = 1 \text{ at } \eta = 0, \tag{2.22a}$$

$$f' = 1, \theta = 0, \phi = 0 \quad \text{as} \quad \eta \to \infty, \tag{2.22b}$$

where primes denote differentiation with respect to the variable $\eta$. The dimensionless parameters appeared into the above equations are defined as follows:

$\Pr = \dfrac{\upsilon}{\alpha}$ is the Prandtl number,

$Sc = \dfrac{\upsilon}{D_m}$ is the Schmidt number,

$\mathrm{Re} = \dfrac{U_\infty x}{\upsilon}$ is the local Reynolds number,

$M = \dfrac{\sigma B_0^2 x}{\rho U_\infty}$ is the local Magnetic parameter,

$Sr = \dfrac{D_m k_T (T_w - T_\infty)}{T_m \upsilon (C_w - C_\infty)}$ is the Soret number,

$Df = \dfrac{D_m k_T (C_w - C_\infty)}{c_s c_p \upsilon (T_w - T_\infty)}$ is the Dufour number,

$Gr = \dfrac{g\beta(T_w - T_\infty)x^3}{\upsilon^2}$ is the local temperature Grashof number,

$Gm = \dfrac{g\beta^*(C_w - C_\infty)x^3}{\upsilon^2}$ is the local mass Grashof number,

$g_t = \dfrac{Gr}{\mathrm{Re}^2}$ is the temperature buoyancy parameter

and $g_c = \dfrac{Gm}{\mathrm{Re}^2}$ is the mass buoyancy parameter.

## Skin-friction coefficient, Nusselt number and Sherwood number:

The parameters of engineering interest for the present problem are the local skin-friction coefficient, local Nusselt number and the local Sherwood number which indicate physically wall shear stress, rate of heat transfer and rate of mass transfer respectively.

The equation defining the wall skin-friction is

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= \mu U_\infty\sqrt{\frac{U_\infty}{\upsilon x}}\,f''(0) \qquad (2.23)$$

Hence the skin-friction coefficient is given by

$$C_f = \frac{2\tau_w}{\rho U_\infty^2}$$

$$\text{or, } C_f = 2(\mathrm{Re})^{-\frac{1}{2}} f''(0)$$

$$\text{i.e., } \frac{1}{2}C_f(\mathrm{Re})^{\frac{1}{2}} = f''(0) \qquad (2.24)$$

Now the heat flux ( $q_w$ ) and the mass flux ( $M_w$ ) at the wall are given by

$$q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = -k\Delta T\sqrt{\frac{U_\infty}{\upsilon x}}\theta'(0),$$

$$\text{and } M_w = -D_m\left(\frac{\partial C}{\partial y}\right)_{y=0} = -D_m\,\Delta C\sqrt{\frac{U_\infty}{\upsilon x}}\phi'(0),$$

where $\Delta T = T_w - T_\infty$ and $\Delta C = C_w - C_\infty$.

Hence the Nusselt number ( $Nu$ ) and Sherwood number ( $Sh$ ) are obtained as

$$Nu = \frac{xq_w}{k\,\Delta T} = -(\mathrm{Re})^{\frac{1}{2}}\theta'(0)$$

$$\text{i.e., } Nu(\mathrm{Re})^{-\frac{1}{2}} = -\theta'(0) \qquad (2.25)$$

and

$$Sh = \frac{xM_w}{D_m\,\Delta C} = -(\mathrm{Re})^{\frac{1}{2}}\phi'(0)$$

$$\text{i.e., } Sh(\mathrm{Re})^{-\frac{1}{2}} = -\phi'(0). \qquad (2.26)$$

These coefficients are then obtained numerically and are sorted in table-2.1 and table-2.2.

## 2.3 Method of Numerical Solution:

The systems of equation (2.19)-(2.21) together with the boundary conditions (2.22) are non-linear and coupled. It is difficult to solve them analytically. Hence we adopt a procedure to obtain the solution numerically. Here we use the standard initial-value solver shooting method namely Nachtsheim-Swigert iteration technique (guessing the missing value) and Runge-Kutta Merson method, in collaboration with Runge-Kutta shooting method.

In a shooting method, the missing (unspecified) initial condition at the initial point of the interval is assumed, and the differential equation is then integrated numerically as an initial value problem to the terminal point. The accuracy of the assumed missing initial condition is then checked by comparing the calculated value of the dependent variable at the terminal point with its given value there. If a difference exists, another value of the missing initial condition must be assumed and the process is repeated. This process is continued until the agreement between the calculated and the given condition at the terminal point is within the specified degree of accuracy. For this type of iterative approach, one naturally inquires whether or not there is a systematic way of finding each succeeding (assumed) value of the missing initial condition.

The Nachtsheim-Swigert iteration technique thus needs to be discussed elaborately. The boundary condition (2.22) associated with the non-linear ODEs (2.19)-(2.21) are the two-point asymptotic class. Two-point boundary conditions have values of the dependent variable specified at two different values of independent variable. Specification of an asymptotic boundary condition implies that the first derivative (and higher derivatives of the boundary layer equations, if exist) of the dependent variable approaches zero as the outer specified value of the independent variable is approached.

The method of numerically integrating a two-point asymptotic boundary-value problem of the boundary-layer type, the initial-value method is similar to an initial-value problem. Thus it is necessary to estimate as many boundary conditions at the surface as were (previously) given at infinity. The governing differential equations are then integrated with these assumed surface boundary conditions. If the required outer boundary condition is satisfied, a solution has been achieved. However, this is not generally the case. Hence, a method must be devised to estimate logically the new surface boundary conditions for the next trial integration. Asymptotic boundary value problems such as those governing the boundary-layer equations are further complicated by the fact that the outer boundary condition is specified at infinity. In the trial integration infinity is numerically approximated by some large value of the independent variable. There is no a priori general method of estimating these values. Selecting too small a maximum value for the independent variable may not allow the solution to asymptotically converge to the required accuracy. Selecting large a value may result in divergence of the trial integration or in slow convergence of surface boundary conditions. Selecting too large a value of the independent variable is expensive in terms of computer time.

Nachtsheim-Swigert (1965) developed an iteration method to overcome these difficulties. Extension of the Nachtsheim-Swigert iteration scheme to the system of equation (2.19)-(2.21) and the boundary conditions (2.22) is straightforward. In equation (2.22) there are three asymptotic boundary conditions and hence three unknown surface conditions $f''(0)$, $\theta'(0)$ and $\phi'(0)$.

Within the context of the initial-value method and Nachtsheim-Swigert iteration technique the outer boundary conditions may be functionally represented as

$$f'(\eta_{max}) = f'(f''(0), \theta'(0), \phi'(0)) = \delta_1, \tag{2.27}$$

$$\theta(\eta_{max}) = \theta(f''(0), \theta'(0), \phi'(0)) = \delta_2, \tag{2.28}$$

$$\phi(\eta_{max}) = \phi(f''(0), \theta'(0), \phi'(0)) = \delta_3, \tag{2.29}$$

with the asymptotic convergence criteria given by

$$f''(\eta_{max}) = f''(f''(0), \theta'(0), \phi'(0)) = \delta_4, \tag{2.30}$$

$$\theta'(\eta_{max}) = \theta'(f''(0), \theta'(0), \phi'(0)) = \delta_5, \tag{2.31}$$

$$\phi'(\eta_{max}) = \phi'(f''(0), \theta'(0), \phi'(0)) = \delta_6. \tag{2.32}$$

Choosing $f''(0) = g_1, \theta'(0) = g_2$ and $\phi'(0) = g_3$ and expanding in a first-order Taylor's series after using equations (2.27)-(2.32) yields

$$f'(\eta_{max}) = f'_C(\eta_{max}) + \frac{\partial f'}{\partial g_1}\Delta g_1 + \frac{\partial f'}{\partial g_2}\Delta g_2 + \frac{\partial f'}{\partial g_3}\Delta g_3 = \delta_1, \tag{2.33}$$

$$\theta(\eta_{max}) = \theta_C(\eta_{max}) + \frac{\partial \theta}{\partial g_1}\Delta g_1 + \frac{\partial \theta}{\partial g_2}\Delta g_2 + \frac{\partial \theta}{\partial g_3}\Delta g_3 = \delta_2, \tag{2.34}$$

$$\phi(\eta_{max}) = \phi_C(\eta_{max}) + \frac{\partial \phi}{\partial g_1}\Delta g_1 + \frac{\partial \phi}{\partial g_2}\Delta g_2 + \frac{\partial \phi}{\partial g_3}\Delta g_3 = \delta_3, \tag{2.35}$$

$$f''(\eta_{max}) = f''_C(\eta_{max}) + \frac{\partial f''}{\partial g_1}\Delta g_1 + \frac{\partial f''}{\partial g_2}\Delta g_2 + \frac{\partial f''}{\partial g_3}\Delta g_3 = \delta_4, \tag{2.36}$$

$$\theta'(\eta_{max}) = \theta'_C(\eta_{max}) + \frac{\partial \theta'}{\partial g_1}\Delta g_1 + \frac{\partial \theta'}{\partial g_2}\Delta g_2 + \frac{\partial \theta'}{\partial g_3}\Delta g_3 = \delta_5, \tag{2.37}$$

$$\phi'(\eta_{max}) = \phi'_C(\eta_{max}) + \frac{\partial \phi'}{\partial g_1}\Delta g_1 + \frac{\partial \phi'}{\partial g_2}\Delta g_2 + \frac{\partial \phi'}{\partial g_3}\Delta g_3 = \delta_6, \tag{2.38}$$

where subscript '$C$' indicates the value of the function at $\eta_{max}$ determined from the trial integration.

Solution of these equations in a least-squares sense requires determining the minimum value of

$$E=\delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2+\delta_5^2+\delta_6^2 \tag{2.39}$$

with respect to $g_1, g_2$ and $g_3$.

Now differentiating $E$ with respect to $g_1$ yields

$$\delta_1\frac{\partial\delta_1}{\partial g_1}+\delta_2\frac{\partial\delta_2}{\partial g_1}+\delta_3\frac{\partial\delta_3}{\partial g_1}+\delta_4\frac{\partial\delta_4}{\partial g_1}+\delta_5\frac{\partial\delta_5}{\partial g_1}+\delta_6\frac{\partial\delta_6}{\partial g_1}=0$$

$$\text{or, }\left(f_c'+\frac{\partial f'}{\partial g_1}\Delta g_1+\frac{\partial f'}{\partial g_2}\Delta g_2+\frac{\partial f'}{\partial g_3}\Delta g_3\right)\frac{\partial f'}{\partial g_1}+\left(\theta_c+\frac{\partial\theta}{\partial g_1}\Delta g_1+\frac{\partial\theta}{\partial g_2}\Delta g_2+\frac{\partial\theta}{\partial g_3}\Delta g_3\right)\frac{\partial\theta}{\partial g_1}$$

$$+\left(\phi_c+\frac{\partial\phi}{\partial g_1}\Delta g_1+\frac{\partial\phi}{\partial g_2}\Delta g_2+\frac{\partial\phi}{\partial g_3}\Delta g_3\right)\frac{\partial\phi}{\partial g_1}+\left(f_c''+\frac{\partial f''}{\partial g_1}\Delta g_1+\frac{\partial f''}{\partial g_2}\Delta g_2+\frac{\partial f''}{\partial g_3}\Delta g_3\right)\frac{\partial f''}{\partial g_1}$$

$$+\left(\theta_c'+\frac{\partial\theta'}{\partial g_1}\Delta g_1+\frac{\partial\theta'}{\partial g_2}\Delta g_2+\frac{\partial\theta'}{\partial g_3}\Delta g_3\right)\frac{\partial\theta'}{\partial g_1}+\left(\phi_c'+\frac{\partial\phi'}{\partial g_1}\Delta g_1+\frac{\partial\phi'}{\partial g_2}\Delta g_2+\frac{\partial\phi'}{\partial g_3}\Delta g_3\right)\frac{\partial\phi'}{\partial g_1}=0$$

$$\text{or, }\left[\left(\frac{\partial f'}{\partial g_1}\right)^2+\left(\frac{\partial\theta}{\partial g_1}\right)^2+\left(\frac{\partial\phi}{\partial g_1}\right)^2+\left(\frac{\partial f''}{\partial g_1}\right)^2+\left(\frac{\partial\theta'}{\partial g_1}\right)^2+\left(\frac{\partial\phi'}{\partial g_1}\right)^2\right]\Delta g_1$$

$$+\left[\frac{\partial f'}{\partial g_2}\frac{\partial f'}{\partial g_1}+\frac{\partial\theta}{\partial g_2}\frac{\partial\theta}{\partial g_1}+\frac{\partial\phi}{\partial g_2}\frac{\partial\phi}{\partial g_1}+\frac{\partial f''}{\partial g_2}\frac{\partial f''}{\partial g_1}+\frac{\partial\theta'}{\partial g_2}\frac{\partial\theta'}{\partial g_1}+\frac{\partial\phi'}{\partial g_2}\frac{\partial\phi'}{\partial g_1}\right]\Delta g_2$$

$$+\left[\frac{\partial f'}{\partial g_3}\frac{\partial f'}{\partial g_1}+\frac{\partial\theta}{\partial g_3}\frac{\partial\theta}{\partial g_1}+\frac{\partial\phi}{\partial g_3}\frac{\partial\phi}{\partial g_1}+\frac{\partial f''}{\partial g_3}\frac{\partial f''}{\partial g_1}+\frac{\partial\theta'}{\partial g_3}\frac{\partial\theta'}{\partial g_1}+\frac{\partial\phi'}{\partial g_3}\frac{\partial\phi'}{\partial g_1}\right]\Delta g_3$$

$$=-\left[f_c'\frac{\partial f'}{\partial g_1}+\theta_c\frac{\partial\theta}{\partial g_1}+\phi_c\frac{\partial\phi}{\partial g_1}+f_c''\frac{\partial f''}{\partial g_1}+\theta_c''\frac{\partial\theta'}{\partial g_1}+\phi_c'\frac{\partial\phi'}{\partial g_1}\right] \tag{2.40}$$

