A STUDY ON NONBONDAGE NUMBER, CONNECTED DOMINATION NUMBER AND TOTAL GLOBAL DOMINATION NUMBER OF GRAPHS

by
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A dissertation submitted in partial fulfillment of the
requirement for the award of the degree
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Candidate's Declaration

I hereby declare that the work which is being presented in this thesis titled "A STUDY ON NONBONDAGE NUMBER, CONNECTED DOMINATION NUMBER AND TOTAL GLOBAL DOMINATION NUMBER OF GRAPHS" submitted in partial fulfillment of the requirement for the award of the degree of MASTER OF PHILOSOPHY in Mathematics, in the Department of Mathematics, Bangladesh University of Engineering and Technology, Dhaka is an authentic record of my own work.

The matter presented in this thesis has not been submitted by me for the award of any other degree in this or any other university.

Date: March 15, 2006

(Taposh Kumar Das)

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INDEX OF SYMBOLS

$V(G)$	- the vertex set of a graph G
$E(G)$	- the edge set of a graph G
P_n, C_n, K_n	- respectively denote the path, cycle and complete graph of n vertices.
$K_{m,n}$	- complete bipartite graph.
$N(u)$	- set of all vertices adjacent to u
$N[u]$	- the closed neighborhood of a vertex u, that is, $\{u\} \cup N(u)$
$d(u,v)$	- the length of the shortest path from u to v.
$N_2(u)$	- set of all vertices v in G with d(u , v) = 2.
$diam(G)$	- the diameter of a connected graph G
$\lceil x \rceil$	- the greatest integer not exceeding x
$\lfloor x \rfloor$	- the least integer not less than x
$\langle S \rangle$	- subgraph induced by a subset S of V(G)
$\overline{G}$	- the complement of a graph G
$\lvert S \rvert$	- the number of elements of a set S
$G-v$	- the graph obtained from G by removing a vertex v
$G-e$	- the graph obtained from G by removing a vertex e
$G+e$	- the graph obtained from G by adding vertex e

$\alpha_0(G)$	- vertex covering number of G.
$\alpha_1(G)$	- edge covering number of G.
$\beta_0(G)$	- the vertex independent number of G
$\beta_1(G)$	- the edge independent number of G
$\delta(G)$	- minimum degree of G
$\Delta(G)$	- maximum degree of G
$\gamma(G)$	- domination number of G
$\gamma_t(G)$	- total domination number of G
$\alpha_{00c}(G)$	- the connected domination number of G
$\gamma_i(G)$	- the independent domination number of G
$\gamma_g(G)$	- global domination number of G
$\gamma_{tg}(G)$	- total global domination number of G
$b(G)$	- bondage number of a graph G
$b_n(G)$	- nonbondage number of a graph G

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ABSTRACT

A set D of vertices in a graph G = (V, E) is a dominating set of G if every vertex in V-D is adjacent to some vertex in D. The domination number of G is the minimum cardinality taken over all minimal dominating sets in G and is denoted by $\gamma(G)$. The nonbondage number of a graph G is the maximum cardinality among all sets of edges $X \subseteq E(G)$ such that $\gamma(G-X) = \gamma(G)$ and it is denoted by $b_n(G)$. In the same way, we can define total domination number, connected domination number, global domination number and total global domination number of graphs. Different types of methods are available depending on types of the problems. Some exact values for the nonbondage number of graphs are found. Upper bounds are obtained for nonbondage number of a graph and the exact values are determined for several classes of graphs. We have illustrated with examples some various results for the connected domination number of graphs of standard graphs with better explanation.The exact values of connected domination number and global domination number for some standard graphs are calculated with the help of methods used by Kulli, Shampathkumar, Janakiram etc. We have also established some theorems related with the total global domination number of graphs.

In order to minimize the direct communication links among the transmitting stations under communication networks where maximum number of links that should be dropped to accomplish this task is the nonbondage of a graph.

In the similar way we can also apply connected domination number and total global domination number in various ways.

CHAPTER ONE

INTRODUCTION

The theory of dominating sets, introduced formally by Ore [27] and Berge [5], is currently receiving much attention in the literature of graph theory. Berge called the domination as external stability and domination number as coefficient of external stability. Ore introduced the word domination in his famous book 'Theory of Graphs' published in 1962. This concept lived almost in hibernation until 1975 when E. J. Cockayne and S. T. Hedetniemi [13] published their paper 'Towards a theory of Domination in Graphs' which appeared in 'Networks' in 1977. This paper brought to light new ideas and potentiality of being applied in variety of areas. A well known problem involving dominating sets (often called the five queen's problem) is to determine the smallest number of queen's which can be placed on a chessboard so that every square is dominated by at least one queen. The evolution of domination in graphs has been supported by hundreds of researchers. Hedetniemi wrote, "It has been said, " fear is to domination as love is to dominion". I have often wondered if we should have changed this term domination, but we accepted Ore's terminology. I can't help thinking that we would have sent a more positive message to researchers in this field had we changed Ore's terminology to the domination number of a graph". S.T.Hedetniemi and R.Laskar attributed the following factors to the growth in the number of domination papers [20]:

a) the diversity of the applications to both real-world and other mathematical covering' or 'location' problems,

b) the wide variety of domination parameters that can be defined,

c) the NP-completeness of the basic domination problem, its close natural' relationships to other NP-complete problems, and the subsequent interest in finding polynomial time solutions to domination problems in special classes of graphs.

Application of domination in communication networks have been discussed by C. L. Liu [25], P. J. Slater [34]. There are numerous papers on various aspects of domination theory.

The domination theory has gained due to the inspiring contributions by eminent graph theorists as C. Berge, E. J. Cockayne, S.T. Hedetniemi, R. C. Laskar, R. B. Allan, P. J. Slater, E. Sampathkumar, V. R. Kulli etc.

In the second chapter, we have discussed about the nonbondage number of graphs. We have extended some proofs of the theorem given in [22]. Some upper as well as lower bounds for nonbondage nnmber of graphs have been obtained. Some exact valnes are also obtained in this chapter.

Chapter three deals with the connected domination number of graphs. In this chapter, we have found some upper aud lower bounds for connected graphs. Some exact values are also found.

In the final chapter, the concept of total global domination number has been introduced. Some alternate proofs of total global domination number for some standard graphs have been obtained.

Now we present the basic definitions and notations which are used in the subsequent chapters. For any nndefined terms, we refer F. Harary [18].

We consider only finite undirected graphs with neither loops nor multiple edges.

Graph:

A graph G consists of a set V of vertices and a collection E (not necessary a set) of unordered pairs of vertices called edges. A graph is symbolically represented as G = (V, E) .The order of a graph is the nnmber of its vertices. and it size is the number of its edges .If u and v are two vertices of a graph and if the unordered pair {u, v} is an edge denoted by e. we say that e joined u and v or that it is an edge between u and v. In this cases, the vertices u and v are said to be incident on e and e is incident to both u and v. Two or more edges that join some pair of distinct vertices are called parallel edges. An edge represented by an unordered pair in which the two elements are not distinct is known as a loop. A graph with no loops is a multigraph. A graph with at least one loop is a pseudograph. A simple graph is a graph with no parallel edges and loops.

Isolated vertex, end vertex and support:

A vertex of a graph G is called an isolated vertex of G if it has degree zero. A vertex of degree 1 is called an end vertex or pendent vertex. Any vertex which is adjacent to a pendant vertex is known as a support.

Adjacent vertices, neighborhood sets:

Two vertices joined by an edge are said to be adjacent or neighbors. The set of all neighbors of a fixed vertex u of a graph G is called the neighborhood set of u and is denoted by N(u).

The open neighborhood of u is

$$N(u) = \{v \in V : u , v \in E\}$$

and the closed neighborhood of u is

$$N[u] = \{u\} \cup N(u).$$

For a set S of vertices, the open neighborhood of S is defined by

$$N(S) = \bigcup_{u \in S} N(u).$$

Subgraphs

Let G be a graph with vertex set V(G) and edge set E(G). Then a graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case, G is called the super graph of H.

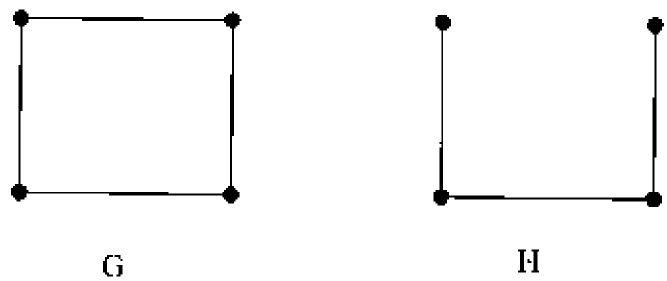


Figure-1.1

H is a subgraph of G and G is the super graph of H.

Proper subgraph:

If $H \subseteq G$ but $V(H) \neq V(G)$ or $E(H) \neq E(G)$, then H is called a proper subgraph of G. From the figure 1.1, we see that H is the proper sub graph of G.

Spanning subgraph:

Let G be a graph. Then H is called a spanning subgraph of G if H has exactly the same vertex set as G. From the figure-1.1, H is a spanning sub graph of G.

Induced subgraph:

Let U be a non-empty subset of the vertex set V of G. Then the subgraph G[U] of G induced by U is a graph having vertex set U and edge set consisting of those edges of G that have both ends in U.

Similarly let F be a non-empty subset of the edge set E of G. Then the Subgraph G [F] of G induced by F is a graph whose vertex set is the set of ends of edges in F and whose edge set is F.

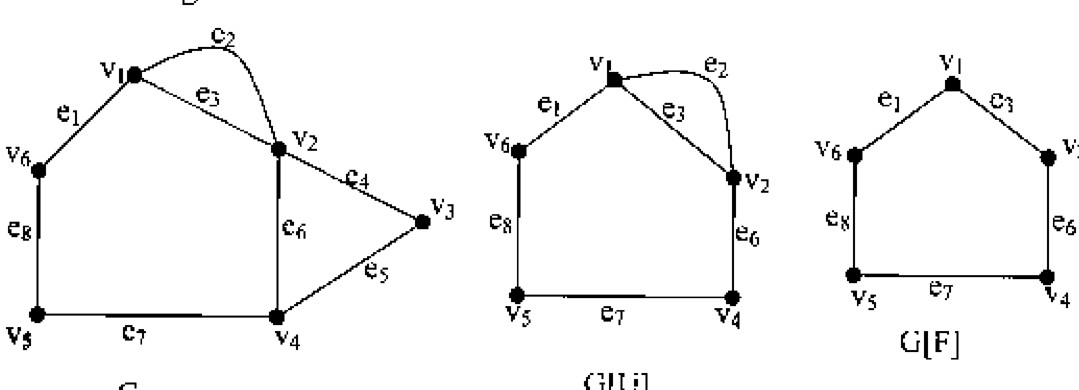


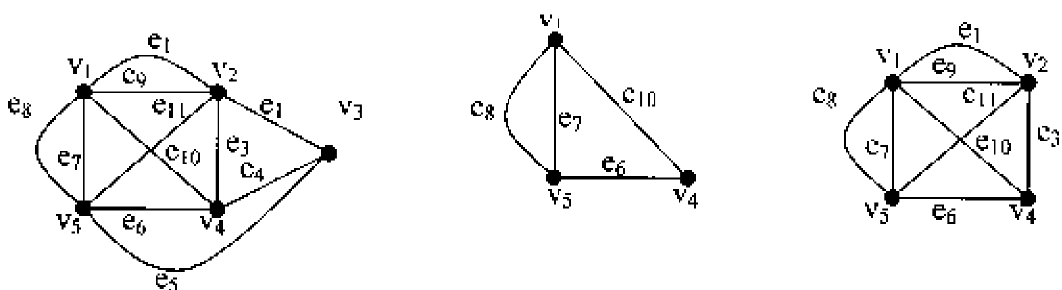
Figure-1.2 , G[U] and G[F] for $U=\{v_1, v_2, v_4, v_5, v_6\}$ and $F = \{e_1, e_3, e_6, e_7, e_8\}$

Vertex deleted and edge deleted subgraph:

Let $u \in V(G)$. Then the induced subgraph $< V(G) - \{u\} >$ denoted by G- u is a subgraph of G obtained by the removal of u.

If $e \in E(G)$,then the spanning subgraph of G with edge set $E(G)- \{e\}$denoted by G - e is the subgraph of G obtained by the removal of e.

For the graph G of Figure-1.3, the followings are the vertex deleted and edge deleted subgraphs.



G (1) G-U (2) G-F

Figure-1.3

(1) G-U , where $U=\{v_2, v_3\}$ and (2) G-F where $F= \{e_2, e_4, e_5\}$

Figure (1) is a vertex deleted and (2) is a edge deleted subgraphs of G.

The minimnm and the maximum degrees of vertices of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively.

Complete graph:

A simple graph G in which each pair of distinct vertices is joined by an edge is called a complete graph of G.

Thus, a graph G with p vertices is complete if it has as many edges as possible provided that there are no loops and no parallel edges.

