GRAEFFE'S ROOT SQUARING METHOD: ITS
SOLVABILITY CONDITIONS, SOFTWARE MODIFICATION
AND EXTENSION

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Graeffe's method determines all the roots of univariate polynomials both real and complex, repeated and non-repeated simultaneously. In this thesis, it is said that this statement is not universally true. It is shown that the method is valid if the algebraic equations satisfy the conditions,

(i) equations with zero coefficient must have at least one pair of equidistant non-zero coefficient from the zero coefficient.

(ii) any transformed equation of a given equation with non-zero coefficient may have zero coefficients but these new coefficients must satisfy (i).

(iii) all the coefficients of non-linear equation must not be unity.

GRAEFFE.BAS program [9] has been modified in these light and has been extended for not solvable equations.
Candidate's Declaration

I hereby declare that the work which is being presented in the thesis entitled "Graeffe's root squaring method: Its solvability conditions, software modification and extension" submitted in partial fulfilment of the requirements for the award of the degree of Master of Philosophy in Mathematics, in the department of Mathematics, Bangladesh University of Engineering and Technology, Dhaka, is an authentic record of my own work.

The matter presented in this thesis has not been submitted by me for the award of any other degree in this or any other University.

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(WAHIDA ZAMAN LOSKOR)
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Chapter 1
Literature survey and objectives

1.1 Introduction

According to Pan in 1997 [1], the classical problem of solving an $n$th degree univariate polynomial and its system has substantially influenced the development of mathematics throughout centuries and still has several important applications to the theory and practice of present-day computing of the computer age.

The $n$th degree nonlinear univariate polynomial (algebraic equation of a single variable $x$) has the form

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0 = 0, \quad a_n \neq 0$$  \hspace{1cm} (1.1)

This type of the equation arises in many occasions as:

(i) auxiliary equation of higher order ODE with constant coefficients,
(ii) characteristic equations of the matrix eigenvalue problems,
(iii) in the area of computer algebra and computing geometry (an active area of modern research)

In general equation (1.1) has $n$ roots (zeros), which are of following types:

(i) real and distinct roots
(ii) real and equal roots
(iii) imaginary or complex roots and complex roots occur in conjugate pair
(iv) combination of (i), (ii) and (iii).

Although problem of solving (1.1) was known to Sumerians (third millennium BC) for particular values of $n$, all the above types of roots were not known to them. After centuries this types of roots become apparent to the present generation. Nowadays many computational problems arising in the sciences, engineering, business management and statistics have been linearised and then solved by using tools from linear algebra, linear programming. Such roots involve the solution of (1.1) for smaller $n$. Computer algebra solves (1.1) for large $n$. There are available software in case of precision, error bound etc. for this causes problems and thus motivation for further research on the design of effective algorithm for solving (1.1) arises.
Thus equation (1.1) retains its major role both as a research problem and a part of practical computational tasks in the highly important area of computing called computer algebra. These information along with others are embodied in Pan [1].

No general algebraic method is available for the solution of equation (1.1) except for very special cases like (a) quadratic (b) cubic and (c) quartic equation. There are some numerical procedures (methods) to determine real roots both distinct and equal only such as

(i) Bisection method
(ii) False-position method
(iii) Newton-Raphson method
(iv) Secant method
(v) Muller’s method
(vi) Bairstow’s method
(vii) Graeffe’s method of roots squaring

Graeffe’s method gives all the roots simultaneously, both real and complex. This method has not received much attention. Modern research based on computer has been carried out by Pan [1] in 1997, Malajovich [8] in 1999 and the references therein around the last part of 20th century and the beginning of 21st century. The lack of popularity of Graeffe’s method are embodied in Malajovich [8] and in sec. 3.3.

This project is on Graeffe’s method. Major work’s contained in chapter 3 which discusses

(i) the conditions for the applicability of the method in sec. 3.2.
(ii) the necessary modification and extension of the software GRAEFFE.BAS in sec. 3.4.
(iii) Modified GRAEFFE.BAS in sec. 3.5.
1.2 History of polynomial root finding and Graeffe's root squaring

Numerical root-finding for a univariate polynomial is a classical problem. The exact starting time of study of this problem is not known. But history reveals that Sumerians in the third B.C. have the root finding knowledge. Its study still continues to present day. Readers are referred to Pan [1.2.3] and Barciss [4] and the references therein for past and recent progress in polynomial root-finding.

Polynomial root-finding occupies its position in the present day computer algebra and computing geometry. It is being widely applied in the study of engineering sciences particularly in network theory, control linear systems and computer aided geometric design.

The basis of Graeffe's method is the "root squaring" process. Controversy prevails in the designation of the method on the issue of a Russian translation which called "Graeffe's method" as "Lobachevski's. This controversy has been discussed in Householder [5] in details and the references therein. Partial discussions are given in Hutchinson [6], Malejovich and Zubelli [7]. According to them, the method is due to Germain Dandelin (1794-1837) in 1826, although the fundamental idea goes back to Edward Waring (1734 -1798) in 1762. Dandelin's paper was not widely circulated and the process goes under the name of Carl Heinrich Graeffe (1799 -1873) in 1837. The method was also suggested independently by Nicholas Ivanovich Lobachevski in 1834. Later contributions were made by Johann Franz Encke (1791-1865) in 1841. Somehow or other the method is popularized under Graeffe.

1.3 Literature survey

Literature reveals that study of root-finding of polynomial in special cases begins in ancient times of Sumerians and Babylonians. With the full fledged growth of number system, solution of univariate polynomial finds its new ways of achievements and still it has been going on with the advancement of computer. Some of early and recent information on Graeffe's method for the study of polynomial root-finding is briefly presented as part of this thesis in the following paragraphs.
In 1946, Bodewig [10] stated many advantages of Graeffe's method such as
(i) no first approximation need be known,
(ii) approximation of all roots are obtained simultaneously,
(iii) not much laborious in comparison for knowing a single root by any other method,
(iv) main advantage of obtaining complex root,
(v) automatically separate close roots,
(vi) it gives multiple roots real and complex,
(vii) determines several pairs of complex roots with same modulus.

Only disadvantage he mentioned, it is not useful in correcting a single root as in other methods. Next he discussed a solution procedure on splitting the transformed equation. Discussing convergence process, he observed that Graeffe's method has greatest efficiency as the error decreases quadratically in each subsequent transformed new equation. Finally he discussed the position of roots with same modulus near a circle in Argand/Gaussian plane.

Bareiss [4] in 1958 presented at the meeting of Association of U.S.A Atomic Energy Commission 'the resultant procedure' for finding simultaneously all zeros of polynomials with real coefficients. The procedure relies on the modification of Graeffe's method. Principles of the procedure are outlined. He also present steps for designing the algorithm. His discussion ended with a summary of history of root squaring method. He derived a formula for determining the minimum number of root squaring.

In 1994 Malajovich [11] investigated approximate Newton iteration for several generalizations of the Newton operator. He generalized some of the results developed by Shub and Smale in 1993 [reference 6 in [11]]. The results have been used to prove complexity theorem on path-following algorithms for solving systems of polynomial equations.

In 1996 Neill and Reif [12] developed an efficient algorithm for the complex roots up to specified precision of a univariate polynomial \( f(x) \) of degree \( n \) with complex coefficients with norms less than \( 2^m \), \( m \) is the smallest integer. Their algorithm requires no assumption for separation of roots. This property makes it remarkable.
In 1997 Pan [1] describes the importance of the equation (1.1), early history of solving polynomial equations, solving equation (1.1) by geometric construction, compares some approaches for solving equation (1.1), the divide and conquer approach to approximate polynomial zeros, balancing problem in splitting the polynomial, discuss the technique of avoiding approximation of the zeros of a higher order derivative, cites three applications of polynomial equation (1.1) by different researchers.

In 1999 Malajovich [8] develops a new variation of Graeffe iteration of modern digital computers. The algorithm is based on (i) classical Graeffe iteration and Newton Diagrams (ii) changes of scale (renormalization). It implemented successfully with a number of numerical experiments. This method computes both the moduli and the argument at all roots under certain generic conditions.

In 2000 Malajovich and Zubeli [7] proposed and implemented a new version of the Graeffe’s algorithms for finding all the roots of univariate complex polynomial. Renormalization idea is used for the construction of the algorithm. They have introduced Newton diagram which is the graph of convex function.


In 2001 Pan [2] studied approximate polynomial GCDs (greatest common devisors) which are important both theoretically and practically in control linear system, network theory and computer aided design.
In 2002 Pan [3] develops two algorithms for approximation of all roots (zeros) of a univariate polynomial. One of them computes a basic well isolated zero free annulus on the complex plane. The other numerically splits input $n$th degree polynomial into two factors balanced in degrees and with the zero sets separated by the basic annulus. The first algorithm uses Graeffe's root squaring step recursively.

1.4 Objective

The main objective of this research is to setup condition(s) for solvability of univariate polynomial using Graeffe's root squaring method in sec. 3.2. On the basis of these criteria GRAEFFE.BAS program developed by Constantinides [9] is to be modified in sec. 3.5, so that the program can identify the problems solvable by Graeffe's method. The modified GRAEFFE.BAS is to be tested in solving few univariate polynomials in sec 3.6.

In the meantime weaknesses of this method will be pointed out and finally some open problems in sec. 4.2 will be stated after conclusion.
Chapter 2
Graeffe's root-squaring method and GRAEFFE.BAS program of Constantinides

2.1 Graeffe's root-squaring method

A manual as well as desk calculator method of Graeffe's root squaring method is described here following a mainly Scarborough [17].

The principle of the Graeffe's method: Literature reveals that Graeffe's root squaring is a direct method of obtaining all types of roots - real distinct, real equal and complex of a univariate polynomial with real coefficients and of no initialization. It transforms a polynomial \( p_n(x) \) into another polynomial of same degree whose roots are the squares of the roots of the original polynomial. Due to squaring, the roots of the transformed equation will be spread apart more widely than the original one. The roots of the transformed equation are said to be separated when the ratio of any root to the next larger is negligible in comparison with unity. This situation is known as "roots are separated". Process continues until the roots are really separated. Finally roots are computed directly from the coefficients using Newton's relations.

The root-squaring process: The transformed equation is obtained by repeated application of a root-squaring process. For first application of this process, given equation have been raised an equation whose roots are the squares of those of the original equation. This second equation is then transformed into a third equation whose roots are the square of those of the second and therefore the fourth powers of those of the original equation. The root-squaring process is continued until the roots of the last transformed equation are completely separated.

Explanation of the method:

Let the given equation be

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0 = 0, \quad a_0 \neq 0 \tag{2.1}
\]

Then if \( x_1, x_2, \ldots, x_n \) be the roots of this equation, the equivalent form is

\[
f (x) = a_n (x - x_1)(x - x_2)(x - x_3) \cdots \cdots (x - x_n) = 0 \tag{2.2}
\]
Thus \((-1)^n f(-x) = (-1)^n a_0(-x-x_1)(-x-x_2)(-x-x_3)\cdots(-x-x_n) = 0\)

\[= a_0(x+x_1)(x+x_2)(x+x_3)\cdots(x+x_n) = 0\]  \hspace{1cm} (2.3)

Now multiplying (2.2) by (2.3), the result is

\((-1)^n f(-x)f(x) = a_0^2(x^2-x_1^2)(x^2-x_2^2)(x^2-x_3^2)\cdots(x^2-x_n^2) = 0\)

(2.4)

Let \((-1)^n f(-x)f(x) = \phi(y)\) and \(x^2 = y\) then (2.4) becomes

\[\phi(y) = a_0^2(y-x_1^2)(y-x_2^2)(y-x_3^2)\cdots(y-x_n^2) = 0\]  \hspace{1cm} (2.5)

The roots of this equation are \(x_1^2, x_2^2, \cdots, x_n^2\) which are the squares of the roots of the given equation (2.2). The equation (2.5) can be described another way round as

\[\phi(y) = a_0^2\left(\sqrt{y} - x_1\right)\left(\sqrt{y} - x_2\right)\cdots\left(\sqrt{y} - x_n\right)\left(\sqrt{y} + x_1\right)\left(\sqrt{y} + x_2\right)\cdots\left(\sqrt{y} + x_n\right)\]

(2.6)

\[\therefore \phi(x) = (-1)^n f(\sqrt{x})f(-\sqrt{x}) \text{ by (2.2) and (2.3)}\]

(2.7)

\[\therefore \phi(x) = (-1)^n f(\sqrt{x})f(-\sqrt{x}) \text{ describes the first iteration (squaring) of the method.}\]

Hence, on multiplying \(f(x) = 0\) by \((-1)^n f(-x)\) an equation whose roots are the squares of those of \(f(x) = 0\) is obtained.

