Study of Cost Allocation Methods and Their Applications

A dissertation submitted in partial fulfillment of
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in Mathematics

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Abstract

Problems arise in joint cost allocation in many practical cases when people decide to work together. Application of a suitable cost allocation method may solve the problem and save cost of every participant. In this study, a relevant cost allocation method is formulated, considering system loss and budgetary constraints of the participants.

The model has been simulated using randomly generated numbers in order to test for workability of the proposed formula Indeed, number of already established methods are in practice to accommodate similar scenario. The model developed by the author may be considered to be an enhancement of all the previous established ones.

Candidate's Declaration

It is hereby declared by the undersigned that the work presented in the thesis titled "Study of Cost Allocation Methods and Their Applications" submitted in partial fulfillment of the requirement for the degree of Philosophy in Mathematics, in the Department of Mathematics, Bangladesh University of Engineering and Technology, Dhaka is an authentic record of work of the undersigned.

The matter presented in this thesis has never been submitted by the undersigned as partial or complete fulfillment of requirement for awarding any other degree in this or any other university.

Anindita Paul

Date: October 19, 2004

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Chapter 1 Introduction

1.1 Background

In the U.S.A., a problem arose among the authorities of navigation, irrigation, and electricity developers as they attempted to work simultaneously in the river of Tennessee. Although the authorities decided to work together, they still failed to formulate proper cost distribution Ultimately in 1930, the engineers and economists of the "The Tennessee Valley Authority" [7, 10, 13] developed a method which was based on cost distribution among them depending on the weight of work of different authorities

Similar problem prevails in every society whenever people tend to work together. Mathematicians have shown interest in the subject considering practical implications of it. The author has found the issue to be a household one and therefore has taken an attempt to add to existing literature on the issue.

1.2 Statement of the Problem

Group of people, authorities, countries often work simultaneously keeping in mind their individual interests In most cases, interests of individuals clash with each other, and therefore the jobs that are supposed to be carried out in a synchronized manner clearly do not get together and result in failure. Prime conflict of interest actually arises out of problem related to cost allocation. Every individual participant can withstand a certain amount of system loss. Considering these factors, cost sharing among the individuals becomes bottleneck.

Mathematicians tasted the urge of solving similar problems and the theoretical approaches addressing the case dates back to the days of Tennessee River Valley conflict of 1930 referred earlier. Attempts have been made to present everyone with win-win situation, or at least minimum loss situation. Game Theory thus came into the picture and different strategies evolved. However, none of the models could really satisfy every individual, who is henceforth referred to as "player" or "participant" Every player usually sets his own priority and tries to hit the jackpot. Such individualistic ideologies never get along with the approach of "optimization" that are developed by mathematicians with a view to satisfy all or at least minimize dissatisfaction of everyone.

The author has gone through the details of the earlier works in line and has drawn priority on two constraints- individual budget and affordable system loss. Then the mathematical translation of the problem really stood at optimizing the scenario Game Theory played the key role in setting the tune of the problem and solution. Indeed, in order to testify the workability of the model, simulation has been performed.

1.3 Objective and Goals

It has already been mentioned that the study revolves around generation of new model that addresses joint cost allocation problem. However, fulfillment of the following specific objectives would take the study toward ultimate goal of formulation of a proposed cost allocation method.

(i) Detail study of existing cost allocation methods

(ii) Assumption and study of constraints / hindrances influencing selection and formulation of any cost allocation method

(iii) Development of a new cost allocation model with a view to minimize individual cost through optimizing constraints specifically identified as system loss and budget

(iv) Simulation of the developed model using random data.

1.4 Limitation of the Study

The aim of the study is to develop a mathematical model based on analytical approach. Simulation is attempted on the model to test its workability. It has been tried to apply the concept in real life situation as well. But, due to unavailability of such actual data, any practical application remains to be done.

2.1 *n*-person Game, Cost Game, and Their Properties

2.1.1 Different Types of Game

The concept of game theory is developed for solving the practical life problem. In real life there are conflicts between peoples, or groups of people, such as political parties, government, business, etc. When the social problems are expressed in mathematical problems, these are called game The participants in a game are called the players. Players may be a single person, or a group of people, or a country, etc. In a game, there must be at least two players. A game with two players is called the two-person game If the number of players in a game is more than two, then it is called n-person game. In n-person game, the players are labeled as 1, 2, 3, … …, n.

The Game Theory consists of ways of analyzing these types of game. See [2].

2.1.2 Properties of *n*-person Game

The coalition of an n-person game can be formed by one or more than one players. {2, 5}, {n}, {1, 2, 3}, {1, 2, 3, ……, n} are some coalition of an n-person game. The numbers of possible coalition in an n-person game are 2^n. The coalition that consists of all the players in a game is usually written as N and all other coalitions are written as S. Always any coalition S is a subset of the coalition N.

Example 2.1.1 (Oil Market Game)

(This problem will be referred to for clarification of every property of n-person game in the later parts)

Country 1 has oil which it can use to run its transport system at a profit of "a" per barrel. Country 2 wants to buy the oil to use in its manufacturing industry, where it gives a profit of "b" barrel, while Country 3 wants it for food manufacturing where the profit is "c" per barrel; $a < b \leq c$. See [13].

Example 2.1.1.a Find the number of players and state all the possible coalition of the Oil Market Game (Example 2.1.1).

There are 3 players in the game, namely, country 1, country 2 and country 3

All possible coalitions of the game are stated below

{ 1 },	{ 2 },	{ 3 },	{ 1, 2 },
{ 2, 3 },	{ 1, 3 },	{ 1, 2, 3 },	φ.

The characteristic function, $v(S)$, of an n-person game assigns to each subset S of the players the maximum values that the coalition S can guarantee itself by coordinating the strategies of its members, no matter what the other players do.

It is standard that the characteristic value of an empty coalition is 0; that is, $v(S) = 0$ when $S = \varphi$.

Example 2.1.1.b Find the characteristic function for the Oil Market Game (Example 2.1.1).

The Characteristic function for the game is stated below.

$v(\varphi) = 0$,	by definition.
$v(1) = a$,	because if 2 and 3 form a coalition against 1, they cannot force him to sell the oil so it is worth "a" to him.

$v(2) = v(3) = v(2,3) = 0$, because any coalition of buyers cannot make the seller sell them the oil.

$v(1,2) = b$, because 1 and 2 can use the oil at a profit of "b" per barrel (1 sells it to 2), and so 3 would have to pay at least "b" to get it.

$v(1,3) = v(1,2,3) = c$, since 1 and 3 can use 1's oil at a profit of c per barrel.

If X_S and Y_{N-S} are the strategies available to the players in S and $N - S$, respectively, then $v(S) = \max_{x \in X_S} \min_{y \in Y_{N-S}} \sum_{i \in S} e_i(x,y)$ where $e_i(x,y)$ is the payoff to player i if x and y are strategies played by the players.