If a complete graph G has p vertices $v_1, v_2, \ldots\ldots v_p$, then

$$G = \{(v_i, v_j): v_i \neq v_j \ ; i,j = 1,2,3,\ldots\ldots\ldots p\}.$$

The complete graph of n vertices is denoted by K_n.

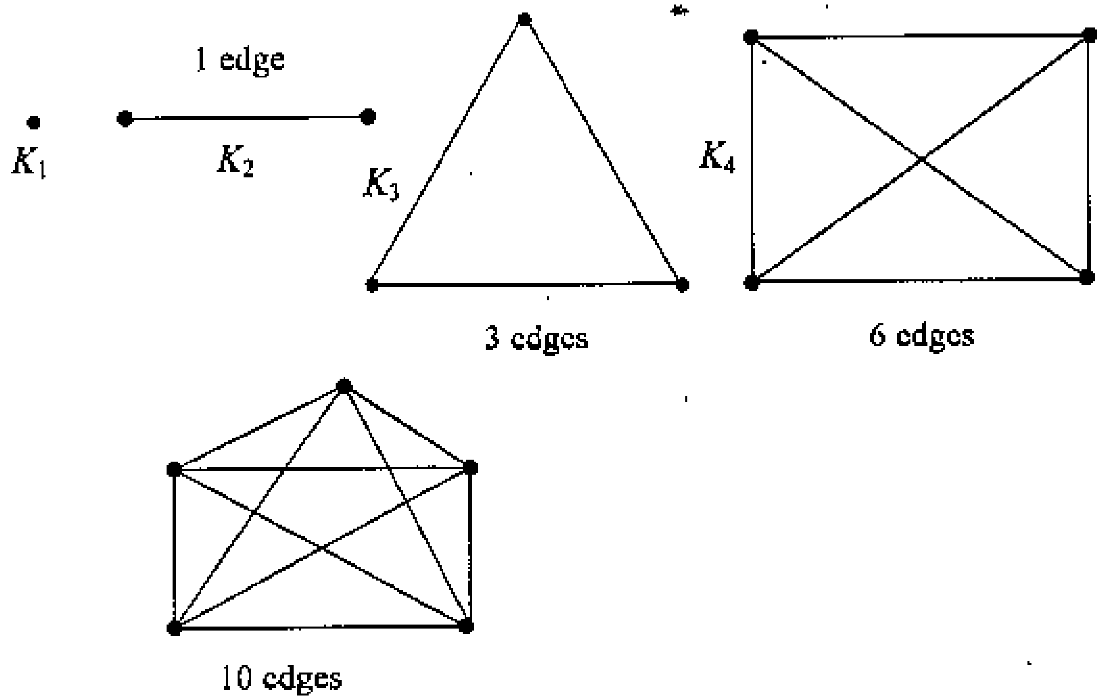


Figure 1.4 The complete graphs on at most 5 vertices.

Null graph:

A graph of order n and size zero is called a null graph or totally disconnected graph, and is denoted by $\overline{K}_n$. Thus $E(\overline{K}_n) = \phi$.

$\overline{K}_1$ The following are the examples of null graph upto the order five.

(1) $\overline{K}_1$: • (2) $\overline{K}_2$: • • (3) $\overline{K}_3$: • • • (4) $\overline{K}_4$: • • • • (5) $\overline{K}_5$: • • • • •

Every vertex of a null graph is an isolated vertex. Further a graph of order n is a null graph if and only if it is a regular graph of regularity zero.

Bipartite Graph:

An empty graph is a graph with no edges. A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint non-empty subsets V_1, and V_2 (i.e., $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$) such that each edge of G has one end in V_1 and one end in V_2 so that no edge in G connects either two vertices in V_1, or two vertices in V_2. The partition $V = V_1 \cup V_2$ is called a bipartition of G.

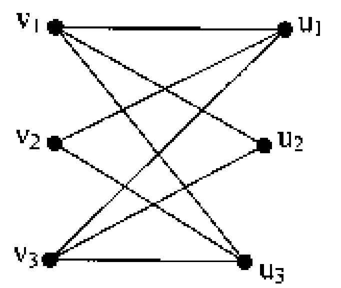


Figure-1.5

Figure-1.5 is a Bipartite graph,where $V_1=\{v_1,v_2,v_3\}$ and $V_2=\{u_1,u_2,u_3\}$

Hamiltonian Graphs:

A Hamiltonian path in a graph G is a path which contains every vertex of G. A Hamiltonian cycle (or Hamiltonian circuit) in a graph G is a cycle which contains every vertex of G. A graph G is called Hamiltonian if it has a Hamiltonian cycle.

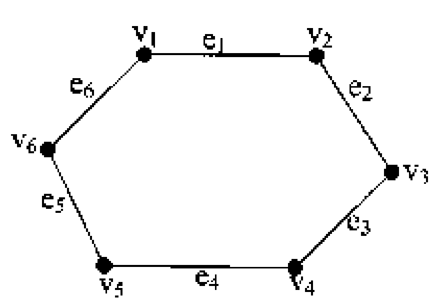


Figure-1.6

From the figure-1.6 , C= $v_1v_2v_3v_4v_5v_6$ is a Hamiltonian cycle and it contains all the vertices of Fig-1.6 , so figure-1.6 is a Hamiltonian cycle.

Complete Bipartite Graph:

A complete bipartite graph is a simple bipartite graph G, with bipartition $V = V_1 \cup V_2$ in which every vertex in V_1. is joined to every vertex in V_2. If V_1 has m vertices and V_2 has n vertices, such a graph is denoted by $K_{m,n}$.

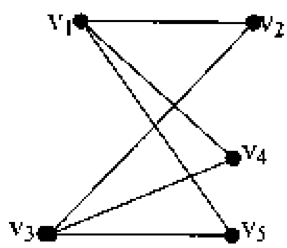


Figure-1.7

The figure-1.7 is a complete bipartite graph with bipartition $V=V_1 \cup V_2$, where $V_1=\{v_1,v_3\}$ and $V_2=\{v_2,v_4,v_5\}$.

Complement of a graph:

The complement $\overline{G}$ of a graph G is the graph with vertex set V (G) such that any two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Connected graph:

A graph G is said to be connected if every two vertices of G are connected. Otherwise, G is a disconnected graph.

Let C(u) denote the set of all vertices in G that are connected to u. Then the subgraph of G induced by u is called the connected component containing u. A maximal connected subgraph of G is a component of G. Thus, a disconnected graph has at least two components. The number of components of G is denoted by $\omega(G)$.

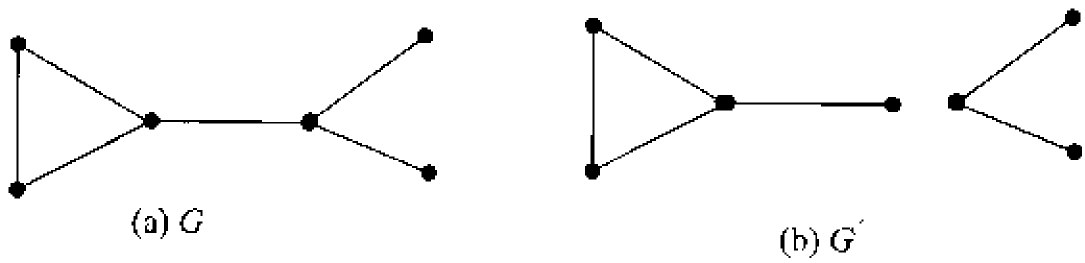


Figure-1.8

In Figure 1.8(a), G is a connected graph and in Figure 1.8(b), G′ is a disconnected graph.

Distance of two vertices:

The distance d(u, v) between two vertices u and v is the length of a shortest distance u-v path in G. If there is no u - v path in G. then we define d(u, v) = 0.

Second neijhborhood:

If v is a vertex of G, then we define the second neighborhood N_2 (v) of v as

$$N_2(v) = \{u : u \in V(G) \text{ and } d(u, v) = 2 \text{ in } G\}.$$

In view of this, we also write N_1 (v) for N(v).

Walk in a graph:

Let G be a graph. Then a walk in G is a finite sequence

$$W = v_0e_1v_1e_2v_2 \ldots \ldots v_{k-1}\, e_k\, v_k$$

whose terms are alternately vertices and edges such that, for i = 1,2......k, the edge e_i has ends v_{i-1}, and v_i.

The above walk W is a walk from origin v_o to terminus v_k. The integer k, the number of edges in the walk, is called the length of W.

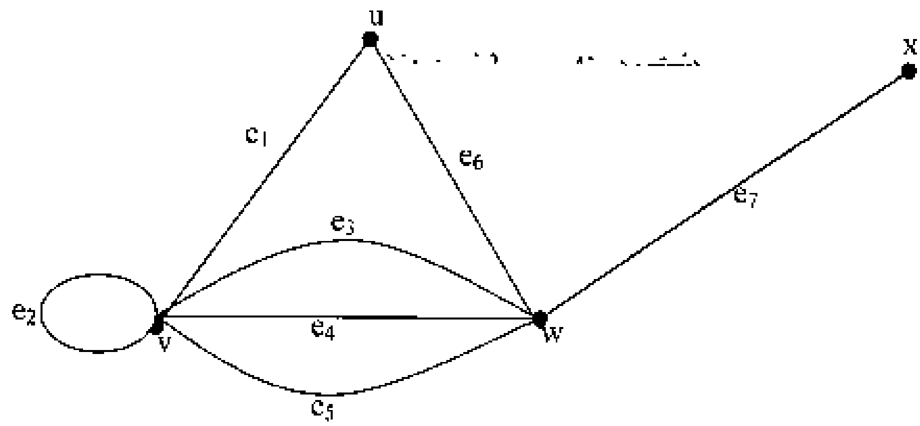


Figure-1.9

For example, in the graph G of Figure 1.9,

$W = ue_1ve_2ve_3we_5ve_4w$, is a walk of length 5 .

In other words, the number of edges in W is called the length of W. If the sequence of W consists solely of one vertex, i.e., $W = v_o$,then W is a trivial walk with length 0.

Trail of a graph:

If the edges $e_1, e_2, \ldots\ldots\ldots\ldots e_k$ of the walk ,

$W = v_0e_1v_1e_2v_2$ ------------ $v_{k-1}e_kv_k$,

are all distinct. then W is called a trail.

In other words. a trail is a walk in which no edge is repeated. In the graph G of Figure 1.9,

$T = xe_7we_5ve_2ve_3w$,

is a trail of length 4.

Paths of a graph:

If the vertices of a walk

$W = v_0 e_1 v_1 e_2 v_2 \ldots\ldots\ldots v_{k-1} e_k v_k,$

are all distinct, then W is called a path. A path with n vertices is denoted by P_n which has length n-1.

In other words, a path is a walk in which no vertex is repeated.

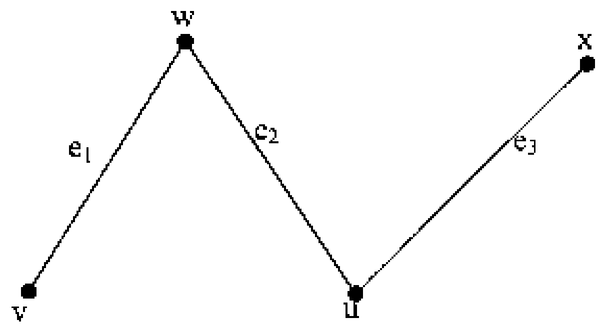


Figure-1.10

From the above graph we have a path P= $ve_1we_2ue_3x$. Thus, in a path no edge can be repeated either, and so every path is a trail. The converse of this statement is not true.

Cycle of a graph:

A non-trivial closed trail in a graph G is called a cycle if its origin and internal vertices are distinct. In detail, the closed trail

$$C = v_1 v_2 \ldots\ldots\ldots v_n v_1$$

is a cycle if

(i) C has at least one edge and

(ii) $v_1, v_2 \ldots\ldots\ldots, v_n$ are all distinct.

A cycle of length k, i.e., with k edges. is called a k -cycle is called odd or even depending on whether k is odd or even. A 3-cycle is often called a triangle.

A cycle with n vertices is denoted by C_n.

Remark: A u-v walk is called closed or open according as u=v or u ≠ v. The vertices $v_1, v_2, \ldots\ldots, v_{k-1}$ in the walk

$$W = v_0 e_1 v_1 e_2 v_2 \ldots\ldots v_{k-1} e_k v_k$$

are called internal vertices. In the graph G of Figure 1.9, $C = ve_4we_6ue_1v$ is a cycle.

Acyclic graph:

A graph G is called acyclic if it has no cycle.

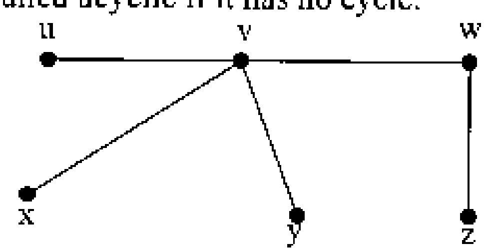


Figure-1.11

Figure 1.11 is an acyclic graph.

Tree of graph:

Let G be a graph. If G is a connected acyclic graph, then it is called a tree.