This multiplication maintains a simple routine which can be best illustrated with lower degree equations. Let a sixth-degree equation be

\[f(x) = a_0x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6 = 0\]

Then \((-1)^6 f(-x) = a_0(-x)^6 - a_1(-x)^5 + a_2(-x)^4 - a_3(-x)^3 + a_4(-x)^2 - a_5(-x) + a_6 = 0\)

On multiplication by \(f(x)\),

\[(-1)^6 f(-x)f(x) = a_0^2x^{12} - a_0^2\begin{vmatrix} x^6 + a_1^2 \hline + 2a_2a_4 \end{vmatrix} \begin{vmatrix} x^8 - a_2^2 \hline - 2a_1a_4 \end{vmatrix} \begin{vmatrix} x^6 + a_2^2 \hline + 2a_4a_6 \end{vmatrix} \begin{vmatrix} x^6 + a_1^2 \hline - 2a_2a_4 \end{vmatrix} \begin{vmatrix} x^6 + a_1^2 \hline + 2a_4a_6 \end{vmatrix} \begin{vmatrix} x^6 + a_2^2 \hline - 2a_1a_4 \end{vmatrix} \begin{vmatrix} x^6 + a_4^2 \hline + 2a_2a_4 \end{vmatrix} \begin{vmatrix} x^6 + a_4^2 \hline + 2a_2a_4 \end{vmatrix} = 0\]

(2.8)

Now let a seventh degree equation be

\[f(x) = a_0x^7 + a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5x^2 + a_6x + a_7 = 0\]
Then \((-1)^3 f(-x) = a_0 x^3 - a_1 x^2 + a_2 x - a_3 + a_4 x^4 - a_5 x^3 + a_6 x - a_7 = 0\)

Multiplying, as before

\[
(-1)^7 f(-x) f(x) = a_0^2 x^{14} + \begin{array}{cccccc}
-a_1^2 + a_2^2 & x^{12} + a_3^2 & -a_4^2 + a_5^2 & x^8 + a_6^2 & -a_7^2 + a_8^2 & x^4 + a_9^2 \\
+2a_1 a_2 & -2a_1 a_3 & +2a_2 a_4 & -2a_3 a_4 & +2a_6 a_7 & -2a_7 a_8 \\
+2a_1 a_5 & -2a_1 a_6 & -2a_2 a_5 & +2a_2 a_7 & +2a_6 a_8 & -2a_7 a_9 \\
+2a_1 a_7 & -2a_2 a_7 & +2a_2 a_8 & +2a_6 a_9 & -2a_7 a_{10} & +2a_7 a_{11}
\end{array}
\]

The equation shows that the law of formation of the coefficients in the squared equation is the same whether the degree of the given equation be even or odd. In practice the multiplication is carried out with detached coefficients as shown below:

\[
\begin{array}{cccccc}
a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\
a_n & -a_1 & a_2 & -a_3 & a_4 & -a_5 \\
a_n^2 & -a_1^2 & a_2^2 & -a_3^2 & a_4^2 & -a_5^2 \\
& +2a_1 a_2 & -2a_1 a_3 & +2a_2 a_4 & -2a_3 a_4 & +2a_4 a_5 \\
& +2a_1 a_5 & -2a_1 a_6 & +2a_2 a_5 & +2a_2 a_6 & -2a_3 a_6 \\
& +2a_1 a_7 & -2a_2 a_7 & +2a_2 a_8 & +2a_6 a_9 & -2a_7 a_{10} \\
& +2a_1 a_{10} & -2a_2 a_{10} & +2a_6 a_{11} & +2a_6 a_{12} & -2a_7 a_{12} \\
b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \\
\end{array}
\]

The coefficients in the new equation (2.10) are the sums \(b_0, b_1, b_2, \ldots, b_n\) of the several columns in the scheme above. The whole scheme is set up as follows:

1. The coefficients of the original equation constitute the first row of the scheme.
2. The second comprises of the first row with alternate positive and negative signs \(a_n\) being positive.
Now the multiplication of these two rows are performed between two lines according to the following rules under the first line:

3. The numbers in the top row are the squares of the coefficients directly above them, with alternating signs – the second, fourth, sixth, etc. square numbers being negative.

4. The quantities directly under these squared numbers are the doubled products of the coefficients equally removed from the one directly overhead, the first being twice the product of the two coefficients adjacent to the one overhead, the second the doubled product of the next two equally removed coefficients, etc.

5. The signs of the doubled products are changed alternately in going along the rows and also in going down the columns, the sign of the first doubled product in each row not being changed.

The above discussions are concentrated to first squaring (first iteration). Let second iteration of

\[ f(y) \]

transforms \( \phi(y) \) to \( \phi(z) \). Thus roots \( y_i = x_i^2 \) and \( z_i = y_i^2 = x_i^4 = (x_i^2)^2 \). Proceeding in this way to \( k \) iterations, the roots of the final transformed equation are, therefore \( r_i = x_i^{2k} \) [19].

**nth iteration formula:**

A general formula for \( n \)th iteration can be written. Let \( f_0(x) \) denote the univariate polynomial (1.1) or (2.2). \( f_1(x) \) denote the \( 1 \)st iterate (2.7) i.e. the relation

\[ f_1(x) = (-1)^k f(\sqrt{x})f(-\sqrt{x}) = f_0(\sqrt{x})f_0(-\sqrt{x}) \]

Similarly, the second iteration

\[ f_2(x) = f_1(\sqrt{x})f_1(-\sqrt{x}) \]

Proceeding in this way, the \( n \)th iteration is

\[ f_{k+l}(x) = f_k(\sqrt{x})f_k(-\sqrt{x}), \quad k = 0, 1, 2, \ldots, m-1 \] (2.11)

**Calculation of roots in three cases**

**Case 1:** Roots all real and unequal

The relations between the roots \( x_0, x_1, x_2, \ldots, x_n \) and the coefficients \( a_0, a_1, a_2, \ldots, a_n \) of the \( n \)th degree general univariate polynomial

\[ f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \] are
Newton's first relation \[
\frac{a_1}{a_0} = -(x_1 + x_2 + x_3 + \cdots + x_n) = \text{sum of roots.}
\]

Newton's second relation \[
\frac{a_2}{a_0} = -(x_1x_2 + x_1x_3 + \cdots) = \text{sum of the products taken two at a time.}
\]

Newton's third relation \[
\frac{a_3}{a_0} = -(x_1x_2x_3 + x_1x_2x_4 + \cdots) = \text{sum of the products taken three at a time.}
\]

Newton's nth relation \[
\frac{a_n}{a_0} = (-1)^n x_1x_2x_3 \cdots x_n = \text{product of the roots.}
\]

In the final transformed equation (i.e., mth iteration in which m is a positive integral power of 2 i.e., \(m = 2^q\), q is a positive integer)

\[
b_0(x^m)^n + b_1(x^m)^{n-1} + \cdots + b_{n-1}x^n + b_n = 0
\]

whose roots \(x_1^m, x_2^m, \cdots x_n^m\) and coefficients \(b_0, b_1, b_2 \cdots b_n\) are connected by the corresponding relations

\[
\frac{b_1}{b_0} = -(x_1^m + x_2^m + \cdots + x_n^m) = -x_1^m \left(1 + \frac{x_2^m}{x_1^m} + \frac{x_3^m}{x_1^m} + \cdots + \frac{x_n^m}{x_1^m}\right),
\]

\[
\frac{b_2}{b_0} = x_1^m x_2^m + x_1^m x_3^m + \cdots = x_1^m x_2^m \left(1 + \frac{x_3^m}{x_1^m} + \frac{x_4^m}{x_1^m} + \cdots\right),
\]

\[
\frac{b_3}{b_0} = -(x_1^m x_2^m x_3^m + x_1^m x_2^m x_4^m + \cdots) = -x_1^m x_2^m x_3^m \left(1 + \frac{x_4^m}{x_1^m} + \cdots\right),
\]

\[
\vdots
\]

\[
\frac{b_n}{b_0} = (-1)^n x_1^m x_2^m x_3^m \cdots x_n^m.
\]

Now if the order of magnitude of the roots is \(|x_1| > |x_2| > |x_3| > |x_n|\), then the ratios \(\frac{x_2^m}{x_1^m}, \frac{x_3^m}{x_2^m}\) etc. are negligible in comparison with unity and can all be made as small as desired by making m large enough. Thus the relations between the roots and the coefficients in the final transformed equation are

\[
\frac{b_1}{b_0} = -x_1^m, \quad \frac{b_2}{b_0} = x_1^m x_2^m, \quad \frac{b_3}{b_0} = -x_1^m x_2^m x_3^m, \cdots, \quad \frac{b_n}{b_0} = (-1)^n x_1^m x_2^m x_3^m \cdots x_n^m.
Dividing each of these equations after the first by the preceding equation, we obtain

\[
\frac{\beta_2}{\beta_1} = -x_1^n, \quad \frac{\beta_3}{\beta_2} = -x_2^n, \ldots, \quad \frac{\beta_n}{\beta_{n-1}} = -x_n^n
\]

Hence, from these and the equation

\[
\frac{\beta_1}{\beta_0} = -x_0^n, \tag{2.13}
\]

one obtains

\[
b_0x_1^n + b_1 = 0, \quad b_1x_2^n + b_2 = 0, \quad b_2x_3^n + b_3 = 0, \ldots, b_{n-1}x_n^n + b_n = 0. \tag{2.14}
\]

Thus the original equation has broken up into \(n\) simple equations and the \(m\)th powered roots can be found from them. Finally, the \(m\)th root of these gives the desired roots.

The number of root squaring to break up the original equation into linear fragments depends upon (1) the ratios of the roots of the given equation and (2) the number of significant figures desired in the computed roots. But it is not possible to determine how many times the root-squaring process is needed because the roots and their ratios are not known in advance. In the root squaring process, the signs of the doubled products will not occur in regular order because the coefficients in the given equation are not in general all positive. In manual practice, the possibilities of making a mistake in the sign are great and for avoiding this mistake one can use some notations like a "c" after each term in which the sign is to be changed and an "n" after each term where the sign is not to be changed. Furthermore, in the transformed equations the coefficients, which are large numbers, these coefficients may be written as simple numbers, multiplied by powers of 10.

Finally, by the repeated root-squaring process, one obtains transformed equation and compute all the roots of an equation. Equations (2.14) can be combined to a signal formula [19]

\[
x_i^n = -\frac{b_i}{b_{i-1}}, i = 1, 2, \ldots, n
\]

Thus \(x_i^n = -\frac{b_i}{b_{i-1}}\). So, \(x_i = 2^m\)th root of \(\left(-\frac{b_i}{b_{i-1}}\right) = 2^m\left(-\frac{b_i}{b_{i-1}}\right)\)

\[
\tag{2.15}
\]

Calahan [18] introduces the symbol \(\lambda\) for the \(j\)th iteration of \(\lambda\) and develops the iterative formula
\[ n! A_i = (-1)^i \left[ A_i^2 + 2 \sum_{j=1}^{i-1} (-1)^j A_{i-j} A_{i-j} \right], \quad 0 \leq i \leq n \]  

for coefficients in (2.12) where \( A_i = a_i \), the initial value of \( A_i \), etc.

**Case II. Complex Roots**

The root squaring process breaks an equation into a product of real linear and quadratic factors, in the presence of complex roots. The real roots are found from linear fragments and the complex roots are found from quadratic fragments, each quadratic factor corresponding to a pair of complex roots. The presence of complex roots is revealed in two ways:

1. The doubled products do not all disappear from the first row and
2. The signs of some of the coefficients fluctuate as the transformations continue.

The reason for these peculiarities are explained below:

Let an equation having two distinct real roots and two pairs of complex roots be \( x_1, r_1 e^{i\theta_1}, r_1 e^{-i\theta_1}, x_3, r_2 e^{i\theta_2}, r_2 e^{-i\theta_2} \) and let the order of their magnitude be \( |x_1| > r_1 > |x_3| > r_2 \).

Then the equation having these roots is

\[
(x - x_1)(x - r_1 e^{i\theta_1})(x - r_1 e^{-i\theta_1})(x - x_3)(x - r_2 e^{i\theta_2})(x - r_2 e^{-i\theta_2}) = 0
\]

(2.18)

The equation whose roots are the \( m \)-th powers of the roots of this equation is therefore

\[
(y - x_1^m)(y - r_1^m e^{i\theta_1})(y - r_1^m e^{-i\theta_1})(y - x_3^m)(y - r_2^m e^{i\theta_2})(y - r_2^m e^{-i\theta_2}) = 0
\]

(2.19)

where \( y = x^m \)

On performing the indicated multiplications in (2.19), then taking out the factors

\[ x_1^m, x_1^m r_1^{2m}, x_1^m r_2^{2m}, x_3^m, x_3^m r_1^{2m}, x_3^m r_2^{2m}, \]

and neglecting the ratio

\[ \frac{r_1^m}{x_1^m}, \frac{x_1^m}{x_1^m}, \frac{r_2^m}{x_1^m}, \frac{r_2^m}{x_2^m}, \frac{x_3^m}{x_3^m}, \frac{x_3^m}{x_3^m} \] for negligible in comparison with unity.

