A STUDY ON DOMINATION AND TOTAL DOMINATION OF GRAPHS

by

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Candidate's Declaration

I hereby declare that the work which is being presented in the thesis titled "A study on domination and total domination of graphs" submitted in partial fulfillment of the requirement for the award of the degree of MASTER OF PHILOSOPHY in Mathematics, in the Department of Mathematics, Bangladesh University of Engineering and Technology, Dhaka is an authentic record of my own work.

The matter presented in this thesis has not been submitted by me for the award of any other degree in this or any other university.

31-10-2004

Date: October 31, 2004 (Khandker Farid Uddin Ahmed)

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INDEX OF SYMBOLS

$V(G)$	- the vertex set of a graph G
$E(G)$	- the edge set of a graph G
P_n, C_n, K_n	- respectively denote the path, cycle and complete graph of n vertices.
$K_{m,n}$	- complete bipartite graph
W_n	- wheel on n vertices
$d(u,v)$	- the length of the shortest path from u to v or infinitely if there is no path from u to v.
$N_1(u)$	- set of all vertices adjacent to u
$N[u]$	- the closed neighborhood of a vertex u, that is, $\{u\} \cup N_1(u)$
$N_2(u)$	- set of all vertices v in G with $d(u,v) = 2$
$dia(G)$	- the diameter of a connected graph G
G^n	- the graph with vertex set $V(G)$ in which u and v are adjacent iff $d(u,v) \leq n$ in G
$\lfloor x \rfloor$	- the greatest integer not exceeding x
$\lceil x \rceil$	- the least integer not less than x
$< S >$	- subgraph induced by a subset S of $V(G)$
$\overline{G}$	- the complement of a graph G
$\lvert S \rvert$	- the number of elements of a set S
$G-v$	- the graph obtained from G by removing a vertex v
$G-e$	- the graph obtained from G by removing an edge e
$G+e$	- the graph obtained from G by adding an edge e

$G+E_0$	- the graph obtained from G by adding a subset E_0 of $E(\overline{G})$
$G-E_0$	- the graph obtained from G by removing a subset E_0 of $E(G)$
$E[V_1,V_2]$	- the set of all edges in G joining the vertices in V_1 and the vertices in V_2, where V_1 and V_2 are subsets of $V(G)$
$\alpha_0(G)$	- vertex covering number of G
$\alpha_1(G)$	- edge covering number of G
$\beta_0(G)$	- the independent number of G
$\beta_1(G)$	- the edge independent number of G
$\overline{\beta_1}(G)$	- the minimum matching number of G
$\Delta(G)$	- maximum degree of G
$\delta(G)$	- minimum degree of G
$\gamma(G)$	- domination number of G
$\gamma_2(G)$	- the two-domination number of G
$\gamma_t(G)$	- total domination number of G
$\gamma_c(G)$	- the connected domination number of G
$\gamma_i(G)$	- the independent domination number of G
$cb(G)$	- cobondage number of a graph G
$(cb)_2(G)$	- two-cobondage number of a graph G
$(cb)_t(G)$	- total cobondage number of a graph G
$b(G)$	- bondage number of a graph G
$b_2(G)$	- two-bondage number of a graph G
$b_t(G)$	- total bondage number of a graph G

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ABSTRACT

A set D of vertices in a graph $G=(V,E)$ is a dominating set of G if every vertex in $V-D$ is adjacent to some vertex in D. The domination number of G is the minimum cardinality taken over all minimal dominating sets in G and is denoted by $\gamma(G)$. For a graph G with $\gamma(G)>1$, the cobondage number $cb(G)$ of G is defined by $cb(G)=\min\left\{|E_0|:E_0\subset E(\overline{G})\ \mathit{and}\ \gamma(G+E_0)<\gamma(G)\right\}$. The bondage number $b(G)$ of a graph G is the minimum cardinality of a set of edges of G whose removal from G results in a graph with domination number larger than that of G. In the same way we can define two cobondage number, total cobondage number, two bondage number and total bondage number. Different types of methods are available depending on types of problems. Sharp upper bounds are obtained for cobondage number of a graph and the exact values are determined for several classes of graphs. The exact values of total cobondage number for some standard graphs are calculated with the help of the methods used by Cockayne, Hedetniemi, Hartnell, Rall, Kulli, etc. An alternative proof of a theorem for total bondage number of Kulli for a complete graph with at least five vertices is provided. Finally, some operations on two bondage number are developed.

CHAPTER ONE

INTRODUCTION

This thesis is devoted to the domination theory in graphs. The concept of dominating sets introduced by Ore and Berge currently receives more attention in Graph Theory. There has been a rapid growth of research in this area and a wide variety of domination parameters have been introduced, after the publication of the paper "Towards a theory of domination in graphs" by E.J. Cockayne and S.T. Hedetniemi [8]. S.T. Hedetniemi and R.C Laskar attributed the following factors to the growth in the number of domination papers [19]:

a) the diversity of the applications to both real-world and other mathematical 'covering' or 'location' problems,

b) the wide variety of domination parameters that can be defined,

c) the NP-completeness of the basic domination problem, its close and 'natural' relationships to other NP-complete problems, and the subsequent interest in finding polynomial time solntions to domination problems in special classes of graphs.

A brief survey of the literature reveals the following sample of applications of the concept of a dominating set.

Ore [24] mentions the problem of placing a minimum number of queens on a chessboard so that each square is controlled by at least one queen.

Berge [3] mentions the problem of keeping all points in a network under surveillance by a set of radar stations.

Application of domination in communication networks have been discussed by C.L. Liu [23], P.J. Slater [29]. There are numerous papers on various aspects of domination theory. The domination theory has gained popularity and remains as a major area of research due to the inspiring contributions by eminent graph theorist like as C. Berge, E.J. Cockayne, S.T. Hedetniemi, R.M. Dawes, B.

Bollobas, R.C. Laskar, R.B. Allan, P.J. Slater, P.L. Hammer, R.C. Brigham, M.A. Henning, Douglas F. Rall, J.F. Fink, B.L. Hartnell, E. Sampathkumar, B.D. Acharya, H.B. Walikar.

We consider only finite undirected graphs with neither loops nor multiple edges.

Graph:

A graph $G = (V(G), E(G))$ consists of two finite sets. The first set $V(G)$ is a non-empty set of elements called vertices of G, and the second set $E(G)$ is an edge set of G, such that each edge $e \in E$ is assigned an ordered pair of vertices (u, v), called the end vertices of e.

Order and Size of a graph:

The cardinality of the vertex set $V(G)$ is denoted by n and is called the order of G. The cardinality of the edge set is said to be the size of G and is denoted by q.

Loop and parallel edges:

An edge e having identical end vertices, i.e., a vertex v joined to itself by an edge, is called a loop.

Parallel edges:

If two or more edges of a graph G have the same end vertices, then these edges are called parallel.

A graph $G = (V, E)$ is called simple if it has no loops and no parallel edges.

Isolated vertex, end vertex and support:

A vertex of a graph G is called an isolated vertex of G if it has degree zero. A vertex of degree 1 is called an end vertex or pendent vertex. Any vertex which is adjacent to a pendent vertex is known as a support.

Adjacent vertices, neighborhood sets:

Two vertices joined by an edge are said to be adjacent or neighbors. The set of all neighbors of a fixed vertex u of a graph G is called the neighborhood set of u and is denoted by $N_1(u)$.

The open neighborhood of u is

$$N_1(u) = \{v \in V : uv \in E\}$$

and the closed neighborhood of u is

$$N[u] = \{u\} \cup N_1(u).$$

For a set S of vertices, the open neighborhood of S is defined by

$$N(S) = \bigcup_{u \in S} N_1(u).$$

Subgraphs:

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Then a graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case, G is called the supergraph of H.

Proper subgraph:

If $H \subseteq G$ but $V(H) \neq V(G)$ or $E(H) \neq E(G)$, then H is called a proper subgraph of G.

Spanning subgraph:

Let G be a graph. Then H is called a spanning subgraph of G if H has exactly the same vertex set as G.

Induced subgraph: Let S be a non-empty subset of the vertex set V of G. Then the subgraph $G[S]$ of G induced by S is a graph having vertex set S and edge set consisting of those edges of G that have both ends in S.

Similarly, let F be a non-empty subset of the edge set E of G. Then the subgraph $G[F]$ of G induced by F is a graph whose vertex set is the set of ends of edges in F and whose edge set is F.

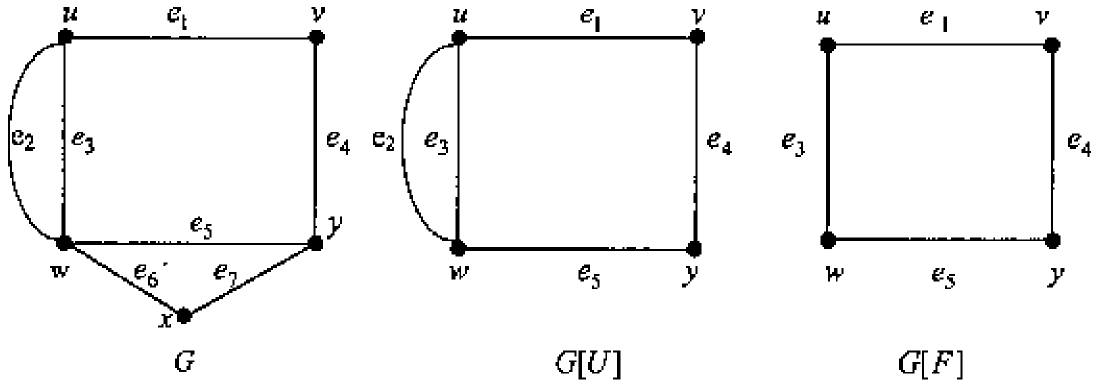


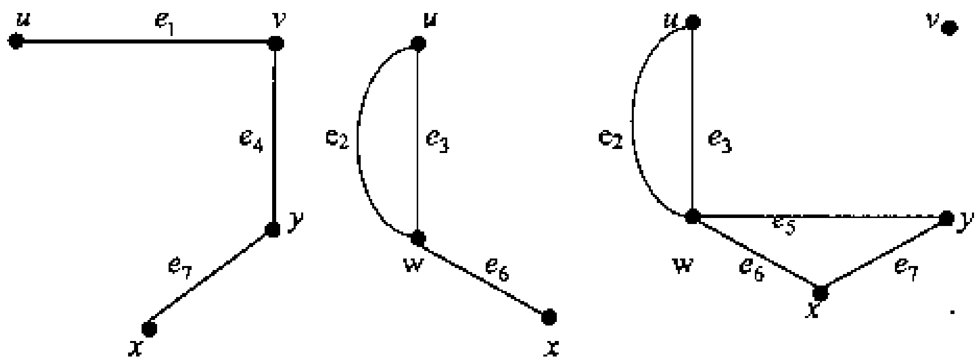
Figure 1.1: $G[U]$ and $G[F]$ for $U = \{u, v, w, y\}$, $F = \{e_1, e_3, e_4, e_5\}$.

Vertex deleted and edge deleted subgraphs:

Let $u \in V(G)$. Then the induced subgraph $< V(G) - \{u\} >$ denoted by $G - u$ is a subgraph of G obtained by the removal of u.

If $e \in E(G)$, then the spanning subgraph of G with edge set $E(G) - \{e\}$ denoted by $G - e$ is the subgraph of G obtained by the removal of e.

For the graph G of Figure 1.1, the followings are the vertex deleted and edge deleted subgraphs.



(i) $G - w$ (ii) $G - U$, where $U = \{v, y\}$ (iii) $G - F$, where $F = \{e_1, e_4\}$

Fig. 1.2: Vertex deleted and edge deleted subgraphs.

The minimum and the maximum degrees of vertices of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively.

Complete graph:

A simple graph G in which each pair of distinct vertices is joined by an edge is called a complete graph of G.

Thus, a graph with n vertices is complete if it has as many edges as possible provided that there are no loops and no parallel edges.

If a complete graph G has n vertices $v_1, v_2, \ldots, v_n$, then

$$G = \{(v_i, v_j) : v_i \neq v_j; i, j = 1, 2, 3, \ldots, n\}.$$

The complete graph of n vertices is denoted by K_n.

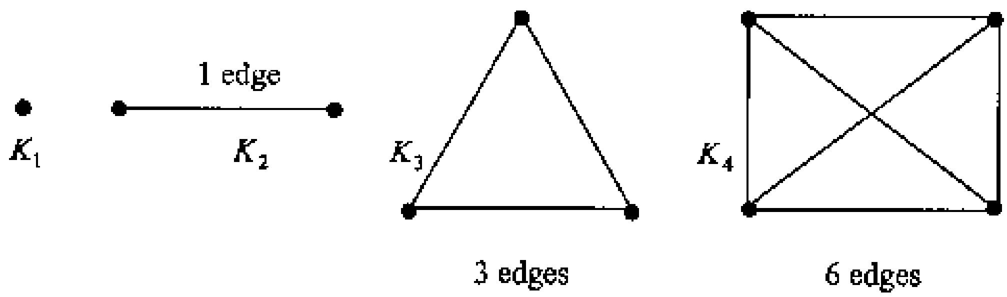


Figure 1.3: The complete graphs on at most 4 vertices.

Bipartite Graph:

An empty graph is a graph with no edges. A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint non-empty subsets V_1 and V_2 (i.e., $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$) such that each edge of G has one end in V_1 and one end in V_2 so that no edge in G connects either two vertices in V_1 or two vertices in V_2. The partition $V = V_1 \cup V_2$ is called a bipartition of G.

Complete Bipartite Graph:

A complete bipartite graph is a simple bipartite graph G, with bipartition $V = V_1 \cup V_2$, in which every vertex in V_1 is joined to every vertex in V_2. If V_1 has m vertices and V_2 has n vertices, such a graph is denoted by $K_{m,n}$.

Complement of a graph:

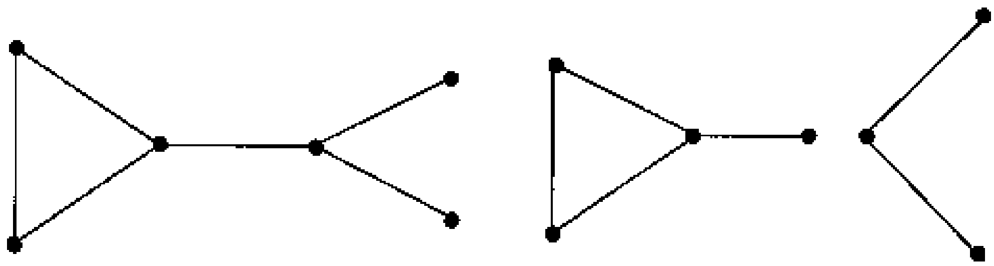
The complement $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ such that any two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Connected graph:

A graph G is said to be connected if every two vertices of G are connected. Otherwise, G is a disconnected graph.

Let $C(u)$ denote the set of all vertices in G that are connected to u. Then the subgraph of G induced by u is called the connected component containing u.

A maximal connected subgraph of G is a component of G. Thus, a disconnected graph has at least two components. The number of components of G is denoted by $\omega(G)$.



(a) G (b) G'

Figure 1.4: (a) Connected graph, (b) Disconnected graph

Distance of two vertices:

The distance $d(u,v)$ between two vertices u and v is the length of a shortest distance $u-v$ path in G. If there is no $u-v$ path in G, then we define $d(u,v)=0$.

Second neighborhood:

If v is a vertex of G, then we define the second neighborhood $N_2(v)$ of v as

$$N_2(v)=\{u : u \in V(G) \text{ and } d(u,v)=2 \text{ in } G\}.$$

Walk in a graph:

Let G be a graph. Then a walk in G is a finite sequence

$$W = v_0e_1v_1e_2v_2\cdots v_{k-1}e_kv_k,$$

whose terms are alternately vertices and edges such that, for $i=1,2,\cdots,k,$ the edge e_i has ends v_{i-1} and v_i.

The above walk W is a walk from origin v_0 to terminus v_k. The integer k, the number of edges iu the walk, is called the length of W.

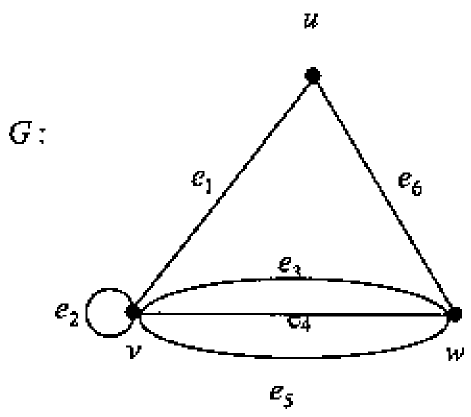


Figure 1.5: $ue_1ve_2ve_3we_5ve_4w$ is a walk.

In other words, the number of edges in W is called the length of W. If the sequence of W consists solely of one vertex, i.e., $W=v_0$, then W is a trivial walk with length 0.

Trail of a graph:

If the edges $e_1, e_2, \ldots, e_k$ of the walk

$$W = v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k,$$

are all distinct, then W is called a trail.

In other words, a trail is a walk in which no edge is repeated.

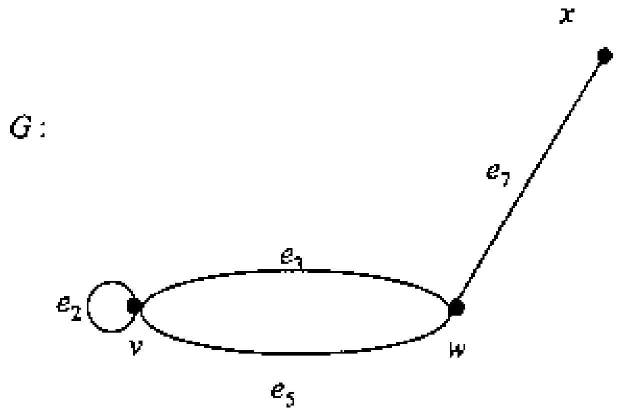


Figure 1.6: $xe_7we_5ve_2ve_3w$ is a trail.