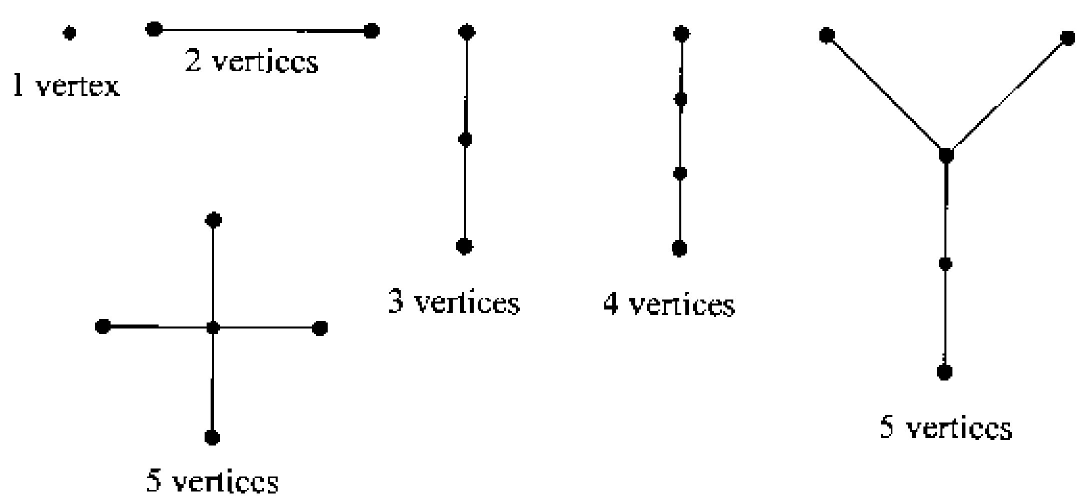


Figure 1.12: Trees with at most five vertices.

A tree on n vertices is denoted by T_n, which has exactly two pendent vertices.

Join of a graph:

Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their join $G_1 + G_2$ is a graph whose vertex set is $V_1 \cup V_2$ and edge set $E_1 \cup E_2 = \{uv: u \in V_1 \ \& \ v \in V_2\}$.

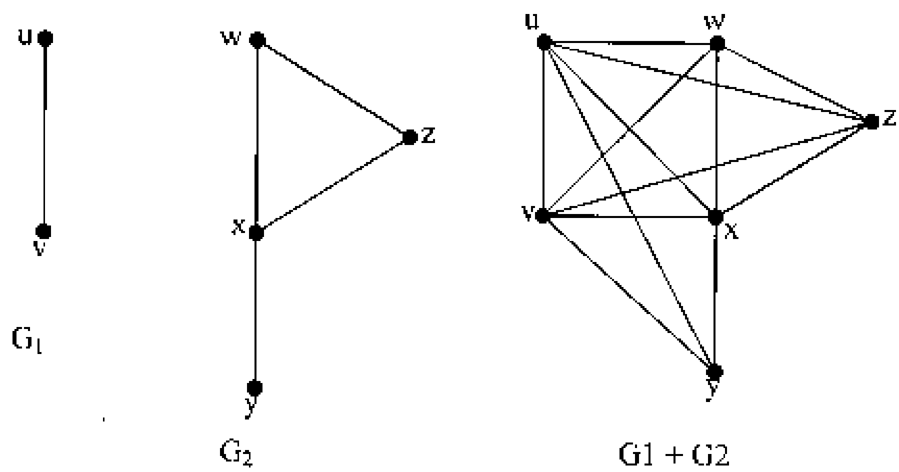


Figure-1.13

Wheel of a Graph:

A wheel is a graph obtained from a cycle by adding a new vertex and edges joining it to all the vertices of the cycle.

A wheel with n vertices is denoted by W_n, and $W_n = K_1 + C_{n+1}$

Connectivity of a Graph:

The connectivity k of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. A graph G is said to be n - connected if $k \geq n$.

Edge Connectivity:

The edge connectivity λ of a graph G is the minimum number of edges whose removal results in a disconnected graph. A graph G is said to be n–edge connected if $\lambda \geq n$.

Matching of a Graph:

A subset M of edges of G, is called a matching if for any two edges e and f in M, the two end vertices of e are both different from the two end vertices of f.

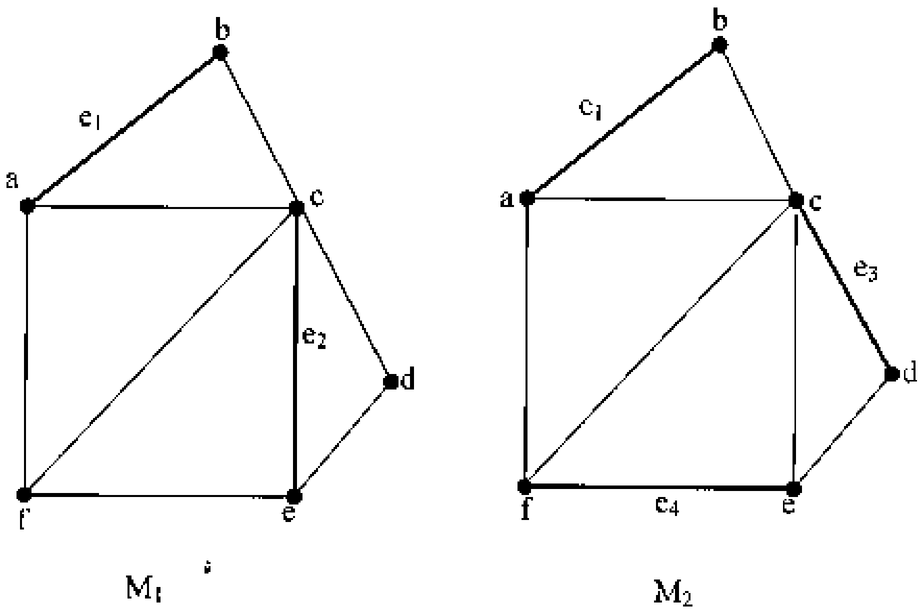


Figure-1.14

In the graph G of Figure 1.14, $M_1 = \{e_1, e_2\}$ and $M_2 = \{e_1, e_3, e\}$ are both matching.

Saturation:

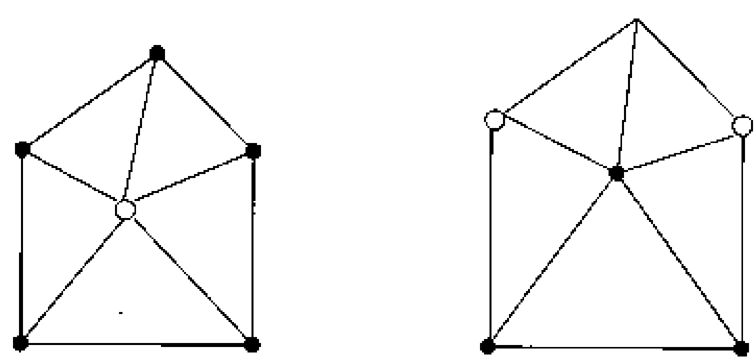
Let G be a graph and let $v \in V(G)$. Then if v is the end vertex of some edge in the matching M, then v is said to be M - saturated and we say " M saturates V. " Otherwise, V is M - unsaturated. Thus, in Figure 1.12, a, b, c and e are all M_1 - saturated while f and d are both M_1 -unsaturated; every vertex of G is M_2 - saturated.

Perfect Matching:

If M is a matching in G such that every vertex of G is M -saturated, then M is called a perfect matching. The matching $M_2 = \{e_1, e_3, e_4\}$ of Figure 1.12, is a perfect matching.

Independent Sets:

A subset S of vertices in a graph G is said to be an independent set of G if no two vertices of S are adjacent in G. An independent set is maximum if G has no independent set S' with $|S'| > |S|$.



An independent set   A maximum independent set

Figure 1.15 Independent set and maximum independent set.

A set S of edges of G is said to be independent if no two of the edges in S are adjacent.

Independent Number:

The maximum number of vertices in an independent set is called the independent number of G and is denoted by β_0 (G).

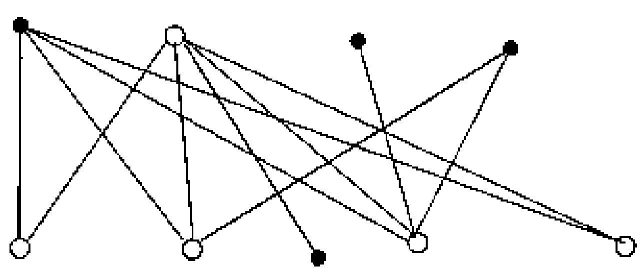
Edge Independent Number:

The maximum cardinality of an independent set of edges of G is called the edge independent number of G and is denoted by β_1 (G), which is also called the matching number of G. The minimum matching number $\beta_1(G)$ of G, is-the minimum number of edges in a maximal independent edge set.

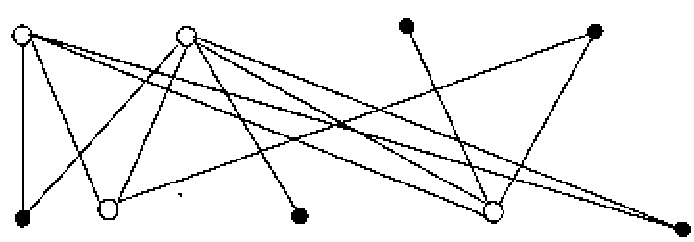
An edge analogue of an independent set is a set of links no two of which are adjacent, i.e., a matching.

Covering of a Graph:

A subset K of vertices in a graph G such that every edge of G has at least one end in K is called a covering of G .The number of vertices in a minimum covering of G is called the covering number of G and is denoted by $\alpha_0(G)$. The edge analogue of a covering is called an edge covering.



(1)



(2)

Figure-1.16

In Fig-1.16, (1) A covering and (2) a maximum covering (Shown by the white vertices)

An edge covering of a graph G is a subset L of edges of G such that each vertex of G is an end of some edge in L. The edge coverings do not always exist. The number of edges in a minimum edge covering of G is denoted by $\alpha_1(G)$. The number $\alpha_1(G)$ is called the edge covering number of G.

Definition:

If x is a real number. $\lceil x \rceil$ and $\lfloor x \rfloor$ denote respectively the least integer not less than x and the greatest integer not greater than x.

Now we present the following definitions of various types of domination in a graph.

Dominating Set:

A set $D \subseteq V$ is said to be a dominating set in G if every vertex in V - D is adjacent to some vertex in D. The domination number of G is the minimum cardinality taken over all minimal dominating sets in G and is denoted by $\gamma(G)$.

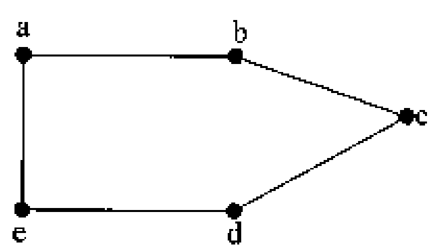


Figure-1.17

In figure 1.17, D={a,c} is a dominating set.

Independent Dominating Set:

A dominating set D of a graph G is called an independent dominating set of G if D is independent in G. The cardinality of the smallest independent dominating set of G is called the independent domination number of G and is denoted by $\gamma_i(G)$.

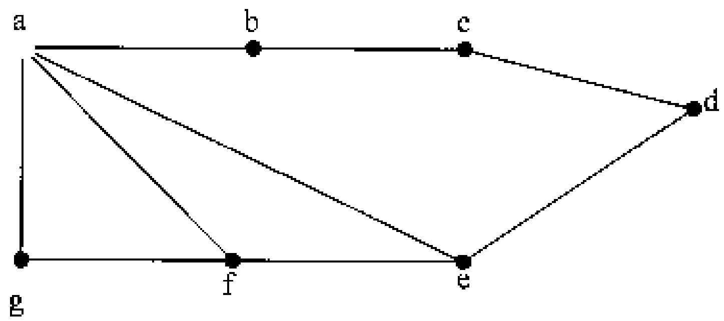


Figure-1.18.

Here D={a,d} is an independent dominating set.

Total Dominating Set:

A dominating set D of a graph G without isolated vertices is called a total dominating set of G if the subgraph G[D] induced by D has no isolated vertices. The cardinality of the smallest total dominating set of G is called the total domination number of G and is denoted by $\gamma_t(G)$.

Connected Dominating Set:

A dominating set D of a connected graph G is called a connected dominating set of G if G[D] is connected. The cardinality of the smallest connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$.

For any connected graph G with $\Delta(G) < n - 1$.

$\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$ .Total dominating sets were first defined and studied by Cockayne. Dawes and Hedetniemi [16]. In addition to several new results involving total domination.

Bondage Number:

The bondage number b(G) of a nonempty graph G is the minimum cardinality among all sets of edges E for which

$$\gamma(G - E) > \gamma(G).$$

Thus. the bondage number of G is the smallest nnmber of edges where removal will render every minimum dominating set in G a "non-dominating" set in the resultant spanning subgraph.

Since the domination number of every spanning subgraph of a non-empty graph G is at least as great as $\gamma(G)$, the bondage number of a non-empty graph is well-defined.

