Similarity Solutions of Unsteady Natural Convection Boundary Layer Flow with Suction Over a Vertical Flat Plate using Group-Theory Method

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Abstract

The group theory is applied to present an analysis of the problem of unsteady laminar natural convection from a vertical porous flat plate. Assuming suction for the porous plate, analytical expressions for flow characteristics are obtained. The application of one-parameter groups reduces the number of independent variables by one and the system of governing partial differential equations with boundary conditions reduces to a system of ordinary differential equations with appropriate boundary conditions. The possible forms of surface temperature variations are derived. The one set of ordinary differential equations are solved numerically using shooting method based on sixth order Runge-Kutta scheme along with Nachtshiem-Swigert iteration technique. Programming software FORTRAN 90 is used to implement Runge-Kutta method and visualisation software TECPLOT is used to create graphs. The effect of suction parameter on the velocity, temperature, the skin friction and the rate of heat transfer for fixed Prandtl number is discussed with the help of graphs. Also the heat transfer characteristics for varying values of Prandtl number and fixed suction parameter are presented.

Candidate's Declaration

I hereby declare that the work presented in this dissertation entitled "Similarity Solutions of Unsteady Natural Convection Boundary Layer Flow with Suction Over a Vertical Flat Plate Using Group Theory Method" which is submitted in partial fulfillment of the requirements for the award of the degree of Master of Philosophy in Mathematics in the Department of Mathematics, Bangladesh University of Engineering and Technology (BUET), Dhaka-1000 is an authentic record of my own work.

It has not been submitted elsewhere (Universities or Institutions) for the award of any other degree.

Harekrishna Das.
(Harekrishna Das)

Date: 20-09-2006

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Contents

Nomenclature

Dimensional variables

Symbol	Quantity	Unit
x^*	Distance along the plate	m
y^*	Distance perpendicular to the plate	m
u^*	Velocity component in the boundary layer along the plate	$\frac{m}{s}$
v^*	Velocity component in the boundary layer normal to the plate	$\frac{m}{s}$
T	Temperature	K
g	Gravitational acceleration	$\frac{m}{s^2}$
L	Reference length	m
$q = -G'(0)$	Heat transfer rate	$\frac{J}{m^2 s}$
U	Characteristic velocity	$\frac{m}{s}$

Dimensionless variables

Symbol	Quantity
Gr	Grashof number
F	Dimensionless stream function
Pr	Prandtl number
Ra	Rayleigh number = $GrPr$
x	Distance along the plate

y	Distance perpendicular to the plate
u	Velocity component in the boundary layer along the plate
v	Velocity component in the boundary layer normal to the plate
$\theta = G = \frac{T}{T_w}$	Temperature
t	Time

Greek symbols

Symbol	Quantity	Unit
α	Thermal diffusivity	$\frac{m^2}{s}$
β	Coefficient of volumetric expansion	$\frac{1}{K}$
μ	Dynamic viscosity	$\frac{kg}{ms}$
ρ	Density of the fluid	$\frac{kg}{m^3}$
τ	Local shear stress	$\frac{kg}{ms^2}$
k	Thermal conductivity	$\frac{J}{mks}$
θ	Dimensionless temperature function	
ψ	Stream function	
η	Similarity variables	
$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	Real numbers	

Subscripts

w = Surface condition

α = Conditions in the ambient

F_{η} =Derivative of stream function w.r.to

similarity variables η

List of Figures:

Chapter 1

1.1 General Introduction With Review of Previous Work

The Phenomenon of natural convection arises in the fluid when temperature changes cause density variation leading to buoyancy forces acting on the fluid elements. This can be seen in our everyday life in the atmospheric flow which is driven by temperature differences. When the vertical plate is being heated, the air layer adjacent to the wall expands and rises due to buoyancy. A natural convection boundary layer is formed adjacent to the wall surface.

Recently, unsteady conditions of motion and heating of bodies in fluids have become increasingly important in certain applications for some engineering fields of aerodynamics and hydrodynamics. Also a natural convection flow has been generated due to the temperature difference inside the plastic green houses. Mankabadi (1988) considered two pumping systems that can utilize a usable power about 200 W for pumping underground water for irrigation purposes. Therefore it becomes necessary to pay more attention to this problem.

Obviously, the introduction of time as the third independent variable in the unsteady problem increases the complexity of the problem. Many attempts were made to find analytical and numerical solutions applying certain special conditions and using different mathematical approaches. Illingworth (1950) studied the problem of unsteady laminar flow of gas near an infinite flat plate. He obtained solutions which are available only with Prandtl number unity and under transient conditions of step change in the surface temperature. The problem of transient free convection at the heated surface has been studied extensively. Siegel (1958) investigated the transient free convection from a vertical plate. Free convective flow past vertical plate has been studied extensively by Ostrach (1953,54) and many others

The free convective heat transfer on a vertical semi-infinite plate has been investigated by Berezovsky (1977) . Martynenko(1984) investigated the laminar free convection from a vertical plate. In all these papers, the plate was assumed to be maintained at a constant temperature which is also the temperature of the surrounding stationary fluid. Soundalgekar (1977) studied the unsteady free convection flow past an infinite vertical plate with constant suction and mass transfer. It was assumed that the plate temperature oscillates in such a way that its amplitude is small.

Possible similarity solutions for laminar free convection on vertical plates and cylinders have been studied by Yang (1960). He established some necessary and sufficient conditions for which similarity solutions are possible. Hansen (1958) investigated possible similarity solutions of three dimensional laminar incompressible boundary layer equations. Zakerullah (2001) has derived similarity solutions of some possible cases of unsteady mixed convection by group theory without suction. He also investigated steady natural convection by group theory method without suction. Abdel-el-Malek (1990) investigated unsteady free convective laminar boundary layer flow on a non-isothermal vertical flat plate without assuming suction at porous plate in this case.

The mathematical technique used in the present analysis is one parameter group transformation which leads to a similarity representation of the problem. The fundamental simplicity and power of this method are well known. Morgan (1952) presented a theory which led to improvements over earlier similarity methods.

Michal (1952) extended Morgans theory. Group Methods, as a class of methods which lead to a reduction of the number of independent variables, were first introduced by Birkoff (1948,1960). He made use of one parameter group transformations to reduce a system of partial differential equations in one independent variable. This technique has been applied intensively by Abdel-el- Malek (1990,1991) , Ames(1985) and many others.

Moran and Gaggioli (1966,1968) presented a general systematic group formalism for similarity analysis. They utilized elementary group theory for the purpose of reducing a given system of partial differential equations in a single variable. Similarity analysis has been applied intensively by Gabbert (1967) .For additional discussions on group transformations, one consults Ames (1972,1985), Eisenhart,Bluman and Cole (1974), Moran and Gaggioli(1967) .

In this work we point out the effect of suction in three cases derived by Yang (1960). We present a general procedure for reducing the number of independent variables in the governing equations from three to one independent variable. The technique used is the one-parameter group transformation which is applied to both the governing partial differential equations and the boundary conditions to assure the invariance conditions. One set of the resultant system of ordinary differential equations with appropriate boundary conditions is then solved numerically using shooting method based on sixth order Runge-Kutta scheme along with Nachtshiem-Swigert iteration technique to find boundary layer flow characteristics due to suction effect and for varying Prandtl number when suction parameter is fixed. Programming software FORTRAN 90 is used to implement Runge-Kutta method and visualisation software TECPLOT is used to create graphs.

Chapter 2

2.1 Group-Theory and Important Terms

2.1.1 Group Theory

One of the most mathematically sophisticated methods of determining similarity solutions of partial differential equations is based on concepts derived from the theory of continuous transformation groups.

Group Concepts: An algebraic group is a collection of 'elements' which has some sorts of operation defined between the elements. In addition, a certain set of rules or statements regarding the elements and the defined operation must be satisfied. The elements can be almost anything we choose to define, such as integers, complex numbers, vectors, matrices, transformations, etc. One important criterion, however, is the definition of an operation for these elements. Typical operations for the classes of elements just mentioned are integer addition, complex number multiplication, vector addition, matrix multiplication and successive transformations.

The rules which a set of elements must obey under a given operation are given below. (The symbol $\circ$ will denote the given operation between two elements; e.g., $a \circ b$.)

1. The sets of elements is closed under the given operation. If a and b are two elements of the sets, then $a \circ b = c$ is also a unique element of the set.
2. There exists an identity element I such that $a \circ I = I \circ a = a$
3. Given any element a, there exists an element a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = I$
4. The associative law is valid for the given operation $a \circ (b \circ c) = (a \circ b) \circ c$

Transformation group:

A complete set of transformation is taken as group elements. Successive transformations employing various sets of function are considered to be "operation" between elements

Invariant Solution:

We consider a system of partial differential equations for which the y^j are dependent variables and x^i are independent variables. Solutions of a system of equations for which y^j are exactly the same functions of the x^i as the Y^j are of the X^i are called invariant solutions.

Conformally invariant, Constant conformally invariant , absolute invariant:

The differential form $\phi(z^1,......,z^p)$ is defined to be conformally invariant under a one-parameter transformation group $Z = T_a z$ if under the group transformations

$\phi(Z^1,......,Z^p) = F(z^1,......,z^p;a)\phi(z^1,.........,z^p)$

where $F(z^1,.... ,z^p;a)$ is some function of the z^k and the single group parameter, a.

If $\phi(Z^1,... ...,Z^p) = F(a)\phi(z^1,.........,z^p)$,

the function ϕ is said to be constant conformally invariant under the group transformations .If $F(a)=1$ so that $\phi(Z^1,.........,Z^p) = \phi(z^1, ... ,z^p)$

the function ϕ is said to be absolute invariant under the transformation group.

Group Theory Method:

Let us consider a single partial differential equation in two independent variables. We seek transformations that will reduce the number of independent variables by one, i.e., lead to an ordinary differential equation .The steps are to

1 Select a one-parameter transformation group. If x and y are the independent variables, a reasonable first choice might be

$$X = a^{n} x$$

$$Y = a^{m} y$$

$$\bar{u} = a^{p} u$$

2. Find an absolute invariant which is a function of the independent variables alone. For example

$$\eta = yx^{s}$$

Establish relation between n, m and s such that

$$yx^{s} = YX^{s}$$

The absolute invariant η will be the new independent variable.

3. Find a second absolute invariant g, chosen in such a way that g involves the dependent variable u. For example

$$g = ux^{r}$$

Find relation such that

$$ux^{r} = \bar{u}X^{r}$$

4. Set $g = F_1(\eta)$. Then

$$u = F_1(\eta)x^{-r}$$

and $F_1(\eta)$ is the new dependent variable.

5 Substituting the transformation for u into the given equation and employing the definition of η should reduce the given partial differential equation to a ordinary differential equation

If more dependent or independent variables are involved , the above procedure remains essentially the same , except that a group of independent variables $\eta_1, \eta_2, \ldots$, are sought from the original independent variables and are one less in number . The η_i are absolute invariants.

For each dependent, an absolute invariant g_i is sought which involves the dependent variable. A good choice is $g_i = u_i h(x_1, \ldots\ldots\ldots\ldots, x_m)$ where u_i is the dependent variable. The function g_i is then equated to a function

$F_1(\eta_1, \eta_2 \ldots\ldots \eta_{n-1}) = u_i h(x_1 \ldots\ldots, x_m)$

then

$$u_i = \frac{F_i(\eta_1, \eta_2, \ldots\ldots)}{h_i(x_1, x_2, \ldots\ldots x_m)}$$

is the dependent variable transformation . Substituting the various transformations into the original system of equations should lead to a new system with the number of independent variables reduced by one.

Evaluation of the Group-Theory Method:

The advantages of the group theory method should be obvious. The first is that the method is rather simple to apply. We merely pick a transformation and proceed. There is no concern about boundary conditions, choices for various functions, etc. Second, in reducing the number of independent variables by one it is possible to obtain a new system of partial differential equations without continuing to obtain ordinary differential equations. The possible advantage of stopping short of a system of ordinary differential equations is that it may be possible to solve wider variety of problems in this manner. It would be very interesting to explore this possibility in solving the boundary layer equations.

On the other hand. there are two evident disadvantages to employing group-theory methods. The first is that boundary conditions are not taken into account in any way until the entire analysis is completed. The second is the uncertainty in choosing a proper transformation group. If we should fail to simplify a system under one choice of transformation group. this does not mean that another, which would prove to be adequate, does not exist. Fortunately, the types of group employed in the examples seem to be adequate for yielding the classes of similarity transformations obtained by other methods.