Paths of a graph:

If the vertices of a walk

$$W = v_0 e_1 v_1 e_2 v_2 \ldots\ldots v_{k-1} e_k v_k,$$

are all distinct, then W is called a path. A path with n vertices is denoted by P_n, which has length $n-1$.

In other words, a path is a walk in which no vertex is repeated.

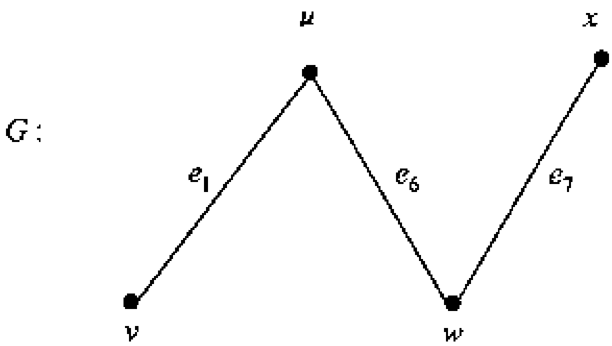


Figure 1.7: $xe_7we_6ue_1v$ is a path.

Thus, in a path no edge can be repeated either, and so every path is a trail. The converse of this statement is not true.

Cycle of a graph:

A non-trivial closed trail in a graph G is called a cycle if its origin and internal vertices are distinct. In detail, the closed trail

$$C = v_1 v_2 \ldots v_n v_1$$

is a cycle if

(i) C has at least one edge and

(ii) $v_1, v_2, \ldots, v_n$ are all distinct.

A cycle of length k, i.e., with k edges, is called a $k-$cycle. A $k-$cycle is called odd or even depending on whether k is odd or even. A 3-cycle is often called a triangle.

A cycle with n vertices is denoted by C_n.

Remark: A $u-v$ walk is called closed or open according as $u = v$ or $u \neq v$. The vertices $v_1, v_2, \ldots v_{k-1}$ in the walk

$$W = v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k,$$

are called internal vertices. In the graph G of Figure 1.5, $C = ve_4 we_6 ue_1 v$ is a cycle.

Acyclic graph:

A graph G is called acyclic if it has no cycle.

Tree of a graph:

Let G be a graph. If G is a connected acyclic graph, then it is called a tree.

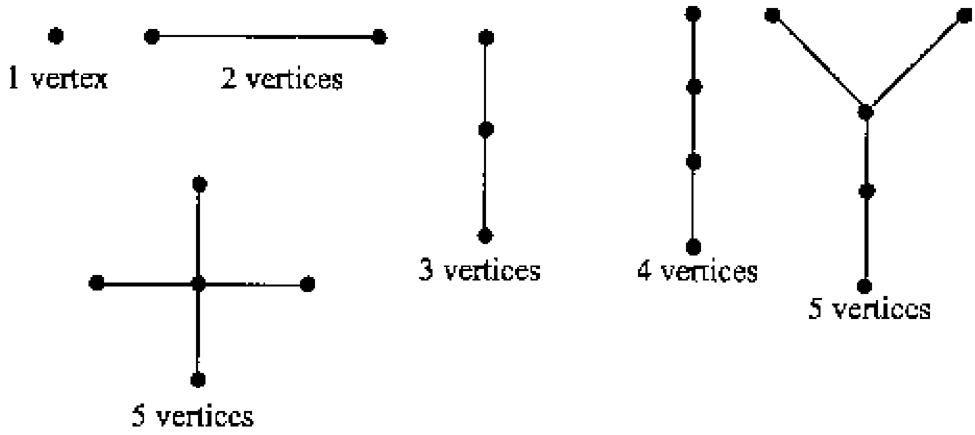


Figure 1.8: Trees with at most five vertices

A tree on n vertices is denoted by T_n, which has exactly two pendent vertices.

Join of a graph:

Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their join $G_1 + G_2$ is a graph whose vertex set is $V_1 \cup V_2$ and edge set

$$E_1 \cup E_2 = \{uv : u \in V_1 \wedge v \in V_2\}.$$

If $1 \le n_1 \le n_2 \le \cdots\cdots\cdots \le n_k$, then

$$K_{n_1, n_2, \ldots, n_k} = \overline{K_{n_1}} + \overline{K_{n_2}} + \cdots\cdots\cdots + \overline{K_{n_k}}.$$

Wheel of a Graph:

A wheel is a graph obtained from a cycle by adding a new vertex and edges joining it to all the vertices of the cycle.

A wheel with n vertices is denoted by W_n, and $W_n = K_1 + C_{n-1}$.

Connectivity of a Graph: The connectivity k of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. A graph G is said to be n – connected if $k \ge n$.

Edge Connectivity:

The edge connectivity λ of a graph G is the minimum number of edges whose removal results in a disconnected graph. A graph G is said to be $n-$edge connected if $\lambda \geq n$.

Subdivision Graph:

An edge $e = uv$ of a graph G is said to be subdivided if e is replaced by the edges uw and wv for some vertex w not in $V(G)$.

The graph obtained from G by subdividing each edge of G exactly once is called the subdivision graph of G and is denoted by $S(G)$.

Matching of a Graph:

A subset M of edges of G, is called a matching if for any two edges e and f in M, the two end vertices of e are both different from the two end vertices of f.

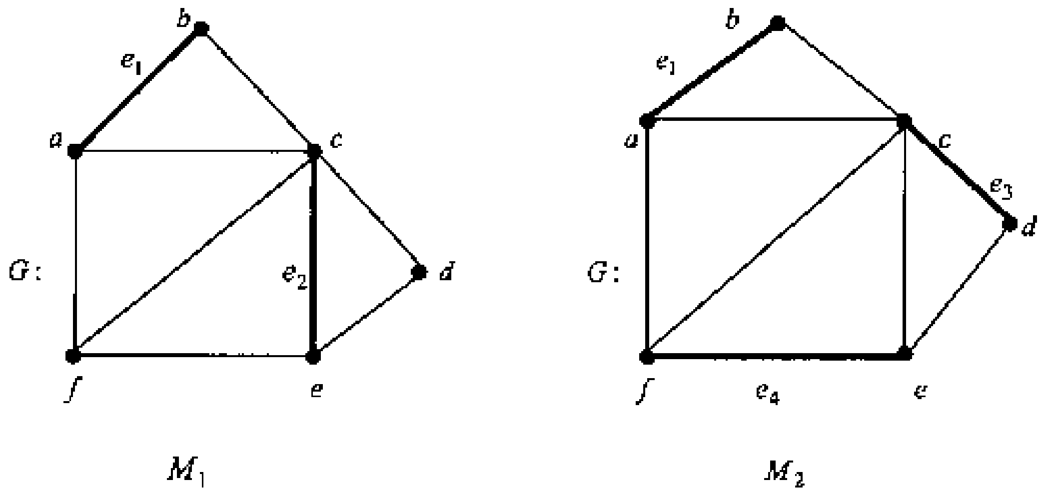


Figure 1.9: Two matchings M_1 and M_2.

In the graph G of Figure 1.9, $M_1 = \{e_1, e_2\}$ and $M_2 = \{e_1, e_3, e_4\}$ are both matchings.

Saturation:

Let G be a graph and let $v \in V(G)$. Then if v is the end vertex of some edge in the matching M, then v is said to be M – saturated and we say "M saturates v." Otherwise, v is M – unsaturated. Thus, in Figure 1.9, a, b, c and e are all M_1 – saturated while f and d are both M_1 – unsaturated; every vertex of G is M_2 – saturated.

Perfect Matching:

If M is a matching in G such that every vertex of G is M – saturated, then M is called a perfect matching. The matching $M_2 = \{e_1, e_3, e_4\}$ of Figure 1.9, is a perfect matching.

Maximum Matching:

A matching M of a graph G is called a maximum matching if G has no matching M' with a greater number of edges than M has.

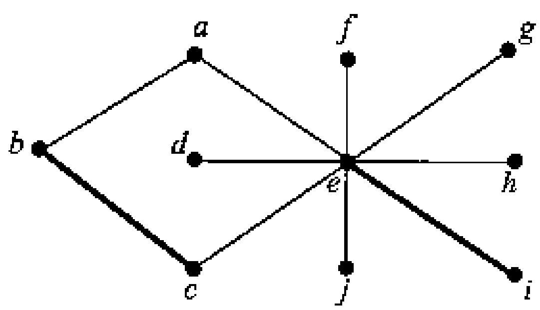
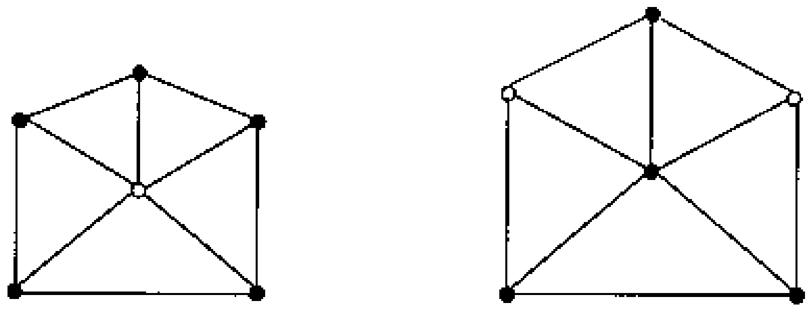


Figure 1.10: Maximum matching

Any perfect matching is a maximum matching. The matching of the graph in Figure 1.10, shown by the bold lines is maximum but not perfect.

Independent Sets: A subset S of vertices in a graph G is said to be an independent set of G if no two vertices of S are adjacent in G. An independent set is maximum if G has no independent set S' with $|S'| > |S|$.



(a) (b)

Figure 1.11: (a) Independent set and (b) Maximum independent set.

A set S of edges of G is said to be independent if no two of the edges in S are adjacent.

Independent Number:

The maximum number of vertices in an independent set is called the independent number of G and is denoted by $\beta_0(G)$.

Edge Independent Number:

The maximum cardinality of an independent set of edges of G is called the edge independent number of G and is denoted by $\beta_1(G)$, which is also called the matching number of G. The minimum matching number $\beta_1^-(G)$ of G, is the minimum number of edges in a maximal independent edge set.

An edge analogue of an independent set is a set of links no two of which are adjacent, i.e., a matching.

Covering of a Graph:

A subset K of vertices in a graph G such that every edge of G has at least one end in K is called a covering of G. The number of vertices in a minimum covering of G is called the covering number of G and is denoted by $\alpha_0(G)$. The edge analogue of a covering is called an edge covering.

An edge covering of a graph G is a subset L of edges of G such that each vertex of G is an end of some edge in L. The edge coverings do not always exist. The number of edges in a minimum edge covering of G is denoted by $\alpha_1(G)$. The number $\alpha_1(G)$ is called the edge covering number of G. For terminology and notations not given here, the reader is referred to [5, 13, 25].

Definition:

If x is a real number, $\lceil x \rceil$ and $\lfloor x \rfloor$ denote respectively the least integer not less than x and the greatest integer not greater than x.

Now we present the following definitions of various types of domination in a graph.

Dominating Set:

A set $D \subseteq V$ is said to be a dominating set in G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number of G is the minimum cardinality taken over all minimal dominating sets in G and is denoted by $\gamma(G)$.

Independent Dominating Set:

A dominating set D of a graph G is called an independent dominating set of G if D is independent in G. The cardinality of the smallest independent dominating set of G is called the independent domination number of G and is denoted by $\gamma_i(G)$.

Total Dominating Set:

A dominating set D of a graph G without isolated vertices is called a total dominating set of G if the subgraph $G[D]$ induced by D has no isolated vertices.

The cardinality of the smallest total dominating set of G is called the total domination number of G and is denoted by $\gamma_t(G)$.

Connected Dominating Set:

A dominating set D of a connected graph G is called a connected dominating set of G if $G[D]$ is connected. The cardinality of the smallest connected dominating set of G is called the connected domination nnmber of G and is denoted by $\gamma_c(G)$.

For any connected graph G with $\Delta(G) < n-1$,

$$\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G).$$

Total dominating sets were first defined and studied by Cockayne, Dawes and Hedetniemi [9]. In addition to several new results involving total domination, Allan, Laskar and Hedetniemi [2] have studied several new inequalities for the domination nnmber of a graph.

Theorem 1.1 to Theorem 1.4 are cited from [9].

Theorem 1.1

If G is a connected graph with $n \geq 3$ vertices, then

$$\gamma_t(G) \leq \frac{2n}{3}.$$

Theorem 1.2

(i) If G has n vertices and no isolated vertices, then

$$\gamma_t(G) \leq n - \Delta(G) + 1.$$

(ii) If G is connected and $\Delta(G) < n-1$, then

$$\gamma_t(G) \leq n - \Delta(G).$$

Theorem 1.3

If G has n vertices, no isolated vertices and $\Delta(G) < n-1$, then

$$\gamma_t(G) + \gamma_t(\overline{G}) \leq n+2,$$

with equality if and only if G or $\overline{G} = mK_2$.

Theorem 1.4

If G has n vertices and no isolated vertices, then

$$\gamma_i(G) \leq n+1-\left\lceil \frac{n-\gamma_t(G)}{\gamma_t(G)} \right\rceil - \frac{\gamma_t(G)}{2}.$$

Robyn Dawes has proposed the following conjectures:

For any connected graph $G=(V,E)$ with $|V| \geq 2$, then

(i) $\gamma(G)+\gamma_t(G) \leq n$ and

(ii) $\gamma_t(G)+\gamma_t(G) \leq n$, where $|V(G)| = n > 2$.

R.B. Allan et al. [2] have settled these conjectures by proving the following theorem.

Theorem 1.5

If $G=(V,E)$ is a graph with $|V| = n$ such that each component has at least 3 vertices, then

$$\gamma_t(G)+\gamma_t(G) \leq n.$$

Furthermore, in [2], Allan et al. have proved the following results:

Proposition 1.6

For any graph G without isolated vertices, then

$$\gamma(G)+\gamma_i(G) \leq n.$$

Theorem 1.7

For any connected graph G with $|V| \geq 2$ vertices, $\gamma_t \geq 2\beta_1^-$.

The following results were obtained by B. Bollobas and E.J. Cockayne [4] concerning $\gamma(G)$ and $\gamma_i(G)$.

Theorem 1.8

If G has no induced subgraph isomorphic to $K_{1,k+1}$ $(k \geq 2)$, then

$$\gamma_i(G) \leq \gamma(G)(k-1)-(k-2).$$

Theorem 1.9

If G has no isolated vertex and $|V| = n$, then

$$\gamma_i(G) \leq n-\gamma(G)+1-\left\lceil \frac{n-\gamma(G)}{\gamma(G)} \right\rceil.$$

In [27], E. Sampathkumar and H.B. Walikar have found out connected domination number for some particular classes of graphs.

The connected domination numbers of some standard graphs from [27] are given as follows:

(i) $\gamma_c(K_n) = 1$.

(ii) $\gamma_c(K_n + G) = 1$, for any graph G.

(iii) $\gamma_c(K_{m,n}) = \begin{cases} 1, \textit{ if either } m \textit{ or } n = 1. \\ 2, \textit{ if } m, n \geq 2. \end{cases}$

(iv) $\gamma_c(C_n) = n-2$.

(v) For any tree T of order n,

$$\gamma_c(T) = n-e$$

where e is the number of pendent vertices (i.e. vertices of degree 1) in T.

Theorem 1.10

For any connected (n,q)-graph G with maximum degree $\Delta(G)$,

$$\left\lfloor \frac{n}{\Delta(G)+1} \right\rfloor \leq \gamma_c(G) \leq 2q-n.$$

E. Sampathkumar [28] has introduced $(1,k)$-domination number of a graph. The $(1,k)$-domination number γ_{1k} of G is the minimum cardinality of a dominating set D such that $|N_1(v) \cap V - D| \leq k$ for all $v \in D$. In [28], Sampathkumar has introduced this new concept and obtained some bounds for γ_k and for γ_{1k}.

A set $U \subset V$ is a k-dominating set if every point of $V - U$ is adjacent to at least k-points in U. The k-domination number γ_k of G is the minimum cardinality of a k-dominating set.

Some results from [28] are given bellow.

Proposition 1.11

If $\delta \geq k$, then

$$\frac{\gamma_k(G)}{k} \leq \gamma_{1k}(G).$$

Proposition 1.12

If $k < \delta$, then

$$\gamma_{1k}(G) \leq \gamma_t(G).$$

Proposition 1.13

If G has no isolated vertices and $|V(G)| = n$, then

$$\gamma_{1k}(G) + \overline{\gamma_{1k}(G)} \leq \begin{cases} n+1+\Delta-1, & \text{if } \delta \leq k < \Delta. \\ n+2, & \text{if } k < \delta. \\ n+1, & \text{if } \Delta \leq k \leq n-1. \end{cases}$$

Bondage Number:

The bondage number $b(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges E for which

$$\gamma(G-E)>\gamma(G).$$

Thus, the bondage number of G is the smallest number of edges where removal will render every minimum dominating set in G a "non-dominating" set in the resultant spanning subgraph.

Since the domination number of every spanning subgraph of a non-empty graph G is at least as great as $\gamma(G)$, the bondage number of a non-empty graph is well-defined.

Cobondage Number:

The cobondage number $cb(G)$ of a graph G is the minimum cardinality among the sets of edges $X \subseteq P_2(V)-E$, where

$$P_2(V)=\{X \subseteq V : |X|=2\}$$

such that $\gamma(G+X)<\gamma(G)$. A γ-set is a minimum dominating set.

If we compare $\gamma(G)$ and $\gamma(H)$, when H is a spanning subgraph of G, it is immediate that $\gamma(H)$ cannot be less than $\gamma(G)$. Every connected graph G has a spanning tree T with $\gamma(G)=\gamma(T)$ and so, in general, a graph will have non-empty sets of edges $F \subseteq E$ for which $\gamma(G-F)=\gamma(G)$. Such a set F will be called an inessential set of edges in G.