Cobondage Number:

The cobondage number cb(G) of a graph G is the minimum cardinality among the sets of edges $X \subseteq P_2(V) - E$, where

$$P_2(V) = \{X \subseteq V : |X| = 2\}$$

such that $\gamma(G + X) < \gamma(G)$. A γ - set is a minimum dominating set.

If we compare $\gamma(G)$ and $\gamma(H)$, when H is a spanning subgraph of G, it is immediate that $\gamma(H)$ cannot be less than $\gamma(G)$. Every conuected graph G has a spanning tree T with $\gamma(G) = \gamma(T)$ and so, in general, a graph will have non-empty sets of edges $F \subseteq E$ for which $\gamma(G - F) = \gamma(G)$. Sucb a set F will be called an inessential set of edges in G.

However, many graphs also possess single edges e for which $\gamma(G-e) > \gamma(G)$.

The bondage number b(G) of a graph G is the minimum cardinality of a set of edges of G whose removal from G results in a graph with domination number larger than that of G.

J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts [8], introduced the bondage number b(G) of a graph G. In [8], Fink et al. have obtained sharp bounds for b(G) and the exact values of b(G) for several classes of graphs have also been determined.

Nonbondage Number:

The nonbondage number of a graph G is the maximum cardinality among all sets of edges $X \subseteq E(G)$ such that $\gamma(G-X)= \gamma(G)$ and it is denoted by $b_n(G)$.

Total Global Dominating Set and Total Global Domination Number:

A total dominating set T of G is a total global dominating set (t.g.d set) if T is also a total dominating set of $\overline{G}$. The total global domination number $\gamma_{tg}(G)$ of G is the minimum cardinality of a t.g.d. set.

V.R Kulli and B. Janakiram [22] have obtained the following theorem and corollary of some standard graphs.

Theorem 1.1 For any graph G,

$$b_n(G) = q - p + \gamma(G).$$

Theorem 1.2 :For any graph,

$$\gamma(G) \leq p - \Delta(G).$$

Corollary 1.1: For any graph G,

$$b_n(G) \leq q - \Delta(G)$$

When $\Delta(G)$ is the maximum degree of G.

Theorem 1.3: For any sub-graph H of G,

$$b_n(H) \leq b_n(G) .$$

Lemma 1.1: For any connected graph G,

$$\left\lceil \frac{diam(G)-2}{3} \right\rceil \leq b_n(G)$$

Where diam(G) is the diameter of G and $\lceil x \rceil$ is the least positive integer not less than x.

Corollary 1.2: If G is a Hamiltonian graph.

$$\text{then } b_n(G) \geq \left\lceil \frac{p}{3} \right\rceil.$$

Theorem 1.4: Let G be a unicyclic graph if $\gamma(G) = \frac{p}{2}$

$$\text{then } b_n(G) \geq \Delta(G).$$

E.Sampathkumar and H.B. Walikar [31] have obtained the following theorem, proposition and corollary of some standard graphs.

Proposition 1.1: For any connected graph G,

$$\gamma(G) \leq \alpha_{ooc}(G).$$

Proposition 1.2: Let G be any graph and H be any spanning subgraph of G. Then every dominating set of H is also a dominating set of G, and consequently $\gamma(G) \leq \gamma(H)$.

Corollary 1.3: Let G be a connected graph and H be any connected spanning subgraph of G. Then every connected dominating set of H is also a connected dominating set of G, and hence $\alpha_{ooc}(G) \le \alpha_{ooc}(H)$.

Propositions 1.3: For any connected graph G of order $p \ge 3$,

$\alpha_{ooc}(G) \le p-2$ and the bound is best possible.

Lemma 1.2: For any connected graph G of order p with maximum degree Δ,

$$\gamma(G) \ge \left[\frac{p}{\Delta+1}\right]$$ Where [x] denotes the greatest integer ≤ x.

Theorem 1.5: For any connected (p,q) graph G with maximum degree Δ,

$$\left[\frac{p}{\Delta+1}\right] \le \alpha_{ooc}(G) \le 2q-p \text{ -------------------- } (1)$$

The lower bound in (1) is attained if and ouly if G has a vertex of full degree (i.e. a vertex of degree p-1), and the upper bound is attained if and only if G is a path.

Theorem 1.6: Let G be a connected graph of order $p \ge 4$ such that both G and $\overline{G}$ are connected.Then , $\alpha_{ooc}(G) + \alpha_{ooc}(\overline{G}) \le p(p-3)$ ....(a) The bound is attained if and only if $G=P_4$

V.R.Kulli and B.Janakiram[23[has obtained the following theorem of some standard graphs.

Theorem 1.7: A total domiuating set T of G is a t.g.d. set if and only if for each vertex $v \in V$ there exists a vertex $u \in T$ such that v is not adjacent to u.

Theorem 1.8: Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex. Then,

(i) $\gamma_{tg}(G) = \gamma_{tg}(\overline{G})$;

(ii) (ii) $\gamma_t(G) \le \gamma_{tg}(G)$;

(iii) $\gamma_g(G) \leq \gamma_{tg}(G)$;

(iv) $\{\gamma_t(G) + \gamma_t(\overline{G})\}/2 \leq \gamma_{tg}(G) \leq \gamma_t(G) + \gamma_t(\overline{G})$.

Theorem 1.9 : Let G be a graph which such that ueither G nor $\overline{G}$ have an isolated vertex. Then $\gamma_{tg}(G) = p$ (p is the number of vertices of G)

if and only it $G = P_4$ (a path on 4 vertices) or mk_2 or $m\bar{k}_2$ where $m \geq 2$.

Theorem1.10 : Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex and T be a γ_t-set of G such that each **x** in T has non-neighbor in T. If there exists a vertex $u \in V-T$ which is adjacent only to vertices in T then ,

$\gamma_{tg}(G) \leq \gamma_t(G) + 2$.

CHAPTER TWO

THE NONBONDAGE NUMBER OF A GRAPH

This chapter deals with the nonbondage nnmber of a graph. We have found some exact values of b_n for any graph G and we have given alternate proof of some standard graphs for nonbondage number of a graph.

Introduction:

First we define the nonbondage number of a graph.

Definition:

A set D of a vertices in a graph G = (V.E) is a dominating set of G if every vertex in V\D is adjacent to at least one vertex in D. The domination number of G is the minimum cardinality of a dominating set of G and we represent it by $\gamma(G)$.

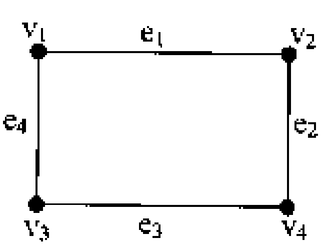
The nonbondage number of a graph G is the maximum cardinality among all sets of edges $X \subseteq E(G)$ snch that $\gamma(G-X) = \gamma(G)$ and it is denoted by $h_n(G)$.

Example:

1.If G is a path of four vertices then $b_n(G) = 1$.

When the vertices of a path is less than or equal to three, then $b_n(G) = 0$.

2. From the following graph we have the domination number $\gamma(G) = 1$.



If we remove the edges e_2 and e_3 then the domination number has no change. But if we remove any other edge with the edges e_2 and e_3 then the domination number is greater than one. Therefore, when $X=\{e_1, e_2\}$ then $\gamma(G-X) = \gamma(G)$. Hence $b_n(G) = 2$.

There are various applications of the nonbondage number of a graph .The one that is discussed most often concerns communication networks .This is an arrangement of establishing a link between two or more sites come under some region. We wish to select the smallest set of sites at which to set up transmitting stations so that every site in the network that does not have a transmitter should receive communication by a direct communication link to one that does have a transmitter. Let the sites represent the vertices of a graph and let the communication links between the sites represent the edges in the graph. By keeping the transmitting station fixed minimize the direct communication links in the network. The maximum number of such links that should be dropped to accomplish this task is the nonbondage number of a graph.

We have illustrated already established proof with better explanation for various standard graphs from [22] of V. R. Kulli and B. Janakiram with examples.

Theorem 2.1: For any graph G,

$$b_n(G) = q - p + \gamma(G)$$

Where q is the number of edges and p is the number of vertices of G.

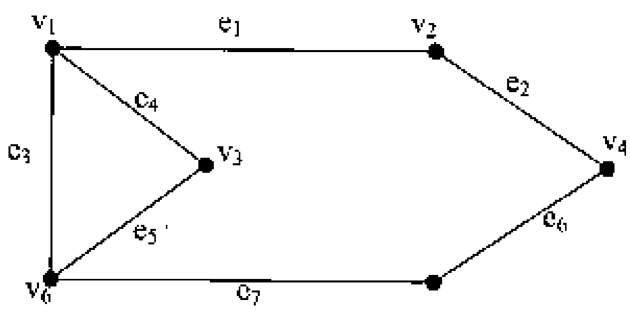
Proof: Let X be a γ-Set of G. For each vertex $u \in V \setminus X$, choose exactly one edge which is incident to u and to a vertex in X. Let E_1 be the set of all such edges. Then clearly $E-E_1$ is a b_n-Set of G. Since number of edge of E_1 is equal to the number of vertices $u \in V \setminus X$. So we have number of edge of

$$E_1 = V - X = p - \gamma(G).$$

b_n = number of edges of E – number of edges of E_1 = q– {p-γ(G)}

= q – p + γ(G). □

Example:



The above graph has seven edges and six vertices and the domination number is two,then nonbondage number $b_n(G) = q - p + \gamma(G) = 7 - 6 + 2 = 3$.If we remove the edges e_1, e_4, and e_6 from the graph then the domination number will not be changed .

Corollary 2.1.1: For any graph G . $b_n(G) \leq q - \Delta(G)$

where Δ(G) is the maximum degree of G.

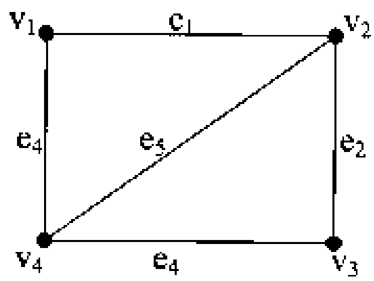
Proof: Since Δ(G) is the maximum degree of G so to get the domination number,we can take various dominating sets . Among all the dominating sets to get domination number, vertex of maximum degree must be included. Again for each vertex $v \in V \setminus D$, choose exactly one edge which is incident to v and to a vertex in dominating set D. Let E_1 be the set of all such edges. Then clearly $E - E_1$ is a nonbondage set of G and Δ (G) ≤ number of edges of E_1 .

$\therefore \quad \Delta(G) \leq |E_1|$

Again, since $b_n = |E| - |E_1|$

$\therefore$ $b_n \leq q - \Delta(G)$ [Since $\Delta(G) \leq |E_1|$ and $q = |E|$] □

Example:
We have from the following graph,



the number of edges q = 5,the nonbondage number, $b_n(G) = 2$ and the maximum degree $\Delta(G) = 3$.

Therefore, 2= 5-3

$$\therefore\ b_n(G) = q - \Delta(G).$$

Theorem 2.2: For any graph,

$$\gamma(G) \leq p - \Delta(G).$$

Proof: We have from the Theorem 2.1 and Corollary 2.2

$$q - p + \gamma(G) \leq q - \Delta(G)$$

$$\text{or}\quad -p + \gamma(G) \leq -\Delta(G)$$

$$\text{or}\quad p - \gamma(G) \geq \Delta(G)$$

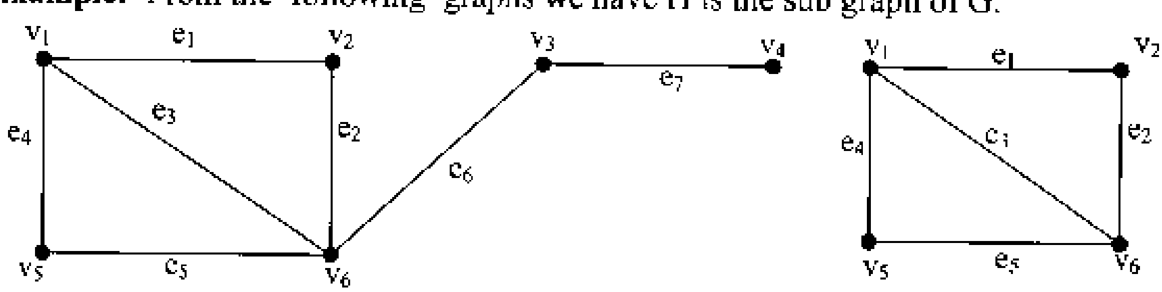
therefore , $\gamma(G) \leq p - \Delta(G)$. □

Theorem 2.3: For any sub-graph H of G,

$$b_n(H) \leq b_n(G).$$

Proof: Since every nonbondage set of H is also a nonbondage set of G. We know $b_n(G)$ is the maximum cardinality among the all sets of edges $X \subseteq E$ such that $V(G-X) = V(G)$. Since H is the snbset of G so, $b_n(H) \leq b_n(G)$.□

Example: From the following graphs we have H is the sub graph of G.



