# The Steady Natural Convection Flow on a Horizontal Circular Disc with Transpiration.

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by



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# Declaration

None of the materials contained in this thesis will be submitted in support of any other degree or diploma at any other university or institution other than publications.

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# Abastract

The present study deals with the effects of transpiration (either suction or blowing) on skin friction and heat transfer coefficients for the steady laminar free convection boundary-layer flow generated by heated horizontal circular disc. The Boussinesq approximation is employed firstly to deal with the two possible steady cases. Secondly, the numerical solutions are displayed for different values of the established parameters.

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# Chapter-1

## General introduction with review of previous research

Fluid mechanics is a subject of widespread interest to researchers and it becomes an obvious challenge for the scientists, engineers as well as users to understand more about fluid motion. Au important contribution to the fluid dynamics is the concept of boundary-layer introduced first by L. Prandtl (1904). The concept of the boundary-layer is the consequence of the fact that flows at high Reynolds numbers can be divided into unequally spaced regions. A very thin layer (called boundary-layer) in the vicinity (of the object) in which the viscous effects domiuate, must be taken into account, and for the bulk of the flow region, the viscosity can be neglected aud the flow corresponds to the inviscid outer flow. Although the boundary-layer is very thin, it plays a vital role in the fluid dynamics. Boundary-layer theory has become an essential study now-a-days in analysing the complex behaviors of real fluids. The concept of a boundary-layer can be utilized to simplify the Navier-Stokes' equations to such an extent that the viscous effects of flow parameters are evaluated, and these are useable in many practical problems (viz. the drag on ships and missiles, the efficiency of compressors and turbines in jet engines, the effectiveness of air intakes for ram and turbojcts and so on).

Further, the boundary-layer effect caused by free convection is frequently observed in our environmental happenings and engineering devices. We know that if externally induced flow is provided and flows arising naturally solely due to the effect of the differences in density, caused by temperature or concentration differences in the body force field (such as gravitatinnal field), then thesc types of flow are called 'free convection' or 'natural convection' flows. The density difference causes buoyancy effects and these effects act as 'driving forces' due to which the flow is generated. Hence free convection is the process of heat transfer which occurs due to movement of the flnid particles by density differences associated with temperature differential in a fluid. In such cases, the free stream velocity falls away, in deed, no reference velocity does a priori exist. If the density in the vicinity of the object is kept constant, a natural convection flow can not form. Thus this is an effect of variable properties, where there is a mutual coupling between momentum and heat transport. The direct origin of the formation of natural convection flows is a heat transfer via conduction through the fixed surfaces surrounding the fluid. If the surface temperature is greater than that of ambient fluid, the heat transfer from the plate to the fluid leads to an increase of the temperature of the fluid close to the surfaces and to a change in the density, because it is temperature dependent. If the density decreases with increasing temperature, buoyancy forces arise close to the surface and warmer fluid moves upwards. Such buoyant forces are proportional to the coefficient of thermal expansion $\beta_T$, defined as $\beta_T = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{p=\text{constant}}$, where $\rho$, $T$ and $p$ are density, temperature and pressure respectively. It is observed that $\beta_T = \frac{1}{T}$ for a perfect gas and we see that stream is well approximated by the perfect-gas result $\beta_T T = 1$ at low pressure and high temperature. Also $\beta_T < \frac{1}{T}$ for a liquid and may even be negative, and $\beta_T > \frac{1}{T}$ for imperfect gas, particnlarly at high pressure. $\beta_T$ is also useful in estimating the dependence of enthalpy '$h$' on pressure, from the thermodynamic relation $dh = c_p dT + (1 - \beta_T T)\frac{dp}{\rho}$, where of course $T$ must be absolute temperature. For the perfect gas, the second term vanishes, so that $h = h(T)$ only.