A game is called super additive if for disjoint coalitions S and T,

$$v(S \cup T) \geq v(S) + v(T).$$

In some games, for all subsets S and T of N,

$$v(S \cup T) = v(S) + v(T), \text{ if } S \cap T = \varphi. \qquad 2.1.1$$

These games, where v is additive, *i.e.*, satisfies 2.1.1, are called inessential game and imply trivially that

$$v(N) = \sum_{i=1}^{n} v(\{i\})$$

The game that is not inessential is called essential.

There are two important and interrelated questions when analyzing n-person games. So far it has discussed what coalitions are likely to form, or rather given a measure to the strengths of the possible coalitions. The second question is. When a coalition does form, how does it share its reward between the individual members of the coalition? It is assumed that the important part of an n-person game is the pre-play negotiations, where coalitions form and the rewards from the game (which can be calculated) shared out. Obviously the distribution of the rewards affected the formation of the coalition because some players might offer a large reward to another player to join a particular favorable coalition. Each individual would tend to join the coalition that offered and could guarantee him most. The second question, which is mentioned previously. How are the rewards shared out between the individual players? The "reasonable" share outs of the rewards are called imputations.

An imputation in an n-person game with characteristic function v is a vector $x = (x_1, x_2, \ldots \ldots, x_n)$ satisfies:

(i) $\sum_{i=1}^{n} x_i = v(N)$,

(ii) $x_i \geq v(i)$ for $i = 1, 2, 3, \ldots \ldots, n$.

where x_i is the i-th player's reward. Here $v(N)$ is the most the players can get out the game when they all work together.

The set of all imputations in a game is usually denoted by $E(v)$. In an inessential game, there is only one imputation, but for essential games there are lots.

Example 2.1.1.c Find the set of imputations in the Oil Market Game (Example 2.1.1).

In the Oil Market Game the set of all imputations is stated below.

$$E(v) = \{(x_1, x_2, x_3) : x_1 \geq a, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 = c\} \qquad 2.1.2$$

If x and y be two imputations, then x dominates y over S (denoted by $x \succ y$), if

(i) $x_i > y_i$, for all $i \in S$,

(ii) $\sum_{i \in S} x_i \leq v(S)$.

x is said to dominate y $(x \succ y)$ if x dominates y for some coalition S.

The core of a game v, denoted by $C(v)$, is the set of imputations, which are not dominated for any coalition.

Theorem 2.1.1

The core of a game v is the set of imputations, which are not dominated for any coalition. That is, x is in the core if and only if

(i) $$\sum_{i=1}^{n} x_i = v(N), \qquad 2.1.3$$

(ii) $$\sum_{i \in S} x_i \leq v(S), \text{ for all } S \subset N. \qquad 2.1.4$$

Lemma 2.1.1

If v is the characteristic function of an essential constant-sum game, then $C(v) = \varphi$.

Example 2.1.1.d Find the core for the Oil Market Game (Example 2.1.1).

If $x = (x_1, x_2, x_3)$ is in the core then 2.1.3 requires $x_1 + x_2 + x_3 = c$, whereas 2 1 4 gives the following inequalities:

$x_1 \geq a$ $\quad (S = \{1\})$;

$x_2 \geq 0$ $\quad (S = \{2\})$;

$x_3 \geq 0$ $\quad (S = \{3\})$;

$x_1 + x_2 \geq b$ $\quad (S = \{1, 2\})$;

$x_2 + x_3 \geq 0$ $\quad (S = \{2, 3\})$,

$x_1 + x_3 \geq c$ $\quad (S = \{1, 3\})$.

Now, $x_1 + x_3 \geq c$, $x_2 \geq 0$ and $x_1 + x_2 + x_3 = c$ imply $x_2 = 0$, $x_1 + x_3 = c$ and substituting this into $x_1 + x_2 \geq b$ gives $x_1 \geq b$ Thus, core for the game is

$$C(v) = \{(x, 0, c - x) \cdot b \leq x \leq c\}$$

One of the more recent ideas put forward as a solution to an n-person game is that of the nucleolus introduced by Schmeidler [11]. It has two very useful properties.

(a) every game has one and only one nucleolus, and

(b) if the core exists, the nucleolus is part of it.

Nucleolus is based on the idea of making the most unhappy coalition under it happier than the most unhappy coalition under any other imputation. For any imputation x and any coalition S, let $x(S) = \sum_{i=S} x_i$. Then each coalition looks at $v(S) - x(S)$ and the larger this number the more unhappy the coalition is with that imputation It is the difference between what they could get by themselves and what they actually get. Define $\theta(x)$ to be the 2^n values $v(S) - x(S)$ for all coalitions S (including N and φ) written in decreasing numerical order That is, if two imputations x and y are compared by looking at the coalition which is the unhappiest under each and calculating $v(S) - x(S)$, $v(S') - y(S')$ for these two coalitions, the smallest value is the better imputation. If these numbers are the same, look at the pair of second most unhappy coalitions and compare these and so on. So, if $\theta(x) = \left(\theta(x)_1, \theta(x)_2, \ldots \ldots, \theta(x)_{2^n}\right)$ and $\theta(y) = \left(\theta(y)_1, \theta(y)_2, \ldots \ldots, \theta(y)_{2^n}\right)$, then $\theta(x)$ and $\theta(y)$ are called ordering lexicographically, denoted by $\theta(x) < \theta(y)$,

if $\theta(x)_1 < \theta(y)_1$,

or if $\theta(x)_k = \theta(y)_k$ for $k = 1, 2, 3, \ldots \ldots, i-1$ and $\theta(x)_i < \theta(y)_i$.

The nucleolus, $N(v)$, is the smallest imputation under the ordering defined by $N(v) = \{x \in E(v) | \theta(x) < \theta(y) \text{ for all } y \in E(v)\}$.

Example 2.1.1.e Find the nucleolus for the Oil Market Game (Example 2.1.1).

The imputation is stated in 2.1.2. A typical imputation for the game is

$$(a + x_1, x_2, c - a - x_1 - x_2);\ x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq c - a \qquad 2.1.5$$

For 2.1.5, calculating $v(S) - x(S)$ for all coalitions S:

$$\left.\begin{aligned}
&v(1) - x(1) = a - (a + x_1) = -x_1, \\
&v(2) - x(2) = 0 - x_2 = -x_2, \\
&v(3) - x(3) = 0 - (c - a - x_1 - x_2) = x_1 + x_2 - c + a; \\
&v(1,2) - x(1,2) = b - (a + x_1 + x_2) = b - a - x_1 - x_2; \\
&v(2,3) - x(2,3) = 0 - (c - a - x_1) = x_1 - c + a; \\
&v(1,3) - x(1,3) = c - (c - x_2) = x_2; \\
&v(1,2,3) - x(1,2,3) = c - c = 0;
\end{aligned}\right] \qquad 2.1.6$$

$$v(\varphi) - x(\varphi) = 0$$

Since $x_1 \geq 0$, $x_2 \geq 0$ and $x_1 + x_2 \leq c - a$, the only entries in 2.1.6 which can be positive are x_2 and $b - a - x_1 - x_2$. Obviously, to make these as low as possible it must take $x_2 = 0$ and $x_1 > b - a$ If $x_2 = 0$, the entries in 2.1.6 become $-x_1$, 0, $x_1 - c + a$, $b - a - x_1$, $x_1 - c + a$, 0, 0, 0. In the values that vary with x_1, $b - a - x_1 > -x_1$ so that the largest element is either $b - a - x_1$ or $x_1 - c + a$. As functions of x_1 these look like the figure 2.1.1.