Here $b_n(G) = 3$ and $b_n(H) = 2$, therefore $b_n(H) < b_n(G)$.

Lemma 2.1: For any connected graph G,

$$\left\lceil \frac{diam(G)-2}{3} \right\rceil \leq b_n(G)$$

Where diam(G) is the diameter of G

Proof: We know the diameter of G is the max $\{d(u,v) : u,v \in V\}$

Let P_k be the path of diam(G) +1 vertices

So k = diam(G) +1 -------------------------------- (a)

Then by Theorem 2.1 the nonbondage number of P_k is

$b_n(P_k) = q-p+\gamma(P_k)$.

$= -1 + \gamma(P_k)$----------- [Since for any path p = q+1]

Again we have from Theorem 2.14, $\gamma(P_k) = \left\lceil \frac{k}{3} \right\rceil$

Hence $b_n(P_k) = -1 + \left\lceil \frac{k}{3} \right\rceil$

$= -1 + \left\lceil \frac{diam\,G+1}{3} \right\rceil$ [by (a)]

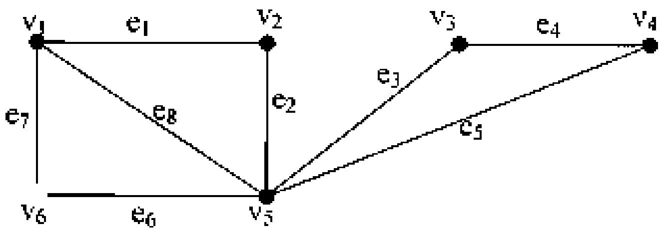
$\therefore b_n(P_k) = \left\lceil \frac{diam(G)-2}{3} \right\rceil$-----------------(b)

Since P_k is the subgraph of G so by the Theorem 2.3 we have,

$b_n(P_k) \leq b_n(G)$

Therefore, $\left\lceil \frac{diam(G)-2}{3} \right\rceil \leq b_n(G)$ [from-b] □

Example:



G

From the above graph, we have diam(G) = 2

Therefore $\frac{diam(G)-2}{3} = \frac{2-2}{3} = 0$

Here nonbondage number $b_n(G) = 3$

$\therefore \left\lceil \frac{diam(G)-2}{3} \right\rceil \le b_n(G)$.

Corollary 2.3.1: If G is a Hamiltonian graph,

then $b_n(G) \ge \left\lceil \frac{P}{3} \right\rceil$

Proof: A graph G is called Hamiltonian graph if it has a Hamiltonian cycle.Let C_p be the Hamiltonian cycle [p is the number of vertex of C] therefore C_p is the spanning subgraph of G.

Since C_p is a subgraph of G so we have from the Theorem[2.3],

$b_n(C_p) \le b_n(G)$ ---------------- (a)

Again by the Theorem 2.1, we have

$b_n(C_p) = q\text{-}p + \gamma(C_p)$ [Since for any cycle p = q]

$\therefore b_n(C_p) = \gamma(C_p)$ -------------------------(b)

But we know from Theorem 2.14, $\gamma(C_p) = \left\lceil \frac{P}{3} \right\rceil$

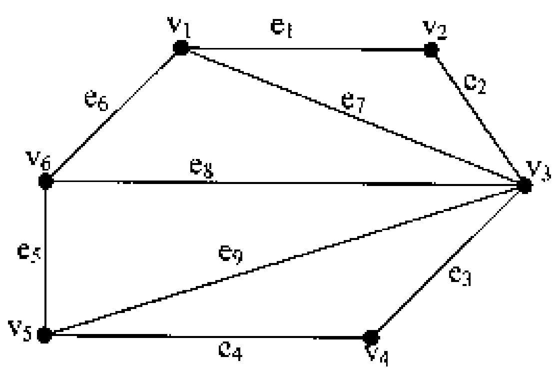
$$\text{Hence } b_n(C_p) = \left\lceil \frac{P}{3} \right\rceil \qquad \text{[from (b)]}$$

So from (a) and (b) we have ,

$$b_n(G) \geq b_n(C_p) = \left\lceil \frac{P}{3} \right\rceil$$

$$\therefore\ b_n(G) \geq \left\lceil \frac{P}{3} \right\rceil \quad \square$$

Example:



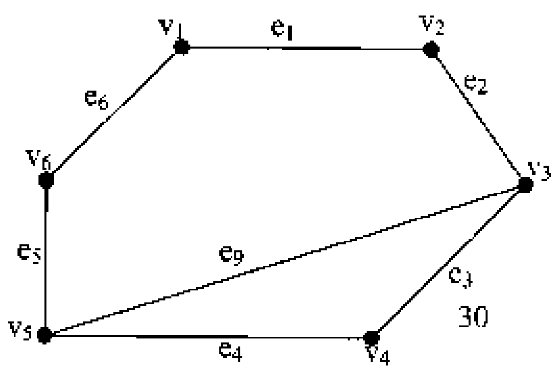
G

Case-I : Here the graph G is a Hamiltonian graph with 6 vertices and $b_n(G) = 5$

and $\left\lceil \frac{P}{3} \right\rceil = \left\lceil \frac{6}{3} \right\rceil = 2.$

So. $b_n(G) > \left\lceil \frac{P}{3} \right\rceil$.

Case-II: If we remove the edges e_7 and e_8 from the graph G then we have the following graph G_1,

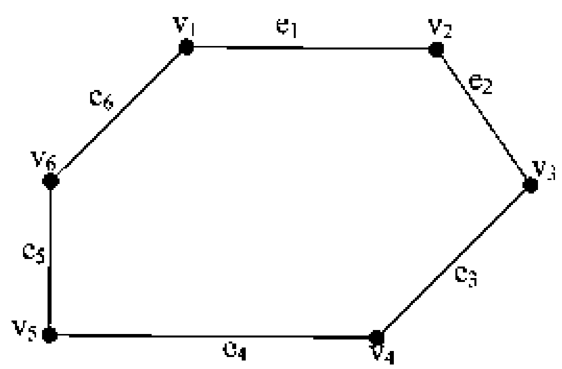


and this graph G_1 is a Hamiltonian graph its nonbondage number $b_n(G_1) = 3$.

But $\left\lceil \frac{P}{3} \right\rceil = \left\lceil \frac{6}{3} \right\rceil = 2$.

Therefore, $b_n(G_1) > \left\lceil \frac{P}{3} \right\rceil$.

CaseIII: If we remove the edges e_7 ,e_8 and e_9 from the graph G then we have the following graph G_2.



G_2

and this graph G_2 is a Hamiltonian graph its nonbondage number $b_n(G_2) = 2$.

But $\left\lceil \frac{P}{3} \right\rceil = \left\lceil \frac{6}{3} \right\rceil = 2$.

Therefore, $b_n(G_2) = \left\lceil \frac{P}{3} \right\rceil$.

So,we have from the above three cases, $b_n(G) \geq \left\lceil \frac{P}{3} \right\rceil$.□

Theorem 2.4: Let G be a unicyclic graph if $\gamma(G) = \frac{p}{2}$

then $b_n(G) \geq \Delta(G)$.

Proof: We know $b_n(G) = q - p + \gamma(G)$ [by Theorem 2.1]

$= \gamma(G)$ [Since for unicyclic graph p = q]

$$= \frac{p}{2} = \frac{q}{2}$$

Therefore $q = b_n(G) + b_n(G)$,

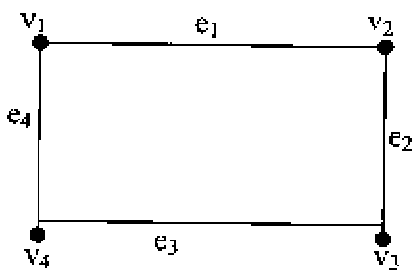
Suppose $b_n(G) < \Delta(G)$,

then $q < b_n(G) + \Delta(G) \leq b_n(G) + q - b_n(G)$ [by cor:2.3.1]

$\therefore \quad q < q$ a contradiction

Hence , $b_n(G) \geq \Delta(G)$. □

Example:



From the above graph we have the domination number $\gamma(G)=2=\frac{4}{2}=\frac{p}{2}$ and it is a unicyclic graph.

Here $b_n(G)=2$ and $\Delta(G) = 2$

Therefore $b_n(G) \geq \Delta(G)$.

New developed theorems:

Lemma 2.2: For any cycle C_p,

$$\gamma(C_p) = b_n(C_p).$$

Proof: Since for any cycle $p = q$

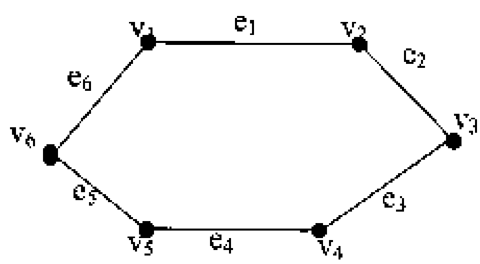
$$\therefore p - q = 0$$

But we have from Theorem[2.1]

$$b_n(C_p) = q - p + \gamma(C_p)$$

$$\therefore b_n(C_p) = \gamma(C_p) \quad \square$$

Example:



C_6

From the above graph we have $\gamma(C_6) = 2$, $b_n(C_6) = 2$

$$\gamma(C_6) = b_n(C_6).$$

Lemma 2.3: For any Hamiltonian graph G with p vertices then,

$$\gamma(G) \geq \frac{4p}{3} - q \text{, [where q is the number of edge of G]}$$

Proof: We have from Theorem[2.1]

$$b_n(G) = q - p + \gamma(G) \text{---------------------------------(a)}$$

But we know when G is Hamiltonian then

$$b_n(G) \geq \left\lceil \frac{p}{3} \right\rceil \text{ [from corollary 2.3.1]}$$

$\therefore\ b_n(G) \geq \frac{p}{3}$ ------------------------ (b)

Therefore by (a) and (b),

$$\frac{p}{3} \leq q - p + \gamma(G)$$

$$\text{or} \quad p \leq 3q - 3p + 3\gamma(G)$$

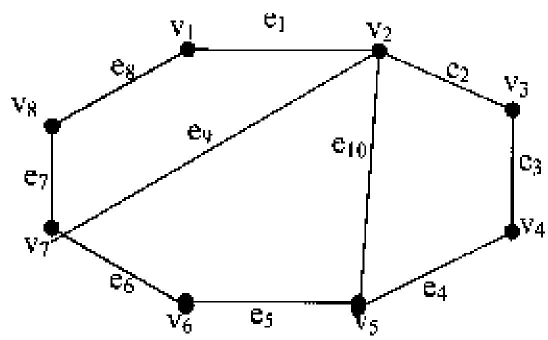
$$\text{or} \quad 4p \leq 3q + 3\gamma(G)$$

$$\therefore \quad p \leq \frac{3}{4}\{q + \gamma(G)\}$$

$$\therefore \quad q + \gamma(G) \geq \frac{4p}{3}$$

$$\therefore \quad \gamma(G) \geq \frac{4p}{3} - q \quad \square$$

Example:



We have from the above graph the domination number $\gamma(G)=4$, vertices number $p=8$ and number of edges $q = 10$.

$$\therefore\ \frac{4p}{3} - q = \frac{4 \times 8}{3} - 10 = \frac{2}{3}$$

Therefore , $\gamma(G) \geq \frac{4p}{3} - q$.

Theorem 2.5: For any unicycle graph $\Delta(G) \le \frac{q}{2}$, when $\gamma(G) = \frac{p}{2}$

Proof: We have from corollary 2.1.1, for any graph G

$$b_n(G) \le q - \Delta(G) \text{ --------------------- (a)}$$

We know for any unicyclic graph p = q

Again $b_n(G) = q - p + \gamma(G)$ (From theorem[2.1])

$$= q - p + \frac{p}{2} \text{ [when } \gamma(G) = \frac{p}{2}\text{]}$$

$$= p - p + \frac{p}{2} \text{ [Since G is unicyclic graph, so p = q]}$$

$$= \frac{p}{2}$$

$$\therefore \quad b_n(G) = \frac{p}{2} = \frac{q}{2}$$

Hence $q = b_n(G) + b_n(G)$ ------------------------ (b)

Suppose $b_n(G) < \Delta(G)$

$\therefore q < b_n(G) + \Delta(G) \le b_n(G) + q - b_n(G)$ [from (a)]

$\therefore q < q$ a contradiction

$\therefore$ $b_n(G)$ is not less than $\Delta(G)$.

Hence $b_n(G) \ge \Delta(G)$

$\therefore$ $\Delta(G) \le b_n(G) \le q - \Delta(G)$ -----[from (a)]

or $\Delta(G) \le q - \Delta(G)$

$\therefore$ $\Delta(G) \le \frac{q}{2}$ □

Theorem 2.6: An edge e = uv is in every b_n set of G and D be the any dominating set know of G, then either $\{u,v\} \subseteq D$ <u>or</u> $\{u, v\} \subseteq V\backslash D$.

Proof: We, the nonbondage number $b_n(G)$ of a graph G is the maximum cardinality among all sets of edges $X \subseteq E$ such that $\gamma(G - X) = \gamma(G)$.

Here, $e = uv \in b_n(G)$.

