Phase Shift Analysis for Nuclear Scattering

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in Mathematics

By

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ABSTRACT

The purpose of this work is to discuss the problems of phase shift analysis and to construct a method for resolving these problems by using the zeros of scattering amplitude. In particular, this work traces Yukawa potential which describes nucleon-nucleon scattering. First, the phase shifts and the differential cross-sections are computed and then compared with those of the Born results. The comparison shows a fair agreement between the two methods which results in feeling free to compute the zeros of scattering amplitude. The zero trajectories are then displayed graphically in the momentum transfer plane. A study of the nature of the zero trajectories show that the ambiguities of phase shift analysis are reduced to some extent for weak potential strengths.

Contents:

Introduction

Phase shift analysis has been found a well-known problem of recent years. In the past it was often assumed that if the measurements were accurately made, the phase shifts could also be accurately determined so that in case of perfect measurements a unique scattering amplitude might be expected. In case of purely elastic scattering for spinless particles at an energy where no other processes than elastic scattering is allowed, Martin [1] shows that it is possible to deduce sufficient conditions on the differential cross-sections for it to correspond to a unique amplitude. However, the conditions being extremely restrictive it holds only for zero scattering energy in most practical cases. Phase shift analysis have been studied by Chrichton [2], Gersten [3-5], Bowcock [6-8], Hohler [9] and others and the possibilities of resolving the ambiguities arising in phase shift analysis have also been studied by them from different angles. Gersten [4] discussed the possibilities of reconstructing the scattering amplitude fairly accurately with the aid of the zeros of the differential cross-section and has shown that the treatment of the ambiguities in phase shift analysis is greatly simplified if one considers the complex zeros in the cosine of the scattering angle plane These zeros can save to parameterize the differential cross-section or the scattering amplitude as the case may be. Barrelet [10] showed how the scattering amplitude could be parameterized in terms of their zeros and pointed out that these zeros move along smooth trajectories as the scattering energy is varied It has also been seen [8] that the smoothness of the zero trajectories may be a useful criterion for selecting from among several acceptable amplitudes.

In this work we give emphasis on mathematical formulation and numerical techniques in formulating the scattering amplitude and in obtaining its zeros. Then making use of the complex zeros of the scattering amplitude we propose a method for reducing the problem of phase shift analysis The work is designed in the following fashion

In chapter one, we give a brief review of the scattering of spinless particles using the method of partial waves. We also explain a high energy approximation method, the Born approximation.

In chapter two, we choose a simple model of potential scattering which describes scattering of nuclear particles without taking into account the effect of spin. Having a brief discussion on phase shifts we first calculate the phase shifts for the model potential and the differential cross-sections using the method of partial waves. To check the reliability of our results we compare both the phase shifts and the differential cross-sections obtained from the partial wave method with those obtained from the Born approximation method.

In chapter three, we have a detailed discussion on phase shift analysis for potential scattering. We choose a model for a particular potential scattering and find the complex zeros of the cross-section in the complex $\cos\theta$ - plane. The behaviour of the zero trajectories is then studied in the complex momentum transfer plane. Finally, we discuss the ambiguities, which arises in the phase shift analysis for this particular potential. Then following Gersten [4], we design a method which may resolve these ambiguities and hence reduce the problem of phase shift analysis.

Chapter – 1

Theory of Scattering

In this chapter we discuss the simplest collision problem: the theory of non-relativistic scattering of two particles which interact through a potential $V(\mathbf{r})$ depending only on their relative coordinates. First, we study the method of partial wave analysis to derive expressions for the scattering amplitude and the differential cross-section A detailed discussion on the boundary conditions have been done for solving the Schödinger wave equation Then we go on to study the integral equation in scattering theory which is followed to derive the scattering amplitude in the Born approximation method as an expansion in different order Born amplitude.

1.1 The partial wave analysis

We know that the scattering amplitude and the differential cross section can be found from the asymptotic behaviour of the stationary scattering wave function When the potential is central i.e. depends only on the magnitude r of the vector r, the Schrödinger wave equation may be separated in spherical polar coordinates and a simple connection between the radial solutions and the asymptotic form of the stationary scattering wave function may be found. This procedure is called the method of partial waves

Let us consider the non-relativistic scattering of a spinless particle of mass m by a potential $V(\mathbf{r})$. The time-dependent Schrödinger equation of the system [11]

$$\left(-\frac{\hbar^2}{2m}\nabla_r^2 + V(\mathbf{r},t)\right)\psi(\mathbf{r},t) = \mathrm{E}\psi(\mathbf{r},t) \qquad (1.1)$$

admits stationary state solutions of the form:

$$\psi(\mathbf{r},t) = \psi(\mathbf{r})e^{-iEt/\hbar}$$

where the wave function $\psi(\mathbf{r})$ is a solution of the time-independent Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\nabla_r^2 + V(\mathbf{r})\right)\psi(\mathbf{r}) = \mathrm{E}\psi(\mathbf{r}) \tag{1.2}$$

and the energy E of the particle has the definite value

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} = \frac{mv^2}{2} \tag{1.3}$$

Here $\mathbf{p}_i = \hbar\mathbf{k}_i = m\mathbf{v}_i$ is the initial momentum of the particle, $\mathbf{k}_i$ its initial wave vector and $\mathbf{v}_i$ its initial velocity and the magnitudes of these vectors are given by p, k and v respectively.

We shall assume that the potential $V(\mathbf{r})$ tends to zero faster than r^{-1} as $r \to \infty$. We may then look for a particular solution of equation (1.2) which we shall call the stationary scattering wave function This function satisfies the asymptotic boundary condition

$$\psi(\mathbf{r}) \xrightarrow[r\to\infty]{} A\left(\exp(i\mathbf{k}_i.\mathbf{r}) + f(\theta,\phi)\frac{e^{ikr}}{r}\right) \tag{1.4}$$

Here A is independent of r and the angles θ and ϕ with our choice of the polar z-axis along the direction of the incident wave vector $\mathbf{k}_i$.

We may easily verify that for any function $f(\theta,\phi)$ the expression (1 4) satisfies equation (1 2) asymptotically through terms of order $1/r$ in the region where $V(\mathbf{r})$ can be neglected, provided that the potential $V(\mathbf{r})$ vanishes faster than r^{-1} as $r \to \infty$

Let us now consider the potential $V(\mathbf{r})$ to be spherically symmetric or central so that $V(\mathrm{r}) = V(r)$ Introducing the spherical polar coordinates with the z-axis along the incident direction we find the Schrödinger time-independent equation (1.2) for the stationary scattering wave function $\psi(\mathbf{r})$ in the following form:

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(\mathbf{r})+V(r)\psi(\mathbf{r})=E\psi(\mathbf{r}) \tag{1.5}$$

Considering the orbital angular momentum operator $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ we see that the operator "square of the orbital angular momentum" can be expressed as

$$L^2=L_x^2+L_y^2+L_z^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{1.6}$$

Using the commutation relations $[L^2,\mathrm{L}]=0$ and $\mathbf{L}\times\mathbf{L}=i\hbar^2\mathbf{L}$ we can find the eigenfunctions which are common to the operators L^2 and one of the components of $\mathbf{L}$ These are the spherical harmonics $Y_{lm}(\theta,\phi)$ such that

$$L^2Y_{lm}(\theta,\phi)=l(l+1)\hbar^2Y_{lm}(\theta,\phi) \tag{1.7}$$

and

$$L_zY_{lm}(\theta,\phi)=m\hbar Y_{lm}(\theta,\phi) \tag{1.8}$$

where l and m are called respectively the orbital angular momentum quantum number and the magnetic quantum number.

We now return to the Hamiltonian $H=-\frac{\hbar^2}{2m}\nabla_r^2+V(r)$ which can be expressed with the help of equations (1.5) and (1.6) as

$$H=-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{L^2}{\hbar^2r^2}\right]+V(r) \tag{1.9}$$

so that

$$[H,\quad L^2]=[H,\quad L_z]=0 \tag{1.10}$$

Because the operators H, L^2 and L_z all commute, we can look for eigenfuentions common to these three operators. We may therefore expand the scattering wave function ψ in partial waves corresponding to given values of the quantum numbers l and m as

$$\psi(k,\mathbf{r})=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}C_{lm}(k)R_{lm}(k,r)Y_{lm}(\theta,\phi) \tag{1.11}$$

Here we have explicitly displayed the dependence of the function ψ, of the radial functions R_{lm} and of the expansion coefficients C_{lm} on the wave number $k = (2mE)^{1/2}/\hbar$. Using the expansion (1.11) in the Schrödinger equation (1.5) and making use of equations (1.7) and (1.9) we obtain for every radial function the equation

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{l(l+1)}{r^2}\right]R_l(k,r)+V(r)R_l(k,r)=ER_l(k,r) \tag{1.12}$$

Here we have written $R_l(k,r)$ instead of $R_{lm}(k,r)$ since there is no dependence on the magnetic quantum number m. It is convenient to use the new function

$$u_l(k,r)=rR_l(k,r) \tag{1.13}$$

and introduce the reduced potential

$$U(\mathbf{r})=2mV(\mathbf{r})/\hbar^2 \tag{1.14}$$

The new radial equation which we obtain from equation (1.12) is then

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-U(r)\right]u_l(k,r)=0 \tag{1.15}$$

In order to solve equation (1.15) for the radial wave functions u_l, it is necessary to specify the boundary conditions which must be satisfied by these functions. Before we do this, however, let us first examine the solutions of equation (1.15) for $U(r)=0$, namely

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}\right]y_l(k,r)=0 \tag{1.16}$$

where we have replaced $u_l(k,r)$ by $y_l(k,r)$ so that equation (1.16) is simply the radial equation for a free particle. Changing variables to $\rho=kr$ and defining the function $f_l(\rho)=y_l/\rho$ we see that the analogue of equation (1.12) without interaction potential reads

$$\left[\frac{d^2}{d\rho^2}+\frac{2}{\rho}\frac{d}{d\rho}+\left(1-\frac{l(l+1)}{\rho^2}\right)\right]f_l(\rho)=0 \tag{1.17}$$

which is the "spherical Bessel differential equation". Particular solutions of this equation which are often used in scattering theory are the spherical Bessel function

j_l, the spherical Neumann function n_l and the spherical Hankel functions $h_l^{(1)}$ and $h_l^{(2)}$ The general solution of equation (1.17) is then a linear combination of two linearly independent particular solutions. Since the pairs of functions (j_l, n_l) and $\left(h_l^{(1)}, h_l^{(2)}\right)$ are linearly independent solutions of equation (1.16), we may write the general solution y_l of equation (1.17) as

$$y_l(k,r) = kr\left[C_l^{(1)}(k) j_l(kr) + C_l^{(2)}(k) n_l(kr)\right] \tag{1.18}$$

or

$$y_l(k,r) = kr\left[D_l^{(1)}(k) h_l^{(1)}(kr) + D_l^{(2)}(k) h_l^{(2)}(kr)\right] \tag{1 19}$$

where the two pairs of "integration constants" $\left(C_l^{(1)}, C_l^{(2)}\right)$ and $\left(D_l^{(1)}, D_l^{(2)}\right)$ may still, of course, depend on k. The examination of the behaviour of the spherical Bessel function $j_l(\rho)$ as $\rho \to 0$ shows that this function is regular at the origin. where it is proportional to ρ^l. The other functions n_l, $h_l^{(1)}$ and $h_l^{(2)}$ have a pole of order $(l+1)$ at $\rho = 0$ and are called irregular solutions of equation (1 17). By analogy, a function $v_l(k,r)$ which is given (up to a k-dependent multiplicative factor) by

$$v_l(k,r) \sim r j_l(kr) \tag{1.20}$$

is called a regular solution of equation (1.16). We note that it vanishes at the origin, namely $v_l(k,0) = 0$. More precisely, we have

$$v_l(k,r) \underset{r\to 0}{\sim} r^{l+1} \tag{1.21}$$

Solutions of equation (1.16) which fail to vanish at $r = 0$, for example $rn_l(kr)$, $rh_l^{(2)}(kr)$ or $rh_l^{(2)}(rk)$ are called irregular solutions of that equation.

1.2 The boundary conditions

In this section, we will examine the boundary conditions with the help of the radial equation (1 15) which we must impose upon the radial functions $u_l(k,r)$.

On the basis of the above discussion we would expect that outside the range 'a' of the potential we may use equation (1 18) to express $u_l(k,r)$ as

$$u_l(k,r) = kr\left[C_l^{(1)}(k) j_l(kr) + C_l^{(2)}(k) n_l(kr)\right], \quad r >> a \tag{1 22}$$

Let us assume that r is so large that the terms $U(r)$ and $l(l+1)/r^2$ may be neglected in equation (1 15) An "asymptotic" solution is then obviously of the form $e^{\pm ikr}$. More precisely, we may write for large r

$$u_l(k,r) = F_l(k,r) e^{\pm ikr} \tag{1.23}$$

where $F_l(k,r)$ is a slowly varying function of r. Substituting equation (1 23) into equation (1 15) we find that

$$\frac{F_l''}{F_l} \pm \frac{2ikF_l'}{F_l} = W_l(r) \tag{1 24}$$

where we have set $W_l(r) = U(r) + \dfrac{l(l+1)}{r^2}$ and we have written $F_l' = \dfrac{dF_l}{dr}$ and $F_l'' = \dfrac{d^2F_l}{dr^2}$. Since the function F_l is slowly varying, we may drop the term $\dfrac{F_l''}{F_l}$ in equation (1.24) and write $\dfrac{\pm 2ikF_l'}{F_l} \simeq W_l(r)$, so that for large r

$$F_l(k,r) = \exp\left\{\pm \frac{1}{2ik}\int^r W_l(r')dr'\right\} \tag{1 25}$$

Therefore, if

$$\lim_{r\to\infty} |U(r)| < \frac{M}{r^{1+\varepsilon}} \tag{1.26}$$

where M is some constant and ε is greater than zero, we deduce from equation (1 25) that the function F_l is independent of r for $r \to \infty$. Thus, if the condition (1 26) is satisfied, the general solution of equation (1.15) for large r is given by

$$u_l(k,r) = B_l^{(1)}(k)e^{ikr} + B_l^{(2)}(k)e^{-ikr} \tag{1.27}$$

where $B_l^{(1)}(k)$ and $B_l^{(2)}(k)$ are independent or r Using the fact that

$$j_l(x) \underset{x\to\infty}{\to} \frac{1}{x}\sin\left(x - \frac{1}{2}l\pi\right), \qquad n_l(x) \underset{x\to\infty}{\to} -\frac{1}{x}\cos\left(x - \frac{1}{2}l\pi\right) \tag{1 28a}$$

$$h_l^{(1)}(x) \underset{x\to\infty}{\to} -i\frac{e^{i\left(x-\frac{1}{2}l\pi\right)}}{x}, \qquad h_l^{(2)}(x) \underset{x\to\infty}{\to} i\frac{e^{-i\left(x-\frac{1}{2}l\pi\right)}}{x} \tag{1 28b}$$

we may also rewrite equation (1.27) in the form of equations (1.18) or (1 19).

Thus we have, in accordance with equation (1 22)

$$u_l(k,r) \underset{r\to\infty}{\to} kr\left[C_l^{(1)}(k)j_l(kr) + C_l^{(2)}(k)n_l(kr)\right]. \tag{1 29}$$

or

$$u_l(k,r) \underset{r\to\infty}{\to} kr\left[D_l^{(1)}(k)h_l^{(1)}(kr) + D_l^{(2)}(k)h_l^{(2)}(kr)\right] \tag{1 30}$$

From equations (1 28a), we obtain the boundary condition for large r

$$u_l(k,r) \underset{r\to\infty}{\to} A_l(k)\sin\left[kr - l\pi/2 + \delta_l(k)\right] \tag{1 31}$$

with

$$A_l(k) = \left\{\left[C_l^{(1)}(k)\right]^2 + \left[C_l^{(2)}(k)\right]^2\right\}^{1/2} \tag{1 32a}$$

and

$$\tan\delta_l(k) = -\frac{C_l^{(2)}(k)}{C_l^{(1)}(k)} \tag{1.32b}$$

We note that equations (1 13), (1.29) and (1 32b) also imply that we may write

$$R_l(k,r) \underset{r\to\infty}{\to} \hat{A}_l(k)\left[j_l(kr) - \tan\delta_l(k)n_l(kr)\right] \tag{1.33}$$

where $\hat{A}_l(k)$ is independent of r. We note that the quantities δ_l, which are called the **phase shifts**, display the influence of the interaction. Indeed, in the absence of interaction, the regular solution $v_l(k,r)$ of the radial equation (1.16) (i.e. the analogue of equation (1 15) for $U(r) = 0$) is just given by $v_l(k,r) \sim rj_l(kr)$ as shown in equation (1.20) so that

$$v_l(k,r) \underset{r\to\infty}{\sim} \sin\left(kr - l\pi/2\right) \tag{1 34}$$

Upon comparison of equations (1.31) and (1 34) we see that the interaction is clearly responsible for the presence of the phase shifts δ_l. It is also convenient to express the boundary condition for $u_l(k,r)$ as $r \to \infty$ in the form of equation (1 27) or (1.30), i.e. in terms of radially incoming $\exp(-ikr)$ and outgoing $\exp(ikr)$ waves. Upon comparison with equation (1.31) we may write for example

$$u_l(k,r) \underset{r\to\infty}{\to} \tilde{A}_l(k)\left[-(-)^l e^{-ikr} + S_l(k) e^{ikr}\right]$$

with $\tilde{A}_l(k) = A_l(k) i^l e^{-i\delta_l(k)}(-)^l / 2i$, while the coefficient of the outgoing wave is given by

$$S_l(k) = e^{2i\delta_l(k)} \tag{1 35}$$

and is called an S-matrix element.