However, many graphs also possess single edges e for which $\gamma(G-e)>\gamma(G)$.

The bondage number $b(G)$ of a graph G is the minimum cardiuality of a set of edges of G whose removal from G results in a graph with domination number larger than that of G.

J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts [11], introduced the bondage number $b(G)$ of a graph G. In [11], Fink et al. have obtained sharp bounds for $b(G)$ and the exact valnes of $b(G)$ for several classes of graphs have also been determined.

The following results have been established in [11].

Proposition 1.14

The bondage number of the complete graph $K_n (n \geq 2)$ is

$$b(K_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Theorem 1.15

The bondage number of the n – cycle is

$$b(C_n) = \begin{cases} 3, \text{ if } n \equiv 1 (mod\, 3) \\ 2, \text{ otherwise.} \end{cases}$$

For $n \geq 2$,

$$b(P_n) = \begin{cases} 2, \text{ if } n \equiv 1 (mod\, 3) \\ 1, \text{ otherwise.} \end{cases}$$

Theorem 1.16

If $G = K(n_1, n_2, \ldots\ldots, n_t)$, where $n_1 \leq n_2 \leq \cdots\cdots\cdots \leq n_t$, then

$$b(G) = \begin{cases} \left\lceil \frac{m}{2} \right\rceil, \text{ if } n_m = 1 \text{ and } n_{m+1} \geq 2, \text{ for some } m, 1 \leq m < t. \\ 2t - 1, \text{ if } n_1 = n_2 = \cdots\cdots = n_t = 2. \\ \sum_{i=1}^{t-1} n_i, \text{ otherwise.} \end{cases}$$

Theorem 1.17

If T is a non-trivial tree, then $b(T) \leq 2$.

Corollary 1.18

If any vertex of a tree T is adjacent with two or more end-vertices, then $b(T) = 1$.

Theorem 1.19

If F is a forest, then F is an induced subgraph of a tree S with $b(S) = 1$ and a tree T with $b(T) = 2$.

Theorem 1.20

If G is a connected graph of order $n \geq 2$, then $b(G) \leq n - 1$.

Theorem 1.21

If G is a non empty graph, then

$$b(G) \leq \min\{\deg(u) + \deg(v) - 1;\ u \textit{ and } v \textit{ are adjacent vertices}\}.$$

Theorem 1.22

If G is a connected graph of order $n \geq 2$, then

$$b(G) \leq n - \gamma(G) + 1.$$

The following conjectures were made in [11].

Conjecture 1.23

If G is a non-empty graph, then $b(G) \leq \Delta(G) + 1$.

But in [15], Hartnell and Rall have given a counter example of the above conjecture.

The following results have been obtained in [15].

Theorem 1.24

If G is a non empty graph, then

$$b(G) \leq \min_{u \in V,\ x \in N_1(u)} \{\deg(u) + e(\{x\}, V - N[u])\}.$$

Theorem 1.25

If G has edge connectivity λ, then

$$b(G) \leq \Delta(G) + \lambda - 1.$$

A vertex v of a tree T will be called a level vertex of T if $\gamma(T - v) = \gamma(T)$ and a down vertex if $\gamma(T - v) < \gamma(T)$.

In [14], Bert L. Hartnell and Douglas F. Rall have studied the effect of removing edges in the domination number $\gamma(G)$. An edge is essential if $\gamma(G - e) > \gamma(G)$ and not essential otherwise, that is, if $\gamma(G - e) = \gamma(G)$.

In [11], Fink et al. were posed an open problem to classify trees of bondage number 2.

Hartnell and Rall [14] have presented a constructive characterization of the trees for which the bondage number is 2.

In [14], they have demonstrated how to build larger trees with bondage number 2 from existing ones. They defined four types of operations on a tree.

Type (1)

Attach a P_2 to T at v where v is a level vertex of T belonging to at least one γ – set of T.

Type (2)

Attach a P_3 to T at v where v is a down vertex of T.

Type (3)

Attach H_1 to T at v where v belongs to at least one γ – set of T.

Type (4)

Attach H_n, $n \geq 2$, to T at v, where v can be any vertex of T.

Finally, let $C = \{T : T$ is a tree and $T = K_1$, P_4 or H_n, for some $n \geq 2$ or T can be obtained from P_4 or $H_n (n \geq 2)$ by a finite sequence of operations of type (1), (2), (3) or (4)$\}$.

ε_0 and C are shown identical in [14], where $\varepsilon_0 = \{T : T$ is a tree and $b(T) = 2\}$.

Here the tree H_n refers to the tree given in the following figure.

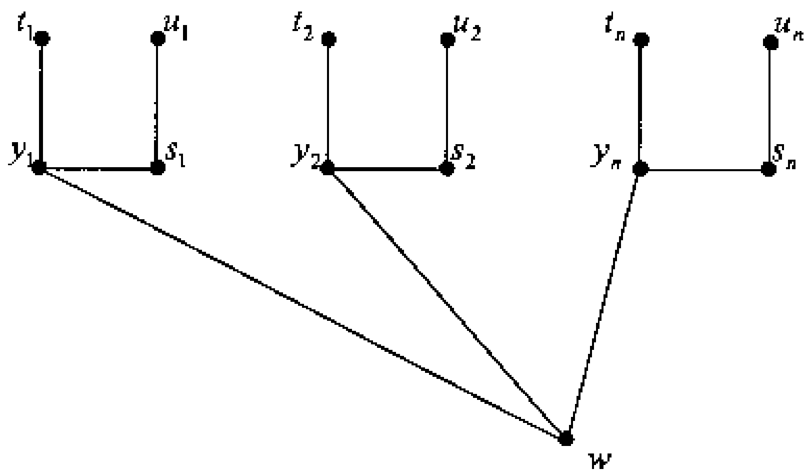


Figure 1.12: The tree H_n.

To determine the bondage number of a graph, we omit certain edges of G in order to obtain a spanning subgraph H of G such that $\gamma(H) > \gamma(G)$. Another direction is to study the domination number of G after adding some edges.

In other words, our aim is to study the relation betweeu $\gamma(G + E_0)$ and $\gamma(G)$, where $E_0 \subset E(\overline{G})$. Can there be any graph H for which G is a spanning subgraph such that $\gamma(H) < \gamma(G)$? If there are some such graphs H, what is the minimum value of $\{E(H) - E(G)\}$? This leads to the cobondage number of G.

V.R. Kulli and B. Janakiram [22] have iuitiated the cobondage number of a graph.

Two-domination number:

Let G be a graph. A subset D of $V(G)$ is said to be a two-dominating set for G if to each $u \in V - D$, $d(u,v) \leq 2$, for some $v \in D$. A minimal cardinality of a two-dominating set is said to be two-domination number of G and is denoted by $\gamma_2(G)$. A two-dominating set D of G is said to be a γ_2 - set if $|D| = \gamma_2(G)$.

Examples:

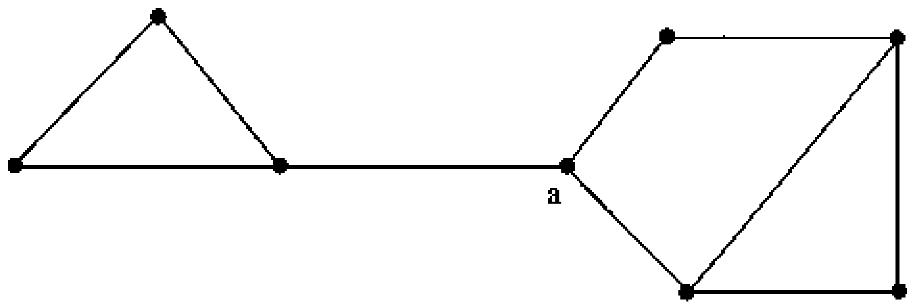


Figure 1.13: G_1

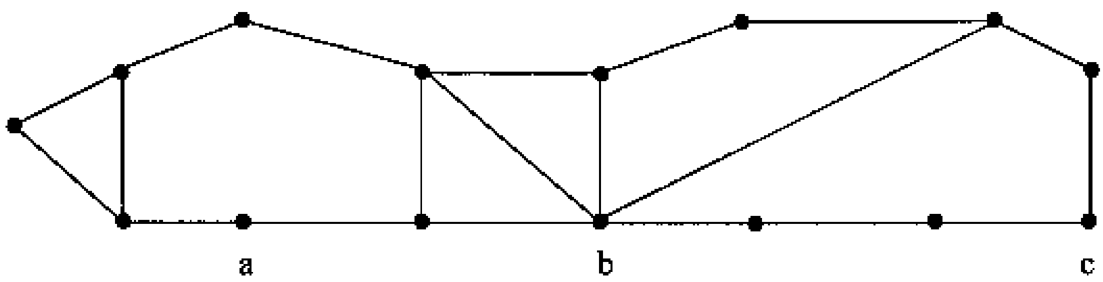


Figure 1.14: G_2.

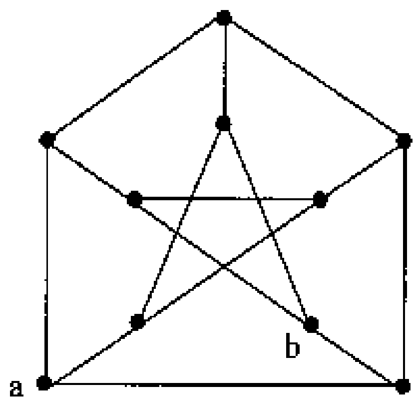


Figure 1.15: G_3.

For G_1, $\{a\}$ is a γ_2 – set.

For G_2, $\{a,b,c\}$ is a γ_2 – set.

For G_3, $\{a,b\}$ is a γ_2 – set.

Remark:

$\gamma_2(G) \leq \gamma(G)$ and $\gamma_2(G) = \gamma(G^2)$, where $V(G^2) = V(G)$ and $u, v \in V$ are adjacent in G^2 if and only if $d(u,v) \leq 2$ in G.

CHAPTER TWO

THE COBONDAGE NUMBER OF A GRAPH INCLUDING TWO DOMINATION

For a graph $G=(V(G),E(G))$ with $\gamma(G)>1$, the cobondage number $cb(G)$ of G is defined by

$$cb(G)=\min\left\{|E_0|:E_0\subset E(\overline{G})\ and\ \gamma(G+E_0)<\gamma(G)\right\}$$

If $\gamma(G)=1$, then the cobondage number is not defined. The following results regarding $cb(G)$ are obtained in [22].

If the order of G is n, then

$$cb(G)\le n-1-\Delta(G) \tag{2.1}$$

$$cb(G)\le \Delta(G)+1 \tag{2.2}$$

$$cb(G)\le n-1 \tag{2.3}$$

If $n\ge 4$ and neither G nor $\overline{G}$ is $2K_2$, then

$$cb(G)+cb(\overline{G})\le 2(n-3) \tag{2.4}$$

In this chapter we improve the result (2.4) and prove

$$cb(G)+cb(\overline{G})<n.$$

Also we characterize the graphs which satisfy the equality

$$cb(G)+cb(\overline{G})=n-1.$$

Let $G=(V(G),E(G))$ be a graph with $\gamma(G)\ge 2$. If D is a minimum dominating set in G with $|D|=\gamma(G)\ge 2$, then to each vertex $u\in D$, let

$$V_u=\{v\in V(G):N_1(v)\cap D=\{u\}\ and\ v\notin D\}.$$

If $V_u = \phi$ for some vertex $u \in D$, then $\gamma(G+e) < \gamma(G)$ whenever $e = uv$ and $v \in D - \{u\}$. In this case, $cb(G) = 1$. Now we assnme that $V_u \neq \phi$ for all vertices $u \in D$. Clearly, if $u_1, u_2 \in D$, $u_1 \neq u_2$, then $V_{u_1} \cap V_{u_2} = \phi$. Hence, if n is the order of G, then

$$\min\{\,|V_u| : u \in D\,\} \leq \frac{n - \gamma(G)}{\gamma(G)} \tag{2.5}$$

Select one vertex $u_0 \in D$ such that

$$|V_{u0}| = \min\{\,|V_u| : u \in D\,\}.$$

Select another vertex $v \neq u$ in D. If

$$E_1 = \{vw : w \in V_{u0}\} \cup \{vu_0\},$$

then $\qquad \gamma(G + E_1) \leq \gamma(G) - 1$

and $\qquad cb(G) \leq |E_1| \leq \dfrac{n - \gamma(G)}{\gamma(G)} + 1 = \dfrac{n}{\gamma(G)}.$

Thus, we have

Lemma 2.1

If G is a graph with order n and $\gamma(G) \geq 2$, then

$$cb(G) \leq \frac{n}{\gamma(G)}. \tag{2.6}$$

Corollary 2.2

If G is a graph of order n and $\gamma(G) \geq 2$, then

$$cb(G) = n - 1 \Leftrightarrow G = \overline{K_2}.$$

Proof. If $\gamma(G) \geq 2$, then $n \geq 2$ and if $cb(G) = n - 1$, by (2.6),

$$n - 1 \leq \frac{n}{\gamma(G)} \leq \frac{n}{2}.$$

Then $n = 2$ and $G = \overline{K_2}$.

Conversely, if $G = \overline{K_2}$, then obviously $cb(G) = n - 1$.

Corollary 2.3

If order of G is n and $\gamma(G) \geq 2$, then $cb(G) = n-2$ if and only if $G = 2K_2$ or $\overline{K_3}$ or $K_2 \cup K_1$.

Proof. From (2.6), $cb(G) \leq \frac{n}{\gamma(G)} \leq \frac{n}{2}$. Therefore,

$$cb(G) = n-2$$

$$\Rightarrow 1 \leq n-2 \leq \frac{n}{2}$$

$$\Rightarrow 2 \leq 2n-4 \leq n$$

$$\Rightarrow 3 \leq n \leq 4.$$

If $n = 3$ and $\gamma(G) \geq 2$, then G is not connected and G is either $\overline{K_3}$ or $K_2 \cup K_1$.

If $n = 4$, then $cb(G) = 2$ and so G is $2K_2$.

Conversely, if $G = 2K_2$ or $\overline{K_3}$ or $K_2 \cup K_1$, then $cb(G) = n-2$.

Lemma 2.4

If G is a graph of order n with $1 \leq \delta(G) \leq \Delta(G) \leq n-2$, then

$$cb(G) + cb(\overline{G}) < n. \tag{2.7}$$

Proof. As $1 \leq \delta(G) \leq \Delta(G) \leq n-2$, $\gamma(G) \geq 2$ and $\gamma(\overline{G}) \geq 2$.

(So both $cb(G)$ and $cb(\overline{G})$ exist).

By Lemma 2.1,

$$cb(G) + cb(\overline{G}) \leq \frac{n}{\gamma(G)} + \frac{n}{\gamma(\overline{G})} \leq \frac{n}{2} + \frac{n}{2} = n. \tag{2.8}$$

If $cb(G) + cb(\overline{G}) = n$, then in (2.8) all the inequalities must be replaced by equalities and we have $cb(G) = cb(\overline{G}) = \frac{n}{2}$.

From (2.1), it follows that $\frac{n}{2} \leq n - \Delta(G) - 1$ and $\frac{n}{2} \leq n - \Delta(\overline{G}) - 1$.

So, $\Delta(G)+\Delta(\overline{G})\leq n-2$, which is a contradiction as

$$n-1=\Delta(G)+(n-1-\Delta(G))=\Delta(G)+\delta(\overline{G})\leq\Delta(G)+\Delta(\overline{G}).$$

Then $cb(G)+cb(\overline{G})\neq n$ and we obtain the required result.

Example:

For the following graphs, we have

$$cb(G)+cb(\overline{G})=n-1.$$

C_4, $\overline{C_4}=2K_2$, C_5, C_6, $\overline{C_6}$, $K_{m,m}$, $2K_m$.

Let us now characterize the graphs for which

$$cb(G)+cb(\overline{G})=n-1.$$

If $n\leq 6$, we can list those graphs.

When $n=4$, $G=C_4$ or $\overline{C_4}=2K_2$.

When $n=5$, $G=C_5$ (note that $\overline{C_5}=C_5$).

When $n=6$, $G=C_6$ or $\overline{C_6}$.

Let a given graph G with order $n\geq 7$ satisfy the equality

$$cb(G)+cb(\overline{G})=n-1. \tag{2.9}$$

As $cb(G)+cb(\overline{G})\leq(n-\Delta(G)-1)+(n-\Delta(\overline{G})-1)$, we have

$$n-1\leq 2n-\Delta(G)-\Delta(\overline{G})-2 \text{ and } \Delta(G)+\Delta(\overline{G})\leq n-1.$$

As $n-1=\Delta(G)+\delta(\overline{G})\leq\Delta(G)+\Delta(\overline{G})\leq n-1$, we have $\delta(\overline{G})=\Delta(\overline{G})$, i.e., $\overline{G}$ is regular. Hence, G is regular.

Claim: $\gamma(G)=\gamma(\overline{G})=2$.

If either $\gamma(G)\geq 3$ or $\gamma(\overline{G})\geq 3$, then

$$n-1=cb(G)+cb(\overline{G})\leq\frac{n}{\gamma(G)}+\frac{n}{\gamma(\overline{G})}\leq n(\frac{1}{2}+\frac{1}{3})$$

implying that $6n-6\leq 5n$ which contradicts that $n\geq 7$. Thus, $\gamma(G)=2=\gamma(\overline{G})$.

Let $D = \{x, y\}$ be a $\gamma -$set in G. Then every vertex in $V(G) - (N_1(x) \cup \{x, y\})$ is adjacent to y. As there are at least $n - \Delta(G) - 2$ vertices adjacent to y,

$$n - \Delta(G) - 2 \le \deg y \le \Delta(G)$$

$$i.e., \ n - 2 \le 2\Delta(G).$$

Similarly,

$$n - 2 \le 2\Delta(\overline{G}) = 2\delta(\overline{G}) = 2(n - 1 - \Delta(G))$$

$$i.e., \ 2\Delta(G) \le n.$$

Then $$\frac{n}{2} - 1 \le \Delta(G) \le \frac{n}{2}.$$

These observations prove the necessary part of the following characterization theorem.

