GRAEFFE'S ROOT SQUARING METHOD: ITS SOLVABILITY CONDITIONS, SOFTWARE MODIFICATION AND EXTENSION

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Contents

Abstract

Graeffe's method determines all the roots of univariate polynomials both real and complex, repeated and non-repeated simultaneously. In this thesis, it is said that this statement is not universally true. It is shown that the method is valid if the algebraic equations satisfy the conditions,

(i) equations with zero coefficient must have at least one pair of equidistant non-zero coefficient from the zero coefficient.

(ii) any transformed equation of a given equation with non-zero coefficient may have zero coefficients but these new coefficients must satisfy (i).

(iii) all the coefficients of non-linear equation must not be unity.

GRAEFFE.BAS program [9] has been modified in these light and has been extended for not solvable equations.

Candidate's Declaration

I hereby declare that the work which is being presented in the thesis entitled " **Graeffe's root squaring method : Its solvability conditions, software modification and extension** " submitted in partial fulfilment of the requirements for the award of the degree of Master of Philosophy in Mathematics, in the department of Mathematics, Bangladesh University of Engineering and Technology, Dhaka, is an authentic record of my own work.

The matter presented in this thesis has not been submitted by me for the award of any other degree in this or any other University.

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Date: 09.11.2004. **(WAHIDA ZAMAN LOSKOR)**

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Chapter 1

Literature survey and objectives

1.1 Introduction

According to Pan in 1997 [1], the classical problem of solving an nth degree univariate polynomial and its system has substantially influenced the development of mathematics throughout centuries and still has several important applications to the theory and practice of present-day computing of the computer age .

The nth degree nonlinear univariate polynomial (algebraic equation of a single variable x) has the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots\cdots\cdots + a_{n-1} x + a_n = 0, \quad a_0 \neq 0 \tag{1.1}$$

This type of the equation arises in many occasions as

(i) auxiliary equation of higher order ODE with constant coefficients,

(ii) characteristic equations of the matrix eigenvalue problems,

(iii) in the area of computer algebra and computing geometry (an active area of modern research)

In general equation (1.1) has n roots (zeros), which are of following types:

(i) real and distinct roots

(ii) real and equal roots

(iii) imaginary or complex roots and complex roots occur in conjugate pair

(iv) combination of (i), (ii) and (iii).

Although problem of solving (1.1) was known to Sumerians (third millennium BC) for particular values of n, all the above types of roots were not known to them. After centuries this types of roots become apparent to the present generation. Now a days many computational problems arising in the sciences, engineering, business management and statistics have been linearised and then solved by using tools from linear algebra, linear programming. Such roots involve the solution of (1.1) for smaller n. Computer algebra solves (1.1) for large n. There are available software in case of precision, error bound etc. for this causes problems and thus motivation for further research on the design of effective algorithm for solving (1.1) arises.

Thus equation (1.1) retains its major role both as a research problem and a part of practical computational tasks in the highly important area of computing called *computer algebra*. These information along with others are embodied in Pan [1].

No general algebraic method is available for the solution of equation (1.1) except for very special cases like (a) quadratic (b) cubic and (c) quartic equation. There are some numerical procedures (methods) to determine real roots both distinct and equal only such as

(i) Bisection method

(ii) False-position method

(iii) Newton-Raphson method

(iv) Secant method

(v) Muller's method

(vi) Bairstow's method

(vii) Graeffe's method of roots squaring

Graeffe's method gives all the roots simultaneously, both real and complex. This method has not received much attention. Modern research based on computer has been carried out by Pan [1] in 1997, Malajovich [8] in 1999 and the references therein around the last part of 20th century and the beginning of 21st century. The lack of popularity of Graeffe's method are embodied in Malajovich [8] and in sec. 3.3.

This project is on Graeffe's method. Major work's contained in chapter 3 which discusses

(i) the conditions for the applicability of the method in sec. 3.2.

(ii) the necessary modification and extension of the software GRAEFFE.BAS in sec. 3.4.

(iii) Modified GRAEFFE.BAS in sec. 3.5.

1.2 History of polynomial root finding and Graeffe's root squaring

Numerical root-finding for a univariate polynomial is a classical problem. The exact starting time of study of this problem is not known. But history reveals that Sumerians in the third B.C. have the root finding knowledge. Its study still continues to present day. Readers are referred to Pan [1,2,3] and Bareiss [4] and the references therein for past and recent progress in polynomial root-finding.

Polynomial root-finding occupies its position in the present day computer algebra and computing geometry. It is being widely applied in the study of engineering sciences particularly in network theory, control linear systems and computer aided geometric design.

The basis of Graeffe's method is the "root squaring" process. Controversy prevails in the designation of the method on the issue of a Russian translation which called "Graeffe's method" as "Lobacevskii's. This controversy has been discussed in Householder [5] in details and the references therein. Partial discussions are given in Hutchinson [6], Malajovich and Zubelli [7]. According to them, the method is due to Germinal Dandelin (1794-1837) in 1826, although the fundamental idea goes back to Edward Waring (1734 -1798) in 1762. Dandelin's paper was not widely circulated and the process goes under the name of Carl Heinrich Graeffe (1799 -1873) in 1837. The method was also suggested independently by Nicholaus Ivanovich Lobacevskii in 1834. Later contributions were made by Johann Franz Encke (1791-1865) in 1841. Somehow or other the method is popularized under Graeffe.

1.3 Literature survey

Literature reveals that study of root-finding of polyinomial in special cases begins in ancient times of Sumerians and Babilonians. With the full fledged growth of number system, solution of univariate polynomial finds its new ways of achievements and still it has been going on with the advancement of computer. Some of early and recent information on Graeffe's method for the study of polynomial root-finding is briefly presented as part of this thesis in the following paragraphs.

In 1946, Bodewing [10] stated many advantages of Graeffe's method such as

(i) no first approximation need be known,

(ii) approximation of all roots are obtained simultaneously.

(iii) not much laborious in comparison for knowing a single root by any other method,

(iv) main advantage of obtaining complex root.

(v) automatically separate close roots,

(vi) it gives multiple roots real and complex,

(vii) determines several pairs of complex roots with same modulus.

Only *disadvantage* he mentioned, it is not useful in correcting a single root as in other methods. Next he discussed a solution procedure on splitting the transformed equation. Discussing convergence process, he observed that Graeffe's method has greatest efficiency as the error decreases quadratically in each subsequent transformed new equation. Finally he discussed the position of roots with same modulus near a circle in Argand/Gaussian plane

Bareiss [4] in 1958 presented at the meeting of Association of U.S.A Atomic Energy Commission 'the resultant procedure' for finding simultaneously all zeros of polynomials with real coefficients. The procedure relies on the modification of Graeffe's method. Principles of the procedure are outlined He also present steps for designing the algorithm. His discussion ended with a summary of history of root squaring method. He derived a formula for determining the minimum number of root squaring.

In 1994 Malajovich [11] investigated approximate Newton iteration for several generalizations of the Newton operator. He generalized some of the results developed by Shub and Smale in 1993 [reference 6 in [11]]. The results have been used to prove complexity theorem on path-following algorithms for solving systems of polynomial equations.

In 1996 Neff and Reif [12] developed an efficient algorithm for the complex roots up to specified precision of a univariate polynomial $f(x)$ of degree n with complex coefficients with norms less than 2^m, m is the smallest integer. Their algorithm requires no assumption for separation of roots. This property makes it remarkable.

In 1997 Pan [1] describes the importance of the equation (1.1), early history of solving polynomial equations, solving equation (1.1) by geometric construction, compares some approaches for solving equation (1.1), the divide and conquer approach to approximate polynomial zeros, balancing problem in splitting the polynomial, discuss the technique of avoiding approximation of the zeros of a higher order derivative, cites three applications of polynomial equation (1.1) by different researchers.

In 1999 Malajovich [8] develops a new variation of Graeffe iteration of modern digital computers. The algorithm is based on (i) classical Graeffe iteration and Newton Diagrams (ii) changes of scale (renormalization). It implemented successfully with a number of numerical experiments. This method computes both the moduli and the argument at all roots under certain generic conditions.

In 2000 Malajovich and Zubeli [7] proposed and implemented a new version of the Graeffe's algorithms for finding all the roots of univariate complex polynomial. Renormalization idea is used for the construction of the algorithm. They have introduced Newton diagram which is the graph of convex function.

In 2000 Hazra and Loskor [14] contradicts the universal advantages of Graeffe's method about giving the roots of algebraic equation both real and complex repeated and non-repeated simultaneously stated by many authors like Hutchinson [6] in 1935, Cronvich [15] in 1939 Bodewing [10] in 1946, Kopal [16] in 1961, Scarborough [17] in 1966, Carnahan [18] in 1969, Constantinides [9] in 1987 and Balagurusamy [19] in 1999 Finally the authors pointed out the conditions of validity of the Graeffe's root squaring method and suggested an algorithm modifying the BASIC software designed by Constantinides [9].

In 2001 Pan [2] studied approximate polynomial GCDs (greatest common devisors) which are important both theoretically and practically in control linear system, network theory and computer aided design.

In 2002 Pan [3] develops two algorithms for approximation of all roots (zeros) of a univariate polynomial. One of them computes a basic well isolated zero free annulus on the complex plane. The other numerically splits input nth degree polynomial into two factors balanced in degrees and with the zero sets separated by the basic annulus. The first algorithm uses Graeffe's root squaring step recursively.

1.4 Objective

The main objective of this research is to setup condition(s) for solvability of univariate polynomial using Graeffe's root squaring method in sec. 3.2. On the basis of these criteria GRAEFFE.BAS program developed by Constantinides [9] is to be modified in sec. 3.5. so that the program can identify the problems solvable by Graeffe's method. The modified GRAEFFE.BAS is to be tested in solving few univariate polynomials in sec 3.6.

In the meantime weaknesses of this method will be pointed out and finally some open problems in sec. 4.2 will be stated after conclusion.

Chapter 2

Graeffe's root-squaring method and GRAEFFE.BAS program of Constantinides

2.1 Graeffe's root-squaring method

A manual as well as desk calculator method of Graeffe's root squaring method is described here following a mainly Scarborough [17].

The principle of the Graeffe's method: Literatures reveal that Graeffe's root squaring is a direct method of obtaining all types of roots – real distinct, real equal and complex of a univariate polynomial with real coefficients and of no initialization. It transforms a polynomial $p_n(x)$ into another polynomial of same degree whose roots are the squares of the roots of the original polynomial. Due to squaring, the roots of the transformed equation will be spread apart more widely than the original one. The roots of the transformed equation are said to be separated when the ratio of any root to the next larger is negligible in comparison with unity. This situation is known as "roots are separated". Process continues until the roots are really separated. Finally roots are computed directly from the coefficients using Newton's relations.

The root-squaring process: The transformed equation is obtained by repeated application of a root-squaring process. For first application of this process, given equation have been raised an equation whose roots are the squares of those of the original equation. This second equation is then transformed into a third equation whose roots are the square of those of the second and therefore the fourth powers of those of the original equation. The root-squaring process is continued until the roots of the last transformed equation are completely separated.

Explanation of the method:

Let the given equation be

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_{n-1}x + a_n = 0, \quad a_0 \neq 0 \tag{2.1}$$

Then if $x_1, x_2, \cdots\cdots, x_n$ be the roots of this equation, the equivalent form is

$$f(x) = a_0(x - x_1)(x - x_2)(x - x_3)\cdots\cdots(x - x_n) = 0 \tag{2.2}$$

Thus $(-1)^n f(-x) = (-1)^n a_0(-x-x_1)(-x-x_2)(-x-x_3)\cdots\cdots\cdots(-x-x_n) = 0$

$$= a_0(x+x_1)(x+x_2)(x+x_3)\cdots\cdots\cdots(x+x_n) = 0 \tag{2.3}$$

Now multiplying (2.2) by (2.3), the result is

$$(-1)^n f(-x)f(x) = a_0{}^2(x^2-x_1^2)(x^2-x_2^2)(x^2-x_3^2)\cdots\cdots\cdots(x^2-x_n^2) = 0 \tag{2.4}$$

Let $(-1)^n f(-x)f(x) = \phi(y)$ and $x^2 = y$ then (2.4) becomes

$$\phi(y) = a_0{}^2(y-x_1^2)(y-x_2^2)(y-x_3^2)\cdots\cdots\cdots(y-x_n^2) = 0 \tag{2.5}$$

The roots of this equation are $x_1^2, x_2^2, \cdots\cdots\cdots, x_n^2$ which are the squares of the roots of the given equation (2.2). The equation (2.5) can be described other way round as

$$\phi(y) = \left\{a_0\left(\sqrt{y}-x_1\right)\left(\sqrt{y}-x_2\right)\cdots\cdots\cdots\left(\sqrt{y}-x_n\right)\right\}\left\{a_0\left(\sqrt{y}+x_1\right)\left(\sqrt{y}+x_2\right)\cdots\left(\sqrt{y}+x_n\right)\right\}$$

$$\therefore\ \phi(x) = \left\{a_0\left(\sqrt{x}-x_1\right)\left(\sqrt{x}-x_2\right)\cdots\cdots\cdots\left(\sqrt{x}-x_n\right)\right\}\left\{a_0\left(\sqrt{x}+x_1\right)\left(\sqrt{x}+x_2\right)\cdots\left(\sqrt{x}+x_n\right)\right\} \tag{2.6}$$

$$= (-1)^n f(\sqrt{x})f(-\sqrt{x}) \text{ by (2.2) and (2.3)} \tag{2.7}$$

$\therefore\ \phi(x) = (-1)^n f(\sqrt{x})f(-\sqrt{x})$ describes the first iteration (squaring) of the method.

Hence, on multiplying $f(x)=0$ by $(-1)^n f(-x)$ an equation whose roots are the squares of those of $f(x)=0$ is obtained.