Let us now examine the boundary condition which must be satisfied by the radial function $u_l(k,r)$ at the origin $r = 0$. This boundary condition is determined by the requirement that all the possible physical states are described by a complete, orthogonal set of wave functions. A detailed examination of this equation shows that two physically allowed solutions $u_l(k,r)$ and $u_l(k',r)$ must satisfy the condition

$$\lim_{r\to 0}\left\{u_l(k,r)\frac{d}{dr}u_l^*(k',r) - u_l^*(k',r)\frac{d}{dr}u_l(k,r)\right\} = 0 \tag{1.36}$$

For a large class of potentials the above requirement may be simplified as follows. Let us assume that near the origin the interaction has the form

$$U(r) = r^p(a_0 + a_1 r + a_2 r^2 + \cdots) \tag{1.37}$$

where p is an integer such that $p \geq -1$. We then expand the solution u_l in the vicinity of $r = 0$ as

$$u_l = r^s(c_0 + c_1 r + \cdots), \quad c_0 \neq 0 \tag{1.38}$$

Upon substitution of equations (1.37) and (1 38) in the radial equation (1.15) we find by looking at the coefficient of the lowest power of r ($i.e. r^{s-2}$) that the quantity s must satisfy the indicial equation

$$s(s-1) - l(l+1) = 0 \tag{1.39}$$

so that $s = l+1$ or $s = -l$. The choice $s = -l$ corresponds to irregular solutions which do not satisfy the condition (1.36). The other choice $s = l+1$ corresponds to regular solutions which are physically allowed These solutions are therefore such that the condition (1.36) simplifies to

$$u_l(k,0) = 0 \tag{1.40}$$

with

$$u_l(k,r) \underset{r\to 0}{\sim} r^{l+1} \tag{1.41}$$

We now consider the case for which the potential is more singular than r^{-1} at the origin. If the interaction is repulsive near $r = 0$, there is no special difficulty and we may continue to impose the simple condition (1.40) since obviously $R_l(k,0) = 0$ in this case On the other hand, if the potential is attractive in the neighbourhood of the origin the nature of the singularity is important.

1.3 Scattering amplitude and differential cross-section

We recall that the asymptotic form of the scattering wave function $\psi(k,\mathbf{r})$ is given by equation (1.4), i.e.,

$$\psi(k,\mathbf{r}) \xrightarrow[r\to\infty]{} A(k)\left[\exp(i\mathbf{k}_i.\mathbf{r}) + f(k,\theta,\phi)\frac{\exp(ikr)}{r}\right] \tag{1.42}$$

where we have displayed explicitly the k-dependence of all the quantities. Choosing the z-axis along the direction of k_i, so that, $\exp(i\mathbf{k}_i.\mathbf{r}) = \exp(ikr\cos\theta)$ is independent of the azimuthal angle ϕ we obtain the well known formula

$$\exp(i\mathbf{k}_i\,\mathbf{r}) = \exp(ikz) = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta) \tag{1.43}$$

which is the partial wave expansion of the plane wave $\exp(ikz)$ where $P_l(\cos\theta)$ are the Legendre polynomials

$$P_l(\cos\theta) = \left(\frac{4\pi}{2l+1}\right)^{1/2} Y_{l,0}(\theta) \tag{1.44}$$

Using equations (1 43) and (1.28a) we may rewrite equation (1.42) in the following form

$$\psi(k,\mathbf{r}) \xrightarrow[r\to\infty]{} A(k)\Bigg[\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}[4\pi(2l+1)]^{1/2} i^l \times \frac{\exp\{i(kr-\frac{1}{2}l\pi)\}-\exp\{-i(kr-\frac{1}{2}l\pi)}{2ikr} Y_{lm}(\theta,\phi)\delta_{m,0} + f(k,0,\phi)\frac{e^{ikr}}{r}\Bigg] \qquad (1.45)$$

On the other hand, we may also consider the partial wave expansion (1.11) for large r Using the fact that $R_{lm}(k,r) \equiv R_l(k,r) = r^{-1}u_l(k,r)$ together with equation (1.31) we obtain the relation

$$\psi(k,\mathbf{r}) \xrightarrow[r\to\infty]{} \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} C_{lm}(k)A_l(k)\frac{1}{2ir}\Big[\exp\{i(kr-\tfrac{1}{2}l\pi+\delta_l) - \exp\{-i(kr-\tfrac{1}{2}l\pi+\delta_l)\}\Big]Y_{lm}(\theta,\phi) \qquad (1.46)$$

Upon comparison of the coefficients of the incoming spherical waves in equations (1.45) and (1.46) we have

$$C_{lm}(k) = \frac{A(k)}{kA_l(k)}[4\pi(2l+1)]^{1/2}\ i^l \exp(i\delta_l)\delta_{m,0} \qquad (1.47)$$

By matching the coefficients of the outgoing spherical waves, and using equations (1 47) and (1 44) we find that the scattering amplitude is independent of ϕ and given by

$$f(k,\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\Big[e^{2i\delta_l(k)}-1\Big]P_l(\cos\theta) \qquad (1.48)$$

This equation can also be rewritten in the form

$$f(k,\theta) = \sum_{l=0}^{\infty}(2l+1)a_l(k)P_l(\cos\theta) \qquad (1\ 49)$$

where the partial wave amplitudes $a_l(k)$ are such that

$$a_l(k) = \frac{1}{2ik}\Big[e^{2i\delta_l(k)}-1\Big] = \frac{1}{2ik}\big[S_l(k)-1\big] = \frac{1}{k}e^{i\delta_l(k)}\sin\delta_l(k) \qquad (1.50)$$

Thus the knowledge of the phase shifts enables us to obtain the scattering amplitude

The differential scattering cross section is given by

$$\frac{d\sigma}{d\Omega}(k,\theta) = |f(k,\theta)|^2 = \frac{1}{k^2}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}(2l+1)(2l'+1)\exp\{i[\delta_l(k)-\delta_{l'}(k)]\}$$

$$\times \sin\delta_l(k)\sin\delta_{l'}(k)P_l(\cos\theta)P_{l'}(\cos\theta) \qquad (1.51)$$

The total cross section is then obtained as

$$\sigma_{tot}(k) = 2\pi\int_0^{\pi}\frac{d\sigma}{d\Omega}(k,\theta)\sin\theta\, d\theta \qquad (1\ 52)$$

which becomes

$$\sigma_{tot}(k) = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l(k) = \sum_{l=0}^{\infty}\sigma_l(k) \qquad (1.53)$$

where each partial wave cross section $\sigma_l(k)$ is given by

$$\sigma_l(k) = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l(k) \qquad (1.54)$$

We note that the maximum contribution of each partial wave to the total cross section is given by

$$\sigma_l^{max}(k) = \frac{4\pi}{k^2}(2l+1) \qquad (1.55)$$

which occurs when

$$\delta_l(k) = \left(n+\frac{1}{2}\right)\pi, \quad n = 0, \pm 1, \pm 2\cdots \qquad (1\ 56)$$

On the contrary, when $\delta_l(k) = n\pi$, there is no contribution to the scattering from that partial wave at the energy $E = \frac{\hbar^2k^2}{2m}$

It is apparent from the above formulae that the method of partial waves is most useful when only a small number of partial waves contribute to the scattering This situation arises at low incident energies More precisely, if 'a' is the range of the potential and k the wave number of the particle, only those partial waves will be important for which

$$l \lesssim ka \qquad (1.57)$$

We note that the first and most important maximum of the free radial wave function $j_l(kr)$ occurs approximately at $r_0 = l/k$, while for small r the function j_l is small and increases as r^l. Therefore, if $a \ll l/k$, the function j_l will be very small in the scattering region and the corresponding phase shift δ_l will be negligible. We may then cut off the partial wave series approximately at $l_{max} = ka$ Thus, if $ka < 1$ we need only calculate a small number of phase shifts in order to obtain the scattering amplitude.

Let us now return to the partial wave expansions (1.11) of the scattering wave function ψ We note that the expression for $c_{lm}(k)$ in equation (1.47) implies that we may rewrite the partial wave expansion (1.11) as

$$\psi(k,\mathbf{r}) = A(k)\sum_{l=0}^{\infty}\frac{\sqrt{4\pi(2l+1)}}{kA_l(k)}i^l e^{i\delta_l}R_l(k,r)Y_{l,0}(\theta)$$

which after using (1.43) and in terms of the radial wave functions $u_l(k,r)$ becomes

$$\psi(k,\mathbf{r}) = A(k)\sum_{l=0}^{\infty}\frac{2l+1}{A_l(k)}i^l e^{i\delta_l}\frac{u_l(k,r)}{kr}P_l(\cos\theta) \tag{1.58}$$

where the functions $u_l(k,r)$ exhibit the asymptotic behaviour (1.31). Hence

$$\psi\ (k,\mathbf{r}) \xrightarrow[r\to\infty]{} A(k)\sum_{l=0}^{\infty}(2l+1)i^l e^{i\delta_l}\frac{\sin(kr-\frac{1}{2}l\pi+\delta_l)}{kr}P_l(\cos\theta) \tag{1.59}$$

and we see that the coefficients $A_l(k)$ have no influence on the scattering. They merely fix the normalization of the radial functions The following normalizations are often used

i) $$u_l(k,r) \xrightarrow[r\to\infty]{} \frac{1}{k}\sin(kr-\tfrac{1}{2}l\pi+\delta_l) \tag{1.60a}$$

ii) $$u_l(k,r) \xrightarrow[r\to\infty]{} \frac{1}{k}\left[\sin(kr-\tfrac{1}{2}l\pi)+\cos(kr-\tfrac{1}{2}l\pi)\tan\delta_l)\right] \tag{1.60b}$$

so that

$$R_l(k,r) \xrightarrow[r\to\infty]{} \frac{1}{kr}\sin(kr-\tfrac{1}{2}l\pi+\delta_l)\,, \tag{1.61a}$$

or $$R_l(k,r) \xrightarrow[r\to\infty]{} j_l(kr)-\tan\delta_l n_l(kr) \tag{1.61b}$$

1.4 The integral equation in scattering theory

Let us rewrite the Schrödinger wave equation (1.2) in terms of the reduced potential (1.14) as

$$\left[\nabla^2 + k^2\right]\psi(\mathbf{r}) = U(\mathbf{r})\psi(\mathbf{r}) \tag{1.62}$$

where the right hand side is considered as an inhomogeneous term. The general solution of this equation can be written as [11]

$$\psi(\mathbf{r}) = \Phi(\mathbf{r}) + \int d\mathbf{r}' G_0^+(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\psi(\mathbf{r}') \tag{1.63}$$

where $\Phi(\mathbf{r})$ is a solution of the homogeneous equation

$$\left[\nabla^2 + k^2\right]\Phi(\mathbf{r}) = 0 \tag{1.64}$$

and $G_0^+(\mathbf{r},\mathbf{r}')$ is a Green's function corresponding to the operator ∇^2 and the wave number k. That is

$$[\nabla^2 + k^2]G_0^+(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') \tag{1.65}$$

with the Green's function $G_o^+(\mathbf{r},\mathbf{r}')$ given by

$$G_0^+(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} \tag{1.66}$$

or in the integral representations

$$G_o^+(\mathbf{r},\mathbf{r}') = -(2\pi)^{-3}\lim_{\epsilon\to 0}\int d\mathbf{k}'\frac{\exp\{\mathbf{k}'.(\mathbf{r}-\mathbf{r}')\}}{k'^2 - k^2 - i\epsilon} \tag{1.67}$$

In the scattering problem that we are considering, the function $\Phi(\mathbf{r})$ is simply the incident plane wave $\exp(i\mathbf{k}_i.\mathbf{r}) = \exp(ikz)$ with the z-axis chosen along $\mathbf{k}_i$. We shall denote this plane wave by $\Phi_{\mathbf{k}_i}(\mathbf{r})$ and 'normalize' it in such a way that.

$$\Phi_{\mathbf{k}_i}(\mathbf{r}) = (2\pi)^{-3/2}\exp(i\mathbf{k}_i.\mathbf{r}) \tag{1.68}$$

Returning to equation (1.63) and using equation (1.68) we may write

$$\psi(\mathbf{r}) = (2\pi)^{-3/2}\exp(i\mathbf{k}_i.\mathbf{r}) + \int d\mathbf{r}' G_0^+(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\psi(\mathbf{r}') \tag{1.69}$$

This is known as the integral equation or Lippmann-Schwinger equation of potential scattering. It replaces the Schrödinger equation (1.2) plus the boundary condition (1.4) which is incorporated in equation (1.70) through the Green's function $G_o^+(\mathbf{r},\mathbf{r}')$. Using equation (1.66) it gives

$$\psi(\mathbf{r}) = (2\pi)^{-3/2}\exp(i\mathbf{k}_i.r) - \frac{1}{4\pi}\int d\mathbf{r}'\frac{\exp(i\mathbf{k}|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}U(\mathbf{r}')\psi(\mathbf{r}') \qquad (1.70)$$

It may be easily verified that the solution of the Lippmann-Schwinger equation (1.70) has the correct asymptotic behaviour (1.4). The first term on the right of equation (1.70) has already the required form, so that we only need to analyze the large-r behaviour of the integral

$$J = \int d\mathbf{r}'\frac{\exp\{ik|\mathbf{r}-\mathbf{r}'|\}}{|\mathbf{r}-\mathbf{r}'|}U(\mathbf{r}')\psi(\mathbf{r}') \qquad (1.71)$$

Since

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 - 2\mathbf{r}.\mathbf{r}' + r'^2} \underset{r\to\infty}{\to} r - \hat{\mathbf{r}}\,\mathbf{r}' + \frac{1}{2r}(\hat{\mathbf{r}}\times\mathbf{r}')^2 + \cdots \qquad (1.72)$$

where $\hat{\mathbf{r}}$ is the unit vector along r, we may write

$$\frac{\exp\{ik|\mathbf{r}-\mathbf{r}'|\}}{|\mathbf{r}-\mathbf{r}'|} \underset{\mathbf{r}\to\infty}{\to} \frac{\exp(ikr)\exp(-ik\hat{\mathbf{r}}.\mathbf{r}')}{r}\times\left[1 + \frac{\hat{\mathbf{r}}.\mathbf{r}'}{r} + \frac{ik}{2r}(\hat{\mathbf{r}}\times\mathbf{r}')^2 + v(r^{-2})\right] \qquad (1.73)$$

provided that r' and k remain finite. Now, for potentials having a finite range 'a' the contribution to the integral (1.71) is negligible when r' becomes somewhat larger 'a', since $U(\mathbf{r}') = 0$ in that region. Let us set $r'_{max} = a$ as a crude estimate of the highest value of r' to be considered in equation (1.71). We may then use equation (1.73) to deduce that for $r >> a$ and $r >> ka^2$ the integral (1.71) is given by

$$J \simeq \frac{\exp(ikr)}{r}\int d\mathbf{r}'\exp(-ik\hat{\mathbf{r}}.\mathbf{r}')U(\mathbf{r}')\psi(\mathbf{r}')$$

Hence, returning to the stationary scattering wave function (1.70) we have

$$\psi(\mathbf{r}) \underset{r\to\infty}{\to} (2\pi)^{-3/2}\exp(i\mathbf{k}_i\,\mathbf{r}) + \frac{\exp(ikr)}{r}\times\left[-\frac{1}{4\pi}\int d\mathbf{r}'\exp(-i\mathbf{k}_f.\mathbf{r}')U(\mathbf{r}')\psi(\mathbf{r}')\right] \quad (1.74)$$

where we have defined the final wave vector $\mathbf{k}_f = k\hat{\mathbf{r}}$ which points in the direction of the detector and has therefore spherical polar coordinates (k,θ,ϕ). It is worth noting that our derivation of equation (1 74) does not apply to a potential which has an infinite range

We now return to the boundary condition (1 4) in which we choose $A = (2\pi)^{-3/2}$ Thus we may write

$$\psi(\mathbf{r}) \underset{r\to\infty}{\to} (2\pi)^{-3/2}\left[\exp(i\mathbf{k}_i.\mathbf{r}) + f(k,\theta,\phi)\frac{\exp(ikr)}{r}\right] \quad (1.75)$$

Comparing with equation (1.74) we obtain the integral representation of the scattering amplitude

$$f = -\frac{(2\pi)^{-3/2}}{4\pi}\int d\mathbf{r}'\exp(-i\mathbf{k}_f.\mathbf{r}')U(\mathbf{r}')\psi(\mathbf{r}') \quad (1\ 76)$$

or

$$f = -2\pi^2 < \Phi_{\mathbf{k}_f}|U|\psi > \quad (1.77)$$

where we have introduced the plane wave $\Phi_{\mathbf{k}_f}$ corresponding to the final wave vector $\mathbf{k}_f$, namely

$$\Phi_{\mathbf{k}_f}(\mathbf{r}) = (2\pi)^{-3/2}\exp(i\mathbf{k}_f.\mathbf{r}) \quad (1\ 78)$$