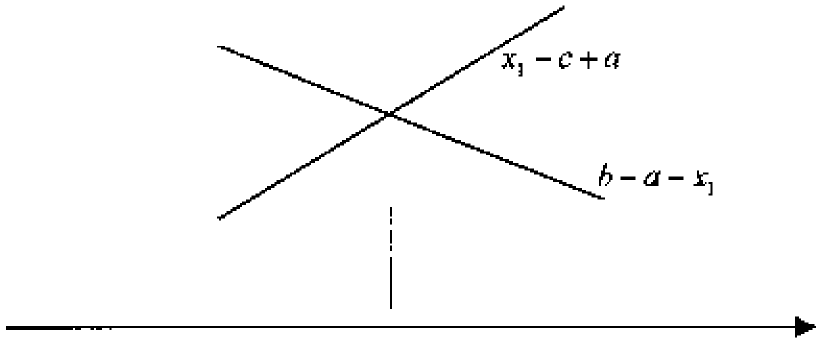


Figure 2.1.1- Maximizing the minimum of two lines

To make the largest of these as small as possible, x_1 is chosen so that the two functions are the small value, i. e., $b-a-x_1 = x_1 - c + a$ This gives $x_1 = \frac{c+b}{2} - a$, and substituting this and $x_2 = 0$ into 2.1.5 gives $\left(\frac{c+b}{2}, 0, \frac{c-b}{2}\right)$. This is the nucleolus and $\theta(x)$ for this imputation is $\left(0,0,0,0,\frac{b-c}{2},\frac{b-c}{2},\frac{b-c}{2},a-\frac{c+b}{2}\right)$

Shapley [11], looked at what each player could reasonably expect to get before the game has begun. He put forward three axioms, which he felt $\theta_i(v)$, player i's expectation in a game with characteristic function v, should satisfy.

Axiom 2.1.a: $\theta_i(v)$ is independent of the labeling of the players. If π is a permutatiou of 1, 2, 3, … …, n and πv is the characteristic function of the game, with the players numbers permuted by π, then $\varphi_{\pi(i)}(\pi) = \varphi_i(v)$.

Axiom 2 1 b: The sum of the expectations should equal the maximum available from the game, so $\sum_{i=1}^{n} \varphi_i(v) = v(N)$.

Axiom 2.1.c: If u and v are characteristic functions of two games, $u+v$ is the characteristic function of the game of playing both the games together φ should satisfy $\varphi_i(u+v)=\varphi_i(u)+\varphi_i(v)$.

Given these assumptions, Shapley proved the following theorem.

Theorem 2.1.2

There is only one function, which satisfies all the three axioms (axiom 2 1.2.a to 2 1.2 c) namely:

$$\varphi_i(v)=\sum_{S:i\in S}\frac{(\#S-1)!(n-\#S)!}{n!}\left(v(S)-v(S-\{i\})\right) \tag{2.1.7}$$

where the summation is over all coalitions S which contain player i and $\#S$ is the number of the players in coalition S. $\varphi_i(v)$ is called the Shapley value.

Example 2.1.1.f Find the Shapley values for the Oil Market Game (Example 2.1.1).

To find $\varphi_1(v)$ sum over the coalitions S = {1}, {1, 2}, {1, 3} and {1, 2, 3}, and substituting into 2.1.7 gives:

$$\varphi_1(v) = \frac{0!2!}{3!}(a-0)+\frac{1!1!}{3!}(b-0)+\frac{1!1!}{3!}(c-0)+\frac{2!0!}{3!}(c-0)$$

$$=\frac{c}{2}+\frac{a}{3}+\frac{b}{6}$$

To find $\varphi_2(v)$ sum over the coalitions S = {2}, {1, 2}, {2, 3} and {1, 2, 3}, and substituting into 2.1.7 gives:

$$\varphi_2(v) = \frac{0!2!}{3!}(0-0)+\frac{1!1!}{3!}(b-a)+\frac{1!1!}{3!}(0-0)+\frac{2!0!}{3!}(c-c)$$

$$= \frac{b}{6}-\frac{a}{6}$$

To find $\varphi_3(v)$ sum over the coalitions S = {3}, {1, 3}, {2, 3} and {1, 2, 3}, and substituting into 2.1.7 gives:

$$\varphi_3(v) = \frac{0!2!}{3!}(0-0)+\frac{1!1!}{3!}(c-a)+\frac{1!1!}{3!}(0-0)+\frac{2!0!}{3!}(c-b)$$

$$= \frac{c}{2}-\frac{a}{6}-\frac{b}{3}$$

2.1.3 Cost Game and It's Properties

When the participants or players work together then cost allocation is important and a cost allocation problem is called a cost game. In a cost game, characteristic function is called cost function. So, the cost function $C(S)$ assigns to any coalition S the minimal cost which should be involved if the individuals in S should work together in order to serve their own purposes Obviously in a cost game the cost function is always subadditive; that is, $C(S_1 \cup S_2) \le C(S_1) + C(S_2)$ where $S_1 \subset N$, $S_2 \subset N$ and $S_1 \cap S_2 = \varphi$.

A cost allocation method on A_n is a map $M : A_n \to R^n$. For each cost function $C \in A_n$, the cost vector is $V(C) = (V_1(C), V_2(C), \ldots \ldots, V_n(C))$, where $V_i(C)$ represents the i-th player's charge in this method. For a cost allocation method the desirable properties are

(i) efficiency,

(ii) individually rational,

(iii) stable,

(iv) dummy player property,

(v) anonymity property, and

(vi) aggregate monotonic. See [1].

Though all these properties are desirable, but there is no any cost allocation method that satisfies all these stated properties

A map $M . A_n \to R^n$ is called efficient cost allocation method if for each $C \in A_n$, $\sum_{i \in N} V_i(C) \leq C(N)$.

A map $M : A_n \to R^n$ is called individually rational cost allocation method if for each $C \in A_n$, $V_i(C) \leq C(\{i\})$ for all $i \in N$.