Finally, one obtains

\[
y^6 - x_1^m x_3^m y^2 + 2 x_1^m r_1^m \cos m \theta_1 y^4 - x_1^m r_1^{2m} y^3 + x_1^m x_3^m x_1^o y^2 = 0
\]

(2.20)

Thus the given equation has been broken up into the linear and quadratic fragments
\[
\begin{align*}
y^6 - x_1^m y^1 &= 0 \\
x_1^m y^4 + 2x_1^m x_2^m - x_1^m x_2^m y^4 &= 0 \\
x_1^m x_2^m y^2 + x_1^m x_2^m x_3^m y^2 &= 0 \\
x_1^m x_2^m x_3^m y^2 - 2x_1^m x_2^m x_3^m x_4^m y^2 &= 0
\end{align*}
\]

(2.21)

from which the original roots can be obtained.

The root-squaring process is applied to (2.19) once more, as shown below

<table>
<thead>
<tr>
<th>(y^6)</th>
<th>(y^5)</th>
<th>(y^4)</th>
<th>(y^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y^6)</td>
<td>(-x_1^m)</td>
<td>(2x_1^m r_1^m \cos m \theta_1)</td>
<td>(-x_1^m r_2^m)</td>
</tr>
<tr>
<td>(y^5)</td>
<td>(-x_1^m r_1^m)</td>
<td>(+4x_1^m r_1^m r_2^m \cos^2 m \theta_1)</td>
<td>(-x_1^m r_2^m)</td>
</tr>
<tr>
<td>(y^4)</td>
<td>(+4x_1^m r_1^m r_2^m \cos^2 m \theta_1)</td>
<td>(-2x_1^m r_1^m r_2^m)</td>
<td>(+4x_1^m r_1^m r_2^m \cos m \theta_1)</td>
</tr>
<tr>
<td>(y^3)</td>
<td>(-2x_1^m r_1^m r_2^m)</td>
<td>(+2x_1^m r_1^m r_2^m x_1^m)</td>
<td>(-4x_1^m r_1^m r_2^m \cos m \theta_2)</td>
</tr>
</tbody>
</table>

Thus it is observed that all these products from the above table are negligible except two in the first row. The doubled product in the first row does not all disappear when the complex roots
are present. Since $2\cos^2 \theta - 1 = \cos 2\theta$, the coefficient of $y^4$ and $y$ becomes $2x_1^2 m r_1^{2m} \cos 2m \theta_1$ and $2x_1^{2m} r_1^{4n} x_2^{2m} r_2^{2n} \cos 2m \theta_2$ respectively. Thus the last transformed equation is

$$1 - x_1^{2m} + 2x_1^{2m} r_1^{2m} \cos 2m \theta_1 - x_1^{2m} r_1^{4n} + x_1^{2m} r_1^{4m} x_1^{2m} r_1^{2m} = 0$$

(2.22)

The equation (2.22) shows that each root squaring doubles the amplitude of the complex roots. The cosines $\cos 2m \theta_1$ and $\cos 2m \theta_2$ fluctuates in value and even in of these amplitudes signs. For this reason, the presence of complex roots shows fluctuation of signs. The complex roots by solving the resulting quadratic equations for $x^n$ and then extraction the $m$th root of the results are obtained by means of De Moivre’s theorem. But in this process the ambiguities of signs in the roots arise and these cannot be easily removed. To obtain the complex roots without ambiguity as to signs, some further relations between roots and coefficients are derived below.

**Relations between the coefficients of an algebraic equation and the reciprocals of its roots:** In the general equation $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0 = 0$

let $x = 1/y$. The result, after clearing of fractions, is

$$a_n y^n + a_{n-1} y^{n-1} + a_{n-2} y^{n-2} + \cdots + a_2 y^2 + a_1 y + a_0 = 0$$

Hence from the relations between roots and coefficients,

$$\frac{a_{n-1}}{a_n} = -(y_1 + y_2 + \cdots + y_n)$$

$$\frac{a_{n-1}}{a_n} = y_1 y_2 + y_1 y_3 + \cdots + y_2 y_3 + \cdots$$

$$\frac{a_{n-2}}{a_n} = (-1)^r y_1 y_2 y_3 \cdots y_r$$

or, since $y = 1/x$. 

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The relations between the coefficients and the reciprocals of the roots will help to avoid ambiguities of sign in the computation of complex roots.

Case III. Roots real and numerically equal

If one of the double products will always remain in the first row and will be just half the squared term, then the root-squaring process can never break up into linear fragments. For this reason the equation has two roots are numerically equal.

Let a third degree equation having three real roots \( x_1, x_2, x_3 \) be

\[
x^3 + a_1x^2 + a_2x + a_3 = 0
\]

Then the equation whose roots are the \( m \)th powers of (2.24) is

\[
(y - x_1^m)(y - x_2^m)(y - x_3^m) = 0,
\]

where \( y = x^m \),

\[
y^4 - (x_1^m + x_2^m + x_3^m)y^3 + (x_1^m x_2^m + x_1^m x_3^m + x_2^m x_3^m)y^2 - x_1^m x_2^m x_3^m y = 0
\]

or

\[
y^3 - x_1^m \left( 1 + \frac{x_2^m}{x_1^m} + \frac{x_3^m}{x_1^m} \right)y^2 + x_1^m x_2^m \left( 1 + \frac{x_3^m}{x_2^m} + \frac{x_3^m}{x_2^m} \right)y - x_1^m x_2^m x_3^m = 0
\]

(2.25)

Now let \( x_2 = x_3 \) and let \( |x_1| > |x_2| \). Then for sufficiently large values of \( m \) the ratios \( \frac{x_2^m}{x_1^m} \) is negligible in comparison with unity and (2.25) reduces to

\[
y^3 - x_1^m y^2 + 2x_1^m x_2^m y - x_1^m x_2^m = 0
\]

(2.26)

The roots of the given equation have now been separated as much as they can ever be but applying the root-squaring process to (2.26) using only the coefficients, we have.
It can be noticed that the first doubled product is negligible in comparison with the squared term above it, whereas the second is of the same order of magnitude as the squared term above and just half as large. Further, in the equation for the $2m$th powers of the roots all the coefficients except one are the squares of those in the preceding equation. This remaining one is only half of the square of the corresponding coefficient in the preceding equation. These peculiarities enable one to detect equal real roots immediately.

2.2 GRAEFFE.BAS Program

Constantinides [9] designed GRAEFFE.BAS program in BASIC programming language for IBM PC for solving univariate non-linear polynomial equation using Graeffe's root squaring method. The program consists of a main program and 6 (six) subroutines. The flow chart of the main program is constructed as follows:

<table>
<thead>
<tr>
<th>$m$th power</th>
<th>$-x_1^m$</th>
<th>$2x_1^m x_2^m$</th>
<th>$-x_1^m x_2^{2m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2m$th power</td>
<td>$-x_1^{2m}$</td>
<td>$-x_1^{2m} x_2^{2m}$</td>
<td>$-x_1^{2m} x_2^{4m}$</td>
</tr>
</tbody>
</table>

| $1$ | $-x_1^{2m}$ | $+ 4x_1^{2m} x_2^{2m}$ | $- 2x_1^{2m} x_2^{2m}$ |
| $2$ | $-x_1^{2m}$ | $+ 2x_1^{2m} x_2^{2m}$ | $- x_1^{2m} x_2^{4m}$ |
Main program: Input
(i) degree of polynomial
(ii) EPS, Convergence criteria of 0.001, changeable by user

Factor coefficient, into mantissa and exponent

Print original coeffs.

Graeffe's root squaring beginning of iteration, R=1. Changeable by user

Calculate new coefficients

\[ \text{FACT}>999 \]

(F) Possibility of repeated or complex root

Determine whether coeffs. are pure square or not

Print status of coeffs (pure square or non square)

Count the number of pure squares

All coeffs. are pure square

Yes

Roots real distinct evaluate them and sum them using
\[ x_k = \pm \sqrt[k]{-a_k}, \quad k = 1, 2, \ldots n \]

Reduce iteration counter by one, i.e. R = R - 1

Calculate sum of product of real roots using subroutine 2

Print all roots → END

No

Possibility of complex or repeated roots

Call subroutine 1 for checking sign of root

Find location of non squares = NSQUARES

(i) If NSQUARES = 1, use sub-routine 4 to calculate one pair of complex or repeated roots.
(ii) If NSQUARES = 2, branch to sub-routine 5 to calculate 2 pairs of complex or repeated roots with different moduli
(iii) If NSQUARES = 3 branch to sub-routine 6 to calculate 2 pairs of complex or repeated roots with identical moduli.
(iv) If NSQUARES >3, Program cannot be used.
Sequence of calling six subroutines by the main program and subroutines themselves is shown below:

- Subroutine 1 for checking signs of real roots
- Subroutine 4 for calculating one pair of complex or repeated roots
- Subroutine 5 for calculating two pair of complex or repeated roots with different moduli
- Subroutine 6 for calculating two pair of complex or repeated roots with identical moduli
- Subroutine 2 calculates sum of products of real roots
- Subroutine 3 checks validity of complex roots by direct substitution in original eqn. Warning if conv. not satisfied by comp. root
2.3 Procedure for identifying roots in GRAEFFE.BAS

Due to squaring, the square terms grow more rapidly than the sum of product terms in the transformed polynomial. As a result, new coefficients \( rA_k \) are pure squares of \( \pm A_k \). However, presence of complex or repeated roots prevents these coefficients from becoming pure squares, i.e., some coefficients remain non-square. Thus Graeffe’s root squaring method provides a very powerful procedure for identifying different categories of roots as follows:

(a) Real and distinct roots correspond to “all coefficients, \( rA_k \) being pure squares.”

(b) One pair of complex roots or repeated real roots correspond to “a nonsquare surrounded by two pure squares” in the form [9]

\[
\begin{array}{ccc}
\text{Pure square} & rA_k & \text{Pure square} \\
\text{Pure square} & rA_k & \text{Pure square} \\
\end{array}
\]

and the roots \( \alpha \pm i\beta \) are determined by solving \( \alpha^2 + \beta^2 = \frac{r A_{k+1}}{A_{k-1}} \) and Newton’s first relation \( \sum_{i=1}^{n} x_i = -\frac{a_{n-1}}{a_n} \) applied to original polynomial. If \( \beta = 0 \), then the two roots are real repeated.

(c) Two pairs of complex roots or repeated real roots with different moduli correspond to “two nonsquares each is surrounded by two pure squares” as

\[
\begin{array}{cccccc}
\text{Pure square} & rA_{k+1} & rA_k & rA_{k-1} & \cdots & rA_{n+1} \\
\text{Pure square} & rA_k & rA_{k-1} & rA_k & \text{Pure square} & \text{Pure square} \\
\end{array}
\]

and the roots \( \alpha_i \pm i\beta_i \), \( \beta_i \pm i\beta_j \) are determined by solving

\[
\alpha_i^2 \pm \beta_i^2 = \frac{r A_{k+1}}{r A_{k-1}}, \quad \alpha_j^2 \pm \beta_j^2 = \frac{r A_{k+1}}{r A_{k+1}}
\]

and Newton’s first and second relations \( \sum_{i=1}^{n} x_i = -\frac{a_{n-1}}{a_n}, \sum_{i=1}^{n} x_i x_j = \frac{a_{n-2}}{a_n} \). If \( \beta_1 = \beta_2 = 0 \), then the four roots are real repeated.
(d) Two pairs of complex or repeated real roots with identical moduli correspond to "three neighboring non-squares surrounded by two pure squares" as

\[
\begin{array}{c}
\text{pure square} \\
A_{r_{-1}}, \quad A_r, \quad A_{r_{+1}}, \quad A_{r_{+2}} \quad \text{pure square}
\end{array}
\]

and roots \( \alpha_x \pm i\beta_x, x = 1, 2, \) are obtained by solving

\[
R^2 = (\alpha_x^2 + \beta_x^2) = 2n \frac{A_{r_{+1}}}{A_{r_{-1}}},
\]

where \( R \) is the common modulus, and first 3 Newton's relations

\[
\sum_{i=1}^{s} x_i = -\frac{a_{r_{-1}}}{a_n}
\]

\[
\sum_{i,j=1 \atop i \neq j}^{s} x_i x_j = \frac{a_{r_{+2}}}{a_n}
\]

\[
\sum_{i,j,k=1 \atop i \neq j \neq k}^{s} x_i x_j x_k = -\frac{a_{r_{+3}}}{a_n}
\]

If \( \beta_x = 0, s = 1, 2, \) the roots are real repeated.
Chapter 3
Solvability conditions, weaknesses and modification

3.1 Observation

None of the researchers in the references discussed the following observations except a few. Their comments about the method regarding advantages and disadvantages are correct to their point of view.