Although it has been pointed out that one advantage of the group- theory methods is to reduce a system of partial differential equations in n independent variables to a system in $n-1$ variables without continuing to ordinary differential equations, the same type of results could be achieved with a modification of free parameter method. We would simply introduce functions of more than one parameter. We would begin an analysis by assuming that the independent variables were expressible in terms of $(n-1)$ dependent variables instead of a single parameter η. While such an approach has not been exploited in literature, it would be quite simple to investigate this possibility. However, the group-theory methods should yield complete results with less effort.

2.1.2 Important Terms

Porous plate:

By porous plate we mean that the plate possesses very fine holes distributed uniformly over the entire surface of the plate through which fluid can flow freely.

Plate with Suction and Injection:

The plate from which the fluid enters into the flow region is known as plate with injection and the plate from which the fluid leaves out the flow region is known as plate with suction. Sometimes it is necessary to control the boundary layer flows by injecting or withdrawing fluid through a heated boundary layer wall. Since this can enhance heating (or cooling) of the system, can help delay the transition from the laminar to turbulent flow. Boundary layer suction is used to control laminar and turbulent separations by removing flow of the low momentum. The technique is used in air wings, some wind tunnels to remove boundary layer. Blowing (injection) a boundary layer on high temperature components can maintain a thin layer of colder flow that allows the system to function with very high fluid velocity.

Isothermal:

It is a process if temperature is held constant during the process. i,e, $\Delta T = 0$ or, having equal or constant temperature with respect to either space or time.

Similar Solutions' of the boundary layer equation:

Boundary layer equations are more simpler than the original momentum equations, yet they are non-linear partial differential equations. We, therefore, simplify them further by reducing them into ordinary differential equations. To this end, we propose to change, if possible, the independent and dependent variables in such a way so as to transform the partial differential equations of the boundary layer equations into ordinary differential equations. Whenever such a transformation exists, we say that 'similar solution' exists.

Similarity variable:

Similarity variable is a function of original independent variables.

Thermal conductivity: The well-known Fourier's heat conduction law states that the conductive heat flow per unit area (or, heat flux) q_n is proportional to the temperature decrease per unit distance in a direction normal to the area through which the heat is flowing. Thus, mathematically

$q_n \propto -\frac{\partial T}{\partial n}$ so that $q_n = -k\frac{\partial T}{\partial n}$ where k is said to be the thermal conductivity.

Viscosity: Viscosity of fluids is due to cohesion and interaction between particles. Vis--cosity introduces resistance to motion by developing shearing and frictional stress between the fluid layers and the boundary. Existence of shearing resistance in real fluid causes the fluids to adhere to the solid boundary, and hence there is no relative motion or slip between the fluid layers immediately in contact with the solid boundary. Viscosity causes the flow to occur in two different modes, namely, laminar and turbulent flow.

Kinematic Viscosity, Dynamic Viscosity: In Fluid problems, the coefficient of dynamic viscosity μ usually occurs together with mass density ρ in the form $\frac{\mu}{\rho}$. In such problems, it is convenient to use another coefficient called the coefficient of kinetic-viscosity ν (nu). It is the ratio of the coefficient of dynamic viscosity to mass density.

$$\nu = \frac{\mu}{\rho}$$

The dimension of ν

$$[\nu] = [L^2/T]$$

In CGS units , it is measured in stoke. which is equal to 1 $cm^2/\sec$.

A parameter μ defined such that [shear stress]= μ [strain rate]

Dynamic viscosity is related to kinetic viscosity ν by $\mu = \rho\nu$, where ρ is the density.

Incompressible, Compressible Fluid: A fluid is said to be incompressible if it can not be compressed easily. The density of an incompressible fluid is almost constant.

A fluid is said to be compressible if it can be easily compressed. Compressible fluids have variable density.

Steady Flow: The flow is said to be steady when the flow characteristics, such as velocity, density, pressure, temperature do not change with time. A flow will be steady when the rate of change of these characteristics is zero. For example, if V is the velocity at any point, the flow will be steady if $\frac{dV}{dt} = 0$

Water flowing through a tap at constant rate is an example of steady flow.

Unsteady Flow: The flow is unsteady if the velocity and other hydraulic characteristics change with time. Mathematically, $\frac{dV}{dt} \neq 0$. If the water is flowing at a changing rate, as is the case when tap is just opened, the flow is unsteady.

Uniform Flow: The flow is said to be uniform when the velocity and other characteristics are constant in a particular reach. A flow will be uniform if the rate of change of these characteristics with respect to distance along the path is zero.

For example, if V is the velocity at any point, the flow will be uniform if $\frac{dV}{dS} = 0$,where S is the distance measured from some fixed point on the path of flow.

In the other words, velocity is constant in the reach. A liquid flowing through a long straight pipe of uniform diameter at constant rate is an example of uniform flow, as the velocity is the same at all sections in the reach.

Non-Uniform Flow: The flow is non- uniform when the flow characteristics change at various points along the path. For example, if V is the velocity at any point, the flow will be non-uniform if $\frac{dV}{dS} \neq 0$. If the diameter of the pipe changes, i.e., the pipe is either converging or diverging, the velocity at different sections in the reach is not constant and the flow is non-uniform.

Stream Function: Stream function ψ is a function which describes the form of pattern of flow. If the stream function ψ is expressed as a function of x and y, the streamlines can be plotted The component of velocity at any point may be obtained by taking partial derivatives of ψ,

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

Conversely, if u and v are given, the stream function ψ can be obtained by integration.

Since, $d\psi = \frac{\partial \psi}{\partial x} \cdot dx + \frac{\partial \psi}{\partial y} \cdot dy$

Therefore, $\psi = \int\left(\frac{\partial \psi}{\partial x}\right)dx + \int\left(\frac{\partial \psi}{\partial y}\right)dy + C$

$$\psi = \int -v dx + \int u dy + C.$$

Skin friction: Plastic drag is drag caused by moving a solid object through a fluid. In Aerodynamics, skin friction is the component of plastic drag arising from the friction of the fluid against the skin of the object that is moving through it. Skin friction is a function of the interaction between the fluid and the skin of the body, as well as the wetted area, or the area of the surface of the body that would become wet if sprayed with water flowing in the wind.

Coefficient of skin friction: It is denoted by C_f and is defined as the ratio of the shear stress τ_o to the quantity $(\rho U^2)/2$.

Thus, $$C_f = \frac{\tau_o}{(1/2)\rho U^2}$$

Laminar and Turbulent Flows: A flow, in which each fluid particle traces out a definite curve and the curves traced out by any two different fluid particles do not intersect, is said to be laminar .On the other hand, a flow, in which each fluid particle does not trace out a definite curve and the curves traced out by fluid particles intersect, is said to be turbulent. The flow of thick oil through a small tube is an example of laminar flow. The flow in rivers at the time of floods is turbulent.

Grashof Number: It is a dimensionless group which represents the ratio of buoyancy force to viscous force in free convection. One of the parameters which must be the same in two free convecting systems for them to be dynamically similar .It is defined by ΔT

$$Gr = \frac{g\alpha \Delta T L^3}{\nu^2}$$

where g is gravity, α is thermal expansion coefficient, ΔT is the temperature difference, L is the length scale and ν is the kinematic viscosity.

Prandtl Number: Prandtl number is a dimensionless parameter of a convecting system that characterizes the regime of convection. It is defined as

$$Pr = \frac{\nu}{\kappa}$$

where ν is the kinematic viscosity and κ is the thermal diffusivity. Prandtl number is the ratio of viscous force to the thermal force.

Reynolds Number: The effect of viscosity of a fluid phenomenon is usually expressed in terms of a non-dimensional parameter called Reynolds Number(Re) given by

$$Re = \frac{VL}{\nu}$$

where V denotes the characteristic velocity , ν is the kinematic viscosity and L denotes the characteristic length. This is the ratio of the inertia force to the viscous force.

Two flows of incompressible viscous fluid about similar geometrical bodies are dynamically similar when Reynold's numbers for the flows are equal. A small Reynold's number implies that viscosity is predominant whereas a large Reynold's number implies that viscosity is small. If the value of Reynold's number exceeds a certain critical value (namely 2,800) the flow ceases to be laminar and the flow becomes turbulent. When $Re < 2000$, the flow is laminar.

Natural Convection: Convection is the study of conduction in a fluid as enhanced by its "convective transport' that is, its velocity with respect to a solid surface. It thus combines the energy equation, or first law of thermodynamics, with the continuity and momentum relations of fluid mechanics.

In natural convection fluid motion is due solely to local buoyancy differences caused by the presence of the hot or cold body surface. Most fluids near a hot wall, for example, will have their density decreased, and an upward near- wall motion will be induced.

Natural convection velocities are relatively gentle and the resultant wall heat flux will generally be less than in forced motion.

Boundary Layer: The layer of reduced velocity in fluids, such as air and water, that is immediately adjacent to the surface of a solid past which the fluid is flowing. In other words, the boundary layer is the layer of fluid in the immediate vicinity of a bounding body. In the atmosphere the boundary layer is the air layer near the ground affected by diurnal heat, moisture or momentum transfer to or from the surface. On an aircraft wing the boundary layer is the part of the flow close to the wing. The boundary layer effect occurs at the field region in which all changes occur in the flow pattern. The boundary layer distorts surrounding non-viscous flow.

The boundary layer is a very thin sheet of air lying over the surface of the wing (and, for that all the surfaces of the airplane). Because air has viscosity, the layer of the air tends to adhere to the wing. As the wing move forward through the air, the boundary layer at first flows smoothly over the streamlined shape of the airfoil. Here the flow is called the laminar layer.

As the boundary layer approaches to the centre of the wing, it begins to lose speed due to skin friction and it becomes thicker and turbulent. Here it is called turbulent layer. The point at which the boundary layer changes from laminar to turbulent is called the transition point. Where the boundary layer becomes turbulent, drag due to skin friction is relatively high. As speed increases, the transition point tends move forward. As the angle of attack increases, the transition point also tends to move forward. The boundary layer is particularly important in aerodynamics because it is responsible for a considerable amount of drag. In high-performance designs, such as sailplanes and commercial transport aircraft, much attention is paid to controlling the behavior of the boundary layer to minimize drag.

Two effects need to be considered. First, the boundary layer adds to the effective thickness of body, hence increasing the pressure drag. Second, the shear forces at the surface of the wing create skin friction drag.

Chapter 3

3.1 Formulation of the problem and the governing equation

Let us consider a laminar free-convective boundary layer adjacent to a semi- infinite vertical flat plate. The non-isothermal porous plate is heated in an unsteady manner, consequently the temperature distribution over the plate, T_w^* will be a function of the vertical distance x and the time t. There is a normal velocity of suction at the porous plate. The fluid is of constant temperature T_∞^* far from the plate such that $T_w^* > T_\infty^*$ in Fig.1

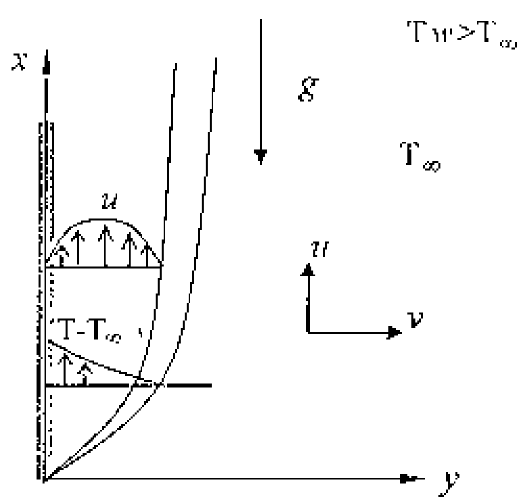

Figure 1:Physical model and co-ordinate system

If we take L as some arbitrary reference length, $\frac{L}{U}$ as a typical time, where $U = \left\{g\beta L(T^*_{ref} - T^*_\infty)\right\}^{1/2}$ is a typical velocity with 'g' the acceleration due to gravity, 'β' is the volumetric coefficient of thermal expansion and T^*_{ref} is some arbitrary reference temperature, along with the application of the Boussinesq and boundary layer approximation, the equations of motion may be written as

Continuity equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ (3.1)

Momentum equation. $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = T + \frac{\partial^2 u}{\partial y^2}$ (3.2)

Energy equation: $\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{Pr}\frac{\partial^2 T}{\partial y^2}$ (3.3)

with the boundary conditions

$$
\begin{aligned}
&u = 0,\ v = v_w(x,t),\ T = T_w(x,t) \quad \text{at } y = 0 \\
&u = 0,\ v = 0, \qquad T = 0 \qquad \text{at } y = \infty
\end{aligned}
\qquad (3.4)
$$

where $x = \frac{x^*}{L}$, $y = \frac{y^*(Gr)^{1/4}}{L}$

$u = \frac{u^*}{U}$, $v = \frac{v^*(Gr)^{1/4}}{U}$

$Gr = \dfrac{g\beta L^3 (T^*_{ref} - T^*_{\infty})}{\nu^2}$ is the Grashof number, ν is the kinematic viscosity,

$Pr = \dfrac{\nu}{\alpha}$ is the Prandtl number and α is the thermal diffusivity.