A vector y is in the core if and only if (i) $\sum_{i=1}^{n} y_i = C(N)$ and (ii) $\sum_{i \in S} y_i \leq C(S)$ for all $S \subset N$ and $S \neq \varphi$. A map $M : A_n \to R^n$ is called stable cost allocation method if $V(C) \neq \text{Core}(C)$ for each $C \in A_n$ with $\text{Core}(C) \neq \varphi$.

A map $M \cdot A_n \to R^n$ is called to possess a dummy player property if for all $C \in A_n$ and $i \in N$, $V_i(C) = C(\{i\})$, for which $C(S \cup \{i\}) - C(S) = C(\{i\})$ for all $S \subset N - \{i\}$

A map $M : A_n \to R^n$ is called to possess an anonymity property if for each $C \in A_n$ and each permutation $\theta : N \to N$ with $\theta_C \in A_n$, $V_{\theta(i)}(\theta_C) = V_i(C)$ for all $i \in N$, where θ_C is the cost function given by $(\theta_C)(\theta S) = C(S)$ for all $S \subset N$.

A map $M : A_n \to R^n$ is called an aggregate monotonic cost allocation method if for each $C1, C2 \in A_n$, with $C1(S) = C2(S)$ for all $S \neq N$ and $C1(N) \leq C2(N)$ then $V_i(C1) \leq V_i(C2)$ for all $i \in N$.

Many other properties are possessed by the different cost allocation methods. Note that for many cost allocation methods cost functions $C \in C_N$ its core is empty. Further for any cost allocation method on the set of cost functions with a non empty core, stability implies efficiency as well as individual rationality. See [4].

2.2 Existing Cost Allocation Methods

The cost allocation problem is comparable with the problem in social choice theory to find a suitable welfare function or social choice rule It is also comparable with the problem of choosing a suitable statistical test in statistics. In specific cost allocation situations, persons involved or arbitrators have to decide which method is most suitable for their purposes. They have to put priorities on the list of desirable properties for the cost allocation method. If one wants a method with too many properties, then impossibilities arise just as in social choice theory See [3, 15]

There are some established cost allocation methods, which are described below.

2.2.1 The Egalitarian Method

In the egalitarian method $E(C)$, See [15], the total cost $C(N)$ is equally distributed among all the participants. So, the weight of all the participants is equal, that is, 1. Therefore, this method is defined by

$$E(C) = \frac{C(N)}{n} \text{ for all } C(N) \in C_N .$$

This method is efficient, aggregate monotonic and possesses anonymity property

2.2.2 Shapley Cost Allocation Method

In an n-person game, for any cost function $C(N) \in \Gamma_N$, the i-th player's cost is $C(\{i\})$. The marginal cost of the i-th participant is defined by $m_i = C(N) - C(N - \{i\})$ for all $i \in N$. This means that the marginal cost of the i-th participant is the cost that increases the cost if player i joins the coalition $N - \{i\}$. Consider that the marginal cost is $M = \{m_1, m_2, \ldots \ldots, m_n\}$. This method is only for that cost functions where the sum of the marginal costs of the participants is less than that when all the participants (*i.e.* when number of participants is n) work together. The smallest marginal cost is defined by

$$m_i = \min_{S, i \in S} C_i(S) \text{ for all } i \in N.$$

And for all subadditive cost function C and for all $i \in N$,

$$m_i = \min_{S \, i \in S} C_i(S) \leq C(\{i\}) - C(\varphi) = C(\{i\})$$

Let the cost function $C \in A_n$ and each permutation $\theta : N \to N$ with $\theta_C \in A_n$. With respect to θ, the participants form the coalition N by entering one by one in the order $\theta 1, \theta 2, \ldots \ldots, \theta n$. So, the marginal cost $m_{\theta i}$ of the participant θi is

$$m_{\theta i} = C(\theta 1, \theta 2, \ldots \ldots, \theta i) - C(\theta 1, \theta 2, \ldots \ldots, \theta(i-1))$$

The Shapley cost allocation method φ, see [14], is defined by

$$\varphi = \frac{\sum_{\theta} M}{n}$$

where $M = (m_{\theta 1}, m_{\theta 2}, \ldots \ldots, m_{\theta n})$ and the sum is taken over all permutations θ on N.

There exists various axiomatic characterizations of the Shapley method. The most well-known characterization is due to Shapley [12] himself and is as follows:

A cost allocation method $M : A_n \to R^n$ is efficient, anonymous, additive and possesses the dummy player property if and only if $M = \varphi$.

Here additivity means that $M\left(C^1 + C^2\right) = M\left(C^1\right) + M\left(C^2\right)$ for all $C^1, C^2 \in A_n$.

Driessen [5] proved that for some special cases Shapley method possesses weak dummy player property and equal individuality property.

Young [17] showed that Shapley method is efficient, anonymous and strongly monotonic if and only if $M = \varphi$.

2.2.3 Cost Allocation Method depending on minimizing the maximum unhappiness

Pioneering in this field was Schmeidler [11] with his paper in which he introduced the game theoretical concept nucleolus. Later many other authors [15] modified the ideas of Schmeidler and introduced related concepts

For the cost function $C \in A_n$, the set of imputation $I(C)$ those are efficient and individually rational, $I(C)$ is defined by

$$I(C) = \left\{ y \in R^n; \sum_{i \in N} y_i = C(N) \text{ and } y_i \le C(\{i\}) \text{ for all } i \in N \right\},$$

which is given in [1].

Given a cost function $C \in A_n$ such that $I(C) \neq \varphi$, the key idea is to look for a function $u : S \times I(C) \to R^n$, where $S \subset N$ and where for a coalition S and for an imputation $y \in I(C)$ the unhappiness, of the coalition S with respect to the cost allocation y, is represented by a number $u(S, y)$. So, the main target is to minimize the maximum unhappiness. Thus any cost allocation $y_1 \in I(C)$ is better than the cost allocation $y_2 \in I(C)$ if

$$\max\{u(S, y_1)\} < \max\{u(S, y_2)\} \text{ for } S \subset N.$$

For any $u: S \times I(C) \rightarrow R^n$ and $y \in I(C)$, let $\theta(u, y)$ be the vector in R^{2^n} whose coordinates are the numbers u arranged in non-increasing order. For given C and u, cost allocation $y_1 \in I(C)$ is better than the cost allocation $y_2 \in I(C)$ if $\theta(u, y_1)$ is smaller than $\theta(u, y_2)$, *i.e.*, $\theta(u, y_1) \underset{L}{\leq} \theta(u, y_2)$. So, the u-nucleolus of the cost function is defined by the set

$$N_u(C) = \left\{ y_1 \in I(C) \middle| \theta(u, y_1) \underset{L}{\leq} \theta(u, y_2) \text{ for all } y_2 \in I(C) \right\}.$$