(i) It is observed that there are equations, which are not transformable by root squaring into a different one with non-zero coefficients from where the roots of the original equation are calculated. It is found that the odd degree equations set like

\[
\begin{align*}
x^4 + a &= 0, \\
x^5 + a &= 0, \\
x^7 + a &= 0, \\
x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1 &= 0
\end{align*}
\]  

(3.1)

etc. cannot be solved by the Graeffe's root squaring method manually as well as using GRAEFFELAS of Constantinides [9]. All these equations (3.1) transform to the form \(x^n + a = 0, n = 3, 5, 7\) after first iteration and GRAEFFELAS shows overflow at the statement 1080 of Constantinides [9]. It is observed that

\[
\begin{align*}
x^4 + x + b &= 0, \quad x^4 + a = 0, \quad x^7 + ax^2 + b &= 0, \\
x^7 + ax^6 + b &= 0, \quad x^4 + 2x^3 + 2x^2 + 2x + 2 &= 0
\end{align*}
\]  

(3.2)

etc. can be solved by both manual procedures and using program of Constantinides [9]. In this case zero coefficients revive and nonzero coefficients do not vanish. As a result final transformed equation in each case provides solution. Householder [21] passes a similar remark for \(x^n - 1 = 0\) in which all \(n\) roots have unit modulus and the Graeffe method fails for such equations without deriving any solvability condition. Similarly, Wilkinson [22] called for
equation \( x^2 - 1 = 0 \) well-condition zeros \( x = \pm 1 \), because on squaring the transformed equation becomes \( x^2 - 2x + 1 = 0 \) and deteriorates to coincident ill-conditioned root 1.

(ii) Actual computation reveals another fact that 4th, 6th, 8th, 10th etc. degree non-linear equations with positive unit coefficients like

\[ x^4 + x^3 + x^2 + x + 1 = 0 \]  

will never stop the procedure because the coefficients of the transformed equation remain unity i.e. \((-1)^4 f(\sqrt{x}) f(-\sqrt{x})\) remains invariant.

### 3.2 Solvability conditions

The observations in sec. 3.1 leads condition of solvability. In Graeffe’s method, roots of the original equation are obtained from the coefficients of the last transformed equation. If some of these intermediate coefficients between the first and the last one are zero, the roots may be infinite or even indeterminate. So, the first and foremost criteria of Graeffe’s root squaring method to be successful is that the coefficients of the last transformed equation must be non-zero which in turn depends on some or all of the non-zero coefficients of the original equation.

Let the \( n \)th degree algebraic equation be

\[ a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_k x^k + \cdots + a_{n-1} x + a_n = 0, \quad a_0 \neq 0 \]

The scheme of the process is as follows:
<table>
<thead>
<tr>
<th>Variable</th>
<th>$x^n$</th>
<th>$x^{n-1}$</th>
<th>$x^{n-2}$</th>
<th>\ldots</th>
<th>$x^k$</th>
<th>\ldots</th>
<th>$x^1$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficients</td>
<td>$a_0$</td>
<td>$a_1$</td>
<td>$a_2$</td>
<td>\ldots</td>
<td>$a_k$</td>
<td>\ldots</td>
<td>$a_{n-1}$</td>
<td>$a_n$</td>
</tr>
<tr>
<td>$a_0^2 - a_1^2$</td>
<td>$+ a_2^2$</td>
<td>\ldots</td>
<td>$(-1)^k a_k^2$</td>
<td>\ldots</td>
<td>$(-1)^{n-1} a_{n-1}^2$</td>
<td>$(-1)^n a_n^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2a_0a_2$</td>
<td>$- 2a_1a_1$</td>
<td>\ldots</td>
<td>$(-1)^{k+1} 2a_{k-1}a_k$</td>
<td>\ldots</td>
<td>$(-1)^n 2a_{n-1}a_n$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+ $2a_0a_k$</td>
<td>\ldots</td>
<td>$(-1)^{k+2} 2a_{k-2}a_{k+2}$</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$2^{\text{st}}$ power

$|A_0| |A_1| |A_2| \ldots |A_k| \ldots |A_{n-1}| A_n$

$2^{\text{th}}$ power

$|A_0| |A_1| |A_2| \ldots |A_k| \ldots |A_{n-1}| A_n$

where $\rho_i A_k = (-1)^i \left[ a_k^2 + 2 \sum_{i=1}^{k} (-1)^j a_{i-1}a_k a_{j+i} \right]$, $0 \leq k \leq n$, $k - l \geq 0$, $k + l \leq n$

is a modification of what is given by Carnahan [18] and $\rho A_k = a_k$. The equation suppose that the root squaring process stops after $j$th squaring according to the stopping conditions suggested in the scheme.

The conditions of solvability are discussed for the following two cases:

**CASE A. Equations with zero coefficients**

Let $a_k = 0$, $0 \leq k \leq n$. If $\rho_i A_i = 0$ for $0 \leq k \leq n$ and for any $j$ then the method will continue indefinitely without satisfying any stopping conditions suggested in the references and so the method fails. But if $\rho_i A_i \neq 0$ at some stages of $j$ then the method is successful in giving roots.

Let $j = 0$, so

$\rho_i A_k = (-1)^i \left[ a_k^2 + 2 \sum_{j=1}^{k} (-1)^j a_{j-1}a_k a_{j+i} \right] \neq 0$ and $a_k = 0$
\[
2 \sum_{l=1}^{i} (-1)^l a_{i-l} a_{k+l} \neq 0
\]

i.e. \( a_{k+j} \neq 0 \) and \( a_{i+l} \neq 0 \).

i.e. from the zero coefficients \( a_k = 0 \), equidistant coefficients \( a_{k+l} \) and \( a_{i+l} \) are non-zero.

Once \( A_k \) become non-zero then the subsequent transformation will produce \( A_k \neq 0 \). This fact may be observed from the equation set (3.2) with zero coefficient in sec.3.1. But if it happens that \( A_k = 0, j > 0 \), then the solvability conditions follow the next case B.

**CASE B. Equation with non-zero coefficients**

Let \( A_k \neq 0 \). The case when \( A_k \neq 0 \) is of less interest of discussion. Because of such case the method is quite successful but if it happens that \( A_k = 0 \) then the conditions will be same as the following.

Let \( A_k = 0, 0 < k \leq n \), also \( j = 0 \). Then \( A_k = (-1)^k \left[ a_k^2 + 2 \sum_{l=1}^{k} (-1)^l a_{i-l} a_{k+l} \right] = 0 \)

\[
\therefore (-1)^k a_k^2 + 2 \sum_{l=1}^{k} (-1)^l a_{i-l} a_{k+l} = 0
\]

i.e. the coefficients of the second powers of the roots in the transformed equation are zero. If these coefficients satisfy, the conditions like CASE A then the given equation is solvable under this method otherwise not. Similarly in any stages of transformation \( A_k = 0 \) but satisfy the conditions like CASE A then the given equation is solvable. Consider the equation

\[
x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1 = 0
\]

in set (3.1) for which \( A_k = 0 \), \( 0 < k < 5 \) and which do not satisfy the criteria in CASE A. So, for this equation Graefel's method fails whereas the method is successful for the equation

\[
x^4 + 2x^3 + 2x^2 + 2x + 2 = 0
\]
3.3 Weaknesses and recovery of Graeffe’s method

The most important advantages of Graeffe’s root squaring method mentioned in most of the references are (i) the method gives all the roots – real, complex, equal and unequal, simple and multiple; (ii) the method needs no initial guess as in other methods. Although Balagurusamy [19], Constantinides [9] mentioned some disadvantage of coefficient growth during iteration, Malajovich and Zubelli [7] clearly and specifically point on the reasons for lack of popularity of Graeffe’s method for a pretty long time. They stress two main weaknesses – (i) Coefficient growth, (ii) the method returns the moduli of roots but not the actual roots. Specifically reasons for Graeffe’s root squaring method for lack of popularity are:

(i) its traditional form leads to exponents that easily exceed the maximum allowed by floating point arithmetic;

(ii) Chaotic behaviour of the arguments of the roots of iterates.

To overcome the weakness (i), they introduce renormalization and to overcome weakness (ii), they differentiate the Graeffe iteration operator, \( Q f(x) = (-1)^d f(\sqrt{x})f(-\sqrt{x}) \) [where \( d \) is the degree of polynomial] and its effect is to square each root of \( f \), [7].

Constantinides [9] in his program alleviate serious limitation (i) above by the following 4 steps.

(a) Coefficient of the original polynomial are separated into mantissa and exponent parts and stored into two separate matrices. The ranges of mantissa used are ±10.

(b) After each iteration mantissas part of coefficients are tested. If their absolute values are greater than 10 (less than 1) these values are divided by 10 (multiplied by 10) and the corresponding exponents are increased (decreased) by 1.

(c) The upper limit of exponent factor is chosen as 999 as maximum values of them may be i.e. \( 10 \times 10^{13109} \) which is not necessary for Graeffe’s method. If one of the exponent factors exceed this limit, the iteration is terminated and the roots are evaluated.

(d) Both the matrices are treated together in the programme, so that exponent part of each coefficient is always accounted for.
Haara and Losker [14] observe that the algebraic equation set (3.1) is not solvable manually as well as GRAEFFE.BAS software in [9]. Thus they conclude that

(i) equation with zero-coefficient must have at least one pair of equidistant non-zero coefficient from the zero-coefficient;
(ii) any transformed equation of a given equation with non-zero coefficient may have zero coefficients but these new coefficients must satisfy (i);
(iii) all the coefficients of non-linear algebraic equation must not be unity;
(iv) GRAEFFE.BAS needs modification in the light of (i), (ii) and (iii).

They suggest a testing procedure which identifies solvability of the equation. Thus software in [9] has been modified for identifying those types of problems and actually identified with the modified software.

Illustration of (i)

Odd degree equations of form \( x^n + a = 0 \) does not have a pair of equidistant non-zero coefficients from the zero coefficients as is seen from first three equations in (3.1); whereas even degree equation of the form \( x^n + a = 0 \) does have a pair of equidistant non-zero coefficient from the zero coefficients as is seen from 2nd equation \( x^4 + a = 0 \) in (3.2). Also the 3rd and 4th equations possess a pair of non-zero coefficient from the zero coefficient. Hence these equations in (3.2) are solvable by both manually and using GRAEFFE.BAS software. The last equation in (3.1) has no zero coefficient but has the coefficients \( a_0 = a_n = 1 \) and other coefficients are equal. This type transform to one with zero coefficients and possesses no equidistant pair of non-zero coefficients from the zero coefficients. But the last equation in (3.2) with \( a_0 \neq a_n \) and other coefficients being equal transforms into one with zero coefficients having at least a pair of non-zero coefficients from the zero coefficients. Hence this type is solvable by Graeffe’s method.

In other words, all the equations in the set (3.1) reduce to the form \( x^n + 1 = 0 \) on the first iteration \( f_1(x) = f_0(\sqrt[2]{x})f_0(-\sqrt[2]{x}) \) and the zero-coefficient of the new transformed equation does not have a pair of non-zero coefficient from the zero coefficients. That set (3.2) does not have the above discussed problem and is solvable by Graeffe’s root squaring method by both manually and using GRAEFFE.BAS software.
3.4 Modification of GRAEFFE.BAS

It is observed that there are equations, which are not transformable by root squaring into a new one with non-zero coefficients from where the roots of the original equation are calculated. It is found that the equations like

\[ \begin{align*}
x^3 + a &= 0, \\
x^5 + a &= 0, \\
x^7 + a &= 0, \\
x^9 + 2x^8 + 2x^7 + 2x^6 + 2x^2 + 2x + 1 &= 0
\end{align*} \tag{3.3} \]

etc. While running the problems with original program [9] shows “Overflow” at statement

1080: \( X(NR) = (A(R, K-1) / A(R, K)) \times (1/ M) \times 10 \times ((FACT(R, K-1) - FACT(R, K)) / M) \)

To overcome this difficulty, we have to modify and extend the original program [9]. So we introduced following statements and generate a new subroutine 7. Statement numbers in [9] are 10, 20, 30, ..., 530. The modified statement numbers are fitted in between. The following are the modified statements with new statement numbers different from the original statement numbers. The new subroutine 7 follows next.