Similarly differentiating $E$ with respect to $g_2$ and $g_3$, we obtain respectively

$$\delta_1\frac{\partial\delta_1}{\partial g_2}+\delta_2\frac{\partial\delta_2}{\partial g_2}+\delta_3\frac{\partial\delta_3}{\partial g_2}+\delta_4\frac{\partial\delta_4}{\partial g_2}+\delta_5\frac{\partial\delta_5}{\partial g_2}+\delta_6\frac{\partial\delta_6}{\partial g_2}=0$$

$$\text{or, }\left[\left(\frac{\partial f'}{\partial g_2}\right)^2+\left(\frac{\partial\theta}{\partial g_2}\right)^2+\left(\frac{\partial\phi}{\partial g_2}\right)^2+\left(\frac{\partial f''}{\partial g_2}\right)^2+\left(\frac{\partial\theta'}{\partial g_2}\right)^2+\left(\frac{\partial\phi'}{\partial g_2}\right)^2\right]\Delta g_2$$

$$+\left[\frac{\partial f'}{\partial g_1}\frac{\partial f'}{\partial g_2}+\frac{\partial\theta}{\partial g_1}\frac{\partial\theta}{\partial g_2}+\frac{\partial\phi}{\partial g_1}\frac{\partial\phi}{\partial g_{22}}+\frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_2}+\frac{\partial\theta'}{\partial g_1}\frac{\partial\theta'}{\partial g_2}+\frac{\partial\phi'}{\partial g_1}\frac{\partial\phi'}{\partial g_2}\right]\Delta g_1$$

$$+\left[\frac{\partial f'}{\partial g_3}\frac{\partial f'}{\partial g_2}+\frac{\partial \theta}{\partial g_3}\frac{\partial \theta}{\partial g_2}+\frac{\partial \phi}{\partial g_3}\frac{\partial \phi}{\partial g_2}+\frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_2}+\frac{\partial \theta'}{\partial g_3}\frac{\partial \theta'}{\partial g_2}+\frac{\partial \phi'}{\partial g_3}\frac{\partial \phi'}{\partial g_2}\right]\Delta g_3$$

$$=-\left[f_C'\frac{\partial f'}{\partial g_2}+\theta_c\frac{\partial \theta}{\partial g_2}+\phi_C\frac{\partial \phi}{\partial g_2}+f_C''\frac{\partial f''}{\partial g_2}+\theta_C''\frac{\partial \theta'}{\partial g_2}+\phi_C'\frac{\partial \phi'}{\partial g_2}\right]. \tag{2.41}$$

and

$$\delta_1\frac{\partial \delta_1}{\partial g_3}+\delta_2\frac{\partial \delta_2}{\partial g_3}+\delta_3\frac{\partial \delta_3}{\partial g_3}+\delta_4\frac{\partial \delta_4}{\partial g_3}+\delta_5\frac{\partial \delta_5}{\partial g_1}+\delta_6\frac{\partial \delta_6}{\partial g_3}=0$$

or, $$\left[\left(\frac{\partial f'}{\partial g_3}\right)^2+\left(\frac{\partial \theta}{\partial g_3}\right)^2+\left(\frac{\partial \phi}{\partial g_3}\right)^2+\left(\frac{\partial f''}{\partial g_3}\right)^2+\left(\frac{\partial \theta'}{\partial g_3}\right)^2+\left(\frac{\partial \phi'}{\partial g_3}\right)^2\right]\Delta g_3$$

$$+\left[\frac{\partial f'}{\partial g_1}\frac{\partial f'}{\partial g_3}+\frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_3}+\frac{\partial \phi}{\partial g_1}\frac{\partial \phi}{\partial g_3}+\frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_3}+\frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_3}+\frac{\partial \phi'}{\partial g_1}\frac{\partial \phi'}{\partial g_3}\right]\Delta g_1$$

$$+\left[\frac{\partial f'}{\partial g_2}\frac{\partial f'}{\partial g_3}+\frac{\partial \theta}{\partial g_2}\frac{\partial \theta}{\partial g_3}+\frac{\partial \phi}{\partial g_2}\frac{\partial \phi}{\partial g_3}+\frac{\partial f''}{\partial g_2}\frac{\partial f''}{\partial g_3}+\frac{\partial \theta'}{\partial g_2}\frac{\partial \theta'}{\partial g_3}+\frac{\partial \phi'}{\partial g_2}\frac{\partial \phi'}{\partial g_3}\right]\Delta g_2$$

$$=-\left[f_C'\frac{\partial f'}{\partial g_3}+\theta_c\frac{\partial \theta}{\partial g_3}+\phi_C\frac{\partial \phi}{\partial g_3}+f_C''\frac{\partial f''}{\partial g_3}+\theta_C''\frac{\partial \theta'}{\partial g_3}+\phi_C'\frac{\partial \phi'}{\partial g_3}\right]. \tag{2.42}$$

We can write equations (2.40)-(2.42) in a system of linear equations as follows:

$$a_{11}\Delta g_1+a_{12}\Delta g_2+a_{13}\Delta g_3=b_1, \tag{2.43}$$

$$a_{21}\Delta g_1+a_{22}\Delta g_2+a_{23}\Delta g_3=b_2, \tag{2.44}$$

$$a_{31}\Delta g_1+a_{32}\Delta g_2+a_{33}\Delta g_3=b_3. \tag{2.45}$$

Here

$$a_{11}=\left[\left(\frac{\partial f'}{\partial g_1}\right)^2+\left(\frac{\partial \theta}{\partial g_1}\right)^2+\left(\frac{\partial \phi}{\partial g_1}\right)^2+\left(\frac{\partial f''}{\partial g_1}\right)^2+\left(\frac{\partial \theta'}{\partial g_1}\right)^2+\left(\frac{\partial \phi'}{\partial g_1}\right)^2\right],$$

$$a_{12}=\frac{\partial f'}{\partial g_2}\frac{\partial f'}{\partial g_1}+\frac{\partial \theta}{\partial g_2}\frac{\partial \theta}{\partial g_1}+\frac{\partial \phi}{\partial g_2}\frac{\partial \phi}{\partial g_1}+\frac{\partial f''}{\partial g_2}\frac{\partial f''}{\partial g_1}+\frac{\partial \theta'}{\partial g_2}\frac{\partial \theta'}{\partial g_1}+\frac{\partial \phi'}{\partial g_2}\frac{\partial \phi'}{\partial g_1},$$

$$a_{13}=\frac{\partial f'}{\partial g_3}\frac{\partial f'}{\partial g_1}+\frac{\partial \theta}{\partial g_3}\frac{\partial \theta}{\partial g_1}+\frac{\partial \phi}{\partial g_3}\frac{\partial \phi}{\partial g_1}+\frac{\partial f''}{\partial g_3}\frac{\partial f''}{\partial g_1}+\frac{\partial \theta'}{\partial g_3}\frac{\partial \theta'}{\partial g_1}+\frac{\partial \phi'}{\partial g_3}\frac{\partial \phi'}{\partial g_1},$$

$$a_{21}=\frac{\partial f'}{\partial g_2}\frac{\partial f'}{\partial g_1}+\frac{\partial \theta}{\partial g_2}\frac{\partial \theta}{\partial g_1}+\frac{\partial \phi}{\partial g_2}\frac{\partial \phi}{\partial g_1}+\frac{\partial f''}{\partial g_2}\frac{\partial f''}{\partial g_1}+\frac{\partial \theta'}{\partial g_2}\frac{\partial \theta'}{\partial g_1}+\frac{\partial \phi'}{\partial g_2}\frac{\partial \phi'}{\partial g_1},$$

$$a_{22}=\left[\left(\frac{\partial f'}{\partial g_2}\right)^2+\left(\frac{\partial \theta}{\partial g_2}\right)^2+\left(\frac{\partial \phi}{\partial g_2}\right)^2+\left(\frac{\partial f''}{\partial g_2}\right)^2+\left(\frac{\partial \theta'}{\partial g_2}\right)^2+\left(\frac{\partial \phi'}{\partial g_2}\right)^2\right],$$

$$a_{23}=\frac{\partial f'}{\partial g_3}\frac{\partial f'}{\partial g_2}+\frac{\partial \theta}{\partial g_3}\frac{\partial \theta}{\partial g_2}+\frac{\partial \phi}{\partial g_3}\frac{\partial \phi}{\partial g_2}+\frac{\partial f''}{\partial g_3}\frac{\partial f''}{\partial g_2}+\frac{\partial \theta'}{\partial g_3}\frac{\partial \theta'}{\partial g_2}+\frac{\partial \phi'}{\partial g_3}\frac{\partial \phi'}{\partial g_2},$$

$$a_{31}=\frac{\partial f'}{\partial g_3}\frac{\partial f'}{\partial g_1}+\frac{\partial \theta}{\partial g_3}\frac{\partial \theta}{\partial g_1}+\frac{\partial \phi}{\partial g_3}\frac{\partial \phi}{\partial g_1}+\frac{\partial f''}{\partial g_3}\frac{\partial f''}{\partial g_1}+\frac{\partial \theta'}{\partial g_3}\frac{\partial \theta'}{\partial g_1}+\frac{\partial \phi'}{\partial g_3}\frac{\partial \phi'}{\partial g_1},$$

$$a_{32}=\frac{\partial f'}{\partial g_3}\frac{\partial f'}{\partial g_2}+\frac{\partial \theta}{\partial g_3}\frac{\partial \theta}{\partial g_2}+\frac{\partial \phi}{\partial g_3}\frac{\partial \phi}{\partial g_2}+\frac{\partial f''}{\partial g_3}\frac{\partial f''}{\partial g_2}+\frac{\partial \theta'}{\partial g_3}\frac{\partial \theta'}{\partial g_2}+\frac{\partial \phi'}{\partial g_3}\frac{\partial \phi'}{\partial g_2},$$

$$a_{33}=\left[\left(\frac{\partial f'}{\partial g_3}\right)^2+\left(\frac{\partial \theta}{\partial g_3}\right)^2+\left(\frac{\partial \phi}{\partial g_3}\right)^2+\left(\frac{\partial f''}{\partial g_3}\right)^2+\left(\frac{\partial \theta'}{\partial g_3}\right)^2+\left(\frac{\partial \phi'}{\partial g_3}\right)^2\right],$$

$$b_1=-\left[f'_C\frac{\partial f'}{\partial g_1}+\theta_C\frac{\partial \theta}{\partial g_1}+\phi_C\frac{\partial \phi}{\partial g_1}+f''_C\frac{\partial f''}{\partial g_1}+\theta'_C\frac{\partial \theta'}{\partial g_1}+\phi'_C\frac{\partial \phi'}{\partial g_1}\right],$$

$$b_2=-\left[f'_C\frac{\partial f'}{\partial g_2}+\theta_C\frac{\partial \theta}{\partial g_2}+\phi_C\frac{\partial \phi}{\partial g_2}+f''_C\frac{\partial f''}{\partial g_2}+\theta'_C\frac{\partial \theta'}{\partial g_2}+\phi'_C\frac{\partial \phi'}{\partial g_2}\right],$$

and

$$b_3==-\left[f'_C\frac{\partial f'}{\partial g_3}+\theta_C\frac{\partial \theta}{\partial g_3}+\phi_C\frac{\partial \phi}{\partial g_3}+f''_C\frac{\partial f''}{\partial g_3}+\theta'_C\frac{\partial \theta'}{\partial g_3}+\phi'_C\frac{\partial \phi'}{\partial g_3}\right].$$

Now solving the equations (2.43)-(2.45) by using Cramer's rule, we have

$$\Delta g_1=\frac{\det A_1}{\det A},\ \Delta g_2=\frac{\det A_2}{\det A}\ \text{and}\quad \Delta g_3=\frac{\det A_3}{\det A}$$

where

$$\det A_1=\begin{vmatrix} b_1 & a_{12} & a_{13}\\ b_2 & a_{22} & a_{23}\\ b_3 & a_{32} & a_{33}\end{vmatrix}=b_1\left(a_{22}a_{33}-a_{32}a_{23}\right)+b_2\left(a_{32}a_{13}-a_{12}a_{33}\right)+b_3\left(a_{12}a_{23}-a_{22}a_{13}\right),$$

$$\det A_2=\begin{vmatrix} a_{11} & b_1 & a_{13}\\ a_{21} & b_2 & a_{23}\\ a_{31} & b_3 & a_{33}\end{vmatrix}=b_1\left(a_{31}a_{23}-a_{21}a_{33}\right)+b_2\left(a_{11}a_{33}-a_{31}a_{13}\right)+b_3\left(a_{21}a_{13}-a_{11}a_{23}\right),$$

$$\det A_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} = b_1\left(a_{21}a_{32} - a_{31}a_{22}\right) + b_2\left(a_{31}a_{12} - a_{11}a_{32}\right) + b_3\left(a_{11}a_{22} - a_{21}a_{12}\right),$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{21}\left(a_{32}a_{13} - a_{12}a_{33}\right) + a_{31}\left(a_{12}a_{23} - a_{22}a_{13}\right).$$

Then we obtain the missing (unspecified) values $g_1$, $g_2$ and $g_3$ as

$g_1 = g_1 + \Delta g_1$,

$g_2 = g_2 + \Delta g_2$,

$g_3 = g_3 + \Delta g_3$.

Thus adopting the numerical technique aforementioned, the solution of the equations (2.19)-(2.21) with boundary conditions (2.22) are obtained together with sixth-order implicit Runge-Kutta initial value solver and determine the velocity, temperature and concentration as a function of the coordinate $\eta$. In the process of integration the skin-friction coefficient $f''(0)$, heat transfer rate $-\theta'(0)$ and mass transfer rate $-\phi'(0)$ are also calculated.