1.5 The Born approximation

In this section we shall express the Born series as a perturbation-type expansion of the wave function or the scattering amplitude in powers of the interaction potential We first attempt to solve the Lippmann-Schwringer equation (1.70) by iteration, starting with the plane wave $\Phi_{\mathbf{k}_i}(\mathbf{r}) = (2\pi)^{-3/2}\exp(i\mathbf{k}_i.\mathbf{r})$ as the zero order approximation. We obtain in this way the sequence of functions

$$\psi_0(\mathbf{r}) = \Phi_{\mathbf{k}_i}(\mathbf{r})$$

$$\psi_1(\mathbf{r}) = \Phi_{\mathbf{k}_i}(\mathbf{r}) + \int d\mathbf{r}' G_0^+(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\Phi_{\mathbf{k}_i}(\mathbf{r}')$$

$$\psi_2(\mathbf{r}) = \Phi_{\mathbf{k}_i}(\mathbf{r}) + \int d\mathbf{r}' G_0^+(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\psi_1(\mathbf{r}')$$

$$\vdots$$

$$\psi_n(\mathbf{r}) = \Phi_{\mathbf{k}_i}(\mathbf{r}) + \int d\mathbf{r}' G_0^+(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\psi_{n-1}(\mathbf{r}') \qquad (1.79)$$

where $G_0^+(\mathbf{r},\mathbf{r}')$ is the Green's function. In case of a free particle, the value of the Green's function is given by equation (1.66) and we have assumed that the interaction potential is real and local. We may also write the function ψ_n in the form

$$\psi_n(\mathbf{r}) = \sum_{m=0}^{n}\Phi_m(\mathbf{r}) \qquad (1.80)$$

where

$$\Phi_0(\mathbf{r}) = \psi_0(\mathbf{r}) = \Phi_{\mathbf{k}_i}(\mathbf{r}) \qquad (1.81)$$

$$\Phi_1(\mathbf{r}) = \int d\mathbf{r}' G_0^+(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\Phi_{\mathbf{k}_i}(\mathbf{r}') = \int d\mathbf{r}' K_1(\mathbf{r},\mathbf{r}')\Phi_{\mathbf{k}_i}(\mathbf{r}')$$

with

$$K_1(\mathbf{r},\mathbf{r}') = G_0^+(\mathbf{r},\mathbf{r}')U(\mathbf{r}') \qquad (1.82)$$

and

$$\Phi_m(\mathbf{r}) = \int d\mathbf{r}' K_m(\mathbf{r},\mathbf{r}')\Phi_{\mathbf{k}_i}(\mathbf{r}'), \quad m \geq 2 \qquad (1.83)$$

with

$$K_m(\mathbf{r},\mathbf{r}') = \int d\mathbf{r}'' K_1(\mathbf{r},\mathbf{r}'')K_{m-1}(\mathbf{r}'',\mathbf{r}'), \quad m \geq 2 \qquad (1.84)$$

The Born series for the scattering wave function is obtained by letting $n \to \infty$ in equation (1.80) We see from equations (1.81) to (1.84) that it is a perturbation series in powers of the potential. Assuming that the sequence (1.79) converges towards the exact solution ψ of the scattering problem, we may then write equation (1 80) as

$$\psi(\mathbf{r}) = \sum_{m=0}^{\infty} \Phi_m(\mathbf{r}) \tag{1.85}$$

Let us now consider the integral representation (1.77) of the scattering amplitude. If we replace in it the exact scattering wave function successively by the functions of the sequence (1 79) we obtain the corresponding sequence

$$f_{B1} = -2\pi^2 < \Phi_{\mathbf{k}_f} | U | \Phi_{\mathbf{k}_i} > \tag{1 86}$$

$$f_{B2} = -2\pi^2 < \Phi_{\mathbf{k}_f} | U | \psi_1 > \tag{1.87}$$

$$\vdots$$

$$f_{Bn} = -2\pi^2 < \Phi_{\mathbf{k}_f} | U | \psi_{n-1} > \tag{1 88}$$

The quantities $f_{B1}, f_{B2}, \cdots, f_{Bn}$ are called respectively the first Born approximation, the second Born approximation, ---- nth Born approximation to the scattering amplitude Using equation (1.69) and setting $j = m+1$ we may also write equation (1.88) as

$$f_{Bn} = \sum_{j=1}^{n} \bar{f}_{Bj} \tag{1 89}$$

where the expression $\bar{f}_{Bj}$ is given by

$$\bar{f}_{Bj} = -2\pi^2 < \Phi_{\mathbf{k}_f} | U | \Phi_{j-1} >, \quad j \geq 1 \tag{1.90}$$

or using equations (1 81) to (1.84)

$$\bar{f}_{Bj} = -2\pi^2 < \Phi_{\mathbf{k}_f} | U G_0^{(+)} U \cdots G_0^{+} U | \Phi_{\mathbf{k}_i} > \tag{1 91}$$

or more explicitly in the coordinate representation as

$$\bar{f}_{Bj} = -\frac{1}{4\pi} \int d\mathbf{r}_1 \cdots d\mathbf{r}_j \exp(-i\mathbf{k}_f \cdot \mathbf{r}_1) U(r_1) G_0^{(+)}(\mathbf{r}, \mathbf{r}_2)$$

$$U(\mathbf{r}_2) \cdots G_0^{(+)}(\mathbf{r}_{j-1}, \mathbf{r}_j) U(\mathbf{r}_j) \exp(i\mathbf{k}_i \cdot \mathbf{r}_j) \tag{1.92}$$

Thus the Born scattering amplitude is useful to analyze the expression of $\bar{f}_{Bj}$ in momentum space also.

Using the integral representation of the Green's function in equation (1.67) and defining

$$<\mathbf{q}|U|\mathbf{q}'>=(2\pi)^{-3}\int d\mathbf{r}\exp\{i(\mathbf{q}'-\mathbf{q}).\mathbf{r}\}U(\mathbf{r}) \tag{1.93}$$

we find that

$$G_0^+(\mathbf{r},\mathbf{r}')=-(2\pi)^{-3}\lim_{\varepsilon\to 0}\int dk'\frac{\exp\{\mathbf{k}'.(\mathbf{r}-\mathbf{r}')\}}{k'^2-k^2-i\varepsilon} \tag{1.94}$$

and

$$f_{B1}=\bar{f}_{B1}=-2\pi^2<\Phi_{\mathbf{k}_f}|U|\Phi_{\mathbf{k}_i}>=-2\pi^2<\mathbf{k}_1|U|\mathbf{k}_i> \tag{1.95}$$

$$\bar{f}_{Bj}=-2\pi^2\int d\mathbf{k}_1 d\mathbf{k}_2\cdots d\mathbf{k}_{j-1}<\mathbf{k}_f|U|\mathbf{k}_{j-1}>\frac{1}{k^2-k_{j-1}^2+i\varepsilon}$$

$$<\mathbf{k}_{j-1}|U|\mathbf{k}_{j-2}>\frac{1}{k^2-k_{j-2}^2+i\varepsilon}<\mathbf{k}_{j-2}|U|\mathbf{k}_{j-3}>$$

$$\cdots<\mathbf{k}_2|U|\mathbf{k}_1>\frac{1}{k^2-k_1^2+i\varepsilon}<\mathbf{k}_1|U|\mathbf{k}_i> \tag{1.96}$$

The formulae (1.92) and (1.96) justify the interpretation of the Green's function as a propagator, while the quantities $\mathbf{k}_1,\mathbf{k}_2,\cdots\mathbf{k}_{j-1}$ are called "intermediate momenta". We see that the quantity $\bar{f}_{Bj}$ contains j times the potential and (j-1) times the free Green's function $G_0^{(+)}$. We shall call it the term of order j in the sum given by equation (1.89). For example

$$\bar{f}_{B1}=f_{B1}=-2\pi^2<\Phi_{\mathbf{k}_f}|U|\Phi_{\mathbf{k}_i}> \tag{1.97}$$

and

$$\bar{f}_{B2}=-2\pi^2<\Phi_{\mathbf{k}_f}|UG_0^{(+)}U|\Phi_{\mathbf{k}_i}> \tag{1.98}$$

so that

$$f_{B2}=\bar{f}_{B1}+\bar{f}_{B2}=f_{B1}+\bar{f}_{B2} \tag{1.99}$$

If we analyze more closely the first Born approximation f_{B1} we find that the matrix element (1.86), evaluated in the coordinate representation, is given by

$$f_{B1} = -2\pi^2 < \Phi_{\mathbf{k}_f} | U | \Phi_{\mathbf{k}_i} > = -\frac{1}{4\pi} \int d\mathbf{r} \exp\{i(\mathbf{k}_i - \mathbf{k}_f).\mathbf{r}\} U(\mathbf{r}) \qquad (1\ 100)$$

Let us now introduce the wave vector transfer $\mathbf{q}$ which is defined by

$$\mathbf{q} = \mathbf{k}_i - \mathbf{k}_f \qquad (1\ 101)$$

Returning to equation (1 100), we see that

$$f_{B1}(k,\theta) = -\frac{1}{4\pi} \int d\mathbf{r} \exp(i\,\mathbf{q}.\mathbf{r}) U(\mathbf{r}) \qquad (1.102)$$

so that the first Born scattering amplitude for a given direction (0, ϕ) is propotional to the Fourier transform of the potential corresponding to the wave vector transferred during the collision. The differential cross section in the first Born approximation is evidently given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{B1} = |f_{B1}|^2 \qquad (1.103)$$

We shall now introduce the second Born approximation given by equation (1.99) where the quantity $\bar{f}_{B2}$, is defined by the matrix element (1.98) We see that $\bar{f}_{B2}$ may be written more explicitly in the coordinate representation as

$$\bar{f}_{B2} = -\frac{1}{4\pi} \int d\mathbf{r} d\mathbf{r}' \exp(-i\mathbf{k}_f.\mathbf{r})\ U(r) G_0^{(+)}(\mathbf{r},\mathbf{r}') U(\mathbf{r}') exp(i\mathbf{k}_i.\mathbf{r}') \qquad (1.104)$$

and in the momentum space as

$$\bar{f}_{B2} = -2\pi^2 \int d\boldsymbol{\kappa} < \mathbf{k}_f | U | \boldsymbol{\kappa} > \frac{1}{k^2 - k^2 + i\in} < \boldsymbol{\kappa} | U | \mathbf{k}_i > \qquad (1.105)$$

The differential cross-section in the second Born approximation is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{B2} = |f_{B2}|^2 = |f_{B1} + \bar{f}_{B2}|^2 \qquad (1.106)$$

To derive the validity of the first Born approximation we write the exact scattering wave function $\psi(\mathbf{r})$ as

$$\psi(\mathbf{r}) = \Phi_{k_i}(\mathbf{r}) + \psi_{sc}(\mathbf{r}) \tag{1 107}$$

Assuming that $|\psi_{sc}(\mathbf{r})|$ reaches its maximum value at the "center" $r = 0$ of the potential, we require in order for our perturbation approach to be valid that

$$|\psi_{sc}(0)| << |\Phi_{k_i}(\mathbf{r})| = (2\pi)^{-3/2} \tag{1.108}$$

To obtain an estimate of $\psi_{sc}(\mathbf{r})$, we return to equation (1.79) and assume that the exact scattering wave function $\psi(\mathbf{r})$ may be approximated by the function $\psi_I(\mathbf{r})$ Then

$$\psi_{sc}(\mathbf{r}) \approx -\frac{1}{4\pi}\int d\mathbf{r}' \frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} U(\mathbf{r}')\Phi_{k_i}(\mathbf{r}') \tag{1.109}$$

so that equation (1 108) may be written as

$$\frac{1}{4\pi}\left|\int d\mathbf{r}' \frac{\exp(ikr')}{r'} U(\mathbf{r}')\exp(i\mathbf{k}_i.\mathbf{r}')\right| << 1 \tag{1 110}$$

For a central potential of "strengths" $|U_0|$ and range 'a', the angular integrations are easily performed and we find that

$$\frac{|U_0|}{4k^2}\left|\exp(2ika) - 2ika - 1\right| << 1 \tag{1 111}$$

For low energies $(ka \to 0)$ this condition becomes $|U_0|a^2/2 << 1$. Remembering that a square well binds a particle when $|U_0|a^2/2 > \frac{1}{8}\pi^2 \approx 1$. We verify that the first Born approximation is valid at low energies only if the potential is very 'weak'. At high energies, the condition given in equation (1 111) yields $|U_0|\frac{a}{2k} << 1$. Hence the first Born approximation is correct for sufficiently large incoming energies.

Chapter – 2

Phase shifts and differential cross-sections for a Yukawa potential

This chapter concerns with the numerical computation of phase shifts and differential cross-sections for a particular potential scattering. We first have a short review on phase shifts and then we choose a Yukawa potential describing nucleon-nucleon scattering where we have neglected the effect of spin for simplicity. We find the phase shifts for this potential scattering using the method of partial waves and to check the correctness of the results we compare them with those obtained from the first Born approximation method. Having reproduced the phase shifts correctly we go on to find the scattering amplitudes and hence the cross-sections numerically using the method of partial waves. The differential cross-sections are also calculated from the first and the second order Born approximations and then compared with those obtained from the partial wave method.

2.1 The phase shifts

We have shown in section 1.3 that the knowledge of the phase shifts allows one to obtain the scattering amplitude by means of the important relation (1.48). In this section we shall study these phase shifts in more detail.

Let us first establish some general relationships between the phase shifts and the interaction potential. For this purpose, we consider the scattering by two reduced potentials $U(r)$ and $\overline{U}(r)$, with respective radial equations:

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-U(r)\right]u_l(r)=0 \qquad (2.1a)$$

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-\overline{U}(r)\right]\overline{u}_l(r)=0 \tag{2.1b}$$

We shall assume that the functions $u_l(r)$ and $\overline{u}_l(r)$ are "normalized" according to equation (1.60b), namely

$$u_l(r)\underset{r\to\infty}{\to}\frac{1}{k}\left[\sin\left(kr-\frac{1}{2}l\pi\right)+\cos\left(kr-\frac{1}{2}l\pi\right)\tan\delta_l\right] \tag{2.2a}$$

$$\overline{u}_l(r)\underset{r\to\infty}{\to}\frac{1}{k}\left[\sin\left(kr-\frac{1}{2}l\pi\right)+\cos\left(kr-\frac{1}{2}l\pi\right)\tan\overline{\delta}_l\right] \tag{2.2b}$$

The Wronskian of the two solutions u_l and $\overline{u}_l$ is defined as

$$W(u_l,\overline{u}_l)=u_l\overline{u}_l'-u_l'\overline{u}_l \tag{2.3}$$

where the prime denotes a derivative with respect to the variable r. From equation (2.1) we obtain

$$\overline{u}_l u_l''-u_l\overline{u}_l''-(U-\overline{U})u_l\overline{u}_l=0 \tag{2.4}$$

or

$$\frac{d}{dr}W(u_l,\overline{u}_l)=-(U-\overline{U})u_l\overline{u}_l \tag{2.5}$$

Upon integration over the variable r in the interval (a, b), we deduce that

$$\left[W(u_l,\overline{u}_l)\right]_a^b=-\int_a^b dr\,\overline{u}_l(r)\left[U(r)-\overline{U}(r)\right]u_l(r) \tag{2.6}$$

choosing $a=0$ and $b=\infty$ and remembering that $u_l(0)=\overline{u}_l(0)=0$ we find with the help of equations (2.2) that

$$\tan\delta_l-\tan\overline{\delta}_l=-k\int_0^\infty dr\,\overline{u}_l(r)\left[U(r)-\overline{U}(r)\right]u_l(r) \tag{2.7}$$

provided that $U(r)$ and $\overline{U}(r)$ tend to zero faster than r^{-1} when $r\to\infty$. We also require that the potentials $U(r)$ and $\overline{U}(r)$ should not be more singular than r^{-2} at the origin , since $u_a(r)\sim r$ as $r\to 0$. For the particular case $\overline{U}=0$ we deduce from equation (2.7) the important integral representation

$$\tan\delta_l=-k\int_0^\infty dr\,j_l(kr)U(r)R_l(r)\,r^2 \tag{2.8}$$

where the radial function $R_l(r)$ is normalized according to equation (1.61b).

To study the behaviour of the phase shifts for large l we see that an increase in the value of l (for fixed k) tends to diminish the importance of a given potential of finite range because of the centrifugal barrier term $l(l+1)/r^2$ appearing in the radial equation (1.15) . Thus we expect that the phase shifts $\delta_l(k)$ will tend to zero (modulo π) as $l \to \infty$ (for fixed k) Another way of investigating the behaviour of the phase shifts for $l>>ka$ is to use the integral representation (2.8). Indeed, for a potential of finite range, we have already shown that the radial function R_l will differ little from the corresponding free wave j_l when $l>>ka$. Hence we may write

$$\tan\delta_l \simeq (\tan\delta_l)_{Bl} = -k\int_0^\infty dr[\,j_l(kr)]^2 U(r) r^2, \quad l >> ka \tag{2.9}$$

The quantity $(\tan\delta_l)_{Bl}$ is called the first Born approximation to $\tan\delta_l$.

To have an actual computation of the phase shifts we have to solve the radial equations (1.12) or (1.15) numerically subject to the boundary conditions discussed in section 1.2. In particular, the solution obtained inside the range of the potential must go over smoothly to the "asymptotic" solution, valid outside the range of the interaction.

When the potential has a strict finite range, i e vanishes for $r > a$, one can divide the domain of the variable r into an internal region ($r < a$) and an external region ($r>a$) The boundary conditions at $r = a$ are then that both R_l and dR_l/dr [or u_l and du_l/dr] be continuous at $r = a$. Now the exterior solution may be written as

$$R_l(k,r) = \hat{A}_l(k)\left[j_l(kr) - \tan\delta_l n_l(kr)\right] \tag{2 10}$$

Thus, if we denote by

$$\gamma_l(k) = \left[R_l^{-1}\left(\frac{dR_l}{dr}\right)\right]_{r=a} \tag{2 11}$$

the value of the logarithmic derivative of the interior solution $R_l(k,r)$ at $r = a$, we find that

$$\gamma_l(k) = \frac{k\left[j_l'(ka) - \tan\delta_l(k) n_l'(ka)\right]}{j_l(ka) - \tan\delta_l(k) n_l(ka)} \tag{2.12}$$

where we have defined $j_l'(ka) = \left[\frac{dj_l(x)}{dx}\right]_{x=ka}$ and $n_l'(ka) = \left[\frac{dn_l(x)}{dx}\right]_{x=ka}$

Hence

$$\tan\delta_l(k) = \frac{kj_l'(ka) - \gamma_l(k)\, j_l(ka)}{kn_l'(ka) - \gamma_l(k) n_l(ka)} \tag{2.13}$$

If the potential does not vanish identically beyond a certain value of r, but has nevertheless a range 'a', one chooses a distance $d \gtrsim a$ at which the influence of the potential is negligible The value of the logarithmic derivative of the interior solution is then matched at $r = d$ with that of the exterior (free) solution so that in this case

$$\tan\delta_l(k) = \frac{kj_l'(kd) - \gamma_l(k) j_l(kd)}{kn_l'(kd) - \gamma_l(k) n_l(kd)} \tag{2.14}$$

It is clear that in performing calculations of this type we must check that the phase shifts so obtained are insensitive to any increase in the quantity d In what follows, we shall assume for simplicity that $d = a$ The potentials having a strict finite range and those having a "range" will then be treated on the same footing.