603: IF ABS(A(R, I)) = 0 THEN GOTO 605 ELSE GOTO 610

605: IF ABS(A(R, I-1)) < 0 AND ABS(A(R, I+1)) < 0 THEN GOTO 1350

1075: IF C$(N-I) <> PS$ AND C$(1) <> PS$ THEN GOTO 1110

1350: GOSUB 7000.

4065: IF C$(K) <> PS$ AND C$(K) <> NS$ THEN GOTO 5230

5085: IF C$(K) <> PS$ AND C$(K) <> NS$ THEN GOTO 5230

SUBROUTINE 7

7010: FOR I = N-1 TO 1 STEP -1

7020: IF ABS(A(R, I)) = 0 THEN GOTO 7030

7030: IF ABS(A(R, I-1)) < 0 AND ABS(A(R, I+1)) < 0 THEN GOTO 7050

7040: NEXT I

7050: FOR K = 1 TO N

7060: IF X(K) + X(K) = 0 THEN GOTO 7090

7070: IF X(K) + X(K) < 0 THEN GOTO 1350

7080: NEXT K

7090: PRINT: PRINT “EQUATION IS NOT SOLVABLE BY GRAEFFE’S ROOT
SQUARING METHOD”

7100: .
The difficulty of the problems (3.3) coefficients $C$(N-1) and $C$(1) do not show PURE SQUARE. If $C$(1) do not show PURE SQUARE then from statement 1070 and statement 1110, we obtain $K=0$, which is out of range of $K$ at statement 1050. So we introduce new statement 1075 among them as follows:

1050: FOR $K= N$ TO 1 STEP -1
1070: IF $C$(K-1)<>PSS THEN $K = K - 1 :$GOTO 1110
1074: New statement
1075: IF $C$(N-1)<>PSS AND $C$(1)<>PSS :GOTO 1110
1110: NEXT $K$

Also some of the problems (3.1) such as $x^2 + x = 0$ do not show PURE SQUARE and NON SQUARE when $K$ is N-1 to 1. In the original program it shows "overflow" at statement 4070 and statement 5090. So we introduce statement 4065 and statement 5085 among them as follows:

4065: IF $C$(K)<>PSS AND $C$(K)<>NS$ THEN GOTO 4170
4070: $R(K) = (A(R,K-1)/A(R,K+1)) \times (1/M) \times 10 \times ((FAC1(R,K-1) - FACT(R,K+1))/M$
5085: IF $C$(K)<>PSS AND $C$(K)<>NS$ THEN GOTO 5230
5090: $R(K) = \text{ABS}(A(R,K-1)/A(R,K+1)) \times (1/M) \times 10 \times ((\text{FACT}(R,K-1)-\text{FACT}(R,K+1))/M$

The program [9] will be modified identifying the inability of solving some of the non-linear algebraic equations of the form (3.1). The modified GRAEFPEB.BAS is developed using the following algorithm.

**Algorithm**

1. Identify (the presence of any zero coefficients)
2. or (all the coefficients are unity)
3. if (there is no pair of non-zero coefficients from the zero coefficient)
4. or (all coefficients are unity)
5. then go to print "equation is not solvable by graeffe's root squaring method"
6. Repeat the above steps for each subsequent new transformed coefficients.
3.5 Modified GRAEFEF.BAS

10 PRINT "******************************************************************************************************************"
20 PRINT "*"""
30 PRINT "* GRAEFFE'S ROOT-SQUARING METHOD ""
40 PRINT "*""
50 PRINT "* (Modified GRAEFFE.BAS) ""
60 PRINT "*******************************************************************************************************************"
70 "******************** MAIN PROGRAM ****************************""
80 
90 
100 "Define the polynomial"
110 
120 PRINT " DEGREE OF POLYNOMIAL " ; : INPUT N
130 DIM A(20, N), C$(N), SUM(N), FACT(20, N), LC(N), R(N), X(N), XI(N), ROOTS(N)
140 FOR K = N TO 0 STEP -1
150 PRINT " COEFFICIENT " ; K:
160 INPUT A(0, K)
170 NEXT K
180 PRINT " GIVE THE CONVERGENCE VALUE OF F " ; : INPUT EPS
190 " Extract factor of ten and reduce coefficients by this factor."
200 " in order to be able to handle very large numbers."
210 " Keep account of factors, and increase or decrease accordingly."
220 
230 FOR K = N TO 0 STEP -1
240 IF A(0, K) = 0 GO TO 300
250 FACT(0, K) = INT(LOG(ABS(A(0, K)))) / LOG(10)
260 A(0, K) = A(0, K) / (10 ^ FACT(0, K))
270 NEXT K
310 PRINT: PRINT "ROOT SQUARING PROCESS:" "
320 ' Print the original coefficients with the factors.
330 ' PRINT:PRINT " r" ;:FOR K=N TO 0 STEP-1 ;PRINT" A " ;:NEXT K; PRINT
340 FOR K = N TO 0 STEP -1 : PRINT " r" ; K ; :NEXT K : PRINT
350 PRINT USING "##" ; R ;
360 FOR K = N TO 0 STEP -1
370 PRINT USING "######"; A(0, K); PRINT"L"; :PRINT USING "### ": FACT(0,K).
380 NEXT K
390 PRINT
400 ' Beginning of major iteration
410 ' FOR K=N TO 0 STEP -1
420 ' FACT (R, I) = 2 * FACT(R-1, I)
430 SUM(I) = 0
440 FOR L = I TO 1
450 IF (1 - L) < 0 OR (I + L) > N THEN GOTO 590
460 FA = FACT(R-1, I+L) + FACT(R-1, I-L)
470 FB = FACT(R, I) - FA
480 IF FB > 20 GOTO 590
490 SUM(I) = SUM(I) + 2*(-I)^L * A(R-1, I+L) * A(R-1, I-L) / (10^FB)
500 NEXT L
510 A(R, I) = A(R-1, I)^2 + SUM(I)
520 ' New statements
603 IF ABS(A(R,1)) = 0 THEN GOTO 605 ELSE GOTO 610
605 IF ABS(A(R, I-1)) <> 0 AND ABS(A(R, I+1)) <> 0 THEN GOTO 1345
606 IF ABS(A(R,1)) > 10 THEN A(R, 1) = A(R, 1)/10; FACT(R, 1) = FACT(R, 1) + 1; GOTO 610
610 IF ABS(A(R, 1)) < 1 THEN A(R, 1) = A(R, 1)*10; FACT(R, 1) = FACT(R, 1) - 1
620 PRINT USING “###.###”, A(R, 1); PRINT “E”; PRINT USING “###” ; FACT(R, 1);
630 IF ABS(FACT(R, 1)) > 999 THEN GOTO 880
640 IF ABS(FACT(R, 1)) > 999 THEN GOTO 880
650 NEXT I
660 ' Check whether coefficients are pure squares
670 ' PS$ = "PURE SQUARE" ; NS$ = "non-square"
680 CS$(N) = PS$ ; CS$(0) = PS$
690 FOR I = N-1 TO 1 STEP -1
700 IF SUM(1) = 0 GOTO 750
710 W = (A(R-1, 1)^2) / SUM(1)
720 IF ABS(W) > 1000 THEN CS$(1) = PS$ ELSE CS$(1) = NS$
730 NEXT I
740 PRINT
750 SQUARES = 0
760 FOR I = N TO 0 STEP -1
770 IF CS$(I) = PS$ THEN SQUARES = SQUARES + 1
780 NEXT I
790 NSQUARES = N + 1 - SQUARES
800 IF SQUARES = N + 1 THEN GOTO 900
810 NEXT R
820 ' End of major iteration.
830 ' PRINT:PRINT "ITERATIONS EXCEEDED. POSSIBILITY OF COMPLEX OR REPEATED ROOTS."
840 GOTO 890
PRINT ; PRINT "FACTOR EXCEEDS 999. POSSIBILITY OF COMPLEX OR
REPEATED ROOTS."

R = R-1

PRINT ; PRINT "THE COEFFICIENTS ARE:" ; PRINT

PRINT ; ; FOR I = N TO 0 STEP -1 : PRINT C$(I); "," ; NEXT : PRINT

' Find location of non-squares

FOR K = N TO 0 STEP -1

IF C$(K) <> PS$ THEN LC(K) = K

NEXT K

' Evaluate the real roots and the sum of these roots.

M = 2 ^ R

PRINT ; PRINT "THE NUMBER OF SQUARING: r = " ; R ; "THE POWER: m = " ; M

PRINT ; PRINT "CALCULATION OF ROOTS:" 

SUMRT = 0

FOR K = N TO 1 STEP -1

NR = N + 1 - K

IF C$(K-1) <> PS$ THEN K = K - 1 : GOTO 1110

' New statement

IF C$(N-1) <> PS$ AND C$(1) <> PS$ THEN GOTO 1110

' X(NR) = (A(R, K-1)/A(R, K)) ^ (1/M) * 10 ^ ((FACT(R, K-1) - FACT(R, K)) / M)

GOSUB 1500 ' Check the sign of the root

SUMR1 = SUMRT + X(NR)