## 2.4 Results and Discussion

The system of non-linear ordinary differential equations (2.19)-(2.21) together with the boundary conditions (2.22) have been solved numerically by using sixth order Runge-Kutta shooting method. Numerical computations have been carried out for different values of magnetic parameter $M$ and for fixed values of Prandtl number $Pr$, Schmidt number $Sc$. The value of Prandtl number $Pr$ is taken equal to 0.71 that corresponds physically to air. The value of Schmidt number $Sc$ = 0.22 has been chosen to represent hydrogen at approx. $T_m = 25^0$C and 1 atm. The values of Soret number $Sr$ and Dufour number $Df$ are chosen in such a way that their product is constant according to their definition provided that the mean temperature $T_m$ is kept constant as well. The dimensionless parameter $g_s = \dfrac{Gr}{\mathrm{Re}^2}$ is used to represent the free, forced and combined (free-forced) convection regimes. The case $g_s \ll 1$ corresponds to pure forced convection, $g_s=1$ corresponds to combined free-forced convection and $g_s \gg 1$ corresponds to pure free convection. As the local mass Grashof number $Gr$ is a measure of the buoyancy forces (due not temperature but to concentration differences) to the viscous forces, the dimensionless parameter $g_c$ has the same meaning as the parameter $g_s$. The dimensionless parameter $g_s$ takes the values 0.1, 1 and 10 which correspond to three different flow regimes as already mentioned above. The corresponding parameter $g_c$ takes the values 0.05, 0.10 and 0.20.

First, to verify the proper treatment of the problem, the present solution for $M = D_f = Sr = 0$ has been compared with that of Kafoussias (1990), see Fig.2.2 and Table 2.1. It can be seen from the Fig.2.2 and Table 2.1 that present results are in excellent agreement with Kafoussias (1990).

Now within the above-mentioned flow parameters, the results are displayed in Figs. 2.3-2.11, for the velocity, temperature and concentration profiles. In Fig.2.3, velocity profiles are shown for different values of $g_s$ and $g_c$. We observe that velocity increases with the increase of $g_s$. This increment is greater for higher values of $g_c$ and in the case of pure forced convection ($g_s \ll 1$). The velocity reaches maximum inside the boundary layer for pure free convection ($g_s=10$, $g_c=0.20$). The variations of temperature and concentration fields for different values of $g_s$ and $g_c$ are displayed in Figs. 2.4 and 2.5, respectively. As would be expected, both fields exhibit the same behavior. The influence of $g_c$ on the temperature and concentration field is not so much evident for higher values of $g_s$.

In Fig.2.6, the effects of magnetic parameter $M$ for different values of $g_s$ are shown. From this figure we see that the increase of magnetic field leads to the decrease the velocity field indicating that the magnetic field retards the flow field. On the other hand, in Figs.2.7 and 2.8 we see that an increase in the magnetic field leads to rise the temperature and concentration distributions respectively both.

The influence of Soret number $Sr$ and Dufour number $Df$ on the velocity, temperature and concentration profiles are shown in Figs. 2.9, 2.10 and 2.11 respectively. From Fig. 2.9, we see that quantitatively, when $\eta = 2$ and $Sr$ decreases from 2 to 0.4 (or $Df$ increases from 0.03 to 0.15) there is 0.33% increase in the velocity value, whereas the corresponding increase is 3.64% when $Sr$ decreases from 0.4 to 0.1. From Fig. 2.10, when $\eta = 3$ and $Sr$ decreases from 2 to 0.4 (or $Df$ increases from 0.03 to 0.15) there is 10.53% increase in the temperature value, whereas the corresponding increase is 32.19% when $Sr$ decreases from 0.4 to 0.1. From Fig. 2.11, when $\eta = 3$ and $Sr$ decreases from 2 to 0.4 (or $Df$ increases from 0.03 to 0.15) there is 38.22% decrease in the concentration value, whereas the corresponding decrease is 10.14% when $Sr$ decreases from 0.4 to 0.1.

Finally, table 2.2 shows the numerical values of local skin-friction coefficients, local Nusselt number and Sherwood number for some values of the parameters $M$, $Sr$ and $Df$ when $g_s = 1$. From this table it is evident that for fixed $g_s$, $Sr$ and $Df$ ; $C_f$, $Nu$ and $Sh$ decrease as $M$ increases. Finally we see that the local Nusselt number increses, while the local Sherwood number decreases as $Df$ decreases and $Sr$ increases.

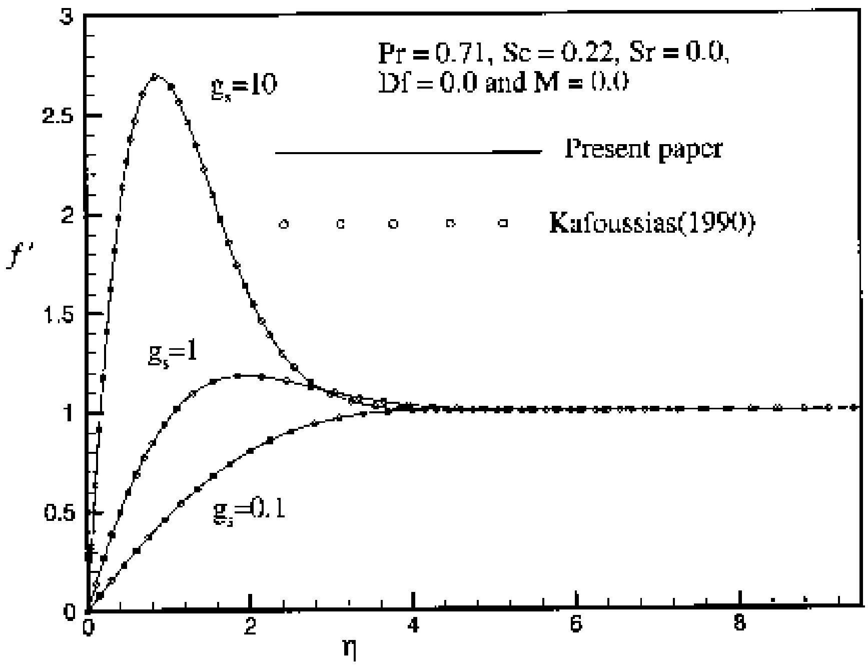


Fig.2.2: Comparison of velocity profiles for different values of $g_s$.

**Table 2.1:** Comparison of skin-friction and local Nusselt number for $M = Df = Sr = 0$.

| $g_s$ | $g_t$ | Kafoussias(1990) Skin-friction | Present Skin-friction | Kafoussias(1990) Nusselt number | Present Nusselt number |
|---|---|---|---|---|---|
| 0.10 | 0.05 | 0.5538 | 0.5538 | 0.3296 | 0.3296 |
| 0.10 | 0.10 | 0.6317 | 0.6315 | 0.3404 | 0.3404 |
| 0.10 | 0.20 | 0.7776 | 0.7772 | 0.3589 | 0.3589 |
| 1.00 | 0.05 | 1.4452 | 1.4451 | 0.4129 | 0.4129 |
| 1.00 | 0.10 | 1.5007 | 1.5001 | 0.4179 | 0.4178 |
| 1.00 | 0.20 | 1.6096 | 1.6081 | 0.4274 | 0.4272 |
| 10.0 | 0.05 | 6.8389 | 6.8385 | 0.6449 | 0.6450 |
| 10.0 | 0.10 | 6.8715 | 6.8712 | 0.6461 | 0.6463 |
| 10.0 | 0.20 | 6.9366 | 6.9356 | 0.6487 | 0.6488 |

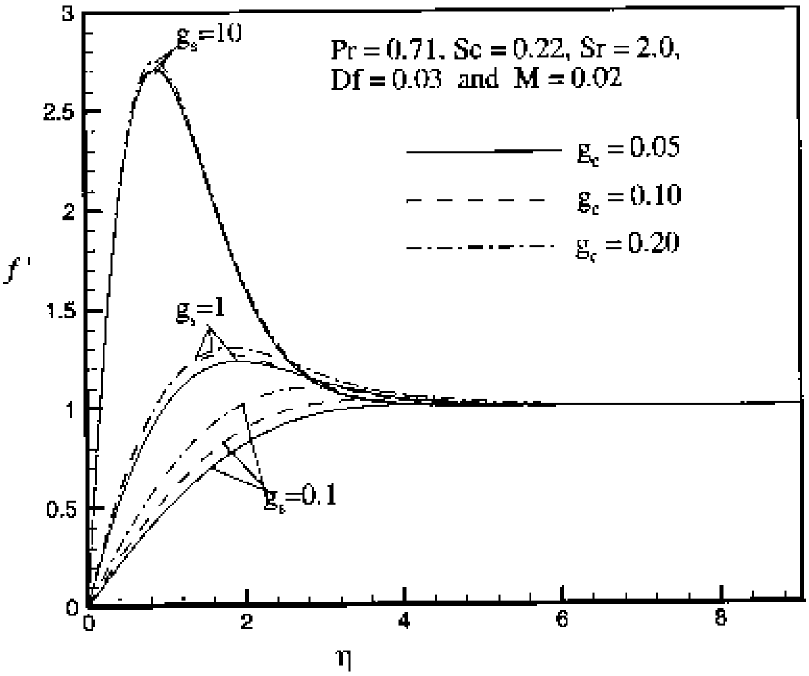


Fig.2.3: Velocity profiles for different values of $g_s$ and $g_c$.

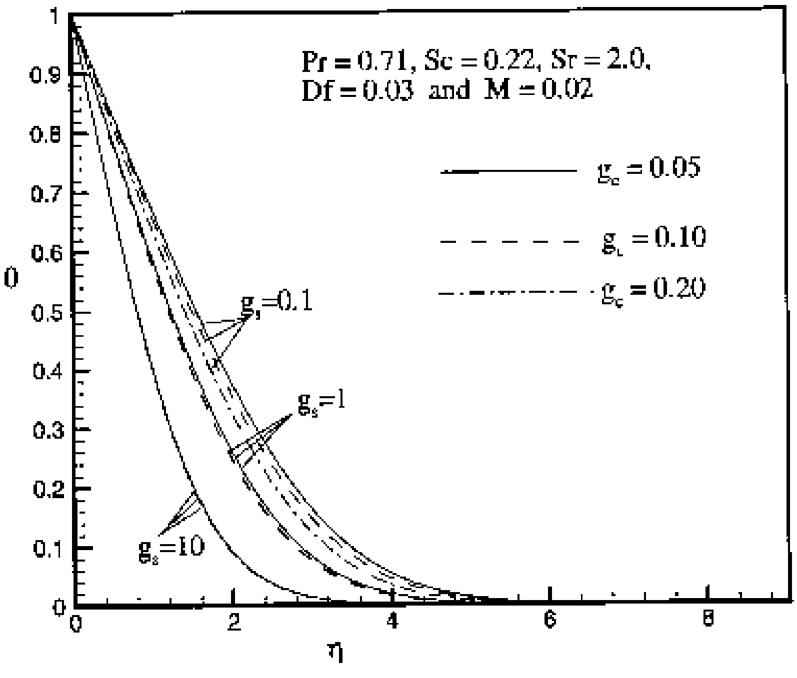


Fig.2.4: Temperature profiles for different values of $g_s$ and $g_c$.

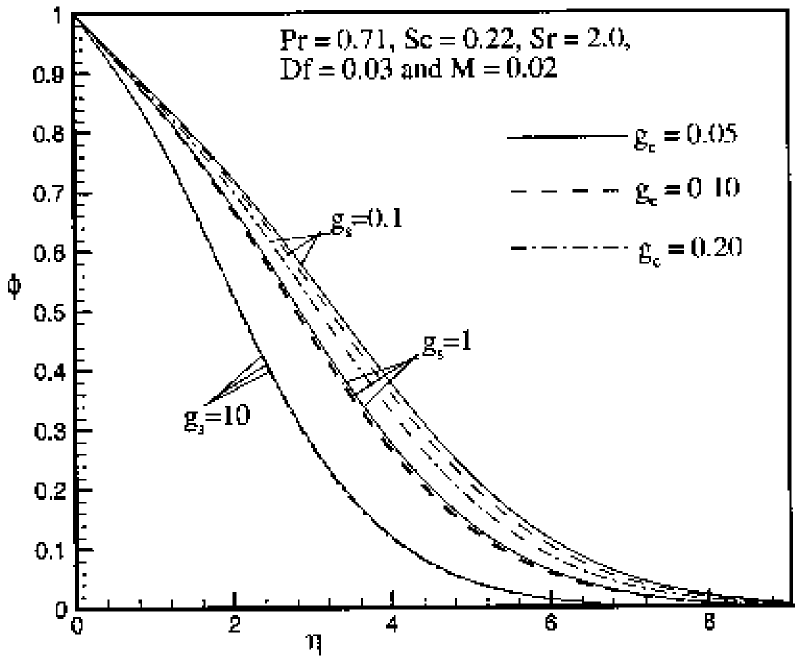


Fig.2.5: Concentration profiles for different values of $g_s$ and $g_c$.