From the continuity equation (3.1), there exists a non-dimensional stream function $\psi(x, y, t)$ such that $u = \dfrac{\partial \psi}{\partial y}$, $v = -\dfrac{\partial \psi}{\partial x}$ which satisfies (3.1) identically. We introduce the non-dimensional temperature defined by $\theta = \dfrac{T}{T_w}$

Equation (3.2) and (3.3) become

$$\phi_1 \equiv \frac{\partial^2 \psi}{\partial y \partial t} + \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - \theta\, T_w - \frac{\partial^3 \psi}{\partial y^3} = 0 \tag{3.5}$$

$$\phi_2 \equiv \left(T_w \frac{\partial \theta}{\partial t} + \theta \frac{\partial T_w}{\partial t} \right) + \frac{\partial \psi}{\partial y}\left(T_w \frac{\partial \theta}{\partial x} + \theta \frac{\partial T_w}{\partial x} \right) - T_w \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} - \frac{1}{Pr} T_w \frac{\partial^2 \theta}{\partial y^2} = 0$$

i.e.,

$$\phi_2 \equiv \frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} + \theta\left[\frac{\partial}{\partial t}(\ln T_w) + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\ln T_w) \right] - \frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2} = 0 \tag{3.6}$$

with boundary conditions:

$$\begin{aligned} &\frac{\partial \psi}{\partial y} = 0,\ -\frac{\partial \psi}{\partial x} = v_w,\ \theta = 1 \quad \text{at } y = 0 \\ &\frac{\partial \psi}{\partial y} = 0,\ \frac{\partial \psi}{\partial x} = 0,\ \theta = 0 \quad \text{at } y = \infty \end{aligned} \tag{3.7}$$

Our method of solution depends on the application of one parameter group transformation to partial differential equations (3.5) and (3.6). Under this transformation, the three independent variables x, y, t will be reduced by one and the differential equations (3.5) and (3.6) will be transformed into ordinary differential equations in only one independent variable, which is a similarity variable.

3.1.1 Case I: $T_w \propto (x+bt)^{-1}$ and suction velocity $\propto (x+bt)^{-1/2}$

Unsteady natural convection with surface temperature varying inversely as a linear combination of x and t and the suction velocity varying inversely as a square root of the linear combination of x and t

Finding the similarity solutions of the equations (3.5) and (3.6) are equivalent to determine the invariant solutions of these equations under a particular continuous one-parameter group.

In order to seek invariant solution to this set of PDE we search a transformation group from one parameter transformation defined by the following group (G1):

$$\left.\begin{aligned} x' &= a^{\alpha_1} x \\ y' &= a^{\alpha_2} y \\ t' &= a^{\alpha_3} t \\ \psi' &= a^{\alpha_3}\psi \\ T_w' &= a^{\alpha_4} T_w \\ \theta' &= \theta \end{aligned}\right] \qquad (3.1.1)$$

Here $a \neq 0$ is the parameter of the group and α 's the arbitrary real numbers whose interrelationship will be determined by the subsequent analysis. We now investigate the relationship among the exponent α 's such that

$$\begin{aligned} &\phi_j\left(x', y', t', u', v', \ldots\ldots, \frac{\partial^3\psi'}{\partial y'^3}\right) \\ &= H_j\left(x, y, t, u, v, \ldots\ldots, \frac{\partial^3\psi}{\partial y^3}; a\right)\phi_j\left(x, y, t, u, v, \ldots\ldots, \frac{\partial^3\psi}{\partial y^3}\right) \end{aligned} \qquad (3.1.2)$$

for this is the requirement that the differential forms ϕ_1 , ϕ_2 be conformally invariant under the transformation group (3.1.1). Substituting the transformations (3.1.1) in (3.5) and (3.6) we have

$$\phi_1 \equiv \frac{\partial^2 \psi'}{\partial y' \partial t'} + \frac{\partial \psi'}{\partial y'}\frac{\partial^2 \psi'}{\partial y' \partial x'} - \frac{\partial \psi'}{\partial x'}\frac{\partial^2 \psi'}{\partial y'^2} - \theta' T_w' - \frac{\partial^3 \psi'}{\partial y'^3} = 0$$

$$= a^{\alpha_3-\alpha_2-\alpha_1}\frac{\partial^2 \psi}{\partial y \partial t} + a^{2\alpha_3-2\alpha_2-\alpha_1}\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - a^{2\alpha_3-2\alpha_2-\alpha_1}\frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} \qquad (3.1.3)$$

$$-a^{\alpha_4}\theta T_w - a^{\alpha_3-3\alpha_2}\frac{\partial^3 \psi}{\partial y^3}$$

$$\phi_2 \equiv \frac{\partial \theta'}{\partial t'} + \frac{\partial \psi'}{\partial y'}\frac{\partial \theta'}{\partial x'} - \frac{\partial \psi'}{\partial x'}\frac{\partial \theta'}{\partial y'} + \theta'\left[\frac{\partial}{\partial t'}(\ln T_w') + \frac{\partial \psi'}{\partial y'}\frac{\partial}{\partial x'}(\ln T_w')\right] - \frac{1}{Pr}\frac{\partial^2 \theta'}{\partial y'^2}$$

$$= a^{-\alpha_1}\frac{\partial \theta}{\partial t} + a^{\alpha_3-\alpha_2-\alpha_1}\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - a^{\alpha_3-\alpha_2-\alpha_1}\frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} \qquad (3.1.4)$$

$$+\theta\left[a^{-\alpha_1}\frac{\partial}{\partial t}(\ln T_w) + a^{\alpha_3-\alpha_2-\alpha_1}\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\ln T_w)\right] - \frac{1}{Pr}a^{-2\alpha_2}\frac{\partial^2 \theta}{\partial y^2}$$

Equating the various exponents of 'a' from equations (3.1.3) and (3.1.4) leads to the following equations

$$\left.\begin{aligned} &\alpha_3 - \alpha_1 - \alpha_2 = 2\alpha_3 - \alpha_1 - 2\alpha_2 = \alpha_4 = \alpha_3 - 3\alpha_2 \\ &-\alpha_1 = \alpha_3 - \alpha_1 - \alpha_2 = -2\alpha_2 \end{aligned}\right] \qquad (3.1.5)$$

Solving the equation (3.1.5) we get the following relationship between the exponents

$\alpha_1 = 2\alpha_2$; $\alpha_3 = \alpha_2$; $\alpha_4 = -2\alpha_2$

or, $\dfrac{\alpha_1}{\alpha_2} = 2$ or, $\dfrac{\alpha_3}{\alpha_2} = 1$ or, $\dfrac{\alpha_4}{\alpha_2} = -2$

It follows that ϕ_1 and ϕ_2 are conformally invariant under the following transformation group.

$$\left.\begin{aligned}
y' &= a^{\alpha_2} y = By \\
x' &= a^{\alpha_1} x = \left(a^{\alpha_2}\right)^{\frac{\alpha_1}{\alpha_2}} x = B^2 x \\
t' &= a^{\alpha_1} t = \left(a^{\alpha_2}\right)^{\frac{\alpha_1}{\alpha_2}} t = B^2 t \\
\psi' &= a^{\alpha_3} \psi = \left(a^{\alpha_2}\right)^{\frac{\alpha_3}{\alpha_2}} \psi = B\psi \\
T_w{}' &= a^{\alpha_4} T_w = \left(a^{\alpha_2}\right)^{\frac{\alpha_4}{\alpha_2}} T_w = B^{-2} T_w \\
\theta' &= \theta
\end{aligned}\right] \qquad (3.1.6)$$

We shall now show that ϕ_1, ϕ_2 can be expressed in terms of new independent variable η (similarity variable). dependent variables F, G, I and their derivatives w.r.to η. The solution of the new system will be a particular set of invariant solutions of the original system in terms of x, y, u, v etc. The variable η is to be an absolute invariant of the subgroup of the transformation of the independent variables.

In otherworlds, η is to be a function such that $\eta(x', y', t') = \eta(x, y, t)$ where

$$\left.\begin{aligned}
x' &= B^2 x \\
y' &= By \\
t' &= B^2 t
\end{aligned}\right] \qquad (3.1.7)$$

The way of seeking absolute invariant is not well defined. From the boundary layer conceptions, it would be a good guess to assume that η might be written in terms of power of x and t.

Variable Transformation

Independent Variable Transformation:

We assume that

$$\eta = y(x+bt)^{p}. \qquad (3.1.8)$$

is an absolute invariant of group (G1), where p is a real number. Now restriction might be placed on p in order that η would be invariant under (3.1.6).

So we must have, $\eta' = y'(x'+bt')^{p} = B^{1+2p}\, y(x+bt)^{p}$

For absolute invariant we put $1+2p=0 \Rightarrow p=-\frac{1}{2}$

So, $\eta = y(x+bt)^{-\frac{1}{2}}$ is an absolute invariant.

Dependent Variable Transformation:

We now express all dependent variables in terms of η. Since there are three dependent variables, we seek three functions $g_i\,(i=1,2,3)$ which are absolutely invariant under (3.1.1).

We select

$$\left.\begin{aligned} g_1 &= \psi(x+bt)^{q} \\ g_2 &= T_w(x+bt)^{r} \\ g_3 &= \theta = G(\eta) \end{aligned}\right] \qquad (3.1.9)$$

where q, r are real numbers.

Employing expression (3.1.6) in g_i gives

$$\left.\begin{aligned} g_1 &= \psi(x+bt)^{q} = B^{-1-2q}\psi'(x'+b't')^{q} \\ g_2 &= T_w(x+bt)^{r} = B^{2-2r}T_w'(x'+bt')^{r} \\ g_3 &= \theta = \theta' = G(\eta) \end{aligned}\right] \qquad (3.1.10)$$

For constant conformally invariant we must have ,

$$\left.\begin{aligned} -1-2q=0 \Rightarrow q=-\frac{1}{2} \\ 2-2r=0 \Rightarrow r=1 \end{aligned}\right] \tag{3.1.11}$$

For invariant solutions of the equations (3.5) and (3.6) will be expressed in terms of η and the functions F, G and I

$$\left.\begin{aligned} \psi &= (x+bt)^{\frac{1}{2}} F(\eta) \\ \therefore T_w &= (x+bt)^{-1} I(\eta) \\ \theta &= G(\eta) \end{aligned}\right] \tag{3.1.12}$$

$$u = \frac{\partial \psi}{\partial y} = F_\eta$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = -\frac{1}{2}\eta(x+bt)^{-1} F_{\eta\eta}$$

$$\frac{\partial^2 \psi}{\partial y \partial t} = -\frac{1}{2} b\,\eta(x+bt)^{-1} F_{\eta\eta}$$

$$\frac{\partial \psi}{\partial y}\cdot\frac{\partial^2 \psi}{\partial x \partial y} = -\frac{1}{2}\eta(x+bt)^{-1} F_\eta F_{\eta\eta}$$

$$-v = \frac{\partial \psi}{\partial x} = \left[\frac{1}{2}F - \frac{1}{2}\eta F_\eta\right](x+bt)^{-\frac{1}{2}}$$

$$\frac{\partial^3 \psi}{\partial y^3} = (x+bt)^{-1} F_{\eta\eta\eta}$$

$$-\frac{\partial \psi}{\partial x}\cdot\frac{\partial^2 \psi}{\partial y^2} = \left[\frac{1}{2}F - \frac{1}{2}\eta F_\eta\right] F_{\eta\eta}(x+bt)^{-1}$$

$$\frac{\partial \theta}{\partial t} = G_\eta \frac{\partial \eta}{\partial t} = G_\eta\left[-\frac{1}{2}\eta b(x+bt)^{-1}\right]$$

$$\frac{\partial \theta}{\partial y} = G_\eta \frac{\partial \eta}{\partial y} = G_\eta (x+bt)^{-1}$$

$$\frac{\partial^2\theta}{\partial y^2} = (x+bt)^{-\frac{1}{2}} G_{\eta\eta}(x+bt)^{-\frac{1}{2}} = G_{\eta\eta}(x+bt)^{-1}$$

$$\frac{\partial\theta}{\partial x} = G_\eta \frac{\partial\eta}{\partial x} = G_\eta\left(-\frac{1}{2}\right)\eta(x+bt)^{-1}$$

$$\frac{\partial\psi}{\partial y}\cdot\frac{\partial\theta}{\partial x} = G_\eta \frac{\partial\eta}{\partial x} = -\frac{1}{2}\eta G_\eta F_\eta (x+bt)^{-1}$$

$$-\frac{\partial\psi}{\partial x}\cdot\frac{\partial\theta}{\partial y} = -\left[\frac{1}{2}FG_\eta - \frac{1}{2}\eta F_\eta G_\eta\right](x+bt)^{-1}$$

$$-Pr^{-1}\frac{\partial^2\theta}{\partial y^2} = -Pr^{-1}G_{\eta\eta}(x+bt)^{-1}$$

$$\theta\frac{\partial}{\partial t}(\ln T_w)\frac{\partial\theta}{\partial y} = G\left[-b-\frac{1}{2}\frac{\eta b I_\eta}{I}\right](x+bt)^{-1}$$

$$\theta\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\ln T_w) = GF_\eta\left[-1-\frac{1}{2}\frac{\eta I_\eta}{I}\right](x+bt)^{-1}$$