NEXT K

' Calculating the sum of the product of the real roots

GOSUB 2000
1150 ' Branch to subroutine for complex or repeated roots
1160 ON NSQUARES GOSUB 4000, 5000, 6000
1170 IF NSQUARES > 3 THEN PRINT " ***PROGRAM CAN NOT DETERMINE
MORE THAN TWO PAIRS OF COMPLEX ROOTS ***"
1180 ' Print all the roots
1190 PRINT: PRINT " THE"; N: "ROOTS ARE : "
1200 FOR K = 1 TO N
1210 YY = 1000
1220 IF ABS (X(K) * YY) > 32000 THEN YY = YY / 10: GOTO 1220
1230 IF X(K) = 0 THEN GOTO 1260
1240 IF ABS (X(K) * YY) < 1000 THEN YY = YY * 10: GOTO 1240
1250 X(K) = CINT (X(K) * YY) / YY
1260 YY = 1000
1270 IF ABS (XI(K) * YY) > 32000 THEN YY = YY / 10: GOTO 1270
1280 IF XI(K) = 0 THEN GOTO 1310
1290 IF ABS (XI(K) * YY) < 1000 THEN YY = YY * 10: GOTO 1290
1300 XI(K) = CINT (XI(K) * YY) / YY
1310 PRINT TAB (20); X(K); 
1320 IF XI(K) > 0 THEN PRINT " + "; XI(K); " i "
1330 IF XI(K) < 0 THEN PRINT " - "; -XI(K); " i "
1340 NEXT K
1345 GOSUB 7000 ' New statement
1350 END
1360 ' 
1500 ' ************ Subroutine 1 : Checking sign of real roots ************
1510 ' 
1520 FUP = A(0, 0) * (10 ^ FACT (0, 0)); FUN = A(0, 0) * (10 ^ FACT (0, 0))
1530 FOR I = N TO 1 STEP -1
1540 FUP = FUP + A(0, I) * (10 ^ FACT (0, I)) * X(NR) ^ I
1550 FUN = FUN + A(0, I) * (10 ^ FACT (0, I)) * (-X(NR)) ^ I
1560 NEXT I
1570 PRINT:PRINT " FUNCTION WITH POSITIVE VALUE OF(" ;X (NR) ; ")=": FUP
1580 PRINT " FUNCTION WITH NEGATIVE VALUE OF(" ;X(NR) ; ")=": FUN
1590 IF ABS (FUP) > EPS AND ABS (FUN) > EPS THEN PRINT " WARNING : CONVERGENCE NOT SATISFIED BY REAL ROOT *
1600 ' choose root which gives lowest value of function
1610 IF ABS (FUN) < ABS (FUP) THEN X(NR) = -X(NR)
1620 RETURN
1630 '
2000 ' * Subroutine 2: Calculation of the sum of the product of the real roots*.
2010 '
2020 PRODRT = 0
2030 FOR K = 1 TO N-1
2040 FOR KK = K+1 TO N
2050 PRODRT = PRODRT + X(K) * X(KK)
2060 NEXT KK
2070 NEXT K
2080 RETURN
2090 '
3000 ' ** Subroutine 3: Checking complex roots **********
3010 '
3020 FOR KK = 0 TO N : SUMI (KK) = 0 : NEXT KK
3030 FOR KK = 0 TO N : SUMC (KK) = 0 : NEXT KK
3040 SUMC (0) = 1
3050 FOR I = 1 TO N+1
3060 SUMC (I) = 1
3070 FOR KK = 2 TO N
3080 SUMC (KK) = SUMC (KK) + SUMC (KK-1)
3090 NEXT KK
3100 FOR J = 0 TO N-1-I
3110 SUMI (J)=SUMI(J)+ A(0, I-1+J)*(10^FACT (0, I-1+J))+(ALPHA^I-1)*SUMC (J)
3120 NEXT J
3130 NEXT I
3140 SUMII = 0
3150 FOR I = 1 TO N STEP 2
3160 SUMII = SUMII + (BL1A^1) * (SUMI(I)) * (-1)^(I-1) / 2
3170 NEXT I
3180 SUMII2 = 0
3190 FOR I = 2 TO N STEP 2
3200 SUMII2 = SUMII2 + (BETA^1) * (SUMI(I)) * (-1)^(I/2)
3210 NEXT I
3220 SUMALL = SUMI(0) + SUMII + SUMII2
3230 IF ABS(SUMALL) < EPS THEN RETURN
3240 PRINT** WARNING: CONVERGENCE NOT SATISFIED BY COMPLEX ROOT**
3250 PRINT "** VALUE OF FUNCTION = " ; SUMALL ; " > " ; EPS ; " **"
3260 RETURN
3270 
4000 **** Subroutine 4: One pair of complex or repeated roots ****
4010 
4020 PRINT: PRINT "CALCULATION IF ONE PAIR OF COMPLEX OR REPEATED ROOTS:"
4030 ALPHA = (-A(0,N-1)/A(0,N)) * 10^FACT(0,N-1)-FACT(0,N)-SUMRT/2
4040 FOR K = N TO 1 STEP -1
4050 NR = N + 1 - K
4060 IF LC(K) = 0 THEN GOTO 4170
4070 R(K) = (A(R, K-1)/A(R, K+1))^((1/M) * 10^((FACT(R, K-1)-FACT(R, K+1))/M)
4080 BETA = SQR(ABS(R(K) - ALPHA^2))
4090 ' Check for repeated roots
4100 PRINT "CHECK FOR REPEATED ROOTS:"
4110 X(NR) = SQR(R(K)): GOSUB 1500
4120 IF ABS(FUP) > EPS AND ABS(FUN) > EPS THEN PRINT "* ROOTS ARE COMPLEX *" : GOTO 4190
4130 IF ABS(FUP) < EPS AND ABS(FUN) < EPS THEN PRINT "** ROOTS ARE REAL AND REPEATED BUT OF OPPOSITE SIGN**: X(NR-1) = -X(NR)
4140 IF ABS(FUP) < EPS AND ABS(FUN) > EPS THEN PRINT "** ROOTS ARE REAL AND THE SAME SIGN**: X(NR-1) = X(NR)
4150 IF ABS(FUP) > EPS AND ABS(FUN) < EPS THEN PRINT "** ROOTS ARE REAL AND REPEATED AND OF THE SAME SIGN**: X(NR-1) = X(NR)
4160 GOTO 4210
4170 NEXT K
4180 ' Check convergence with complex roots
4190 GOSUB 3000
4200 X(NR) = ALPHA : XI(NR) = -BETA : X(NR-1) = ALPHA : XI(NR-1) = BETA
4210 RETURN
4220 '
5000 ' Subroutine 5: Two pair of complex or repeated roots with different moduli:
5010 '
5020 RSUM = 0
5030 PRINT: PRINT "TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH DIFFERENT MODULI:"
5040 B = ( (A(0, N-1) / A(0, N) ) * 10 ^ (FACT(0, N-1) - FACT(0, N)) + SUMRT ) / 2
5050 PRINT "CHECK FOR REPEATED ROOTS:"
5060 FOR K = N TO 1 STEP -1
5070 NR = N + 1 - K
5080 IF LC(K) = 0 THEN GOTO 5230
5085 IF CS(K) <> PS$ AND CS(K) <> NS$ GOTO 5230 ' New statement
5090 R(K) = (ABS(A(R, K-1) / A(R, K))) ^ (1/M) * 10 ^ ((FACT(R, K-1) - FACT(R, K+1))/M)
5100 RSUM = RSUM + R(K)
5110 ' Check for repeated roots
5120 X(NR) = SQR(R(K)) : GOSUB 1500
5130 IF ABS(FUP) > EPS AND ABS(FUN) > EPS THEN PRINT "* ROOTS ARE COMPLEX": GOTO 5230
IF \( |FUP| < \varepsilon \) AND \( |FUN| < \varepsilon \) THEN PRINT "*** ROOTS ARE REAL AND REPEATED BUT OF OPPOSITE SIGN***": \( X(NR-1) = -X(NR) \)

IF \( |FUP| < \varepsilon \) AND \( |FUN| > \varepsilon \) THEN PRINT "*** ROOTS ARE REAL AND REPEATED AND OF THE SAME SIGN***": \( X(NR-1) = X(NR) \)

IF \( |FUP| > \varepsilon \) AND \( |FUN| < \varepsilon \) THEN PRINT "*** ROOTS ARE REAL AND REPEATED AND OF THE SAME SIGN***": \( X(NR-1) = X(NR) \)

' Cancel the nonsquare corresponding to the repeated pair
' and goto the subroutine for one pair of roots

\[ LC(K) = 0 \]

\[ SUMRT = SUMRT + X(NR) + X(NR-1) \]

GOSUB 4000

GOTO 5400

NEXT K

\[ C = \left( \frac{(A(0, N-2)/A(0, N)) \times 10^8 (FACT(0, N-2) - FACT(0, N)) + 2 \times D \times SUMRT - PRODRT - RSUM}{4} \right) \]

\[ ALPHEL1 = \left( \frac{-B + \sqrt{ABS(D^2 - 4 \times C)}}{2} \right) \]

\[ ALPHEL2 = -B - ALPHEL1 \]

\[ COUNT = 0 \]

FOR K = N TO 1 STEP -1

\[ NR = N + 1 - K \]

IF \( LC(K) = 0 \) THEN GOTO 5390

COUNT = COUNT + 1

IF COUNT = 1 THEN BETAL1 = SQRT(ABS(R(K) - ALPHEL1^2))

' Check convergence with complex roots

ALPHA = ALPHEL1; BETA = BETAL1; GOSUB 3000

IF ABS(SUMALL) > \varepsilon \) THEN PRINT "** SWAP THE VALUE OF ALPHAL AND ALPHAL2 AND TRY AGAIN **": SWAP ALPHAL1, ALPHAL2; GOTO 5270

IF COUNT = 1 THEN \( X(NR) = ALPHAL1; XI(NR) = -BETAL1; X(NR-1) = ALPHAL1; \)

\[ XI(NR-1) = BETAL1 \]

IF COUNT = 2 THEN BETAL2 = SQRT(ABS(R(K) - ALPHAL2^2))
5380 IF COUNT = 2 THEN X(NR) = ALPHA2: XI(NR) = -BETA2: X(NR-1) = ALPHA2: XI(NR-1) = BETA2
5390 NEXT K
5400 RETURN
5410 ' subroutine 6: Two pairs of complex or repeated roots with identical moduli
6000 ' Check for neighboring nonsquares
6030 W = 0
6040 FOR K = N TO 1 STEP -1
6050 IF LC(K) <> 0 AND LC(K-1) <> 0 THEN W = 1
6060 NEXT K
6070 IF W = 0 THEN GOTO 6520
6080 PRINT: PRINT "TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH IDENTICAL MODULI:"
6090 NR = (NSQUARES+1) / 2
6100 B = ((A(N, N-1) A(0, N)) *10 ^ (FACT(0, N-1) - FACT(0, N)) + SUM(R) / 2
6110 RSUM = 0
6120 PRINT " CHECK FOR REPEATED ROOTS:"
6130 FOR K = N TO 1 STEP -1
6140 NR = N + 1 - K
6150 IF LC(K) = 0 THEN GOTO 6230
6160 R(K) = (A(R, K+1-2*P) / A(R, K+1)) ^ (1 / (P*M)) * 10 ^((FACT(R, K+1-2*P) -FACT(R, K+1)) / (P*M))
6170 RSUM = RSUM + P * R(K)
6180 X(NR) = SQR(R(K)): GOSUB 1500
6190 IF ABS(FUP) > EPS AND ABS(FUN) > E:PS THEN PRINT " * ROOTS ARE COMPLEX *": GOTO 6240
6200 IF ABS(FUP)<EPS AND ABS(FUN) < EPS THEN PRINT " ** ROOTS ARE REAL AND REPEATED BUT OF OPPOSITE SIGN**: X(NR-1) = -X(NR): GOTO 6380
6210 IF ABS(FUP) < EPS AND ABS(FUN) > EPS THEN PRINT "* ROOTS ARE REAL AND REPEATED AND OF THE SAME SIGN***": GOTO 6240
6220 IF ABS(FUP) > EPS AND ABS(FUN) < EPS THEN PRINT "* ROOTS ARE REAL AND REPEATED AND OF THE SAME SIGN***": GOTO 6240
6230 NEXT K
6240 'Repeated roots of same sign or complex roots:
6250 C = ((A(0,N-2)/A(0,N)) * 10^-10 - FACT(0,N-2)/FACT(0,N)^2) * SUMRT - PRODRT - RSUM) / 4
6260 ALPHA1 = (-B + SQR(ABS(B^2 - 4*C))) / 2
6270 ALPHA2 = -B - ALPHA1
6280 FOR K = N TO 1 STEP -1
6290 NR = N + 1 - K
6300 IF LC(K) = 0 THEN GOTO 6360
6310 BETA1 = SQR(ABS(R(K) - ALPHA1^2))
6320 X(NR) = ALPHA1: X(NR) = -BETA1: X(NR-1) = ALPHA1: X(NR-1) = BETA1
6330 BETA2 = SQR(ABS(R(K) - ALPHA2^2))
6340 X(NR+2) = ALPHA2: X(NR+2) = -BETA2: X(NR+1) = ALPHA2: X(NR+1) = BETA2
6350 GOTO 6370
6360 NEXT K
6370 RETURN
6380 'Repeated roots of opposite sign or complex roots with zero real part
6390 COUNT = 0
6400 ALPHA2 = -B
6410 FOR K = N TO 1 STEP -1
6420 NR = N + 1 - K
6430 IF LC(K) = 0 THEN GOTO 6500
6440 BETA2 = SQR(ABS(R(K) - ALPHA2^2))
6450 ALPHA = ALPHA2: BETA = BETA2: GOSUB 3000
6460 IF ABS(SUMALL)>EPS THEN PRINT "SWAP THE VALUE OF ALPHA2 AND BETA2 AND TRY AGAIN. SWAP ALPHA2,BETA2: COUNT= COUNT+1: GOTO 6450
6470 X(NR+2)=ALPHA2: XI(NR+2)=-BETA2; X(NR+1) = ALPHA2: XI(NR+1) = BETA2
6480 IF BETA2 =0 AND COUNT > 0 THEN X(NR+2) =ALPHA2: X(NR+1) = -ALPHA2
6490 GOTO 6510
6500 NEXT K
6510 RETURN
6520 PRINT " ** PROGRAM CANNOT DETERMINE MORE THEN TWO PAIRS OF COMPLEX ROOTS **"
6530 RETURN
7000 "Subroutine 7: " for not solvable"
7010 FOR I=N-1 TO 1 STEP -1
7020 IF ABS (A(R, I)) = 0 THEN GOTO 7030
7030 IF ABS(A (R, I-1))<> 0 AND ABS(A (R, I+1)) <> 0 THEN GOTO 7050
7040 NEXT I
7050 FOR K=1 TO N
7060 IF X(K) + XI(K)=0 THEN GOTO 7090
7070 IF X(K) + XI(K) <>0 THEN GOTO 1350
7080 NEXT K
7090 PRINT: PRINT " EQUATION IS NOT SOLVABLE BY GRAEFFE'S ROOT SQUARING METHOD"
7100
DEGREE OF POLYNOMIAL: 5

COEFFICIENT 5 IS 1

COEFFICIENT 4 IS -5

COEFFICIENT 3 IS -15

COEFFICIENT 2 IS 85

COEFFICIENT 1 IS -26

COEFFICIENT 0 IS -120

GIVE THE CONVERGENCE VALUE OF \( F = 0.002 \)

ROOT-SQUARING PROCESS:

\[
\begin{array}{cccccccc}
& r & 5 & r & 4 & r & 3 & r & 2 \\n0 & 1.000E+0 & -5.000E+0 & -1.500E+1 & 8.500E+1 & -2.600E+1 & -1.200E+2 \\
1 & 1.000E-1 & 55.000E+0 & 10.230E+1 & 76.450E+2 & 21.076E+3 & 14.400E+4 \\
3 & 1.000E-1 & 46.298E+2 & 28.712E+1 & 17.531E+2 & 43.173E+3 & 42.998E+4 \\
4 & 1.000E-1 & 15.693E+2 & 66.212E+1 & 28.254E+2 & 18.489E+3 & 18.488E+4 \\
5 & 1.000E-1 & 23.302E+2 & 42.954E+2 & 79.587E+3 & 34.182E+5 & 34.182E+5 \\
6 & 1.000E-1 & 54.210E+3 & 18.447E+2 & 63.340E+3 & 11.684E+5 & 11.684E+5 \\
7 & 1.000E-1 & 29.388E+3 & 34.028E+2 & 40.119E+3 & 13.652E+5 & 13.652E+5 \\
\end{array}
\]