2.2 Phase shifts for a Yukawa potential

We now turn to the actual computation of the phase shifts for a Yukawa potential

$$U(r) = -U_0 \frac{e^{-r/a}}{r} \tag{2.15}$$

where U_0 is the potential strength and 'a' is the range of the potential. Using this Yukawa potential in the expression (2.9), we get the Born phase shifts in the following form:

$$\tan\delta_{Bl} = kU_0 \int_0^\infty dr\left[J_l(kr)\right]^2 re^{-r/a} \tag{2.16}$$

We evaluate this integral numerically by using Rhomberg method. The computer program is so written that it reads the potential parameters U_0, upper and lower bounds on k and then evaluates the integral giving the values of $\tan\delta_{Bl}$ for different values of l and k.

To find the phase shifts in partial wave method we see that the radial equation for the Yukawa potential given in (2.15) becomes

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}+U_0\frac{e^{-r/a}}{r}\right]R_l(k,r)=0 \tag{2.17}$$

Since it is not possible to solve this equation analytically we solve this numerically. We follow the shooting method to have a numerical solution of this integral equation by using the boundary conditions

$$R_l(k,r)\underset{r\to 0}{\sim} r^l \tag{2 18a}$$

$$R_l(k,r)\underset{r\to\infty}{\to}\frac{1}{kr}\sin\left(kr-\frac{1}{2}l\pi+\delta_l\right) \tag{2 18b}$$

where the phase shift $\tan\delta_{Bl}$ calculated from the first Born approximation may be used for δ_l in the asymptotic form of $R_l(k,r)$ in equation (2 18b).

Since this potential does not vanish identically beyond a certain value of r, we choose a distance $d\gtrsim a$. We now find the phase shifts by using the expression for $\tan\delta_l$ given in equation (2 14) We also check the insensitivity of the phase shifts to any increase in the quantity d

Our computer program starts by reading the potential parameters, upper and lower bounds on k from a data file The program then solves equation (2.17) for $R_l(k,r)$ and then it calculates γ_l by using equation (2 11) for a fixed k and different l The program also calculates the Bessel and Neumann functions $j_l(ka)$ and $n_l(ka)$ and their derivatives for fixed k and different l. Using the formula for phase shifts $\tan\delta_l(k)$ given in equation (2 14) the program calculates the partial wave phase shifts δ_{pwl} for different values of l.

In Tables (2 1-2.3) we have compared the first Born results for $\tan\delta_l$ with the exact ones, that is, from partial wave method obtained by integrating numerically the radial equation (2 17) and using equation (2 14) for six attractive Yukawa potentials (2.15) having a unit 'range' a = 1*fm* and strength potentials U_0= 0 001, 0 005, 1.0, 5.0, 10.0, 20.0 fm^{-1} respectively. The comparison is made for various values of l and for a wave number $k = 5$ fm^{-1} which corresponds to a sufficiently large energy 3.5 GEV.

Table – 2.1

Comparison of phase shifts δ_l from partial wave (PW) method and from the first Born approximation (FBA) for the Yukawa potential (2.15) with "range" $a = 1.0\ fm$ and wave number $k = 5.0\ fm^{-1}$. The notation 2.308(-4) means 2.308×10^{-4}.

	$U_0 = 0.001\ fm^{-1}$		$U_0 = 0.005\ fm^{-1}$	
l	$\delta_l(PW)$	$\delta_l(FBA)$	$\delta_l(PW)$	$\delta_l(FBA)$
0	2.308(-4)	2.329(-4)	1.154(-3)	1.165(-3)
1	1.355(-4)	1.380(-4)	6.770(-4)	6.900(-4)
2	9.180(-5)	9.172(-5)	4.588(-4)	4.586(-4)
3	6.575(-5)	6.573(-5)	3.286(-4)	3.286(-4)
4	4.851(-5)	4.848(-5)	2.425(-4)	2.424(-4)
5	3.647(-5)	3.646(-5)	1.823(-4)	1.823(-4)
6	2.777(-5)	2.776(-5)	1.388(-4)	1.388(-4)
7	2.135(-5)	2.135(-5)	1.067(-4)	1.067(-4)
8	1.652(-5)	1.653(-5)	8.266(-5)	8.267(-5)
9	1.286(-5)	1.288(-5)	6.438(-5)	6.439(-5)
10	1.006(-5)	1.006(-5)	5.037(-5)	5.032(-5)
11	7.893(-6)	7.917(-6)	3.956(-5)	3.959(-5)
12	6.212(-6)	6.240(-6)	3.117(-5)	3.120(-5)
13	4.901(-6)	4.933(-6)	2.463(-5)	2.466(-5)
14	3.873(-6)	3.907(-6)	1.950(-5)	1.954(-5)
15	3.065(-6)	3.100(-6)	1.547(-5)	1.550(-5)
16	2.428(-6)	2.466(-6)	1.230(-5)	1.233(-5)
17	1.924(-6)	1.966(-6)	9.785(-6)	9.831(-6)
18	1.524(-6)	1.568(-6)	7.795(-6)	7.842(-6)
19	1.206(-6)	1.252(-6)	6.217(-6)	6.259(-6)
20	9.540(-7)	1.001(-6)	4.961(-6)	5.003(-6)

Table – 2.2

Comparison of phase shifts δ_l from partial wave (PW) method and from the first Born approximation (FBA) for the Yukawa potential (2.15) with "range" $a = 1.0\ fm$ and wave number $k = 5.0\ fm^{-1}$. The notation 2.381(-1) means 2.381×10^{-1}.

	$U_0 = 1.0\ fm^{-1}$		$U_0 = 5.0\ fm^{-1}$	
l	$\delta_l(PW)$	$\delta_l(FBA)$	$\delta_l(PW)$	$\delta_l(FBA)$
0	2.381(-1)	2.329(-1)	2.534	1.165
1	1.377(-1)	1.380(-1)	8.631(-1)	6.900(-1)
2	9.283(-2)	9.172(-2)	5.200(-1)	4.586(-1)
3	6.629(-2)	6.573(-2)	3.547(-1)	3.286(-1)
4	4.882(-2)	4.848(-2)	2.552(-1)	2.424(-1)
5	3.665(-2)	3.646(-2)	1.890(-1)	1.823(-1)
6	2.788(-2)	2.776(-2)	1.426(-1)	1.388(-1)
7	2.142(-2)	2.135(-2)	1.090(-1)	1.067(-1)
8	1.658(-2)	1.653(-2)	8.397(-2)	8.267(-2)
9	1.291(-2)	1.288(-2)	6.519(-2)	6.439(-2)
10	1.010(-2)	1.006(-2)	5.088(-2)	5.032(-2)
11	7.929(-3)	7.917(-3)	3.990(-2)	3.959(-2)
12	6.247(-3)	6.240(-3)	3.140(-2)	3.120(-2)
13	4.936(-3)	4.933(-3)	2.478(-2)	2.466(-2)
14	3.910(-3)	3.907(-3)	1.961(-2)	1.954(-2)
15	3.104(-3)	3.100(-3)	1.556(-2)	1.550(-2)
16	2.468(-3)	2.466(-3)	1.237(-2)	1.233(-2)
17	1.966(-3)	1.966(-3)	9.848(-3)	9.831(-3)
18	1.568(-3)	1.568(-3)	7.854(-3)	7.842(-3)
19	1.252(-3)	1.252(-3)	6.272(-3)	6.259(-3)
20	1.002(-3)	1.001(-3)	5.015(-3)	5.003(-3)

Table – 2.3

Comparison of phase shifts δ_l from partial wave (PW) method and from the first Born approximation (FBA) for the Yukawa potential (2.15) with "range" $a = 1.0\ fm$ and wave number $k = 5.0\ fm^{-1}$. The notation 1.077(+1) means 1.077×10.

	$U_0 = 10.0\ fm^{-1}$		$U_0 = 20.0\ fm^{-1}$	
l	$\delta_l(PW)$	$\delta_l(FBA)$	$\delta_l(PW)$	$\delta_l(FBA)$
0	- 1 038	2 329	2 209	4 658
1	1 078(+1)	1.380	-9.232(-2)	2 760
2	1.557	9.172(-1)	-1.551	1.834
3	8.548(-1)	6.573(-1)	1.970(+1)	1 315
4	5.664(-1)	4.848(-1)	1 955	9 696(-1)
5	4 031(-1)	3 646(-1)	1 046	7 292(-1)
6	2 975(-1)	2 776(-1)	6 897(-1)	5 553(-1)
7	2.243(-1)	2.135(-1)	4.917(-1)	4.270(-1)
8	1.715(-1)	1 653(-1)	3.644(-1)	3.307(-1)
9	1.324(-1)	1.288(-1)	2.761(-1)	2 576(-1)
10	1.030(-1)	1.006(-1)	2.122(-1)	2.013(-1)
11	8.051(-2)	7.917(-2)	1.646(-1)	1.583(-1)
12	6 323(-2)	6 240(-2)	1 286(-1)	1.248(-1)
13	4.984(-2)	4 933(-2)	1.010(-1)	9.865(-2)
14	3.940(-2)	3.907(-2)	7.957(-2)	7 815(-2)
15	3 123(-2)	3.100(-2)	6 294(-2)	6.200(-2)
16	2 481(-2)	2 466(-2)	4 991(-2)	4 933(-2)
17	1.974(-2)	1.966(-2)	3.967(-2)	3.933(-2)
18	1.574(-2)	1 568(-2)	3.160(-2)	3.137(-2)
19	1.256(-2)	1 252(-2)	2.520(-2)	2.504(-2)
20	1.004(-2)	1.001(-2)	2 014(-2)	2 001(-2)

We observe that for weak potential strengths U_0= 0.001, 0.005 fm^{-1} the first Born values are very accurate even for the lowest partial waves. We note that for the case $U_0 = 1.0\,fm^{-1}$ for which the $|U_0|a/2k = 1/10$, the exact values of the phase shifts agree with the Born values satisfactorily both for small and large values of the partial waves. For the stronger coupling cases U_0= 5 0, 10.0, and 20 0 fm^{-1} for which $|U_0|a/2k = 1/2,1$, and 2 respectively the first Born phase shifts and the exact phase shifts disagree for the lowest l–values, but become progressively close together as l increases. In fact, for l = 10, we see that the first Born and the exact results already agree very well which is in accordance with the discussion for the case $l >> ka$ in the previous sections.

2.3 Differential cross-sections for a Yukawa potential

2.3.1 Born approximation method

Using the expressions in equation (1 102) we obtain the scattering amplitude from the first Born approximation for the Yukawa potential (2 15) as

$$f_{B1} = \frac{1}{4\pi}\int d\mathbf{r} \exp(i\,\mathbf{q}.\mathbf{r})\; U_0 \frac{e^{-r/a}}{r} \tag{2.19}$$

Introducing spherical polar coordinates we get

$$f_{B1} = \frac{U_0}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dr \exp(i\,\mathbf{q}.\mathbf{r})\; e^{-r/a} r \tag{2.20}$$

Performing the integrations over the angular variables we obtain

$$f_{B1} = \frac{U_0}{q}\int_0^{\infty} dr\, e^{-r/a} \sin(qr) \tag{2 21}$$

Evaluating the integral in (2.21) we finally get

$$f_{B1} = \frac{U_0}{\alpha^2 + q^2} \quad , \quad \text{where } \alpha = 1/a \tag{2.22}$$

Now we go on to find the scattering amplitude from the second Born approximation for the Yukawa potential. We rewrite equation (1.105) as

$$\bar{f}_{B2} = 2\pi^2 \int d\boldsymbol{\kappa} < \mathbf{k}_f|U|\boldsymbol{\kappa} > \frac{1}{\kappa^2 - k^2 - i\in} < \boldsymbol{\kappa}|U|\mathbf{k}_i > \tag{2.23}$$

Using the definition of $< \mathbf{q}|U|\mathbf{q}' >$ given in equation (1.93) and the result

$$\int d\mathbf{r} \exp(i\,\mathbf{q}.\mathbf{r}) e^{-\alpha r}/r = \frac{4\pi}{\alpha^2 + q^2}$$

we can write

$$< \mathbf{k}_f|U|\boldsymbol{\kappa} > = \frac{U_0}{2\pi^2} \frac{1}{\alpha^2 + |\boldsymbol{\kappa} - \mathbf{k}_f|^2}$$

$$< \boldsymbol{\kappa}|U|\mathbf{k}_i > = \frac{U_0}{2\pi^2} \frac{1}{\alpha^2 + |\boldsymbol{\kappa} - \mathbf{k}_i|^2}$$

where we have used $U(\mathbf{r}) = -U_0 e^{-r/a}/r = -U_0 e^{-\alpha r}/r,\ (1/a = \alpha)$ for the Yukawa potential Therefore

$$\bar{f}_{B2} = (2\pi^2)^{-1} U_0^2 \int d\boldsymbol{\kappa} \frac{1}{(\kappa^2 - k^2 - i\in)(\alpha^2 + |\boldsymbol{\kappa} - \mathbf{k}_f|^2)(\alpha^2 + |\boldsymbol{\kappa} - \mathbf{k}_i|^2)} \tag{2.24}$$

Let us now consider Dalitz integrals [12] of the type

$$I_{m,n}(\alpha, \beta; \mathbf{k}_i, \mathbf{k}_f; k) = \int d\boldsymbol{\kappa} \frac{1}{(\kappa^2 - k^2 - i\in)(\alpha^2 + |\boldsymbol{\kappa} - \mathbf{k}_i|^2)^m (\beta^2 + |\boldsymbol{\kappa} - \mathbf{k}_f|^2)^n}$$

$$(m, n = 1, 2, \ldots\ldots\ldots\ldots\ldots) \tag{1.25}$$

and the Feynman integral representations

$$\frac{1}{a^m b^n} = \frac{(m+n-1)!}{(m-1)!(n-1)!} \int_0^1 dt \frac{t^{m-1}(1-t)^{n-1}}{[at + b(1-t)]^{m+n}} \tag{2.26}$$

If we set

$$a = \alpha^2 + |\boldsymbol{\kappa} - \mathbf{k}_i|^2 \quad \text{and} \quad b = \beta^2 + |\boldsymbol{\kappa} - \mathbf{k}_f|^2$$

We can show that

$$[at + b(1-t)] = (\alpha^2 + |\boldsymbol{\kappa} - \mathbf{k}_i|^2)t + (\beta^2 + |\boldsymbol{\kappa} - \mathbf{k}_f|^2)(1-t) = \Gamma^2 + |\boldsymbol{\kappa} - \Lambda|^2 \tag{2.27}$$

where

$$\Gamma^2 = \alpha^2 t + \beta^2(1-t) + t(1-t)|\mathbf{k}_i - \mathbf{k}_f|^2 \tag{2.28}$$

and

$$\Lambda = t\mathbf{k}_i + (1-t)\mathbf{k}_f \tag{2.29}$$

so that

$$\frac{1}{a^m b^n} = \frac{1}{(\alpha^2 + |\kappa - \mathbf{k}_i|^2)^m (\beta^2 + |\kappa - \mathbf{k}_f|^2)^n} = \frac{(m+n-1)!}{(m-1)!(n-1)!}\int_0^1 dt \frac{t^{m-1}(1-t)^{n-1}}{(\Gamma^2 + |\kappa - \Lambda|^2)^{m+n}} \tag{2.30}$$

Now, if we set $m = n = 1$ and $\alpha = \beta$ we obtain

$$\frac{1}{(\alpha^2 + |\kappa - \mathbf{k}_i|^2)(\alpha^2 + |\kappa - \mathbf{k}_f|^2)} = \int_0^1 dt \frac{1}{(\Gamma^2 + |\kappa - \Lambda|^2)^2} \tag{2.31}$$

where

$$\Gamma^2 = \alpha^2 + (t - t^2)q^2 \quad , \quad \mathbf{q} = \mathbf{k}_i - \mathbf{k}_f \tag{2.32}$$

so $\bar{f}_{B2}$ now becomes

$$\bar{f}_{B2} = (2\pi^2)^{-1} U_0^2 \int_0^1 dt \int d\kappa \frac{1}{(\kappa^2 - k^2 - i\epsilon)(\Gamma^2 + |\kappa - \Lambda|^2)^2} \tag{2.33}$$

Apart from one dimensional integral on the t variable, the calculation of $I_{m,n}(\alpha, \beta; \mathbf{k}_i, \mathbf{k}_f; k)$ therefore reduces to the evaluation of integrals of the type

$$L_s(k, \Gamma, \Lambda) = \int d\kappa \frac{1}{(\kappa^2 - k^2 - i\epsilon)(\Gamma^2 + |\kappa - \Lambda|^2)^s} \tag{2.34}$$

Let us start form the simple case $s = 1$ for which

$$L_1(k, \Gamma, \Lambda) = \int d\kappa \frac{1}{(\kappa^2 - k^2 - i\epsilon)(\Gamma^2 + |\kappa - \Lambda|^2)} \tag{2.35}$$

We take Λ as the z-axis of spherical co-ordinates $(\kappa, \theta_\kappa, \Phi_\kappa)$ in κ space. Performing the integration over the azimuthal angle Φ_κ, we find that

$$I_1 = 2\pi \int_0^{\pi} d\theta_\kappa \sin\theta_\kappa \int_0^{\infty} d\kappa \frac{\kappa^2}{\left(\kappa^2 - k^2 - i\epsilon\right)\left(\Gamma^2 + \kappa^2 + \Lambda^2 - 2\kappa\Lambda\cos\theta_\kappa\right)} \quad (2.36)$$