This multiplication maintains a simple routine which can be best illustrated with lower degree equations. Let a sixth-degree equation be

$$f(x) = a_0x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6 = 0$$

Then $(-1)^6 f(-x) = a_0x^6 - a_1x^5 + a_2x^4 - a_3x^3 + a_4x^2 - a_5x + a_6 = 0$

On multiplication by $f(x)$,

$$(-1)^6 f(-x)f(x) = a_0^2x^{12} \begin{array}{c|c|c|c|} -a_1^2 & x^{10}\ \ +a_2^2 & x^8\ \ -a_3^2 & x^6\ \ +a_4^2 \\ +2a_0a_1 & -2a_1a_3 & +2a_2a_4 & -2a_3a_5 \\ & +2a_0a_4 & -2a_1a_5 & +2a_2a_6 \\ & & +2a_0a_6 & \end{array}\ x^4$$

$$\begin{array}{c|} -a_5^2 \\ +2a_4a_6 \end{array}\ x^2 + a_6^2 = 0 \tag{2.8}$$

Now let a seventh degree equation be,

$$f(x) = a_0x^7 + a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5x^2 + a_6x + a_7 = 0$$

Then $(-1)^7 f(-x) = a_0x^7 - a_1x^6 + a_2x^5 - a_3x^4 + a_4x^3 - a_5x^2 + a_6x - a_7 = 0$

Multiplying, as before

$$
(-1)^7 f(-x)f(x) = a_0^2x^{14}\quad
\begin{array}{r|rr|rr|rr|rl}
-a_1^2 & x^{12} & +a_2^2 & x^{10} & -a_3^2 & x^8 & +a_4^2 & x^6 \\
+2a_0a_2 & & -2a_1a_3 & & +2a_2a_4 & & -2a_3a_5 & \\
 & & +2a_0a_4 & & -2a_1a_5 & & +2a_2a_6 & \\
 & & & & +2a_0a_6 & & -2a_1a_7 &
\end{array}
$$

$$
\begin{array}{r|rr|l}
-a_5^2 & x^4 & +a_6^2 & x^2 - a_7^2 = 0 \\
+2a_4a_6 & & -2a_5a_7 & \\
-2a_3a_7 & & &
\end{array}
\tag{2.9}
$$

The equation shows that the law of formation of the coefficients in the squared equation is the same whether the degree of the given equation be even or odd. In practice the multiplication is carried out with detached coefficients as shown below:

a_0	a_1	a_2	a_3	a_4	$a_5 \cdots$
a_0	$-a_1$	a_2	$-a_3$	a_4	$-a_5 \cdots$
a_0^2	$-a_1^2$	a_2^2	$-a_3^2$	a_4^2	$-a_5^2 \cdots$
	$+2a_0a_2$	$-2a_1a_3$	$+2a_2a_4$	$-2a_3a_5$	$+2a_4a_6 \cdots$
		$+2a_0a_4$	$-2a_1a_5$	$+2a_2a_6$	$-2a_3a_7 \cdots$
			$+2a_0a_6$	$-2a_1a_7$	$+2a_2a_8 \cdots$
				$+2a_0a_8$	$-2a_1a_9 \cdots$
					$+2a_0a_{10} \cdots$
b_0	b_1	b_2	b_3	b_4	$b_5 \cdots$

(2.10)

The coefficients in the new equation (2.10) are the sums $b_0, b_1, b_2, \cdots, b_n$ of the several columns in the scheme above. The whole scheme is set up a follows:

1. The coefficients of the original equation constitute the first row of the scheme.
2. The second comprises of the first row with alternate positive and negative signs a_0 being positive.

Now the multiplication of these two rows are perfomed between two lines according to the following *rules* under the first line:

3 The numbers in the top row are the squares of the coefficients directly above them, with alternating signs – the second, fourth, sixth, etc. square numbers being negative.

4. The quantities directly under these squared numbers are the doubled products of the coefficients equally removed from the one directly overhead, the first being twice the product of the two coefficients adjacent to the one overhead, the second the doubled product of the next two equally removed coefficients, etc.

5. The signs of the doubled products are changed alternately in going along the rows and also in going down the columns, the sign of the first doubled product in each row not being changed.

The above discussions are concentrated to first squaring (first iteration). Let second iteration of (2.6) transforms $\phi(y)$ to $\phi(z)$. Thus roots $y_i = x_i^2$ and $z_i = y_i^2 = x_i^4 = (x_i^2)^2$. Proceeding in this way to *k* iterations, the roots of the final transformed equation are, therefore $r_i = x_i^{2^k}$ [19].

mth iteration formula:

A general formula for *m*th iteration can be written. Let $f_0(x)$ denote the univariate polynomial (1.1) or (2.2). $f_1(x)$ denote the 1st iterate (2.7) i.e. the relation

$$f_1(x) = (-1)^n f(\sqrt{x}) f(-\sqrt{x}) = f_0(\sqrt{x}) f_0(-\sqrt{x}).$$

Similarly, the second iteration

$$f_2(x) = f_1(\sqrt{x}) f_1(-\sqrt{x}).$$

Proceeding in this way, the *m*th iteration is

$$f_{k+1}(x) = f_k(\sqrt{x}) f_k(-\sqrt{x}), \quad k = 0,1,2,\cdots\cdots\cdots, m-1. \tag{2.11}$$

Calculation of roots in three cases

Case 1: Roots all real and unequal

The relations between the roots $x_0, x_1, x_2, \cdots\ \cdots, x_n$ and the coefficients $a_1, a_2, a_3 \cdots\cdots\cdots a_n$ of the *n*th degree general univariate polynomial

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots\cdots\cdots + a_{n-1} x + a_n = 0$ are

Newton's first relation $\frac{a_1}{a_0} = -(x_1 + x_2 + x_3 + \cdots\cdots\cdots + x_n)$ = sum of roots.

Newton's second relation $\frac{a_2}{a_0} = +(x_1x_2 + x_1x_3 + \cdots)$ = sum of the products taken two at a time.

Newton's third relation $\frac{a_3}{a_0} = -(x_1x_2x_3 + x_1x_2x_4 \cdots)$ = sum of the products taken three at a time.

... ...

Newton's nth relation $\frac{a_n}{a_0} = (-1)^n x_1x_2x_3 \cdots\cdots\cdots x_n$ = product of the roots.

In the final transformed equation (i.e. mth iteration in which m is a positive integral power of 2 i.e. $m = 2^q$, q is a positive integer)

$$b_0(x^m)^n + b_1(x^m)^{n-1} + \cdots\cdots\cdots + b_{n-1}x^m + b_n = 0 \tag{2.12}$$

whose roots $x_1^m, x_2^m, \cdots x_n^m$ and coefficients $b_0, b_1, b_2 \cdots b_n$ are connected by the corresponding relations

$$\frac{b_1}{b_0} = -(x_1^m + x_2^m + \cdots\cdots\cdots + x_n^m) = -x_1^m\left(1 + \frac{x_2^m}{x_1^m} + \frac{x_3^m}{x_1^m} + \cdots\cdots\cdots + \frac{x_n^m}{x_1^m}\right),$$

$$\frac{b_2}{b_0} = x_1^m x_2^m + x_1^m x_3^m \cdots\cdots = x_1^m x_2^m\left(1 + \frac{x_3^m}{x_2^m} + \frac{x_4^m}{x_2^m} + \cdots\cdots\right),$$

$$\frac{b_3}{b_0} = -(x_1^m x_2^m x_3^m + x_1^m x_2^m x_4^m + \cdots) = -x_1^m x_2^m x_3^m\left(1 + \frac{x_4^m}{x_3^m} + \cdots\right),$$

... ...

$$\frac{b_n}{b_0} = (-1)^n x_1^m x_2^m x_3^m \cdots\cdots\cdots x_n^m.$$

Now if the order of magnitude of the roots is $|x_1| > |x_2| > |x_3| > |x_n|$, then the ratios $\frac{x_2^m}{x_1^m}, \frac{x_3^m}{x_2^m}$ etc. are negligible in comparison with unity and can all be made as small as desired by making m large enough. Thus the relations between the roots and the coefficients in the final transformed equation are

$$\frac{b_1}{b_0} = -x_1^m, \quad \frac{b_2}{b_0} = x_1^m x_2^m, \quad \frac{b_3}{b_0} = -x_1^m x_2^m x_3^m, \cdots\cdots\cdots, \frac{b_n}{b_0} = (-1)^n x_1^m x_2^m x_3^m \cdots\cdots\cdots x_n^m$$

Dividing each of these equations after the first by the preceding equation, we obtain

$\frac{b_2}{b_1} = -x_2^m, \frac{b_3}{b_2} = -x_3^m, \cdots\cdots\cdots, \frac{b_n}{b_{n-1}} = -x_n^m$ are obtained.

Hence from these and the equation

$$\frac{b_1}{b_0} = -x_1^m, \tag{2.13}$$

one obtains

$$b_0 x_1^m + b_1 = 0, \quad b_1 x_2^m + b_2 = 0, \quad b_2 x_3^m + b_3 = 0, \cdots, b_{n-1} x_n^m + b_n = 0. \tag{2.14}$$

Thus the original equation has broken up into n simple equations and the mth powered roots can be found from them. Finally mth root of these gives the desired roots.

The number of root squaring to break up the original equation into linear fragments depends upon (1) the ratios of the roots of the given equation and (2) the number of significant figures desired in the computed roots. But it is not possible to determine how many times the root-squaring process is needed because the roots and their ratios are not known in advance. In the root squaring process, the signs of the doubled products will not occur in regular order because the coefficients in the given equation are not in general all positive. In manual practice the possibilities of making a mistake in the sign are great and for avoiding this mistake one can use some notations like a "c" after each term in which the sign is to *be changed* and an "n" after each term where the sign is *not to be changed*. Furthermore, in the transformed equations the coefficients, which are large numbers, these coefficients may be written as simple numbers, multiplied by powers of 10.

Finally, by the repeated root-squaring process, one obtains transformed equation and compute all the roots of an equation . Equations (2.14) can be combined to a signal formula [19]

$$x_i^m = -\frac{b_i}{b_{i-1}}, i = 1, 2, \cdots, n$$

Thus $(x_i)^{2^m} = -\frac{b_i}{b_{i-1}}$. So, $x_i = 2^m$ th root of $\left(\left|\frac{b_i}{b_{i-1}}\right|\right) = \sqrt[2^m]{\left|\frac{b_i}{b_{i-1}}\right|}$ (2.15)

Carnahan [18] introduces the symbol $_jA_i$ for the jth iteration of A_i and develops the iterative formula

$$_{j+1}A_i = (-1)^i\left[\,_jA_i^2 + 2\sum_{l=1}^{i}(-1)^l\,_jA_{i+l}\,_jA_{i-l}\right],\ 0 \le i \le n \qquad (2.16)$$

for coefficients in (2.12) where $_0A_i = a_i$, the initial value of A_i etc.

Case II. Complex Roots

The root squaring process breaks an equation into a product of real linear and quadratic factors, in the presence of complex roots. The real roots are found from linear fragments and the complex roots are found from quadratic fragments, each quadratic factor corresponding to a pair of complex roots. The presence of complex roots is revealed in two ways:

(1) The doubled products do not all disappear from the first row and

(2) the signs of some of the coefficients fluctuate as the transformations continue.

The reason for these peculiarities are explained below:

Let an equation having two distinct real roots and two pairs of complex roots be $x_1, r_1e^{i\theta_1}, r_1e^{-i\theta_1}, x_3, r_2e^{i\theta_2}, r_2e^{-i\theta_2}$ and let the order of their magnitude be $|x_1| > r_1 > |x_3| > r_2$.

Then the equation having these roots is

$$(x - x_1)(x - r_1e^{i\theta_1})(x - r_1e^{-i\theta_1})(x - x_3)(x - r_2e^{i\theta_2})(x - r_2e^{-i\theta_2}) = 0 \qquad (2.18)$$

The equation whose roots are the mth powers of the roots of this equation is therefore

$$(y - x_1^m)(y - r_1^m e^{im\theta_1})(y - r_1^m e^{-im\theta_1})(y - x_3^m)(y - r_2^m e^{im\theta_2})(y - r_2^m e^{-im\theta_2}) = 0 \qquad (2.19)$$

where $y = x^m$

On performing the indicated multiplications in (2.19), then taking out the factors $x_1^m r_1^m, x_1^m r_1^{2m}, x_1^m r_1^{2m} x_3^m, x_1^m r_1^{2m} x_3^m r_2^m$. and neglecting the ratio $\frac{r_1^m}{x_1^m}, \frac{x_3^m}{x_1^m}, \frac{r_2^m}{x_1^m}, \frac{x_3^m}{r_1^m}, \frac{r_2^m}{r_1^m}, \frac{r_2^m}{x_3^m}$, for negligible in comparison with unity.

Finally , one obtains

$$y^6 - x_1^m y^5 + 2x_1^m r_1^m \cos m\theta_1 y^4 - x_1^m r_1^{2m} y^3 + x_1^m r_1^{2m} x_3^m y^2$$
$$- 2x_1^m r_1^{2m} x_3^m r_2^m \cos m\theta_2 y + x_1^m r_1^{2m} x_3^m r_2^{2m} = 0 \qquad (2.20)$$

Thus the given equation has been broken up into the linear and quadratic fragments

$$\left.\begin{aligned}
&y^6 - x_1^m y^5 = 0\\
&-x_1^m y^5 + 2x_1^m r_1^m \cos m\theta_1 y^4 - x_1^m r_1^{2m} y^3 = 0\\
&x_1^m r_1^{2m} y^3 + x_1^m r_1^{2m} x_3^m y^2 = 0\\
&x_1^m r_1^{2m} x_3^m y^2 - 2x_1^m r_1^{2m} x_3^m r_2^m \cos m\theta_2 y + x_1^m r_1^{2m} x_3^m r_2^{2m} = 0
\end{aligned}\right\} \tag{2.21}$$

from which the original roots can be obtained.