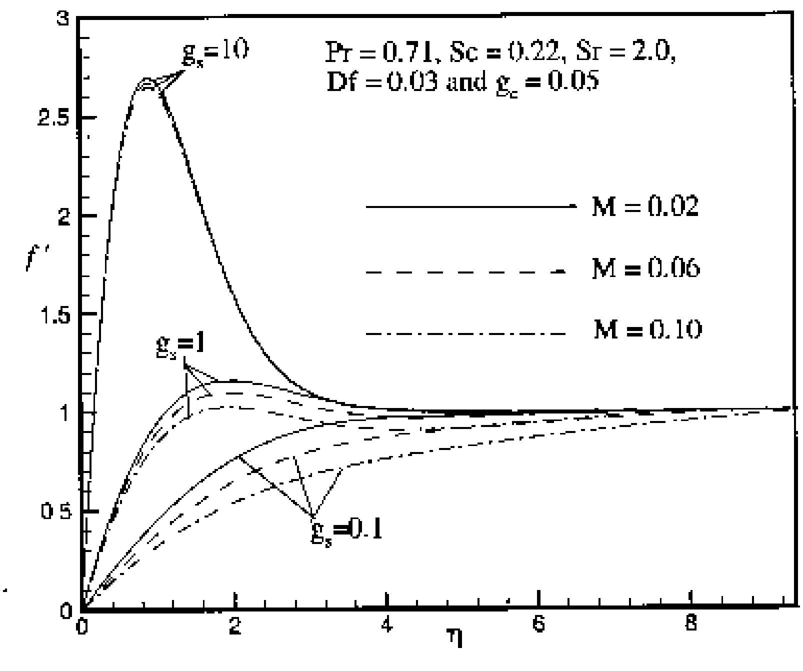


Fig.2.6: Velocity profiles for different values of M.

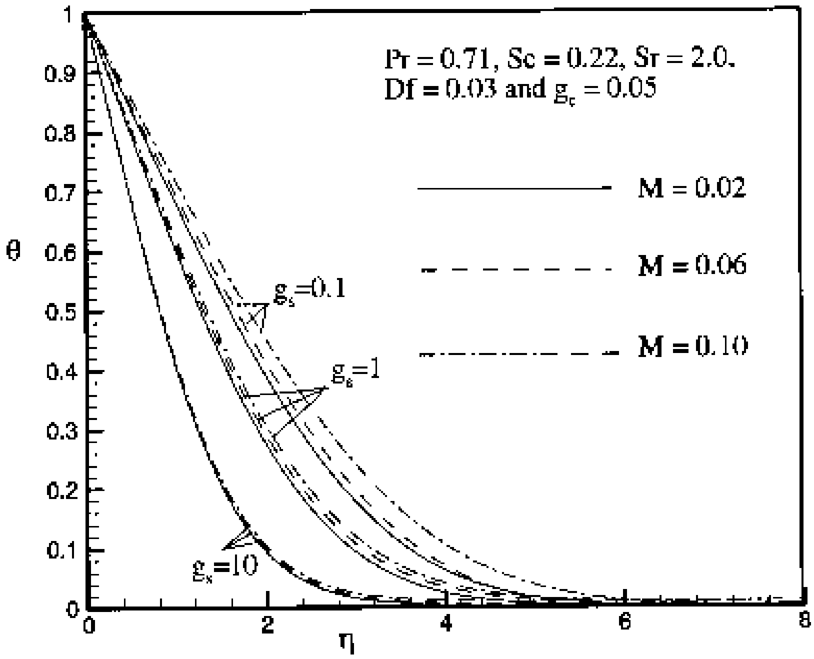


Fig.2.7: Temperature profiles for different values of M.

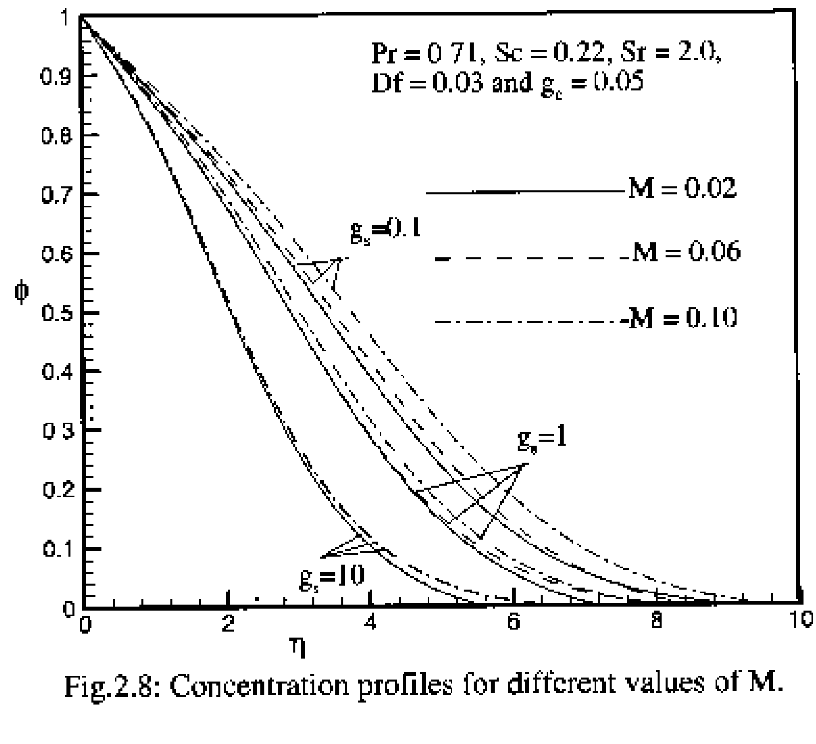


Fig.2.8: Concentration profiles for different values of M.

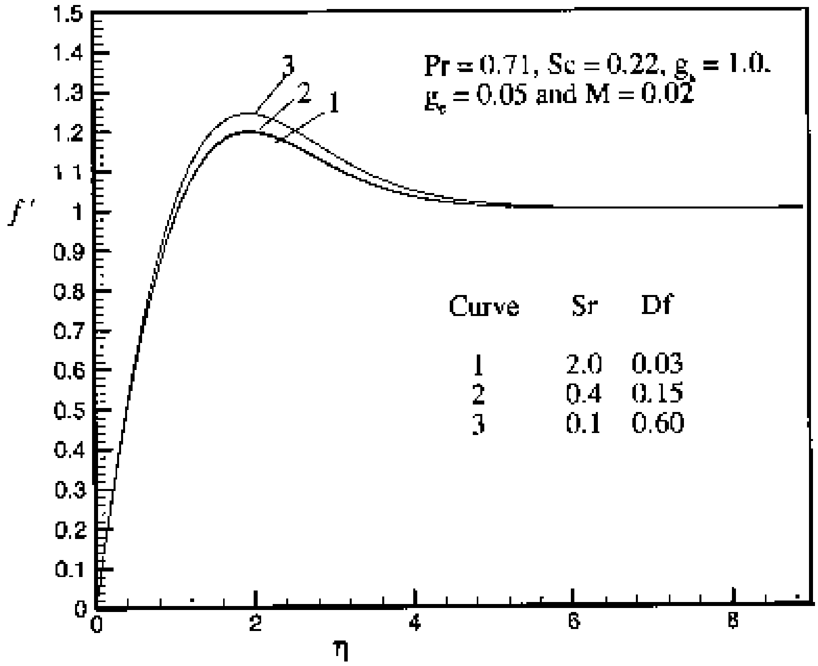


Fig.2.9: Velocity profiles for different values of *Sr* and *Df*.

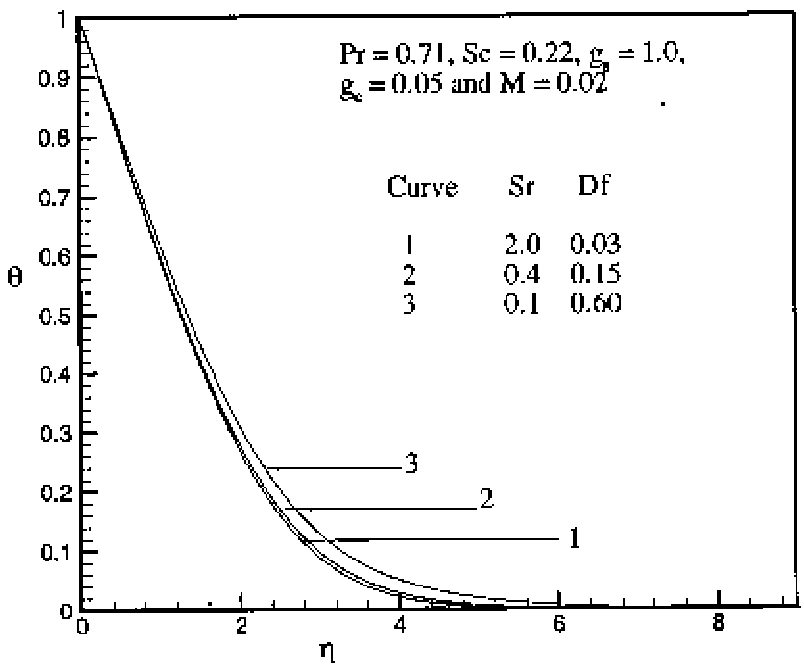


Fig.2.10: Temperature profiles for different values of *Sr* and *Df*.

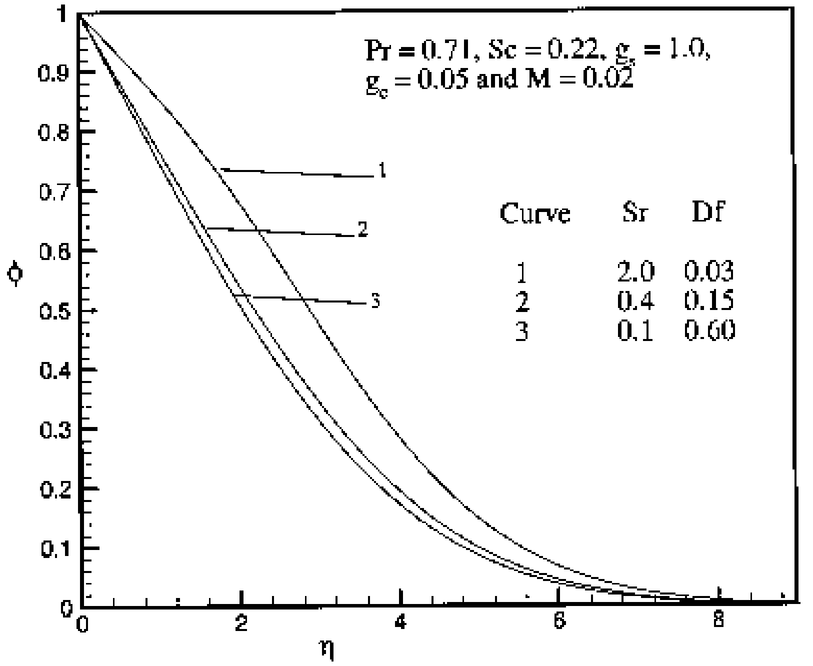


Fig.2.11: Concentration profiles for different values of *Sr* and *Df*.

**Table 2.2:** Numerical values of skin-friction coefficients, Nusselt number and Sherwood number for $Pr = 0.71$, $Sc = 0.22$, $g_s = 1.0$ and $g_c = 0.05$.

| $M$ | $Sr$ | $Df$ | $C_f$ | $Nu$ | $Sh$ |
|---|---|---|---|---|---|
| 0.02 | 2.0 | 0.03 | 1.4222 | 0.4086 | 0.1465 |
| 0.06 | 2.0 | 0.03 | 1.3671 | 0.3978 | 0.1440 |
| 0.10 | 2.0 | 0.03 | 1.3069 | 0.3869 | 0.1388 |
| 0.02 | 2.0 | 0.03 | 1.4222 | 0.4086 | 0.1465 |
| 0.02 | 0.40 | 0.15 | 1.4260 | 0.4012 | 0.2538 |
| 0.02 | 0.10 | 0.60 | 1.4650 | 0.3830 | 0.2545 |

# Chapter-3

## Unsteady MHD free convection and mass transfer flow past an infinite vertical porous plate

The flow of an incompressible viscous fluid, past an impulsively started infinite horizontal plate, was studied first by Stokes (1856) and it is also known as Reyleigh's problem. Because of the significance of this problem in fluid mechanics and aerodynamics, many authors have extended the above problem for different cases. Soundalgeker (1977) studied the free convection flow past an impulsively started infinite vertical plate, when it is cooled or heated by free convection currents. Kafoussias and Daskalakis (1986) studied the hydromagnetic free convection flow of a viscous incompressible and electrically conducting fluid past an infinite vertical porous flat plate, which is moving in its own plane impulsively or uniformly-accelerated. The governing equations of the flow were solved by finite difference method when the Prandtl number $Pr$ is equal to 0.71 and 7. The effects of mass transfer on MHD free convection flow have also been investigated by many researchers, some of them are Haldavnekar and Soundalgeker (1977), Rahman and Sattar (1999) and Mahmud et al. (2001).

In the above studies, some of the cases the thermal-diffusion (Soret) and diffusion-thermo (Dufour) effects on the flow field have not been considered. In Chapter 2 we have studied the thermal-diffusion (Soret) and diffusion-thermo (Dufour) effects for a two-dimensional steady motion. In this Chapter we further investigate these effects on an unsteady MHD free convection and mass transfer flow past an infinite vertical porous plate. Two cases are considered; (I) Impulsively started plate (ISP) moving in its own plane, and (II) Uniformly accelerated plate (UAP). Similarity equations of the momentum, energy and concentration equations are derived by introducing a time dependent scaling factor. The suction velocity is taken to be inversely proportional to the above scale factor. Finally the transformed non-linear ordinary differential equations, which are locally similar, have been solved numerically using Nachtsheim-Swigert shooting iteration technique with Runge-Kutta sixth-order integration method.

## 3.1 Governing equations of the flow:

We consider a model of unsteady MHD free convection and mass transfer flow of an electrically conducting incompressible viscous fluid, along an infinite vertical porous flat plate. The $x$-axis is taken on the infinite plate, and parallel to the free-stream velocity which is vertical and the $y$-axis is taken normal to the plate. A magnetic field of uniform strength is applied transversely to the direction of the flow. Initially the plate and the fluid are at same temperature $T_\infty$ in a stationary condition with concentration level $C_\infty$ at all points. But at time $t > 0$, the plate starts with a velocity $U(t)$ in its own plane, its temperature is raised to $T_w$ and the concentration level at the plate is raised to $C_w$. The physical configuration considered here is shown in the following figure 3.1.