Evaluating the above integral we obtain

$$L_1(k,\Gamma,\Lambda) = \frac{i\pi^2}{\Lambda}\log\left(\frac{k+\Lambda+i\Gamma}{k-\Lambda+i\Gamma}\right) \quad (2.37)$$

To find $L_2(k,\Gamma,\Lambda)$ we differentiate (2.35) with respect to Γ and get

$$\frac{d}{d\Gamma} I_1(k,\Gamma,\Lambda) = -2\Gamma \int d\kappa \frac{1}{(\kappa^2 - k^2 - i\epsilon)(\Gamma^2 + |\kappa - \Lambda|^2)^2} = -2\Gamma L_2(k,\Gamma,\Lambda)$$

so that

$$L_2(k,\Gamma,\Lambda) = -\frac{1}{2\Gamma}\frac{d}{d\Gamma} L_1(k,\Gamma,\Lambda) = -\frac{\pi^2}{\Gamma(k^2 - \Gamma^2 - \Lambda^2 + 2ik\Gamma)} \quad (2.38)$$

Thus

$$\bar{f}_{B2} = -\frac{U_0^2}{2}\int_0^1 dt \frac{1}{\Gamma(k^2 - \Gamma^2 - \Lambda^2 + 2ik\Gamma)} \quad (2.39)$$

Since

$$\Gamma^2 + \Lambda^2 = \alpha^2 + (t - t^2)q^2 + \left|t\mathbf{k}_i + (1-t)\mathbf{k}_f\right|^2 = k^2 + \alpha^2$$

Therefore $\bar{f}_{B2}$ now becomes

$$\bar{f}_{B2} = \frac{U_0^2}{2}\int_0^1 dt \frac{1}{\Gamma(\alpha^2 - 2ik\Gamma)} \quad (2.40)$$

To evaluate the above integral we write in the following form:

$$\begin{aligned}
\bar{f}_{B2} &= \frac{U_0^2}{2}\int_0^1 dt \frac{\alpha^2 + 2ik\Gamma}{\Gamma(\alpha^4 + 4k^2\Gamma^2)} \\
&= \frac{U_0^2}{2}\int_0^1 dt \frac{\alpha^2}{\Gamma(\alpha^4 + 4k^2\Gamma^2)} + U_0^2 ki\int_0^1 dt \frac{1}{(\alpha^4 + 4k^2\Gamma^2)} \\
&= I_1 + I_2 \qquad (2.41)
\end{aligned}$$

where

$$I_1 = \frac{U_0^2}{2}\int_0^1 dt \frac{\alpha^2}{\Gamma(\alpha^4 + 4k^2\Gamma^2)} \quad \text{and} \quad I_2 = U_0^2 ki\int_0^1 dt \frac{1}{(\alpha^4 + 4k^2\Gamma^2)}$$

Evaluating the integrals I_1 and I_2 we get

$$I_1 = \frac{U_0^2}{q\sqrt{\alpha^4 + 4k^2\alpha^2 + k^2q^2}} \tan^{-1} \frac{\alpha q}{2\sqrt{\alpha^4 + 4k^2\alpha^2 + k^2q^2}}$$

$$I_2 = \frac{iU_0^2}{2q\sqrt{\alpha^4 + 4k^2\alpha^2 + k^2q^2}} \log \frac{\sqrt{\alpha^4 + 4k^2\alpha^2 + k^2q^2} + kq}{\sqrt{\alpha^4 + 4k^2\alpha^2 + k^2q^2} - kq}$$

So that the scattering amplitude from the second Born approximation finally takes the following form :

$$\bar{f}_{B2} = \frac{U_0^2}{q\sqrt{\alpha^4 + 4k^2\alpha^2 + k^2q^2}} \left\{ \tan^{-1} \frac{\alpha q}{2\sqrt{\alpha^4 + 4k^2\alpha^2 + k^2q^2}} + \frac{1}{2} i \log \left[\frac{\sqrt{\alpha^4 + 4k^2\alpha^2 + k^2q^2} + kq}{\sqrt{\alpha^4 + 4k^2\alpha^2 + k^2q^2} - kq} \right] \right\} \tag{2.42}$$

Separating the real and imaginary parts of $\bar{f}_{B2}$ we obtain the real part of $\bar{f}_{B2}(k,q)$ as

$$Re\, \bar{f}_{B2}(k,q) = \frac{U_0^2}{q\sqrt{\alpha^4 + 4k^2\alpha^2 + k^2q^2}} \tan^{-1} \frac{\alpha q}{2\sqrt{\alpha^4 + 4k^2\alpha^2 + k^2q^2}}$$

$$= \frac{U_0^2}{qak} \left(1 + \frac{\alpha^4}{a^2k^2} \right)^{-1/2} \tan^{-1} \left\{ \frac{\alpha q}{2ak} \left(1 + \frac{\alpha^4}{a^2k^2} \right)^{-1/2} \right\}; \quad a^2 = 4\alpha^2 + q^2$$

$$= \frac{U_0^2}{qak} \left(1 - \frac{\alpha^4}{2a^2k^2} + \ldots \right) \tan^{-1} \left\{ \frac{\alpha q}{2ak} \left(1 - \frac{\alpha^4}{2a^2k^2} + \ldots \right) \right\}$$

$$= \left(\frac{U_0^2}{qak} - \frac{U_0^2\alpha^4}{2qa^3k^3} + \cdots \right) \left[\frac{\alpha q}{2ak} \left(1 - \frac{\alpha^4}{2a^2k^2} + \cdots \right) - \cdots \right]$$

$$= \frac{U_0^2\alpha}{2a^2k^2} + \text{terms containing higher powers of } k^{-1}$$

$$= A(q)/k^2 + \ldots\ldots \tag{2.43}$$

and the imaginary part of $\bar{f}_{B2}(k,q)$ as

$$Im\,\bar{f}_{B2}(k,q) = \frac{U_0^2}{2q\sqrt{\alpha^4+4k^2\alpha^2+k^2q^2}}\log\frac{\sqrt{\alpha^4+4k^2\alpha^2+k^2q^2}+kq}{\sqrt{\alpha^4+4k^2\alpha^2+k^2q^2}-kq}$$

$$= \frac{U_0^2}{2qak}\left(1+\frac{\alpha^4}{a^2k^2}\right)^{-1/2}\log\frac{\left(\sqrt{a^2+\alpha^4/k^2}+q\right)^2}{a^2+\alpha^4/k^2-q^2}\quad;\qquad a^2=4\alpha^2+q^2$$

$$= \frac{U_0^2}{2qak}\left(1-\frac{\alpha^4}{2a^2k^2}+\ldots\right)[2\log\{q+(a^2+\alpha^4/k^2)^{1/2}\}$$
$$-\log\{(a^2-q^2)+\alpha^4/k^2\}]$$

$$= \frac{U_0^2}{2qak}\left(1-\frac{\alpha^4}{2a^2k^2}+\ldots\right)\left[2\log\left\{q+a\left(1+\frac{\alpha^4}{2a^2k^2}+\ldots\right)\right\}\right.$$
$$\left.-\log(a^2-q^2)+\log\left\{1+\frac{\alpha^4}{k^2(a^2-q^2)}\right\}\right]$$

On expansion of logarithmic series followed by simplification, we have

$$Im\,\bar{f}_{B2}(k,q) = \frac{U_0^2}{2qak}\left(1-\frac{\alpha^4}{2a^2k^2}+\ldots..\right)\left[\left\{\log(a+q)^2+\frac{\alpha^4}{ak^2(q+a)}-\ldots..\right\}\right]$$
$$-\left\{\log(a^2-q^2)+\frac{\alpha^4}{(a^2-q^2)k^2}-\ldots..\right\}$$

$$= \left[\frac{U_0^2}{2qa}\log\frac{(a+q)^2}{(a^2-q^2)}\right]\frac{1}{k}+\text{terms containing higher powers of } k^{-1}$$

$$= B(q)/k+\ldots\ldots \qquad (2.44)$$

and we see that for large k the differential cross-section in the second Born approximation takes the form.

$$\left(\frac{d\sigma}{d\Omega}\right)_{B2} = \left(f_{B1}(q)+\frac{A(q)}{k^2}\right)^2+\frac{B^2(q)}{k^2} \qquad (2.45)$$

or to order k^{-2}

$$\left(\frac{d\sigma}{d\Omega}\right)_{B2} = \left(\frac{d\sigma}{d\Omega}\right)_{B1}+\frac{1}{k^2}[2f_{B1}(q)A(q)+B^2(q)] \qquad (2.46)$$

where the functions $A(q)$ and $B(q)$ only depend on q and we have neglected terms of higher order in k^{-1}.

2.3.2 Partial wave method

To find the scattering amplitude for the Yukawa potential (2 15) from the partial wave method we use the formula derived in equation (1.48). We take the upper bound of l as $l_{max} = 25$ and write a computer program which reads the upper and lower bounds on θ and then calculates the Legender polynomials $P_l(\cos\theta)$ for different l and θ. The program is so written that it reads upper and lower bounds on k also Using the calculated values of the phase shifts $\delta_l(k)$ obtained from the partial wave method for different k, the program then computes the scattering amplitude $f(k,\theta)$ for different values of k and θ We then calculate the differential cross-sections both from the partial wave method using the relation $\frac{d\sigma}{d\Omega} = |f(k,\theta)|^2$ and from the Born approximation method by using the relation given in equation (1.106)

In order to check the accuracy of our results we compare the differential cross-sections from the partial wave method with those of the first and the second Born approximations $\left(\frac{d\sigma}{d\Omega}\right)_{B1}$ and $\left(\frac{d\sigma}{d\Omega}\right)_{B2}$ obtained respectively from equations (1 103) and (1.106) and using equations (2 22) and (2.42). The comparison are shown in the Tables (2 4-2 8) for an attractive Yukawa potential of unit range $a = 1\ 0\ fm$ and having five different potential strengths $U_0 = 0.001,\ 0\ 1,\ 0\ 5,\ 1.0$ and $5.0\ fm^{-1}$. The results are shown for three different values of the wave number $k = 1.0,\ 2.5,\ 5.0\ fm^{-1}$ and for various scattering angles.

Table – 2.4

Differential scattering cross section for the Yukawa potential $U(r) = -U_0 e^{-r/a}/r$ with $U_0 = 0.001 fm^{-1}$. The notation 1.000(-6) means 1.000×10^{-6}.

k	θ (degrees)	First Born approximation	Second Born approximation	Exact
	0	1.000(-6)	1 000(-6)	1 060(-6)
	30	6 220(-7)	6.222(-7)	6.221(-7)
	60	2.500(-7)	2 501(-7)	2 498(-7)
1.0	90	1 111(-7)	1.112(-7)	1.109(-7)
	120	6 250(-8)	6 253(-8)	6.224(-8)
	150	4.466(-8)	4.468(-8)	4.439(-8)
	180	4 000(-8)	4.002(-8)	3.956(-8)
	0	1.000(-6)	1 000(-6)	1.017(-6)
	30	1 398(-7)	1.398(-7)	1.395(-7)
	60	1 902(-8)	1 903(-8)	1.891(-8)
2 5	90	5.487(-9)	5.488(-9)	5.416(-9)
	120	2 564(-9)	2.564(-9)	2.510(-9)
	150	1 690(-9)	1.690(-9)	1.645(-9)
	180	1.479(-9)	1 480(-9)	1.427(-9)
	0	1.000(-6)	1 000(-6)	9.521(-7)
	30	1 687(-8)	1 687(-8)	1 559(-8)
	60	1 479(-9)	1.479(-9)	1.364(-9)
5.0	90	3.845(-10)	3 845(-10)	3 455(-10)
	120	1.731(-10)	1.731(-10)	1 463(-10)
	150	1.125(-10)	1.125(-10)	8.810(-11)
	180	9.803(-11)	9 803(-11)	5 824(-11)

Table – 2.5

Differential scattering cross section for the Yukawa potential $U(r) = -U_0 e^{-r/a}/r$ with $U_0 = 0.1 fm^{-1}$ The notation 1 000(-2) means 1.000×10^{-2}.

k	θ (degrees)	First Born approximation	Second Born approximation	Exact
	0	1 000(-2)	1.021(-2)	1.020(-2)
	30	6.220(-3)	6 374(-3)	6.368(-3)
	60	2.500(-3)	2.586(-3)	2 582(-3)
1.0	90	1 111(-3)	1 161(-3)	1.158(-3)
	120	6.250(-4)	6 578(-4)	6.560(-4)
	150	4 466(-4)	4.722(-4)	4.707(-4)
	180	3.100(-4)	4 235(-4)	4.221(-4)
	0	1.000(-2)	1 004(-2)	1 003(-2)
	30	1.398(-3)	1.409(-3)	1.274(-3)
	60	1.902(-4)	1.926(-4)	1.678(-4)
2.5	90	5.487(-5)	5 571(-5)	4.800(-5)
	120	2 564(-5)	2.607(-5)	2 236(-5)
	150	1 690(-5)	1 720(-5)	1.472(-5)
	180	1.479(-5)	1.506(-5)	1 286(-5)
	0	1.000(-2)	1.001(-2)	9.672(-3)
	30	1.687(-4)	1.693(-4)	1.689(-4)
	60	1.479(-5)	1 486(-5)	1 520(-5)
5.0	90	3 845(-6)	3.865(-6)	3.961(-6)
	120	1 731(-6)	1 741(-6)	1.720(-6)
	150	1.125(-6)	1 131(-6)	1.057(-6)
	180	9.803(-7)	9.861(-7)	7.014(-7)

Table – 2.6

Differential scattering cross section for the Yukawa potential $U(r) = -U_0 e^{-r/a}/r$ with $U_0 = 0.5\ fm^{-1}$. The notation 2.500(-1) means 2.500×10^{-1}.

k	θ (degrees)	First Born approximation	Second Born approximation	Exact
	0	2.500(-1)	2.781(-1)	2.726(-1)
	30	1.555(-1)	1.770(-1)	1.727(-1)
	60	6.250(-2)	7.516(-2)	7.259(-2)
1.0	90	2.778(-2)	3.549(-2)	3.389(-2)
	120	1.562(-2)	2.097(-2)	1.985(-2)
	150	1.116(-2)	1.545(-2)	1.454(-2)
	180	1.000(-2)	1.399(-2)	1.314(-2)
	0	2.500(-1)	2.554(-1)	2.536(-1)
	30	3.495(-2)	3.660(-2)	3.600(-2)
	60	4.756(-3)	5.187(-3)	4.998(-3)
2.5	90	1.372(-3)	1.545(-3)	1.458(-3)
	120	6.409(-4)	7.376(-4)	6.854(-4)
	150	4.225(-4)	4.924(-4)	4.530(-4)
	180	3.698(-4)	4.328(-4)	3.965(-4)
	0	2.500(-1)	2.514(-1)	2.417(-1)
	30	4.218(-3)	4.318(-3)	3.976(-3)
	60	3.698(-4)	3.857(-4)	3.576(-4)
5.0	90	9.612(-5)	1.015(-4)	9.317(-5)
	120	4.328(-5)	4.610(-5)	4.035(-5)
	150	2.811(-5)	3.008(-5)	2.468(-5)
	180	2.451(-5)	2.626(-5)	1.600(-5)

Table – 2.7

Differential scattering cross section for the Yukawa potential $U(r) = -U_0 e^{-r/a}/r$ with U_0 = 1 0 fm^{-1}. The notation 6.220(-1) means 6.220×10^{-1}

k	θ (degrees)	First Born approximation	Second Born approximation	Exact
	0	1 000	1 250	1.137
	30	6.220(-1)	8 173(-1)	7.310(-1)
	60	2 500(-1)	3.703(-1)	3.207(-1)
1.0	90	1.111(-1)	1.880(-1)	1.588(-1)
	120	6 250(-2)	1 177(-1)	9 837(-2)
	150	4.466(-2)	8.987(-2)	7 491(-2)
	180	4.000(-2)	8 231(-2)	6.860(-2)
	0	1.000	1.048	1 019
	30	1 398(-1)	1.560(-1)	1.461(-1)
	60	1.902(-2)	2 381(-2)	2.073(-2)
2 5	90	5.487(-3)	7.548(-3)	6.154(-3)
	120	2 564(-3)	3.761(-3)	2.923(-3)
	150	1 690(-3)	2.571(-3)	1.942(-3)
	180	1.479(-3)	2.278(-3)	1 703(-3)
	0	1.000	1.012	9.711(-1)
	30	1.687(-2)	1.800(-2)	1.720(-2)
	60	1.479(-3)	1.681(-3)	1.562(-3)
5.0	90	3.845(-4)	4.566(-4)	4.083(-4)
	120	1 731(-4)	2 115(-4)	1 783(-4)
	150	1.125(-4)	1 396(-4)	1 104(-4)
	180	9 803(-5)	1.223(-4)	7.636(-5)

Table – 2.8

Differential scattering cross section for the Yukawa potential $U(r) = -U_0 e^{-r/a}/r$ with $U_0 = 5.0\ fm^{-1}$. The notation 2.500(+1) means 2.500×10.

k	θ (degrees)	First Born approximation	Second Born approximation	Exact
	0	2.500(+1)	8.125(+1)	1.136(+1)
	30	1.555(+1)	6.308(+1)	5.495
	60	6.250	4.032(+1)	1.015
1.0	90	2.778	2.752(+1)	6.874(-1)
	120	1.562	2.088(+1)	1.706
	150	1.116	1.768(+1)	2.768
	180	1.000	1.672(+1)	3.199
	0	2.500(+1)	3.582(+1)	1.913(+1)
	30	3.495	8.550	2.378
	60	4.756(-1)	2.413	3.991(-1)
2.5	90	1.372(-1)	1.073	1.620(-1)
	120	6.409(-2)	6.372(-1)	9.929(-2)
	150	4.225(-2)	4.748(-1)	7.731(-2)
	180	3.698(-2)	4.317(-1)	7.161(-2)
	0	2.500(+1)	2.778(+1)	2.232(+1)
	30	4.218(-1)	8.829(-1)	3.730(-1)
	60	3.698(-2)	1.368(-1)	3.685(-2)
5.0	90	9.612(-3)	4.744(-2)	1.008(-2)
	120	4.328(-3)	2.500(-2)	4.749(-3)
	150	2.811(-3)	1.761(-2)	3.248(-3)
	180	2.451(-3)	1.574(-2)	3.389(-3)

In Tables (2.4--2.7) we see that the exact values of the differential cross-sections obtained from the partial wave method agree satisfactorily with those of the Born results for the weak potential strengths $U_0 = 0.001$, 0.1, 0.5, and 1.0 fm^{-1} as expected. So we are free enough to proceed further with these computations for scattering amplitude and hence for differential cross-section.