The root-squaring process is applied to (2.19) once more, as shown below

	y^6	y^5	y^4	y^3
*m*th power	1	$-x_1^m$	$2x_1^m r_1^m \cos m\theta_1$	$-x_1^m r_1^{2m}$
	1	$-x_1^{2m}$ $+4x_1^m r_1^m \cos m\theta_1$	$+4x_1^{2m} r_1^{2m} \cos^2 m\theta_1$ $-2x_1^{2m} r_1^{2m}$ $+2x_1^m r_1^{2m} x_3^m$	$-x_1^{2m} r_1^{4m}$ $+4x_1^{2m} r_1^{3m} x_1^m \cos m\theta_1$ $-4x_1^{2m} r_1^{2m} x_1^m r_2^m \cos m\theta_2$ $+2x_1^m r_1^{2m} x_1^m r_2^{2m}$
2*mth* *power*	1	$-x_1^{2m}$	$+4x_1^{2m} r_1^{2m} \cos^2 m\theta_1$ $-2x_1^{2m} r_1^{2m}$	$-x_1^{2m} r_1^{4m}$

	y^2	y^1	y^0
*m*th power	$x_1^m r_1^{2m} x_3^m$	$-2x_1^m r_1^{2m} x_3^m r_2^m \cos m\theta_2$	$x_1^m r_1^{2m} x_3^m r_2^{2m}$
	$+x_1^{2m} r_1^{4m} x_3^{2m}$ $-4x_1^{2m} r_1^{4m} x_3^m r_2^m \cos m\theta_2$ $+4x_1^{2m} r_1^{3m} x_3^m r_2^{2m} \cos m\theta_1$	$-4x_1^{2m} r_1^{4m} x_3^{2m} r_2^{2m} \cos^2 m\theta_2$ $+2x_1^{2m} r_1^{4m} x_3^{2m} r_2^{2m}$	$+x_1^{2m} r_1^{4m} x_3^{2m} r_2^{4m}$
2*mth* *power*	$+x_1^{2m} r_1^{4m} x_3^{2m}$	$-4x_1^{2m} r_1^{4m} x_3^{2m} r_2^{2m} \cos^2 m\theta_2$ $+2x_1^{2m} r_1^{4m} x_3^{2m} r_2^{2m}$	$+x_1^{2m} r_1^{4m} x_3^{2m} r_2^{4m}$

Thus it is observed that all these products from the above table are negligible except two in the first row. The doubled product in the first row does not all disappear when the complex roots are present. Since $2\cos^2\theta - 1 = \cos 2\theta$, the coefficient of y^4 and y becomes $2x_1^{2m}r_1^{2m}\cos 2m\theta_1$ and $2x_1^{2m}r_1^{4m}x_3^{2m}r_2^{4m}\cos 2m\theta_2$ respectively. Thus the last transformed equation is

$$1 - x_1^{2m} + 2x_1^{2m}r_1^{2m}\cos 2m\theta_1 - x_1^{2m}r_1^{4m} + x_1^{2m}r_1^{4m}x_1^{2m}$$
$$- 2x_1^{2m}r_1^{4m}x_3^{2m}r_2^{2m}\cos 2m\theta_2 + x_1^{2m}r_1^{4m}x_1^{2m}r_2^{4m} = 0 \qquad (2.22)$$

The equation (2.22) shows that each root squaring doubles the amplitude of the complex roots. The cosines $\cos 2m\theta_1$ and $\cos 2m\theta_2$ fluctuates in value and even in of these amplitudes signs. For this reason, the presence of complex roots shows fluctuation of signs. The complex roots by solving the resulting quadratic equations for x^m and then extraction the mth root of the results are obtained by means of De Moivre's theorem. But in this process the ambiguities of signs in the roots arise and these cannot be easily removed. To obtain the complex roots without ambiguity as to signs some further relations between roots and coefficients are derived below.

Relations between the coefficients of an algebraic equation and the reciprocals of its Roots: In the general equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_{n-1}x + a_n = 0$

let $x = 1/y$. The result, after clearing of fractions, is

$$a_ny^n + a_{n-1}y^{n-1} + a_{n-2}y^{n-2} + \cdots\cdots + a_3y^3 + a_2y^2 + a_1y + a_0 = 0$$

Hence from the relations between roots and coefficients,

$$\frac{a_{n-1}}{a_n} = -(y_1 + y_2 + \cdots\cdots + y_n)$$
$$\frac{a_{n-2}}{a_n} = y_1y_2 + y_1y_3 + \cdots\cdots + y_2y_3 + ..$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$\frac{a_0}{a_n} = (-1)^n y_1y_2y_3\cdots\cdots y_n;$$

or, since $y = 1/x$,

$$\left.\begin{array}{l}\dfrac{1}{x_1}+\dfrac{1}{x_2}+\cdots\cdots\cdots+\dfrac{1}{x_n}=-\dfrac{a_{n-1}}{a_n}\\ \dfrac{1}{x_1x_2}+\dfrac{1}{x_1x_3}+\cdots\cdots\cdots+\dfrac{1}{x_2x_3}+\cdots\cdots\cdots+\dfrac{1}{x_{n-1}x_n}=\dfrac{a_{n-2}}{a_n}\\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\\ \dfrac{1}{x_1x_2x_3\cdots\cdots\cdots x_n}=(-1)^n\dfrac{a_0}{a_n}\end{array}\right\} \tag{2.23}$$

These relations between the coefficients and the reciprocals of the roots will help to avoid ambiguities of sign in the computation of complex roots.

Case III. Roots real and numerically equal

If one of the double products will always remain in the first row and will be just half the squared term, then the root-squaring process can never break up into linear fragments. For this reason the equation has two roots are numerically equal.

Let a third degree equation having three real roots x_1, x_2, x_3 be

$$x^3+a_1x^2+a_2x+a_3=0 \tag{2.24}$$

Then the equation whose roots are the *m*th powers of (2.24) is

$$(y-x_1^m)(y-x_2^m)(y-x_3^m)=0,\ \text{where}\quad y=x^m,$$

or $$y^3-(x_1^m+x_2^m+x_3^m)y^2+(x_1^mx_2^m+x_1^mx_3^m+x_2^mx_3^m)y-x_1^mx_2^mx_3^m=0$$

or $$y^3-x_1^m\left(1+\frac{x_2^m}{x_1^m}+\frac{x_3^m}{x_1^m}\right)y^2+x_1^mx_2^m\left(1+\frac{x_3^m}{x_2^m}+\frac{x_3^m}{x_1^m}\right)y-x_1^mx_2^mx_3^m=0 \tag{2.25}$$

Now let $x_2=x_3$ and let $|x_1|>|x_2|$. Then for sufficiently large values of m the ratios $\dfrac{x_2^m}{x_1^m}$ is negligible in comparison with unity and (2.25) reduces to

$$y^3-x_1^my^2+2x_1^mx_2^my-x_1^mx_2^{2m}=0 \tag{2.26}$$

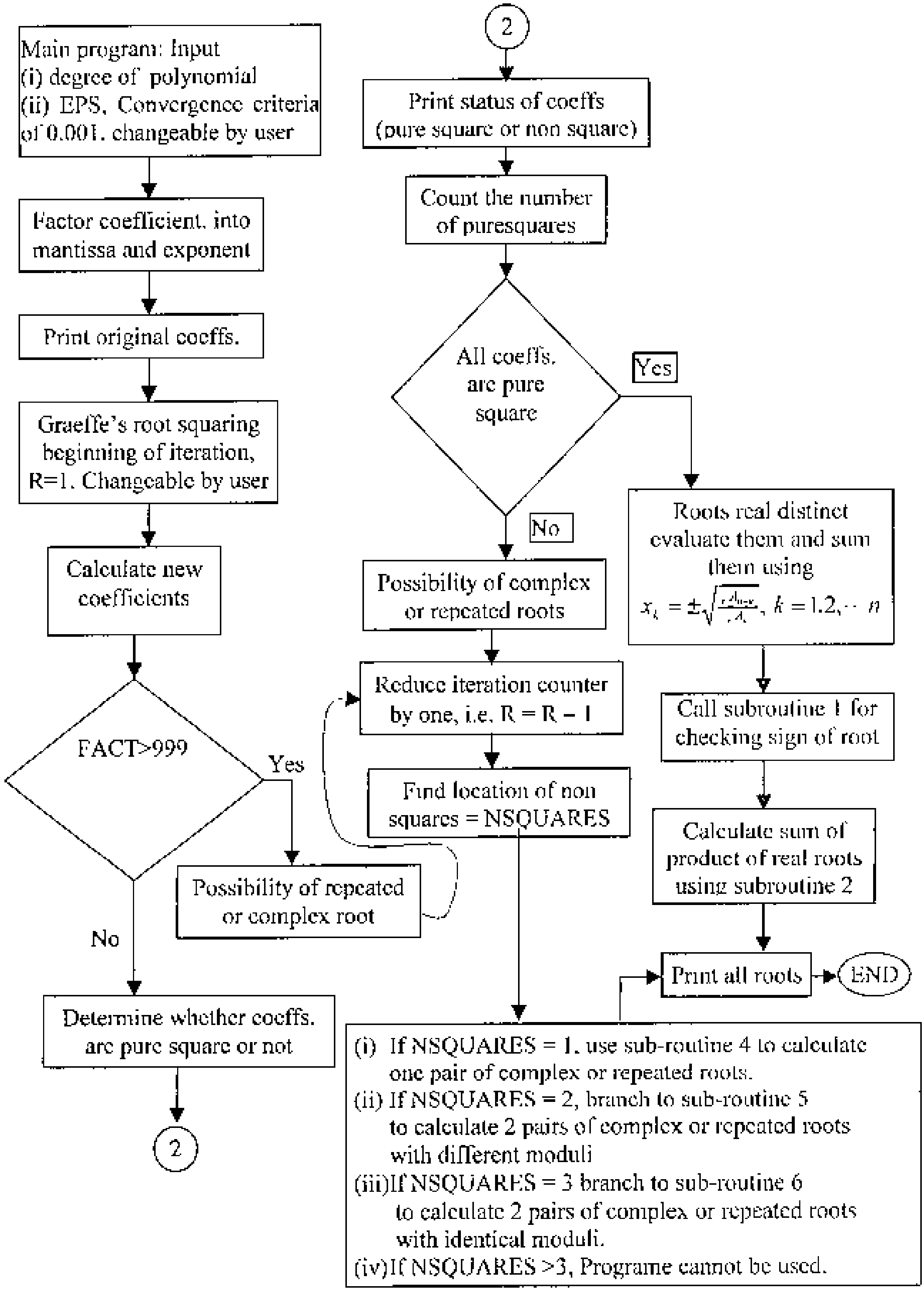
The roots of the given equation have now been separated as much as they can ever be but applying the root-squaring process to (2.26) using only the coefficients, we have,

mth power	1	$-x_1^m$	$2x_1^m x_2^m$	$-x_1^m x_2^{2m}$
	1	$-x_1^{2m}$ $+4x_1^m x_2^m$	$+4x_1^{2m} x_2^{2m}$ $-2x_1^{2m} x_2^{2m}$	$-x_1^{2m} x_2^{4m}$
$2m$th power	1	$-x_1^{2m}$	$+2x_1^{2m} x_2^{2m}$	$-x_1^{2m} x_2^{4m}$

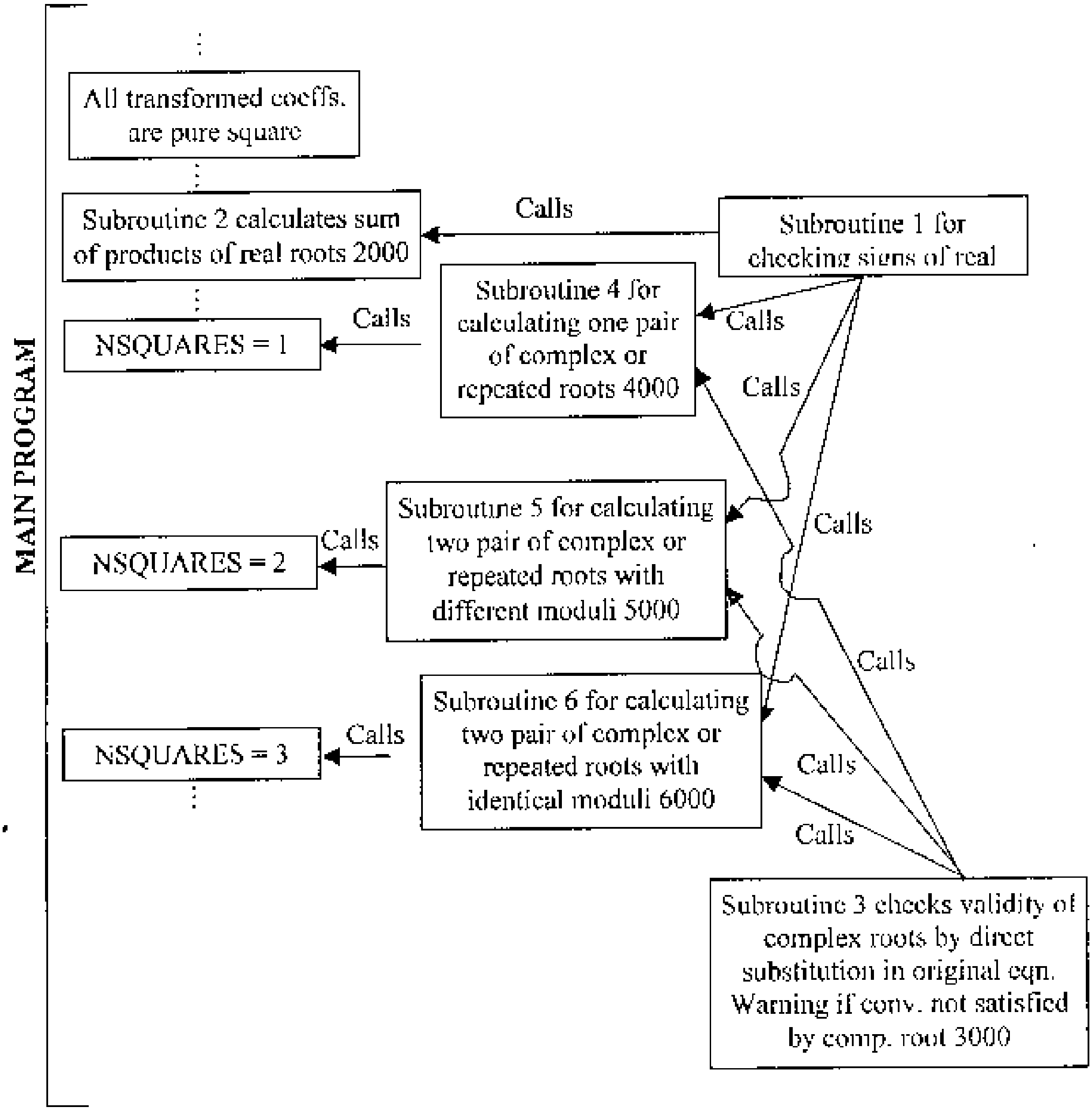
It can be noticed that the first doubled product is negligible in comparison with the squared term above it, whereas the second is of the same order of magnitude as the squared term above and just half as large. Further, in the equation for the $2m$th powers of the roots all the coefficients except one are the squares of those in the preceding equation. This remaining one is only half of the square of the corresponding coefficient in the preceding equation. These peculiarities enable one to detect equal real roots immediately.