The natural convection studies begun in the year 1881 with Lorentz and continued at a relatively constant rate until recently. This mode of heat transfer occurs very commonly, the cooling of transmission lines, electric transformers and rectifiers, the heating of rooms by use of radiators, the heat transfer from hot pipes and ovens surrounded by cooled air, cooling the reactor core (in nuclear power plant) and carry out the heat generated by nuclear fission etc. Bulks of information are now available in literature about the boundary-layer form of natural convection flows over bodies of different shapes.

Schmidt (1932) was apparently the first researcher who investigated experimentally the behavior of the flow near the leading edge above a flat horizontal surface.

The theoretical analysis of the laminar, two-dimensional, steady natural convection boundary-layer flow on a semi-infinite horizontal flat plate was first considered by Stewartson (1958) (later corrected by Gill, Zeh and Del-Casal (1965)). In that analysis he used the Boussinesq approximation to show how the boundary-layer analysis could be incorporated with the natural convection on rectangular plates, which are of high planform aspect ratio.

Rotem and Claassen (1959a) investigated the boundary layer equation over a semi-infinite horizontal surface of uniform temperature and results were presented for some specific values of Prandtl number with its limits from zero to infinity. The effect of buoyancy forces that exist in boundary-layer flow, over a horizontal surface, where the surface temperature differs from that of ambient fluid, was studied by Sparrow and Minkowycz (1962). The free convection above a heated and almost horizontal plate has been treated by Jones (1973).

The boundary-layer type of the natural convection flow, which occurs on the upper surface of heated horizontal, surfaces has been investigated theoretically and experimentally by amongst other, Rotem and Claassen (1959b), Pera and Gebhart (1973) and Goldstein, Sparrow and Jones (1973). It is seen from their experiments and also from the flow visualization of Husar and Sparrow (1968) that a boundary-layer starts from each edge of a plate edge, each boundary-layer having its leading at a straight-side plate edge. The boundary-layer development occurs normal to the corresponding edge so that collisions between opposing boundary-layer flows occur on the plate surface. After collision, the fluid checked in the boundary-layer forms a rising buoyant plume. Most of the above analyses were based on the Buossinesq approximation and have been concerned with the seeking of similarity solutions in which the plate temperature varies with the distance from plate leading edge. In this approximation, thus density, viscosity, thermal conductivity and specific heat variations are ignored except for the necessary inclusion of the density-variation in the body force term.

An analysis is performed by Cheu, Tien, and Armaly (1986) to study the flow and heat transfer characteristic of laminar natural convection in boundary-layer flows from horizontal, inclined and vertical plates with power law variation of the wall temperature.

With a parameter associated with the body shapes a similarity solution on the natural convection flow has also been studied by Pop and Takhar (1993).

In most of the above analyses the boundary-layer of the natural convection flows were considered over vertical, horizontal or near horizontal, semi-infinite or rectangular plates.

The natural convection boundary-layer flows on horizontal circular disc has not yet been taken into consideration with transpiration. Zakerullah and Ackroyd (1979) theoretically investigated the higher order boundary-layer natural convection flow on horizontal circular discs and paid an emphasis on the effects of fluid-property variations. Later Merkin (1983, 1985) obtained series solutions of the similarity equations derived by them (Zakerullah and Ackroyd (1979)) valid near the circumference of the disc. In his analysis it was shown that the solution at the circumference of the disc is basically the same as on a flat plate, with the importance of the curvature effects increasing as the centre of the disc is approached. However, near the centre of the disc, the boundary-layer thickness increases very rapidly and that the solution splits up into two distinct regions, a thin inner viscons region next to the disc in which the temperature is almost constant, and the pressure is large (and negative) and almost uniform, and outside this region is a thick outer inviscid region. In those analyses the boundary-layer flows were considered over heated or uniformly heated horizontal circular discs. The surface is impermeable to the fluid, so that there is no transpiration i.e., suction or blowing velocity normal to the surface. This led to the kinematic boundary condition $w_s = 0$.