The Tables also show that except at the lowest value of k the second Born approximation offers a little improvement over the first Born values. The reason for this may be found by returning to equation (2.46) and noting that this formula does not provide all the corrections of order k^{-2} to the first Born cross section. In fact it may be the real part of the third order term $\bar{f}_{B3}$ of the Born series which gives a contribution of order k^{-2} to the scattering amplitude and therefore also to the differential cross-section. We note that the importance of this missing term with respect to the contribution coming from $\mathrm{Re}\,\bar{f}_{B2}$ should increase with the strength $|U_0|$ of the potential. This is illustrated in Table 2.8, which corresponds to the choice $U_0 = 5.0\,fm^{-1}$ of the potential strength. The second Born results are seen to be disastrous in Table 2.8 while they are much better in Tables 2.4, 2.5 and 2.6.

Chapter – 3

Zeros of the Scattering Amplitude and Phase Shift Analysis

In this chapter, we first have a detailed review of phase shift analysis and the possible ambiguities in case of purely elastic as well as inelastic scattering processes. We then present a mathematical model which greatly facilitates the computation of complex zeros of the scattering amplitude. The model deals with an attractive Yukawa potential describing spin-independent scattering. The behaviour of the zero trajectories is then studied in the complex momentum transfer plane. Finally, we discuss the application of the zeros of scattering amplitude in resolving the probable ambiguities arising in the construction of the scattering amplitude for this Yukawa potential and design a method which may be used to reduce the problem of phase shift analysis.

3.1 Phase shift analysis

The expression for the scattering amplitude in partial wave analysis derived in equation (1.48) may be rewritten in the following form

$$f(\theta)=\frac{1}{2ik}\sum_{l=0}^{L}(2l+1)\left[S_l-1\right]P_l(\cos\theta) \tag{3.1}$$

where we have truncated the infinite series at 'L' since only those partial waves are significant for which $l \le ka$. Below the inelastic threshold, for pure elastic scattering, the elastic unitarity relation requires $|S_l|=\left|e^{2i\delta_l}\right|=1$ giving the real phase shifts δ_l. In the inelastic region unitarity is less powerful and simply requires $|S_l|\le 1$ giving complex phase shifts δ_l with $\mathrm{Im}\,\delta_l \ge 0$. Hence we have $S_l=e^{2i\delta_l}=\varsigma_l e^{2i\mathrm{Re}\delta_l}$, $0\le\varsigma_l\le 1$,

where the quantity ς_l is often called the 'inelasticity' factor. The scattering amplitude then becomes

$$f(\cos\theta)=\frac{1}{2ik}\sum_{l=0}^{L}(2l+1)\left(\varsigma_l e^{2i\mathrm{Re}\delta_l}-1\right)P_l(\cos\theta) \tag{3.2}$$

We note that the special case $\varsigma_l=1\left(\mathrm{Im}\,\delta_l=0\right)$ corresponds to the pure elastic scattering given in equation (3.1).

The problem of phase shift analysis deals with the technique of deducing information on partial wave amplitudes from differential cross-section measurements in scattering processes. Since the scattering amplitude $f(\theta)$ is a complex number and the differential cross-section $d\sigma/d\Omega=\left|f(\theta)\right|^2$ is real, sufficient information does not exist to fix $f(\theta)$ and so ambiguities arise

If we change the sign of the real part of the phase shifts, the scattering amplitude $f(\cos\theta)$ is changed to $-f^*(\cos\theta)$. Thus, if the signs of all the real parts of the phase shifts are changed, the scattering amplitude is changed to its negative complex conjugate; thus its phase is changed, but its absolute value is unchanged This is known as **trivial ambiguity** For pure elastic potential scattering (real phase shifts) of spinless particles it was assumed that the only ambiguity is the trivial ambiguity that is the simultaneous change of the sign of all phase shifts

Chrichton [2] shows that there are some exceptions to the above statement. One readily sees that for S-wave only and for S- and P-waves only, the phase shifts are determined uniquely except for the trivial ambiguity. But for the case when the only non-vanishing phase shifts are for S, P and D waves, the phase shifts are not unique. By crude search method, Chrichton found that for each value of δ_2 there exists two phase shifts δ_0 and δ_1 -- such that a different set of phase shifts $(\delta_0',\delta_1',\delta_2)$ exists which gives the same $f(\theta)$. The amplitudes of Chrichton contained only S, P and D waves and may be written in the form of polynomials in $\cos\theta$ which are related by the complex conjugation of one root Then ambiguous phase

shifts were constructed with S, P, D and F waves and then with S, P, D, F and G waves. In all these cases the equivalent amplitudes are formed by replacing one or more zeros of an amplitude in the complex $\cos\theta$ plane by its complex conjugate. In fact, the expression in equation (3.1) being composed of a sum of polynomials is itself a polynomial of degree L in $\cos\theta$. We may, therefore, choose to write the amplitude as a product over its zeros z_i,

$$f(\cos\theta) = A_L \prod_{i=1}^{L} (\cos\theta - z_i) \tag{3.3}$$

Gersten [3] first noted that this cross-section will be unaffected if we replace any of the roots z_i by its complex conjugate. We can make these changes in 2^L possible ways, as long as we do not bother about unitarity, except for the highest partial wave where it fixes $\sin^2\delta_l$. With brevity we may call these as **Chrichton ambiguity.**

Bowcock and Burkardt [6] argue that for pure elastic scattering, whatever be the value of L, there will be, at most, two possible unitary amplitudes. They start by assuming that there are N unitary amplitudes, out of the possible 2^L, which means that there is N number of F_l, where $F_l = (S_l - 1)/2i$ We denote them by F_l^k, where $k = 1, 2, 3, \ldots, N$

For pure elastic scattering, since, $F_l = e^{i\delta_l}\sin\delta_l$ and $|F_l|^2 = \sin^2\delta_l$ we can write,

$$\text{Im } F_l^k = |F_l^k|^2, \quad \text{where } k = 1, 2, \ldots, N \text{ and } l = 0, 1, \ldots, L-1 \tag{3.4}$$

The highest power of $\cos\theta$ in the amplitude in equation (3.3) is L and the overall coefficient of $\cos^L\theta$ is clearly proportional to F_L. Expanding $P_l(\cos\theta)$ in equation (3.1) in powers of $\cos\theta$ we find the coefficients of $\cos^l\theta$ and then using the relation $F_l = e^{i\delta_l}\sin\delta_l$ and equation (3.3), the scattering amplitude for purely elastic scattering can be expressed as

$$f(\cos\theta) = \frac{1}{k}(2L+1)e^{i\delta_L}\sin\delta_L \frac{(2L)!}{2^L (L!)^2} \prod_{i=1}^{L} (\cos\theta - z_i) \tag{3.5}$$

For a fixed L, it depends on $(2L+1)$ variables -- δ_L, the real parts of the L zeros, and the modulo of their imaginary parts -- each amplitude being a different function of those variables because of the sign changes from the complex conjugation Now for each of the solutions eqnation (3.4) provide a total of NL constraints. If these equations are independent and there is to be a solution, there must not be more equations than variables, so $NL \le 2L+1 \Rightarrow N \le 2$, since N is an integer So, when we are dealing with the scattering of spinless particles at an energy, where no other processes than elastic scattering is allowed, there is a two-fold ambiguity. This is quite apart from and in addition to the so-called trivial ambiguity, which just involves reversing the sign of all phase shifts

In the complex potential scattering, that is, for complex phase shifts, the complex conjugation of the zeros do not end to a two-fold ambiguity as in the case of real potential scattering. The expression in equation (3.3) implies that

$$f(\theta) = f(0)\prod_{i=1}^{L}\frac{(\cos\theta - z_i)}{(1-z_i)} \tag{3.6}$$

For real $\cos\theta$, the replacement of one or more z_i by their complex conjugates will result in the same differential cross-section, $d\sigma/d\Omega = |f(\theta)|^2$ and the same $f(0)$. For fixed L these are the only ambiguities in determining the phase shifts (which are of course, related to z_i) from $d\sigma/d\Omega$ The number of these ambiguities including the unchanged function in equation (3.6) is 2^L. So, we arrive at the result that all scattering amplitudes of the same form as in equation (3.6), which correspond to the same $d\sigma/d\Omega$ and the same $f(0)$ can be obtained from the 2^L functions

$$f_m(\theta) = f(0)\prod_{i=1}^{L}\frac{\cos\theta - z_i(m)}{1-z_i(m)}, \quad m = 0,1,2,\ldots,2^L-1 \tag{3.7}$$

where m is an index labeling the 2^L different possibilities, $z_i(m)$ standing for z_i or its complex conjugate z_i^*. Again $d\sigma/d\Omega = |f(\theta)|^2$ and the optical theorem

$$Im\ f(0) = \frac{k}{4\pi}\sigma_{tot} \tag{3.8}$$

permits us to calculate $\mathrm{Re}\, f(0)$ apart from sign. Also, we observe that if we change the sign of $\mathrm{Re}\, f(0)$ in equation (3.6), we still have the same value of the differential cross-section $|f(\theta)|^2$ and the total cross-section σ_{tot}. Here we mark that all the possibilities of obtaining $f_m(\theta)$ according to equation (3.7) with reversed sign of $\mathrm{Re}\, f(0)$ are contained in the functions obtained by equation (3.7), with the additional transformation.

$$f_m(\theta) \rightarrow -f_m^*(\theta)$$

Such a transformation results in changing the sign of all the real parts of the phase shifts, which is the same as in the case of trivial ambiguities. Therefore, the 2^L possibilities of constructing $f(\theta)$ arising from the complex conjugation of the zeros must be multiplied by 2 again, allowing for the two possible signs of $\mathrm{Re}\, f(0)$. There are no additional ambiguities other than those found here and their total number is 2^{L+1}. We now go on to find the zeros of the scattering amplitude for a particular potential and study the significance of the zeros in phase shift analysis.

3.2 Zeros of the scattering amplitude for a Yukawa Potential

We choose the Yukawa potential as given in equation (2.15) as our model of potential scattering and find the zeros. The scattering amplitude as an expansion in partial waves derived in equation (1.48) may be rewritten as

$$f(k,\theta) = \sum_{l=0}^{\infty} \frac{(2l+1)P_l(\cos\theta)}{k(\cot\delta_l - i)} \tag{3.9}$$

Those partial waves will be significant which satisfy the condition $l \ll ka$ so that we may truncate the infinite series in the above equation at $l = L$, L being finite and write

$$f(k,\theta) = \sum_{l=0}^{L} \frac{(2l+1)P_l(\cos\theta)}{k(\cot\delta_l - i)} \tag{3.10}$$

Since $z=\cos\theta$ permits analytic continuation into the complex $z-$plane, we can find solution for equation (3.10) such that

$$f(k,z)=\sum_{l=0}^{L}\alpha_l(k)P_l(z)=0 \tag{3.11}$$

where

$$\alpha_l(k)=\frac{2l+1}{k(\cot\delta_l-i)} \tag{3.12}$$

The Rodrigue's formula for Legendre polynomial is given by

$$P_n(z)=\frac{1}{2^n n!}\left(\frac{d}{dz}\right)^n(z^2-1)^n \tag{3.13}$$

Also expanding $(z^2-1)^n$ we obtain

$$\left(\frac{d}{dz}\right)^n(z^2-1)^n=\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}(-1)^{n-r}\prod_{k=0}^{n-1}(2r-k)z^{2r-n} \tag{3.14}$$

So $P_n(z)$ now takes the form

$$P_n(z)=\frac{1}{2^n}\sum_{r=0}^{n}\frac{(-1)^{n-r}}{r!(n-r)!}\prod_{k=0}^{n-1}(2r-k)z^{2r-n} \tag{3.15}$$

We note that

$$\prod_{k=0}^{n-1}(2r-k)=0 \quad \text{for } 2r\le n-1 \tag{3.16}$$

Accordingly, this leads to the new expression

$$f(k,z)=\sum_{l=0}^{N}\alpha_l(k)\delta_l^{\,n}\frac{1}{2^n}\sum_{r=n/2,(n+1)/2}^{n}\beta(r,n)z^{2r-n} \tag{3.17}$$

where the lower summation takes the values

$$\begin{aligned} r&=n/2 \qquad \text{for even } n,\\ r&=(n+1)/2 \qquad \text{for odd } n \end{aligned} \tag{3.18}$$

and

$$\beta(r,n)=\frac{(-1)^{n-r}}{r!(n-r)!}\prod_{k=0}^{n-1}(2r-k) \tag{3.19}$$

Thus we can finally write down the coefficients of z^m in equation (3 9) both for even m and odd m in the following forms:

$$\sum_{k=m/2}^{N/2} \alpha_{2k} \frac{1}{2^{2k}} \beta\left(k+m/2, 2k\right) \qquad m \text{ even,} \tag{3.20}$$

$$\sum_{k=(m+1)/2}^{N/2} \alpha_{2k-1} \frac{1}{2^{2k-1}} \beta\left(k+(m+1)/2, 2k-1\right) \qquad m \text{ odd.} \tag{3.21}$$

It should be noted that the above expressions are calculated for even N which is the total number of zeros of the complex polynomial.

We have already computed the phase shifts $\delta_l(k)$ for the Yukawa potential (2 15) by using partial wave method and compared with those obtained from the Born approximation method. In order to calculate numerical zeros of $f(k, z)$ in equation (3.11) we write a computer program which calculates the arrays for $\alpha_i(k)$ and $\beta(r,n)$. We then compute the numerical values of the coefficients of z^m given in equations (3 20) and (3.21) parameterized by the array k A NAG-routine is then called which returns the complex zeros z_i for which the scattering amplitude vanishes identically and at least twenty zero trajectories are calculated. The sorting of the zeros is then done in such a way that for each value of k, the program selects six smallest absolute values out of twenty k parameterized zeros. Then it calculates all the different ways of selecting four numbers out of six. By minimizing the sums of the absolute values of the differences between the k-th and the $k+1$- th points for each permutations possible, the first four zero trajectories closest to the origin are selected on the premises of continuity. Following Hohler [9] we work with $-q^2$ which is equivalent to the Mandelstam [12] variable 't' in the $s-$channel

$$-q^2 = t = -2k^2(1-z) \tag{3.22}$$

For the range of the momentum $1.0 \le k \le 5.0\, fm^{-1}$ the first four zero trajectories as a function of k are shown in figures 1-4 for different potential strengths U_0=1.50, 5.0 and 15.0 fm^{-1}.