2.2 GRAEFFE.BAS Program

Constantinides [9] designed GRAEFFE. BAS program in BASIC programming language for IBM PC for solving univariate non-linear polynomial equation using Graeffe's root squaring method. The program consists of a main program and 6 (six) subroutines. The flow chart of the main program is constructed as follows:


Main program: Input
(i) degree of polynomial
(ii) EPS, Convergence criteria of 0.001, changeable by user
Factor coefficient, into mantissa and exponent
Print original coeffs.
Graeffe's root squaring beginning of iteration, R=1. Changeable by user
Calculate new coefficients
FACT>999
Yes
No
Possibility of repeated or complex root
Determine whether coeffs. are pure square or not
2
2
Print status of coeffs (pure square or non square)
Count the number of puresquares
All coeffs. are pure square
Yes
No
Roots real distinct evaluate them and sum them using
Possibility of complex or repeated roots
Reduce iteration counter by one, i.e. R = R – 1
Find location of non squares = NSQUARES
Call subroutine 1 for checking sign of root
Calculate sum of product of real roots using subroutine 2
Print all roots
END
(i) If NSQUARES = 1, use sub-routine 4 to calculate one pair of complex or repeated roots.
(ii) If NSQUARES = 2, branch to sub-routine 5 to calculate 2 pairs of complex or repeated roots with different moduli
(iii) If NSQUARES = 3 branch to sub-routine 6 to calculate 2 pairs of complex or repeated roots with identical moduli.
(iv) If NSQUARES >3, Programe cannot be used.

Sequence of calling six subroutines by the main program and subroutines themselves is shown below:



2.3 Procedure for identifying roots in GRAEFFE.BAS

Due to squaring, the square terms grow more rapidly than the sum of product terms in the transformed polynomial. As a result new coefficinets ${}_rA_k$ are pure squares of ${}_{r-1}A_k$. However presence of complex or repeated roots prevents these coefficients from becoming pure squares i.e. some coefficients remain non-square. Thus Graeffe's root squaring method provides a very powerful procedure for identifying different categories of roots as follows:

(a) real and distinct roots correspond to "all coefficients, ${}_rA_k$ being pure squares".

(b) One pair of complex roots or repeated real roots correspond to "a nonsquare sorrounded by two pure squares" in the form [9]

${}_rA_{s+1}$	${}_rA_s$	${}_rA_{s-1}$
Pure square	Nonsquare	Pure square

and the roots $\alpha \pm i\beta$ are determined by solving $\alpha^2 + \beta^2 = \sqrt[m]{\frac{{}_rA_{s-1}}{{}_rA_{s+1}}}$ and Newton's first relation $\sum_{i=1}^{n} x_i = -\frac{a_{n-1}}{a_n}$ applied to original polynomial. If $\beta = 0$, then the two roots are real repeated.

(c) Two pairs of complex roots or repeated real roots with *different* moduli correspond to "two nonsquares each is sorrounded by two pure squares" as

${}_rA_{s_1+1}$	${}_rA_{s_1}$	${}_rA_{s_1-1}$	...	${}_rA_{s_2+1}$	${}_rA_{s_2}$	${}_rA_{s_2-1}$
Pure square	Nonsquare	Pure square		Pure square	Nonsquare	Pure square

and the roots $\alpha_{s_1} \pm i\beta_{s_1}$, $\alpha_{s_2} \pm i\beta_{s_2}$ are determined by solving

$$\alpha_{s_1}^2 \pm \beta_{s_1}^2 = \sqrt[m]{\frac{{}_rA_{s_1-1}}{{}_rA_{s_1+1}}}, \qquad \alpha_{s_2}^2 \pm \beta_{s_1}^2 = \sqrt[m]{\frac{{}_rA_{s_2-1}}{{}_rA_{s_2+1}}}$$

and Newton's first and second relations $\sum_{i=1}^{n} x_i = -\frac{a_{n-1}}{a_n}$, $\sum_{\substack{i,j=1 \\ i\neq j}}^{n} x_i x_j = \frac{a_{n-2}}{a_n}$. If $\beta_{s_1} = \beta_{s_2} = 0$, then the four roots are real repeated.

(d) Two pairs of complex or repeated real roots with *identical* moduli correspond to "three neighbouring nonsquares sorrounded by two pure squares" as

$$\underset{\text{pure square}}{{}_rA_{s+1}} \quad \underbrace{{}_rA_s \quad {}_rA_{s-1} \quad {}_rA_{s-2}}_{\text{Nonssquare s}} \quad \underset{\text{puresquare}}{{}_rA_{s-3}}$$

and roots $\alpha_s \pm i\beta_s$, $s = 1,2$, are obtained by solving

$$R^2 = \alpha_s^2 + \beta_s^2 = \sqrt[2m]{\frac{{}_rA_{s-3}}{{}_rA_{s+1}}}.$$

where R is the common modulus, and first 3 Newton's relations

$$\sum_{i=1}^{n} x_i = -\frac{a_{n-1}}{a_n}$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{n} x_i x_j = \frac{a_{n-2}}{a_n}$$

$$\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n} x_i x_j x_k = -\frac{a_{n-3}}{a_n}$$

If $\beta_s = 0$, $s = 1,2$, the roots are real repeated.

Chapter 3

Solvability conditions, weaknesses and modification

3.1 Observation

None of the researchers in the references discussed the following observations except a few. Their comments about the method regarding advantages and disadvantages are correct to their point of view.

(i) It is observed that there are equations, which are *not* transformable by root squaring into a different one with non-zero coefficients from where the roots of the original equation are calculated. It is found that the odd degree equations set like

$$\left.\begin{aligned} &x^3 + a = 0, \\ &x^5 + a = 0, \\ &x^7 + a = 0, \\ &x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1 = 0 \end{aligned}\right\} \quad (3.1)$$

etc cannot be solved by the Graeffe's root squaring method manually as well as using GRAEFFE.BAS of Constantinides [9]. All these equations (3.1) transform to the form $x^n + a = 0,\ n = 3,5,7$ after first iteration and GRAEFFE.BAS shows overflow at the statement 1080 Of constantinides[9]. It is observed that

$$\left.\begin{aligned} &x^2 + x + b = 0, \quad x^4 + a = 0, \quad x^7 + ax^2 + b = 0, \\ &x^7 + ax^6 + b = 0, \quad x^4 + 2x^3 + 2x^2 + 2x + 2 = 0 \end{aligned}\right\} \quad (3.2)$$

etc. can be solved by both manual procedures and using program of Constantinides [9]. In this case zero coefficients revive and nonzero coefficients do not vanish. As a result final transformed equation in each case provides solution. Householder [21] passes a similar remark for $x^n - 1 = 0$ in which all n roots have unit modulus and the Graeffe method fails for such equations without deriving any solvability condition. Similarly, Wilkinson [22] called for equation $x^2 - 1 = 0$ well-condition zeros $x = \pm 1$, because on squaring the transformed equation becomes $x^2 - 2x + 1 = 0$ and deteriorates to coincident ill-conditioned root 1.

(ii) Actual computation reveals another fact that 4th, 6th, 8th, 10th etc. degree non-linear equations with positive unit coefficients like

$$x^4 + x^3 + x^2 + x + 1 = 0 \tag{3.3}$$

will never stop the procedure because the coefficients of the transformed equation remain unity i.e. $(-1)^4 f(\sqrt{x}) f(-\sqrt{x})$ remains invariant.

3.2 Solvability conditions

The observations in sec. 3.1 leads condition of solvability. In Graeffe's method, roots of the original equation are obtained from the coefficients of the last transformed equation. If some of those intermediate coefficients between the first and the last one are zero, the roots may be infinite or even indeterminate. So, the first and foremost criteria of Graeffe's root squaring method to be successful is that the coefficients of the last transformed equation *must* be non-zero which in turn depends on some or all of the non-zero coefficients of the original equation.

Let the nth degree algebraic equation be

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_k x^k + \cdots + a_{n-1} x + a_n = 0, \quad a_0 \neq 0$$

The scheme of the process is as follows:

Variable.	x^n	x^{n-1}	x^{n-2}	......	x^k	......	x	1
	+	−	+		$(-1)^k$		$(-1)^{n-1}$	$(-1)^n$
Coefficients	a_0	a_1	a_2		a_k		a_{n-1}	a_n
	a_0^2	$-a_1^2$	$+a_2^2$	......	$(-1)^k\ a_k^2$	......	$(-1)^{n-1}a_{n-1}^2$	$(-1)^n a_n^2$
		$2a_0a_2$	$-2a_1a_3$	......	$(-1)^{k+1}2a_{k-1}a_{k+1}$	......	$(-1)^n 2a_{n-2}a_n$	
			$+2a_0a_4$	......	$(-1)^{k+2}2a_{k-2}a_{k+2}$	......		
				......	⋮	......		
				......	$(-1)^{k+l}2a_{k-l}a_{k+l}$	......		
				......		......		
2^{jth} power	${}_jA_0$	${}_jA_1$	${}_jA_2$	......	${}_jA_k$	......	${}_jA_{n-1}$	${}_jA_n$
	⋮	⋮	⋮	......	⋮	......	⋮	⋮
	⋮	⋮	⋮	......	⋮	......	⋮	⋮
$2^{j+1}th$ power	${}_{j+1}A_0$	${}_{j+1}A_1$	${}_{j+1}A_2$	......	${}_{j+1}A_k$	......	${}_{j+1}A_{n-1}$	${}_{j+1}A_n$

where ${}_{j+1}A_k = (-1)^k\left[{}_jA_k^2 + 2\sum_{l=1}^{k}(-1)^l\,{}_jA_{k-l}\,{}_jA_{k+l}\right],\quad 0 \le k \le n,\quad k-l \ge 0,\quad k+l \le n$

is a modification of what is given by Carnahan [18] and ${}_0A_k = a_k$. The equation suppose that the root squaring process stops after jth squaring according to the stopping conditions suggested in the scheme.

The *conditions* of solvability are discussed for the following two cases:

CASE A. Equations with zero coefficients

Let $a_k = 0,\quad 0 < k \le n$. If ${}_{j+1}A_k = 0$ for $0 < k \le n$ and for any j then the method will continue indefinitely without satisfying any stopping conditions suggested in the references and so the method *fails*. But if ${}_{j+1}A_k \ne 0$ at some stages of j then the method is successful in giving roots.

Let $j = 0$, so

$${}_1A_k = (-1)^k\left[a_k^2 + 2\sum_{l=1}^{k}(-1)^l.a_{k-l}.a_{k+l}\right] \ne 0 \text{ and } a_k = 0$$

$$\therefore \qquad 2\sum_{l=1}^{k}(-1)^l.a_{k-l}.a_{k+l} \neq 0$$

i.e. $a_{k-l} \neq 0$ and $a_{k+l} \neq 0$.

i.e. from the zero coefficients $a_k = 0$, equidistant coefficients a_{k-l} and a_{k+l}. are non-zero. Once ${}_1A_k$ become non-zero then the subsequent transformation will produce ${}_jA_k \neq 0$. This fact may be observed from the equation set (3.2) with zero coefficient in sec.3.1. But if it happens that ${}_{j+1}A_k = 0, j > 0$, then the solvability conditions follow the next case B.

CASE B. Equation with non-zero coefficients

Let $a_k \neq 0,\ \ 0 < k \leq n$. The case when ${}_{j+1}A_k \neq 0,\ \ j > 0$ is of less interest of discussion. Because of such case the method is quite successful but if it happens that ${}_{j+1}A_k = 0,\ \ j > 0$ then the conditions will be same as the following.