Theorem 2.5

Let G be a graph with order $n \ge 7$ and let $1 \le \delta(G) \le \Delta(G) \le n - 2$. Then G satisfies the equality $cb(G) + cb(\overline{G}) = n - 1$ if and only if

(i) G is regular,

(ii) $\gamma(G) = \gamma(\overline{G}) = 2$ and

(iii) $\frac{n}{2} - 1 \le \Delta(G) \le \frac{n}{2}$.

Proof. In view of the earlier observations, it is enough to prove the sufficient part. Let G be a graph that satisfies (i), (ii) and (iii). If n is even, then

$$\{\Delta(G), \Delta(\overline{G})\} = \left\{\frac{n}{2} - 1, \frac{n}{2}\right\}, \quad cb(G) = n - 1 - \Delta(G)$$

and $cb(\overline{G}) = n - 1 - \Delta(\overline{G})$. This is because $\gamma(G) = 2 = \gamma(\overline{G})$.

Now $cb(G) + cb(\overline{G}) = n - 1 - \Delta(G) + n - 1 - \Delta(\overline{G})$

$$= 2n - 2 - (\Delta(G) + \Delta(\overline{G}))$$

$$= 2n - 2 - \left(\frac{n}{2} - 1 + \frac{n}{2}\right) = n - 1.$$

If n is odd, then

$$\Delta(G) = \Delta(\overline{G}) = \frac{n-1}{2}; \quad cb(G) = cb(\overline{G}) = n - 1 - \frac{n-1}{2}$$

and hence $cb(G) + cb(\overline{G}) = n - 1$.

Given an integer $n \geq 7$, we now describe a method to construct all graphs G of order n, which satisfy the equality (2.9).

Case (i): Let n be even and $\Delta(G) = \frac{n}{2} - 1$.

Observation: If $D = \{x, y\}$ is a $\gamma -$set in G, then y is adjacent only to the vertices in $V(G) - (N_1(x) \cup \{x, y\})$ (in particular, x and y are not adjacent) and $H = G - \{x, y\}$ is $\frac{n}{2} - 2$ regular.

Construction: Let $D = \{x, y, z_1, z_2, \ldots\ldots, z_{n-2}\}$. First form any $\frac{n-4}{2}$ regular graph H with the vertex set $V(H) = \{z_1, z_2, \ldots\ldots, z_{n-2}\}$ and then obtain G from H as follows.

$V(G) = V$ and

$$E(G) = E(H) \cup \{xz_i : i = 1, 2, \ldots\ldots, \frac{n-2}{2}\} \cup \{yz_i : \frac{n-2}{2} < i \leq n-2\}.$$

Obviously, (i) $\Delta(G) = \frac{n}{2} - 1$.

(ii) G is regular.

(iii) $D = \{x, y\}$ is a $\gamma -$set in both G and $\overline{G}$.

(iv) $cb(G) = n - \frac{n}{2} = \frac{n}{2}$.

(v) $cb(\overline{G}) = \frac{n}{2} - 1$

and hence, (vi) $cb(G) + cb(\overline{G}) = n - 1$.

Let $\Im_n$ be the family of graphs with order n, (n is an even number greater than 7), constructed by this method. From the observation made just before the description of the construction, it follows that a graph G with order $n > 7$, n even, $\Delta(G) = \frac{n}{2} - 1$, satisfies the condition (2.9) if and only if $G \in \Im_n$.

Case (ii): Let n be even and $\Delta(G) = \frac{n}{2}$.

As $\Delta(G) = \frac{n}{2}$, $\Delta(\overline{G}) = \frac{n}{2} - 1$ and hence $\overline{G} \in \Im_n$. If $\overline{\Im_n} = \{\overline{G} : G \in \Im_n\}$, then $\overline{\Im_n}$ is the family of graphs with order $n > 7$, where n is even, $\Delta(G) = \frac{n}{2}$ whence $cb(G) + cb(\overline{G}) = n - 1$ holds.

Case (iii) **<u>Observation:</u>**

If $n \geq 7$ is odd, then $n = 4k + 1$ or $4k + 3$ for some positive integer k. As n is odd and $\frac{n}{2} - 1 \leq \Delta(G) \leq \frac{n}{2}$, we have $\Delta(G) = \frac{n-1}{2}$. Since G is a $\frac{n-1}{2}$-regular graph with order n, $\frac{n-1}{2}$ must be even ; i.e., $n = 4k + 1$ (and the possibility $n = 4k + 3$ cannot arise). Let $n = 4k + 1$ and $D = \{x, y\}$ be a γ-set in G. Then x and y are not adjacent to each other.

(If $xy \in E(G)$, each of x and y is adjacent to only $\frac{n-1}{2} - 1$ vertices in $V(G) - \{x, y\}$ and hence $\{x, y\}$ cannot be a dominating set in G).

As $\Delta(\overline{G}) = \frac{n-1}{2}$ and $\gamma(\overline{G}) = 2$, if $D' = \{x', y'\}$ is a γ-set in $\overline{G}$, then $x'y' \notin E(\overline{G})$. So $x'y' \in E(G)$ and $N_1(x') \cap N_1(y') = \phi$ in G. There exists a vertex $z' \in V(G)$ such that $x'z'$, $y'z' \notin E(G)$.

Thus

(i) $N_1(x')\cap\{x,y\}\neq\phi$; $N_1(y')\cap\{x,y\}\neq\phi$ in G;

(ii) $N_1(x)\cap\{x',y'\}\neq\phi$; $N_1(y)\cap\{x',y'\}\neq\phi$ in $\overline{G}$;

(iii) If $x'\in N_1(x)$ in G, then $y'\notin N_1(x)$ in G.

Construction:

Let $V=\{x,y,x',y',z_1,z_2,\ldots\ldots,z_{4k-3}\}$. Construct in any manner a $(2k-2)$-regular graph H with vertex set $V(H)=\{z_1,z_2,\ldots\ldots,z_{4k-3}\}$. Obtain G from H as follows:

$V(G)=V$ and

$$E(G)=E(H)\cup\{xx',x'y',y'y\}\cup\{xz_i:1\le i\le 2k-3\}$$
$$\cup\{yz_i:2k-1\le i\le 4k-3\}\cup\{x'z_i:1\le i\le 2k-2\}$$
$$\cup\{y'z_i:2k\le i\le 4k-3\}.$$

Then clearly, (i) $V(G)=4k+1$,

(ii) $\Delta(G)=\Delta(\overline{G})=2k=\dfrac{n-1}{2}$,

(iii) $D=\{x,y\}$ is a γ-set in G,

(iv) $D'=\{x',y'\}$ is γ-set in $\overline{G}$,

(v) $\gamma(G)=\gamma(\overline{G})=2$ and

(vi) $cb(G)+cb(\overline{G})=n-1$.

Let $\Im_n$, $(n=4k+1,\ k\ge 2)$ be the family of all graphs of order n and constructed by this method. We note that $G\in\Im_n$ if and only if $\overline{G}\in\Im_n$ and hence $\overline{\Im_n}=\Im_n$.

Thus, we have the following theorem.

Theorem 2.6

If $n \geq 7\,; n \not\equiv 3 (\mathrm{mod}\, 4)$, then the set of all graphs G with order n satisfying the equality

$$cb(G) + cb(\overline{G}) = n - 1$$

is the family $\Im_n \cup \overline{\Im_n}$.

Examples:

1. For every integer n, $2K_n \in \Im_{2n}$ and $K_{n,n} \in \overline{\Im_{2n}}$.
2. For $n = 13$, the graph given in Figure 2.1 is a member of $\Im_{13}$.

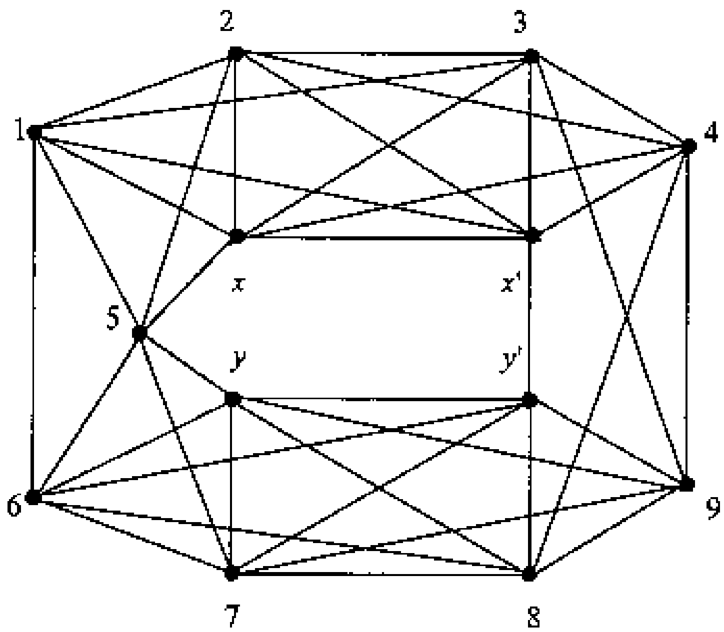


Figure 2.1. A graph $G \in \Im_{13}$.

Remarks:

1. If $n \geq 7$ and $n \not\equiv 3\ (\mathrm{mod}\, 4)$, every graph $G \in \Im_n$ satisfies the equation

$$cb(G) = \left\lfloor \frac{n}{\gamma(G)} \right\rfloor.$$

This shows that there are infinitely many graphs for which $cb(G)$ attains the upper bound given in (2.6). This class includes $\Im_n$ for all even $n \geq 8$.

2. If $n > 7$ and even, every graph $G \in \overline{\Im_n}$ satisfies the equation

$$cb(G) = \frac{n}{\gamma(G)} - 1.$$

3. If $n > 7$ and even, for every $G \in \Im_n$, $cb(G) = \Delta(G) + 1$.
4. If $n \geq 7$ and $n \equiv 1 \pmod 4$, the equality $cb(G) = \Delta(G)$ holds good for every $G \in \Im_n$.
5. Let $n \geq 7$ and $n \not\equiv 3 \pmod 4$. The class $\Im_n \cup \overline{\Im_n} \neq \aleph_n$, where $\aleph_n$ is the family of all regular graphs G with order n, $\frac{n}{2} - 1 \leq \Delta(G) \leq \frac{n}{2}$ and $\gamma(G) = 2$. We can construct a regular graph G with $\gamma(G) = 2$ and $\frac{n}{2} - 1 \leq \Delta(G) \leq \frac{n}{2}$ such that $G \notin \Im_n \cup \overline{\Im_n}$.

Graphs given in Figure 2.2 and in Figure 2.3 are not in $\Im_9$ and $\Im_{10} \cup \overline{\Im_{10}}$ respectively.

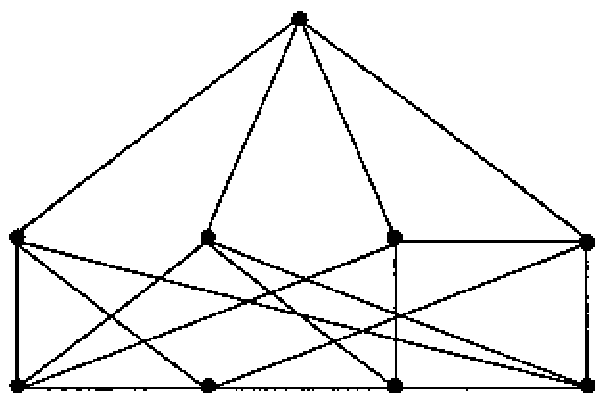


Figure 2.2. A graph $G \notin \Im_9$ with $\overline{G} \in \aleph_9$.

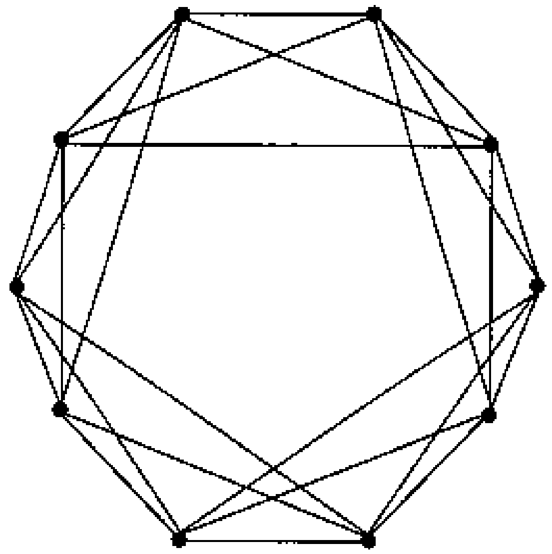


Figure 2.3: A graph $G \notin \Im_{10} \cup \overline{\Im}_{10}$ but $G \in \aleph_{10}$.

In this section we extend the concept of the cobondage number to two domination.

Definitioin:

Let G be a graph and let $\gamma_2(G) \geq 2$. Then the minimum cardinality among the sets $E_1 \subset E(\overline{G})$ such that $\gamma_2(G+E_1) < \gamma_2(G)$ is denoted by $(cb)_2(G)$ and is called the two-cobondage number of G.

A set $E_1 \subset E(\overline{G})$ for which $\gamma_2(G+E_1) < \gamma_2(G)$ is said to be a two-cobondage set and a minimum-two cobondage set is called $(cb)_2$ – set of edges for G.

Two-cobondage number $(cb)_2(G)$ for some standard graphs.

1. If P_n is a path on $n \geq 6$ vertices,

$$(\text{cb})_2(P_n) = \begin{cases} 1 & if \;\; n \equiv 1,2,3 \pmod 5 \\ 2 & if \;\; n \equiv 4,0 \pmod 5. \end{cases}$$

2. If C_n is a cycle on $n \geq 6$ vertices,

$$(cb)_2(C_n) = \begin{cases} 1 & \text{if } n \equiv 1,2,3 \pmod 5 \\ 2 & \text{if } n \equiv 4,0 \pmod 5). \end{cases}$$

We now obtain upper bounds for $(cb)_2(G)$.

Let G be a graph with $\gamma_2(G) \geq 2$. If u is a vertex in G, let

$$A = \{u\} \cup N_1(u) \cup N_2(u)$$

and $B = V - A$. Then

$$1 \leq |B| \leq n - (\Delta(G) + 2) \text{ if } G \text{ is connected}$$

and $\quad 1 \leq |B| \leq n - (\Delta(G) + 1)$ if G is not connected.

If $E_1 = \{ux : x \in B\}$, then $\gamma_2(G + E_1) = 1 < \gamma_2(G)$.

Thus we have the following proposition.

Proposition 2.7

If G is a graph with $\gamma_2(G) \geq 2$, we have

$$(cb)_2(G) \leq n - \Delta(G) - 1.$$

Theorem 2.8

If $\gamma_2(G) \geq 2$, then

(i) $(cb)_2(G) \leq \Delta(G)$ and

(ii) $(cb)_2(G) \leq \dfrac{n}{\gamma_2(G)} - 1 \leq \dfrac{n}{2} - 1$.

Proof: Let $D = \{u_1, u_2, \ldots, u_m\}$ be a γ_2 – set for G, where $m = \gamma_2(G)$. To each $i, 1 \leq i \leq m$, let

$$V_i = N_1(u_i) - \bigcup_{i \neq j} N_1(u_j).$$

If $V_i = \phi$ for some i, then let $E_1 = \{e = u_i x\}$, where x is some fixed vertex in $D - \{u_i\}$. Since $\gamma_2(G + E_1) < \gamma_2(G)$ and hence in this case

$$(cb)_2(G) = |E_1| = 1.$$

If $V_i \neq \phi$ for all i, select one i_0 such that

$$\left|V_{i0}\right| = \min\left\{\left|V_{i0}\right| : i = 1,2,\ldots,m\right\}.$$

Select one $j \neq i_0 \in \{1,2,\ldots,m\}$ and let

$$E_1 = \left\{yu_j : y \in V_{i_0}\right\}.$$

Then we have $\gamma_2(G+E_1) < \gamma_2(G)$ and

$$(cb)_2(G) \leq \left|E_1\right| \leq \frac{n-\gamma_2(G)}{\gamma_2(G)} = \frac{n}{\gamma_2(G)} - 1 \leq \frac{n}{2} - 1.$$

Also, $(cb)_2(G) \leq \left|E_1\right| \leq \Delta(G)$.

Thus $(cb)_2(G) \leq \min\left\{\Delta(G), \frac{n}{\gamma_2(G)} - 1\right\}$ (2.10)

Example:

Graphs G for which $(cb)_2(G) = \Delta(G)$.

(i) If $G = 2K_2$, $(cb)_2(G) = \Delta(G) = \frac{n}{\gamma_2(G)} - 1 = 1.$

The next result improves the inequality

$$(cb)_2(G) \leq \min\left\{\Delta(G), \frac{n}{\gamma_2(G)} - 1\right\}.$$

Theorem 2.9

If G is a graph with $\gamma_2(G) \geq 2$, then either $(cb)_2(G) \leq 1$ or $(cb)_2(G) \leq \min\left\{\Delta(G), \frac{n}{2\gamma_2(G)}\right\}$.

Proof: Let G be a graph with $\gamma_2(G) \geq 2$ and let $D = \{u_1, u_2, \ldots, u_m\}$, where $m = \gamma_2(G)$, be a γ_2-set for G. To each $i, 1 \leq i \leq m$, let

$$X_i = \left\{u \in N_2(u_i) : d(u,u_j) \textit{ is not less than } 3 \textit{ for all } j \neq i, i \in \{1,2,\ldots,m\}\right\}.$$

If $X_i = \phi$, for at least one i, say, for i_0 let $e = u_{i_0} u_j$ for some $j \neq i_0$. Now we have $D - \{u_{i_0}\}$ is a γ_2 – set for $G + e$ and hence in this case $(cb)_2(G) = 1$.