THE COEFFICIENTS ARE:

PURE SQUARE, PURE SQUARE, PURE SQUARE, PURE SQUARE, PURE SQUARE,

PURE SQUARE

THE NUMBER OF SQUARING: \( r = 7 \)       THE POWER: \( m = 128 \)
CALCULATION OF ROOTS:

FUNCTION WITH POSITIVE VALUE OF (5) = 1.602173E-04
FUNCTION WITH NEGATIVE VALUE IS (5) = -2240.001
FUNCTION WITH POSITIVE VALUE OF (4) = -80.00003
FUNCTION WITH NEGATIVE VALUE IS (4) = 1.959801E-04
FUNCTION WITH POSITIVE VALUE OF (3) = 1.525879E-05
FUNCTION WITH NEGATIVE VALUE IS (3) = 480
FUNCTION WITH POSITIVE VALUE OF (2) = 1.478195E-05
FUNCTION WITH NEGATIVE VALUE IS (2) = 280.0001
FUNCTION WITH POSITIVE VALUE OF (1) = -80
FUNCTION WITH NEGATIVE VALUE IS (1) = -8.583069E-06

THE 5 ROOTS ARE:

5
-4
3
2
-1
**Example 02**

*GRAEFLE'S ROOT-SQUARING METHOD* 

**GRAEFLE.EAS**

**DEGREE OF POLYNOMIAL: 5**

<table>
<thead>
<tr>
<th>COEFFICIENT</th>
<th>5 IS 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>COEFFICIENT</td>
<td>4 IS -10</td>
</tr>
<tr>
<td>COEFFICIENT</td>
<td>3 IS 42</td>
</tr>
<tr>
<td>COEFFICIENT</td>
<td>2 IS -102</td>
</tr>
<tr>
<td>COEFFICIENT</td>
<td>1 IS 145</td>
</tr>
<tr>
<td>COEFFICIENT</td>
<td>0 IS -100</td>
</tr>
</tbody>
</table>

**GIVE THE CONVERGENCE VALUE OF** \( F = 0.001 \)

**ROOT-SQUARING PROCESS:**

<table>
<thead>
<tr>
<th>( R )</th>
<th>( A )</th>
<th>( A )</th>
<th>( A )</th>
<th>( A )</th>
<th>( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>( 1.000E\theta )</td>
<td>( -1.000E\theta )</td>
<td>( 4.200E1 )</td>
<td>( -1.020E2 )</td>
<td>( 1.450E2 )</td>
</tr>
<tr>
<td>1</td>
<td>( 10.000E-1 )</td>
<td>( 1.600E1 )</td>
<td>( 1.400E1 )</td>
<td>( 0.224E3 )</td>
<td>( 0.625E3 )</td>
</tr>
<tr>
<td>2</td>
<td>( 10.000E-1 )</td>
<td>( 22.800E1 )</td>
<td>( -57.220E2 )</td>
<td>( 3.527E5 )</td>
<td>( -40.894E5 )</td>
</tr>
<tr>
<td>3</td>
<td>( 10.000E-1 )</td>
<td>( 63.428E3 )</td>
<td>( -13.626E7 )</td>
<td>( 12.318E10 )</td>
<td>( -53.812E12 )</td>
</tr>
<tr>
<td>4</td>
<td>( 10.000E-1 )</td>
<td>( 42.957E8 )</td>
<td>( 28.322E14 )</td>
<td>( 17.776E20 )</td>
<td>( 43.212E25 )</td>
</tr>
<tr>
<td>5</td>
<td>( 10.000E-1 )</td>
<td>( 18.447E18 )</td>
<td>( -72.495E29 )</td>
<td>( 15.712E41 )</td>
<td>( -16.879E52 )</td>
</tr>
<tr>
<td>6</td>
<td>( 10.000E-1 )</td>
<td>( 34.029E37 )</td>
<td>( -54.139E59 )</td>
<td>( 39.046E82 )</td>
<td>( -29.349E104 )</td>
</tr>
<tr>
<td>7</td>
<td>( 10.000E-1 )</td>
<td>( 11.580E76 )</td>
<td>( -23.643E121 )</td>
<td>( 18.874E166 )</td>
<td>( -69.479E210 )</td>
</tr>
<tr>
<td>8</td>
<td>( 10.000E-1 )</td>
<td>( 13.409E153 )</td>
<td>( 12.188E243 )</td>
<td>( 50.851E332 )</td>
<td>( 10.525E422 )</td>
</tr>
<tr>
<td>9</td>
<td>( 10.000E-1 )</td>
<td>( 17.981E307 )</td>
<td>( 12.183E486 )</td>
<td>( 28.833E665 )</td>
<td>( 90.741E843 )</td>
</tr>
</tbody>
</table>

**FACTOR EXCEEDS 999. POSSIBILITY OF COMPLEX OR REPEATED ROOTS.**

**THE COEFFICIENTS ARE:**

PURE SQUARE, PURE SQUARE, NON SQUARE, NON SQUARE, NON SQUARE.
THE NUMBER OF SQUAREING: \( r = 8 \)    THE POWER: \( n = 256 \)

CALCULATION OF ROOTS:
FUNCTION WITH POSITIVE VALUE OF \( (4.000002) = 1.125336E-04 \)
FUNCTION WITH NEGATIVE VALUE IS \( (4.000002) = -8584.013 \)

TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH IDENTICAL MODULI:
CHECK FOR REPEATED ROOTS:
FUNCTION WITH POSITIVE VALUE OF \( (2.236068) = -10.29418 \)
FUNCTION WITH NEGATIVE VALUE IS \( (2.236068) = -1709.705 \)

*WARNING: CONVERGENCE NOT SATISFIED BY REAL ROOT*

*ROOTS ARE COMPLEX*

* THE 5 ROOTS ARE:

\[
\begin{align*}
4 \\
2 + 1i \\
2 - 1i \\
1 + 2i \\
1 - 2i
\end{align*}
\]
GRAEFFE'S ROOT-SQUARING METHOD

DEGREE OF POLYNOMIAL 5

COEFFICIENT 5 IS 1
COEFFICIENT 4 IS 0
COEFFICIENT 3 IS 0
COEFFICIENT 2 IS 0
COEFFICIENT 1 IS 0
COEFFICIENT 0 IS 1

GIVE THE CONVERGENCE VALUE OF F = 0.001

ROOT-SQUARING PROCESS:

<table>
<thead>
<tr>
<th>R</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>r 5</td>
<td>r 4</td>
<td>r 3</td>
<td>r 2</td>
<td>r 1</td>
<td>r 0</td>
</tr>
<tr>
<td>0</td>
<td>1.000E+0</td>
<td>0.000E+0</td>
<td>0.000E+0</td>
<td>0.000E+0</td>
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</tr>
<tr>
<td>1</td>
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<td>0.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
<td>1.000E+1</td>
</tr>
<tr>
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<td>0.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
<td>1.000E+1</td>
</tr>
<tr>
<td>3</td>
<td>1.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
<td>1.000E+1</td>
</tr>
<tr>
<td>4</td>
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<td>0.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
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<td>0.000E+1</td>
<td>0.000E+1</td>
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<td>0.000E+1</td>
<td>1.000E+1</td>
</tr>
<tr>
<td>6</td>
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<td>0.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
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<td>0.000E+1</td>
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</tr>
<tr>
<td>9</td>
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<td>0.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
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</tr>
<tr>
<td>10</td>
<td>1.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
<td>0.000E+1</td>
<td>1.000E+1</td>
</tr>
</tbody>
</table>
FACTOR EXCEEDS 999. POSSIBILITY OF COMPLEX OR REPEATED ROOTS.

THE COEFFICIENTS ARE:
PURE SQUARE, , , , PURE SQUARE

THE NUMBER OF SQUARING: \( r = 9 \) THE POWER: \( m = 512 \)

CALCULATION OF ROOTS:
THE 5 ROOTS ARE:

\[
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}
\]

EQUATION IS NOT SOLVABLE BY GRAEFFE'S ROOT SQUARING METHOD
**GRAEFfE'S ROOT-SQUARING METHOD**

**DEGREE OF POLYNOMIAL: 4**

COEFFICIENT 4 IS 1

COEFFICIENT 3 IS 0

COEFFICIENT 2 IS 0

COEFFICIENT 1 IS 0

COEFFICIENT 0 IS 1

GIVE THE CONVERGENCE VALUE OF \( f = 0.0001 \)

**ROOT-SQUARING PROCESS:**

<table>
<thead>
<tr>
<th>( r )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.000E 0</td>
<td>0.000E -1</td>
<td>0.000E -2</td>
<td>0.000E -3</td>
<td>0.000E -4</td>
</tr>
<tr>
<td>1</td>
<td>10.000E 0</td>
<td>0.000E 0</td>
<td>0.000E 0</td>
<td>0.000E 0</td>
<td>0.000E 0</td>
</tr>
<tr>
<td>2</td>
<td>10.000E -1</td>
<td>20.000E -1</td>
<td>40.000E -1</td>
<td>60.000E -1</td>
<td>80.000E -1</td>
</tr>
<tr>
<td>3</td>
<td>10.000E -2</td>
<td>20.000E -2</td>
<td>40.000E -2</td>
<td>60.000E -2</td>
<td>80.000E -2</td>
</tr>
<tr>
<td>4</td>
<td>10.000E -3</td>
<td>20.000E -3</td>
<td>40.000E -3</td>
<td>60.000E -3</td>
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</tr>
<tr>
<td>5</td>
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<td>20.000E -4</td>
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<td>40.000E -5</td>
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<td>80.000E -5</td>
</tr>
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<td>60.000E -6</td>
<td>80.000E -6</td>
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<td>80.000E -8</td>
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<tr>
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<td>80.000E -12</td>
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<td>60.000E -17</td>
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<td>80.000E -18</td>
</tr>
<tr>
<td>20</td>
<td>10.000E -19</td>
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<td>40.000E -19</td>
<td>60.000E -19</td>
<td>80.000E -19</td>
</tr>
</tbody>
</table>

48
ITERATIONS EXCEEDED. POSSIBILITY OF COMPLEX OR REPEATED ROOTS.

THE COEFFICIENTS ARE:
PURE SQUARE, NON SQUARE, NON SQUARE, NON SQUARE, PURE SQUARE

THE NUMBER OF SQUARING: \( r = 20 \) \hspace{1cm} \text{THE POWER:} \hspace{0.5cm} m = 1048576

CALCULATION OF ROOTS:
TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH IDENTICAL MODULUS:
CHECK FOR REPEATED ROOTS
FUNCTION WITH POSITIVE VALUE OF \((1) = 2\)
FUNCTION WITH NEGATIVE VALUE IS \((1) = 2\)
*WARNING: CONVERGENCE NOT SATISFIED BY REAL ROOT*
* ROOTS ARE COMPLEX *

THE 4 ROOTS ARE:
\[
0.7071 + 0.7071 \ i \\
0.7071 - 0.7071 \ i \\
-0.7071 + 0.7071 \ i \\
-0.7071 - 0.7071 \ i
\]
Example:05

************* GRAEFE'S ROOT-SQUARING METHOD *************

*************** (GRAEFE.BAS) ***************

DEGREE OF POLYNOMIAL 3

COEFFICIENT 3 IS 1

COEFFICIENT 2 IS 0

COEFFICIENT 1 IS 0

COEFFICIENT 0 IS 1

GIVE THE CONVERGENCE VALUE OF \( F = 2 \)

ROOT-SQUARING PROCESS:

<table>
<thead>
<tr>
<th>R</th>
<th>( A_r )</th>
<th>( A_2 )</th>
<th>( A_1 )</th>
<th>( A_0 )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.000E 0</td>
<td>0.000E 0</td>
<td>1.000E 0</td>
</tr>
<tr>
<td>1</td>
<td>1.000E -1</td>
<td>0.000E -1</td>
<td>0.000E -1</td>
<td>1.000E -1</td>
</tr>
<tr>
<td>2</td>
<td>1.000E -1</td>
<td>0.000E -3</td>
<td>0.000E -3</td>
<td>1.000E -3</td>
</tr>
<tr>
<td>3</td>
<td>1.000E -1</td>
<td>0.000E -7</td>
<td>0.000E -7</td>
<td>1.000E -7</td>
</tr>
<tr>
<td>4</td>
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<td>0.000E -15</td>
<td>1.000E -15</td>
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<td>1.000E -31</td>
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<td>1.000E -63</td>
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<tr>
<td>7</td>
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<td>0.000E -127</td>
<td>1.000E -127</td>
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<td>0.000E -255</td>
<td>1.000E -255</td>
</tr>
<tr>
<td>9</td>
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<td>0.000E -511</td>
<td>0.000E -511</td>
<td>1.000E -511</td>
</tr>
<tr>
<td>10</td>
<td>1.000E -1</td>
<td>0.000E -1023</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