Let ${}_{j+1}A_k = 0,\ \ 0 < k \leq n$, also $j = 0$. Then ${}_1A_k = (-1)^k\left[a_k^2 + 2\sum_{l=1}^{k}(-1)^l.a_{k-l}.a_{k+l}\right] = 0$

$$\therefore \ (-1)^k a_k^2 + 2\sum_{l=1}^{k}(-1)^{l+k}.a_{k-l}.a_{k+l} = 0$$

i.e. the coefficients of the second powers of the roots in the transformed equation are zero. If these coefficients satisfy, the conditions like CASE A then the given equation is solvable under this method otherwise not. Similarly in any stages of transformation ${}_{j+1}A_k = 0$ but satisfy the conditions **like CASE A** then the given equation is solvable. Consider the equation

$$x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1 = 0$$

in set (3.1) for which ${}_1A_k = 0,\ \ 0 < k < 5$ and which do not satisfy the criteria in CASE A. So, for this equation Graeffe's method fails whereas the method is successful for the equation

$$x^4 + 2x^3 + 2x^2 + 2x + 2 = 0.$$

3.3 Weaknesses and recovery of Graeffe's method

The most important advantages of Graeffe's root squaring method mentioned in most of the references are (i) the method gives all the roots – real, complex, equal and unequal, simple and multiple; (ii) the method needs no initial guess as in other method. Although Balagurusamy [19], Constantinide [9] mentioned some disadvantage of coefficient growth during iteration, Malajovich and Zubelli [7] clearly and specifically point on the reasons for lack of popularity of Graeffe's method for a pretty long time. They stress two main weakness – (i) Coefficient growth, (ii) the method returns the moduli of roots but not the actual roots. Specifically reasons for Graeffe's root squaring method for lack of popularity are

(i) its traditional form leads to exponents that easily exceed the maximum allowed by floating point arithmatic;

(ii) Chaotic behaviour of the arguments of the roots of iterates.

To overcome the weakness (i), they introduce renormalization and to overcome weakness (ii), they differentiate the Graeffe iteration operator, $Gf(x) = (-1)^d f(\sqrt{x})f(-\sqrt{x})$ [where d is the degree of polynomial] and its effect is to square each root of f, [7].

Constaintinides [9] in his program alleviate serious limitation (i) above by the following 4 steps.

(a) Coefficient of the original polynomial are separated into mantissa and exponent parts and stored into two separate matrices. The ranges of mantissa used are ± 10.

(b) After each iteration mantissas part of coefficients are tested. If their absolute values are greater than 10 (less than 1) these values are divided by 10 (multiplied by 10) and the corresponding exponents are increased (decreased) by 1.

(c) The upper of limit of exponent factor is chosen as 999 as maximum values of them may be i.e. $10 \times 10^{1.7\times10^{4}}$ which is not necessary for Graeffe's method. If one of the exponent factors exceed this limit, the iteration is terminated and the roots are evaluated.

(d) Both the matrices are treated together in the programme, so that exponent part of each coefficient is always accounted for.

Hazra and Loskor [14] observe that the algebraic equation set (3.1) is not solvable manually as well as GRAEFFE.BAS software in [9]. Thus they conclude that

(i) equation with zero-coefficient must have at least one pair of equidistant non-zero coefficient from the zero-coefficient;

(ii) any transformed equation of a given equation with non-zero coefficient may have zero coefficients but these new coefficients must satisfy (i);

(iii) all the coefficients of non-linear albegraic equation must not be unity;

(iv) GRAEFFE.BAS needs modification in the light of (i), (ii) and (iii).

They suggest a testing procedure which identifies solvability of the equation. Thus software in [9] has been modified for identifying those types of problems and actually identified with the modified software.

Illustration of (i)

Odd degree equations of form $x^n + a = 0$ does not have a pair of equidistant non-zero coefficients from the zero coefficients as is seen from first three equations in (3.1); whereas even degree equation of the form $x^n + a = 0$ does have a pair of equidistant non-zero coefficient from the zero coefficients as is seen from 2nd equation $x^4 + a = 0$ in (3.2). Also the 3rd and 4th equations possess a pair of non-zero coefficient from the zero coefficient. Hence these equations in (3.2) are solvable by both manually and using GRAEFFE.BAS software. The last equation in (3.1) has no zero coefficient but has the coefficients $a_0 = a_n = 1$ and other coefficients are equal. This type transform to one with zero coefficients and possesses no equidistant pair of non-zero coefficients from the zero coefficients. But the last equation in (3.2) with $a_0 \neq a_n$ and other coefficients being equal transforms into one with zero coefficients having at least a pair of non-zero coefficients from the zero coefficients Hence this type is solvable by Graeffe's method.

In other words, all the equations in the set (3.1) reduce to the form $x^n + 1 = 0$ on the first iteration $f_1(x) = f_0(\sqrt{x})f_0(-\sqrt{x})$ and the zero-coefficient of the new transformed equation does not have a pair of non-zero coefficient from the zero coefficients. That set (3.2) does not have the above discussed problem and is solvable by Graeffe's root squaring method by both manually and using GRAEFFE.BAS software.

3.4 Modification of GRAEFFE.BAS

It is observed that there are equations, which are not transformable by root squaring into a new one with non-zero coefficients from where the roots of the original equation are calculated. It is found that the equations like

$$x^3 + a = 0, \quad x^5 + a = 0, \quad x^7 + a = 0, \quad x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1 = 0 \tag{3.3}$$

etc. While running the problems with original program [9] shows "Overflow" at statement

1080 : X(NR) = (A(R, K−1) / A(R, K)) ^ (1/ M)*10 ^ ((FACT(R, K−1)− FACT(R, K)) / M).

To overcome this difficulty, we have to modify and extend the original program [9]. So we introduced following statements and generate a new subroutine 7. Statement numbers in [9] are 10,20,30, …, 530. The modified statement numbers are fitted in between. The following are the modified statements with new statement numbers different from the original statement numbers. The new subroutine 7 follows next.

```
603  : IF ABS ( A ( R , I ) ) = 0 THEN GOTO 605 ELSE GOTO 610
605  : IF ABS (A ( R , I−1) ) < > 0 AND ABS ( A ( R, I+1 ) ) < > 0 THEN GOTO 1350
1075: IF C$ (N − 1) < > PS$  AND C$ ( 1 ) < > PS$  THEN GOTO 1110
1350: GOSUB 7000.
4065: IF C$ ( K ) < > PS$  AND C$ ( K ) < > NS$  THEN GOTO 4170
5085: IF C$ ( K ) < > PS$  AND C$ ( K ) < > NS$  THEN GOTO 5230
```

SUBROUTINE 7

```
7010   :  FOR I = N−1  TO  1  STEP −1
7020   :  IF ABS (A(R, I) ) = 0 THEN GOTO 7030
7030   :  IF ABS(A (R, I−1) )<> 0  AND  ABS(A (R, I+1) ) < > 0  THEN GOTO 7050
7040   :  NEXT  I
7050   :  FOR K=1 TO N
7060   :  IF  X(K) + XI(K) =0  THEN GOTO 7090
7070   :  IF  X(K) + XI(K) <>0  THEN GOTO 1350
7080   :  NEXT  K
7090   :  PRINT: PRINT " EQUATION IS NOT SOLVABLE BY GRAEFFE'S ROOT
          SQUARING METHOD"
7100   :  .
```

The difficulty of the problems (3.3) coefficients C$(N−1) and C$(1) do not show PURE SQUARE. If C$(1) do not show PURE SQUARE then from statement 1070 and statement 1110. we obtain K=0. which is out of range of K at statement 1050. So we introduce new statement 1075 among them as follows:

```
1050 : FOR K = N TO 1 STEP −1
1070 : IF C$ ( K−1 ) < > PS$  THEN K = K− 1 : GOTO 1110
1074 : New statement
1075 : IF C$ ( N−1 ) < > PS$ AND C$ (1) < > PS$ : GOTO 1110
1110 : NEXT K
```

Also some of the problems (3.1) such as $x^3 + a = 0$ do not show PURESQUARE and NON SQUARE when K is N−1 to 1. In the original program it shows "overflow" at statement 4070 and statement 5090. So we introduce statement 4065 and statement 5085

among them as follows: .

```
4065 : IF C$ ( K ) < > PS$ AND C$ ( K ) < > NS$ THEN GOTO 4170
4070 : R(K) = (A (R,K−1) / A(R, K+1) ) ^ (1 / M ) * 10 ^ ( (FACT ( R, K−1) −
       FACT(R. K+1) ) / M
5085 : IF C$ ( K ) < > PS$ AND C$ ( K ) < > NS$ THEN GOTO 5230
5090 : R (K)=(ABS(A(R,K-1)/A(R,K+1))) ^ (1/M)*10 ^((FACT(R. K-1)-FACT(R,K+1) ) / M
```

The program [9] will be modified identifying the inability of solving some of the non-linear algebraic equations of the form (3.1). The modified GRAEFFE.BAS is developed using the following algorithm.

Algorithm

Identify (the presence of any zero coefficients)

or (all the coefficients are unity)

if (there is no pair of non zero coefficients from the zero coefficient)

or (all coefficients are unity)

then go to print "equation is not solvable by graeffe's root squaring method"

Repeat the above steps for each subsequent new transformed coefficients.