Assume that $X_i \neq \phi$ for all $i = 1,2,\ldots,m$. Let S_i be a subset of $N_1(u_i)$ with minimum cardinality such that each $u \in X_i$ is adjacent to at least one vertex of S_i.

Let Y_i be a minimum dominating set for $< X_i >$.

We note that $|Y_i| = \gamma(< X_i >)$ and $|S_i| \leq |X_i|$. Let

$$t_i = \min \{1 + |S_i|, |N_1(u_i)|, |Y_i| + 1\}$$

and let $t = \min \{t_i : i = 1,2,3,\ldots,m\}$.

Select one i_0 such that $t = t_{i_0}$. Select some $j \neq i_0 \in \{1,2,3,\ldots,m\}$. Take

$$E_1 = \{u_j x : x \in Y_{i_0}\} \cup \{u_j u_{i_0}\} \text{ if } t = |Y_{i_0}| + 1$$

$$= \{u_j x : x \in S_i\} \cup \{u_j u_{i_0}\} \text{ if } t = |S_i| + 1 \leq |N_1(u_i)|$$

$$= \{u_j x : x \in S_i\} \text{ if } |S_i| = |N_1(u_i)| = t.$$

Then $D - \{u_{i_0}\}$ is a γ_2 – set for $G + E_1$ and hence in this case

$$(cb)_2(G) \leq |E_1| = t \tag{2.11}$$

Let $k = (cb)_2(G)$. From (2.11), we get the following:

If $k \neq 1$, then $k \leq \min \{|S_i| + 1, |Y_i| + 1, |N_1(u_i)|\}$ for $i = 1,2,3,\ldots,m$.

Hence in this case $n \geq 2k\,\gamma_2(G)$. We note that the set S_i, X_i, for $i = 1,2,3,\ldots,m$, are all disjoint.

Therefore, $k \leq \dfrac{n}{2\gamma_2(G)} \leq \dfrac{n}{4}$.

Thus, we have either $(cb)_2(G) \leq 1$ or $(cb)_2(G) \leq \min \left\{\Delta(G), \dfrac{n}{2\gamma_2(G)}\right\}$.

Corollary 2.10 If $n \geq 4$, $(cb)_2(G) \leq \dfrac{n}{4}$.

Corollary 2.11

If each component of G contains at least three vertices, then $\gamma_2(G) \le \frac{n}{3}$ and hence $1 \le \frac{n}{2\gamma_2(G)}$ and $(cb)_2(G) \le \min\left\{\Delta(G), \frac{n}{2\gamma_2(G)}\right\}$.

Remark:

If $\gamma_2(G) \ne 1$, then $\gamma_2(\overline{G}) = 1$ and hence $(cb)_2(\overline{G})$ is not defined.

Let us now analyse the structure of graphs for which $(cb)_2(G) = \Delta(G)$.

Let G be a graph with $\gamma_2(G) \ge 2$ and $(cb)_2(G) = \Delta(G)$.

If $\Delta(G) = 1$ and $n \ge 4$, then $(cb)_2(G) = \Delta(G)$ and $G = mK_2$ $(m \ge 2)$. So, assume that $\Delta(G) \ge 2$.

Let D be any minimum γ_2-set for G. Let $D = \{u_1, u_2, \ldots\ldots, u_m\}$, where $m = \gamma_2(G)$.

First we claim that D is an independent set in G. If possible assume that $u_i u_j$ is an edge in G for some i, j $(1 \le i < j \le m)$.

Then $|N_1(u_i) - D| \le \Delta(G) - 1$ and if $E_1 = \{u_j x : x \in N_1(u_i) - D\}$, it is clear that $D - \{u_i\}$ is a γ_2-set for $G + E_1$ and hence $(cb)_2(G) \le \Delta(G) - 1$, which is a contradiction. Thus, D is an independent set in G.

Next we claim that $N_1(u_i) \cap N_1(u_j) = \phi$ for all $i \ne j \in \{1, 2, \ldots\ldots, m\}$.

For if possible, assume that $N_1(u_i) \cap N_1(u_j) \ne \phi$ for some i, j $(i \ne j)$.

If we take $E_1 = \{u_j x : x \in N_1(u_i) - N_1(u_j)\}$, then $|E_1| \le \Delta(G) - 1$ and $D - \{u_i\}$ is a γ_2-set for $G + E_1$ and hence $(cb)_2(G) \le \Delta(G) - 1$, which is a contradiction.

Thus, $N_1(u_i)\cap N_1(u_j)=\phi$ for all $i\neq j$. We note that $|N_1(u_i)|=\Delta(G)$ for all i. For a fixed i, if $E_1=\{u_j x : x\in N_1(u_i)\}$ for some $i\neq j$, then $\gamma_2(G+E_1)<\gamma_2(G)$ and hence $|E_1|=|N_1(u_i)|\geq\Delta(G)$.

Let $x\in N_1(D)$. Then $x\in N_1(u_i)$ for some i. Let

$$A(x)=\{z\in N_2(D):N_1(z)\cap N_1(D)=\{x\}\}.$$

We claim that $A(x)\neq\phi$.

If $A(x)=\phi$, then by taking $E_1=\{u_j y : y\in N_1(u_i)-\{x\}\}$ for some $i\neq j$, we have $\gamma_2(G+E_1)<\gamma_2(G)$ which is a contradiction.

Thus, $A(x)\neq\phi$, for all $x\in N_1(D)$.

Thus, we have the ineqnality

$$|D|+|N_1(D)|+|N_1(D)|\leq|D|+|N_1(D)|+\sum_{x\in N_1(D)}|A(x)|\leq n \tag{2.12}$$

Hence, $\gamma_2(G)+2\gamma_2(G)\Delta(G)\leq n$. (2.13)

We have the following proposition.

Proposition 2.12

If for a graph G, $\Delta(G)\geq 2$ and $(cb)_2(G)=\Delta(G)$, then

$$\Delta(G)\leq\frac{n-\gamma_2(G)}{2\gamma_2(G)} \tag{2.14}$$

Now we obtain certain properties of a graph G for which

$$(cb)_2(G)=\Delta(G)=\frac{n-\gamma_2(G)}{2\gamma_2(G)}.$$

Let G be a graph with $(cb)_2(G)=\Delta(G)\geq 2$ and

$$\Delta(G)=\frac{n-\gamma_2(G)}{2\gamma_2(G)}.$$

As $\Delta(G)=\dfrac{n-\gamma_2(G)}{2\gamma_2(G)}$, we get

$$n\leq\gamma_2(G)\,(1+2\Delta(G))\leq|D|+|N_1(D)|+\sum|A(x)|\leq n$$

and hence $|A(x)|=1$ for all $x \in N_1(D)$.

Also from the proof of theorem 2.9, the set $<\cup\{A(x): x \in N_1(u_i)\}>$ can contain at most one edge for each $i \in \{1,2,......,m\}$.

Thus we have

1. $V(G)=D \cup N_1(D) \cup (\cup A(x): x \in N_1(D))$ is a partition of $V(G)$.
2. $|A(x)|=1$ for all $x \in N_1(D)$.
3. $<\cup\{A(x): x \in N_1(u_i)\}>$ can contain at most one edge, for each i.
4. D is an independent set in G.
5. If $A(x)=\{y\}$, then $N_1(y) \cap (D \cup N_1(D))=\{x\}$.
6. $\beta_0(G)=$ *independent number of* $G \geq 2\Delta(G)-1$.

We now give an example for a graph G for which $p(G)=36, \gamma_2(G)=4$ and $\Delta(G)=4$ and so $\dfrac{n-\gamma_2(G)}{2\gamma_2(G)}=\dfrac{36-4}{8}=4=\Delta(G)$.

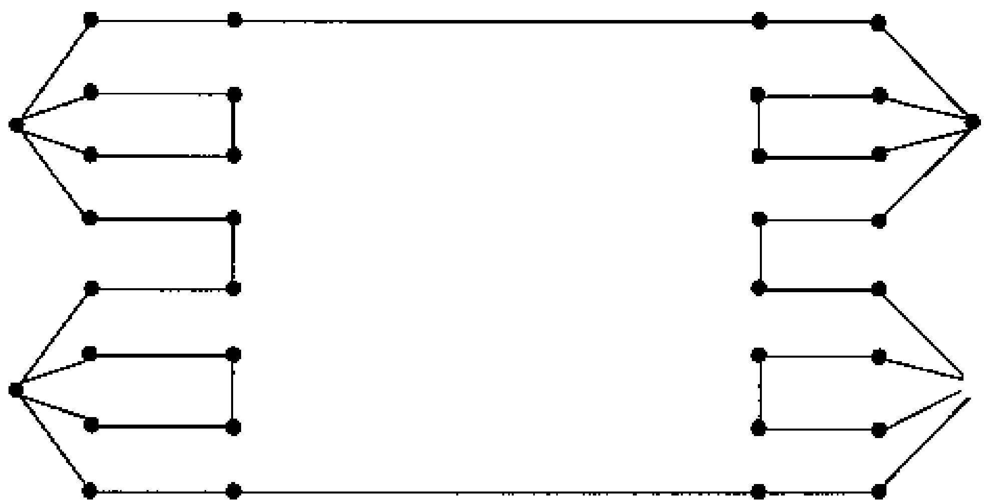


Figure 2.4: G

CHAPTER THREE

THE COBONDAGE NUMBER OF A GRAPH WITH TOTAL DOMINATION

This chapter is devoted to the total cobondage number of a graph. Given a graph G, without isolated vertices, we can find the total domination number $\gamma_t(G)$. If $\gamma_t(G) > 2$, G cannot be the complete graph K_n, where $n = |V(G)|$. As $2 = \gamma_t(K_n) = \gamma_t(G + E(\overline{G})) < \gamma_t(G)$, we can find a subset $E_0 \subset E(\overline{G})$, with least cardinality such that $\gamma_t(G + E_0) < \gamma_t(G)$. We define this least cardinality as the total cobondage number of G.

Definition:

If G is a graph without isolated vertices and $\gamma_t(G) > 2$, then the total cobondage number of G is denoted by $(cb)_t(G)$ and is defined as

$$(cb)_t(G) = \min\ \{\, |E_0| : E_0 \subset E(\overline{G})\ and\ \ \gamma_t(G + E_0) < \gamma_t(G)\}.$$

We can obtain $(cb)_t(G)$ for known standard graphs. First we observe that $(cb)_t(G) \leq \Delta(G)$.

Let D be a minimal total dominating set for G. If the induced graph $<D>$ contains a component with at least three vertices, select such a component $\{x_1, x_2, \cdots\cdots, x_n\}$, $n \geq 3$ of $<D>$. Now

$$E_0 = \{x_2 y : x_1 y \in E(G) \text{ and } x_2 y \in E(\overline{G})\}.$$

Then $D - \{x_1\}$ is a total dominating set for $G + E_0$. If every component of $< D >$ is K_2, select a component $\{x_1, x_2\}$ and select one $x \in D - \{x_1, x_2\}$. Then if

$$E_0 = \{x x_2\} \cup \{x_2 y : x_1 y \in E(G)\ \ and\ \ x_2 y \notin E(G)\}.$$

Then $D-\{x_1\}$ is a total dominating set for $G+E_0$. Hence, $(cb)_t(G) \le \Delta(G)$ in both the cases.

There are infinite number of graphs for which $(cb)_t(G) = \Delta(G)$.

Examples for graphs for which $(cb)_t(G) = \Delta(G)$.

Figure 3.1

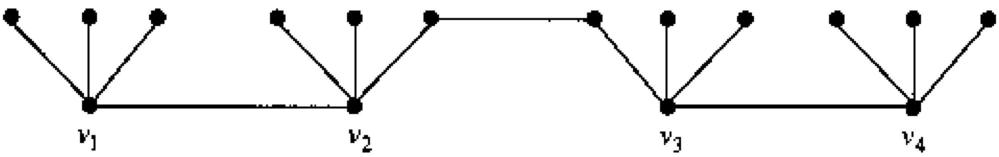


Figure 3.2

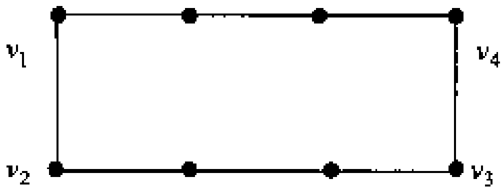


Figure 3.3

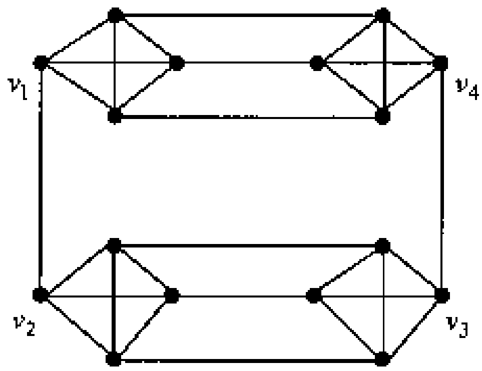


Figure 3.4

For the above four graphs $(cb)_t(G) = \Delta(G)$.

Now we show that for the graph given in Figure 3.4, $(cb)_t(G) = \Delta(G) = 4$. First we note that $n = 16$, G is regular with $\Delta(G) = \delta(G) = 4$, and $D = \{v_1, v_2, v_3, v_4\}$ is a $\gamma_t -$set for G and hence $\gamma_t(G) = 4$. Let $E_1 \subset E(\overline{G})$ such that $\gamma_t(G+E_1) < \gamma_t(G)$. Then either $\gamma_t(G+E_1) = 3$ or $\gamma_t(G+E_1) = 2$.

Case I: $\gamma_t(G+E_1) = 3$.

Let $D = \{z_1, z_2, z_3\}$ be a $\gamma_t -$set for $G+E_1$. As D is a $\gamma_t -$set for $G+E_1$ there is at least one edge from each vertex in $\{V(G) - \{z_1, z_2, z_3\}\}$ to D. As D is a total dominating set, there should be at least two edges in $<D>$ with respect to $G+E_1$. Hence, there are 15 distinct edges which are incident with $\{z_1, z_2, z_3\}$ and $\{\deg(z_1) + \deg(z_2) + \deg(z_3)\} \geq 17$ in $G+E_1$.

Case II: $\gamma_t(G+E_1) = 2$.

Let $D = \{u, v\}$ be a $\gamma_t -$set for $G+E_1$. Again, in $G+E_1$,

$$\deg(u) + \deg(v) \geq 14 + 2 = 16.$$

But in G, $\deg(u) + \deg(v) = 8$. Hence, $|E_1| \geq 4$.

Thus, in all the cases $|E_1| \geq 4$. Therefore, $(cb)_t(G) \geq 4$.

But we can find E_1 such that $|E_1| = 4$ and $D = \{v_1, v_2, v_3\}$ is a $\gamma_t -$set for $G+E_1$. Then $(cb)_t(G) \leq 4$. This proves that $(cb)_t(G) = 4$.

<u>**Example:**</u>

Given any positive integer $n \geq 2$, there exists G with $\Delta(G) = n$, $\gamma_t(G) > 3$ and $(cb)_t(G) = \Delta(G) = n$. It is enough to consider the case $n \geq 5$ (in view of the examples 3.1, 3.2, 3.3, 3.4).

Let $V(G) = \{a_1, a_2, \cdots\cdots, a_n, b_1, b_2, \cdots\cdots, b_n, c_1, c_2, \cdots\cdots, c_n, d_1, d_2, \cdots\cdots, d_n\}$ and

$$E(G) = \{a_j a_k, b_j b_k, c_j c_k, d_j d_k : j \neq k \in \{1,2,3,\cdots\cdots,n\}\} \cup \{a_1 b_1, c_1 d_1\} \cup \{a_j d_j, b_j c_j : j = 2,3,\cdots\cdots,n\}.$$

Then for G, $\gamma_t(G)=4$, $(cb)_t(G)=\Delta(G)=n$. Now we obtain the necessary conditions for a graph G to have $(cb)_t(G)=\Delta(G)$.

Theorem 3.1

Let $G \neq qK_2$ and $\gamma_t(G) \geq 3$.

If $(cb)_t(G)=\Delta(G)$, then every γ_t-set D of G satisfies the following conditions:

(i) $D = qK_2$ and $n = 2q\Delta(G)$ for some integer $q \geq 2$,

(ii) every vertex in D is of maximum degree and

(iii) every vertex in $V-D$ is adjacent to exactly one vertex in D.

Let D be a γ_t-set for G. If C is a component of $<D>$ which contains more than two vertices, C contains at least two vertices x and x' such that both $<C-\{x\}>$ and $<C-\{x'\}>$ will contain no isolated vertex.

If $E_1 = \{x'w : w \in V-D$ and $N_1(w) \cap D = \{x\}\}$, then $D-\{x\}$ is a total dominating set for $G+E_1$ and hence $(cb)_t(G) < |E_1| \leq \deg(x)-1 < \Delta(G)$.

As $(cb)_t(G)=\Delta(G)$, it follows that every component of $<D>$ must contain exactly two vertices. Thus, $\Delta(G)=qK_2$, for some $q \geq 2$ [$q \geq 2$ follows from the fact that $\gamma_t(G) \geq 3$]. Let $u \in D$.

Claim: $\deg(u)=\Delta(G)$.

Let $E_1 = \{vw : w \in V-D \textit{ and } N_1(w) \cap D = \{u\}\}$, where v is the unique vertex in D which is adjacent to u. Let v_2 be a vertex in $D-\{u,v\}$ and $e = vv_2$. Then $D-\{u\}$ is a total dominating set for $G+(E_1+e)$ and hence $\Delta(G)=(cb)_t(G) \leq |E_1|+1 \leq \deg(u)$.