The nucleolus of the cost function C is obtained when

$$u(S, y) = \sum_{i \in S} y_i - C(S) \text{ for } S \neq \varphi$$

and

$$u(\varphi, y) = 0$$

Another term, normalized nucleolus, (introduced by Grotte), is the unhappiness function defined by

$$u(S, y) = \frac{\sum_{i \in S} y_i - C(S)}{|S|} \text{ for } S \neq \varphi$$

and

$$u(\varphi, y) = 0.$$

The disruption nucleoli, are studied by Gately [6], Littlechild and Vaidya [8], and Charmes, Rousseau and Seiford [4]. In Gately [6] and Littlechild and Vaidya [8], the unhappiness function for the cost games with nonempty strict core and is defined by

$$u(S, y) = \frac{\sum_{i \in N-S} y_i - C(N-S)}{\sum_{i \in S} y_i - C(S)} \text{ for } S \neq \varphi \text{ and } N.$$

In Charnes [4] the ratio of these two quantities is replaced by normalized difference Thus this idea gives the following definition:

$$u(S, y) = \frac{C(N-S) - \sum_{i \in N-S} y_i}{|N-S|} - \frac{C(S) - \sum_{i \in S} y_i}{|S|} \text{ for } S \neq \varphi \text{ and } N.$$

Schmeidler [11] proved that the nucleolus of a cost game consists of a single point and hence it gives rise to a cost allocation method on A_n, which turns out to be efficient, individually rational, stable, anonymous and to possess the dummy player property However, as Megiddo [9] showed, the nucleolus cost allocation method is not monotonic in aggregate.

2.2.4 Cost allocation methods based on separable and non-separable costs

For any cost function the total of the separable costs will be less than the cost when all the participants work together. The difference of these two cost give the remaining cost, which is allocated among the participants in some way. So, remaining cost is mathematically defined by

$$g(N) = C(N) - \sum_{i \in N} m_i \, (>0)$$

The weight $W = (w_1, w_2, \ldots \ldots, w_n)$ where all w_is are nonnegative, $\sum_{i=1}^{n} w_i > 0$ and depend on the cost function. The separable cost allocation method with weight vector W assigns to a cost function $C \in A_n$ the cost allocation

$$S(C, W) = M + \frac{g(N)W}{\sum_{i=1}^{n} w_i}$$

So, the separable cost is totally depending on the weight and in different separable cost allocation methods the weights are given below.

(i) Equal charge method (EC-method):

$W = (1, 1, \ldots \ \ldots, 1)$

(ii) Alternative cost avoided method (ACA-method).

$W = (C(\{1\}) - m_1, C(\{2\}) - m_2, \ldots \ \ldots, C(\{n\}) - m_n)$

(iii) Separable cost remaining benefits method (SCRB-method):

$W = \min\{C(\{i\}), b_i\} - m_i$ for all $i \in N$

Here, b_i is the benefit to player i only when his purposes are served

These three cost allocation methods are efficient, anonymous.

2.2.5 The cost gap allocation method

In 1981, Tijs [16] introduced the τ- value, a game theoretical solution concept. This method is called the cost gap method because the allocation of the non-separable cost by this separable method is determined with the aid of a cost gap function.

Let $M = (m_1, m_2, \ldots \ldots, m_n)$ be the marginal vector for the game, whose i-th coordinate is the separable cost of the player i, that is, $m_i = C(N) - C(N - \{i\})$ for all $i \in N$. For each coalition S the cost gap of S in the game is defined by

$$g(S) = C(S) - \sum_{i \in S} m_i \quad \text{if } S \neq \varphi$$

and

$$g(\varphi) = 0.$$

The map $g : 2^N \to R$ is called the cost gap function of the cost function of the cost game. $g(N)$ is equal to the non-separable cost in the cost game.

Let a cost game with nonnegative cost gap function, $i \in N$ and $S \subset N$ such that $i \in S$ As in all separable methods, the separable cost m_i is seen as a lower bound for the cost contribution of player i to the joint cost $C(N)$. For any S with $i \in S$, the number $\min_{S, i \in S}(m_i + g(S))$ or equivalent $m_i + \min_{S, i \in S} g(S)$ can be seen as an upper bound for the cost contribution of player i to the joint cost $C(N)$. Due to this reasoning, for any cost function a corresponding weight vector $W = (w_1, w_2, \ldots \quad \ldots, w_n)$ is defined by

$$w_i = \min_{S: i \in S} g(S) \text{ for all } i \in N$$

The number w_i can be seen as the maximal contribution of player i to the non-separable cost $g(N)$. Assume that the total of these maximal contributions covers the non-separable cost $g(N)$, that is, $\sum_{i \in N} w_i \geq g(N)$.

The cost gap allocation method is now defined as separable method, where the non-separable cost is allocated to the players proportional to the above mentioned weight vector Formally, to any cost function C such that

$$g(N) \geq 0 \text{ for all } S \subset N$$

and

$$\sum_{i \in N} w_i \geq g(N).$$

The cost gap allocation method assigns the cost allocation

$$CGA(C) = \begin{cases} M & \text{if } g(N) = 0 \\ M + \dfrac{g(N)W}{\sum_{i \in N} w_i} & \text{if } g(N) > 0 \end{cases}.$$

The cost gap allocation method is efficient, individually rational and possesses the dummy player property, the anonymity property, the strategic equivalent property and the continuity property. See [15].

Chapter 3 Derived Cost Allocation Method and It's Properties

3.1 Cost Allocation Method, *CRD(C)*

The formula is derived in this study, depends on system loss, where system loss is defined as loss of product in the form of wastage, loss of product due to misuse, and loss of product arising from illegal and unaccounted connection of product with the network. Let b_i is the i-th participant's benefit only when his purpose served and without considering the system loss. Also consider l_i is the i-th participant's system loss. Consider that the benefit, $B=(b_1,b_2,\dots\dots,b_n)$, the system loss, $L=(l_1,l_2,\dots\dots,l_n)$ and therefore the actual benefit $D=(d_1,d_2,\dots\dots,d_n)$ where $d_i=b_i-l_i$ for all $i\in N$. The weight $W=(w_1,w_2,\dots\dots,w_n)$ is the maximal contribution of player i to the non-separable cost, where $w_i\ge 0$ for all $i\in N$ and $\sum_{i=1}^{n} w_i>0$. So the maximum weight of player i is $\max w_i=C(\{i\})-m_i$.

One of the three possibilities can arise after calculating d_i. The value can be positive, or zero, or negative. Remaining cost calculation and cost allocation of the participants depends on the value of actual benefit.