The problem of boundary-layer control has become very important factor; in actual application it is often necessary to prevent separation. The separation of the boundary-layer is generally undesirable, since separated flow causes a great increase in the drag experienced by the body. So it is often necessary to prevent separation in order to reduce pressure drag and attain high lift.

Suction (or blowing) is one of the useful means in preventing boundary-layer separation. The effect of suction consists in the removal of decelerated particles from the boundary-layer before they are given a chance to cause separation. The surface is considered to be permeable to the fluid, so that the surface will allow a non-zero normal velocity and fluid is either sucked or blown through it. In doing this however, no-slip condition $u_s = 0$ at the surface (non-moving) shall continue to remain valid.

In driving the boundary-layer equation, it is anticipated that the $w$-component of the velocity is a small quantity of the order of magnitude $O\left(Re^{-\frac{1}{2}}\right)$ and it is assumed that the suction (or blowing) velocity $w_s$ normal to the surface has its magnitude of order (characteristic Reynolds number)$^{-1/2}$. The consequence of this is that outer flow is independent of $w_s$ and the boundary condition at the surface is given by $z = 0$ ; $u = 0$, $w = w_s(x)$ .

Suction or blowing causes double effects with respect to the heat transfer. On the one hand, the temperature profile is influenced by the changed velocity field in the boundary-layer, leading to a change in the heat conduction at the surface. On the other hand, convective heat transfer occurs at the surface along with the heat conduction for $w_s \neq 0$. A summary of flow separation and its cootrol are found in Chang (1970, 1976).

The boundary-layer suction was first applied by Prandtl (1904) in his fundamental works on boundary-layers on a circular cylinder. The effects of blowing and suction on forced or free convection flow over vertical as well as horizontal plates were analyzed in a symmetric way by Gortler (1957), Sparrow and Cess (1961), Koh and Hartnett (1961), Gersten and Gross (1974), Merkin (1972, 1975), Vedhanaygam, Altenkirch and Eichhorn (1980), Hasio-Tsung and Wen-Shing (1988), Merkin (1994) and Acharya, Shingh and Dash (1999) etc. The effect of transpiration on free convection above heated horizontal surface has been discussed by Clarke and Riley (1975), allowing for variable fluid density. But the effects of suction (or blowing) on free convection flow over a heated horizontal circular disc has received substantially less attention.

In our present study, we confined our discussion about the steady, laminar, free convection boundary-layer flow on axi-symmetric, heated, horizontal circular disc including the effects of suction (or blowing) situated near the edge of the disc. The flow parameters like skin friction and heat transfer co-efficient are also studied.

In order to solve the laminar natural convection boundary-layer equations it is in general the N-S and energy equations are to be transformed into convenient simplified forms like local non-similar solution. At the outset attempts are made to incorporate the idea of similarity analysis. Because, the objectives of seeking similarity solutions are manifold, firstly, the partial differential equations (PDE) governing the flow fields are to reduce ordinary differential equations (ODE) by using self-similar technique. By this means it is possible to obtain a number of exact special solutions either analytically or sometimes even in numerical form. Secondly, the results obtained from similarity equations may be directly usable in solving the local non-similar solutions. Here we adopt the method of classical 'separation of variables' which is of the simplest and most straightforward method of determining similarity solutions. This method was first initiated by Abbott and Kline (1960). In this method, once a specific form of similarity variable is chosen, the given PDE is changed under the selected co-ordinate transformations. The dependent variables are considered to be functions of the new co-ordinates. The dependent variables are to be expressed in terms of the product of separable functions of the new independent variables where each function is dependent on the single variable. Substitution of the product form of the dependent variables into the original PDE generally leads to an equation in which no functions of single variable can be isolated on the two sides of the equation unless certain parameters are to be specified. Usually, these parameters can be specified quite readily and "separation of the variables" is achieved. On this way the separation proceeds until the one side becomes an ODE. Finally, if the complete transformation to ODE is not possible, the local non-similar solutions are derived with some physical background to the remaining independent variable.