Substituting all values in the equations (3.5) and (3.6), we obtain,

$$F_{\eta\eta\eta} + \left[\frac{1}{2}F + \frac{b}{2}\eta\right]F_{\eta\eta} + GI = 0 \qquad (3.1.13)$$

$$Pr^{-1}G_{\eta\eta} + \left(\frac{1}{2}F + \frac{b}{2}\eta\right)G_\eta + \left(\frac{1}{2}\frac{\eta b I_\eta}{I} + b + F_\eta + \frac{1}{2}\frac{\eta I_\eta F_\eta}{I}\right)G = 0 \qquad (3.1.14)$$

$$T_w(x,t) = (x+bt)^{-1}I(\eta) = \omega(x,t)I(\eta)$$

where $\omega(x,t) = (x+bt)^{-1}$

Since $\omega(x,t)$ and $T_w(x,t)$ are independent of y where η depends on y it follows that I must be equal to a constant.

For simplicity, let $I(\eta) = 1 \Rightarrow I'(\eta) = 0$

Now, (3.1.13) and (3.1.14) become

$$F_{\eta\eta\eta} + \left(\frac{1}{2}F + \frac{b}{2}\eta\right)F_{\eta\eta} + G = 0 \tag{3.1.15}$$

$$Pr^{-1}G_{\eta\eta} + \left(\frac{1}{2}F + \frac{b}{2}\eta\right)G_{\eta} + (b + F_{\eta})G = 0 \tag{3.1.16}$$

The boundary conditions are

$$\begin{aligned} &F_{\eta}(0) = 0, F(0) = \text{Fw} \neq 0, G(0) = 1 \text{ at } \eta = 0 \\ &F_{\eta}(\infty) = 0, G(\infty) = 0 \text{ at } \eta \to \infty \end{aligned} \tag{3.1.17}$$

The additional parameter is given in the boundary condition as $F(0) = \text{Fw}$ related to the suction v_w.

Here $-v_w(x,t) = \frac{1}{2}(x+bt)^{-1/2}F(0)$.

$v_w < 0$ signifies suction , $v_w > 0$ signifies injection.

The boundary- layer characteristics for this case are

(i) Vertical velocity

$u = F_{\eta}$

(ii) Horizontal velocity

$v = \frac{1}{2}(x+bt)^{-1/2}(F - \eta F_{\eta})$

(iii) Surface heat flux

$q = \frac{1}{(x+bt)^{3/2}}\left[-G'(0)\right]$

3.1.2 Case II: $T_w \propto (x/t^2)$ and suction velocity $\propto (1/\sqrt{t})$

Unsteady free convection with surface temperature varying directly with a function x and inversely with the square of a function of t and suction velocity varying inversely with the square root of a function of t

Finding the similarity solutions of the equations (3.5) and (3.6) are equivalent to determine the invariant solutions of these equations under a particular continuous one-parameter group.

In order to seek invariant solution to this set of PDE we search a transformation group from one parameter transformation defined by the following group (G2):

$$\left.\begin{aligned} x' &= a^{\alpha_1} x \\ y' &= a^{\alpha_2} y \\ t' &= a^{\alpha_1} t \\ \psi' &= a^{\alpha_3} \psi \\ T_w' &= a^{\alpha_4} T_w \\ \theta' &= \theta \end{aligned}\right] \qquad (3.2.1)$$

Here $a \neq 0$ is the parameter of the group and α 's the arbitrary real numbers whose interrelationship will be determined by the subsequent analysis. We now investigate the relationship among the exponent α 's such that

$$\begin{aligned} &\phi_j\left(x', y', t', u', v', \ldots\ldots, \frac{\partial^3 \psi'}{\partial y'^3}\right) \\ &= H_j\left(x, y, t, u, v, \ldots\ldots, \frac{\partial^3 \psi}{\partial y^3}; a\right)\phi_j\left(x, y, t, u, v, \ldots\ldots, \frac{\partial^3 \psi}{\partial y^3}\right) \end{aligned} \qquad (3.2.2)$$

for this is the requirement that the differential forms ϕ_1 , ϕ_2 be conformally invariant under the transformation group (3.2.1). Substituting the transformations (3.2.1) in (3.5) and (3.6) we have

$$\phi_1 \equiv \frac{\partial^2\psi'}{\partial y'\partial t'} + \frac{\partial\psi'}{\partial y'}\frac{\partial^2\psi'}{\partial y'\partial x'} - \frac{\partial\psi'}{\partial x'}\frac{\partial^2\psi'}{\partial y'^2} - \theta' T'_w - \frac{\partial^3\psi'}{\partial y'^3} = 0$$

$$= a^{\alpha_3-\alpha_2-\alpha_1}\frac{\partial^2\psi}{\partial y\partial t} + a^{2\alpha_3-2\alpha_2-\alpha_1}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x} - a^{2\alpha_3-2\alpha_2-\alpha_1}\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} \qquad (3.2.3)$$

$$-a^{\alpha_4}\theta\, T_w - a^{\alpha_3-3\alpha_2}\frac{\partial^3\psi}{\partial y^3}$$

$$\phi_2 \equiv \frac{\partial\theta'}{\partial t'} + \frac{\partial\psi'}{\partial y'}\frac{\partial\theta'}{\partial x'} - \frac{\partial\psi'}{\partial x'}\frac{\partial\theta'}{\partial y'} + \theta'\left[\frac{\partial}{\partial t'}(\ln T'_w) + \frac{\partial\psi'}{\partial y'}\frac{\partial}{\partial x'}(\ln T'_w)\right] - \frac{1}{Pr}\frac{\partial^2\theta'}{\partial y'^2}$$

$$= a^{-\alpha_1}\frac{\partial\theta}{\partial t} + a^{\alpha_3-\alpha_2-\alpha_1}\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - a^{\alpha_3-\alpha_2-\alpha_1}\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} \qquad (3.2.4)$$

$$+\theta\left[a^{-\alpha_1}\frac{\partial}{\partial t}(\ln T_w) + a^{\alpha_3-\alpha_2-\alpha_1}\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\ln T_w)\right] - \frac{1}{Pr}a^{-2\alpha_2}\frac{\partial^2\theta}{\partial y^2}$$

Equating the various exponents of 'a' in equations (3.2.3) and (3.2.4) leads to the following equations

$$\left.\begin{aligned} &\alpha_3-\alpha_1-\alpha_2 = 2\alpha_3-\alpha_1-2\alpha_2 = \alpha_4 = \alpha_3-3\alpha_2 \\ &-\alpha_1 = \alpha_3-\alpha_1-\alpha_2 = -2\alpha_2 \end{aligned}\right] \qquad (3.2.5)$$

Solving the equation (3.2.5) we get the following relationship between the exponents

$\alpha_1 = 2\alpha_2$: $\alpha_3 = \alpha_2$: $\alpha_4 = -2\alpha_2$

or, $\dfrac{\alpha_1}{\alpha_2} = 2$ or, $\dfrac{\alpha_3}{\alpha_2} = 1$ or, $\dfrac{\alpha_4}{\alpha_2} = -2$

It follows that ϕ_1 and ϕ_2 are conformally invariant under the following transformation group.

$$\left.\begin{aligned}
y' &= a^{\alpha_2} y = By \\
x' &= a^{\alpha_1} x = \left(a^{\alpha_2}\right)^{\frac{\alpha_1}{\alpha_2}} x = B^2 x \\
t' &= a^{\alpha_1} t = \left(a^{\alpha_2}\right)^{\frac{\alpha_1}{\alpha_2}} t = B^2 t \\
\psi' &= a^{\alpha_3}\psi = \left(a^{\alpha_2}\right)^{\frac{\alpha_3}{\alpha_2}} \psi = B\psi \\
T_w' &= a^{\alpha_4} T_w = \left(a^{\alpha_2}\right)^{\frac{\alpha_4}{\alpha_2}} T_w = B^{-2} T_w \\
\theta' &= \theta
\end{aligned}\right] \qquad (3.2.6)$$

We shall now show that ϕ_1, ϕ_2 can be expressed in terms of new independent variable η (similarity variable), dependent variables F, G, I and their derivatives w.r.to η. The solution of the new system will be a particular set of invariant solutions of the original system in terms of x, y, u, v etc. The variable η is to be an absolute invariant of the subgroup of the transformation of the independent variables.

In otherworlds, η is to be a function such that $\eta(y', t') = \eta(y, t)$, Where

$$\left.\begin{aligned}
y' &= By \\
t' &= B^2 t
\end{aligned}\right] \qquad (3.2.7)$$

Variable Transformation

Independent Variable Transformation

We assume that

$$\eta = yt^{p} \tag{3.2.8}$$

is an absolute invariant of group (G2) where p is real number. Now restriction might be placed on p in order that η would be invariant under (3.2.6).

So we must have, $\eta' = y't'^{p} = B^{1+2p} yt^{p}$

For absolute invariant we put $1+2p = 0 \Rightarrow p = -\frac{1}{2}$

So, $\eta = yt^{-\frac{1}{2}}$ is an absolute invariant.

Dependent Variable Transformation:

We now express all dependent variables in terms of η. We select the dependent variables for ψ, T_w and θ in such a way that

$$\left.\begin{aligned} g_1 &= \psi\left(\frac{x^2}{t}\right)^{q} = B^{-1-2q}\psi'\left(\frac{x'^2}{t'}\right)^{q} = F(\eta) \\ g_2 &= T_w\left(\frac{t^2}{x}\right)^{r} = B^{-2-2r}T'_w\left(\frac{t'^2}{x'}\right)^{r} = I(\eta) \\ g_3 &= \theta = \theta' = G(\eta) \end{aligned}\right] \tag{3.2.9}$$

where q, r are real numbers. For invariant of transformation group we must have to put

$$\left.\begin{aligned} -1-2q = 0 &\Rightarrow q = -\frac{1}{2} \\ 2-2r = 0 &\Rightarrow r = 1 \end{aligned}\right] \tag{3.2.10}$$

Now,

$$\left.\begin{aligned} \psi &= \left(\frac{x^2}{t}\right)^{\frac{1}{2}} F(\eta) = \frac{x}{t^{1/2}} F(\eta) \\ \therefore T_w &= \left(\frac{x}{t^2}\right) I(\eta) \\ \theta &= G(\eta) \end{aligned}\right] \tag{3.2.11}$$