3.5 Modified GRAEFFE.BAS

```
10   PRINT "*****************************************************************"
20   PRINT " *                                                              *"
30   PRINT " *           GRAEFFE'S ROOT-SQUARING METHOD                      *"
40   PRINT " *                                                              *"
50   PRINT " *           (Modified GRAEFFE . BAS)                            *"
90   PRINT "*****************************************************************"
60   PRINT "*****************************************************************"
100  '************************* MAIN PROGRAM *************************
110  '
120  ' Define the polynomial
130  '
140  PRINT " DEGREE OF POLYNOMIAL " ; : INPUT N
150  DIM A(20, N), C$(N), SUM(N), FACT(20, N), LC(N), R(N), X(N), XI(N),
     ROOTS(N)
160  FOR K = N TO 0 STEP -1
170  PRINT " COEFFICIENT " ; K :
180  INPUT A(0, K)
190  NEXT K
200  PRINT " GIVE THE CONVERGENCE VALUE OF F " ; : INPUT EPS
210  '
220  ' Extract factor of ten and reduce coefficients by this factor.
230  ' In order to be able to handle very large numbers.
240  ' Keep account of factors, and increase or decrease accordingly.
250  '
260  FOR K = N TO 0 STEP -1
270  IF A (0, K) = 0 GOTO 300
280  FACT (0, K) = INT (LOG (ABS (A (0, K ) ) ) / LOG ( 10 ) )
290  A (0, K) = A (0 ,K) / ( 10 ^ FACT (0, K ) )
300  NEXT K
```

```
310   PRINT: PRINT " ROOT SQUARING PROCESS : "
320   '
330   ' Print the original coefficients with the factors.
340   '
350   PRINT:PRINT "  r" ; :FOR  K= N TO 0 STEP-1 :PRINT"    A   "; :NEXT  K. PRINT
360   FOR K =  N  TO  0  STEP  -1 :  PRINT "       r  "; K ;  : NEXT K  : PRINT
370   PRINT USING  "## " ;  R ;
380   FOR K =  N  TO  0  STEP  -1
390   PRINT USING "###.###"; A(0, K):: PRINT"E"; :PRINT USING "###  "; FACT(0,K),
400   NEXT  K
410   PRINT
420   '
430   ' Beginning of major iteration
440   '
450   FOR R= 1  TO  20
460   PRINT USING "##  " ;  R ;
470   '
480   ' Calculate new coefficients
490   '
500   FOR I =  N  TO  0  STEP  -1
510   FACT ( R, I ) = 2 *  FACT ( R-1, I )
520   SUM( I ) = 0
530   FOR  L = 1  TO  I
540   IF ( I - L ) < 0  OR  (I + L) > N  THEN GOTO  590
550   FA = FACT ( R -1, I + L) +  FACT ( R-1, I-L )
560   FB = FACT  (R, I) - FA
570   IF FB > 20  GOTO  590
580   SUM ( I ) =  SUM( I ) + 2* (-1) ^ L * A ( R-1, I+L ) * A ( R-1, I-L ) / ( 10 ^ FB )
590   NEXT  L
600   A( R , I) = A ( R -1, I) ^ 2 + SUM ( I )
602   ' New statements
```

```
603   IF ABS(A(R,I)) = 0 THEN GOTO 605 ELSE GOTO 610
605   IF ABS(A(R, I-1)) <> 0 AND ABS(A(R, I+1)) <>0 THEN GOTO 1345
606   '
610   IF ABS(A(R,I))>10THEN A(R, I )=A(R, I )/10:FACT (R,I)=FACT(R,I)+1.GOTO  610
620   IF ABS(A( R, I ))<1 THEN A (R, I ) = A (R, I )* 10 : FACT (R, I) = FACT (R, I ) –1
630   PRINT USING "###.###"; A(R, I): : PRINT "E";:PRINT USING "###  "; FACT(R, I):
640   IF ABS (FACT ( R, I ) ) > 999 THEN GOTO 880
650   NEXT I
660   '
670   ' Check whether coefficients are pure squares
680   '
690   PS$ =  " PURE SQUARE" :  NS$ =  "non-square"
700   C$( N ) = PS$  :  C$ ( 0 ) = PS$
710   FOR  I = N–1  TO  1  STEP  –1
720   IF  SUM ( I ) = 0 GOTO 750
730   W =  ( A (R–1,  I ) ^ 2 ) / SUM ( I )
740   IF  ABS (W) > 1000 THEN  C$ ( I ) = PS$  ELSE  C$ ( I ) = NS$
750   NEXT  I
760   PRINT
770   SQUARES = 0
780   FOR I = N  TO  0  STEP  –1
790   IF  C$ ( I ) =  PS$  THEN  SQAURES =  SQUARES  + 1
800   NEXT  I
810   NSQUARES =  N + 1 – SQUARES
820   IF SQUARES =  N + 1 THEN GOTO  900
830   NEXT  R
840   ' End of major iteration.
850   '
860   PRINT:PRINT  "ITERATIONS  EXCEEDED.  POSSIBILITY  OF  COMPLEX  OR
      REPEATED ROOTS ."
870   GOTO  890
```

```
880   PRINT :  PRINT " FACTOR EXCEEDS  999 . POSSIBILITY OF COMPLEX OR
      REPEATED ROOTS."
890   R = R-1
900   PRINT : PRINT " THE COEFFICIENTS ARE : " : PRINT
910   PRINT "    "; : FOR I = N  TO 0 STEP -1 : PRINT C$ ( I ) ; " , " ;  NEXT: PRINT
920   '
930   ' Find location of non-squares
940   '
950   FOR  K =  N  TO  0  STEP -1
960   IF  C$( K ) <> PS$  THEN  LC ( K ) = K
970   NEXT  K
980   '
990   ' Evaluate the real roots and the sum of these roots.
1000  '
1010  M = 2 ^ R
1020  PRINT :PRINT "THE NUMBER OF SQUARING : r = "  ; R ,"THE POWER:m ="; M
1030  PRINT : PRINT " CALCULATION OF ROOTS : "
1040  SUMRT  = 0
1050  FOR  K = N  TO  1  STEP -1
1060  NR =  N + 1 - K
1070  IF  C$ ( K - 1 ) <>  PS$  THEN  K = K - 1 :  GOTO 1110
1074  ' New statement
1075  IF C$(N-1) <> PS$ AND C$(1) <> PS$ THEN GOTO 1110
1076  '
1080  X (NR) =(A(R, K-1)/A (R, K )) ^ (1/M )* 10 ^ (( FACT (R,K-1)-FACT(R, K )) / M)
1090  GOSUB 1500 ' Check the sign of the root
1100  SUMRT =  SUMRT  + X ( NR )
1110  NEXT  K
1120  '
1130  ' Calculating the sum of the product of the real roots
1140  GOSUB  2000
```

```
1150 ' Branch to subroutine for complex or repeated roots
1160 ON NSQUARES GOSUB 4000, 5000, 6000
1170 IF NSQUARES > 3 THEN PRINT " ***PROGRAM CAN NOT DETERMINE
     MORE THAN TWO PAIRS OF COMPLEX ROOTS ***"
1180 'Print all the roots
1190 PRINT : PRINT "THE" ; N ; "ROOTS ARE : "
1200 FOR K = 1 TO N
1210 YY = 1000
1220 IF ABS ( X ( K ) * YY ) > 32000 THEN YY = YY / 10 : GOTO 1220
1230 IF X( K ) = 0 THEN GOTO 1260
1240 IF ABS ( X ( K ) * YY ) < 1000 THEN YY = YY * 10 : GOTO 1240
1250 X ( K ) = CINT ( X ( K ) * YY ) / YY
1260 YY = 1000
1270 IF ABS ( XI ( K ) * YY ) > 32000 THEN YY = YY / 10 : GOTO 1270
1280 IF XI ( K ) = 0 THEN GOTO 1310
1290 IF ABS ( XI ( K ) * YY ) < 1000 THEN YY = YY * 10 : GOTO1290
1300 XI ( K ) = CINT ( XI ( K ) * YY ) / YY
1310 PRINT TAB (20) ; X( K);
1320 IF XI ( K ) > 0 THEN PRINT " + " ; XI ( K ) ; " i "
1330 IF XI (K) < 0 THEN PRINT "–" ; – XI(K) ; " i "
1340 NEXT K
1345 GOSUB 7000 'New statement
1350 END
1360 '
1500 ' ********** Subroutine 1 : Checking sign of real roots *********
1510 '
1520 FUP = A( 0, 0) * (10 ^ FACT ( 0, 0 ) ) : FUN = A (0, 0) * (10 ^ FACT (0, 0) )
1530 FOR I = N TO 1 STEP –1
1540 FUP = FUP + A (0, I ) * (10 ^ FACT (0, I ) ) * X (NR ) ^ I
1550 FUN = FUN + A ( 0, I ) * ( 10 ^ FACT ( 0, I ) ) * ( –X ( NR ) ) ^ I
1560 NEXT I
```

```
1570  PRINT:PRINT "      FUNCTION WITH POSITIVE VALUE OF(" ;X (NR) ; ")="; FUP
1580  PRINT "        FUNCTION WITH NEGATIVE VALUE OF(" : X(NR) : ") = "; FUN
1590  IF ABS (FUP) > EPS AND ABS (FUN) > EPS THEN PRINT " * WARNING :
      CONVERGENCE NOT SATISFIED BY REAL ROOT *"
1600  ' choose root which gives lowest value of function
1610  IF  ABS (FUN) < ABS (FUP) THEN  X(NR) =  -X(NR)
1620  RETURN
1630  '
2000  ' * Subroutine 2 : Calculation of the sum of the product of the real roots*.
2010  '
2020  PRODRT  = 0
2030  FOR  K = 1  TO  N-1
2040  FOR  KK =  K+1  TO  N
2050  PRODRT = PRODRT  + X(K)  * X (KK)
2060  NEXT  KK
2070  NEXT  K
2080  RETURN
2090  '
3000  ' ** Subroutine 3 : Checking complex roots **********
3010  '
3020  FOR  KK = 0  TO  N  :  SUMI (KK)  = 0  : NEXT  KK
3030  FOR  KK =  0  TO  N  :  SUMC (KK) = 0  :  NEXT  KK
3040  SUMC ( 0 ) = 1
3050  FOR  I =  1  TO  N+1
3060  SUMC ( 1 ) = I
3070  FOR  KK =  2  TO  N
3080  SUMC (KK) =  SUMC (KK)  + SUMC (KK-1)
3090  NEXT  KK
3100  FOR  J =  0  TO  N+1-I
3110  SUMI (J)=SUMI(J)+ A(0, I-1+J)*(10^FACT (0, I-1+J ))*(ALPHA^(I-1))*SUMC (J)
3120  NEXT  J
```

```
3130  NEXT I
3140  SUMII = 0
3150  FOR I = 1 TO N STEP 2
3160  SUMII = SUMII + ( BETA ^ I ) * ( SUMI ( I ) ) * (-1 ) ^ ( ( I-1 ) / 2 )
3170  NEXT I
3180  SUMII2 = 0
3190  FOR I = 2 TO N STEP 2
3200  SUMII2 = SUMII2 + ( BETA ^ I ) * ( SUMI ( I ) ) * ( -1 ) ^ ( I / 2 )
3210  NEXT I
3220  SUMALL = SUMI(0) + SUMII + SUMII2
3230  IF ABS ( SUMALL ) < EPS THEN RETURN
3240  PRINT"* WARNING:CONVERGENCE NOT SATISFIED BY COMPLEX ROOT*"
3250  PRINT " * VALUE OF FUNCTION = " ; SUMALL ; " > " ; EPS ; " * "
3260  RETURN
3270  '
4000  ' ****** Subroutine 4 : One pair of complex or repeated roots *****
4010  '
4020  PRINT: PRINT "CALCULATION IF ONE PAIR OF COMPLEX OR REPEATED
      ROOTS:"
4030  ALPHA =((-A( 0, N-1) / A(0, N)) * 10 ^(FACT (0, N-1)-FACT (0, N))-SUMRT )/ 2
4040  FOR K = N TO 1 STEP -1
4050  NR = N + 1 - K
4060  IF LC ( K ) = 0 THEN GOTO 4170
4070  R( K) = (A (R, K-1)/A(R, K+1)) ^ (1/M) * 10 ^ ((FACT(R, K-1)-FACT(R, K+1))/ M)
4080  BETA = SQR ( ABS ( R ( K ) -ALPHA ^ 2 ) )
4090  'Check for repeated roots
4100  PRINT " CHECK FOR REPEATED ROOTS: "
4110  X( NR ) = SQR ( R(K) ): GOSUB 1500
4120  IF ABS(FUP) > EPS AND ABS (FUN) > EPS THEN PRINT " * ROOTS ARE
      COMPLEX *" : GOTO 4190
```

```
4130  IF ABS(FUP) < EPS AND ABS(FUN) < EPS THEN PRINT " ** ROOTS ARE REAL AND REPEATED BUT OF OPPOSITE SIGN**": X(NR-1) = - X(NR)
4140  IF ABS(FUP) < EPS AND ABS(FUN) > EPS THEN PRINT " ** ROOTS ARE REAL AND THE SAME SIGN**": X(NR-1)=X(NR)
4150  IF ABS (FUP) > EPS AND ABS(FUN) < EPS THEN PRINT " ** ROOTS ARE REAL AND REPEATED AND OF THE SAME SIGN**": X(NR-1)=X(NR)
4160  GOTO 4210
4170  NEXT K
4180  ' Check convergence with complex roots
4190  GOSUB 3000
4200  X(NR) = ALPHA : XI(NR) = -BETA : X(NR-1) = ALPHA : XI(NR-1) = BETA
4210  RETURN
4220  '
5000  ' Subroutine 5 : Two pair of complex or repeated roots with different moduli:"
5010  '
5020  RSUM = 0
5030  PRINT: PRINT "TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH DIFFERENT MODULI:"
5040  B = ( (A(0, N-1) / A(0, N) ) * 10 ^ (FACT (0, N-1) - FACT(0,N ) ) +SUMRT ) / 2
5050  PRINT "CHECK FOR REPEATED ROOTS:"
5060  FOR K = N TO 1 STEP -1
5070  NR = N + 1 - K
5080  IF LC(K) = 0 THEN GOTO 5230
5085  IF C$(K) <> PS$ AND C$(K) <> NS$ GOTO 5230 ' New statement
5090  R(K)=(ABS(A(R,K-1)/A(R, K+1)))^(1/M)*10 ^((FACT (R, K-1)-FACT(R, K+1))/M)
5100  RSUM = RSUM + R ( K )
5110  ' Check for repeated roots
5120  X (NR) = SQR ( R ( K ) ) : GOSUB 1500
5130  IF ABS (FUP) > EPS AND ABS (FUN) > EPS THEN PRINT " * ROOTS ARE COMPLEX *": GOTO 5230
```

```
5140  IF ABS (FUP) < EPS  AND  ABS (FUN) < EPS  THEN PRINT " ** ROOTS ARE
      REAL AND REPEATED BUT OF OPPOSITE SIGN**";  X(NR-1) = -X(NR)
5150  IF ABS (FUP) < EPS AND ABS (FUN) > EPS THEN PRINT "** ROOTS ARE
      REAL AND REPEATED AND OF THE SAME SIGN**": X(NR-1)=X(NR)
5160 IF ABS (FUP) > EPS AND ABS (FUN) < EPS THEN PRINT "** ROOTS ARE REAL
      AND REPEATED AND OF THE SAME SIGN**" :  X(NR-1) = X(NR)
5170  'Cancel the nonsquare corresponding to the repeated pair
5180  ' and goto the subroutine for one pair of roots
5190  LC( K) = 0
5200  SUMRT = SUMRT + X(NR) + X(NR-1)
5210  GOSUB  4000
5220  GOTO  5400
5230  NEXT  K
5240  C = ( (A(0, N-2)/A (0, N)) * 10 ^ ( FACT (0, N-2) -FACT(0, N ) ) + 2 * B * SUMRT
      - PRODRT -RSUM ) / 4
5250  ALPHA1 = ( -B + SQR ( ABS ( B ^ 2 - 4 * C ) ) ) / 2
5260  ALPHA2 = -B - ALPHA1
5270  COUNT  = 0
5280  FOR  K =  N  TO 1  STEP  -1
5290  NR = N + 1 - K
5300  IF LC(K) = 0  THEN  GOTO  5390
5310   COUNT =  COUNT + 1
5320   IF COUNT  = 1 THEN  BETA1 =  SQR ( ABS ( R (K) - ALPHA1 ^ 2 ) )
5330  ' Check convergence with complex roots
5340  ALPHA = ALPHA1:  BETA = BETA1:  GOSUB  3000
5350  IF ABS (SUMALL) > EPS THEN PRINT " * SWAP THE VALUE OF ALPHA1
      AND ALPHA2 AND TRY AGAIN *" : SWAP ALPHA1, ALPHA2: GOTO 5270
5360  IF COUNT =1 THEN X(NR)= ALPHA1: XI(NR) = -BETA1: X (NR-1) = ALPHA1 :
      XI(NR-1) = BETA1
5370  IF COUNT = 2 THEN BETA2 = SQR(ABS (R (K) - ALPHA2 ^ 2 ) )
```

```
5380  IF COUNT =2 THEN X(NR) = ALPHA2: XI(NR) = –BETA2: X(NR–1)= ALPHA2:
      XI(NR–1) = BETA2
5390  NEXT K
5400  RETURN
5410  '
6000  ' Subroutine 6 : Two pairs of complex or repeated roots with identical moduli
6010  '
6020  ' Check for neighboring nonsquares
6030  W = 0
6040  FOR K = N TO 1 STEP –1
6050  IF LC(K) < > 0 AND LC(K–1) < > 0 THEN W = 1
6060  NEXT K
6070  IF W = 0 THEN GOTO 6520
6080  PRINT: PRINT "TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH
      IDENTICAL MODULI:"
6090  P = ( NSQUARES +1 ) / 2
6100  B = ( ( A(0, N –1)/A (0, N) ) *10 ^ (FACT(0, N–1) – FACT (0, N ) ) + SUMRT ) / 2
6110  RSUM = 0
6120  PRINT"    CHECK FOR REPEATED ROOTS:"
6130  FOR K = N TO 1 STEP –1
6140  NR = N + 1 – K
6150  IF LC(K) = 0 THEN GOTO 6230
6160  R(K) = (A (R, K+1–2 * P ) / A (R, K+1 ) ) ^( 1 / (P*M) ) * 10 ^((FACT (R, K+1–2*P )
      –FACT(R, K+1) ) / (P*M) )
6170  RSUM = RSUM + P * R(K)
6180  X(NR) = SQR(R(K) ): GOSUB 1500
6190  IF ABS (FUP) > EPS AND ABS (FUN) > EPS THEN PRINT " * ROOTS ARE
      COMPLEX *":GOTO 6240
6200  IF ABS (FUP)<EPS AND ABS (FUN) < EPS THEN PRINT " ** ROOTS ARE REAL
      AND REPEATED BUT OF OPPOSITE SIGN**": X(NR–1)= – X(NR): GOTO 6380
```

```
6210  IF ABS (FUP) < EPS AND ABS (FUN) > EPS THEN PRINT " * ROOTS ARE REAL
      AND REPEATED AND OF THE SAME SIGN**": GOTO 6240
6220  IF ABS (FUP) > EPS AND ABS (FUN) < EPS THEN PRINT " * ROOTS ARE REAL
      AND REPEATED AND OF THE SAME SIGN**": GOTO 6240
6230  NEXT K
6240  ' Repeated roots of same sign or complex roots
6250  C =(( A(0, N-2) / A(0, N)) * 10 ^ (FACT (0, N-2) -FACT (0, N))+2*B* SUMRT -
      PRODRT - RSUM ) / 4
6260  ALPHA1 = (-B + SQR (ABS (B ^ 2 -4 * C ) ) ) / 2
6270  ALPHA2 = -B - ALPHA1
6280  FOR K = N TO 1 STEP -1
6290  NR = N + 1 - K
6300  IF LC(K) = 0 THEN GOTO 6360
6310  BETA1 = SQR (ABS (R (K) - ALPHA1 ^ 2 ) )
6320  X(NR) = ALPHA1: XI(NR) = -BETA1: X(NR-1) = ALPHA1: XI(NR-1) = BETA1
6330  BETA2 = SQR ( ABS ( R (K) - ALPHA2 ^ 2 ) )
6340  X(NR+2)=ALPHA2:XI(NR+2)=-BETA2:X(NR+1) = ALPHA2 : XI(NR+1)= BETA2
6350  GOTO 6370
6360  NEXT K
6370  RETURN
6380  ' Repeated roots of opposite sign or complex roots with zero real part
6390  COUNT = 0
6400  ALPHA2 = - B
6410  FOR K = N TO 1 STEP -1
6420  NR = N + 1 - K
6430  IF LC(K) = 0 THEN GOTO 6500
6440  BETA2 = SQR (ABS (R(K) -ALPHA2 ^ 2 ) )
6450  ALPHA = ALPHA2: BETA = BETA2: GOSUB 3000
```

```
6460  IF ABS(SUMALL)>EPS THEN PRINT"*SWAP THE VALUE OF ALPHA2 AND
      BETA2 AND TRY AGAIN* ": SWAP ALPHA2,BETA2: COUNT= COUNT+1:
      GOTO 6450
6470  X(NR+2)=ALPHA2: XI(NR+2)=-BETA2:X(NR+1) = ALPHA2: XI(NR+1) = BETA2
6480  IF BETA2 =0 AND COUNT > 0 THEN  X(NR+2) =ALPHA2: X(NR+1) = -ALPHA2
6490  GOTO  6510
6500  NEXT  K
6510  RETURN
6520  PRINT " ** PROGRAM CANNOT DETERMINE MORE THEN TWO PAIRS OF
      COMPLEX ROOTS **"
6530  RETURN
7000  'Subroutine 7: " for not solvable "
7011   FOR I = N-1  TO  1  STEP -1
7021   IF ABS (A(R, I) ) = 0 THEN GOTO 7030
7031   IF ABS(A (R, I-1) )< > 0  AND  ABS(A (R, I+1) ) < > 0  THEN GOTO 7050
7041  NEXT  I
7050  FOR K=1 TO N
7060  IF  X(K) + XI(K) =0  THEN GOTO 7090
7070  IF  X(K) + XI(K) <>0  THEN GOTO 1350
7080  NEXT  K
7090  PRINT: PRINT " EQUATION IS NOT SOLVABLE BY GRAEFFE'S ROOT
      SQUARING METHOD"
7100  ,
```