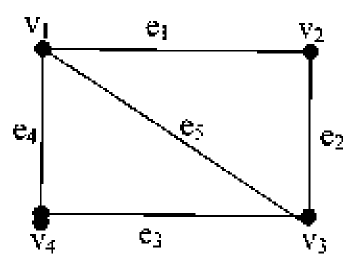
Suppose that $n \in \gamma(G) = D$ and $v \in V\backslash D$.

Since n belongs to the dominating set and v adjacent to u so, when we remove the edge e from G then u and v are not adjacent. For this matter, the domination number will be increase from the previous domination number of G.

Therefore $e \notin b_n(G)$.

So,the edge $e = \{u,v\} \subseteq D$ or $\{u,v\} \subseteq V\backslash D$.

Example: From the following graph,



G

we have the dominating set $D = \{v_1\}$ and the set of nonbondage number $b_n(G)$ is $\{e_2, e_3\}$. Here $e_2 = \{v_2,v_3\} \subset V\backslash D$ and $e_3 = \{v_3, v_4\} \subset V\backslash D$.

We have also obtained the following theorem in addition to the nonbondage number of graphs:

Theorem 2.7: If T be a Spanning tree of a connected graph G then

$$\gamma(T) \geq \gamma(G).$$

Proof: If we remove one or more edges from a connected graph G [Here the vertices of G will be same] then the domination number of the reduced graph will be equal or greater than the domination number of G. If there is no cycle in G then

G will be disconnected when we remove any edge from G . In that place $G = T$, therefore $\gamma(T) = \gamma(G)$.---------------------------------------(1)

Again if G has one or more cycle then to get T we remove some edge from G and hence $\gamma(T) > \gamma(G)$.---(2)

So, we have from (1) and (2)

$\gamma(T) \geq \gamma(G)$. □

Theorem 2.8: If T_1, T_2 ,-------------------T_n are the spanning tree of a connected graph of G then ,$E(T_1) = E(T_2) =$ ----------------------- $= E(T_n)$.

Proof: If we remove any one edge of a cycle then this cycle will be a tree and this tree is called a spanning tree of that cycle . But if we remove two or more edges of a cycle then this cycle will be disconnected graph. So in order to have a spanning tree it is necessary to remove one and only one edge from a cycle.

Now , if G is a connected graph and it has X sub-cycle then to have any spanning tree from G, we must remove exactly X edge from G.

Therefore $E(T_1) = E(G) - X$

$E(T_2) = E(G) - X$

$E(T_n) = E(G) - X$

So, $E(T_1) = E(T_2) =$ ----------------------- $= E(T_n)$. □

Theorem 2.9: For any path or any cycle with vertices k then ,

$$\gamma(c_k) \text{ or } \gamma(p_k) = \lceil k/3 \rceil$$

Where c_k is the cycle of k vertices and p_k is the path of k vertices.

Proof: We know, any vertex of a path dominate maximum two vertices. So, the domination number of P_1, P_2 and P_3 is 1 for each of the three paths. Again the dominating number of P_4, P_5 and P_6 is equal to 2 for each of the three paths. So we see that when the number vertices of path become 1,2 and 3 then the domination number of path is 1 in each cases. When the vertices number becomes 4, 5 and 6 then the domination number of path is 2 and so on. Therefore, we see that when the number of vertices is increased by 3, then in each cases the domination number will be increased by 1 and the domination number will be the same when the number of vertices of the paths exits between the duration which range is three and started from 1 of the first duration.

We have clear idea of the above description from the following table:

Class of vertex number of path.	Domination number of the path .
1——3	1
4——6	2
7——9	3
10——12	4
13——15	5

----------	

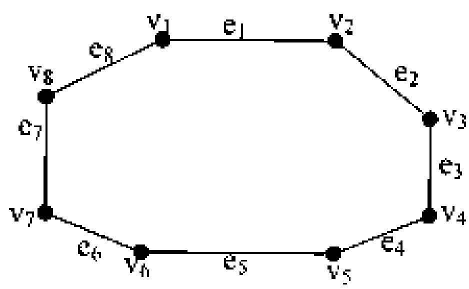
and so on . Here the range of classes of vertices number of path is 3 and the domination number of the paths which number of vertices in any one class is same.

Again $\lceil x \rceil$ is the least positive integer not less then x.

Hence $\gamma(p_k) = \lceil k/3 \rceil$

Similarly we can show that $\gamma(c_k) = \lceil k/3 \rceil$ □

Example:



The above graphs has 8 vertices and 8 edges and the domination number is 3.

Again we have $\lceil k/3 \rceil = \lceil 8/3 \rceil = 3$ = the domination number $\gamma(c_k)$.

CHAPTER THREE

THE CONNECTED DOMINATION NUMBER OF A GRAPH

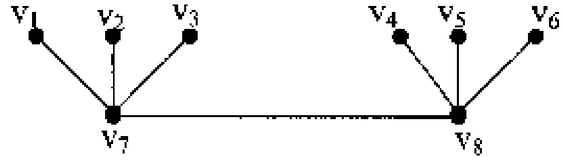
This chapter describes the connected domination number of graphs and we give the exact value of connected domination number. In this chapter, we find the upper and lower bound of some connected graphs and comparing the connected domination number with the domination number of some graphs.

Introduction:

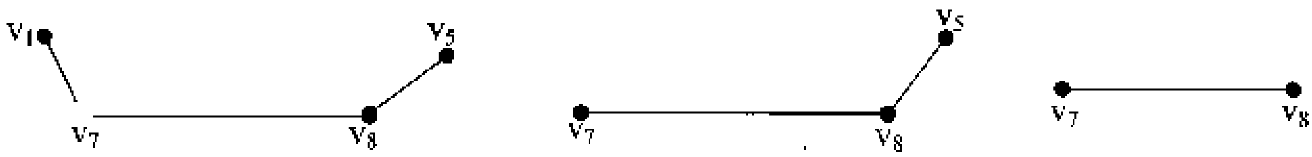
First we define the connected domination number and domination number of a graph.

Definition: A sub set D of the vertex set V(G) of a graph G is said to be dominating set if every vertex of G not in D is adjacent to at least one vertex in D . A dominating set D is said to be a connected dominating set if the subgraph <D> induce by D is connected in G .The minimum of the cardinalities of the connected dominating sets of G is called the connected domination number $\alpha_{ooc}(G)$ of G. Harary ([18]p.97),by regarding each vertex as covering itself and two vertices as cover each other if they are adjacent, denotes by $\alpha oo(G)$ the minimum number of vertices needed to cover V(G).

Example:

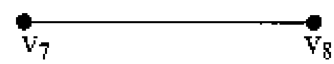


Form the above graph we have more than one connected dominating subgraph, such as



------------- and so on .

But The connected dominating sub graph which contained the minimnm number of vertices is,



Therefore the connected domination nnmber $\alpha_{ooc}(G) = 2$ (i.e., the number of vertices of minimum connected dominating subgraph of G.).

S. T. Hedetniemi suggested a new parameter in domination theory as follows:

A dominating set D is a connected dominating set if it induces a connected snbgraph in G. Since a dominating set must contain at least one vertex from every component of G . it follows that a connected dominating set exists for a graph G if and only if G is connected.

The minimnm of the cardinalities of the connected dominating sets of G is termed as the connected domination number of G, and is denoted $\alpha_{ooc}(G)$ .The connected domination number of some standard graphs can be easily found. and are given as follows:

(i) $\alpha_{ooc}(K_p) = 1$.

(ii) $\alpha_{ooc}(K_p + G) = 1$, for any graph G.

(iii) $\alpha_{ooc}(K_{m,n}) = \begin{cases} 1, \text{if either } m \text{ or } n = 1 \\ 2, \text{if } m, n \geq 2. \end{cases}$

(iv) $\alpha_{ooc}(C_p) = P-2.$

(v) For any tree T of order p, $\alpha_{ooc}(T) = p-e.$ Where e is the number of pendent vertices (i.e., vertices of degree 1) in T.

(vi) $\alpha_{ooc}(P_k) = K-2.$

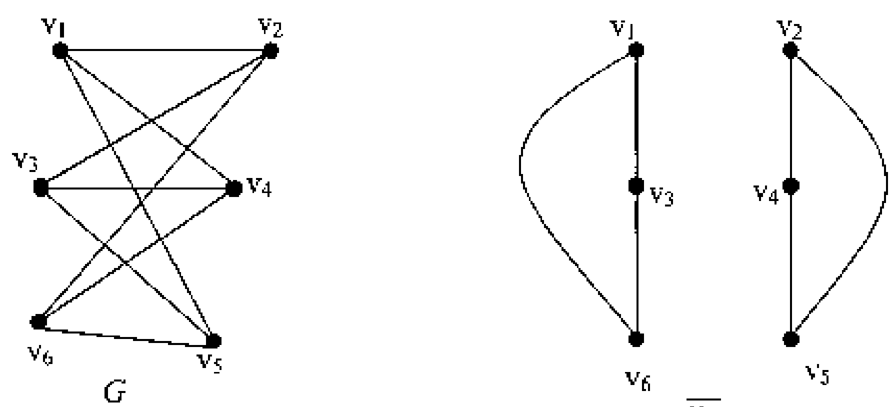
when P_k is a path of k vertices. $k \geq 3$.

(vii) For every complete bipartite graph, domination nnmber and connected domination number are same and is equal or less than two.

viii) For every complete graph the domination number = connected domination number = 1.

(ix) A complement of any complete bipartite graph that will be two complete graph which are disjoint. So the complement of any complete bipartite graph has no connected dominating set and its domination number is two.

Example:



G $\overline{G}$

Here G be a complete bipartite graph and $\overline{G}$ is a complement of G. We see that $\overline{G}$ has no connected dominating set and its domination number is two.

Connected dominating set has been used widely in multi-hop adhoc networks (MANET) by numerous routing, broadcast and collision avoidance protocols.

Although computing minimum connected dominating set is known to be NP-hard, many protocols have been proposed to construct a sub-optimal dominating set. However, these protocols are either too complicated, needing non-local information, or not adaptive to topology charge.

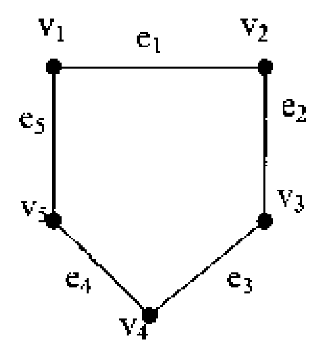
We have illustrated already established proof with better explanation for various standard graphs from [31] of E. Sampathkumar and H. B. Walikar with examples.

Proposition 3.1: For any connected graph G,

$$\gamma(G) \leq \alpha_{ooc}(G).$$

Proof: since any connected dominating set of any connected graph is also a dominating set so $\gamma(G) \leq \alpha_{ooc}(G)$. Conversely, any dominating set may be or not connected dominating set. Hence only $\gamma(G) \leq \alpha_{ooc}(G)$ is true.

Example:



G

We have from the above graph G the domination number of it is two but its connected domination number is three. Therefore, $\gamma(G) \leq \alpha_{ooc}(G)$.

Proposition 3.2: Let G be any graph and H be any spanning subgraph of G. Then every dominating set of H is also a dominating set of G, and consequently $\gamma(G) \leq \gamma(H)$.

Proof: Since H be a spanning subgraph of G so the all vertices of H are equal to the all vertices of G and the number of edge of H is equal or less than the number of edge of G. Therefore every dominating set of H is also a dominating set of G. Now, If the number of edge of H is equal to the number of edge of G then

$\gamma(H) = \gamma(G)$ ------------- --(1)

But if the number of edge of H is less than the number of edge of G then. $\gamma(G) \leq \gamma(H)$ ------------- --(2)

So we have from (1) and (2) for any spanning subgraph H of G, $\gamma(G) \leq \gamma(H)$. □

Corollary 3.2.1: Let G be a connected graph and H be any connected spanning subgraph of G. Then every connected dominating set of H is also a connected dominating set of G, and hence $\alpha_{ooc}(G) \leq \alpha_{ooc}(H)$.

Proof: Since H spanning subgraph of G so all the vertices of H is also the all vertices of G and all the edges of H is also the edges of G but $E(H) \leq E(G)$. Therefore, since connected dominating set of H is dominate all the vertices of H so it also dominate all the vertices of G. Hence any connected dominating set of H is also a connected dominating set of G. But the inverse of it is not true. Because if we remove some edge of G to get H the number of vertices of connected dominating set of H is greater than the number of vertices of connected dominating set of G. Therefore, $\alpha_{ooc}(G) \leq \alpha_{ooc}(H)$.□

Propositions 3.3: For any connected graph G of order $p \geq 3$

$\alpha_{ooc}(G) \leq p-2$

and the bound is best possible.