Here, $u = \dfrac{\partial \psi}{\partial y} = \left(\dfrac{x}{t}\right) F_\eta$

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{t} F_\eta\right) = \frac{1}{t} F_\eta$$

$$\frac{\partial^2 \psi}{\partial y \partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial y}\right) = -\frac{x}{t^2} F_\eta - \frac{1}{2} \cdot \frac{x}{t^2} \eta F_{\eta\eta}$$

$$\frac{\partial \psi}{\partial y} \cdot \frac{\partial^2 \psi}{\partial x \partial y} = \left(\frac{x}{t} F_\eta\right)\left(\frac{1}{t} F_\eta\right) = \frac{x}{t^2} F_\eta^2$$

$$-v = \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{t^{1/2}} F\right) = \frac{1}{t^{1/2}} F$$

$$\frac{\partial^3 \psi}{\partial y^3} = \left(\frac{x}{t^2}\right) F_{\eta\eta\eta}$$

$$\frac{\partial \psi}{\partial x} \cdot \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{t^{1/2}} F \cdot \frac{x}{t} F_{\eta\eta} t^{-1/2} = \frac{x}{t^2} F F_{\eta\eta}$$

$$\theta T_w = \left(\frac{x}{t^2}\right) I(\eta) G(\eta)$$

$$\frac{\partial \theta}{\partial t} = G_\eta \frac{\partial \eta}{\partial t} = \frac{1}{t} G_\eta \left(-\frac{1}{2}\right) \eta$$

$$\frac{\partial \theta}{\partial y} = G_\eta \frac{\partial \eta}{\partial y} = G_\eta \, t^{-1/2}$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{1}{t} G_{\eta\eta}$$

$$\frac{\partial\theta}{\partial x} = 0$$

$$\frac{\partial\psi}{\partial y}\cdot\frac{\partial\theta}{\partial x} = 0$$

$$\frac{\partial\psi}{\partial x}\cdot\frac{\partial\theta}{\partial y} = \frac{1}{t} F G_{\eta}$$

$$Pr^{-1}\frac{\partial^2\theta}{\partial y^2} = Pr^{-1}\cdot\frac{1}{t} G_{\eta\eta}$$

$$\theta\frac{\partial}{\partial t}\left(\ln T_w\right) = \theta\left[\frac{1}{T_w}\frac{\partial T_w}{\partial t}\right] = G\left[\frac{t^2}{xI}\frac{\partial}{\partial t}\left(\frac{x}{t^2} I\right)\right]$$

$$= G\left[\frac{t^2}{xI}\left\{\frac{x}{t^2} I_\eta \frac{\partial\eta}{\partial t} - 2I.x.t^{-3}\right\}\right]$$

$$= G\left[\frac{t^2}{xI}\left\{\frac{x}{t^2} I_\eta \left(-\frac{1}{2}\right)\eta t^{-1} - 2I.x.t^{-3}\right\}\right]$$

$$= G\left[-2 - \frac{1}{2}\frac{\eta I_\eta}{I}\right]\frac{1}{t}$$

$$= \left(-2G - \frac{1}{2}\frac{\eta I_\eta G}{I}\right)\frac{1}{t}$$

$$\theta \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x}(\ln T_w)$$

$$= \theta \frac{\partial \psi}{\partial y}\left[\frac{1}{T_w} \frac{\partial T_w}{\partial x}\right]$$

$$= G \frac{x}{t} F_\eta \left[\frac{t^2}{xI} \frac{\partial}{\partial x}\left(\frac{xI}{t^2}\right)\right]$$

$$= G \frac{x}{t} F_\eta \left[\frac{t^2}{xI}\left\{\frac{x}{t^2} I_\eta \frac{\partial \eta}{\partial x} + I.\frac{1}{t^2}\right\}\right]$$

$$= G \frac{x}{t} F_\eta \left[\frac{t^2}{xI}\left\{\frac{x}{t^2} I_\eta .0 + I.\frac{1}{t^2}\right\}\right]$$

$$= G F_\eta I.\frac{1}{t}$$

Substituting all values in the equations (3.5) and (3.6)

we obtain,

$$F_{\eta\eta\eta} + \left(\frac{1}{2}\eta + F\right)F_{\eta\eta} + F_\eta - F_\eta^2 + GI = 0 \qquad (3.2.12)$$

$$Pr^{-1} G_{\eta\eta} + \left(F + \frac{1}{2}\eta\right)G_\eta + (2 - F_\eta I)G + \frac{1}{2} \cdot \frac{\eta I_\eta G}{I} = 0 \qquad (3.2.13)$$

$$T_w(x,t) = \left(\frac{x}{t^2}\right) I(\eta)$$

Since $\left(\frac{x}{t^2}\right)$ and $T_w(x,t)$ are independent of y where as η depends on y

,it follows that $I(\eta)$ must be equal to a constant.

For simplicity, let $I(\eta) = 1 \Rightarrow I'(\eta) = 0$

Now, (3.1.12) and (3.2.13) become

$$F_{\eta\eta\eta} + \left(\frac{1}{2}\eta + F\right)F_{\eta\eta} + F_\eta - F_\eta^2 + G = 0 \qquad (3.2.14)$$

$$Pr^{-1} G_{\eta\eta} + \left(F + \frac{1}{2}\eta\right)G_\eta + (2 - F_\eta)G = 0 \qquad (3.2.15)$$

The boundary conditions are

$$F_\eta(0)=0, F(0)=\mathrm{Fw}\neq 0, G(0)=1 \text{ at } \eta=0$$
$$F_\eta(\infty)=0, G(\infty)=0 \text{ at } \eta\to\infty \qquad (3.2.16)$$

The additional parameter is given in the boundary condition as $F(0)=\mathrm{Fw}$ related to the suction v_w when $\eta=0$

Here $-v_w(x,t)=t^{-1/2}F(0)$

$v_w<0$ signifies suction, $v_w>0$ signifies injection.

The boundary- layer characteristics for this case are

(i)Vertical velocity

$$u=\frac{x}{t}F_\eta$$

(ii) Horizontal velocity

$$v=-\frac{1}{t^{1/2}}F$$

(iii) Surface heat flux

$$q=\frac{x}{t^{3/2}}\left[-G'(0)\right]$$

And for wall shear stress, $\tau_w=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$

Defining skin friction coefficient

$$c_f=\frac{2\tau_w}{\rho U^2}$$

$$\Rightarrow \frac{1}{2}c_f Gr\left(\frac{\sqrt{t}}{x}\right)=F_{\eta\eta}(0)$$

3.1.3 Case III: $T_w \propto t^r$ and suction velocity variation is zero

Unsteady Free Convection with Uniform but Unsteady surface Temperature variation at Large Distance x

In order to seek invariant solution, we search a transformation group (G3) from one parameter transformation defined by

$$x' = a^{\alpha_1 n} x$$
$$y' = a^{\alpha_2 n} y$$
$$t' = a^{\alpha_3 n} t$$
$$\psi' = a^{\alpha_4 n} \psi$$
$$T'_w = a^{\alpha_5} T_w$$
$$\theta' = \theta$$

Here $a \neq 0$ is the parameter of the group and α 's are the arbitrary real numbers.

Substituting the above transformations in equations (3.5) and (3.6) we have

$$\phi_1 \equiv \frac{\partial^2 \psi'}{\partial y' \partial t'} + \frac{\partial \psi'}{\partial y'}\frac{\partial^2 \psi'}{\partial y' \partial x'} - \frac{\partial \psi'}{\partial x'}\frac{\partial^2 \psi'}{\partial y'^2} - \theta' T'_w - \frac{\partial^3 \psi'}{\partial y'^3} = 0$$

$$= a^{\alpha_4 n - \alpha_2 n - \alpha_3 n}\frac{\partial^2 \psi}{\partial y \partial t} + a^{2\alpha_4 n - 2\alpha_2 n - \alpha_1 n}\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - a^{2\alpha_4 n - 2\alpha_2 n - \alpha_1 n}\frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2}$$

$$- a^{\alpha_5}\theta T_w - a^{\alpha_4 n - 3\alpha_2 n}\frac{\partial^3 \psi}{\partial y^3}.$$

$$\phi_2 \equiv \frac{\partial \theta'}{\partial t'} + \frac{\partial \psi'}{\partial y'}\frac{\partial \theta'}{\partial x'} - \frac{\partial \psi'}{\partial x'}\frac{\partial \theta'}{\partial y'} + \theta'\left[\frac{\partial}{\partial t'}\left(\ln T'_w\right) + \frac{\partial \psi'}{\partial y'}\frac{\partial}{\partial x'}\left(\ln T'_w\right)\right] - \frac{1}{Pr}\frac{\partial^2 \theta'}{\partial y'^2}$$

$$= a^{-\alpha_3 n}\frac{\partial \theta}{\partial t} + a^{\alpha_4 n - \alpha_2 n - \alpha_1 n}\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - a^{\alpha_4 n - \alpha_2 n - \alpha_1 n}\frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y}$$

$$+ \theta\left[a^{-\alpha_3 n}\frac{\partial}{\partial t}\left(\ln T_w\right) + a^{\alpha_4 n - \alpha_2 n - \alpha_1 n}\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\ln T_w\right)\right] - \frac{1}{Pr}a^{-2\alpha_2 n}\frac{\partial^2 \theta}{\partial y^2}.$$

For conformal invariant we have the following set of equations

$$\alpha_4 n - \alpha_2 n - \alpha_3 n = 2\alpha_4 n - 2\alpha_2 n - \alpha_1 n = \alpha_5 = \alpha_4 n - 3\alpha_2 n$$
$$-\alpha_3 n = \alpha_4 n - \alpha_2 n - \alpha_1 n = -2\alpha_2 n$$

We have the following relationship between the exponents

$$\frac{\alpha_1}{\alpha_2} = 2 \qquad , \frac{\alpha_3}{\alpha_2} = 2 \qquad , \frac{\alpha_4}{\alpha_2} = 1 \qquad , \frac{\alpha_5}{\alpha_2} = -2n$$

ϕ_1 and ϕ_2 are conformally invariant under the following transformation group.

$$y' = a^{\alpha_2 n} y = By \quad , \text{ where } \; a^{\alpha_2 n} = B$$

$$x' = a^{\alpha_1 n} x = \left(a^{\alpha_2 n}\right)^{\frac{\alpha_1}{\alpha_2}} x = B^2 x$$

$$t' = a^{\alpha_3 n} t = \left(a^{\alpha_2 n}\right)^{\frac{\alpha_3}{\alpha_2}} t = B^2 t$$

$$\psi' = a^{\alpha_4 n} \psi = \left(a^{\alpha_2 n}\right)^{\frac{\alpha_4}{\alpha_2}} \psi = B\psi$$

$$T_w' = a^{\alpha_5 n} T_w = \left(a^{\alpha_2 n}\right)^{\frac{\alpha_5}{\alpha_2}} x = B^{-2n} T_w$$

Variable Transformation

Independent Variable Transformation:

Let $\eta = yt^p$ is an absolute invariant of group (G3), where p is a real number.

$$\eta' = y' t'^p = B^{1+2p} yt^p$$

For absolute invariant we put $1 + 2p = 0 \Rightarrow p = -\frac{1}{2}$

So, $\eta = yt^{-\frac{1}{2}}$

Dependent Variable Transformation:

We now express all dependent variables in terms of η. We select the dependent variables for ψ, T_w and θ in such a way that

$$g_1 = t^q T_w = B^{-2q+2n} t'^q T_w'$$

$$g_2 = t^s \psi = B^{-2s-1} t'^s \psi'$$

$$g_3 = \theta = \theta' = G(\eta)$$

where q, s are real numbers.

For constant conformally invariant we must have ,

$$-2q + 2n = 0 \Rightarrow q = n$$

$$-2s - 1 = 0 \Rightarrow s = -\frac{1}{2}$$

We must have,

$$g_1 = t^n T_w$$

$$g_2 = t^{-1/2} \psi$$

$$g_3 = \theta = \theta' = G(\eta)$$

Writing $-\frac{1}{2} = -\left(r + \frac{3}{2}\right)$ and $n = -r$

We have

$$g_1 = t^{-r} T_w$$

$$g_2 = t^{-\left(r+\frac{3}{2}\right)} \psi$$

$$g_3 = \theta = \theta' = G(\eta)$$

Therefore,

$$T_w = t^r I(\eta)$$

$$\psi = t^{\left(r+\frac{3}{2}\right)} F(\eta)$$

$$\theta = G(\eta)$$

Now, $u = \frac{\partial \psi}{\partial y} = t^{r+1} F_\eta$

$$-v_w = \frac{\partial \psi}{\partial x} = 0$$

i,e v_w variation is zero.