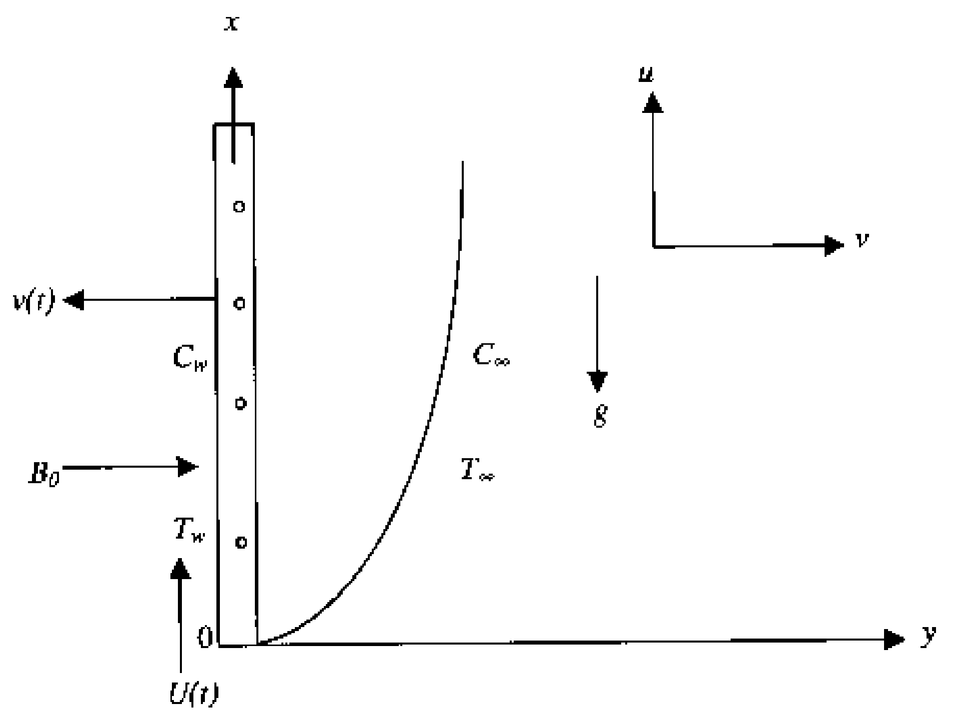


Fig. 3.1: Flow configuration and coordinate system .

Since the plate is considered to be of infinite extent, all derivatives with respect to $x$ vanish. Then under the usual Boussinesq approximation the basic equations relevant to the problem are:

$$\frac{\partial v}{\partial y}=0, \tag{3.1}$$

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}+g\beta(T-T_\infty)+g\beta^*(C-C_\infty)-\frac{\sigma B_0^2 u}{\rho}, \tag{3.2}$$

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{D_m k_T}{c_s c_p}\frac{\partial^2 C}{\partial y^2}, \tag{3.3}$$

$$\frac{\partial C}{\partial t}+v\frac{\partial C}{\partial y}=D_m\frac{\partial^2 C}{\partial y^2}+\frac{D_m k_T}{T_m}\frac{\partial^2 T}{\partial y^2}, \tag{3.4}$$

where the variables and related quantities are defined in the Nomenclature.

The initial and boundary conditions for the above problem are:

for $t\le 0$: $u=v=0, T=T_\infty, C=C_\infty$ for all $y$ (3.5)

for $t>0$:

$u=U(t)=U_0F(t^n), v=v(t), T=T_w, C=C_w$ at $y=0$ (3.6a)

$u=0, v=0, T=T_\infty, C=C_\infty$ as $y\to\infty$ (3.6b)

The above problem is solved for two different values of $n$ signifying two different cases, e.g., when the flat plate is impulsively started, moving in its own plane ( $n$ = 0 or ISP) and when it is uniformly accelerated ( $n$ = 1 or UAP).

## 3.2 Mathematical formulations:

**Case I;** Impulsively started plate (ISP)

In this case setting $n$ = 0 and considering $F(1)$ = 1 [Alam (1995)], we get $u = U_0$ from equation (3.6a). Now in order to obtain similarity solutions to the problem considered, we introduce a time dependent length scale $\delta$ as

$$\delta=\delta(t),\ t>0. \tag{3.7}$$

In terms of this length scale, a convenient solution of the equation (3.1) is considered to be in the following form:

$$v = v(t) = -v_0 \frac{\upsilon}{\delta}, \tag{3.8}$$

where $v_0$ is the suction parameter.

In order to non-dimensionalise the equations (3.2)-(3.4) we introduce the following dimensionless quantities:

$\eta = \frac{y}{\delta}$, $Gr = \frac{g\beta(T_w - T_\infty)\delta^2}{\upsilon U_0}$ (local Grashof number),

$u = U_0 f_i(\eta)$, $Gm = \frac{g\beta^*(C_w - C_\infty)\delta^2}{\upsilon U_0}$ (local Modified Grashof number),

$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}$, $Df = \frac{D_m k_T (C_w - C_\infty)}{c_s c_p \upsilon (T_w - T_\infty)}$ (Dufour number),

$\phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}$, $Sr = \frac{D_m k_T (T_w - T_\infty)}{\upsilon T_m (C_w - C_\infty)}$ (Soret number),

$Sc = \frac{\upsilon}{D_m}$ (Schmidt number), $M = \frac{\sigma B_0^2 \delta^2}{\upsilon \rho}$ (Magnetic parameter),

$\Pr = \frac{\upsilon}{\alpha}$ (Prandtl number). (3.9)

In view of (3.8) and (3.9), equations (3.2) - (3.4) become

$$f_i'' + \eta\left(\frac{\delta}{\upsilon}\frac{d\delta}{dt}\right) f_i' + v_0 f_i' + Gr\theta + Gm\phi - Mf_i = 0 \tag{3.10}$$

$$-\eta\left(\frac{\delta}{\upsilon}\frac{d\delta}{dt}\right)\theta' - v_0\theta' = \frac{1}{\Pr}\theta'' + Df\phi'' \tag{3.11}$$

$$-\eta\left(\frac{\delta}{\upsilon}\frac{d\delta}{dt}\right)\phi' - v_0\phi' = \frac{1}{Sc}\phi'' + Sr\theta'' \tag{3.12}$$

where primes denote differentiation with respect to $\eta$.

The corresponding boundary conditions for $t > 0$ are obtained as:

$$f_i = 1, \theta = 1, \phi = 1 \quad \text{at } \eta = 0 \tag{3.13a}$$

$$f_i = 0, \theta = 0, \phi = 0 \quad \text{as } \eta \to \infty \tag{3.13b}$$

Now the equations (3.10) - (3.12) are locally similar except the term $\left(\frac{\delta}{\upsilon}\frac{d\delta}{dt}\right)$, where $t$ appears explicitly. Thus the local similarity condition requires that $\left(\frac{\delta}{\upsilon}\frac{d\delta}{dt}\right)$ in the equations (3.10) - (3.12) must be a constant quantity.

Hence following the works of Hasimoto (1957), Sattar and Hossain (1992), Sattar (1993) and Sattar and Maleque (2000), one can try a class of solutions of the equations (3.10) - (3.12) by assuming that

$$\left(\frac{\delta}{\upsilon}\frac{d\delta}{dt}\right) = \lambda \text{ (a constant).} \tag{3.14}$$

Integrating (3.14) we have

$$\delta = \sqrt{2\lambda\upsilon t} \tag{3.15}$$

where the constant of integration is determined through the condition that $\delta = 0$ when $t = 0$. From (3.15) choosing $\lambda$ = 2, the length scale $\delta(t) = 2\sqrt{\upsilon t}$ which exactly corresponds to the usual scaling factor for various unsteady boundary layer flows [Schlichting (1968)]. Since $\delta$ is a scaling factor as well as a similarity parameter, any value of $\lambda$ in (3.14) would not change the nature of the solutions except that the scale would be different.

Now introducing (3.14) [with λ=2] in the equations (3.10) - (3.12) respectively, we obtain the following dimensionless ordinary differential equations which are locally similar.

$$f_i'' + (2\eta + v_0)f_i' + Gr\theta + Gm\phi - Mf_i = 0 \tag{3.16}$$

$$\theta'' + \Pr(2\eta + v_0)\theta' + \Pr Df\phi'' = 0 \tag{3.17}$$

$$\phi'' + Sc(2\eta + v_0)\phi' + ScSr\theta'' = 0 \tag{3.18}$$

Subject to the above formulation the boundary conditions follow from (3.13)

$$f_i = 1, \theta = 1, \phi = 1 \quad \text{at } \eta = 0 \tag{3.19a}$$

$$f_i = 0, \theta = 0, \phi = 0 \quad \text{as } \eta \to \infty \tag{3.19b}$$

Case II: Uniformly accelerated plate (UAP)

In this case setting $n = 1$ we get $u = U_0 F(t)$ from equation (3.6). As in case I we consider the same similarity parameter δ as represented by (3.7) and the same solution (3.8) of the equation (3.1).

Now for reasons of similarity, the plate velocity is taken [Alam (1995)] to be

$$U(t) = U_0\, F(t) \tag{3.20}$$

where $F(t)$ is taken to be equal to $\delta_*^2$, with $\delta_* = \dfrac{\delta}{\delta_0}$, such that $\delta = \delta_0$ at $t = t_0$.

We now introduce the following dimensionless quantities:

$$\left.\begin{aligned} \eta &= \frac{y}{\delta}, \\ f_a(\eta) &= \frac{u}{U(t)}, \\ \theta(\eta) &= \frac{T - T_\infty}{T_w - T_\infty}, \\ \phi(\eta) &= \frac{C - C_\infty}{C_w - C_\infty}. \end{aligned}\right\} \tag{3.21}$$

Now substituting the relations (3.7), (3.8) and (3.20) – (3.21) into equations (3.2), (3.3) and (3.4) and introducing the following dimensionless parameters:

$$\Pr = \frac{\upsilon}{\alpha},\; Sc = \frac{\upsilon}{D_m},\; M = \frac{\sigma B_0^2 \delta^2}{\upsilon \rho},\; Sr = \frac{D_m k_T (T_w - T_\infty)}{T_m \upsilon (C_w - C_\infty)},$$

$$Df = \frac{D_m k_T (C_w - C_\infty)}{c_s c_p \upsilon (T_w - T_\infty)},\; Gr = \frac{g\beta (T_w - T_\infty)\delta^2}{\upsilon U_0},\; Gm = \frac{g\beta^* (C_w - C_\infty)\delta^2}{\upsilon U_0}$$

we obtain the following ordinary differential equations:

$$f_a'' + \left(\frac{\delta}{\upsilon}\frac{d\delta}{dt}\right)[\eta f_a' - 2 f_a] + v_0 f_a' + Gr\theta + Gm\phi - Mf_a = 0 \tag{3.22}$$

$$-\eta\left(\frac{\delta}{\upsilon}\frac{d\delta}{dt}\right)\theta' - v_0\theta' = \frac{1}{\Pr}\theta'' + Df\phi'', \tag{3.23}$$

$$-\eta\left(\frac{\delta}{\upsilon}\frac{d\delta}{dt}\right)\phi' - v_0\phi' = \frac{1}{Sc}\phi'' + Sr\theta''. \tag{3.24}$$

Subject to the above equations (3.22) – (3.24) the boundary conditions (3.6) become

$$f_a = 1, \theta = 1, \phi = 1 \quad \text{at } \eta = 0 \tag{3.25a}$$

$$f_a = 0, \theta = 0, \phi = 0 \quad \text{as } \eta \to \infty \tag{3.25b}$$

Now following the arguments in case I. we respectively have

$$f_a'' + (2\eta + v_0) f_a' + Gr\theta + Gm\phi - (4 + M) f_a = 0 \tag{3.26}$$

$$\theta'' + \Pr(2\eta + v_0)\theta' + \Pr Df\phi'' = 0 \tag{3.27}$$

$$\phi'' + Sc(2\eta + v_0)\phi' + ScSr\theta'' = 0 \tag{3.28}$$

where primes denote differentiation with respect to $\eta$. The above equations thus describe the basis of our problem for case II. The solutions of which are now sought subject to the boundary conditions (3.25).

Now it is important to calculate the physical quantities of the primary interest, which are the local wall shear stress, local surface heat flux and the local surface mass flux respectively from the following definitions:

$$\tau_w = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}, \tag{3.29}$$

$$q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0}, \tag{3.30}$$

$$M_w = -D \left( \frac{\partial C}{\partial y} \right)_{y=0}. \tag{3.31}$$

The dimensionless local wall shear stress, local surface heat flux and the local surface mass flux for impulsively started plate respectively obtained as

$$\frac{\tau_w \delta}{\mu U_0} = f'(0), \tag{3.32}$$

$$\frac{q_w \delta}{k(T_w - T_\infty)} = -\theta'(0), \tag{3.33}$$

$$\frac{M_w \delta}{D(C_w - C_\infty)} = -\phi'(0). \tag{3.34}$$

Hence the dimensionless skin-friction coefficient, Nusselt number and Sherwood number for impulsively started plate are given by

$$C_{f_i} = \frac{2\tau_w}{\rho U_0^2} = 2(\mathrm{Re}_\delta)^{-1} f'(0), \text{ where } \mathrm{Re}_\delta = \frac{U_0\delta}{\upsilon}, \tag{3.35}$$

$$Nu_i = \frac{q_w\delta}{k(T_w - T_\infty)} = -\theta'(0), \tag{3.36}$$

and

$$Sh_i = \frac{M_w\delta}{D(C_w - C_\infty)} = -\phi'(0). \tag{3.37}$$

and for the accelerated plate we have the followings

$$C_{f_a} = \frac{2\tau_w}{\rho U^2} = \frac{2}{\mathrm{Re}_\delta \delta_*^2} f'(0), \text{ where } \mathrm{Re}_\delta = \frac{U_0\delta}{\upsilon}, \tag{3.38}$$

$$Nu_a = \frac{q_w\delta}{k(T_w - T_\infty)} = -\theta'(0), \tag{3.39}$$

and

$$Sh_a = \frac{M_w\delta}{D(C_w - C_\infty)} = -\phi'(0). \tag{3.40}$$

Thus the dimensionless values of the local skin-friction coefficient, Nusselt number and Sherwood number for impulsive as well as accelerated plate are obtained from the process of numerical calculations and are sorted in Tables-3.1-3.4.