Proof: Since G is connected, by a well known result, G must have a spanning tree T. Taking H =T then we get

$\alpha_{ooc}(G) \le \alpha_{ooc}(H)$, by corollary3.2.1

But we know for any tree H of order p,

$\alpha_{ooc}(H) = p\text{-}e$, where e is the number of pendant vertex

$\therefore \quad \alpha_{ooc}(G) \le p\text{-}e$

Since $e \ge 2$ for any tree, hence $\alpha_{ooc}(G) \le p\text{-}2$. □

Lemma 3.1: For any connected graph of order P with maximum degree Δ,

$$\gamma(G) \ge \left[\frac{P}{\Delta+1}\right] \text{------------------(a)}$$

Where [x] denotes the greatest integer ≤ x. The bound (a) is attained if and only if there exists a minimnm dominating set (i.e., a dominating set of cardinality $\alpha_{oo}(G)$) D of G satisfying the following three conditions.

C1. D is independent.

C2. For any vertex $u \in V - D$ there exists a unique vertex $v \in D$ such that $N(u) \cap D = \{V\}$, where N(x) denotes the set of vertices adjacent to x.

C3. $d(u) = \Delta$, for every $u \in D$.

Theorem 3.1: For any connected graph G with maximum degree Δ,

$\left[\frac{p}{\Delta+1}\right] \le \alpha_{ooc}(G) \le 2q\text{-}p$ ---(1), where q is the number of edges and p is the number of vertices of G.

The lower bound in (1) is attained if and only if G has a vertex of full degree (i.e. a vertex of degree p-1), and the upper bound is attained if and only if G is a path.

Proof: We know for any connected graph G

$\gamma(G) \le \alpha_{ooc}(G)$,by the proposition 3.1 -----------(2)

Again we have from lemma 3.1, for any connected graph of order p with maximum degree Δ,

$$\gamma(G) \geq \left[\frac{p}{\Delta+1}\right] \text{------------------------}(3)$$

So we have from (2) and (3)

$$\alpha_{ooc}(G) \geq \gamma(G) \geq \left[\frac{p}{\Delta+1}\right]$$

$$\therefore \qquad \alpha_{ooc}(G) \geq \left[\frac{p}{\Delta+1}\right]$$

Again we know for any connected graph of order $p \geq 3$,

$\therefore$ $\alpha_{ooc} \leq p-2$, by proposition 3.3

$= 2(p-1)-p$, since for any connected graph, $q \geq p-1$

$\leq 2q - p$

$\therefore$ $\alpha_{ooc} \leq 2q-p$

We shall now show that $\alpha_{ooc}(G) = 2q-p$ if and only if G is a path.

We know, for any tree T of order p, $\alpha_{ooc}(T) = p-e$, where e is the number of pendant vertexes (i.e.. a vertex of order 1).

Since G is a path so it has exactly two pendent vertices.

Therefore, $\alpha_{ooc}(G) = p-2$

$= 2(p-1)-p$

$= 2q-p$, since G is a path so $p-1 = q$

$\therefore$ $\alpha_{ooc}(G) = 2q-p$.

Conversely, suppose that $\alpha_{ooc}(G) = 2q-p$. Then, since G is connected so,

$\alpha_{ooc}(G) \leq$ p-2, by proposition 3.3

$\therefore$ 2q-p $\leq$ p-2

$\therefore$ q $\leq$ p-1.

Since G is connected, we then see that q = p-1; hence G must be a tree. But ,we know for any tree $\alpha_{ooc}(G)$ = p-e. If e > 2, we get

$\alpha_{ooc}(G)$ = p-e. < p-2 = 2q-p

$\therefore$ $\alpha_{ooc}(G)$ < 2q-p, a contradiction.

Thus e $\leq$ 2. But since G is a tree so, e $\geq$ 2 hence e = 2. This proves that G must be a path. □.

Theorem 3.2: Let G be a connected graph of order p $\geq$ 4 such that both G and $(\overline{G})$ are connected. Then $\alpha_{00c}(G)+\alpha_{00c}(\overline{G}) \leq p(p-3)$ ....(a)

The bound is attained if and only if G = P_4.

Proof: Since G and $\overline{G}$ are both connected. Hence by the Theorem 3.1 we have,

$\alpha_{ooc}(G) \leq$ 2q-p and $\alpha_{ooc}(\overline{G}) \leq 2\overline{q}$ -p.

Where q and $\overline{q}$ denote the number of edges in G and $\overline{G}$ respectively. Thus

$\alpha_{ooc}(G) + \alpha_{ooc}(\overline{G}) \leq$ 2q-p+2$\overline{q}$ -p

$$= 2(q + \overline{q}) - 2p$$

$$= 2\binom{p}{2} - 2p = p(p-3)$$

$\therefore$ $\alpha_{ooc}(G) + \alpha_{ooc}(\overline{G}) \leq$ p(p-3)

It remains to show that the equality (a) holds if and only if G=P_4

If G = P_4, then $\alpha_{ooc}(G) = \alpha_{ooc}(\overline{G})$ =2

Thus $\alpha_{ooc}(G) + \alpha_{ooc}(\overline{G}) = 2 + 2 = 4 = 4(4\text{-}3) = p(p\text{-}3)$

$\therefore \quad \alpha_{ooc}(G) + \alpha_{ooc}(\overline{G}) = p(p\text{-}3)$.

Conversely, if the equality holds in (a). We should have

$\alpha_{ooc}(G) = 2q\text{-}p$ and $\alpha_{ooc}(\overline{G}) = 2\overline{q}\text{-}p$. So, G and $\overline{G}$ are paths, by theorem 3.1.

Siuce G is a path So $q = p\text{-}1$ again $\overline{G}$ is also a path so $\overline{q} = p\text{-}1$.

$\therefore \alpha_{ooc}(G) + \alpha_{ooc}(\overline{G}) = \{2(p\text{-}1)\text{-}p\}+\{2(p\text{-}1)\text{-}p\} = 2p\text{-}4$ ----(b)

Since we consider the equality in (a) hold so

$\alpha_{ooc}(G) + \alpha_{ooc}(\overline{G}) = p(p\text{-}3)$.

or, $\quad 2p - 4 = p(p\text{-}3)$ [by (b)]

or $\quad p^2 - 5p + 4 = 0$

$\therefore \quad p = 4$ or 1. But $p \neq 1$ since $p \geq 4$ given

Therefore, $p = 4$.

So, when G and $\overline{G}$ are equal to P_4 then the equality in (a) hold. This completes the proof.□

New developed theorem:

Theorem 3.3: For any connected graph G,

$\gamma(G) \leq k(G)$. Where k(G) is denoted the vertex covering number of G.

Proof: We know the domination number $\gamma(G)$ is the number of vertices of smallest vertex set D of G such that every vertex of G\D is iucident at least one vertex of D. While, a vertex covering of a graph G is a subset k of vertex set V of G such that every edge of G is incident with at least one vertex in **k** . A covering k is called minimum if there is no covering k' of G with $|k'| < k$ .Then the number of vertices of k is called the covering number of G which is denoted by k(G) in this chapter.

Let v_1, v_2, v_3 be any three vertices of connected graph G and $e_1=\{v_1,v_2\}$, $e_2=\{v_2,v_3\}$ and $e_3=\{v_3,v_1\}$. Then v_1 dominate the vertices v_2 and v_3 .So v_1 is the minimum dominating set of G ,

$$\therefore \gamma(G)= 1.\text{------------------}(a)$$

But v_1 is not the end point of all the edges of G.(v_1 is not the end point of e_2) so to get a minimum covering set we include any one vertex with v_1 i.e. $\{v_1,v_2\}$ or $\{v_1,v_3\}$ is a minimum covering set of G.

Therefore, k(G)=2,-------------------------(b).

Otherwise , if there is no edge e_2 in G then $\{v_1\}$ is the minimum dominating set and also minimum vertex covering set of G.

Therefore for this case k(G)=1-----------------(c).

So, we have from (a),(b) and (c), $\gamma(G)\leq k(G)$. □

Example:

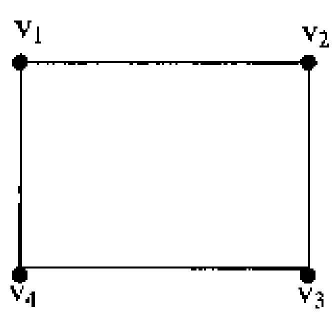


Fig-1

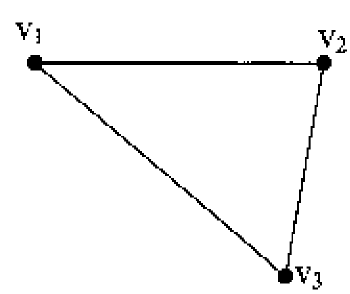


Fig-2

We have from the Fig-1, $\gamma(G)=k(G) = 2$ and from the Fig-2 , $\gamma(G) = 1$ but

$k(G) = 2$,so in Fig-2 $\gamma(G)<k(G)$. Hence for all simple connected graph, we show that $\gamma(G)\leq k(G)$.

CHAPTER FOUR

TOTAL GLOBAL DOMINATION NUMBER OF A GRAPH

A total dominating set T of a graph G=(V,E) is a total global dominating set (t.g.d set) if T is also a total dominating set of $\overline{G}$. The total global dominating number $\gamma_{tg}(G)$ of G is the minimum cardinality of a t.g.d set. In this chapter, we exhibit inequalities involving variations on domination number, total domination number and total global domination number.

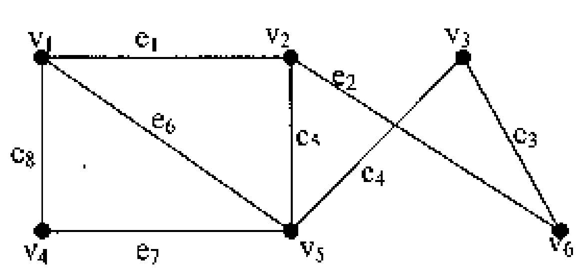
Introduction:

The graph G considered here have order p and size q (i.e, p vertices and q edges) and both G and their complement $\overline{G}$ have no isolates.

Now we define Dominating set, Total Dominating set and Total Global Dominating set of a graph.

Dominating set: A set D of vertices in a graph G = (V, E) is a dominating set of G if every vertex in V-D is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set.

Example:



From the above graph we have the vertices v_1 and v_2 dominate the all vertices of the graph G. So $\{v_1, v_2\}$ is a dominating set of G and this dominating set is the minimum dominating set of all dominating sets of G . So, the domination number of G is $\gamma(G) = 2$.

Total Dominating set: A total dominating set T of G is a dominating set such that the induced subgraph <T> has no isolates. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set.

Total Global Dominating set : A total dominating set T of G is a total global dominating set (t.g.d set) if T is also a total dominating set of $\overline{G}$. The total global domination number $\gamma_{tg}(G)$ of G is the minimum cardinality of a t.g.d. set.

We note that $\gamma(G)$ and $\gamma_g(G)$ are defined for any G while $\gamma_t(G)$ is only defined for G with $\delta(G) \geq 1$ and $\gamma_{tg}(G)$ is only defined for G with $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$, where $\delta(G)$ is the minimum degree of G.

A γ_t - set is a minimum total dominating set. Similarly a γ_g- set and a γ_{tg}- set are defined.

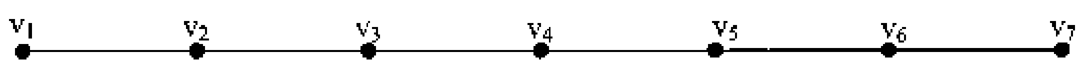
We have illustrated already established proof with better explanation for various standard graphs from [23] of V. R. Kulli and B. Janakiram with examples.

Theorem 4.1: A total dominating set T of G is a t.g.d. set if and only if for each vertex $v \in V$ there exists a vertex $u \in T$ such that v is not adjacent to u.

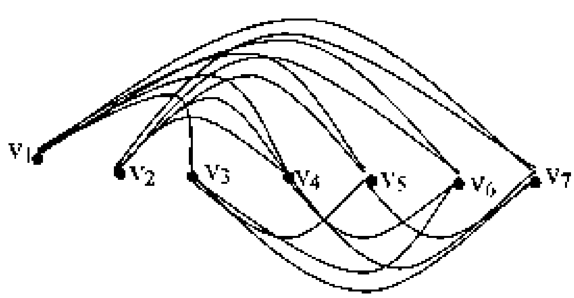
Proof: Since for each vertex $v \in V$ there exist a vertex $u \in T$ such that v is not adjacent to n. Again the all vertex of T belong to V. Therefore we can say that, for every vertex of T, has a non adjacent vertex in T.

Conversely, let there exist a vertex $w \in T$ such that it has no nonadjacent vertex in T. But we know T will be a total global dominating set of G if T also a total dominating set of $\overline{G}$. Since the vertex w has no nonadjacent vertex in T. So, in total dominating set of $\overline{G}$ [i.e this total dominating set is T] has no any adjacent vertex of w. Therefore w will be isolated. But by definition of total dominating set there is no isolated vertex, so if T will be total dominating set then there must be a non adjacent vertex of w in T. So we write that a total dominating set T of G will be a total global dominating set iff for each $v \in T$ there exists a vertex $u \in T$, such that v is not adjacent to u. □

Example:



G



$\overline{G}$

We have , from the graph G the vertex set T= $\{v_2,v_3,v_5,v_6\}$ is a dominating set and this dominating set has no isolated vertex so T is a total dominating set. We also see that for every vertex of G , there exist a nonadjacent vertex in T.so T is a total global dominating set .Exactly we see that T is a total dominating set of $\overline{G}$.