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial x}\left(t^{r+1} F(\eta)\right) = 0$$

$$\frac{\partial^2 \psi}{\partial y \partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial t}\left(t^{r+1} F(\eta)\right)$$

$$= t^{r+1} F_{\eta\eta}\left(-\frac{1}{2}\right)\frac{\eta}{t} + F_\eta (r+1) t^r$$

$$\frac{\partial \psi}{\partial y} \cdot \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

$$\frac{\partial \psi}{\partial x} \cdot \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\frac{\partial^3 \psi}{\partial y^3} = t^r F_{\eta\eta\eta}$$

$$\theta T_w = G t^r I(\eta)$$

$$\frac{\partial \theta}{\partial y} = G_\eta \frac{\partial \eta}{\partial y} = G_\eta t^{-\frac{1}{2}}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{G_{\eta\eta}}{t}$$

$$\frac{\partial \theta}{\partial x} = G_\eta \frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial \psi}{\partial y} \cdot \frac{\partial \theta}{\partial x} = 0$$

$$\frac{\partial \psi}{\partial x} \cdot \frac{\partial \theta}{\partial y} = 0$$

$$\frac{\partial\theta}{\partial t} = G_\eta \frac{\partial\eta}{\partial t} = G_\eta\left(-\frac{1}{2}\right)\frac{\eta}{t}$$

$$P_r^{-1}\frac{\partial^2\theta}{\partial y^2} = P_r^{-1}\frac{G_{\eta\eta}}{t}$$

$$\frac{\partial}{\partial x}(\ln T_w) = 0$$

$$\theta\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\ln T_w) = 0$$

$$\frac{\partial}{\partial t}(\ln T_w) = \frac{r}{t} + t^r I_\eta\left(-\frac{1}{2}\right)\frac{\eta}{t}$$

$$\theta\frac{\partial}{\partial t}(\ln T_w) = G\left[\frac{r}{t} + t^r I_\eta\left(-\frac{1}{2}\right)\frac{\eta}{t}\right]$$

Substituting all values in the equations (3.5) and (3.6) we obtain,

$$F_{\eta\eta\eta} + \frac{1}{2}\eta F_{\eta\eta} - (r+1)F_\eta + GI = 0$$

$$P_r^{-1}G_{\eta\eta} + \frac{1}{2}\eta G_\eta - r\,G + \frac{1}{2}t^r I_\eta \eta G = 0$$

For simplicity, let I = constant=1 $\Rightarrow I_\eta = 0$

We get

$$F_{\eta\eta\eta} + \frac{1}{2}\eta F_{\eta\eta} - (r+1)F_\eta + G = 0$$

$$P_r^{-1}G_{\eta\eta} + \frac{1}{2}\eta G_\eta - r\,G = 0$$

The boundary conditions are

$F_\eta(0) = F(0) = 0 \quad G(0) = 1$ at $\eta = 0$

$F_\eta(\infty) = 0,\ G(\infty) = 0$ at $\eta \to \infty$

The boundary- layer characteristics for this case are

(i) Vertical velocity

$$u = t^{r+1} F_\eta$$

(ii) Horizontal velocity

$$v = 0$$

(iii) Surface heat flux

$$q = t^{\left(\frac{r-1}{2}\right)}\left[-G'(0)\right]$$

The situation is valid at large distance x. Therefore all partial derivatives w.r. to x will be neglected. Since $T_w = t^r I(\eta)$, surface temperature distribution is independent of x, i.e uniform. It is a function of time t. The surface temperature may increase or decrease with time according to r being positive or negative respectively.

Chapter 4

4.1 Results and Discussion

Figure 2 shows that the velocity decreases with increasing suction parameter, Fw. It can also be seen that at each value of Fw , there exists a local maximum value of the velocity profile in the boundary layer region. The maximum values are obtained as 0.540, 0.405, 0.302 at $\eta = 1.778$ and as 0.109, 0.062, 0.046 at η=1.238, 0.347, 0.242 respectively.

For the temperature profile, Figure 3 indicates the occurrence of the decrease in temperature, G near the plate for suction. Also, the thermal boundary layer thickness increases with increasing Fw. It is evident that temperature does not exhibit any defect corresponding to Fw = 2.0 and Fw = 4.0. The temperature decreases slowly near the plate for Fw = 0.1 than Fw = 0, Fw = 0.5 than Fw = 0.2 and so on. Figure 4 indicates that skin friction decreases with increasing suction parameter, Fw. From Figure 5, it is observed that the rate of heat transfer increases with increasing suction parameter, Fw. Figure 6 shows that the velocity increases in the vicinity of the plate and then decreases far from the plate for increasing Prandtl number. The temperature profile in the Figure 7 shows that G becomes negative in a certain region of the boundary layer for varying values of Pr. This phenomenon is known as temperature defect that will vanish for the limiting case $Pr \to \infty$. Physically it is true because the increase in the Prandtl number is due to increase in the viscosity of the fluid. So, skin friction decreases with increasing Pr. The Fig. 8 shows that the skin friction decreases with increasing Pr. In this figure it is seen that skin friction decreases rapidly for Fw = 0.5 than Fw = 0.2.

Figure 9 represents the effect of Prandtl number and suction parameter on the surface heat flux represented by $-G'(0)$. The rate of heat transfer increases with increasing Prandtl number, Pr. The rate of heat transfer increases rapidly for Fw = 0.5 than Fw = 0.2, i.e., at a given Pr increase in the suction results in an increase in heat transfer.

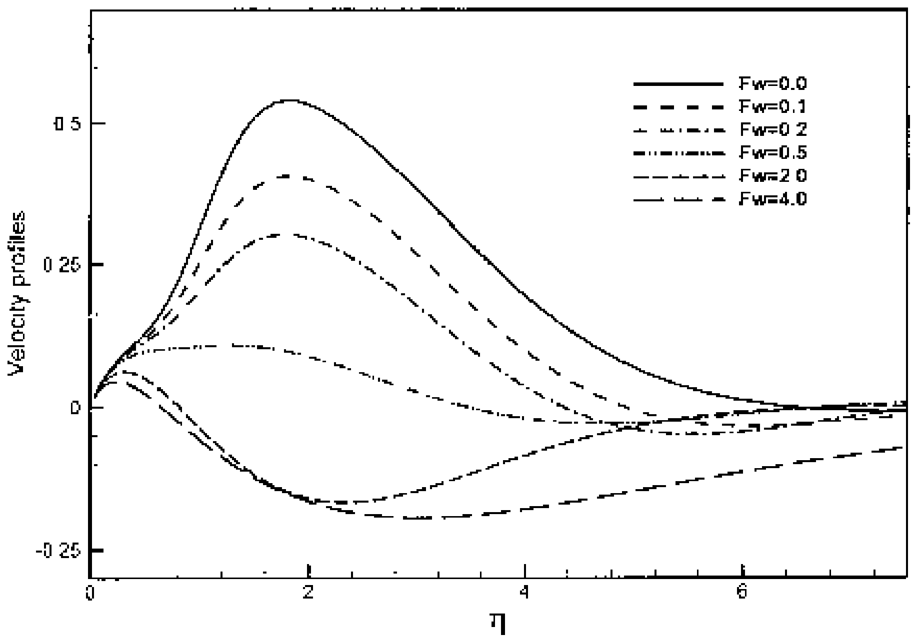

Fig. 2: Velocity profiles for fixed Prandtl number, $Pr=10$ and varying values of Fw.

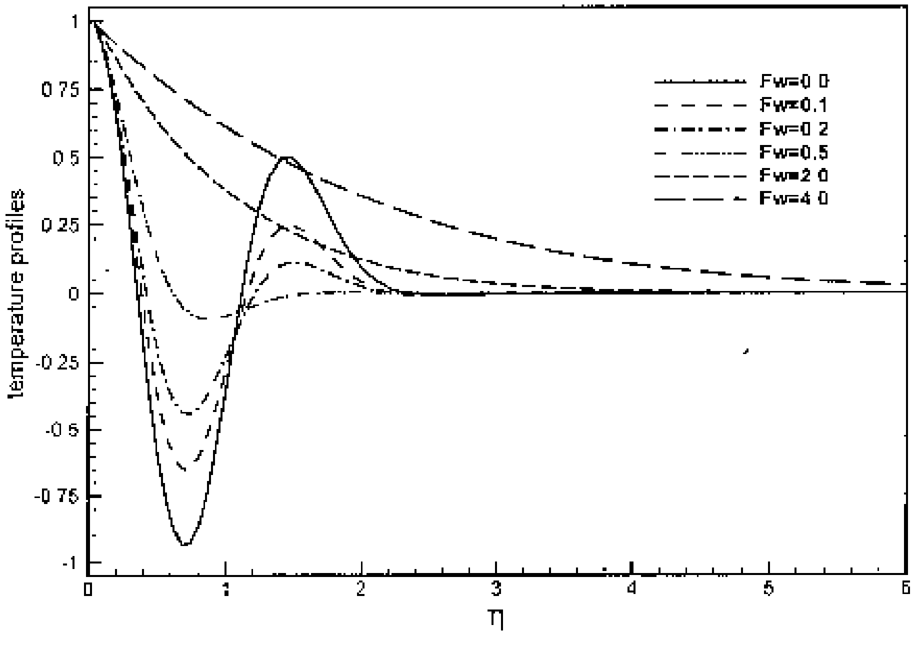

Fig. 3: Temperature profiles for fixed Prandtl number, $Pr=10$ and varying values of Fw.

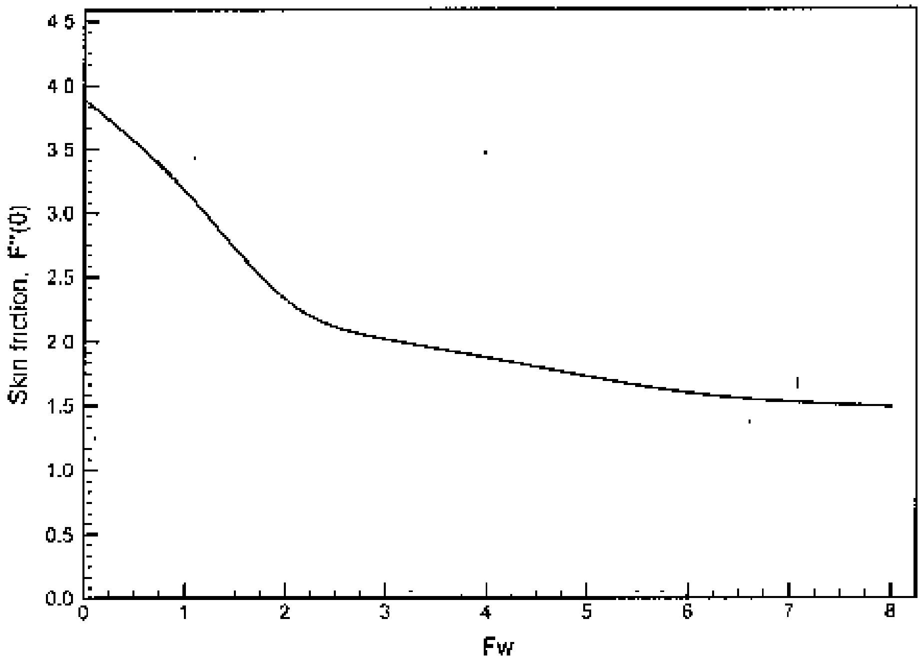

Fig. 4: Effect of Fw on skin friction factor for fixed Prandtl - number, $Pr = 10$.

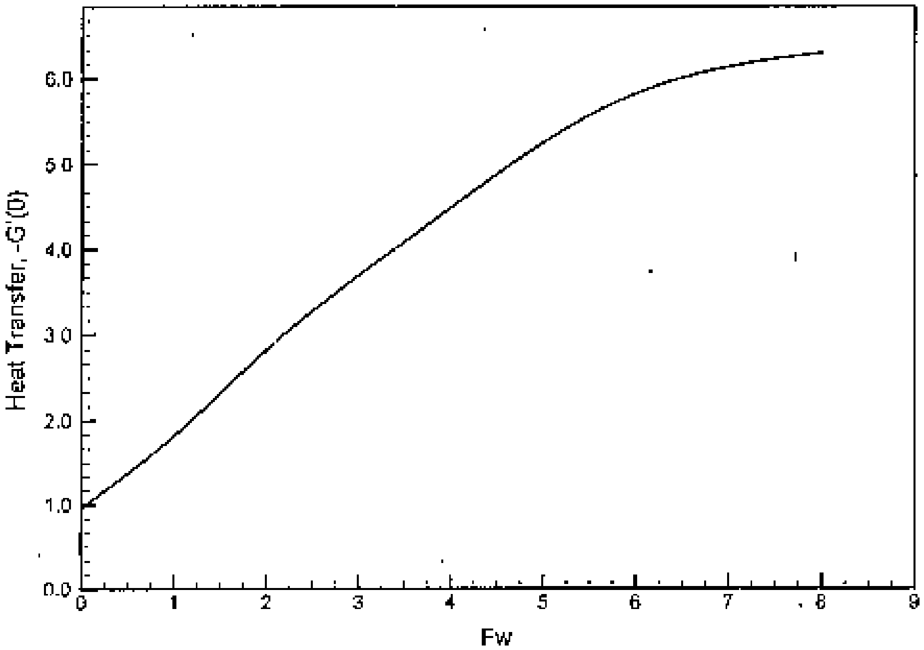

Fig. 5: Effect of Fw on heat transfer factor for fixed Prandtl number, $Pr = 10$.