## 3.3 Method of Solution

The non-linear ordinary differential equations (3.16)-(3.18) for case I and (3.22)-(3.24) for case II with respective boundary conditions have been solved by employing sixth order Runge - Kutta method along with the Nachtsheim - Swigert (1965) shooting iteration technique. The solution procedure has been discussed in details in Chapter-2 and therefore any further discussion is discarded here. The numerical results obtained are presented in the following section.

## 3.4 Results and Discussion

For the purpose of discussing the effects of various parameters on the flow behaviour near the plate, numerical calculations have been carried out for different arbitrary values of suction parameter $v_0$, magnetic parameter $M$ and for fixed values of Prandtl number $Pr$, Schmidt number $Sc$, Grashof number $Gr$ and modified Grashof number $Gm$. The value of Prandtl number $Pr$ is taken equal to 0.71 that corresponds physically to air. The value of Schmidt number $Sc = 0.22$ has been chosen to represent hydrogen at approx. $T_m = 25^0C$ and 1 atm. The values of Grashof number $Gr$ and modified Grashof number $Gm$ are taken to be both positive and negative, since these values represent respectively cooling and heating of the plate. Finally, the values of Soret number $Sr$ and Dufour number $Df$ are chosen in such a way that their product is constant.

**Case-I:** The velocity, temperature and concentration profiles are shown graphically in figs. 3.2-3.9 for both cooling and heating of impulsively started plate. In Fig. 3.2 the effects of magnetic and suction parameters are shown for cooling of the plate. It is seen from this figure that as the magnetic parameter increases the velocity decreases which indicates that the magnetic field retards the fluid motion. The same effect on the velocity profiles is also observed for increasing values of the suction parameter, which is usually expected. In Fig. 3.3 the effects of magnetic and suction parameters are shown for heating of the plate. But in this case the velocity is just reversed in comparison with cooling of the plate.

The effects of Soret and Dufour numbers on the velocity field for cooling and heating of the plate are shown in Figs. 3.4 and 3.5 respectively. We observe that for cooling of the plate, Quantitatively when $\eta = 0.5$ and $Sr$ decreases from 2 to 0.4 (or $Df$ increases from 0.03 to 0.15) there is 22.47% decrease in the velocity value, whereas the corresponding decrease, when $Sr$ decreases from 0.4 to 0.1, is 3.25%. But compared to the case of cooling of the plate, opposite effects is observed in the case of heating of the plate.

The temperature profiles are shown in Figs. 3.6 and 3.7 for cooling of the plate. From Fig. 3.6 we see that the temperature decreases with the increase of suction parameter. From Fig. 3.7,when $\eta = 0.5$ and $Sr$ decreases from 2 to 0.4 (or $Df$ increases from 0.03 to

0.15) there is 50.82% increase in the temperature value, whereas the corresponding increase, when $Sr$ decreases from 0.4 to 0.1, is 15.42%.

In Fig. 3.8 and 3.9, the concentration profiles are shown for cooling of the plate. It is observed from Fig. 3.8 that the concentration increases with the increase of suction parameter close to the wall (approx. $\eta \leq 0.60$) whereas for $\eta \geq 0.60$, the concentration decreases with increase of suction parameter. In Fig. 3.9, the effects of Soret and Dufour numbers on the concentration profiles are shown. It is seen from this figure that for $\eta = 1$ and $Sr$ decreases from 2 to 0.4 (or $Df$ increases from 0.03 to 0.15) there is 35.96% decrease in the concentration value, whereas the corresponding decrease is 7.09% when $Sr$ decreases from 0.4 to 0.1.

In Tables 3.1 and 3.2, numerical values of the skin-friction coefficients, Nusselt number and Sherwood number are given for impulsively started plate (both cooling and heating). From Table 3.1, it appears that for cooling of the plate both the skin-friction coefficients and Sherwood number decrease with the increase of $v_0$ and $M$. On the other hand, the Nusselt number increases with the increase of $v_0$ but it decrases with the increase of $M$. Table 3.2 indicates that for cooling of the plate the skin-friction coefficients and Nusselt number increase with the increase of Soret number, whereas the Sherwood number decreases with the increase of Soret number. But compared to the case of cooling of the plate, opposite effects is observed in the case of heating of the plate.

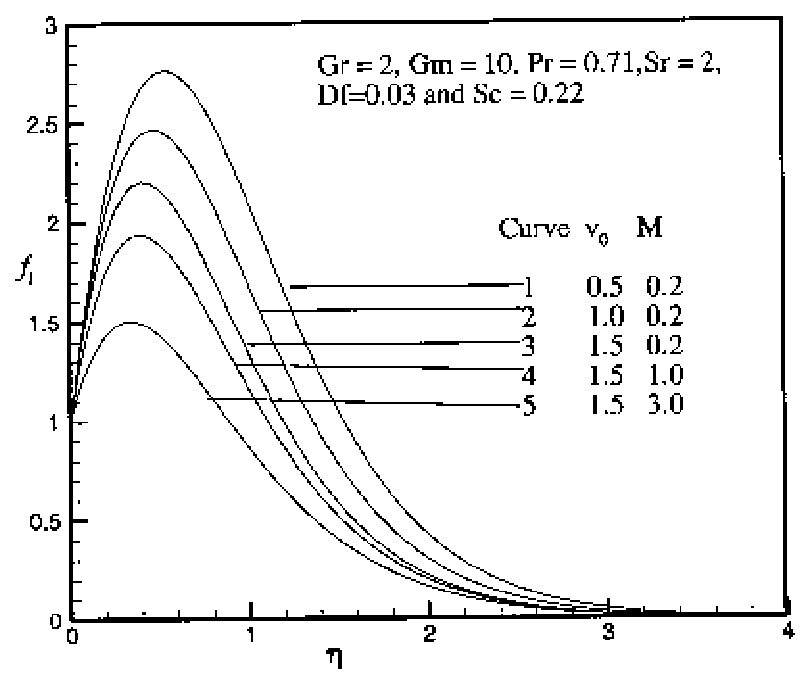


Fig.3.2: velocity profiles due to cooling of impulsively started plate for different values of $v_0$ and M.

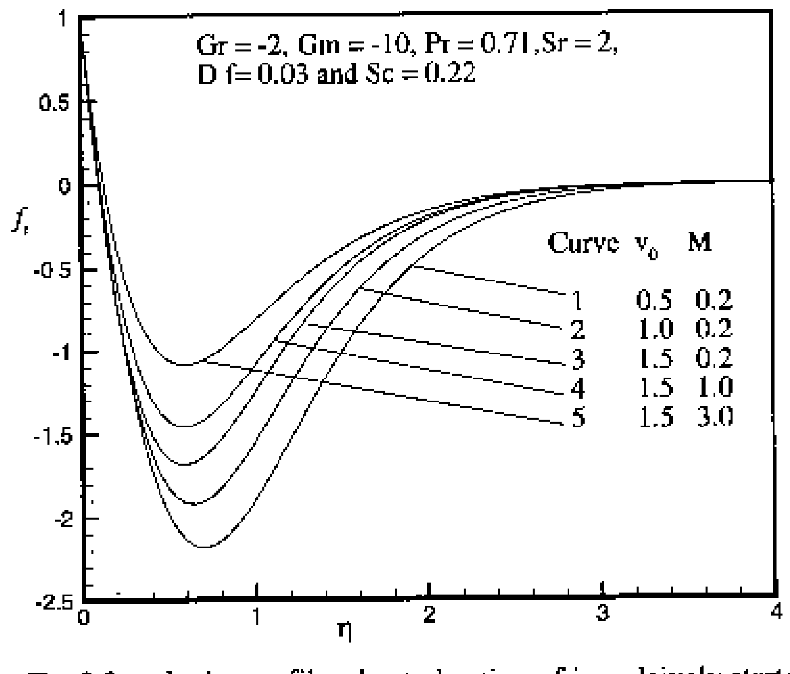


Fig.3.3: velocity profiles due to heating of impulsively started plate for different values of $v_0$ and M.

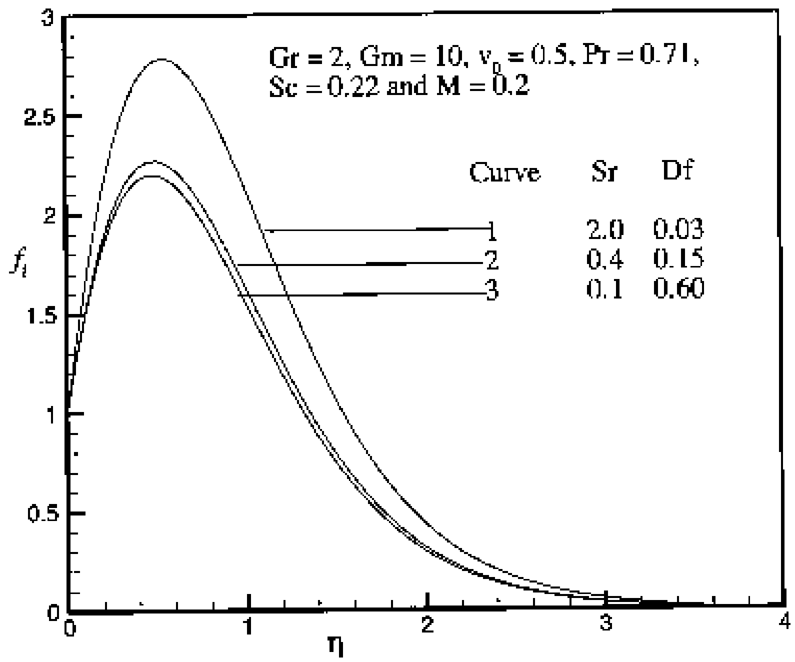


Fig.3.4: velocity profiles due to cooling of impulsively started plate for different values of Sr and Df.

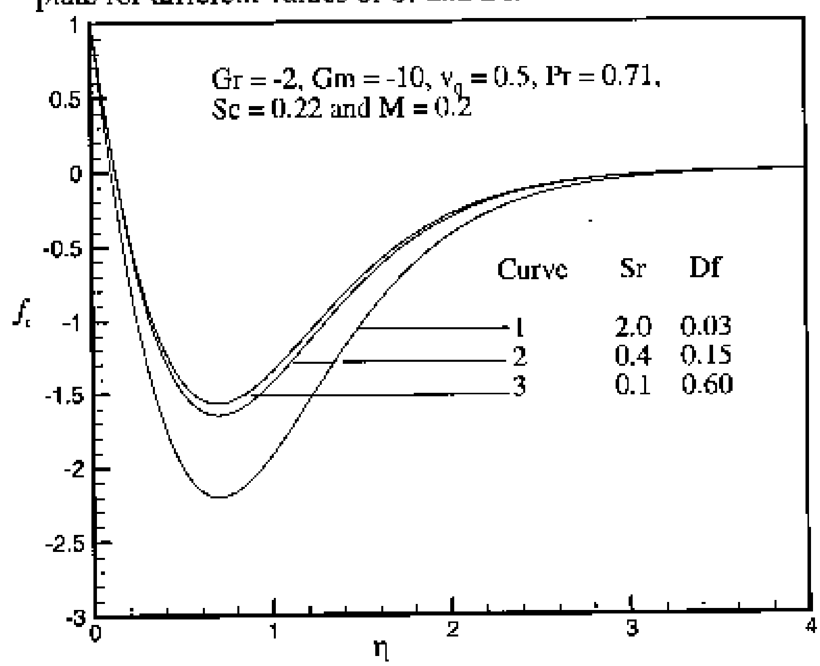


Fig3.5: velocity profiles due to heating of impulsively started plate for different values of Sr and Df.

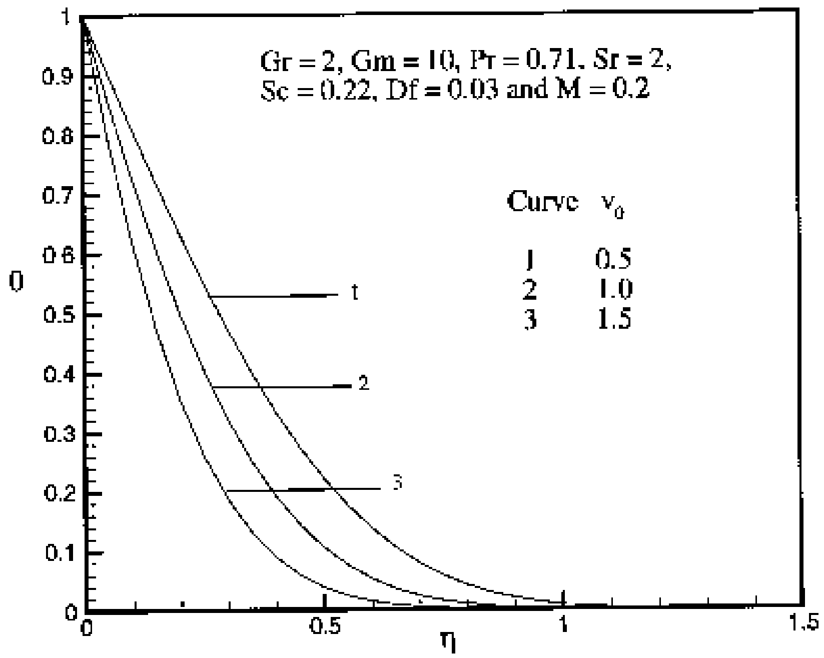


Fig.3.6: Temperature profiles for different values of $v_0$.