***************** GRAEFFE'S ROOT - SQUARING METHOD ********************

************************ (GRAEFFE.BAS) **************************************

DEGREE OF POLYNOMIAL 5

COEFFICIENT 5 IS 1

COEFFICIENT 4 IS -5

COEFFICIENT 3 IS -15

COEFFICIENT 2 IS 85

COEFFICIENT 1 IS -26

COEFFICIENT 0 IS -120

GIVE THE CONVERGENCE VALUE OF F = 0.002

ROOT-SQUARING PROCESS:

R	A_{r5}	A_{r4}	A_{r3}	A_{r2}	A_{r1}	A_{r0}
0	1.000E 0	-5.000E 0	-1.500E 1	8.500E 1	-2.600E 1	-1.200E 2
1	10.000E -1	55.000E 0	10.230E 2	76.450E 2	21.076E 3	14.400E 3
2	10.000E -1	97.900E 1	24.773E 4	16.909E 6	22.402E 7	20.736E 7
3	10.000E -1	46.298E 4	28.712E 9	17.531E 13	43.173E 15	42.998E 15
4	10.000E -1	15.693E 10	66.212E 19	28.254E 27	18.489E 32	18.488E 32
5	10.000E -1	23.302E 21	42.954E 40	79.587E 55	34.182E 65	34.182E 65
6	10.000E -1	54.210E 43	18.447E 82	63.340E 112	11.684E 132	11.684E 132
7	10.000E -1	29.388E 88	34.028E 165	40.119E 226	13.652E 265	13.652E 265

THE COEFFICIENTS ARE:

PURE SQUARE, PURE SQUARE, PURE SQUARE, PURE SQUARE, PURE SQUARE, PURE SQUARE

THE NUMBER OF SQUARING: r = 7 THE POWER: m = 128

CALCULATION OF ROOTS:

FUNCTION WITH POSSITIVE VALUE OF (5) = 1.602173E-04

FUNCTION WITH NEGATIVE VALUE IS (5) = -2240.001

FUNCTION WITH POSSITIVE VALUE OF (4) = -80.00003

FUNCTION WITH NEGATIVE VALUE IS (4) = 1.959801E-04

FUNCTION WITH POSSITIVE VALUE OF (3) = 1.525879E-05

FUNCTION WITH NEGATIVE VALUE IS (3) = 480

FUNCTION WITH POSSITIVE VALUE OF (2) = 1.478195E-05

FUNCTION WITH NEGATIVE VALUE IS (2) = 280.0001

FUNCTION WITH POSSITIVE VALUE OF (1) = -80

FUNCTION WITH NEGATIVE VALUE IS (1) = -8.583069E-06

THE 5 ROOTS ARE:

5

-4

3

2

-1

Example 02

**************** GRAEFFE'S ROOT - SQUARING METHOD ****************

*********************** (GRAEFFE.BAS) ****************************

DEGREE OF POLYNOMIAL 5

COEFFICIENT 5 IS 1

COEFFICIENT 4 IS -10

COEFFICIENT 3 IS 42

COEFFICIENT 2 IS -102

COEFFICIENT 1 IS 145

COEFFICIENT 0 IS -100

GIVE THE CONVERGENCE VALUE OF F = 0.001

ROOT-SQUARING PROCESS:

R	A r 5	A r 4	A r 3	A r 2	A r 1	A r 0
0	1.000E 0	-1.000E 1	4.200E 1	-1.020E 2	1.450E 2	-1.000E 2
1	10.000E -1	1.600E 1	1.400E 1	0.224E 3	0.625E 3	10.000E 3
2	10.000E -1	22.800E 1	-57.220E 2	3.527E 5	-40.894E 5	10.000E 7
3	10.000E -1	63.428E 3	-13.626E 7	12.318E 10	-53.812E 12	10.000E 15
4	10.000E -1	42.957E 8	28.322E 14	17.776E 20	43.212E 25	10.000E 31
5	10.000E -1	18.447E 18	-72.495E 29	15.712E 41	-16.879E 52	10.000E 63
6	10.000E -1	34.029E 37	-54.139E 59	39.046E 82	-29.349E104	10.000E 127
7	10.000E -1	11.580E 76	-23.643E 121	18.874E 166	-69.479E210	10.000E 255
8	10.000E -1	13.409E 153	12.188E 243	50.851E 332	10.525E422	10.000E511
9	10.000E -1	17.981E 307	12.183E 486	28.833E 665	90.741E843	10.000E1023

FACTOR EXCEEDS 999. POSSIBILITY OF COMPLEX OR REPEATED ROOTS.

THE COEFFICIENTS ARE:

PURE SQUARE, PURE SQUARE, NON SQUARE , NON SQUARE, NON SQUARE,

THE NUMBER OF SQUARING: r = 8 THE POWER: m = 256

CALCULATION OF ROOTS:

FUNCTION WITH POSSITIVE VALUE OF (4.000002) = 1.125336E-04

FUNCTION WITH NEGATIVE VALUE IS (4.000002) = -8584.013

TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH IDENTICAL MODULI:

CHECK FOR REPEATED ROOTS:

FUNCTION WITH POSSITIVE VALUE OF (2.236068) = -10.29418

FUNCTION WITH NEGATIVE VALUE IS (2.236068) = -1709.705

WARNING : CONVERGENCE NOT SATISFIED BY REAL ROOT

ROOTS ARE COMPLEX

* THE 5 ROOTS ARE:

4

2 + 1i

2 - 1i

1 + 2i

1 - 2i

***************** GRAEFFE'S ROOT - SQUARING METHOD ********************

*********************** (GRAEFFE.BAS) ************************************

DEGREE OF POLYNOMIAL 5

COEFFICIENT 5 IS 1

COEFFICIENT 4 IS 0

COEFFICIENT 3 IS 0

COEFFICIENT 2 IS 0

COEFFICIENT 1 IS 0

COEFFICIENT 0 IS 1

GIVE THE CONVERGENCE VALUE OF F = 0.001

ROOT-SQUARING PROCESS:

R	A r 5	A r 4	A r 3	A r 2	A r 1	A r 0
0	1.000E 0	0.000E 0	0.000E 0	0.000E 0	0.000E 0	1.000E 0
1	10.000E -1	0.000E -1	0.000E -1	0.000E -1	0.000E -1	10.000E -1
2	10.000E -1	0.000E -3	0.000E -3	0.000E -3	0.000E -3	10.000E -1
3	10.000E -1	0.000E -7	0.000E -7	0.000E -7	0.000E -7	10.000E -1
4	10.000E -1	0.000E -15	0.000E -15	0.000E -15	0.000E -15	10.000E -1
5	10.000E -1	0.000E -31	0.000E -31	0.000E -31	0.000E -31	10.000E -1
6	10.000E -1	0.000E -63	0.000E -63	0.000E -63	0.000E -63	10.000E -1
7	10.000E -1	0.000E -127	0.000E -127	0.000E -127	0.000E -127	10.000E -1
8	10.000E -1	0.000E -255	0.000E -255	0.000E -255	0.000E -255	10.000E -1
9	10.000E -1	0.000E -511	0.000E -511	0.000E -511	0.000E -511	10.000E -1
10	10.000E -1	0.000E%-1023				

FACTOR EXCEEDS 999 . POSSIBILITY OF COMPLEX OR REPEATED ROOTS.

THE COEFFICIENTS ARE:

PURE SQUARE , . , , , PURE SQUARE

THE NUMBER OF SQUARING: r = 9 THE POWER: m = 512

CALCULATION OF ROOTS:

THE 5 ROOTS ARE:

0

0

0

0

0

EQUATION IS NOT SOLVABLE BY GRAEFFE'S ROOT SQUARING METHOD

**************** GRAEFFE'S ROOT - SQUARING METHOD *******************

*********************** (GRAEFFE.BAS) **********************************

DEGREE OF POLYNOMIAL 4

COEFFICIENT 4 IS 1

COEFFICIENT 3 IS 0

COEFFICIENT 2 IS 0

COEFFICIENT 1 IS 0

COEFFICIENT 0 IS 1

GIVE THE CONVERGENCE VALUE OF F = 0.0001

ROOT-SQUARING PROCESS:

R	A_{r4}	A_{r3}	A_{r2}	A_{r1}	A_{r0}
0	1.000E 0	0.000E 0	0.000E 0	0.000E 0	0.000E 0
1	10.000E -1	0.000E -1	20.000E -1	0.000E -1	10.000E -1
2	10.000E -1	-40.000E -1	60.000E -1	-40.000E -1	10.000E -1
3	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
4	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
5	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
6	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
7	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
8	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
9	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
10	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
11	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
12	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
13	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
14	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
15	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
16	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
17	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
18	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
19	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1
20	10.000E -1	40.000E -1	60.000E -1	40.000E -1	10.000E -1

ITERATIONS EXCEEDED. POSSIBILITY OF COMPLEX OR REPEATED ROOTS.