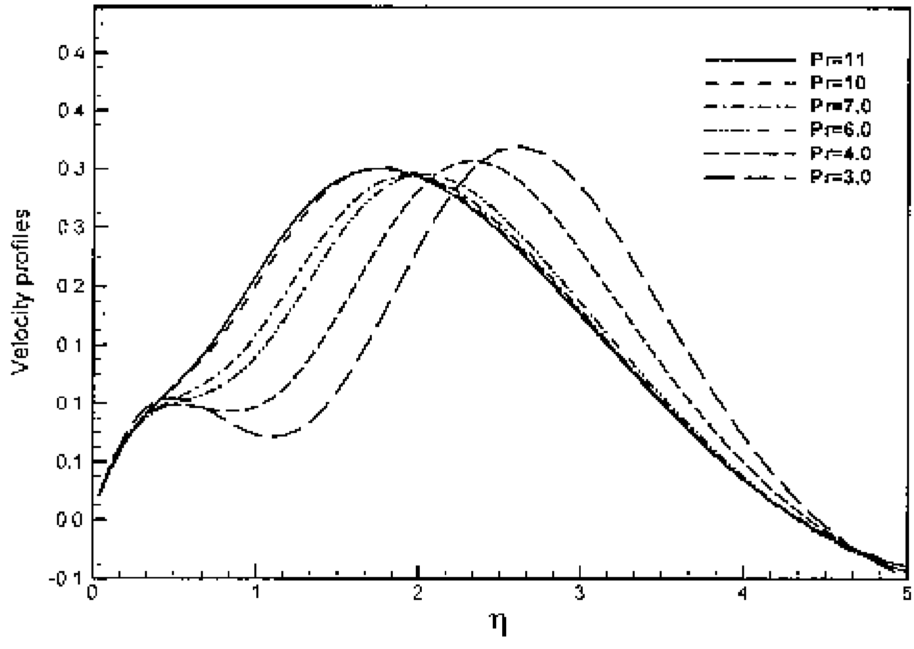

Fig. 6: Velocity profiles for fixed Fw = 0.2 and varying values of Prandtl number ,Pr

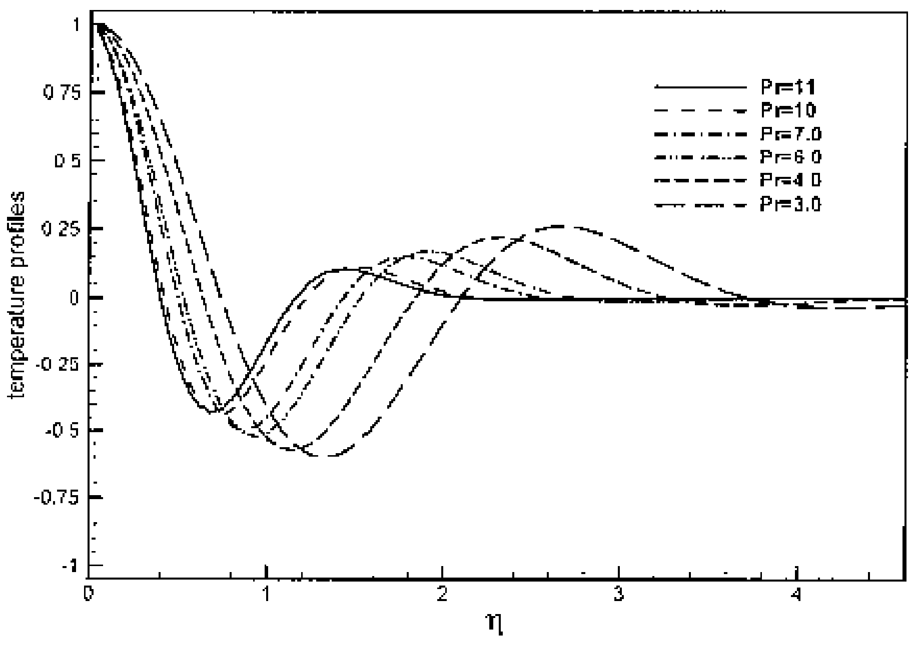

Fig. 7: Temperature profiles for fixed Fw = 0.2 and varying values of Prandtl number ,Pr

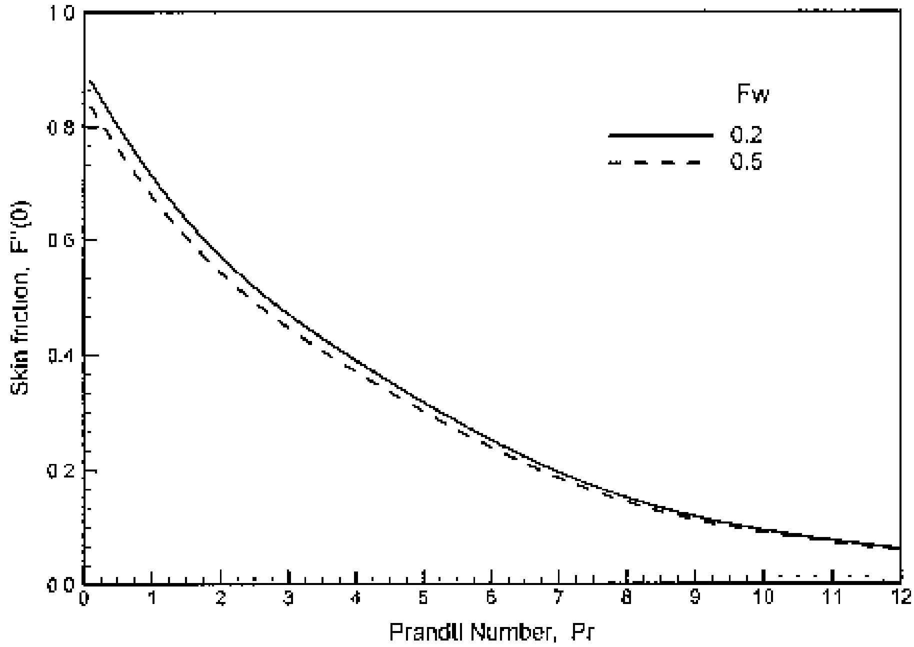

Fig. 8: Effect of Prandtl number, Pr on skin friction factor for fixed Fw.

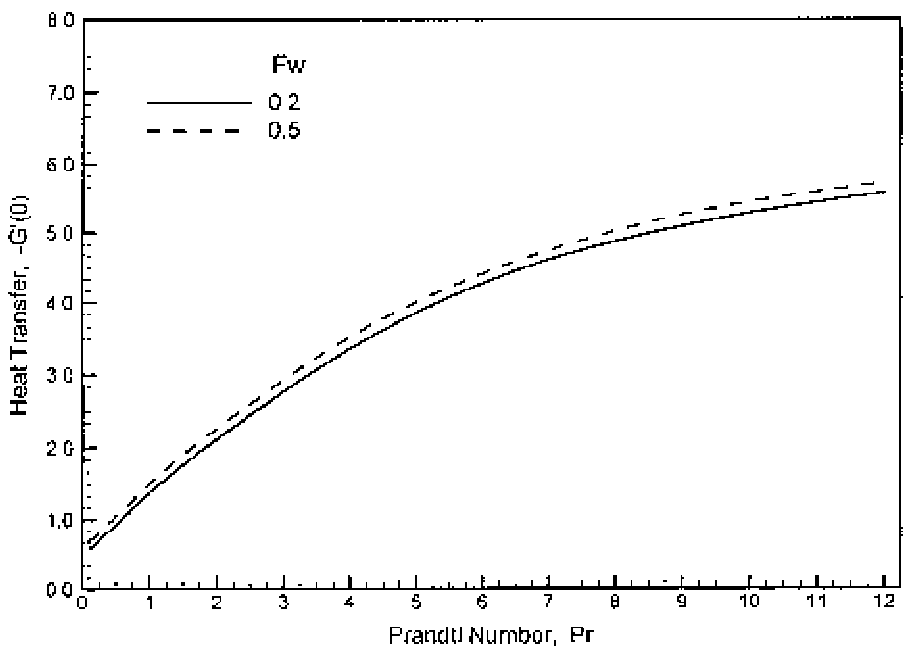

Fig. 9: Effect of Prandtl number, *Pr* on heat transfer factor for fixed Fw.

Numerical results of $F''(0)$ and $-G'(0)$ are presented in the following Table 1, Table 2 and Table 3

Table1: Numerical values of shear stress, $F''(0)$ and the rate of heat transfer, $-G'(0)$ for different values of suction parameter, Fw while $Pr = 7.2$

Fw	$F''(0)$	$-G'(0)$
0	3.88345	0.97461
1	3.16530	1.83355
2	2.31261	2.82467
3	2.00563	3.68462
4	1.85072	4.47856
5	1.71367	5.26384
6	1.58603	5.81176
7	1.53174	6.04768
8	1.45846	6.28258
9	1.41117	6.51869

Table2: Numerical values of shear stress $F''(0)$ and the rate of heat transfer, $-G'(0)$ for different values of Prandtl number, Pr while Fw = 0.4

Pr	$F''(0)$	$-G'(0)$
0.1	0.87050	0.60055
0.7	0.75411	1.12784
1	0.70200	1.38759
2	0.56120	2.12758
4	0.37823	3.36781
6	0.24141	4.28272
7	0.19056	4.59690
8	0.14140	4.88185
10	0.11542	5.24689
12	0.05145	5.58191

Table3: Numerical values of shear stress $F''(0)$ and the rate of heat transfer, $-G'(0)$ for different values of Prandtl number, Pr while Fw = 0.6

Pr	$F''(0)$	$-G'(0)$
0.1	0.826579	0.703205
0.7	0.715988	1.242018
1	0.666493	1.515695
2	0.531272	2.269813
4	0.359833	3.535938
6	0.218922	4.418237
7	0.181469	4.725248
8	0.133919	5.030259
10	0.091173	5.397764
12	0.048421	5.755267

4.2 Conclusion

A successful method of obtaining similarity solutions from partial differential equations is the method of group theory. Here boundary conditions are not taken into account for choosing various functions. So this method is very easy and simple to apply. An analysis is made of the two-dimensional unsteady natural convection boundary layer flow with suction on vertical plate for establishing the conditions under which similarity solutions are possible. Three possible cases have been derived on the basis of these conditions. The case "Unsteady free convection with surface temperature varying directly with function of x and inversely with the square of a function of t and suction velocity varying inversely with the square root of a function of t" has been studied numerically. The velocity and temperature distributions are presented for finite values of suction parameter and Prandtl number. It is shown that skin friction decreases with increasing suction parameter and increases in the suction results in an increases in heat transfer .The effect of Prandtl number on the surface heat flux and skin friction is also found here. The skin friction decreases with increasing Prandtl number and the rate of heat transfer increases with increasing Prandtl number.

4.3 Future work

1. Future work should focus on the application of two-parameter group transformations to reduce the system of governing partial differential equations with boundary conditions to ordinary differential equations with appropriate boundary conditions for the problem.
2. An analysis should carry out to study the effect of magnetic field for the problem, which is applied normal to the surface.
3. Future work in this area consists of identifying the effect of suction for case I.
4. Finite difference method can be used to solve system of ordinary differential equations numerically for comparing with the results obtained by using shooting method based on sixth order Runge-Kutta scheme along with Nachtshiem-Swigert iteration technique.

Appendix

Runge-Kutta Shooting Method along with Nachtshiem-Swigert Iteration Technique

In shooting method, the missing (unspecified) initial conditions at the initial point in the interval is guessed and differential equation is then integrated numerically as an initial value problem to the terminal points by Runge-Kutta method. Calculated values are then compared with the given values at the terminal points, if there is any difference (error) found guessed values must be changed before next iteration. This process is repeated until the agreement between the calculated and the given condition at the terminal point is within the specified tolerance small quantity, ε.

$$F''' + \left(\frac{1}{2}\eta + F\right)F'' + F' - F'^2 + G = 0$$

and $$\frac{G''}{Pr} + \left(F + \frac{\eta}{2}\right)G' + (2 - F')G = 0$$

The procedure of refining the values $F'(0)$, and $G(0)$ by a shooting method based on Nachtsheim-Swigert (1965) iteration technique, where $F'(0)$. $G(0)$ stand for dimensionless velocity, and temperature respectively.