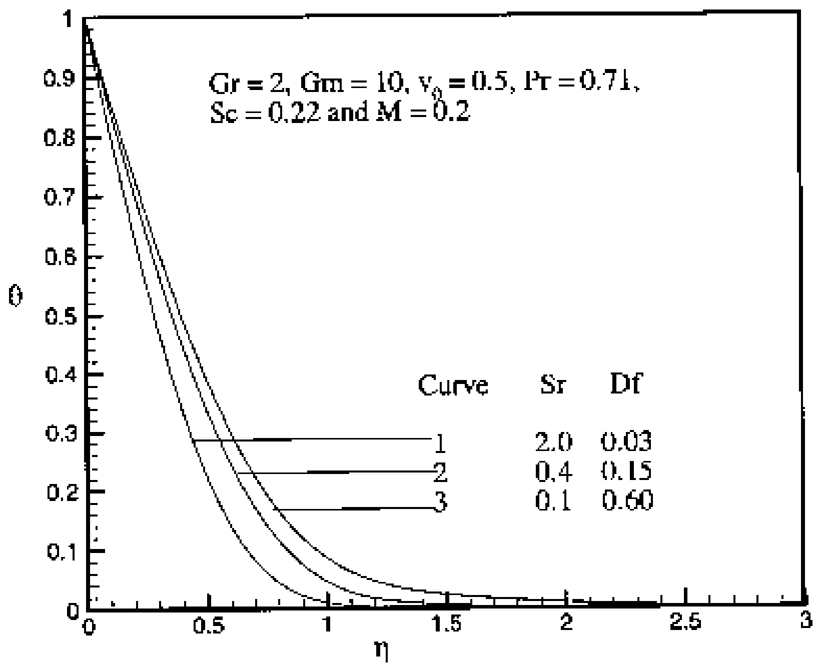


Fig.3.7: Temperature profiles for different values of Sr and Df.

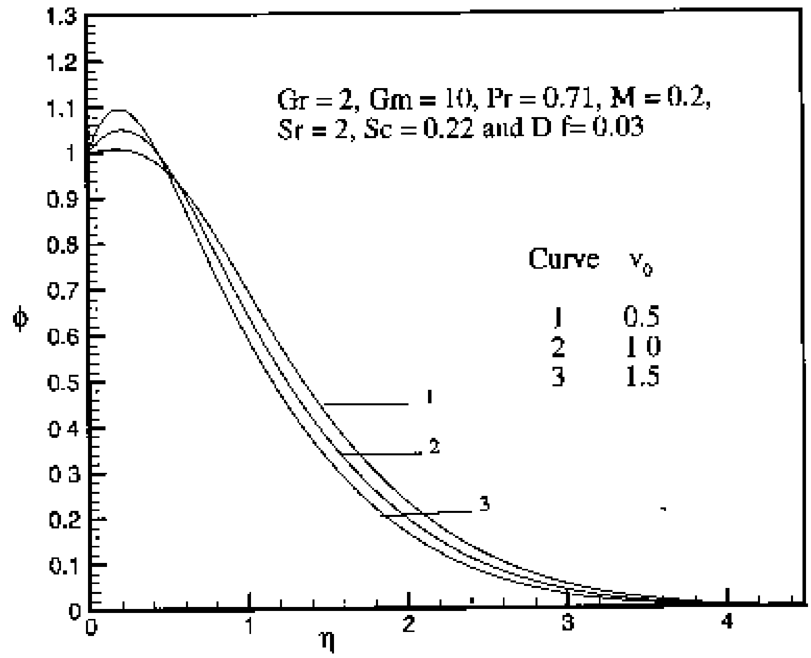


Fig.3.8: Concentration profiles for different values of $v_0$.

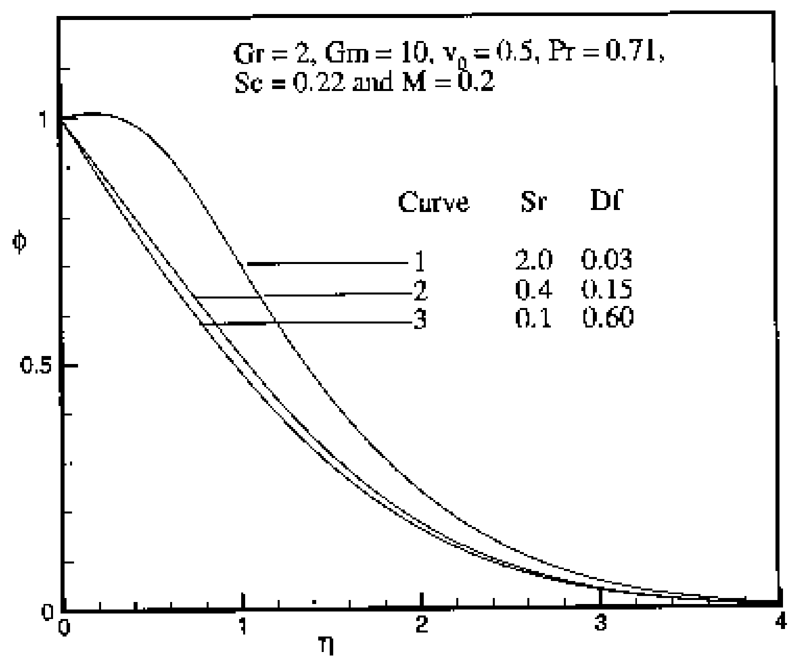


Fig.3.9: Concentration profiles for different values of Sr and Df

**Table 3.1:** Numerical values of skin-friction coefficients, Nusselt number and Sherwood number for $Pr$ =0.71, $Sr$ =2.0, $Df$ =0.03 and $Sc$ =0.22.

| $Gr$ | $Gm$ | $M$ | $v_0$ | $C_{fi}$ | $Nu_i$ | $Sh_i$ |
|---|---|---|---|---|---|---|
| -2 | -10 | 0.2 | 0.5 | -10.227144 | 1.940148 | -0.082728 |
| -2 | -10 | 0.2 | 1.0 | -10.793929 | 2.944274 | -0.452290 |
| -2 | -10 | 0.2 | 1.5 | -11.256771 | 4.445256 | -1.040100 |
| -2 | -10 | 1.0 | 1.5 | -10.518761 | 4.445252 | -1.040102 |
| -2 | -10 | 3.0 | 1.5 | -9.395594 | 4.445239 | -1.040107 |
| +2 | +10 | 0.2 | 0.5 | 7.156127 | 1.940148 | -0.082728 |
| +2 | +10 | 0.2 | 1.0 | 6.998597 | 2.944274 | -0.452290 |
| +2 | +10 | 0.2 | 1.5 | 6.686176 | 4.445256 | -1.040100 |
| +2 | +10 | 1.0 | 1.5 | 5.505187 | 4.445252 | -1.040102 |
| +2 | +10 | 3.0 | 1.5 | 3.435226 | 4.445239 | -1.040107 |

**Table 3.2:** Numerical values of skin-friction coefficients, Nusselt number and Sherwood number for $Pr$ = 0.71, $v_0$=0.5, $M$ =0.2 and $Sc$ =0.22.

| $Gr$ | $Gm$ | $Sr$ | $Df$ | $C_{fi}$ | $Nu_i$ | $Sh_i$ |
|---|---|---|---|---|---|---|
| +2 | +10 | 2.0 | 0.03 | 7.205083 | 1.934014 | 0.087042 |
| +2 | +10 | 0.4 | 0.15 | 5.775135 | 1.517723 | 0.495844 |
| +2 | +10 | 0.1 | 0.60 | 5.581905 | 1.364413 | 0.575167 |
| -2 | -10 | 2.0 | 0.03 | -10.276100 | 1.934014 | 0.087042 |
| -2 | -10 | 0.4 | 0.15 | -8.846151 | 1.517723 | 0.495844 |
| -2 | -10 | 0.1 | 0.60 | -8.652921 | 1.364413 | 0.575167 |

## Case-II

Figs. 3.10-3.17 show the variations of velocity, temperature and concentration profiles due to accelerated motion of the plate when it is cooled and heated by free convection currents respectively. From Fig.3.10 it is observed that the velocity decreases with the increase of both parameters $v_0$ and $M$. In Fig.3.11 the heating effects of the plate on the velocity profiles are shown. As in the case of impulsively started plate (case-I), when the accelerated plate is being heated by free convection currents, a reverse type of flow occurs in comparison to that of the cooling of the plate for different values of $v_0$ and $M$. In Figs. 3.12 and 3.13 the effects of Soret and Dufour numbers on the velocity field are shown for cooling and heating of the plate. Comparing Figs. 3.4 and 3.5 with Figs. 3.12 and 3.13 respectively, the same effects on the velocity profiles are also observed for increasing values of Dufour numbers.

Figs. 3.14 - 3.15 show the variations of temperature profiles for different values of $v_0$, $Sr$ and $Df$. The effects of these parameters on the temperature profiles are similar to those of the impulsively started plate.

Figs. 3.16 - 3.17 show the variations of concentration profiles for different values of $v_0$, $Sr$ and $Df$. The effect of suction parameter on the concentration profiles is similar to that of the impulsively started plate. But the influence of Soret and Dufour number on the concentration field is 18.91% less than that of the impulsively started plate.

From the Tables 3.3 - 3.4, we observe that the effects of the various parameters on the local skin-friction coefficients, rate of heat transfer and rate of mass transfer are similar to those of case-I. But, finally, it is seen from both the cases that the wall shear stress has a larger effect in case of impulsively started plate as compared to the uniformly accelerated plate.

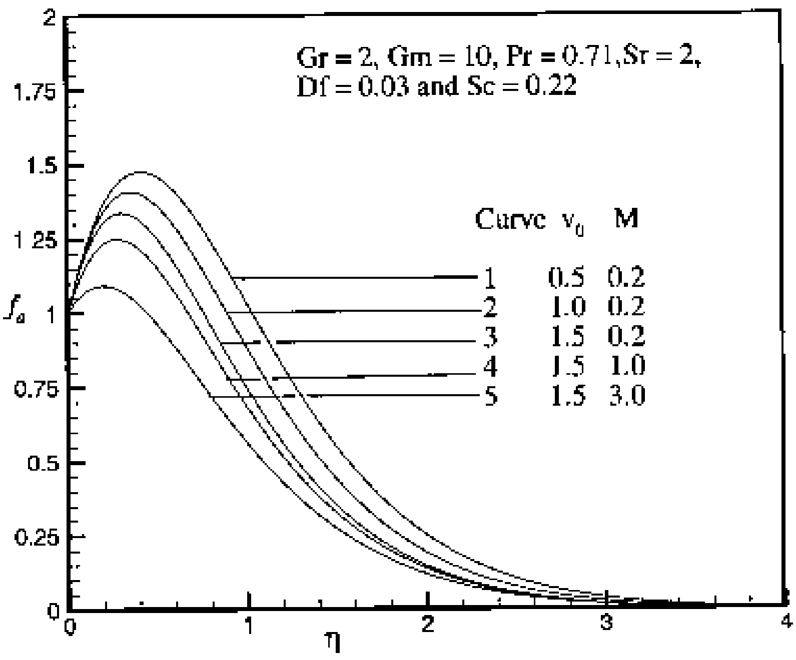


Fig.3.10: velocity profiles due to cooling of uniformly accelerated plate for different values of $v_0$ and M.

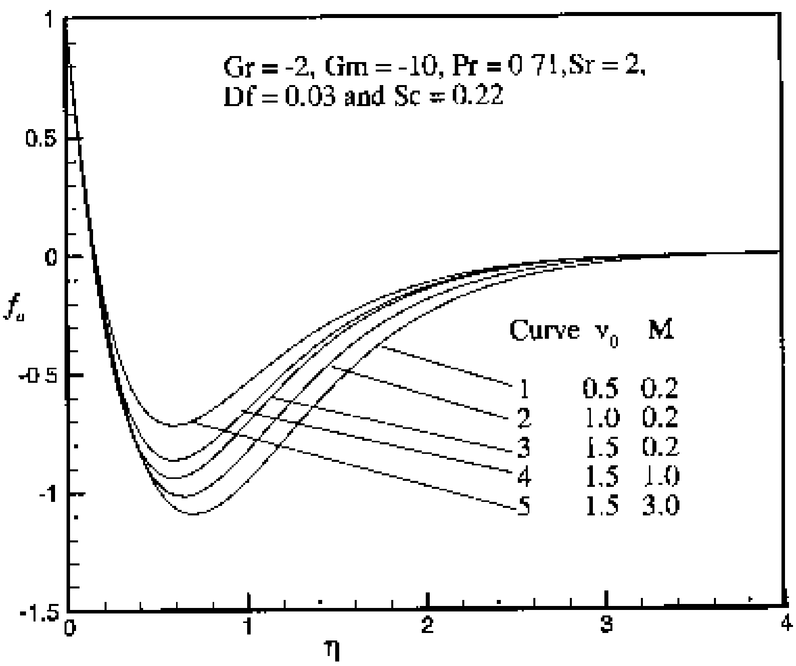


Fig.3.11: velocity profiles due to heating of uniformly accelerated plate for different values of $v_0$ and M.