3.1.1 Case I: When $d_i>0$ for all $i\in N$.

Remaining cost, $R=\left(C(N)-\sum_{i=1}^{n} m_i\right)>0$ 3.1 1

In this case the actual benefit, D is considered to be the weight function, W and the method $CRS(C)=(C_1,C_2,\dots\dots,C_n)$ is given by

$$CRS(C) = M + \frac{RW}{\sum_{i=1}^{n} w_i} \quad 3.1.2$$

3.1.2. Case II: When $d_i \le 0$ for all $i \in N$.

In this case the remaining cost is same as 3.1.1 and the weight of all the players are considered to be same, *i.e.*, $W = (1,1,\dots\dots,1)$. For this case the method is given by

$$CRS(C) = M + \frac{RW}{n} \quad 3.1.3$$

3.1.3. Case III: When $d_j > 0$ for $j = 1,2,\dots\dots,p$ and $d_k \le 0$ for $k = p+1, p+2,\dots\dots,n$.

$$\text{Remaining cost, } R = \left(C(N) - \sum_{j=1}^{p} m_j - \sum_{k=p+1}^{n} C(\{k\}) \right) > 0 \quad 3.1.4$$

In this case the method is given by

$$CRS(C) = \left(\left(m_1 + \frac{Rw'_1}{\sum_{j=1}^{p} w'_j} \right), \left(m_2 + \frac{Rw'_2}{\sum_{j=1}^{p} w'_j} \right), \dots, \left(m_p + \frac{Rw'_p}{\sum_{j=1}^{p} w'_j} \right), C(\{p+1\}), C(\{p+2\}), \dots, C(\{n\}) \right)$$

3.1.5

3.2 Properties

The cost allocation method *CRS*(*C*) satisfies the following basic properties.

3.2.1 Property 1: The *CRS*(*C*) is efficient.

For the first two cases: $\sum_{i=1}^{n} CRS_i(C) = \sum_{i=1}^{n} m_i + R = C(N)$.

For the last case: $$\sum_{i=1}^{n} CRS_i(C) = \sum_{i=1}^{p} m_i + R + \sum_{k=p+1}^{n} C(\{k\}) = C(N)$$

3.2.2 Property 2: The *CRS*(*C*) is individually rational.

For the first two cases

$$
\begin{aligned}
CRS_i(C) &= m_i + \frac{Rw_i}{\sum_{i=1}^{n} w_i} \\
&\le m_i + \frac{Rw_i}{R} \\
&= m_i + w_i \\
&\le m_i + C(\{i\}) - m_i \\
&= C(\{i\})
\end{aligned}
$$

For the last case, when i = 1, 2, ..., p then the same relation is obtained (just converting w_i by w_i'), and when $i = p+1, p+2, \ldots \ldots, n$ then always $CRS_i(C) = C(\{i\})$.

3.2.3 Property 3: The *CRS*(*C*) sometimes possesses dummy player property.

Let D be the set of dummy players. Any player $i \in D$ obviously satisfies the following properties.

(i) $m_i = C(\{i\})$ for all $i \in D$,

(ii) $w_i = 0$ for all $i \in D$.

For the first and the last cases when $w_i = 0$ and $w_i' = 0$ respectively for some $i \in N$ then $CRS_i(C) = m_i = C(\{i\})$. So, always $D \neq \{\ \}$.

3.2.4 Property 4: The *CRS*(*C*) satisfies the anonymity property.

For the first two cases: For all $C \in A_n$ and $\theta_C \in A_n$

$$m_{\theta i} = (\theta_C)(\theta S) - (\theta_C)(\theta(S-1)) \text{ for all } \theta i \in \theta S \text{ and } \theta i \in N$$

$$= C(S) - C(S-1) \text{ for all } S \subset N$$

$$= m_i \text{ for } i \in N,$$

$$w_{\theta i} = \max\left[(\theta_C)(\theta i) - m_{\theta i}\right] \text{ for all } \theta i \in N$$

$$= \max\left[C(\{i\}) - m_i\right] \text{ for } i \in N$$

$$= w_i$$

and

$$R_{\theta C} = C(N) - \sum_{\theta i=1}^{n} m_{\theta i}$$

$$= C(N) - \sum_{i=1}^{n} m_i$$

$$= R,$$

$$\therefore CRS(\theta C) = M_{\theta C} + \frac{R_{\theta C} W_{\theta C}}{\sum_{i=1}^{n} w_{\theta i}}$$