Then, we have $\deg(u)=\Delta(G)$ and $|V_u|=|E_1|=\Delta(G)-1$. As $|V_u|=\Delta(G)-1$ for all $u \in D$, it follows that if $w \in V-D$ is adjacent to u, then $w \in V_u$ and hence

w is not adjacent to any vertex in $D-\{u\}$. Thus every vertex in $V-D$ is adjacent to exactly one vertex in D and

$$V-D=\bigcup_{u\in D} V_u .$$

Therefore,

$$|V-D|=\left|\bigcup_{u\in D} V_u\right|=2q(\Delta(G)-1)$$

and $$n=|D|+|V-D|=2q+2q(\Delta(G)-1)=2q\Delta(G).$$

We recall the following result [9].

Theorem 3.2

If G is a connected graph with order $n\geq 3$, then $\gamma_t(G)\leq\frac{2n}{3}$. Now we obtain the following theorem.

Theorem 3.3

If G is a graph with order n, without an isolated vertex and $\gamma_t(G)\geq 3$, then

$$(cb)_t(G)\leq\min\left\{\Delta(G),\frac{n-2}{2}\right\}.$$

Proof: Let G be a graph without isolated vertices and $\gamma_t(G)\geq 3$. Let S be a total dominating set of smallest cardinality, ($|S|=\gamma_t(G)\geq 3$). By minimality, to each $s\in S$ has at least one of the following two properties:

$P1$: There exists a $w\in V-S$ such that $N_1(w)\cap S=\{s\}$.

$P2$: $<S-\{s\}>$ contains an isolated vertex.

To each $s\in S$, let

$$V_s=\{w\in V-S:\ N_1(w)\cap S=\{s\}\}$$

and $$C_s=\{w\in S:\ w \textit{ is an isolated vertex in } <S-\{s\}>\}.$$

Let $M = \{s \in S : s \text{ has the property } P1\}$ and $N = S - M$. Then $M = \{s \in S : V_s \neq \phi\}$ and $N = \{s \in S : V_s = \phi\}$.

Case (i):

Let $N \neq \phi$, consider one $s' \in N$. Then $|C_{s'}| \leq \deg(s') \leq \Delta(G)$ and $|C_{s'}| \leq |S| - 1 = \gamma_t(G) - 1$. Let $C_{s'} = \{z_1, z_2, \cdots\cdots, z_q\}$. We note that if $q \neq 1$, then $z_i \in M$ and $|C_{s'}| < n - (q+1)$.

We construct E_1 as follows:

(a) If $q = 1, E_1 = \{z_1 s\}$ for some $s \in S - \{z_1, s\}$.

(b) If q is even, $E_1 = \{z_1 z_2, z_3 z_4, \cdots\cdots, z_{q-1} z_q\}$.

(c) If $q \geq 3$ and odd, $E_1 = \{z_1 z_2, z_1 z_3, z_4 z_5, \cdots\cdots, z_{q-1} z_q\}$.

Then $1 \leq |E_1| \leq \left\lceil \frac{C_{s'}}{2} \right\rceil \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$ and $S - \{s'\}$ is a total dominating set for $G + E_1$.

Thus, in this case, $(cb)_t(G) \leq |E_1| \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$ and

$$(cb)_t(G) \leq |E_1| \leq \frac{\gamma_t}{2} \leq \begin{cases} \frac{n}{3}, & \text{if } G \text{ is connected} \\ \frac{n-q}{2}, & \text{if } q \neq 1. \end{cases}$$

Therefore, $$(cb)_t(G) \leq \min\left\{ \left\lceil \frac{\Delta(G)}{2} \right\rceil, \frac{\gamma_t(G)}{2}, \frac{n-2}{2} \right\}. \qquad (3.1)$$

Case (ii):

Let $N = \phi$. Select a vertex $s' \in S$ such that $|V_{s'}| = \min\{|V_s| : s \in S\}$. Then

(a) $1 \leq |V_{s'}| < \deg(s')$.

(b) $|V_{s'}| \leq \frac{|V - S|}{|S|} = \frac{n - \gamma_t(G)}{\gamma_t(G)}$.

(c) $0 \leq |C_{s'}| \leq \deg(s') - |V_{s'}|$.

If $C_{s'} \neq \phi$, let $C_{s'} = \{z_1, z_2, \cdots\cdots, z_q\}$ and construct E_1 as in case (i).

If $C_{s'} = \phi$, put $E_1 = \phi$.

Select some $s \in S - \{s'\}$ and put $E_2 = \{sw : w \in V_{s'}\}$. Then $S - \{s'\}$ is a total dominating set for $G + E_1 + E_2$ and hence,

$$(cb)_t(G) \leq |E_1| + |E_2| \leq \deg(s') \leq \Delta(G).$$

Thus, in this case, $(cb)_t(G) \leq \Delta(G)$.

If $< S >$ is not connected, let $C_1, C_2, \cdots\cdots, C_n$ be the components of S. If there is a component C_1 with $|C_1| \geq 3$, we can find at least two points $s_1, s_2 \in C_1$ such that $< C_1 - \{s_j\} >$ has no isolated vertices for $j = 1,2$.

For at least one s_j, $j \in \{1,2\}$, we have

$$\left|V_{s_j}\right| \leq \frac{[(n - \gamma_t(G)) - (\gamma_t(G) - 2)]}{2}.$$

Say, $\left|V_{s_1}\right| \leq \dfrac{n - 2\gamma_t(G) + 2}{2} \leq \dfrac{n-4}{2}$ as $\gamma_t(G) \geq 3$. Let $E_3 = \{s_2 w : w \in V_{s_1}\}$, then $S - \{s_1\}$ is a total dominating set for $G + E_3$ and hence $(cb)_t(G) \leq |E_3| \leq \dfrac{n-4}{2}$.

Thus, in this case, $\quad (cb)_t(G) \leq \min\left\{\Delta(G), \dfrac{n-4}{2}\right\}$. (3.2)

If there is no component C_i of $< S >$ such that $|C_i| \geq 3$, then $S = qK_2$ for some $q \geq 2$.

Let $S = \{x_j, y_j : j = 1, 2, \cdots\cdots, q\}$ where $x_j y_j \in E\,(< S >)$.

There is a point, say $x_1 \in S$ such that

$$\left|V_{x_1}\right| \leq \frac{|V - S|}{\gamma_t(G)} = \frac{n - \gamma_t(G)}{\gamma_t(G)}.$$

Let $E_4 = \{y_1 w : w \in V_{x_1}\} \cup \{x_1, x_2\}$.

Then $S-\{x_1\}$ is a total dominating set for $G+E_4$ and

$$(cb)_t(G)\le |E_4| = 1+|V_{x_1}| \le 1+\frac{n-\gamma_t(G)}{\gamma_t(G)} = \frac{n}{\gamma_t(G)} \le \frac{n}{4},$$

as $\gamma_t(G)\ge 4$ in this case.

So, we have $(cb)_t(G)\le \min\left\{\Delta(G),\frac{n}{4}\right\}$. As $n\ge 4$, we have $\frac{n}{4}\le\frac{n-2}{2}$.

Thus, in all the cases, we have

$$(cb)_t(G)\le \min\left\{\Delta(G),\frac{n-2}{2}\right\}.$$

Hence the result.

CHAPTER FOUR

THE TOTAL BONDAGE NUMBER

This chapter deals with total bondage number of a graph. We show that for a given positive integer n, there is a tree T whose total bondage number is n. We obtain bounds for total bondage number of a tree T in terms of $|V(T)|$ and $\Delta(T)$. We also obtain complete graphs with total bondage number $2n-5$ for $n \geq 5$ vertices.

1. Introduction:

First we define the total bondage number for a graph.

Definition:

Let G be a graph. If there exists $E_0 \subset E(G)$ such that

(i) there is no isolated vertex in $G - E_0$ and

(ii) $\gamma_t(G - E_0) > \gamma_t(G)$,

then the edge set E_0 is called a total bondage edge set for G.

If there is at least one total bondage edge set for G, we define

$$b_t(G) = \min\{|E_0| : E_0 \textit{ is a total bondage edge set of } G\}.$$

Otherwise, we put $b_t(G) = \infty$.

We call $b_t(G)$, the total bondage number of a graph G.

Example:

1. If G is a path on six vertices, then $b_t(G) = 2$.

2. $b_t(G) = \infty$ for the following graphs:

(a) $K_{1,n}$ (star on $n+1$ vertices),

(b) graphs for which each component is $K_{1,n}$ or K_2.

(c) K_3.

In [21], Kulli and Patwari calculated the exact values of $b_t(G)$ for some standard graphs.

Theorem 4.1

If P_n is the path with $n \geq 4$ vertices, then

$$b_t(P_n) = \begin{cases} 2, & \text{if } n \equiv 2 \pmod 4 \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 4.2

If C_n is the cycle with $n \geq 4$ vertices, then

$$b_t(C_n) = \begin{cases} 3, & \text{if } n \equiv 2 \pmod 4 \\ 2, & \text{otherwise.} \end{cases}$$

Theorem 4.3

If $K_{m,n}$ is the complete bipartite graph with $2 \leq m \leq n$, then $b_t(K_{m,n}) = m$.

Theorem 4.4

If K_n is the complete graph with $n \geq 5$ vertices, then $b_t(K_n) = 2n - 5$.

Theorem 4.5

If G be any graph with $n \geq 5$ vertices, then $b_t(G) \leq 2n - 5$.

In this chapter, we investigate certain properties of the total bondage number of a graph.

2. Bound for total bondage number of a tree.

We know that for a non-trivial tree T, the bondage number $b(T) \leq 2$. But given any positive integer n, we can find a tree T for which the total bondage number $b_t(T) = n$.

For example, if n is the positive integer, consider the following tree on $3n+2$ vertices.

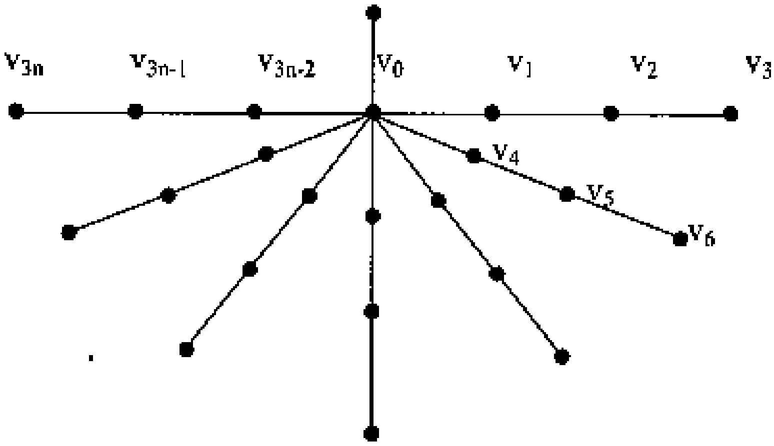


Figure 4.1: Tree H_n.

We denote this tree by H_n.

Theorem 4.6

For the tree H_n, $b_t(H_n) = n$.

Proof. First we note that $\gamma_t(H_n) = 2n+1$. Let

$$E_0 = \{v_0v_1, v_0v_4, \ldots\ldots, v_0v_{3n-2}\}.$$

Then $\gamma_t(H_n - E_0) = 2n+2 > \gamma_t(H_n)$ and hence

$$b_t(H_n) \le |E_0| = n. \tag{4.1}$$

Let $E_1 \subseteq E(H_n)$ and $|E_1| < n$ and there is no isolated vertex in $H_n - E_1$.

Then we note that

(i) $v_{3j}v_{3j-1} \notin E_1$ for all $j = 1, 2, \ldots\ldots, n$;

(ii) if $v_0v_{3j-2} \in E_1$, then $v_{3j-2}v_{3j-1} \notin E_1$ and

if $v_{3j-2}v_{3j-1} \in E_1$, then $v_0v_{3j-2} \notin E_1$;

(iii) as $|E_1| < n$, there is at least one k $(1 \le k \le n)$ such that

$v_0v_{3k-2}, v_{3k-2}v_{3k-1}, v_{3k-1}v_{3k} \notin E_1$.

Then the set $D = \{v_0, v_{3k-1}, v_{3k-2}, v_{3j}, v_{3j-1};\ j \in \{1,2,3,......,n\} - \{k\}\}$ is a total dominating set for $H_n - E_1$.

As $|D| = 2n+1$, we have

$$\gamma_t(H_n - E_1) = \gamma_t(H_n) = 2n+1.$$

Then $\gamma_t(H_n - E_1) = \gamma_t(H_n)$, whenever $|E_1| < n$.

Thus, $b_t(H_n) \geq n$ (4.2)

From (4.1) and (4.2), we have $b_t(H_n) = n$.

We now obtain an upper bound for $b_t(T)$, where T is a tree.

Theorem 4.7

If T is a tree on n vertices and $T \neq K_{1,n-1}$, then $b_t(T) \leq \min\{\Delta(G), \frac{n-1}{3}\}$.

Proof: Let T be a tree which is not a star. Let $P = v_1, v_2, v_3, ..., v_k\ (k \geq 4)$ be a longest path in T. If $\gamma_t(T) \leq 3$ or $k = 5$, $\gamma_t(T - e_1) > \gamma_t(T)$, where $e_1 = v_2 v_3$. In this case, $E_0 = \{e_1\}$ and $b_t(T) = 1$. Now we assume that $\gamma_t(T) \geq 4$ and $k \geq 6$, let $e_1 = v_2 v_3$.

If $v_3 \in D$ for some minimum total dominating set D, for T_3, where T_3 is the component of $T - e_1$ which contains the vertex v_3, then $D \cup \{v_2\}$ is a total dominating set for T while $|D| + 2$ is the $\gamma_t(T - e_1)$. In this case, $\gamma_t(T - e_1) > \gamma_t(T)$. Hence, $b_t(T) = 1$.

So, assume that v_3 is not a vertex of any minimum total dominating set D for T_3. Then clearly, $\deg(v_3) = 1$ in T_3 and $v_4 \in D$ for every minimum total dominating set for T_3. If $\deg(v_4) = 2$ in T_3, then $v_5 \in D$ for all minimum total dominating set for T_3.

We can take $E_1 = \{v_2v_3, v_4v_5\}$ to get $\gamma_t(T - E_1) > \gamma_t(T)$. Hence, in this case, $\gamma_t(T) \leq 2$. Now we consider the case in which $\deg(v_4) \geq 3$.

Let S be the set of all paths P_1 in T such that P_1 starts from the point v_4 and $P \cap P_1 = \{v_4\}$.

Let $P_1 \in S$. If there is no $P_2 \in S, P_2 \neq P_1$ such that P_1 is a part of P_2, then we say that P_1 is a maximal path in T starting from the vertex v_4 such that $P \cap P_1 = \{v_4\}$.

Let P_1 be a maximal path in T starting from the point v_4 such that $P \cap P_1 = \{v_4\}$. Then P_1 is of length at most 3. If length of $P_1 = 2$, then e_2 is the edge in P_1 incident with v_4. If e_2 is a part of a path P_2 of length 3, when $P \cap P_2 = \{v_4\}$, let e_3 be the edge of P_2 which is incident with e_2, then $\gamma_t(T - e_3) > \gamma_t(T)$. If e_2 is not a part of any path P_2 of length 3, when $P \cap P_2 = \{v_4\}$, then $\gamma_t(T - e_1 - e_2) > \gamma_t(T)$ and $\gamma_t(T) \leq 2$.

So, assume that there is no maximal path P_1 of length 2 starting from v_4 such that $P \cap P_1 = \{v_4\}$. If length of $P_1 = 1$ for all $P_1 \in S$, then as $v_3 \notin D$ for any minimal dominating set of T_3, we get $v_5 \in D$. In this case, let $e_2 = v_4v_5$, so $b_t(T) \leq 2$, as $\gamma_t(T - e_1 - e_2) > \gamma_t(T)$.

So, assume that there are maximal paths P_1 in S with length 3. Let $Q_1, Q_2, \ldots\ldots, Q_n$ be a maximal collection of distinct such paths. If $v_5 \notin D$, for all minimal total dominating set of T_3, then let

$$E_0 = \{v_2v_3\} \cup \{\textit{central edges of } Q_i : i = 1,2,\ldots\ldots,n\}.$$

If v_5 is in some minimum total dominating set for T_3, then take

$$E_0 = \{v_2v_3, v_4v_5\} \cup \{\textit{all central edges of } Q_i : i = 1,2,\ldots\ldots,n\}.$$

Then $\gamma_t(T - E_0) > \gamma_t(T)$. Therefore, $b_t(T) \leq |E_0| \leq \frac{n-1}{3}$.

Also, as $|E_0| \leq \deg(v_4)$, then we have $b_t(T) \leq \Delta(G)$.

Remark:

Let G be a connected graph other than K_1, K_2, K_3 and $K_{1,n}$. As $G \neq K_1, K_2, K_3$ and $K_{1,n}$, there is a path of length 3 in G. Then we can find $E_1 \subseteq E$ such that $G - E_1$ is a spanning tree T for G containing a path of length 3. ($|E_1| = q - n + 1$). For the tree T we find $E_2 \subseteq E(G - E_1)$ such that $\gamma_t(G - E_1 - E_2) > \gamma_t(G - E_1) \geq \gamma_t(G)$. So, $b_t(G) \leq |E_1| + |E_2|$.

Thus, for every connected graph G other than K_1, K_2, K_3 and $K_{1,n}$, $b_t(G)$ is finite.

3. Total bondage number of K_n.

If S and T are disjoint subsets of $V(G)$, by $[S,T]$, we mean the set of all edges in G which have one end in S and the other end in T. Thus, $e = uv \in [S,T]$ if and only if $|\{u,v\} \cap S| = 1 = |\{u,v\} \cap T|$.

In [21], Kulli has proved that $b_t(K_n) = 2n - 5$, if $n \geq 5$.

We give an another proof for this result.

Theorem 4.8

Let K_n be a complete graph with $n \geq 5$ vertices, then $b_t(K_n) = 2n - 5$.