The Boussinesq approximation is employed first in chapter-3 to deal with the two possible steady cases. Numerical solution with graphs and tables are presented in chapter-4 and 5.

# Chapter-2

**Basic equations and their order analyses**

The generalilised Navier-Stokes' (N-S) equation (i.e., continuity and momentum equations) and energy equation for an axially symmetrical steady natural convection flow are given:

continuity equation,

$$\nabla.(\rho x\vec{\mathbf{q}}) = 0 \quad (2.1)$$

( As was described by Shih-I Pai (1958))

momentum equation,

$$\rho(\vec{\mathbf{q}}.\nabla)\vec{\mathbf{q}} = \vec{\mathbf{F}} - \nabla\tilde{p} + \nabla.(\mu\nabla)\vec{\mathbf{q}} \quad (2.2)$$

and energy equation

$$\rho c_p(\vec{\mathbf{q}}.\nabla)T = \nabla.(\kappa\nabla T) + (\vec{\mathbf{q}}.\nabla)\tilde{p} + \Phi . \quad (2.3)$$

Here,

$\vec{\mathbf{q}} = \vec{\mathbf{q}}(u, w)$ be the velocity vector of the fluid,

$\vec{\mathbf{F}} = (\rho - \rho_0)\vec{\mathbf{g}} = (\rho - \rho_0)\vec{\mathbf{g}}(g_x, g_z)$ is the gravitational body force per unit volume, where $\vec{\mathbf{g}}$ is the vector acceleration of gravity,

and $\Phi$ denotes the 'dissipation function' involving the viscous stresses and it represents the rate at which energy is being dissipated per unit volume through the action of viscosity. In fact the dissipation of energy is that energy which is dissipated in a viscous fluid in motion on account of the internal friction given by-

$$\Phi = \mu\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right] + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)^2$$

which is always positive since all the terms are quadratic. Here $\lambda$ is associated only with volume expansion, called the 'coefficient of bulk viscosity', may actually be negative. Stokes' simply resolved the issue by an assumption:

$$\lambda + \frac{2}{3}\mu = 0$$

i.e., $\lambda = -\frac{2}{3}\mu$

(Stokes' hypothesis (1845))

Thus we obtain

$$\Phi = \mu\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)^2\right] \tag{2.4}$$

Hence the above equations (2.1) to (2.3) can be reduced to most simplified forms as:

continuity equation,

$$\frac{\partial}{\partial x}(\rho x u) + \frac{\partial}{\partial z}(\rho x w) = 0 \tag{2.5}$$

$u$-momentum equation,

$$\rho\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = (\rho - \rho_0)g_x - \frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) \tag{2.6}$$

$w$-momentum equation,

$$\rho\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) = (\rho - \rho_0)g_z - \frac{\partial \bar{p}}{\partial z} + \frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right), \tag{2.7}$$

and energy equation,

$$\rho c_p\left(u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z}\right) = \frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial z}\left(\kappa\frac{\partial T}{\partial z}\right) + \left(u\frac{\partial \bar{p}}{\partial x} + w\frac{\partial \bar{p}}{\partial z}\right)$$
$$+ \mu\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)^2\right] \tag{2.8}$$

In the analysis we considered only the pressure perturbation $\widetilde{p}$ which is related to the absolute pressure $p$ by-

$$p = \widetilde{p} + p_0 ,$$

where $p_0$ satisfies the hydrostatic condition $\dfrac{\partial p_0}{\partial x} = \rho_0 g_x$ , $\dfrac{\partial p_0}{\partial z} = \rho_0 g_z$ (2.9)

Suffix '0' refers to conditions in a fluid at rest. Here, the gradients of the hydrostatic pressure $p_0$ are balanced by the body force terms. Hence $\widetilde{p}$ is called here the motion pressure. In general the functions of state $p_0$, $\rho_0$, $T_0$ vary with altitude.