$$= M + \frac{RW}{\sum_{i=1}^{n} w_{\theta i}}$$

$$= CRS(C)$$

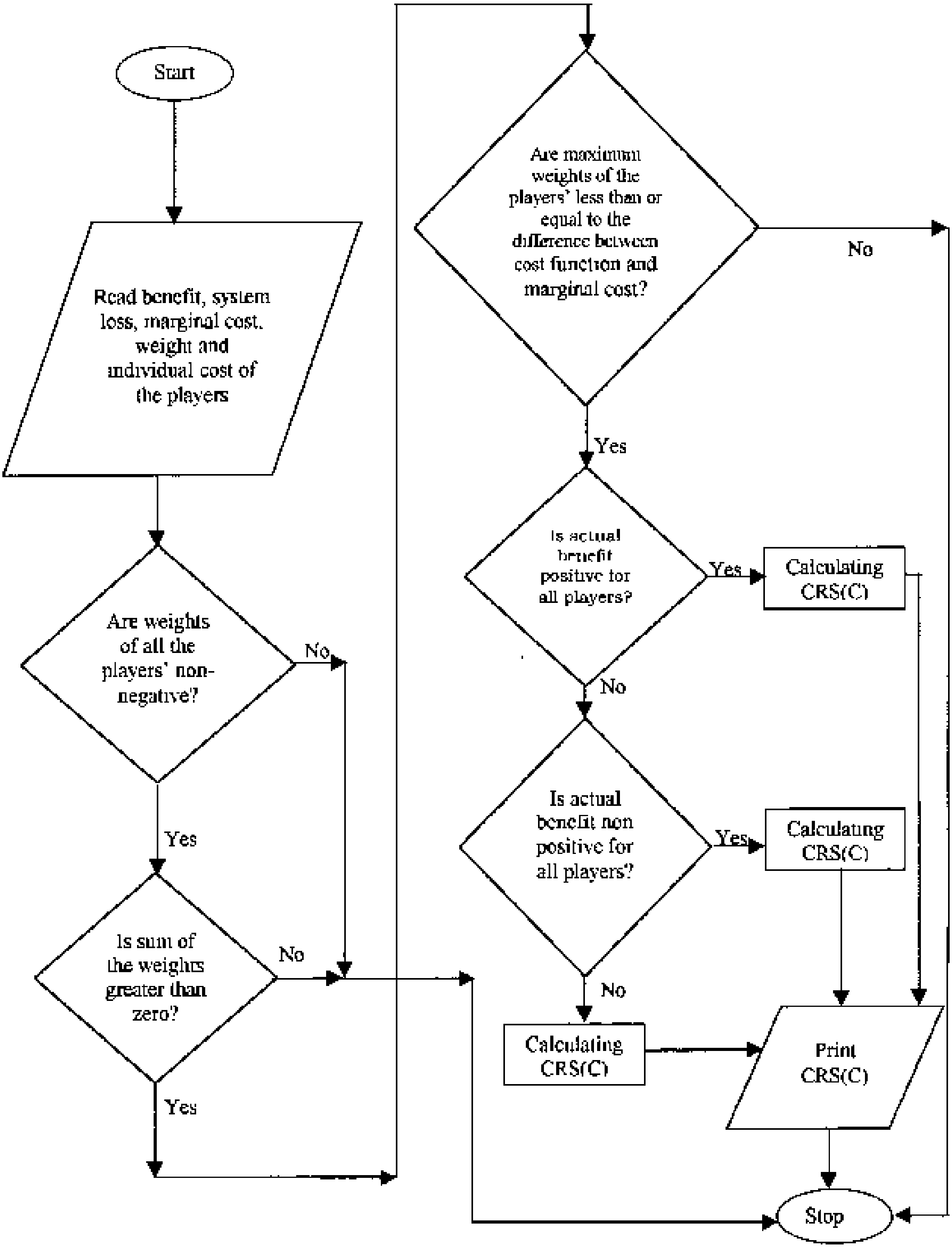
For the last case: Rewrite 3.1.5 as

$$CRS(C) = \left((m_1 + q_1), (m_2 + q_2), \ldots \ldots, (m_p + q_p), (m_{p+1} + q_{p+1}), \right.$$
$$\left. (m_{p+2} + q_{p+2}), \ldots \ldots, (m_n + q_n) \right)$$

where $q_i = \frac{Rw'_i}{\sum_{i=1}^{p} w'_i}$ when $i = 1, 2, \ldots, p$ and $q_j \geq 0$ when $j = p+1, p+2, \ldots, n$

Using the method for the first two cases $CRS(\theta C) = CRS(C)$ is obtained.

3.3 Flow Chart



4.1 Examples

In this chapter examples are given and $CRS(C)$ is calculated from the given data All the values in the examples are taken randomly. Calculations have been done using programs in C. The program is given in the appendix.

The followings are examples of the derived cost allocation method, $CRS(C)$, where all the values are assumed. In every example the number of player is 5 and cost function is 485 unit.

4.1.1 Example on Case I

Player	m_i	$C(\{i\})$	b_i	l_i	w_i
1	70	100	10	8	30
2	60	90	8	3	30
3	95	125	21	15	30
4	50	100	17	10	50
5	60	90	11	9	30

$CRS(C) = (96.47, 86.47, 121.47, 94\ 12, 86.47)$

4.1.2 Example on Case II

Player	m_i	$C(\{i\})$	b_i	l_i	w_i
1	70	100	10	11	1
2	60	90	12	14	1

Player	m_i	$C(\{i\})$	b_i	l_i	w_i
3	95	125	17	20	1
4	50	100	22	25	1
5	60	90	12	13	1

$CRS(C) = (100, 90, 125, 80, 90)$

4.1.3 Example on Case III

4.1.3.a When $d_j > 0$ for $j = 1, 2, 3, \ldots\ldots, p$ and $d_k = 0$ for $k = p + 1, p + 2, \ldots \quad \ldots, n$

Player	m_i	$C(\{i\})$	b_i	l_i	w_i
1	70	100	10	10	10
2	60	90	20	15	30
3	95	125	15	10	30
4	50	100	15	12	50
5	60	90	15	5	30

$CRS(C) = (100, 85.71, 120.71, 92.86, 85.71)$

4.1.3.b When $d_j > 0$ for $j = 1, 2, 3, \ldots\ldots, p$ and $d_k < 0$ for $k = p + 1, p + 2, \ldots \quad \ldots, n$

Player	m_i	$C(\{i\})$	b_i	l_i	w_i
1	70	100	8	10	10

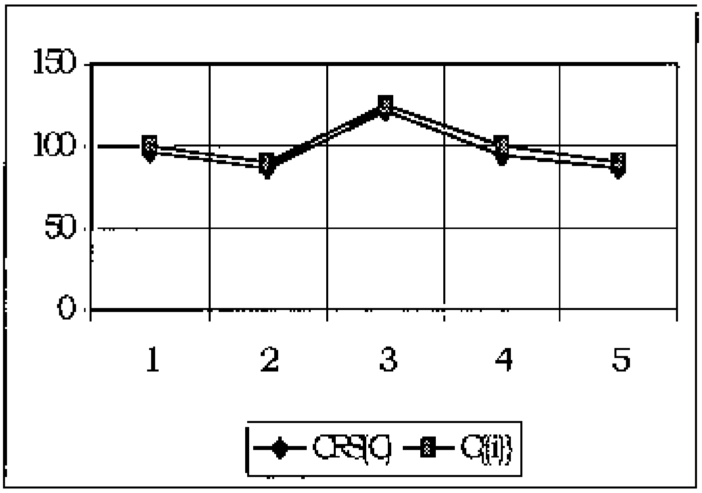
Player	m_i	$C(\{i\})$	b_i	l_i	w_i
2	60	90	10	15	10
3	95	125	15	10	30
4	50	100	20	12	50
5	60	90	10	5	30

$CRS(C) = (100, 90, 119\ 55, 90\ 91, 84.55)$

4.2 Line Diagram

The results obtained in the examples are presented in the form of line diagrams that clearly shows that the cost of every player either remains same or decrease if the cost allocation method $CRS(C)$ is applied