THE COEFFICIENTS ARE:

PURE SQUARE , NON SQUARE, NON SQUARE , NON SQUARE, PURE SQUARE

THE NUMBER OF SQUARING: r = 20 THE POWER: m = 1048576

CALCULATION OF ROOTS:

TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH IDENTICAL MODULI:

CHECK FOR REPEATED ROOTS

FUNCTION WITH POSITIVE VALUE OF (1) = 2

FUNCTION WITH NEGATIVE VALUE IS (1) = 2

WARNING: CONVERGENCE NOT SATISFIED BY REAL ROOT

* ROOTS ARE COMPLEX *

THE 4 ROOTS ARE:

0.7071 + 0.7071 i

0.7071 - 0.7071 i

- 0.7071 + 0.7071 i

- 0.7071 - 0.7071 i

Example:05

****************** GRAEFFE'S ROOT - SQUARING METHOD ************************

************************ (GRAEFFE.BAS) ************************************

DEGREE OF POLYNOMIAL 3

COEFFICIENT 3 IS 1

COEFFICIENT 2 IS 0

COEFFICIENT 1 IS 0

COEFFICIENT 0 IS 1

GIVE THE CONVERGENCE VALUE OF F = 2

ROOT-SQUARING PROCESS:

R	A r 3	A r 2	A r 1	A r 0
0	1.000E 0	0.000E 0	0.000E 0	1.000E 0
1	10.000E -1	0.000E -1	0.000E -1	10.000E -1
2	10.000E -1	0.000E -3	0.000E -3	10.000E -1
3	10.000E -1	0.000E -7	0.000E -7	10.000E -1
4	10.000E -1	0.000E -15	0.000E -15	10.000E -1
5	10.000E -1	0.000E -31	0.000E -31	10.000E -1
6	10.000E -1	0.000E -63	0.000E -63	10.000E -1
7	10.000E -1	0.000E -127	0.000E -127	10.000E -1
8	10.000E -1	0.000E -255	0.000E -255	10.000E -1
9	10.000E -1	0.000E -511	0.000E -511	10.000E -1
10	10.000E -1	0.000E%-1023		

FACTOR EXCEEDS 999 , POSSIBILITY OF COMPLEX OR REPEATED ROOTS

THE COEFFICIENTS ARE:

PURE SQUARE , , ,PURE SQUARE

THE NUMBER OF SQUARING: r = 9 THE POWER: m = 512

CALCULATION OF ROOTS:

TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH DIFFERENT MODULI:

CHECK FOR REPEATED ROOTS:

THE 3 ROOTS ARE:

0

0

0

EQUATION IS NOT SOLVABLE BY GRAEFFE'S ROOT SQUARING METHOD

Example:06

***************** GRAEFFE'S ROOT - SQUARING METHOD *********************

*********************** (GRAEFFE.BAS) ******************************

DEGREE OF POLYNOMIAL 5

COEFFICIENT 5 IS 1

COEFFICIENT 4 IS 2

COEFFICIENT 3 IS 2

COEFFICIENT 2 IS 2

COEFFICIENT 1 IS 2

COEFFICIENT 0 IS 1

GIVE THE CONVERGENCE VALUE OF F = 0.0001

ROOT-SQUARING PROCESS:

R	A r 5	A r 4	A r 3	A r 2	A r 1	A r 0
0	1.000E 0	2.000E 0	2.000E 0	2.000E 0	2.000E 0	1.000E 0
1	10.000E -1	0.000E -1	0.000E -1	0.000E -1	0.000E -1	10.000E -1
2	10.000E -1	0.000E -3	0.000E -3	0.000E -3	0.000E -3	10.000E -1
3	10.000E -1	0.000E -7	0.000E -7	0.000E -7	0.000E -7	10.000E -1
4	10.000E -1	0.000E -15	0.000E -15	0.000E -15	0.000E -15	10.000E -1
5	10.000E -1	0.000E -31	0.000E -31	0.000E -31	0.000E -31	10.000E -1
6	10.000E -1	0.000E -63	0.000E -63	0.000E -63	0.000E -63	10.000E -1
7	10.000E -1	0.000E -127	0.000E -127	0.000E -127	0.000E -127	10.000E -1
8	10.000E -1	0.000E -255	0.000E -255	0.000E -255	0.000E -255	10.000E -1
9	10.000E -1	0.000E -511	0.000E -511	0.000E -511	0.000E -511	10.000E -1
10	10.000E -1	0.000E%-1023				

FACTOR EXCEEDS 999 . POSSIBILITY OF COMPLEX OR REPEATED ROOTS.

THE COEFFICIENTS ARE:

PURE SQUARE, NON SQUARE, NON SQUARE, NON SQUARE, NON SQUARE,
PURE SQUARE

THE NUMBER OF SQUARING: r = 9 THE POWER: m = 512

CALCULATION OF ROOTS:

THE 5 ROOTS ARE:

0

0

0

0

0

EQUATION IS NOT SOLVABLE BY GRAEFFE'S ROOT SQUARING METHOD

***************** GRAEFFE'S ROOT - SQUARING METHOD ********************

************************* (GRAEFFE.BAS) ***********************************

DEGREE OF POLYNOMIAL 4

COEFFICIENT 4 IS 1

COEFFICIENT 3 IS 7

COEFFICIENT 2 IS 12

COEFFICIENT 1 IS -4

COEFFICIENT 0 IS -16

GIVE THE CONVERGENCE VALUE OF F = 0.0001

ROOT-SQUARING PROCESS:

R	$A_{r\,4}$	$A_{r\,3}$	$A_{r\,2}$	$A_{r\,1}$	$A_{r\,0}$
0	1.000E 0	7.000E 0	1.200E 1	-4.000E 0	-1.600E 1
1	10.000E -1	25.000E 0	16.800E 1	40.000E 1	25.600E 1
2	10.000E -1	28.900E 1	87.360E 2	73.984E 3	65.536E 3
3	10.000E -1	66.049E 3	33.686E 6	43.286E 8	42.950E 8
4	10.000E -1	42.951E 8	56.296E 13	18.447E 18	18.447E 18
5	10.000E -1	18.447E 18	15.846E 28	34.028E 37	34.028E 7
6	10.000E -1	34.028E 37	12.555E 57	11.579E 76	11.579E 6
7	10.000E -1	11.579E 76	78.818E 114	13.408E 153	13.408E153
8	10.000E -1	13.408E 153	31.072E 230	17.977E 307	17.977E307
9	10.000E -1	17.977E 307	48.341E 461	32.318E 615	32.318E615
10	10.000E -1	32.316E 615	11.749E 924	10.445E%1232	

FACTOR EXCEEDS 999 . POSSIBILITY OF COMPLEX OR REPEATED ROOTS

THE COEFFICIENTS ARE:

PURE SQUARE , PURE SQUARE, NON SQUARE . PURE SQUARE, PURE SQUARE

THE NUMBER OF SQUARING: r = 9 THE POWER: m = 512

CALCULATION OF ROOTS:

FUNCTION WITH POSITIVE VALUE OF (4) = 863.9999

FUNCTION WITH NEGATIVE VALUE IS (4) = -9.536743E-07

FUNCTION WITH POSITIVE VALUE OF (1) = 4.768372E-07

FUNCTION WITH NEGATIVE VALUE IS (1) = -6

ONE PAIR OF COMPLEX OR REPEATED ROOTS:

CHECK FOR REPEATED ROOTS:

FUNCTION WITH POSITIVE VALUE OF (2) = 96

FUNCTION WITH NEGATIVE VALUE IS (2) = 1.907349E-06

* ROOTS ARE REAL AND REPEATED AND OF THE SAME SIGN +

THE 4 ROOTS ARE:

-4

-2

-2

1

Example:08

***************** GRAEFFE'S ROOT - SQUARING METHOD ***********************

************************ (GRAEFFE.BAS) **************************************

DEGREE OF POLYNOMIAL 4

COEFFICIENT 4 IS 1

COEFFICIENT 3 IS -8

COEFFICIENT 2 IS 42

COEFFICIENT 1 IS -80

COEFFICIENT 0 IS 125

GIVE THE CONVERGENCE VALUE OF F = 0.0001

ROOT-SQUARING PROCESS:

R	A r 4		A r 3		A r 2		A r 1		A r 0	
0	1.000E	0	-8.000E	0	4.200E	1	-8.000E	1	1.250E	2
1	10.000E	-1	-20.000E	0	73.400E	1	-41.000E	2	15.625E	3
2	10.000E	-1	-10.680E	2	40.601E	4	-61.275E	5	24.414E	7
3	10.000E	-1	32.861E	4	15.224E	10	-16.070E	13	59.605E	15
4	10.000E	-1	-19.650E	10	23.283E	21	76.756E	26	35.527E	32
5	10.000E	-1	-79.553E	20	54.210E	43	-10.652E	55	12.622E	66
6	10.000E	-1	-10.209E	44	29.387E	88	-23.379E	110	15.931E	133
7	10.000E	-1	45.451E	88	86.360E	177	-88.167E	222	25.379E	167
8	10.000E	-1	33.865E	177	74.580E	356	33.899E	446	64.411E	535
9	10.000E	-1	-13.749E	357	55.621E	714	18.836E	893	41.488E%	1072

FACTOR EXCEEDS 999. POSSIBILITY OF COMPLEX OR REPEATED ROOTS.

THE COEFFICIENTS ARE:

PURE SQUARE, NON SQUARE, PURE SQUARE, NON SQUARE, PURE SQUARE.

THE NUMBER OF SQUARING: r = 8 THE POWER: m = 256

CALCULATION OF ROOTS:

TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH DIFFERENT MODULI.

CHECK FOR REPEATED ROOTS:

FUNCTION WITH POSITIVE VALUE OF (5) = 399.9998

FUNCTION WITH NEGATIVEVALUE OF (5) = 3199.999

* WARNING : CONVERGENCE NOT SATISFIED BY REAL ROOT *

* ROOTS ARE COMPLEX *

FUNCTION WITH POSITIVE VALUE OF (2.236068) = 91.67185

FUNCTION WITH NEGATIVEVALUE OF (2.236068) = 628.3282

* WARNING : CONVERGENCE NOT SATISFIED BY REAL ROOT *

* ROOTS ARE COMPLEX *

THE 4 ROOTS ARE:

$3 + 4i$

$3 - 4i$

$1 + 2i$

$1 - 2i$

Chapter 4

Conclusion and open problems

4.1 Conclusion

Literature survey of the past and the recent has been carried out. Classical method of Graeffe's root squaring method has been discussed. Some problems not solvable manually and using GRAEFFE.BAS software [9] have been discovered under observations. The weaknesses of the method and their recoveries have been discussed.

On the observations in sec. 3.1 the following solvability conditions of Graeffe's root squaring method has been derived in sec. 3.2.

(i) equation with zero-coefficient must have at least one pair of equidistant non-zero coefficient from the zero-coefficient;

(ii) any transformed equation of a given equation with non-zero coefficient may have zero coefficients but these new coefficients must satisfy (i);

(iii) all the coefficients of non-linear albegraic equation must not be unity;

(iv) GRAEFFE.BAS needs modification in the light of (i), (ii) and (iii).

Under these solvability conditions GRAEFFE.BAS software [9] has been modified and subroutine 7 has been developed and extended the software to identify the polynomials which are not solvable by this software. Finally some open problems in sec. 4.2 have been put forward for further research of both computer science and applied mathematics.

4.2 Open problems

Although Graeffe's root squaring finds its way in the first part of twentieth century, still it needs more study. So, some open problems are cited for further research and study

(i) Development of a general software capable of solving any higher order univariate polynomial having different types of roots repeated any number of times.

(ii) Barciss [4] resultant procedure and his ALGOL 60 may be studied to solve the equations set (3.1).

(iii) Open problems of Neff and Reif [12], Pan [1,2], Demmel [13] are good research problems for both computer science and applied mathematics.

References:

[1] Victor. Y. Pan, Solving a polynomial equation: some history and recent progress, SIAM REV. 39(2), 187–220, June 1997.

[2] Victor. Y. Pan, Computation of approximate polynomial GCDs and an extension, Information and Computation 167, 71–85, 2001.

[3] Victor. Y. Pan, Univariate polynomials: nearly optimal alogorithms for numerical factorization and root-finding, J. Symbolic Computation, 33, 701–733, 2002

[4] Erwin H. Bareiss, Resultant procedure and the mechanization of the Graeffe process, presented at the meeting of the Association of US. Atomic Energy Commission, 346–386, June 11–13, 1958.

[5] Alston S. Householder, Dandeline, Lobaćevskii or Graeffe?, Amer. Math. Monthly, 66, 464–466, June–July 1959.

[6] C. A. Hutchinson, On Graeffe's method for the numerical solution of algebraic equations, Amer, Math. Monthly, 42, 149–160, 1935.

[7] Gregorio Malajovich and Jorge P. Zubelli, On the geometry of Graeffe iteration. Informes de Mathemática Série B-119, IMPA, 1–27, 2000.

[8] Gregorio Malajovich and Jorge P. Zubelli, On the geometry of Graeffe iteration, Tangent Graeffe Iteration, Informes de Mathemática Série B-119, IMPA, 1–34. 1998 Revised in August, 1999.

[9] A. Constantinides. Applied Numerical Methods with Personal Computers, McGraw Hill Book Company, 1987.

[10] E. Bodewing, On Graeffe's method for solving algebraic quations, Quart. Appl. Math, Vol. 4, 177–190, 1946.

[11] Gregorio Malajovich, On generalized Newton algorithms: Quadratic convergence, path-following and error analysis. Theoretical Computer Science 133, 65–84, 1994.

[12] C. Andrew Neff and John H. Reif, An efficient Algorithm for the complex roots problem, J. of Complexity 12, 81–115, 1996.

[13] James Demmel,, Benjamin Diament and Gregorio Malajovich. On the complexity of computing error bounds, Foundations of Computational Mathematics 1(1), 101–125. 2000.

[14] A. K. Hazra and W. Z. Loskor, "Solvability conditions of algebraic equations using Graeffes root squaring method". Proceedings of international conference on mathematics and its applications in the new millennium, Department of Mathematics, Universiti Putra, Malaysia, P 444-447. 18–19, July 2000.

[15] L. L. Cronvich. On Graeffe's method of solution of equations, Am. Math. Monthly, 46, 185–190, 1939.

[16] Z. Kopal, Numerical Analysis, 2/e, Chapman Hill, 1961.

[17] J. B. Scarborough, Numerical Mathematical Analysis, 6/e, Oxford and IBM publishing Co., New Delhi, 1966.

[18] B. Carnahan, A. Luther and J. O. Wilkes, Applied Numerical Methods, John Wiley and sons, Inc., New York, 1969

[19] E. Balagurusamy, Numerical Methods. Tata McGraw-Hill publishing Company Limited, New Delhi, 1999.

[20] W. S. Burnside and A.W. Panton, "The theory of equations, vol. 1, 13/e. S.Chand and Co. (Pvt) Limited, New Delhi, 1972.

[21] A. S. Householder, Principle of Numerical Analysis, McGraw Hill, 1953.

[22] J. H. Wilkinson, Rounding errors in algebraic process prentice-Hall, 1963.