In forming the governing boundary-layer equations from the above equations (2.6) to (2.8) together with the continuity equation (2.5), we introduce the following non-dimensional variables (dashes):

$$\left.\begin{aligned} & x' = \frac{x}{\ell_r},\ z' = R_e^{\frac{1}{2}}\frac{z}{\ell_r},\ u' = \frac{u}{\widetilde{U}},\ w' = R_e^{\frac{1}{2}}\frac{w}{\widetilde{U}},\ p' = \frac{\widetilde{p}}{\rho_r \widetilde{U}^2},\ \rho' = \frac{\rho}{\rho_r},\ \mu' = \frac{\mu}{\mu_r}, \\ & \kappa' = \frac{\kappa}{\kappa_r},\ c_p' = \frac{c_p}{c_{p_r}},\ T' = \frac{T - T_r}{T_s - T_r} = \frac{T - T_r}{\Delta T},\ g_x' = \frac{g_x}{g} \text{ and } g_z' = \frac{g_z}{g} \end{aligned}\right\} \tag{2.10}$$

Here $\ell_r$ is the characteristic length of the boundary-layer, $\widetilde{U}$ is the convenient characteristic velocity, $R_e\left(=\dfrac{\widetilde{U}\ell_r}{\upsilon_r}\right)$ is a characteristic Reynolds number based on $\widetilde{U}$ and $\ell_r$, and suffix 'r' is used to denote convenient constant reference quantities evaluated in the fluid at rest far from the boundary-layer.

Now substituting the above dimensionless quantities with primes, the non-dimensional forms of the equations (2.5) to (2.8) become,

Continuity equation,

$$\frac{\partial}{\partial x'}(\rho' x' u') + \frac{\partial}{\partial z'}(\rho' x' w') = 0 \tag{2.11}$$

$u$-momentum equation,

$$u'\frac{\partial u'}{\partial x'}+w'\frac{\partial u'}{\partial z'}=\left(1-\frac{\rho_0'}{\rho'}\right)\frac{g\ell_r}{\tilde{U}^2}g_x'-\frac{1}{\rho'}\frac{\partial p'}{\partial x'}+\frac{\upsilon_r}{\tilde{U}\ell_r}\frac{1}{\rho'}\frac{\partial}{\partial x'}\left(\mu'\frac{\partial u'}{\partial x'}\right)$$
$$+\frac{\upsilon_r}{\tilde{U}\ell_r}\frac{R_e}{\rho'}\frac{\partial}{\partial z'}\left(\mu'\frac{\partial u'}{\partial z'}\right)$$

or, $$u'\frac{\partial u'}{\partial x'}+w'\frac{\partial u'}{\partial z'}=\left(1-\frac{\rho_0'}{\rho'}\right)\frac{g_x'}{F_r}-\frac{1}{\rho'}\frac{\partial p'}{\partial x'}+\frac{1}{R_e}\frac{1}{\rho'}\frac{\partial}{\partial x'}\left(\mu'\frac{\partial u'}{\partial x'}\right)+\frac{1}{\rho'}\frac{\partial}{\partial z'}\left(\mu'\frac{\partial u'}{\partial z'}\right)$$

or, $$u'\frac{\partial u'}{\partial x'}+w'\frac{\partial u'}{\partial z'}=\frac{1-\dfrac{\rho_0'}{\rho'}}{F_r}g_x'-\frac{1}{\rho'}\frac{\partial p'}{\partial x'}+\frac{1}{\rho'}\frac{\partial}{\partial z'}\left(\mu'\frac{\partial u'}{\partial z'}\right)+O(\varepsilon^2) \tag{2.12}$$