Theorem 4.2: Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex. Then,

(i) $\gamma_{tg}(G) = \gamma_{tg}(\overline{G})$;

(ii) $\gamma_t(G) \leq \gamma_{tg}(G)$;

(iii) $\gamma_g(G) \leq \gamma_{tg}(G)$;

(iv) $\{\gamma_t(G) + \gamma_t(\overline{G})\}/2 \leq \gamma_{tg}(G) \leq \gamma_t(G) + \gamma_t(\overline{G})$.

(i) $\gamma_{tg}(G) = \gamma_{tg}(\overline{G})$;

Proof: By definition we have $\gamma_t(G)$ will be $\gamma_{tg}(G)$ if it also total domination number of $\overline{G}$. Suppose that $\gamma_t(G)$ is a total global dominating number then we have . $\gamma_t(G) = \gamma_t(\overline{G}) = \gamma_{tg}(G)$ --------(1)

Similarly $\gamma_t(\overline{G})$ will be $\gamma_{tg}(\overline{G})$ if it also total domination number of G [since the complement of $\overline{G}$ is G]

Therefore we can write

$\gamma_t(\overline{G}) = \gamma_t(G) = \gamma_{tg}(\overline{G})$ ------------(2)

So we have from (1) and (2)

$\gamma_{tg}(G) = \gamma_{tg}(\overline{G})$ □

(ii) $\gamma_t(G) \le \gamma_{tg}(G)$;

proof: We know the total domination number $\gamma_t(G)$ is the minimum cardinality of the total dominating set.

Therefore, $\gamma_t(G) \le$ number of element of each total dominating set

But we know that any total dominating set of G will be total global dominating set if it also a total dominating set of $\overline{G}$. Let T = $\gamma_t(G)$ is the minimum cardinality of the total dominating set. so, if this T = $\gamma_t(G)$ is also a total dominating set of $\overline{G}$ then $\gamma_t(G)=\gamma_{tg}(G)$----(1) and the number of element of any other total dominating set is greater then $\gamma_t(G)$. So. if any total dominating set of the other t.d. set of G is also a t.d. set of $\overline{G}$ then $\gamma_t(G) < \gamma_{tg}(G)$ ----------- (2)

So, we have from (1) and (2), $\gamma_t(G) \le \gamma_{tg}(G)$ □

(iii) $\gamma_g(G) \le \gamma_{tg}(G)$;

Proof : We know a dominating set of G will be a global dominating set if it is also a dominating set of $\overline{G}$.

So, $\gamma_g(G)$ must be a dominating set of G. But we know $\gamma(G) \le \gamma_t(G)$ [For any connected graph, here G and $\overline{G}$ are connected graph]

$$\therefore \quad \gamma_g(G) \le \gamma_t(G) \text{ --------------------- } (4)$$

Therefore by (ii) and (4) we have,

$$\gamma_g(G) \le \gamma_t(G) \le \gamma_{tg}(G)$$

$$\therefore \gamma_g(G) \le \gamma_{tg}(G) \quad \square$$

(iv) $\{\gamma_t(G) + \gamma_t(\overline{G})\}/2 \leq \gamma_{tg}(G) \leq \gamma_t(G) + \gamma_t(\overline{G})$.

Proof : We have from (ii) $\gamma_t(G) \leq \gamma_{tg}(G)$------------------ (1)

Therefore $\gamma_t(\overline{G}) \leq \gamma_{tg}(\overline{G})$------------------ (2)

Bnt from (i) we have $\gamma_{tg}(G) = \gamma_{tg}(\overline{G})$

∴ By (2) and (i) we can write $\gamma_t(\overline{G}) \leq \gamma_{tg}(G)$ ----------- (3)

∴ by (1) + (3) we have

$$\gamma_t(G) + \gamma_t(\overline{G}) \leq \gamma_{tg}(G) + \gamma_{tg}(G) = 2\gamma_{tg}(G)$$

$$\therefore \quad \{\gamma_t(G) + \gamma_t(\overline{G})\}/2 \leq \gamma_{tg}(G) \quad \square$$

Theorem 4.3: Let G be a graph which such that neither G nor $\overline{G}$ have an isolated vertex. Then $\gamma_{tg}(G) = p$ (p is the number of vertices of G)

if and only it $G = P_4$ (a path on 4 vertices) or mk_2 or $m\overline{k}_2$ where m≥2.

Proof: Any connected graph G has a connected complement if for each vertex of G has a nonadjacent vertex in G. Becanse, if a vertex u∈G has no nonadjacent vertex in G then this vertex u will be isolated in $\overline{G}$ and for this matter $\overline{G}$ will be disconnected. So, for connected $\overline{G}$ u must has a nonadjacent vertex in G.

Now, for p_1 there is no total dominating set and $\gamma_t(P_2) = 2$, So, in total dominating set of P_2 has all vertices of P_2 and each vertex of $\gamma_t(P_2)$ has no nonadjacent vertex in $\gamma_t(P_2)$. Therefore $\gamma_t(P_2)$ will not be total global dominating set by theorem 4.1.

Again the number of vertex of the total dominating set of P_3 will 2 or 3. But by theorem 4.1 the both two t. d. set will not be t. g. d. set.

Now, the number of vertices of the t.d. set of P_4 will be 2,3, or 4. But by theorem 1, the first two t.d. set will be not t.g.d. set and the last one will be t.g.d. set because for every vertex of it has a non adjacent vertex in it.

Therefore $\gamma_{tg}(P_4) = 4 = p$ = number of vertices of P_4.

Again, the number of vertices of t.d. set of P_5 which will be t.g.d. set are 4 and 5. But $\gamma_{tg}(P_5)$ will be minimum of all t.g.d. set.

So, $\gamma_{tg}(P_5) = 4 < 5$ = number of vertex of P_5.

Similarly we can proof that $\gamma_{tg}(P_n) < n = p$ when $n > 4$.

So, we have for only P_4,

$\gamma_{tg}(P_4) = p$.

Again, in total dominating set, has no isolated vertex and each vertex of mk_2 or $m\bar{k}_2$ ($m \geq 2$) has only one adjacent vertex. So all vertices of mk_2 or $m\bar{k}_2$ will be included in t. d. set of mk_2 or $m\bar{k}_2$. Since $m \geq 2$, So, of every vertex of t .d. set of mk_2 or $m\bar{k}_2$ has a non adjacent vertex in this t.d. set. Hence by theorem 4.1, this total dominating set will be total global dominating set.

Hence $\gamma_{tg}(G) = p$ when G is mk_2 or $m\bar{k}_2$ and $m \geq 2$.

But when m=1 then mK_2 will be a path p_2 and we see from the above,that there is no total global dominating set.

Hence $\gamma_{tg}(G)=p$ if and only if $G=p_4$ or mk_2 or $m\bar{k}_2$ when $m \geq 2$.□

Theorem 4.4: Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex and T be a γ_t-set of G such that each vertex x in T has non-neighbor in T. If there exists a vertex u∈V-T which is adjacent only to vertices in T, then $\gamma_{tg}(G) \leq \gamma_t(G) + 2$.

Proof: Since each vertex x∈T has a non-neighbor in T and T is a γ_t-set of G. So, T has minimum 4 vertices. Because we know in total dominating set has no isolated vertex. Again if T has two or three vertices then of each vertex of T has no nonadjacent vertex in T. So T has minimum four vertices.

Case 1: If V-T = {u}, then there exists a vertex v∈T such that v is not adjacent to u. Because if u is adjacent to all vertices of T then the number of vertices of T will be two it is contradiction since T has minimum four vertices. So, must u has a nonadjacent vertex iu T.

Therefore for each vertex v∈V there exists a vertex u∈T such that v is not adjacent to u. Hence by theorem 4.1, T is a total global dominating set of G.

So, $\gamma_{tg} < \gamma_t(G) + 2$ [Since T= $\gamma_t(G)$] --------------------- (1)

Case II: If V-T ≠ {u}, then there exists a vertex v∈V-T. This u and v are not adjacent to all vertices of T. If u and v are adjacent to all vertices of T then the number of vertices of T will be 2 [if u adjacent to v] or 3 [if u is not adjacent to v]. It is impossible since T has minimum 4 vertices.

Hence T∪{u,v} is a total global dominating set (by theorem 4.1).

So, $\gamma_{tg}(G) = T + 2$

or $\gamma_{tg}(G) = \gamma_t(G) + 2$ ------------------- (2)

So, we have from (1) and (2)

$\gamma_{tg}(G) \leq \gamma_t(G) + 2$ □

New developed theorem :

Theorem 4.5: A total dominating set T of G is a total global dominating set of G if for each vertex v ∈ T there exist a vertex u ∈ T such that v is not adjacent to u.

Proof: We know the all vertices of T is also the vertices of V(here V is the vertex set of G).Again we have from the theorem 4.1, A total dominating set T of G is a t.g.d. set if and only if for each vertex v ∈ V there exists a vertex u ∈ T such that v is not adjacent to u.Since any vertex of T is also the vertex of V so, A total dominating set T of G is a total global dominating set of G if for each vertex v ∈ T there exist a vertex u ∈ T such that v is not adjacent to u.□

Theorem 4.6: A total dominating set T will be a total global dominating set then the number of vertices of T ≥ 4.

Proof: We know from the theorem 4.5 that a total dominating set T of G is a total global dominating set if and only if for each vertex v ∈ T there exist a vertex n ∈ T snch that v is not adjacent to u. Since T is a t.d.set so T has at least two vertices which are adjacent. So by theorem 4.5 it is not t.g.d.set. If T has three vertices then they are connected because T has no isolated vertex. So there is one vertex which has no nonadjacent vertex is T. So by theorem 4.5 . T is not a t.g.d.set. So the t.d.set T will be t.g.d.set if and only if number of vertices of T≥4.

Case1: Since T is a total global dominating set so by definition of t.d. set,the induced subgraph <T> has no isolated vertex.Hence the vertex nnmber of T is greater than one.

Case2: Let T has two vertices. Since the induced subgraph <T> has no isolated vertex so the two vertices are adjacent to each other. But we know from the theorem 4.5 ,a total dominating set T is a total global dominating set if and only if for each vertex $v \in T$ there exist a vertex $u \in T$ such that v is not adjacent to u,which is contradiction .So, the number of vertex of T is greater than two.

Case3: Let T has three vertices which are u, v and w. Since the induced subgraph <T> has no isolated vertex so, any one of the three vertices is not isolated vertex. Again, let u is not adjacent to v then w is must adjacent to v and u otherwise there exist a isolated vertex.Which is contradicts the theorem 4.5.Hence,the number of vertices of T is greater than three.But, if T has 4 vertices then we can easily see that any vertex of T has a nonadjacent vertex in T.

Therefore, we say that from the above three cases, the number of vertices of

$T \geq 4$.□

Theorem 4.7: Any connected graph G has a connected complement if each vertex of G has at least one non adjacent vertex in G and at least one vertex in G which has at least two non adjacent vertex in G. When $p > 3$.

Proof: Case(1): Let any vertex u of G, which has no nonadjacent vertex in G. Therefore u has not any adjacent vertex in $\overline{G}$. Hence u will be isolated vertex in $\overline{G}$ and $\overline{G}$ will be disconnected.

Case(2):Again, any connected graph [p>3] has at least one vertex which order is two. Now, if G has no any vertex which has at least two non adjacent vertex then in $\overline{G}$ there is no any vertex which has order two. So, $\overline{G}$ will be disconnected.

Therefore we have from the above two cases, any connected graph G has a connected complement if each vertex of G has at least one nonadjacent vertex in G and at least one vertex in G which has at least two non adjacent vertex in G. When $p > 3$.

CONCLUSION

This thesis is devoted to the domination theory in various aspects in graphs. The concept of dominating sets introduced by Ore and Berge currently receives more attention in Graph Theory. The domination theory has gained due to the inspiring contributions by eminent graph theorists as E. J. Cockayne, S. T. Hedeniemi, R. C. Laskar, P. J. Slater, E. Sampathkumar, V. R. Kulli, B. Janakiram etc.

In the first chapter, we have presented the necessary graph theoretic definitions and earlier works on the domination theory.

In the second chapter. we have obtained a relation between the domination number of cycles with the nonbondage number of cycles. By using various relations of nonbondage number of graphs we have also established some relations among the domination number, degrees of various graphs and trees of spanning subgraphs. We have also extended some graphs by illustration for some standard graphs.

The concept of connected domination number has been introduced in chapter three. In this chapter we have compared some graphs between the connected domination number & the domination number of graphs .

For example: $\gamma(G) \leq \alpha_{00c}(G)$. Also we have found out some bounds for connected graphs.

The fourth chapter deals with the total global domination number of graphs. We have exhibited the various relations among domination number total domination number, global domination nnmber and total global domination number of graphs.

In future, one can proceed about connected domination number. global domination number and total global domination nnmber of graphs by using algorithms .

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