Proof: First we prove that $b_t(K_n) \geq 2n - 5$. Let $E_0 \subseteq E(K_n)$ with $|E_0| \leq 2n - 6$, such that there is no isolated vertex in $K_n - E_0$. We claim that $\gamma_t(K_n - E_0) = 2$. Assume that $\gamma_t(K_n - E_0) > 2$. Hereafter we denote the graph $K_n - E_0$ by H.

Let $s = \min\{n - 1 - \deg_H(u) : u \in V(H)\}$. As $|E_0| \leq 2n - 6$, we have $0 \leq s \leq 3$.

If $s = 0$, then $\deg_H(u_0) = n - 1$ for some $u_0 \in V(H)$ and hence $\{u_0, u_1\}$ is a total dominating set for H, when $u_1 \neq u_0 \in V(H)$, which is a contradiction.

If $s=1$, there are vertices $u,w \in V(H)$ such that $\deg_H(u)=n-2$, the edge $uw \in E_0$. As w is not an isolated vertex in H, $wv \in E(H)$ for some v (as $v \neq w, uv \notin E_0$). Now $\{u,v\}$ is a total dominating set for H, which is a contradiction.

Assume that $s=2$. Select u,v_1,v_2 three distinct vertices in H such that $\deg_H(u)=n-3$ and the edges $uv_1, uv_2 \in E_0$. Let $S'=V(H)-\{u,v_1,v_2\}$. Then we observe the following:

(i) $|S'|=n-3$.

(ii) there is no vertex $w \in S'$ such that both v_1w and $v_2w \notin E_0$.

(If such a $w \in S'$ exists, then $\{u,w\}$ is a total dominating set for H).

(iii) let $S=\{w \in S' : v_1w \notin E_0\}$, $T=\{w \in S' : v_2w \notin E_0\}$ and $U=\{w \in S' :$ both v_1w and $v_2w \in E_0\}$. Then $S'=S \cup T \cup U$, $S \neq \phi$, $T \neq \phi$ and the subsets S,T,U are disjoint to each other.

Assume that $S \neq \phi$. Then $v_1v_2 \notin E_0$, as v_1 is not isolated in H. If $T=\phi$, then $|E_0| \geq 2+2(n-3)=2n-4$, which is a contradiction.

Hence, $T \neq \phi$. To each $w \in T$ there is at least one $z \in U$ such that $wz \in E_0$. (Otherwise $\{v_2,w\}$ is a total dominating set for H). This implies that $|E_0| \geq 2+2|U|+2|T|=2+2(n-3)$, as $S=\phi$, which is a contradiction.

Thus, $S \neq \phi$. Similarly, $T \neq \phi$.

As to each $w \in S'$, either $wv_1 \in E_0$ or $wv_2 \in E_0$, we have

$$\left|\,[\{v_1,v_2\},S'] \cap E_0\right| = n-3+|U| \tag{4.3}$$

(iv) fix one vertex $w_1 \in S$ and a vertex $w_2 \in T$. To each $w \in T$, either $w_1w \in E_0$ or there is a vertex $z \in S'$ such that both wz and $w_1z \in E_0$. (Otherwise $\{w,w_1\}$ is a total dominating set for H).

In the case $w_1w \in E_0$, select w_1w, otherwise select wz. By this process, by varying w in T, we get a collection $E_1 \subseteq E_0$ and $|E_1| = |T|$. Now to each $w \in S$, either $w_2w \in E_0$ or there is a vertex $z \in S'$ such that both wz and $w_2z \in E_0$. Select w_2w if it is in E_0, otherwise select wz. We get a collection $E_2 \subseteq E_0$ and $|E_2| = |S|$. We note that

$$|E_1 \cup E_2| \geq |S| + |T| - 1 \text{ and } (E_1 \cup E_2) \cap [\{v_1, v_2\}, S'] = \phi \qquad (4.4)$$

By (4.3) and (4.4), we have

$$|E_0| \geq n - 3 + |U| + |S| + |T| - 1 + 2 = n - 3 + n - 3 - 1 + 2 = 2n - 5,$$

which is a contradiction as $|E_0| \leq 2n - 6$. Thus, $s \neq 2$. Hence the only possibility is $s = 3$.

Assume that $s = 3$. Let u, v_1, v_2, v_3 be four vertices in H such that uw is an edge in H, for all $w \neq u, v_1, v_2, v_3$ and $uv_j \in E_0$ for $j = 1,2,3$. Let $S' = V(H) - \{u, v_1, v_2, v_3\}$. There is no vertex $w \in S'$ such that $v_jw \notin E_0$ for all $j = 1,2,3$ (as $\{u, w\}$ is not a total dominating set for H).

Let $S_j = \{w \in S' : |[\{v_1, v_2, v_3\}, \{w\}]| = 3 - j \text{ in } H\}$ for $j = 1,2,3$.

As $s = 3$, to each $w \in S_1$, there are at least two vertices w_1 and w_2 in S' such that $ww_1, ww_2 \in E_0$ (w_j may be in S_1). Now it is possible to select $|S_1|$ disjoint edges in E_0 such that each one of selected edge has one end in S_1 and the other in S'.

Thus,
$$\begin{aligned} |E_0| &\geq 3 + 2|S_1| + 2|S_2| + 2|S_3| \\ &\geq 3 + 2(|S_1| + |S_2| + |S_3|) \\ &= 3 + 2(n-4) \\ &= 2n - 5, \end{aligned}$$

which is a contradiction.

Thus, in all the cases, we get a contradiction.

Thus, $$b_t(K_n) \geq 2n - 5 \qquad (4.5)$$

Now we show that $b_t(K_n) \le 2n-5$.

Select four vertices $u_1, u_2, u_3, u_4 \in V(K_n)$.

$$E_0 = \{wu_2, wu_3, u_1u_4, u_1u_3, u_2u_4 \mid w \in V(K_n) \text{ and } w \ne u_1, u_2, u_3, u_4\}.$$

Then $|E_0| = 2(n-4)+3 = 2n-5$ and $\gamma_t(K_n - E_0) = 3$ and $\{u_1, u_2, u_3\}$ is a total dominating set for $K_n - E_0$. Thus,

$$b_t(K_n) \le |E_0| \le 2n-5 \qquad (4.6)$$

By (4.5) and (4.6), it follows that $b_t(K_n) = 2n-5$, if $n \ge 5$.

Remark:

If $G = K_4$, then $b_t(K_4) = 4 = 2n-4$ and if $G = K_3$, then $b_t(K_3) = \infty$.

CHAPTER FIVE

THE BONDAGE NUMBER FOR TWO DOMINATION

One measure of the stability of the domination number G under edge removal is the bondage number $b(G)$. It was defined by Fink et al. [11]. The bondage number $b(G)$ of a non-empty graph G is the minimum cardinality among all sets of edges E for which $\gamma(G-E) > \gamma(G)$. Thus, the bondage number $b(G)$ of G is the smallest number of edges, whose removal will render every minimum dominating set of G a non-dominating set in the resulting spanning subgraph. The bondage number is denoted by $b(G)$.

Now we define the bondage number for two domination.

Definition:

Let G be a graph with at least one edge. The bondage number $b_2(G)$ for two domination of G is the smallest cardinality of an edge set $E_0 \subset E(G)$ for which $\gamma_2(G-E_0) > \gamma_2(G)$. We also call $b_2(G)$, the two bondage number of a graph G.

Now we obtain some results on $b_2(G)$.

Lemma 5.1

If any vertex of a tree T is adjacent with two or more end vertices, then $b_2(T) = 1$.

Proof: Assume that u is a vertex in T and two end vertices x and y are adjacent to u in T. Take $e = ux$. If D is a two-dominating set for $T - x$, then D is a two-dominating set for T also.

Hence, $\gamma_2(T-x)=\gamma_2(T)$. But $\gamma_2(T-e)=1+\gamma_2(T-x)$.

Thus, $\gamma_2(T-e)>\gamma_2(T)$ and so $b_2(T)=1$.

Lemma 5.2

Let T be a tree. If there is an end vertex u of T such that there is no γ_2 – set for T containing u, then $b_2(T)=1$.

Proof: Let u be an end vertex of T such that whenever D is a γ_2 – set for T, then $u \notin D$. Let e be the edge incident with u.

Claim: $\gamma_2(T-u)=\gamma_2(T)$.

If $\gamma_2(T-u)<\gamma_2(T)$, let D_0 be a γ_2 – set for $T-u$. Then $D_0 \cup \{u\}$ is a γ_2 – set for T, which is a contradiction. Hence, $\gamma_2(T-u)=\gamma_2(T)$.

But $\gamma_2(T-e)=\gamma_2(T-u)+1=\gamma_2(T)+1$.

Thus, $\gamma_2(T-e)>\gamma_2(T)$ and so $b_2(T)=1$.

Remark:

Converse of the theorem 5.2 is not true. There are trees T with $b_2(T)=1$, but for every given end vertex u of T there is one γ_2 – set of T such that $u \in D$.

Example:

For $T=P_7$, a path on 7 vertices, $b_2(T)=1$, and for every end vertex u of P_7, there is one γ_2 – set of T such that $u \in D$.

Now we obtain an upper bound for $b_2(G)$, where G is a tree.

Lemma 5.3

For every tree T, with $n \geq 3$, $b_2(T) \leq 2$.

Proof: Let T be a tree and $P = u_0 u_1 u_2 ...... u_m$ be a longest path in T. Let $e_1 = u_0 u_1$ and $e_2 = u_1 u_2$. Let $H = T - e_1 - e_2$.

If T_1 is the component of H containing u_2 and A is a γ_2-set for T_1, then $\gamma_2(H) = \gamma_2(T_1) + 2$, whereas $\gamma_2(T) \le \gamma_2(T_1) + 1$ [as $A \cup \{u_1\}$ is a γ_2-set for T.] Thus, $b_2(T) \le 2$.

Example for a tree for which $b_2(T) = 2$.

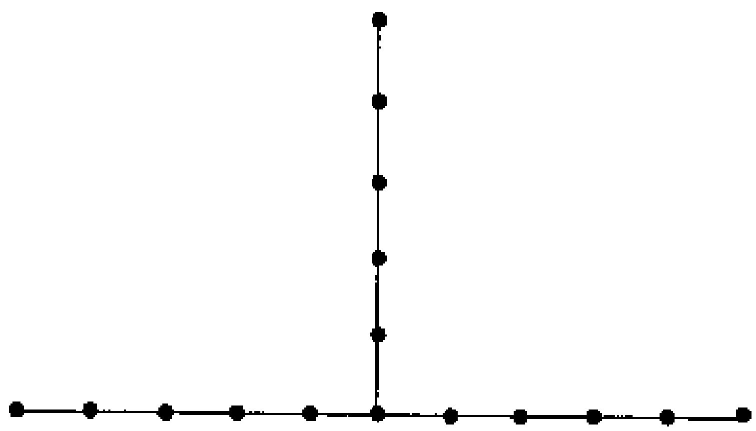


Figure 5.1: A tree T with $b_2(T) = 2$.

Now we obtain exact values of $b_2(G)$ for some standard graphs G.

Lemma 5.4

The bondage number $b_2(G)$ for two-domination of the complete graph K_n, $n \ge 3$, is $b_2(K_n) = n - 1$.

Proof: Let E_1 be a set of $n-2$ edges of K_n and $H = K_n - E_1$. Let u_0 be a vertex of H with least degree. Take $\deg(u_0) = m < n-1$, and $N_1(u_0) = \{u_1, ..., u_m\}$. If $v \in V(G)$ and $v \ne u_0, ..., u_m$, then $\deg_H(v) > (n-1) - (m-1)$ and hence

$\deg_H(v) > n - m$, and v is adjacent to at least one of $u_0, \ldots\ldots, u_m$. Then $\{u_0\}$ is a γ_2 – set for H and $\gamma_2(K_n) = \gamma_2(H) = \gamma_2(K_n - E) = 1$ whenever $|E| \leq n - 2$.

Thus, $b_2(K_n) \geq n - 1$ (5.1)

Fix one vertex u of $V(G) = V(K_n)$. Let E_1 be the set of all $n-1$ edges of G incident with u. Then for $H = V - E_1$, $\gamma_2(H) = 2 > \gamma_2(K_n)$.

Thus, $b_2(K_n) \leq n - 1$ (5.2)

From (5.1) and (5.2), we can conclude that

$$b_2(K_n) = n - 1.$$

The $b_2(G)$, where G is a path, cycle, wheel are given in the following theorems.

Theorem 5.5

The bondage number $b_2(G)$ for two-domination of the path of order n ($n \geq 2$), is given by

$$b_2(P_n) = \begin{cases} 1 & \text{if } n \not\equiv 1 (\text{mod} 5) \\ 2 & \text{if } n \equiv 1 (\text{mod} 5). \end{cases}$$

Theorem 5.6

If C_n is the cycle on n vertices, $b_2(C_n)$ is given by

$$b_2(C_n) = \begin{cases} 2 & \text{if } n \not\equiv 1 \,(\text{mod} 5) \\ 3 & \text{if } n \equiv 1 \,(\text{mod} 5) \end{cases}$$

Theorem 5.7 $b_2(K_{m,n}) = \min(m, n)$.

Theorem 5.8

If W_n is a wheel on n vertices, then $b_2(W_n) = 3$, for all $n \geq 4$.

We now obtain an upper bound for $b_2(G)$, where G is a connected graph.

Theorem 5.9

For every connected graph G with $n \geq 2$ vertices,

$$b_2(G) \leq n - \gamma_2(G).$$

Proof: Let G be a connected graph with $n \geq 2$ vertices. Let $u \in V(G)$ and uv be an edge in G. Let E_1 be the set of edges in G incident with u.

Case (i):

Assume that $b_2(G) > |E_1| = \deg(u)$. Then $\gamma_2(G) = \gamma_2(G - E_1)$ and $\gamma_2(G) = 1 + \gamma_2(G - u)$. If D is a minimum two-dominating set for $G - u$, then $(N_1(u) \cup N_2(u)) \cap D = \phi$ in G, otherwise D is a two-dominating set for G and hence $\gamma_2(G) = \gamma_2(G - u)$, which is a contradiction.

Let S be the union of all minimum two-dominating sets for $G - u$ and let $E_v = [\{v\}, N(S)]$ in $G - u$. If $\gamma_2(G - u - E_v) \leq \gamma_2(G - u)$, then any minimum two-dominating set D' for $G - u - E_v$ contains a vertex w such that $d(w, v) \leq 2$. This shows that $w \notin S$. But as $|D'| = \gamma_2(G - u), D' \subseteq S$ and $w \in S$, we get a contradiction.

Hence, $\gamma_2(G - u - E_v) = 1 + \gamma_2(G - u) = \gamma_2(G - E_1) = \gamma_2(G)$

$$\gamma_2(G - E_1 - E_v) = 1 + \gamma_2(G - u - E_v) = 1 + \gamma_2(G)$$

$$b_2(G) \leq |E_1 \cup E_v| = |E_1| + |E_v| = \deg(u) + |E_v|.$$

In G, the set $N_1(u)$ and $N(S)$ are disjoint [otherwise if $y \in N_1(u) \cap N(S)$, there exists $w \in S$ such that wy is an edge and a minimum two-dominating set D' for $G - u$ which contains w. Then D' is also a two-dominating set for G as $d(w, u) \leq 2$, which is a contradiction as $\gamma_2(G) = 1 + \gamma_2(G - u)$].

Thus, $|E_v| \leq (n-1) - \deg(u) - |S|$

$$\leq n - 1 - \deg(u) - \gamma_2(G - u) = n - 1 - \deg(u) - (\gamma_2(G) - 1).$$

Hence, $b_2(G) \le \deg(u) + n - 1 - \deg(u) - (\gamma_2(G) - 1)$

i.e., $b_2(G) \le n - \gamma_2(G)$ (5.3)

Case (ii):

If $b_2(G) \le |E_1|$, then

$$b_2(G) \le \deg(u) \le n - \gamma_2(G) \tag{5.4}$$

Thus, in all the cases, we have

$$b_2(G) \le n - \gamma_2(G).$$

This completes the proof of the theorem.

Remark:

In a connected graph G, for any vertex, $\deg(u) \le n - \gamma_2(G)$.

If $\gamma_2(G) = 1$, then $\deg(u) \le n - 1$.

If $\gamma_2(G) \ne 1$, let $D = \{u_1, \ldots, u_t\}$ be a minimum two-dominating set for G, where $t = |\gamma_2(G)|$. To each u_i, there is a vertex $w_i \in V(G)$ such that $d(u_j, w_i) \le 2$ only for $j = i$. Let $I = \{i : 1 \le i \le t \text{ and } u_i \in N_1(u)\}$.

Then either $|I| \le 1$ or $w_i \notin N_1(u)$ for all $i = 1, 2, \ldots\ldots, n$.

Therefore, $\deg(u) \le n - t = n - \gamma_2(G)$.

Fink, Jacobson, Kinch and Roberts introduced the bondage number of a graph and proved that any tree has bondage number either 1 or 2 and posed as an open problem to classify the trees of bondage number 2.

In [14], B.L. Hartnell and D.F. Rall gave structural characterisation of class of trees for which the bondage number is 2. We now demonstrate that their techniques with some modification is valid to characterise the class of trees for which the bondage number for two-domination $b_2(T)$ is 2.

By P_n, we mean a path on n vertices. When we say that a path $P_n[x_1,\ldots,x_n]$ is attached to a vertex v in a tree we refer the operation of joining the vertices v and x_1 by an edge. We refer the graphs given in Figure 5.2 and Figure 5.3, respectively as H_1 and H_n.