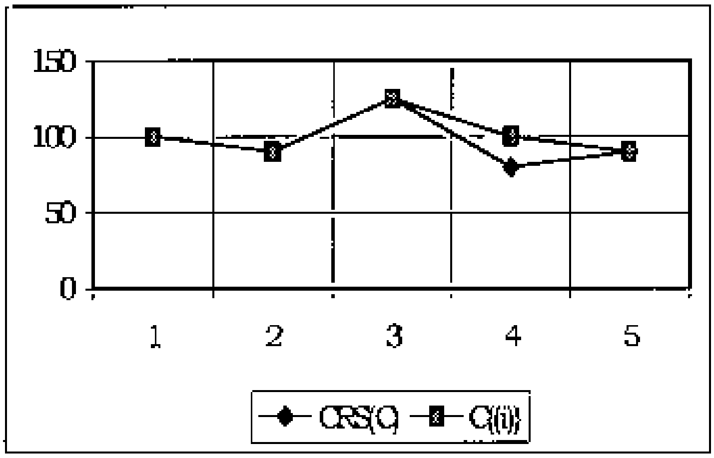
4.2.3 Line Diagram of Example 4.1.1



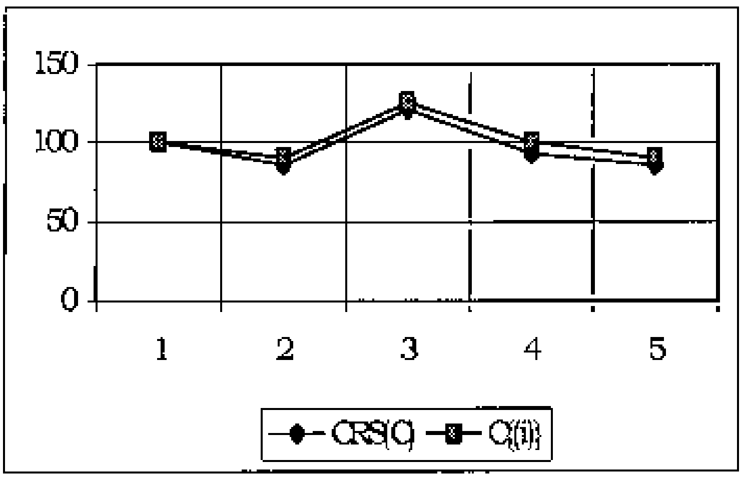
The graph shows that the cost of all the players decreases, depending on their actual benefit and weight. The percentage of the actual benefit and the weight of the Player 4 are higher than that of other four players. The graph shows that the difference between the cost of Player 4 when he plays individually and the cost of same player when he played together is higher than that of the others.

4.2.4 Line Diagram of Example 4.1.2

In this case no player is able to make actual benefit and the weight of the players are equal. But also the cost of Player 4, when he plays all together, decreases; because the remaining cost of the player is bigger than that of all other participating players in the game.



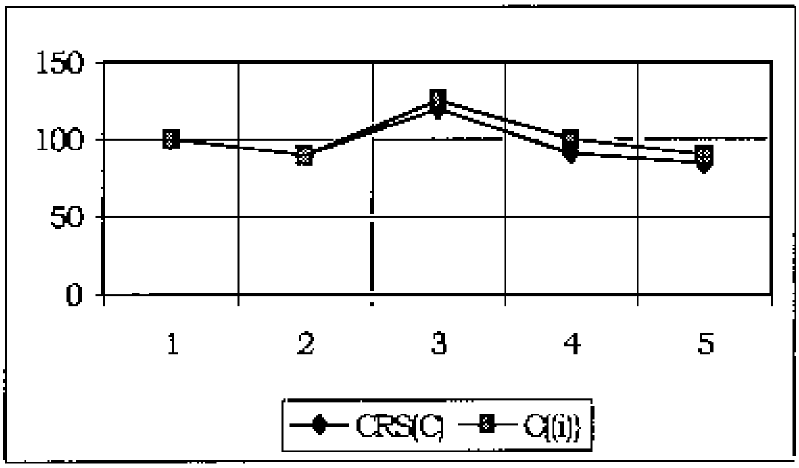
4.2.5 Line Diagram of Example 4.1.3.a



The graph shows that depending on system loss, weight and remaining cost the cost of the individual players vary when they work alone and work together.

4.2.4 Line Diagram of Example 4.1.3.b

Player 1 and 2 are unable to make actual benefit, and the graph shows that they have to pay the same amount in both of the cases, but all other



players make actual benefit and their cost when they work together are lower than that when they work alone

Chapter 5 Conclusion

Cost allocation methods are designed to optimally distribute the cost among players incurred in a joint-project with a view to satisfy players considering one's strengths and weaknesses Certain properties are mathematically defined pertaining to cost allocation methods whose satisfaction by a method ensures true optimality of the method. Unfortunately no method satisfies all of the desired properties (defined and identified earlier) and therefore scope is always there to add a new method to the collection. Proposed and developed method by the author satisfies four of the properties (efficient, individually rational, possess dummy player property and anonymity property) and therefore optimizes allocation to a certain extent in consideration of earlier defined constraints. The author believes that the developed method could play some role in real application level.

In our country, unplanned and multiple road digging by different authorities is a common problem that could be helped out employing the developed formula Since data collection in true and accurate form is a big deal of problem in Bangladesh, therefore, the method could not be tried upon real road digging situation However, simulated result with random data on the model speaks of work ability of it.

The developed formula is planned to be tested in the future using real life data.

Appendix

```
#include<stdio.h>
#include<conio.h>
void main()
	{
	int n,i,pv = 0,nv = 0;
	float cf,r,sm = 0,sw = 0,sc = 0,*m,*c,*b,*l,*w,*d,*crs;
	clrscr(),
	printf("No. of players are  "),
	scanf("%d",&n);
	printf("\nCost function is  ");
	scanf("%f",&cf),
	printf("\nm[i]\t\tc[i]\t\tb[i]\t\tl[i]\t\tw[i]\n");
	for(i = 0;i<n;i++)
		{
		printf("\n");
		scanf("%f%f%f%f%f",&m[i],&c[i],&b[i],&l[i],&w[i]),
		d[i] = b[i] - l[i];
		sm = sm + m[i];
		if((w[i]<0)||(w[i]<d[i])||(w[i]>(c[i]-m[i])))
			{
			break;
			}
		}
	r = cf-sm,
	for(i = 0;i<n;i++)
		{
		if(d[i]>0)
			{
```

```
                pv = pv + 1;
                }
        else
                {
                nv = nv + 1;
                }
        }
if(pv = = n)
        {
        for(i = 0;i<n;i++)
                {
                sw = sw + w[i],
                }
        printf("\nCRD(C) = (");
        for(i = 0;i<n;i++)
                {
                crs[i] = m[i] + ((r*w[i])/sw);
                if(i<n -1)
                        {
                        printf(" %.2f,",crs[i]);
                        }
                else
                        {
                        printf(" %.2f",crs[i]),
                        }
                }
        printf(")");
        }
 else if(nv = = n)
        {
```

```
        printf("\nCRD(C) = (");
    for(i =0;i<n;i++)
            {
            crs[i] = m[i] + ((r*w[i])/n);
            if(i<n -1)
                {
                printf(" %.2f,",crs[i]),
                }
            else
                {
                printf(" %.2f",crs[i]);
                }
            }
        printf(")");
        }
else
        {
        sm = 0;
        sw = 0;
        for(i = 0;i<n;i++)
            {
            if(d[i]>0)
                {
                sm = sm + m[i];
                sw = sw + w[i];
                }
            else
                {
                sc = sc + c[i],
                }
```

```
            }
        sm = sm + sc;
        r = cf - sm;
        printf("\nCRD(C)=(");
        for(i = 0,i<=n -1,i++)
            {
            if(d[i]>0)
                {
                crs[i] = m[i] + ((r*w[i])/sw);
                }
            else
                {
                crs[i] = c[i];
                }
            if(i<n -1)
                {
                printf(" %.2f,",crs[i]);
                }
            else
                {
                printf(" %.2f",crs[i]),
                }
            }
        printf(")");
        }
    getch();
    }
```

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