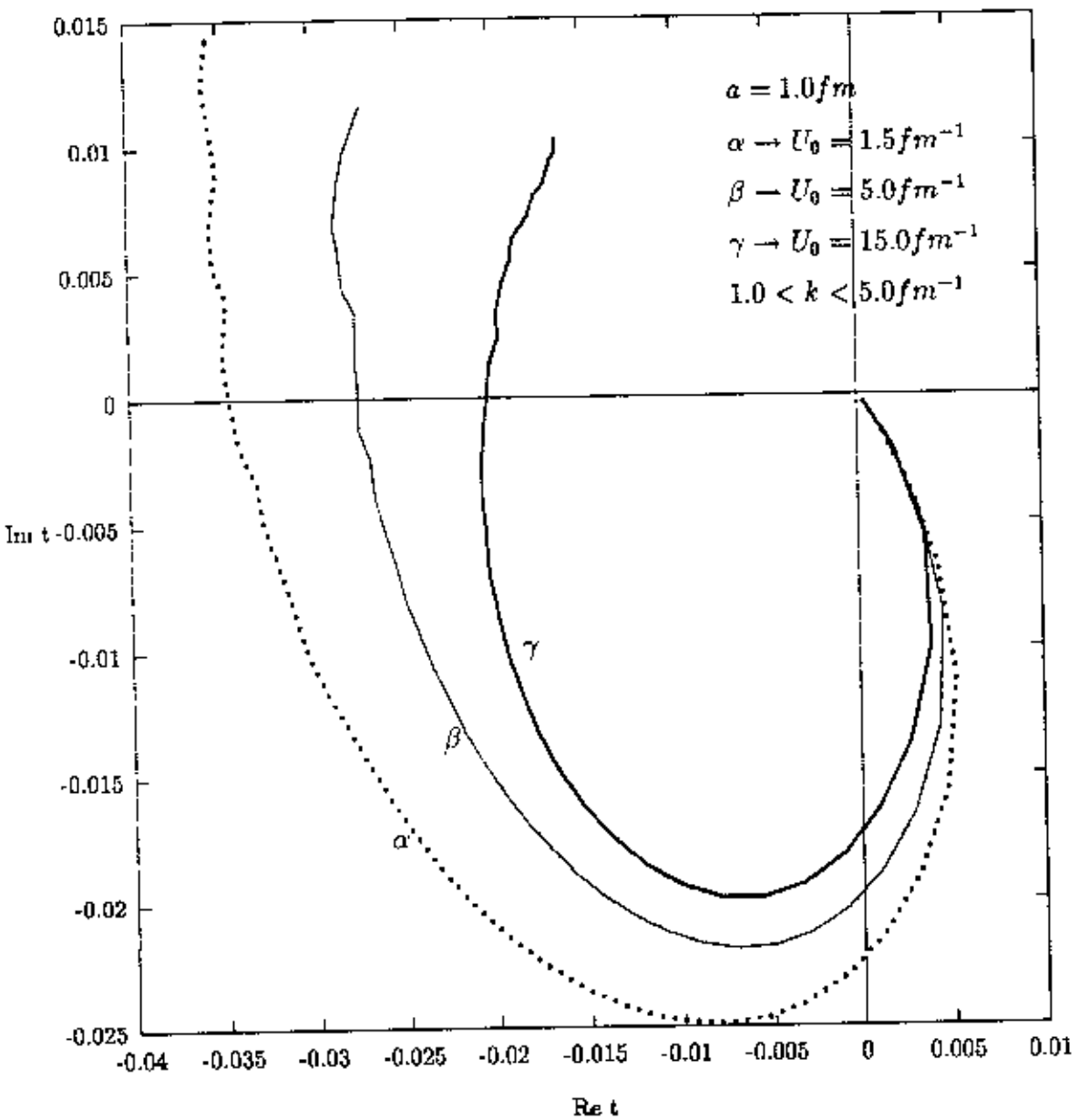

Figure 1: First zero trajectories for the Yukawa potential with $U_0 = 1.50, 5.0$ and $15.0fm^{-1}$

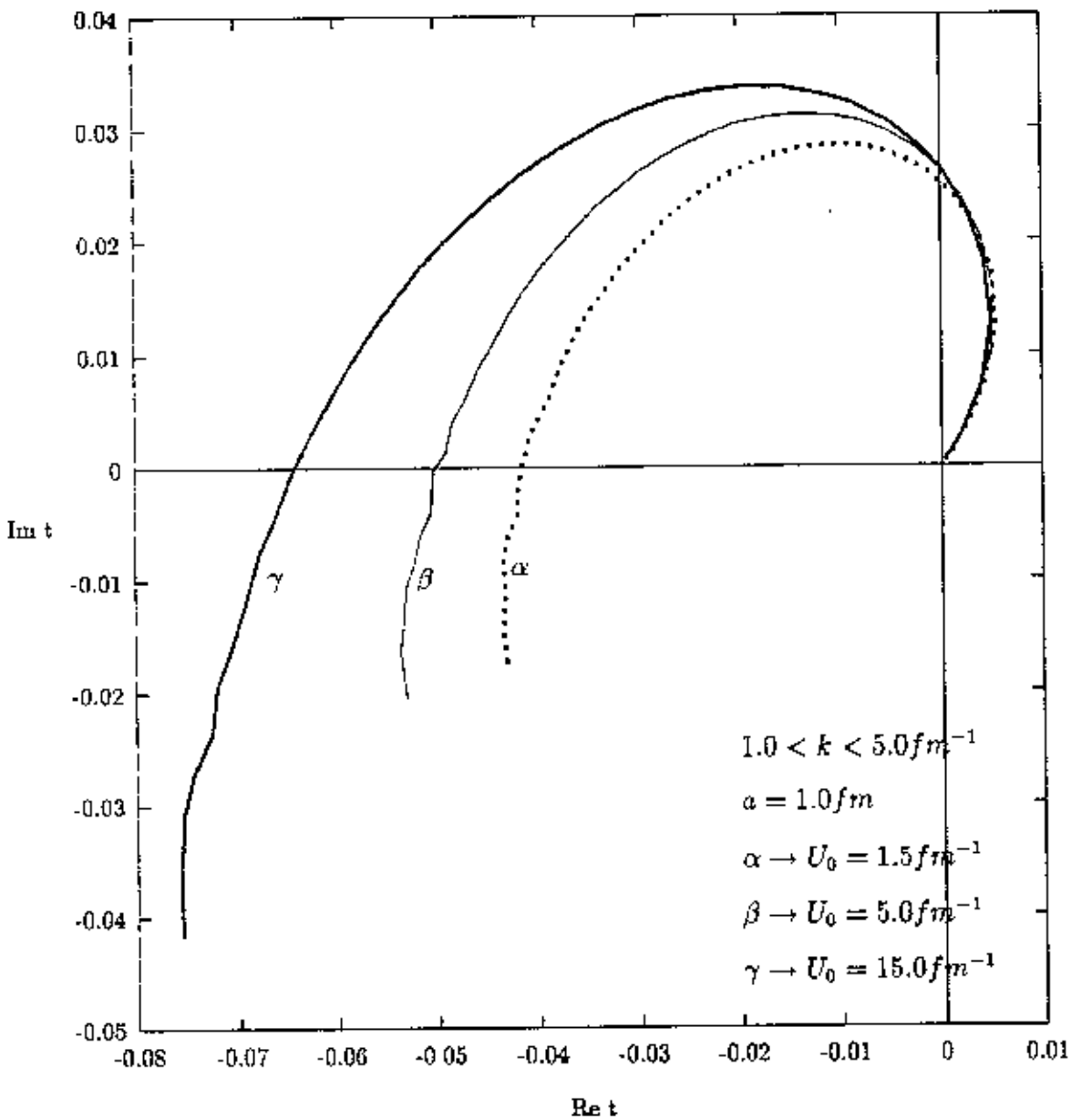

Figure 2: Second zero trajectories for the Yukawa potential with $U_0 = 1.50, 5.0$ and $15.0 fm^{-1}$

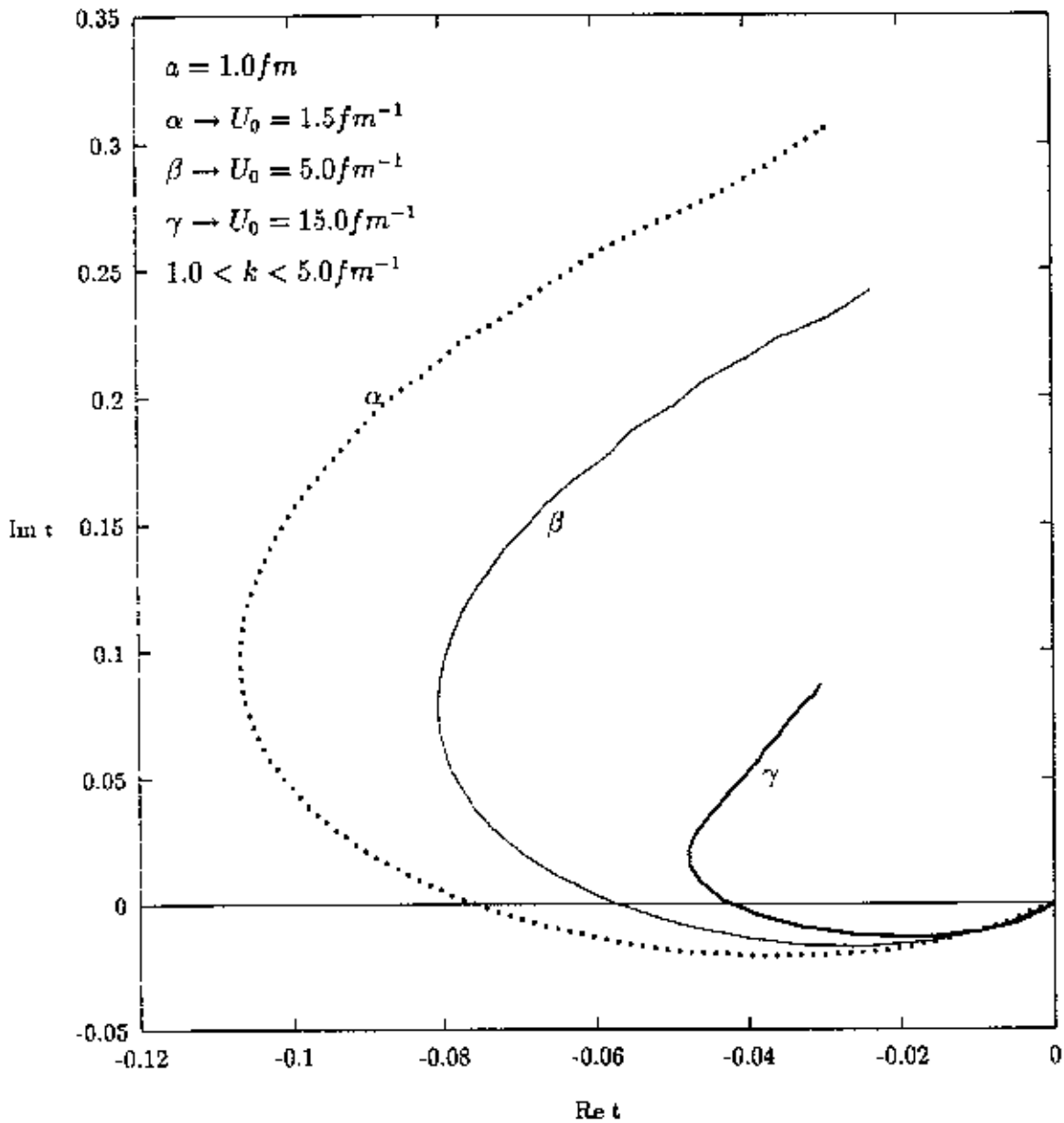

Figure 3: Third zero trajectories for the Yukawa potential with $U_0 = 1.50, 5.0$ and $15.0fm^{-1}$

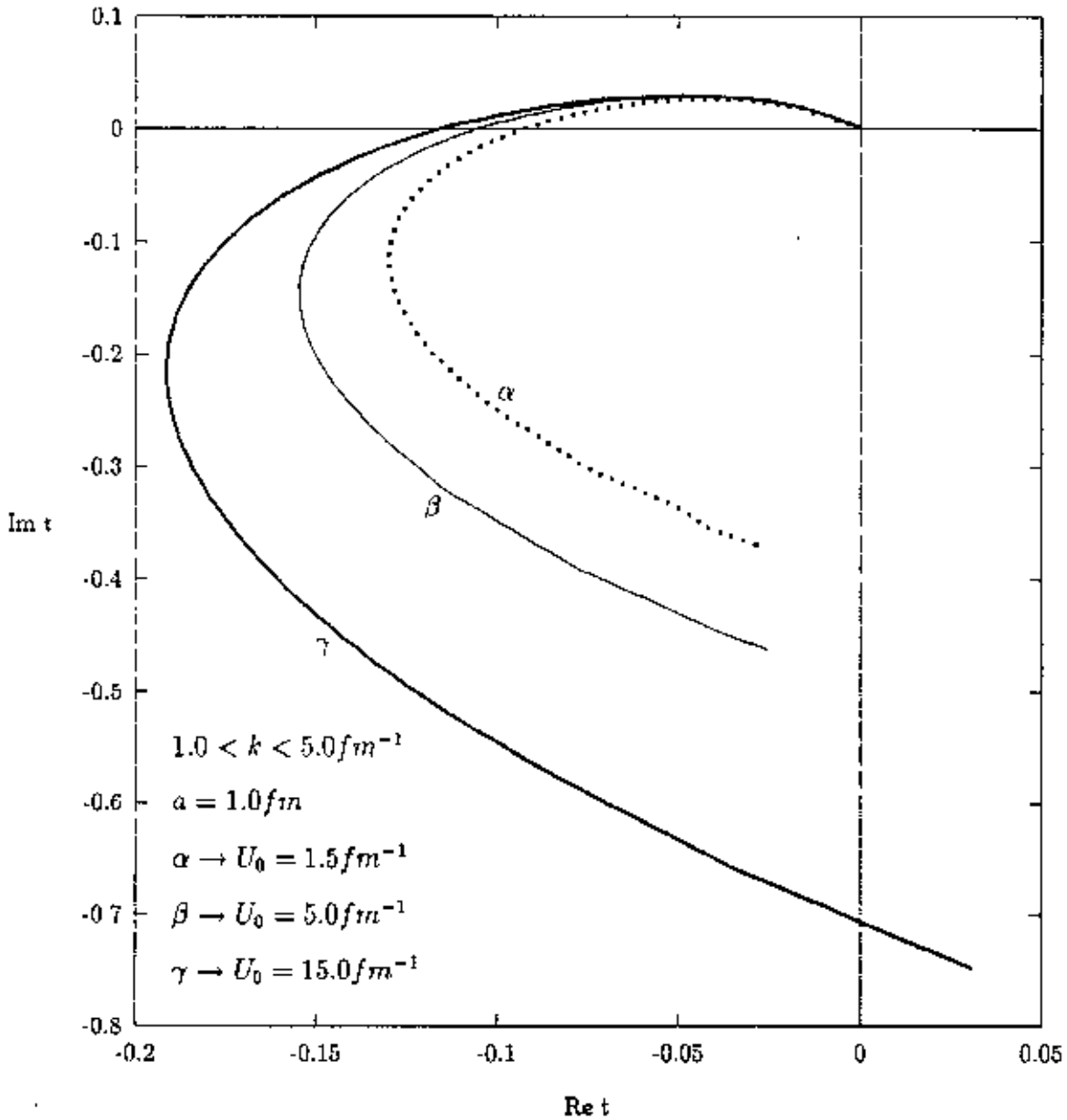

Figure 4: Fourth zero trajectories for the Yukawa potential with $U_0 = 1.50, 5.0$ and $15.0 fm^{-1}$

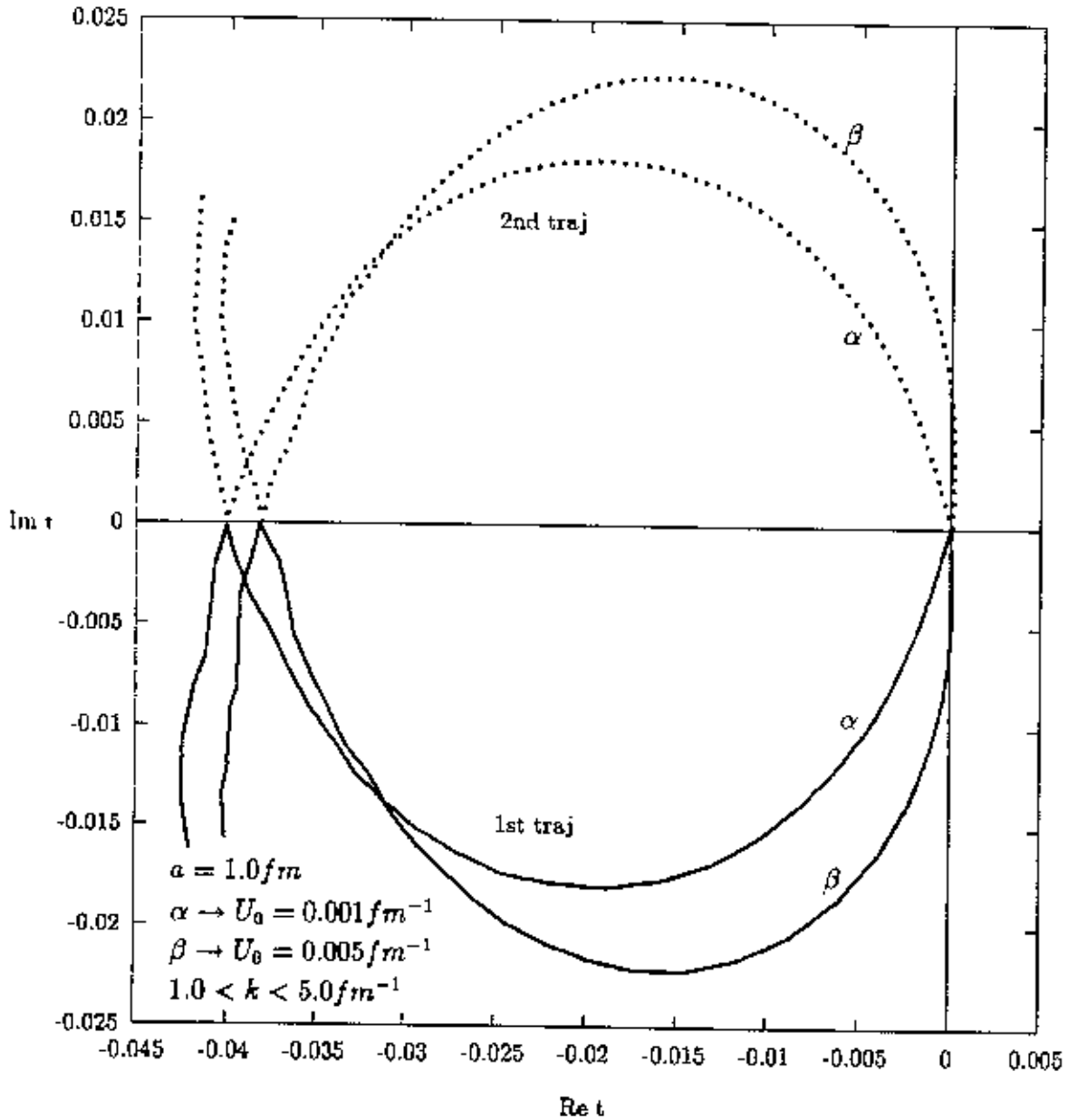

Figure 5: 1st and 2nd zero trajectories for Yukawa potential with $U_0 = 0.001$ and $0.005 fm^{-1}$