FACTOR EXCEEDS 999. POSSIBILITY OF COMPLEX OR REPEATED ROOTS

THE COEFFICIENTS ARE:

PURE SQUARE , , , PURE SQUARE
THE NUMBER OF SQUARING: \( r = 9 \)    THE POWER: \( m = 512 \)

CALCULATION OF ROOTS:
TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH DIFFERENT MODULI
CHECK FOR REPEATED ROOTS:

THE 3 ROOTS ARE:

0
0
0

EQUATION IS NOT SOLVABLE BY GRAEFFE'S ROOT SQUARING METHOD
Example: 06

----------------------------GRAEFFE'S ROOT-SQUAREING METHOD----------------------------

----------------------------GRAEFFE BAS----------------------------

DEGREE OF POLYNOMIAL: 5

COEFFICIENT 5 IS 1

COEFFICIENT 4 IS 2

COEFFICIENT 3 IS 2

COEFFICIENT 2 IS 2

COEFFICIENT 1 IS 2

COEFFICIENT 0 IS 1

GIVE THE CONVERGENCE VALUE OF \( F = 0.0001 \)

ROOT-SQUAREING PROCESS:

<table>
<thead>
<tr>
<th>R</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>r 5</td>
<td>r 4</td>
<td>r 3</td>
<td>r 2</td>
<td>r 1</td>
<td>r 0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1.000E 0</td>
<td>2.000E 0</td>
<td>2.000E 0</td>
<td>2.000E 0</td>
<td>2.000E 0</td>
<td>1.000E 0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>10.000E -1</td>
<td>0.000E -1</td>
<td>0.000E -1</td>
<td>0.000E -1</td>
<td>0.000E -1</td>
<td>10.000E -1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10.000E -1</td>
<td>0.000E -3</td>
<td>0.000E -3</td>
<td>0.000E -3</td>
<td>0.000E -3</td>
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<td></td>
</tr>
<tr>
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<td>10.000E -1</td>
<td></td>
</tr>
<tr>
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<td>0.000E -127</td>
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<td>8</td>
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<td>0.000E -255</td>
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<td>9</td>
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<td>0.000E -511</td>
<td>0.000E -511</td>
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<td>10.000E -1</td>
<td></td>
</tr>
<tr>
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<td>10.000E -1</td>
<td>0.000E-1023</td>
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</tbody>
</table>

FACTOR EXCEEDS 999. POSSIBILITY OF COMPLEX OR REPEATED ROOTS.
THE COEFFICIENTS ARE:
PURE SQUARE, NON SQUARE, NON SQUARE, NON SQUARE, NON SQUARE, PURE SQUARE

THE NUMBER OF SQUARING: \( r = 9 \)  \hspace{1cm} THE POWER: \( m = 512 \)

CALCULATION OF ROOTS:

THE 5 ROOTS ARE:

\[
\begin{align*}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{align*}
\]

EQUATION IS NOT SOLVABLE BY GRAEFFE'S ROOT SQUARING METHOD
**Graeffe's Root-Squaring Method**

**GRAEFFE'S ROOT-SQUARING METHOD**

**Graeffe's Root-Squaring Method** (GRAEFFE.DAS)

**Degree of Polynomial**: 4

**Coefficient 4 is**: 1

**Coefficient 3 is**: 7

**Coefficient 2 is**: 12

**Coefficient 1 is**: -4

**Coefficient 0 is**: -16

**Give the convergence value of F = 0.0001**

**Root-Squaring Process**:

<table>
<thead>
<tr>
<th>R</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
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<tr>
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<td>1.000E+00</td>
<td>7.000E-01</td>
<td>1.200E+01</td>
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<tr>
<td>2</td>
<td>1.000E+00</td>
<td>2.890E+01</td>
<td>8.730E+02</td>
<td>7.398E+03</td>
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<tr>
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<td>6.049E+01</td>
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<tr>
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<td>4.951E+01</td>
<td>5.629E+03</td>
<td>1.844E+04</td>
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<tr>
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<td>1.000E+00</td>
<td>1.847E+02</td>
<td>1.586E+03</td>
<td>3.402E+03</td>
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<tr>
<td>6</td>
<td>1.000E+00</td>
<td>3.028E+02</td>
<td>1.255E+05</td>
<td>1.157E+06</td>
<td>1.157E+06</td>
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<tr>
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<td>1.000E+00</td>
<td>1.579E+03</td>
<td>7.818E+05</td>
<td>1.340E+06</td>
<td>1.340E+06</td>
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<tr>
<td>8</td>
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<td>1.408E+03</td>
<td>3.107E+06</td>
<td>1.797E+07</td>
<td>1.797E+07</td>
</tr>
<tr>
<td>9</td>
<td>1.000E+00</td>
<td>1.797E+03</td>
<td>4.834E+06</td>
<td>3.231E+07</td>
<td>3.231E+07</td>
</tr>
<tr>
<td>10</td>
<td>1.000E+00</td>
<td>3.216E+03</td>
<td>1.749E+07</td>
<td>1.045E+08</td>
<td>1.045E+08</td>
</tr>
</tbody>
</table>

54
FACTOR EXCEEDS 999. POSSIBILITY OF COMPLEX OR REPEATED ROOTS

THE COEFFICIENTS ARE:
PURE SQUARE, PURE SQUARE, NON SQUARE, PURE SQUARE, PURE SQUARE

THE NUMBER OF SQUARING: \( n = 9 \) \hspace{1cm} \text{THE POWER:} \hspace{0.2cm} m = 512

CALCULATION OF ROOTS:
FUNCTION WITH POSITIVE VALUE OF \((4) = 863.9999\)
FUNCTION WITH NEGATIVE VALUE IS \((4) = -9.536743 \times 10^{-7}\)
FUNCTION WITH POSITIVE VALUE OF \((1) = 4.768372 \times 10^{-7}\)
FUNCTION WITH NEGATIVE VALUE IS \((1) = -6\)

ONE PAIR OF COMPLEX OR REPEATED ROOTS:
CHECK FOR REPEATED ROOTS:
FUNCTION WITH POSITIVE VALUE OF \((2) = 96\)
FUNCTION WITH NEGATIVE VALUE IS \((2) = 1.907349 \times 10^{-6}\)

* ROOTS ARE REAL AND REPEATED AND OF THE SAME SIGN

THE 4 ROOTS ARE:

-4
-2
-2
1
Example: 08

GRAFFE'S ROOT-SQUAREING METHOD

DEGREE OF POLYNOMIAL 4

COEFFICIENT 4 IS 1
COEFFICIENT 3 IS -8
COEFFICIENT 2 IS 42
COEFFICIENT 1 IS -80
COEFFICIENT 0 IS 125

GIVE THE CONVERGENCE VALUE OF F = 0.0001

ROOT-SQUAREING PROCESS:

<table>
<thead>
<tr>
<th>R</th>
<th>( A )</th>
<th>( A )</th>
<th>( A )</th>
<th>( A )</th>
<th>( A )</th>
<th>( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>r 4</td>
<td>r 2</td>
<td>r 2</td>
<td>r 1</td>
<td>r 1</td>
<td>r 0</td>
</tr>
<tr>
<td>0</td>
<td>1.000E 0</td>
<td>-8.000E 0</td>
<td>4.200E 1</td>
<td>-8.000E 1</td>
<td>1.230E 2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>10.000E -1</td>
<td>-20.000E 0</td>
<td>73.400E 1</td>
<td>-41.000E 2</td>
<td>15.625E 3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10.000E -1</td>
<td>-10.680E 2</td>
<td>40.601E 4</td>
<td>-61.275E 5</td>
<td>24.414E 7</td>
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<tr>
<td>3</td>
<td>10.000E -1</td>
<td>32.861E 4</td>
<td>15.224E 10</td>
<td>-16.070E 13</td>
<td>59.605E 15</td>
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<td>-19.650E 10</td>
<td>23.283E 21</td>
<td>76.756E 26</td>
<td>35.527E 32</td>
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<td>5</td>
<td>10.000E -1</td>
<td>-79.553E 20</td>
<td>54.210E 43</td>
<td>-10.652E 55</td>
<td>12.622E 66</td>
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<tr>
<td>6</td>
<td>10.000E -1</td>
<td>-10.209E 44</td>
<td>29.387E 88</td>
<td>-23.379E 110</td>
<td>15.931E 133</td>
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</tr>
<tr>
<td>7</td>
<td>10.000E -1</td>
<td>45.451E 88</td>
<td>86.360E 177</td>
<td>-88.167E 222</td>
<td>25.379E 167</td>
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<tr>
<td>8</td>
<td>10.000E -1</td>
<td>33.865E 177</td>
<td>74.580E 356</td>
<td>33.899E 446</td>
<td>64.411E 535</td>
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<td>10.000E -1</td>
<td>-13.749E 357</td>
<td>55.621E 714</td>
<td>18.836E 893</td>
<td>41.488E 1072</td>
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</tbody>
</table>

FACTOR EXCEEDS 999. POSSIBILITY OF COMPLEX OR REPEATED ROOTS.

THE COEFFICIENTS ARE:

PURE SQUARE, NON SQUARE, PURE SQUARE, NON SQUARE, PURE SQUARE.
THE NUMBER OF SQUARING: \( r = 8 \)

THE POWER: \( m = 256 \)

CALCULATION OF ROOTS:

TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH DIFFERENT MODULI.

CHECK FOR REPEATED ROOTS:

FUNCTION WITH POSITIVE VALUE OF \( (5) = 399.9998 \)

FUNCTION WITH NEGATIVE VALUE OF \( (5) = 3199.999 \)

* WARNING: CONVERGENCE NOT SATISFIED BY REAL ROOT *

* ROOTS ARE COMPLEX *

FUNCTION WITH POSITIVE VALUE OF \( (2.236068) = 91.67185 \)

FUNCTION WITH NEGATIVE VALUE OF \( (2.236068) = 628.3282 \)

* WARNING: CONVERGENCE NOT SATISFIED BY REAL ROOT *

* ROOTS ARE COMPLEX *

THE 4 ROOTS ARE:

\[ 3 + 4i \]
\[ 3 - 4i \]
\[ 1 + 2i \]
\[ 1 - 2i \]
Chapter 4

Conclusion and open problems

4.1 Conclusion

Literature survey of the past and the recent has been carried out. Classical method of Graeffe's root squaring method has been discussed. Some problems not solvable manually and using GRAEFFE.BAS software [9] have been discovered under observations. The weaknesses of the method and their recoveries have been discussed.

On the observations in sec. 3.1 the following solvability conditions of Graeffe's root squaring method has been derived in sec. 3.2.

(i) equation with zero-coefficient must have at least one pair of equidistant non-zero coefficient from the zero-coefficient;
(ii) any transformed equation of a given equation with non-zero coefficient may have zero coefficients but these new coefficients must satisfy (i);
(iii) all the coefficients of non-linear algebraic equation must not be unity;
(iv) GRAEFFE.BAS needs modification in the light of (i), (ii) and (iii).

Under these solvability conditions GRAEFFE.BAS software [9] has been modified and subroutine 7 has been developed and extended the software to identify the polynomials which are not solvable by this software. Finally some open problems in sec. 4.2 have been put forward for further research of both computer science and applied mathematics.

4.2 Open problems

Although Graeffe's root squaring finds its way in the first part of twentieth century, still it needs more study. So, some open problems are cited for further research and study.

(i) Development of a general software capable of solving any higher order univariate polynomial having different types of roots repeated any number of times.
(ii) Barciss [4] resultant procedure and his ALGOL 60 may be studied to solve the equations set (3.1).
(iii) Open problems of Nell and Reif [12], Pan [1.2], Demmel [13] are good research problems for both computer science and applied mathematics.
References:

[1] Victor. Y. Pan. Solving a polynomial equation; some history and recent progress, SIAM

[2] Victor. Y. Pan. Computation of approximate polynomial GCDs and an extension, Infor-


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[8] Gregorio Malajovich and Jorge P. Zubelli, On the geometry of Graeffe iteration,
Tangent Graeffe iteration, Informes de Matemática Serie B-119, IMPA, 1-34, 1998
Revised in August, 1999.


Vol. 4, 177-190, 1946.


[12] C. Andrew Neff and John H. Reif, An efficient Algorithm for the complex roots

[13] James Demmel, Benjamin Diament and Gregorio Malajovich, On the complexity of
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