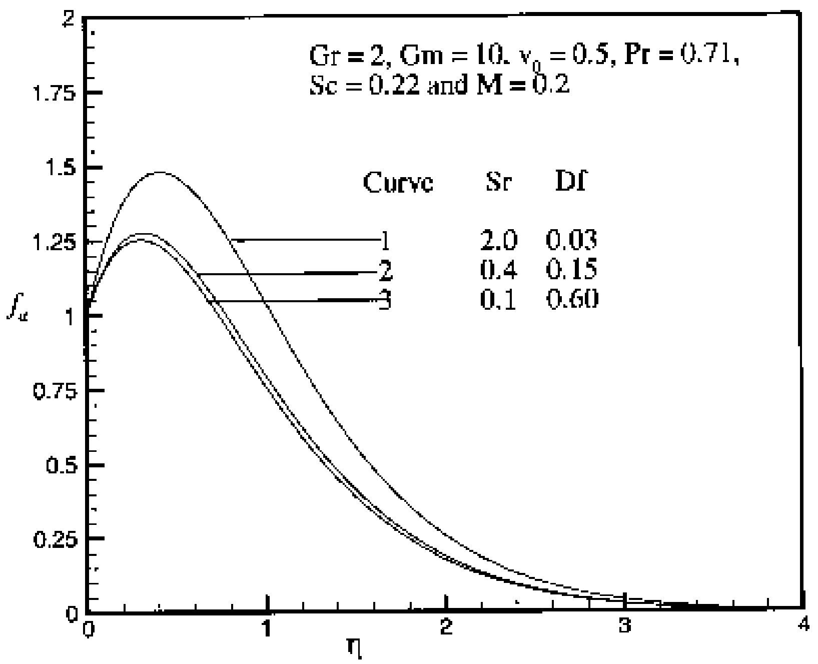


Fig.3.12: velocity profiles due to cooling of uniformly accelerated plate for different values of Sr and Df.

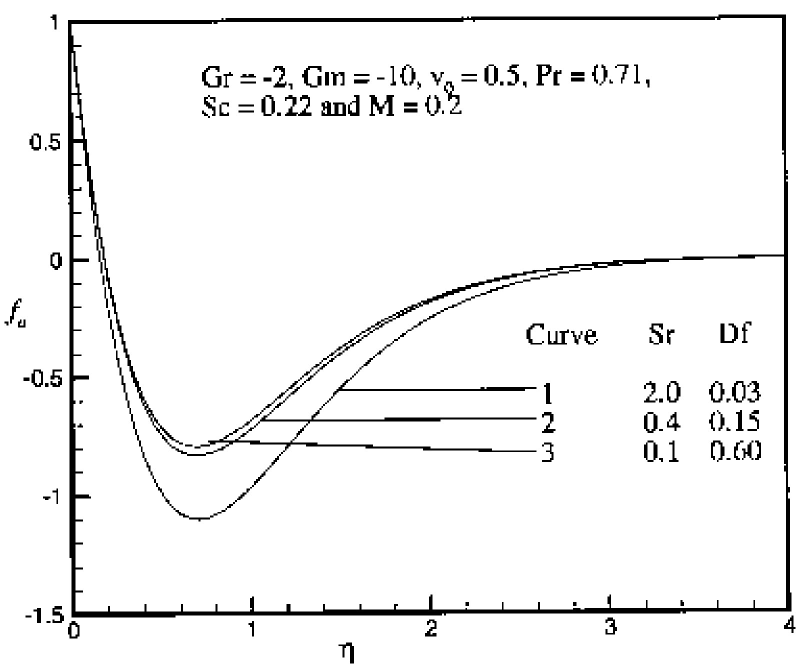


Fig.3.13: velocity profiles due to heating of uniformly accelerated plate for different values of Sr and Df.

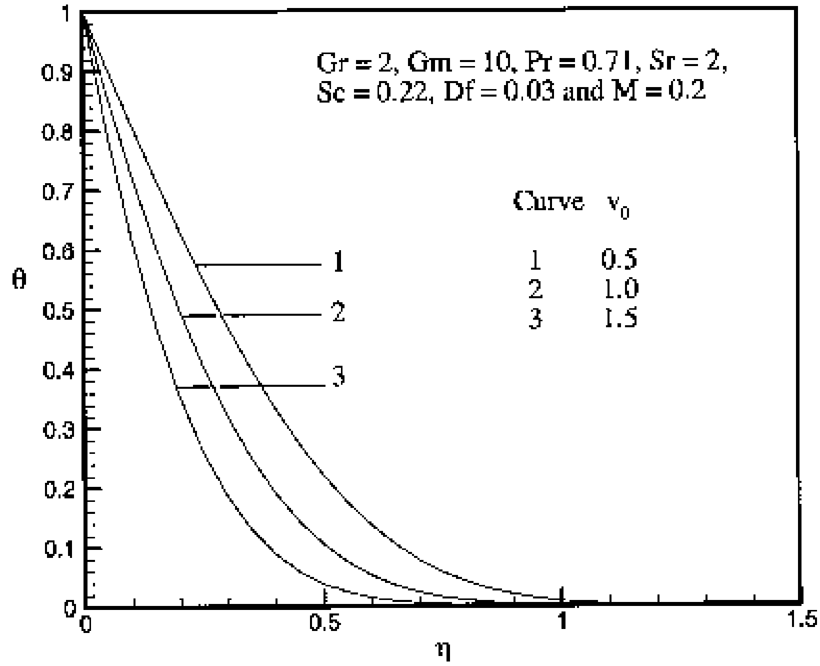


Fig.3.14: Temperature profiles for different values of $v_0$.

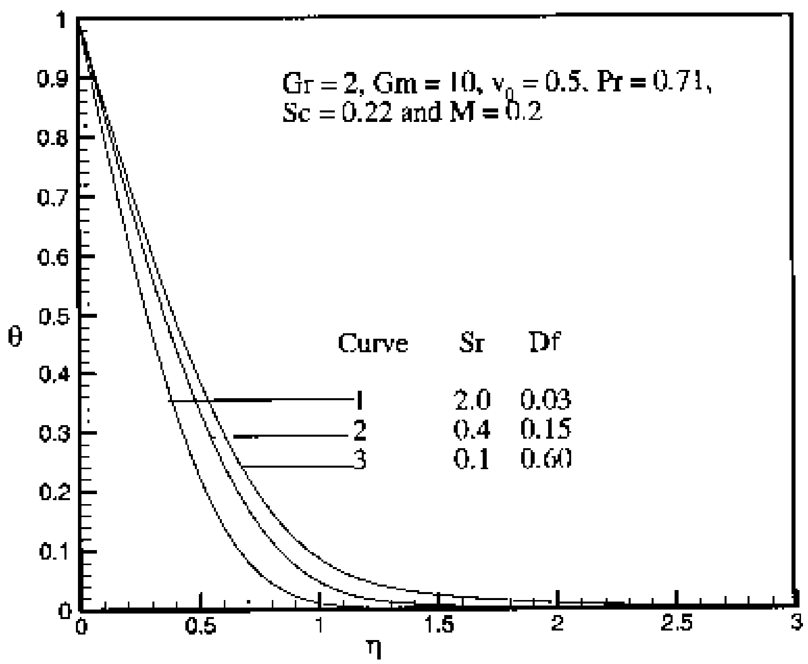


Fig.3.15: Temperature profiles for different values of Sr and Df.

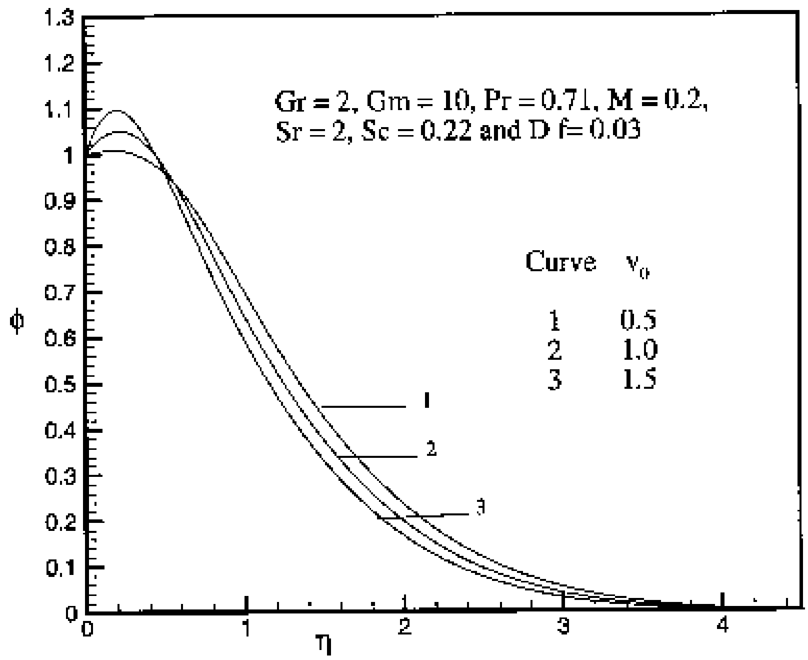


Fig.3.16: Concentration profiles for different values of $v_0$.

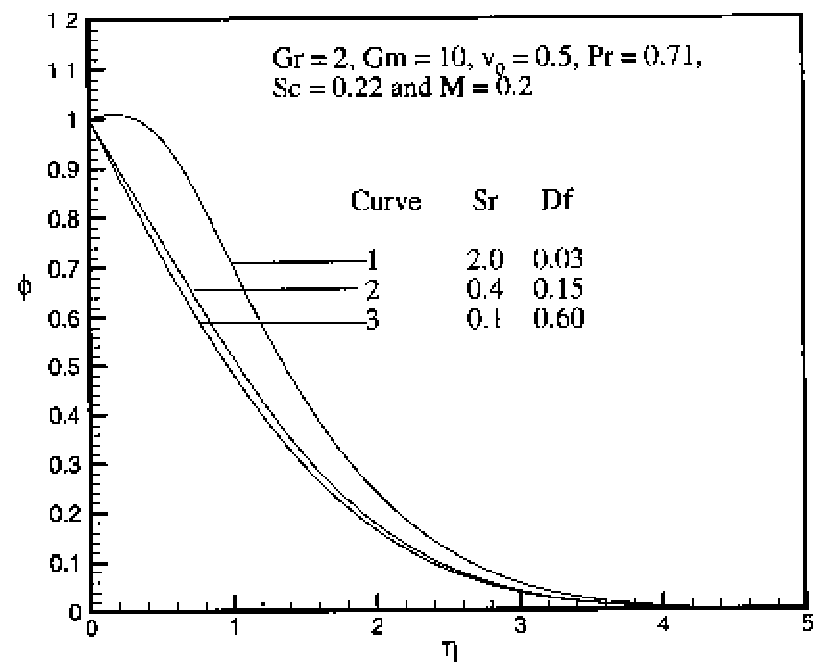


Fig.3.17: Concentration profiles for different values of Sr and Df.

**Table 3.3:** Numerical values of skin-friction coefficients, Nusselt number and Sherwood number for $Pr = 0.71$, $Sr = 2.0$, $Df = 0.03$ and $Sc = 0.22$.

| $Gr$ | $Gm$ | $M$ | $v_0$ | $C_{fa}$ | $Nu_a$ | $Sh_a$ |
|---|---|---|---|---|---|---|
| -2 | -10 | 0.2 | 0.5 | -7.848545 | 1.940135 | -0.082743 |
| -2 | -10 | 0.2 | 1.0 | -8.442502 | 2.944255 | -0.452304 |
| -2 | -10 | 0.2 | 1.5 | -9.003997 | 4.445230 | -1.040112 |
| -2 | -10 | 1.0 | 1.5 | -8.812949 | 4.445223 | -1.040115 |
| -2 | -10 | 3.0 | 1.5 | -8.495821 | 4.445206 | -1.040122 |
| +2 | +10 | 0.2 | 0.5 | 2.678352 | 1.940135 | -0.082743 |
| +2 | +10 | 0.2 | 1.0 | 2.653647 | 2.944255 | -0.452304 |
| +2 | +10 | 0.2 | 1.5 | 2.552066 | 4.445230 | -1.040112 |
| +2 | +10 | 1.0 | 1.5 | 2.056558 | 4.445223 | -1.040115 |
| +2 | +10 | 3.0 | 1.5 | 1.041530 | 4.445206 | -1.040122 |

**Table 3.4:** Numerical values of skin-friction coefficients, Nusselt number and Sherwood number for $Pr = 0.71, v_0 = 0.5$, $M = 0.2$ and $Sc = 0.22$.

| $Gr$ | $Gm$ | $Sr$ | $Df$ | $C_{fa}$ | $Nu_a$ | $Sh_a$ |
|---|---|---|---|---|---|---|
| +2 | +10 | 2.0 | 0.03 | 2.694813 | 1.934002 | 0.087057 |
| + 2 | +10 | 0.4 | 0.15 | 2.010203 | 1.517713 | 0.495823 |
| +2 | +10 | 0.1 | 0.60 | 1.924497 | 1.364407 | 0.575145 |
| -2 | -10 | 2.0 | 0.03 | -7.865007 | 1.934002 | 0.087057 |
| -2 | -10 | 0.4 | 0.15 | -7.180397 | 1.517713 | 0.495823 |
| -2 | -10 | 0.1 | 0.60 | -7.094691 | 1.364407 | 0.575145 |

# Chapter-4

## Conclusions

Magnetohydrodynamics (MHD) heat and mass transfer flow of a viscous incompressible fluid past a vertical flat plate have been studied applying different (steady and unsteady) flow conditions. Two types of convection, free convection and combined (free-forced) convection have been considered. The thermal-diffusion and diffusion-thermo effects have been studied extensively in each flow conditions.

Using usual similarity transformations, the governing equations have been transformed into non-linear ordinary differential equations. The similarity solutions are obtained numerically by using Nachtsheim-Swigert shooting iteration technique with Runge-Kutta sixth-order integration method. Since no experimental results of the corresponding studies are available, so the obtained numerical results are compared with that of established results. As for example, qualitative agreement of our results with Kafoussias (1990) is excellent.

All of the models studied in this dissertation show a decreasing effect on velocity, temperature and concentration profiles as the suction parameter increases. On the other hand, the magnetic parameter shows decreasing effects on velocity profiles and increasing effect on temperature as well as concentration profiles. The presented analysis has also shown that the flow field is appreciably influenced by the Soret and Dufour effects. Therefore, we can conclude that for fluids with medium molecular weight ($H_2$, air), the thermal-diffusion and diffusion-thermo effects should not be neglected.

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