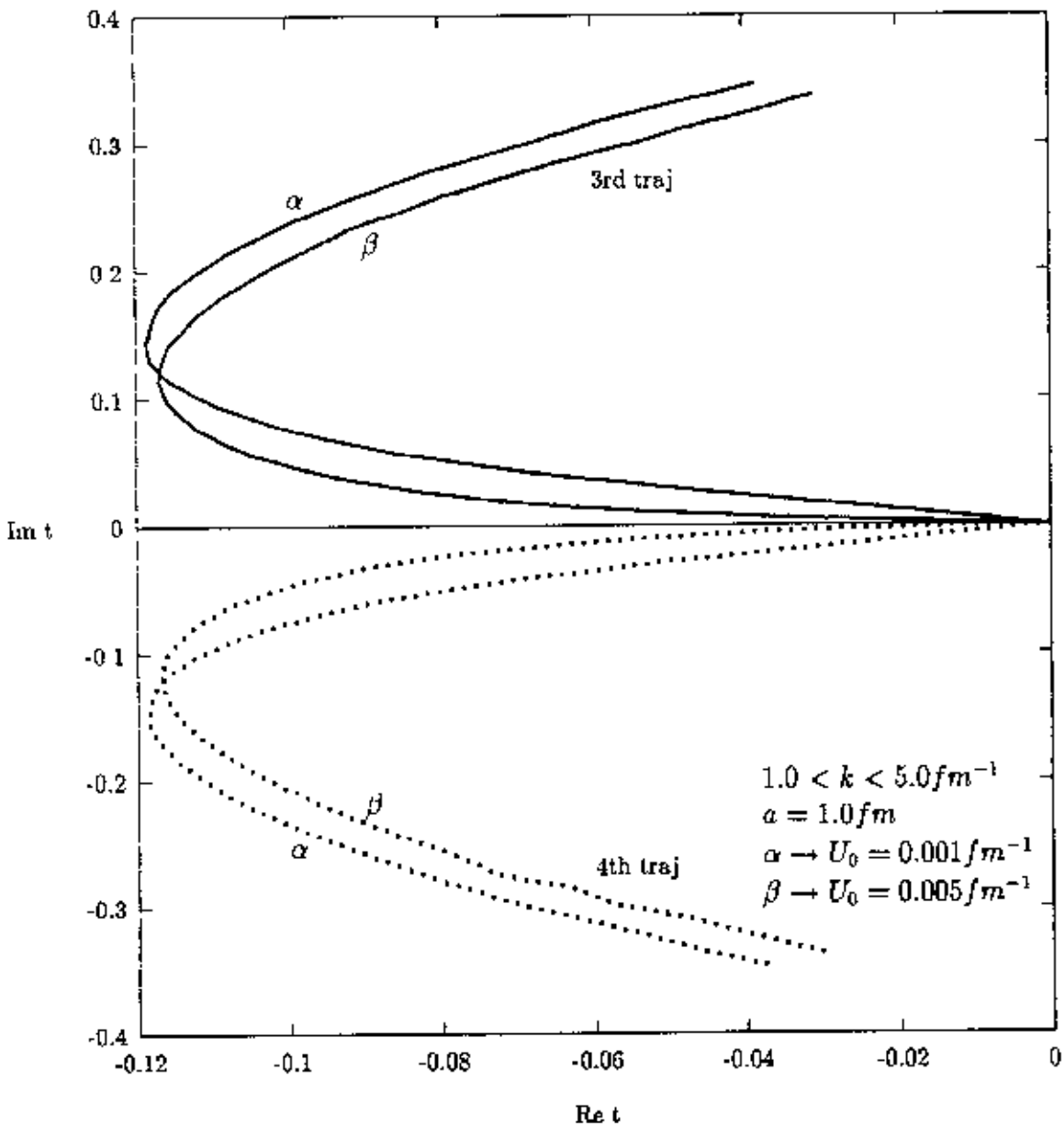

Figure 6: 3rd and 4th zero trajectories for Yukawa potential with $U_0 = 0.001$ and $0.005 fm^{-1}$

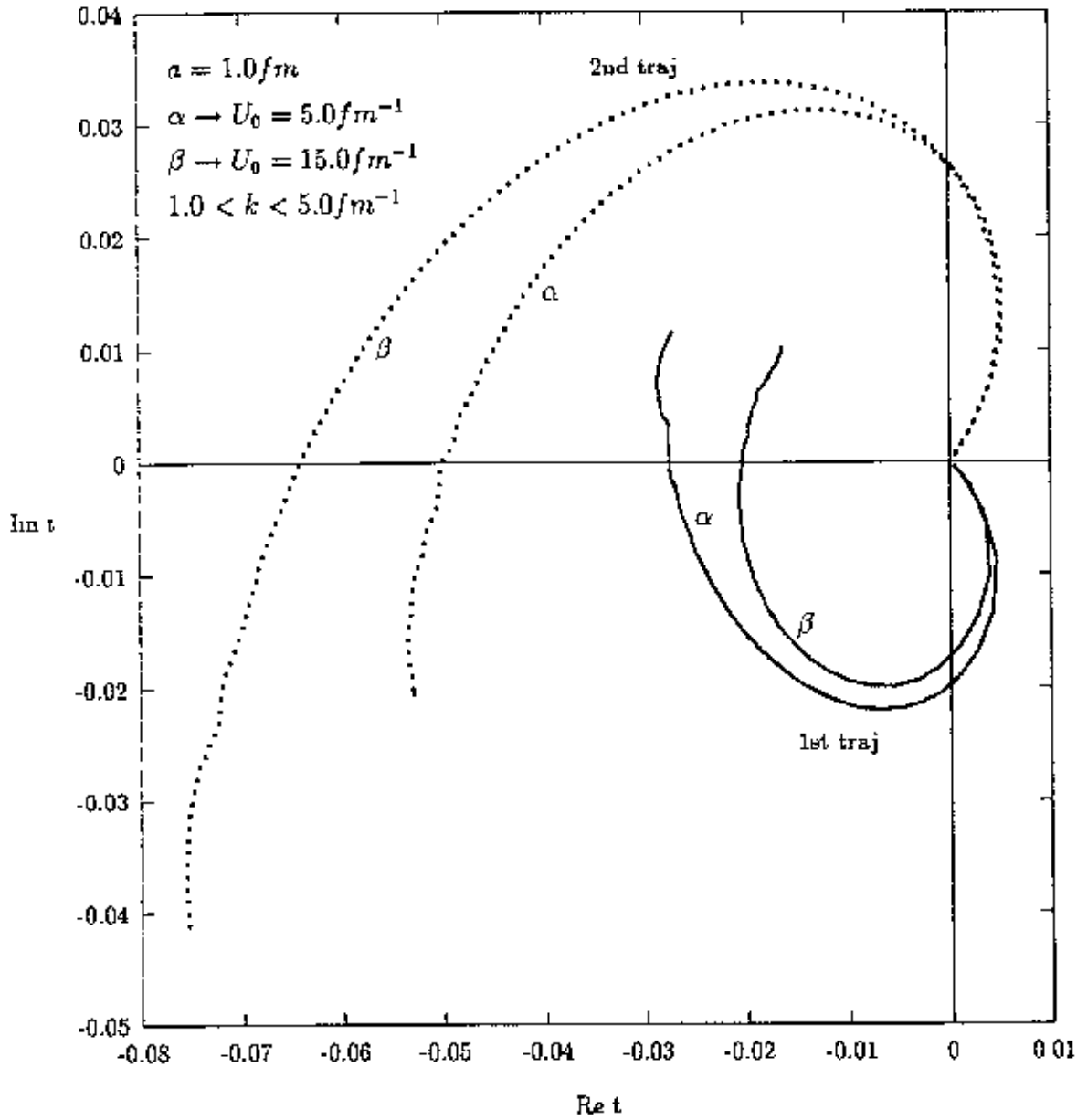

Figure 7: 1st and 2nd zero trajectories for Yukawa potential with $U_0 = 5.0$ and $15.0fm^{-1}$

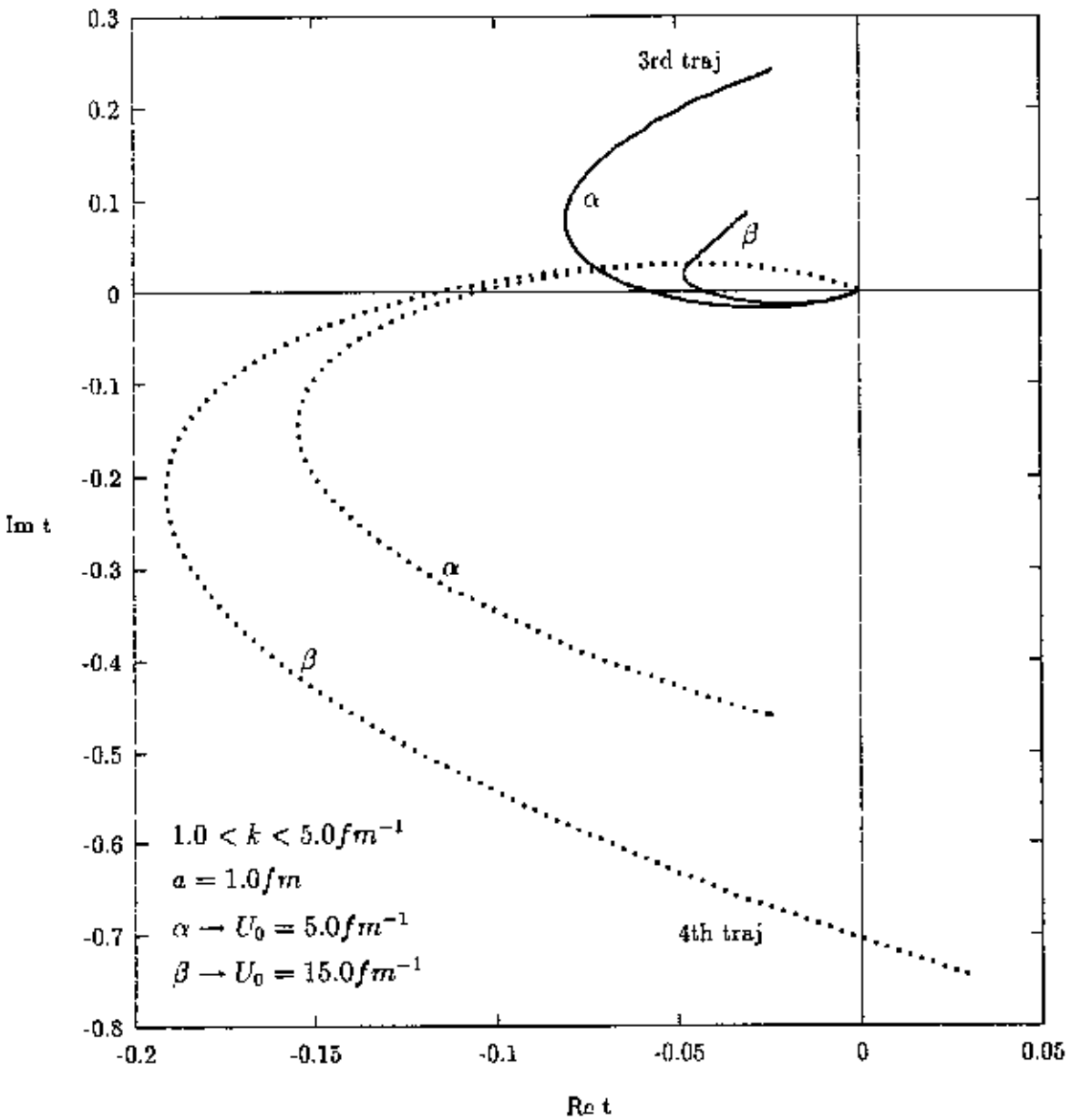

Figure 8: 3rd and 4th zero trajectories for Yukawa potential with $U_0 = 5.0$ and $15.0 fm^{-1}$

Numbering is done according to the proximity to the physical region $|z| \leq 1$, $z = \cos\theta$. The figures show that all the four zero trajectories converge towards the origin at low energies. Also we observe that as the potential strength increases the first and third zero trajectories become more squeezed while the second and fourth trajectories become more extended In figure-5 we observe that for weak potential strengths $U_0 = 0.001$ and $0.005\,fm^{-1}$ each of the first and second zero trajectories appear in a way, which indicates that the zeros in one trajectory are just the complex conjugates of the zeros in the other one The third and fourth zero trajectories also behave like the first two trajectories for the same weak potential strengths, which are shown in figure-6 For higher potential strengths $U_0 = 5.0$ and $15.0\,fm^{-1}$ the first and second zero trajectories are shown in figure- 7 while the third and fourth trajectories are shown in figure-8 The figures show that for higher potential strengths the zeros in either of the first or second trajectories are no more the complex conjugates of the other and same is the case with the third and fourth zero trajectories.

We now apply the zeros of the scattering amplitude to construct a method which can be used for resolving the probable ambiguities of phase shift analysis.

3.3 Application of the Zeros of Scattering Amplitude in phase shifts analysis

We now discuss a method which makes an extensive use and application of the zeros of scattering amplitude for resolving the ambiguities in phase shift analysis in the case of elastic scattering with no absorption from the Yukawa potential The phase shifts δ_l calculated for this potential from the partial wave method discussed in section-2 2 can be used to compute the scattering matrix elements $S_l = e^{2i\delta_l}$ Using these values of S_l in equation (3 1), the complex roots of $f(\theta)$ in the complex $\cos\theta$ -plane can be calculated for a fixed value of L, say, $L = 6$. Now the calculated values of S_l can also be used in equation (3.1), to obtain the numerical values of the scattering amplitudes $f(\theta)$ for at least fifty one values of $\theta = (\pi/50)r$ including $f(0)$ for $\theta = 0$ which fixes $\mathrm{Im}\, f(0)$ through the optical theorem given in equation (3 8) The differential cross-sections $d\sigma/d\Omega$ are then computed for all values of θ as stated above along with total cross-section σ_{tot} Then we approximate the numerical values of $d\sigma/d\Omega$ by a polynomial of degree, say, L=4 following the method of least squares Since $d\sigma/d\Omega > 0$, we must construct the polynomials, which cannot assume negative values in the physical domain The least square fit is to be done in a way such that $f(\theta)$ for $\theta = 0$ coincides with $f(0)$ calculated from the polynomial fit to the numerical values of $d\sigma/d\Omega$ at $\theta = 0$. We then find the complex zeros z_i $(i = 1,2,3,4)$ of the fitted polynomial and their complex conjugates z_i^* and then according to equation (3 7). the $2^4 = 16$ possible amplitudes $f_m(\theta)$ can be constructed which are all polynomials in cosθ of degree L. Now the amplitude $f_m(\theta)$ can also be expanded into Legendre polynomials in the following form.

$$f_m(\theta) = \frac{1}{2ik}\sum_{l=0}^{L}(2l+1)\left[S_l(m)-1\right]P_l(\cos\theta) \qquad (3\ 23)$$

where $m=0$ case is the initial scattering amplitude given by equation (3 1). The right hand side of equation (3 23) can be expressed in the form of polynomials in $\cos\theta$ of degree L

$$f_m(\theta)=\sum_{l=0}^{L} C_l(\cos\theta)^l \tag{3.24}$$

where the coefficients C_l contains $S_l(m)$ The above relation being an identity allows us to equate the coefficients of equal powers of $\cos\theta$ and hence to find the $S-$matrix elements $S_l(m)$. The possible combinations of the complex zeros z_i and their complex conjugates z_i^* may be ordered in the following way. We start with the zeros z_i $(i=1,2,3,4)$ and mark this combination with the number 1234 Then we take the complex conjugate of z_1 and combine with the remaining $z_i(i=2,3,4)$ and mark this combination with the number $\bar{1}234$ In this way, the complex conjugation of any one zero is denoted by a 'bar' on the respective number For example, $1\bar{2}\bar{3}\bar{4}$ represents the combination of z_1, z_2^*, z_3 and z_4^* In this way, all the possible $2^4=16$ matrix elements $|S_l(m)|$ for $l=0,1,2,...4$ may be obtained.

Further additional sets of the matrix elements $S_l(m)$ and hence the phase shifts $\delta_l(m)$ are obtained by changing the sign of the real parts of $f(0)$ in equation (3 7) For all sets of phase shifts obtained, the total cross-section remains unchanged, because $d\sigma/d\Omega=|f(\theta)|^2$ is unchanged We note that all the ambiguous phase shifts are not physically acceptable. Since for an absorptive scattering center the unitarity condition requires $|S_l|\le 1$ while for pure elastic scattering the unitarity condition requires $|S_l|=1$. by checking with the condition $|S_l|$ almost nearly equal to unity for $l=0,1,2,..4$ we can sift out the unacceptable sets and reconstruct the scattering amplitude and hence reduce the problems of phase shift analysis.

Conclusion

In chapter one, we have reviewed the necessary backgrounds of spin-independent scattering theory. We have discussed the partial wave analysis and the Born approximation theory in detail along with the formulation of the scattering amplitude and the differential cross section

In chapter two, we have chosen a Yukawa potential as the model potential and found the phase shifts and differential cross-sections both from the partial wave method and from the Born approximation method for different potential strengths We have seen that the results of the phase shifts obtained from the partial wave method agree most satisfactorily with those of the first Born results for weaker potential strengths and large momentum as expected The differential cross-sections obtained from the partial wave method also agree well with those obtained form the Born approximation method But the results also indicate that we need to consider the third order Born approximation also, for better agreement.

In chapter three, we first have a detailed discussion on phase shift analysis and also on the probable ambiguities of phase shift analysis where we have marked that the zeros of the scattering amplitude plays a significant role We then find the zero trajectories for the Yukawa potential. We approximately truncate the infinite series of the partial wave expansion of the scattering amplitude satisfying $l >> ka$ At least twenty zeros of the amplitude have been evaluated in the complex momentum transfer plane We then examine the behaviour of the first four zero trajectories closest to the physical region for a variety of potential strengths. We observe that for higher potential strengths all the four zero trajectories start moving from the origin and spread out as the momentum k increases from zero and they vary smoothly with the energy.

We also observe that for weak potential strengths the zeros in the first and second trajectories are just the complex conjugates of each other and the same holds for the third and fourth zero trajectories. Therefore we may conclude that the ambiguities due to the complex conjugation of the zeros are resolved at least to some extent for weak potential strengths. However, for higher potential strengths the above conclusion does not hold and we remain with all the possible ambiguities. Lastly, we have proposed a method for reducing these ambiguities which makes use of the zeros of scattering amplitude

Our next work would be to apply and follow the method as designed here for reconstructing the scattering amplitude and hence resolving the probable ambiguities in phase shift analysis

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