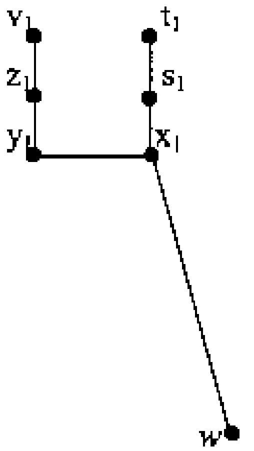


Figure 5.2: H_1

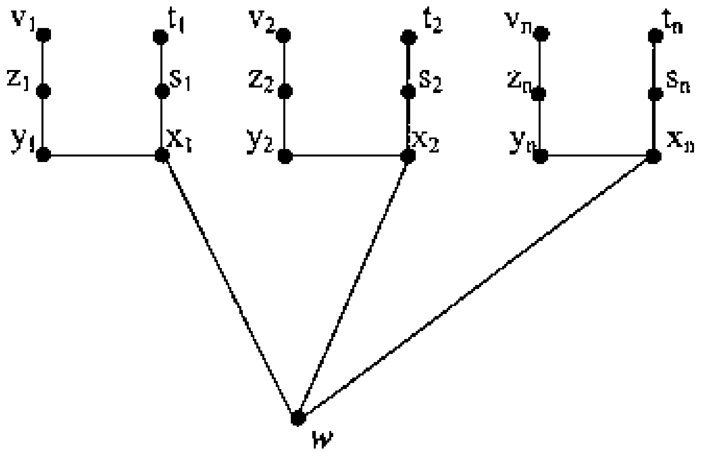


Figure 5.3: H_n

When we say that we attach H_n (or H_1) to a vertex u in a tree T, we mean that the vertex u and the vertex w of H_1 are joined by an edge.

Definition:

A vertex v of a tree T is called a **down vertex** of T if $\gamma_2(T-v)<\gamma_2(T)$ and **level vertex** of T if $\gamma_2(T-v)=\gamma_2(T)$.

If $\gamma_2(T-v)\geq\gamma_2(T)$, then v is said to be a **non-down vertex.** In the same way we say that the vertex v of T is

1. a **neighborhood level vertex** if

   $\gamma_2(T-(N_1(v)\cup\{v\}))=\gamma_2(T)$.

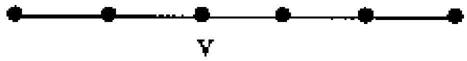
2. a **neighborhood down vertex** if

$$\gamma_2(T-(N_1(v)\cup\{v\}))<\gamma_2(T).$$

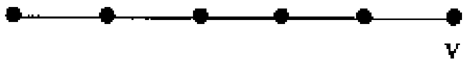
3. a **neighborhood non-down vertex** if

$$\gamma_2(T-(N_1(v)\cup\{v\}))\geq\gamma_2(T).$$

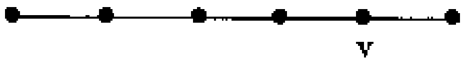
Examples:



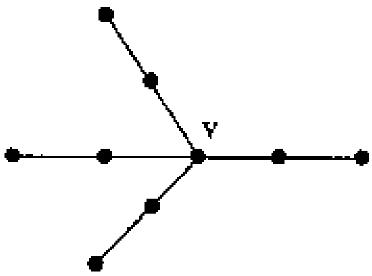
(a)



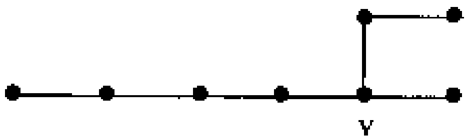
(b)



(c)



(d)



(e)

Figure 5.4: (a), (b), (c), (d), (e)

Figure 5.4	The vertex v is
(a)	Level vertex and neighborhood level vertex
(b)	Down vertex and neighborhood level vertex
(c)	Level vertex but neighborhood down vertex
(d)	Non-down vertex and neighborhood non-down vertex
(e)	Non-down vertex and neighborhood level vertex

We have already noted that for any tree $b_2(T) \leq 2$.

Let ε_0 be the collection of trees T for which $b_2(T) \neq 1$.

We devote the remaining section of this chapter to describe the class ε_0.

Thus,

$$\varepsilon_0 = \{T : T \text{ is a tree and } \gamma_2(T-e) = \gamma_2(T) \text{ for all } e \in E(T)\}$$

Lemma 5.10

Let T be a tree and let u be a vertex in T. If either u is adjacent to two end vertices or u is adjacent to an end vertex and a path P_2 is attached in T at u, or at least two paths P_2 are attached in T at u, then $T \notin \varepsilon_0$ and $b_2(T) = 1$.

Proof: Let T be a tree and let u be a vertex in T. Assume that u is adjacent to two end vertices, say x_1 and y_1 or u is adjacent to an end vertex x_2 and a path $P_2[x_1, y_1]$ is attached at u or two paths $P_2[x_1, y_1]$ and $P_2[x_2, y_2]$ are attached at u as shown in the Figure 5.5.

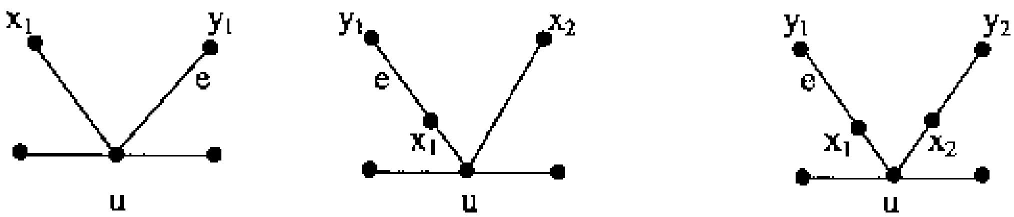


Figure 5.5: (a), (b), (c)

Let D be a γ_2 – set for $T - x_2$. Then D is also a two-dominating set for T and $D \cup \{y_1\}$ is a γ_2 – set for $T - e$, where $e = u\,y_1$ in the first case, while $e = x_1\,y_1$ for the other two cases. Hence, $b_2(T) = 1$ and $T \notin \varepsilon_0$.

If T is a given tree, we consider the following operations.

Type (i)

Attach a path P_3 to T at a vertex v, where v is a non-neighborhood down vertex in T and v belongs to at least one γ_2 – set of T.

Type (ii)

Attach a path P_4 to T at a vertex v, where v is a neighborhood down vertex in T.

Type (iii)

Attach H_1 to T at v if there is at least one γ_2 – set D for T such that

$$D \cap (N_1(v) \cup \{v\}) \neq \phi$$

and v is not a down vertex of T.

Type (iv)

Attach H_n $(n \geq 2)$ to T at v, where v is not a down vertex of T.

We now prove that if $T \in \varepsilon_0$ and S is a tree obtained from T by any one of the four types of operations, then $S \in \varepsilon_0$.

Lemma 5.11

If $T \in \varepsilon_0$ and S is a tree obtained from T by **Type (i)** operation, then $S \in \varepsilon_0$.

Proof: Let $T \in \varepsilon_0$ and $v \in V(T)$ such that

$$\gamma_2(T - (N_1(v) \cup \{v\})) \geq \gamma_2(T)$$

and v belongs to at least one γ_2 – set of T. Let S be the tree obtained from T by attaching a P_3, say $[x, y, z]$ at v.

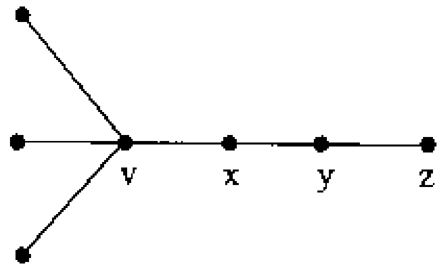


Figure 5.6

As $\gamma_2(T-(N_1(v)\cup\{v\}))\geq\gamma_2(T)$, it follows that $\gamma_2(S)=\gamma_2(T)+1$. Let $e\in E(T)$ and D be a γ_2 – set for $T-e$. Then $D\cup\{x\}$ is a γ_2 – set for $S-e$ and

$$\gamma_2(S-e)\leq|D\cup\{x\}|=|D|+1=\gamma_2(T)+1=\gamma_2(S).$$

Hence, $\gamma_2(S-e)=\gamma_2(S)$ whenever $e\in E(T)$. Let D' be a γ_2 – set for T which contains v. Then $D'\cup\{z\}$ is a γ_2 – set for $S-e$, where $e=ux$ or xy or yz and $|D'\cup\{z\}|=\gamma_2(T)+1=\gamma_2(S)$.

Thus, $\gamma_2(S-e)=\gamma_2(S)$ is true even if $e\in E(S)-E(T)$.

Therefore, $S\in\varepsilon_0$.

Lemma 5.12

If $T\in\varepsilon_0$ and S is a tree obtained from T by a **Type (ii)** operation, then $S\in\varepsilon_0$.

Proof: Let $T\in\varepsilon_0$ and let v be a neighborhood down vertex in T. Then

$$\gamma_2(T-(N_1(v)\cup\{v\}))<\gamma_2(T).$$

Let S be a tree obtained from T by attaching a P_4, say $[x, y, z, w]$ at v as shown in the Figure 5.7.

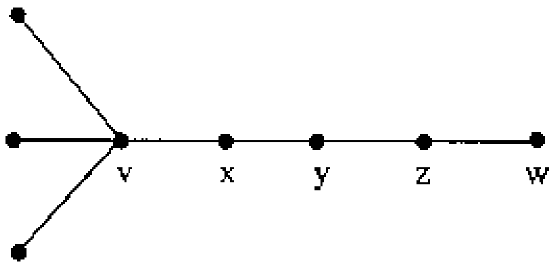


Figure 5.7

If D is a γ_2 − set for $T-(N_1(v)\cup\{v\})$, then $D\cup\{v\}$ is a γ_2 − set for T and hence

$$\gamma_2(T-(N_1(v)\cup\{v\}))+1=\gamma_2(T).$$

Also, $D\cup\{v,w\}$ is a γ_2 − set for S and any γ_2 − set for S should contain at least one vertex of $\{y,z,w\}$. Hence we have $\gamma_2(S)=\gamma_2(T)+1$.

If $e=vx$ or xy or yz, then whenever D is a γ_2 − set for $T-(N_1(v)\cup\{v\})$, $D\cup\{v,z\}$ is a γ_2 − set for $S-e$. If $e=zw$, then $D\cup\{x,w\}$ is a γ_2 − set for $S-e$. If $e\in E(T)$, then let D be a γ_2 − set for $T-e$. Then $D\cup\{y\}$ is a γ_2 − set for $S-e$ and

$$\gamma_2(S-e)\le|D\cup\{y\}|=|D|+1=\gamma_2(S).$$

Thus, in all the cases, $\gamma_2(S-e)=\gamma_2(S)$ and hence $S\in\varepsilon_0$.

Lemma 5.13

Let $T\in\varepsilon_0$ and let S be a tree obtained from T by a **Type (iii)** operation. Then $S\in\varepsilon_0$.

Proof: Let v be a vertex in T such that there is at least one γ_2 − set D for T such that $D\cap(N_1(v)\cup\{v\})\neq\phi$ and v is not a down vertex of T. Let S be the tree obtained by attaching H_1 at v as shown in the Figure 5.8.

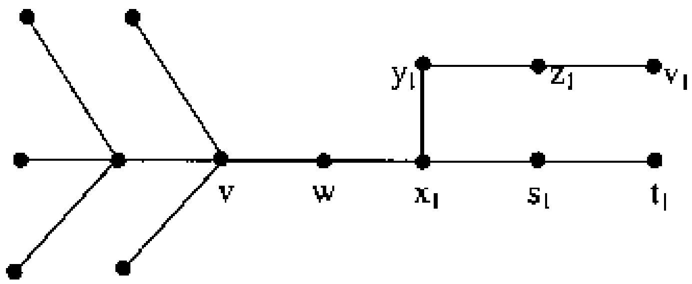


Figure 5.8

As $\gamma_2(T-v) \geq \gamma_2(T)$, we have $\gamma_2(S) = \gamma_2(T)+2$. If D is any γ_2-set for T such that $D \cap (N_1(v) \cup \{v\}) \cap V(T) \neq \phi$, then

(i) $D \cup \{t_1, y_1\}$ is a γ_2-set for $S-e$, where $e = s_1 t_1$ or $x_1 s_1$.

(ii) $D \cup \{v_1, x_1\}$ is a γ_2-set for $S-e$, where $e = y_1 x_1$ or $z_1 v_1$ or $x_1 y_1$.

(iii) $D \cup \{x_1, y_1\}$ is a γ_2-set for $S-e$, where $e = uw$ or wx_1 or $x_1 y_1$.

If $e \in E(T)$, let D be a γ_2-set for $T-e$. Then $D \cup \{x_1, y_1\}$ is a γ_2-set for $S-e$ and $\gamma_2(S-e) \leq |D \cup \{x_1, y_1\}|$

$$= |D|+2$$

$$= \gamma_2(T-e)+2$$

$$= \gamma_2(T)+2$$

$$= \gamma_2(S)$$

Therefore, $S \in \varepsilon_0$.

Lemma 5.14

If $T \in \varepsilon_0$ and S is a tree obtained from T by a **Type (iv)** operation, then $S \in \varepsilon_0$.

Proof: Let T be a tree and $T \in \varepsilon_0$. Let u be a vertex in T such that u is not a down vertex for T. Hence, $\gamma_2(T-u) \geq \gamma_2(T)$. Let H_n be attached to u, as shown in the Figure 5.9 and let the resulting tree be S.

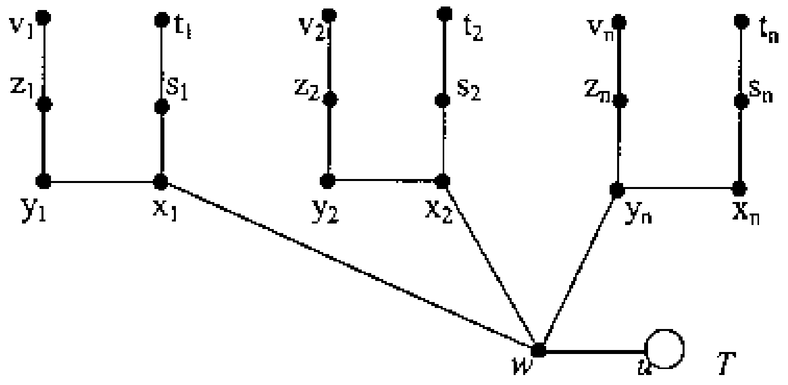


Figure 5.9

As $\gamma_2(T-u) \geq \gamma_2(T)$, it follows that whenever D is a γ_2-set for T, $D \cup \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ is a γ_2-set for S.

[If D' is a γ_2-set for S, then if $u \in D'$, then $D' \cap V(T)$ is a γ_2-set for T and if $u \notin D'$, then $D' \cap V(T)$ is a γ_2-set for $T-u$ and hence $D' \geq 2n + \gamma_2(T)$].

If $e \in E(T)$, let D be a γ_2-set for $T-e$, then $D' = D \cup \{x_i, y_i : i = 1, 2, \ldots\ldots, n\}$ is a γ_2-set for S and $\gamma_2(S-e) \leq |D'| = \gamma_2(T-e) + 2n = \gamma_2(T) + 2n = \gamma_2(S)$ and hence $\gamma_2(S-e) = \gamma_2(S)$.

We note that if D is a γ_2-set for T, then

(i) $D \cup \{x_i, y_i : i = 1, 2, \ldots\ldots, n\}$ is a γ_2-set for $S-e$, where $e = x_j y_j$ or $x_j w$ or wu for some j.

(ii) $D \cup \{v_i, x_i : i = 1, 2, \ldots\ldots, n\}$ is a γ_2-set for $S-e$, where $e = y_j z_j$ or $z_j v_j$ for some j.

(iii) $D \cup \{x_i, y_i : i \neq j\} \cup \{y_j, t_j\}$ is a γ_2-set for $S-e$, where $e = t_j s_j$ or $s_j x_j$ for some j.

Thus, in all the cases, we have

$$\gamma_2(S-e) = \gamma_2(S)$$

for all $e \in E(S)$. Therefore, $S \in \varepsilon_0$.

CONCLUSION

This thesis is devoted to the domination theory in graphs. The concept of dominating sets introduced by Ore and Berge currently receives more attention in Graph Theory. A rapid growth of research in this area and a wide variety of domination parameters have been introduced after the investigations on the theory of domination in graphs by Cockayne and Hedetniemi. This thesis concentrates mainly on the two cobondage number, the total cobondage number, the total bondage number and the two bondage number of a graph.

This thesis contains five chapters.

In the first chapter, we present necessary graph-theoretic definitions and earlier works on the domination theory.

In the second chapter, we have found a best upper bound for $cb(G)+cb(\overline{G})$, the sum of the cobondage number of G and the cobondage number of the complement of G. We have characterised the graph G for which $cb(G)+cb(\overline{G})=n-1$, where $|V(G)|=n$. A constructional method is also developed to obtain all these graphs. If G is a graph with $\gamma_2(G)\geq 2$, the minimum cardinality among the sets $E_1\subseteq E(\overline{G})$ such that $\gamma_2(G+E_1)<\gamma_2(G)$ is denoted by $(cb)_2(G)$ and is called the two cobondage number of G. This chapter also deals with the cobondage number for two-domination. Upper bounds for $(cb)_2(G)$ are obtained and a structural theorem for the graphs for which $(cb)_2(G)=\Delta(G)$ has been proved.

Chapter three deals with the total cobondage number of a graph. An upper bound for the total cobondage number has been obtained.

In the fourth chapter, we have proved that for any given positive integer n, we can find a tree T for which the total bondage number is n. An upper bound for $b_t(T)$, where T is a tree, is obtained. A rigorous proof was given to prove that the total bondage number of the complete graph K_n is $2n-5$.

In the last chapter, the concept of bondage number for two-domination is introduced and a study on it has been initiated. If G is a graph with at least one edge, the bondage number $b_2(G)$ for two-domination is the smallest cardinality of an edge set $E_0 \subset E(G)$ for which $\gamma_2(G-E_0) > \gamma_2(G)$. The exact values of $b_2(G)$ are obtained for known families of graphs. An upper bound for $b_2(G)$, where G is a connected graph has been obtained. It is also proved that $b_2(T) \leq 2$ whenever T is a tree. A structural characterization of the class of trees for which $b_2(T) = 2$ has been obtained.

In future, any one can proceed with this work by using algorithms.

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