The boundary conditions associated with non-dimensional ordinary differential equations of the boundary type are of the two-points asymptotic class. Two-point boundary conditions have values of the dependent variable specified at two different values of the independent variable. Specification of an asymptotic boundary condition implies the velocities F' and G tend to unity as the independent variable tends to outer specified value. The method numerically integrates two-point asymptotic boundary value problem of the boundary layer type, the initial value method, requires that it be recast as an initial value problem. Thus it is necessary to guess as many boundary conditions at the surface as were given at infinity. The solution has been achieved ashen the required outer boundary conditions are satisfied after the integration of governing differential equations by the assumed surface conditions. If this is not satisfied, another new surface boundary condition is estimated for the next trial integration. But this is not so easy, because selecting a value may result in the divergence of the trial integration or in slow convergence of surface boundary conditions required satisfying the asymptotic outer boundary condition, Moreover selecting too large a value of the independent variable is expensive in terms of computer time. Nachtsheim-Swigert developed an iteration method, which overcomes these difficulties. In equation there are three asymptotic boundary conditions and hence two unknown surface conditions $F''=g_1$ and $G'=g_2$. Within the context of the initial value method and the Nachtsheim-Swigert iteration technique the outer boundary conditions may be functionally represented as

$$F(\eta_{max}) = F(g_1,\ g_2) = \delta_1 \quad (1.1)$$

$$G(\eta_{max}) = G(g_1, g_2) = \delta_2 \quad (1.2)$$

$$F'(\eta_{max}) = F'(g_1, g_2) = \delta_3 \quad (1.3)$$

With the asymptotic convergence criteria given by

$$F''(\eta_{max}) = F''(g_1, g_2) = \delta_4 \quad (1.4)$$

$$G'(\eta_{max}) = G(g_1,\ g_2) = \delta_5 \quad (1.5)$$

Expanding the equation (1.1)-(1.5) in a first order Taylor's series gives

$$F(\eta_{max}) = F_c(\eta_{max}) + \frac{\partial F}{\partial g_1}\Delta g_1 + \frac{\partial F}{\partial g_2}\Delta g_2 = \delta_1 \quad (1.6)$$

$$G(\eta_{max}) = G_c(\eta_{max}) + \frac{\partial G}{\partial g_1}\Delta g_1 + \frac{\partial G}{\partial g_2}\Delta g_2 = \delta_2 \quad (1.7)$$

$$F'(\eta_{max}) = F_c'(\eta_{max}) + \frac{\partial F'}{\partial g_1}\Delta g_1 + \frac{\partial F'}{\partial g_2}\Delta g_2 = \delta_3 \quad (1.8)$$

$$F''(\eta_{max}) = F_c''(\eta_{max}) + \frac{\partial F''}{\partial g_1}\Delta g_1 + \frac{\partial F''}{\partial g_2}\Delta g_2 = \delta_4 \quad (1.9)$$

$$G'(\eta_{max}) = G_c'(\eta_{max}) + \frac{\partial G'}{\partial g_1}\Delta g_1 + \frac{\partial G'}{\partial g_2}\Delta g_2 = \delta_5 \quad (1.10)$$

The subscript c indicates the value of the function at η_{max} determined from trial integration. Solution of these equations in a least square sense requires determining the minimum value of

$$E = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 + \delta_5^2 \tag{1.11}$$

Differentiating E with respect to g_1 and g_2 respectively differentiation yields a system of linear equations in the following from as

$$a_{11}\Delta g_1 + a_{12}\Delta g_2 = b_1 \tag{1.12}$$

$$a_{21}\Delta g_1 + a_{22}\Delta g_2 = b_2 \tag{1.13}$$

Where the coefficients a_{ij} and b_j are shown in (A4).

From equations (1.12) and (1.13), we have

$$\Delta g_1 = \frac{\det A_1}{\det A}, \quad \Delta g_2 = \frac{\det A_2}{\det A}$$

Where, $\det \; A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$\det \; A_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$,

And $\det \; A_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$

Now, the guess values are revised by Δg_1 and Δg_2 respectively,

i,e $\quad g_1 = g_1 + \Delta g_1$
$\quad\quad g_2 = g_2 + \Delta g_2$

Thus adopting the numerical technique described above, a computer program was set up for the solutions of the basic non-linear differential equations (1.12) and (1.13) with the boundary conditions

$F(\eta) = \text{Fw} \neq 0, \; F'(\eta) = 0, G(\eta) = 1$, for $\eta = 0$

$F'(\eta) = 0, \; G(\eta) = 0$, at $\eta \to \infty$

Of our problem where the integration technique was adopted as the sixth order implicit Range-Kutta of integration. Based on the integration done with the above numerical technique, the results obtained are presented in the following section

$$E = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 + \delta_5^2 \quad \text{(A1)}$$

Differentiating (A1) with respect to g_1, we have

$$2\delta_1 \frac{\partial \delta_1}{\partial g_1} + 2\delta_2 \frac{\partial \delta_2}{\partial g_1} + 2\delta_3 \frac{\partial \delta_3}{\partial g_1} + 2\delta_4 \frac{\partial \delta_4}{\partial g_1} + 2\delta_5 \frac{\partial \delta_5}{\partial g_1} = 0$$

$$\Rightarrow \delta_1 \frac{\partial \delta_1}{\partial g_1} + \delta_2 \frac{\partial \delta_2}{\partial g_1} + \delta_3 \frac{\partial \delta_3}{\partial g_1} + \delta_4 \frac{\partial \delta_4}{\partial g_1} + \delta_5 \frac{\partial \delta_5}{\partial g_1} = 0$$

$$\Rightarrow \left(F_c + \frac{\partial F}{\partial g_1}\Delta g_1 + \frac{\partial F}{\partial g_2}\Delta g_2 \right)\frac{\partial F}{\partial g_1} +$$

$$\left(G_c + \frac{\partial G}{\partial g_1}\Delta g_1 + \frac{\partial G}{\partial g_2}\Delta g_2 \right)\frac{\partial G}{\partial g_1} + \left(F_c' + \frac{\partial F'}{\partial g_1}\Delta g_1 + \frac{\partial F'}{\partial g_2}\Delta g_2 \right)\frac{\partial F'}{\partial g_1} +$$

$$\left(G_c' + \frac{\partial G'}{\partial g_1}\Delta g_1 + \frac{\partial G'}{\partial g_2}\Delta g_2 \right)\frac{\partial G'}{\partial g_1} + \left(F_c'' + \frac{\partial F''}{\partial g_1}\Delta g_1 + \frac{\partial F''}{\partial g_2}\Delta g_2 \right)\frac{\partial F''}{\partial g_1} = 0$$

$$\text{or, }\left[\left(\frac{\partial F}{\partial g_1}\right)^2 + \left(\frac{\partial F'}{\partial g_1}\right)^2 + \left(\frac{\partial F''}{\partial g_1}\right)^2 + \left(\frac{\partial G'}{\partial g_1}\right)^2 + \left(\frac{\partial G'}{\partial g_1}\right)^2 \right]\Delta g_1$$

$$+ \left(\frac{\partial F}{\partial g_1}\frac{\partial F}{\partial g_2} + \frac{\partial G}{\partial g_1}\frac{\partial G}{\partial g_2} + \frac{\partial F'}{\partial g_1}\frac{\partial F'}{\partial g_2} + \frac{\partial G'}{\partial g_1}\frac{\partial G'}{\partial g_2} + \frac{\partial F''}{\partial g_1}\frac{\partial F''}{\partial g_2} \right)\Delta g_2$$

$$= -\left(F_c\frac{\partial F}{\partial g_1} + G_c\frac{\partial G}{\partial g_1} + F_c'\frac{\partial F'}{\partial g_1} + G_c'\frac{\partial G'}{\partial g_1} + F_c''\frac{\partial F''}{\partial g_1} \right)$$

$$\Rightarrow a_{11}\Delta g_1 + a_{12}\Delta g_2 = b_1 \quad \text{(A2)}$$

Where, the coefficients a_{11}, a_{12} and b_1 are shown in (A4).

Differentiating (A1) with respect to g_2, we have

$$2\delta_1 \frac{\partial \delta_1}{\partial g_2} + 2\delta_2 \frac{\partial \delta_2}{\partial g_2} + 2\delta_3 \frac{\partial \delta_3}{\partial g_2} + 2\delta_4 \frac{\partial \delta_4}{\partial g_2} + 2\delta_5 \frac{\partial \delta_5}{\partial g_2} = 0$$

$$\Rightarrow \delta_1 \frac{\partial \delta_1}{\partial g_2} + \delta_2 \frac{\partial \delta_2}{\partial g_2} + \delta_3 \frac{\partial \delta_3}{\partial g_2} + \delta_4 \frac{\partial \delta_4}{\partial g_2} + \delta_5 \frac{\partial \delta_5}{\partial g_2} = 0$$

$$\Rightarrow \left(F_c + \frac{\partial F}{\partial g_1}\Delta g_1 + \frac{\partial \theta}{\partial g_2}\Delta g_2\right)\frac{\partial F}{\partial g_2} + \left(G_c + \frac{\partial G}{\partial g_1}\Delta g_1 + \frac{\partial G}{\partial g_2}\Delta g_2\right)\frac{\partial G}{\partial g_2} +$$

$$\left(F_c' + \frac{\partial F'}{\partial g_1}\Delta g_1 + \frac{\partial F'}{\partial g_2}\Delta g_2\right)\frac{\partial F'}{\partial g_2} + \left(G_c' + \frac{\partial G'}{\partial g_1}\Delta g_1 + \frac{\partial G'}{\partial g_2}\Delta g_2\right)\frac{\partial G'}{\partial g_2} +$$

$$\left(F_c'' + \frac{\partial F''}{\partial g_1}\Delta g_1 + \frac{\partial F''}{\partial G_2}\Delta g_2\right)\frac{\partial F''}{\partial g_2} = 0$$

$$\Rightarrow \left(\frac{\partial F}{\partial g_1}\frac{\partial F}{\partial g_2} + \frac{\partial G}{\partial g_1}\frac{\partial G}{\partial g_2} + \frac{\partial F'}{\partial g_1}\frac{\partial F'}{\partial g_2} + \frac{\partial G'}{\partial g_1}\frac{\partial G'}{\partial g_2} + \frac{\partial F''}{\partial g_1}\frac{\partial F''}{\partial g_2}\right)\Delta g_1 +$$

$$\left[\left(\frac{\partial F}{\partial g_2}\right)^2 + \left(\frac{\partial G}{\partial g_2}\right)^2 + \left(\frac{\partial F'}{\partial g_2}\right)^2 + \left(\frac{\partial G'}{\partial g_2}\right)^2 + \left(\frac{\partial F''}{\partial g_2}\right)^2\right]\Delta g_2 +$$

$$= -\left(F_c\frac{\partial F}{\partial g_2} + G_c\frac{\partial G}{\partial g_2} + F_c'\frac{\partial F'}{\partial g_2} + G_c'\frac{\partial G'}{\partial g_2} + F_c''\frac{\partial F''}{\partial g_2}\right)$$

$$a_{21}\Delta g_1 + a_{22}\Delta g_2 = b_2 \qquad \text{(A3)}$$

The coefficients a_{21}, a_{22} and the constant b_2 are shown in (A4).

$$a_{11} = \left(\frac{\partial F}{\partial g_1}\right)^2 + \left(\frac{\partial F'}{\partial g_1}\right)^2 + \left(\frac{\partial F''}{\partial g_1}\right)^2 + \left(\frac{\partial G}{\partial g_1}\right)^2 + \left(\frac{\partial G'}{\partial g_1}\right)^2$$

$$a_{12} = \left(\frac{\partial F}{\partial g_1}\frac{\partial F}{\partial g_2} + \frac{\partial G}{\partial g_1}\frac{\partial G}{\partial g_2} + \frac{\partial F'}{\partial g_1}\frac{\partial F'}{\partial g_2} + \frac{\partial G'}{\partial g_1}\frac{\partial G'}{\partial g_2} + \frac{\partial F''}{\partial g_1}\frac{\partial F''}{\partial g_2}\right) = a_{21}$$

$$a_{22} = \left[\left(\frac{\partial F}{\partial g_2}\right)^2 + \left(\frac{\partial G}{\partial g_2}\right)^2 + \left(\frac{\partial F'}{\partial g_2}\right)^2 + \left(\frac{\partial G'}{\partial g_2}\right)^2 + \left(\frac{\partial F''}{\partial g_2}\right)^2\right] \qquad \text{(A4)}$$

$$b_1 = -\left(F_c\frac{\partial F}{\partial g_1} + G_c\frac{\partial G}{\partial g_1} + F_c'\frac{\partial F'}{\partial g_1} + G_c'\frac{\partial G'}{\partial g_1} + F_c''\frac{\partial F''}{\partial g_1}\right)$$

$$b_2 = -\left(F_c\frac{\partial F}{\partial g_2} + G_c\frac{\partial G}{\partial g_2} + F_c'\frac{\partial F'}{\partial g_2} + G_c'\frac{\partial G'}{\partial g_2} + F_c''\frac{\partial F''}{\partial g_2}\right)$$

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