$w$-momentum equation,

$$\frac{1}{R_e}\left(u'\frac{\partial w'}{\partial x'}+w'\frac{\partial w'}{\partial z'}\right)=\frac{1}{\sqrt{R_e}}\left(1-\frac{\rho_0'}{\rho'}\right)\frac{g\ell_r}{\tilde{U}^2}g_z'-\frac{1}{\rho'}\frac{\partial p'}{\partial z'}+\frac{\upsilon_r}{\tilde{U}\ell_r R_e}\frac{1}{\rho'}\frac{\partial}{\partial x'}\left(\mu'\frac{\partial w'}{\partial x'}\right)$$
$$+\frac{\upsilon_r}{\tilde{U}\ell_r}\frac{1}{\rho'}\frac{\partial}{\partial z'}\left(\mu'\frac{\partial w'}{\partial z'}\right)$$

or, $$\frac{1}{R_e}\left(u'\frac{\partial w'}{\partial x'}+w'\frac{\partial w'}{\partial z'}\right)=\frac{1}{\sqrt{R_e}}\left(1-\frac{\rho_0'}{\rho'}\right)\frac{g_z'}{F_r}-\frac{1}{\rho'}\frac{\partial p'}{\partial z'}+\frac{1}{R_e^2}\frac{1}{\rho'}\frac{\partial}{\partial x'}\left(\mu'\frac{\partial w'}{\partial x'}\right)$$
$$+\frac{1}{R_e}\frac{1}{\rho'}\frac{\partial}{\partial z'}\left(\mu'\frac{\partial w'}{\partial z'}\right)$$

or, $$\varepsilon^2\left(u'\frac{\partial w'}{\partial x'}+w'\frac{\partial w'}{\partial z'}\right)=\frac{\varepsilon}{F_r}\left(1-\frac{\rho_0'}{\rho'}\right)g_z'-\frac{1}{\rho'}\frac{\partial p'}{\partial z'}+O(\varepsilon^2), \tag{2.13}$$

and the energy equation,

$$\rho'c_p'\left(u'\frac{\partial T'}{\partial x'}+w'\frac{\partial T'}{\partial z'}\right)=\frac{\kappa_r}{\mu_r c_{p_r}}\frac{\upsilon_r}{\tilde{U}\ell_r}\left\{\frac{\partial}{\partial x'}\left(\kappa'\frac{\partial T'}{\partial x'}\right)+R_e\frac{\partial}{\partial z'}\left(\kappa'\frac{\partial T'}{\partial z'}\right)\right\}$$
$$+\frac{\tilde{U}^2}{c_{p_r}\Delta T}\left(u'\frac{\partial p'}{\partial x'}+w'\frac{\partial p'}{\partial z'}\right)+\frac{\tilde{U}^2}{c_{p_r}\Delta T}\frac{\upsilon_r}{\tilde{U}\ell_r}\left[\mu'\left\{2\left(\left(\frac{\partial u'}{\partial x'}\right)^2+\left(\frac{\partial w'}{\partial z'}\right)^2\right)\right.\right.$$
$$\left.\left.+\frac{1}{R_e}\left(R_e\frac{\partial u'}{\partial z'}+\frac{\partial w'}{\partial x'}\right)^2-\frac{2}{3}\left(\frac{\partial u'}{\partial x'}+\frac{\partial w'}{\partial z'}\right)^2\right\}\right]$$

or, $$\rho' c'_p\left(u'\frac{\partial T'}{\partial x'}+w'\frac{\partial T'}{\partial z'}\right)=\frac{1}{P_r}\frac{1}{R_e}\left\{\frac{\partial}{\partial x'}\left(\kappa'\frac{\partial T'}{\partial x'}\right)+R_e\frac{\partial}{\partial z'}\left(\kappa'\frac{\partial T'}{\partial z'}\right)\right\}$$
$$+E_c\left(u'\frac{\partial p'}{\partial x'}+w'\frac{\partial p'}{\partial z'}\right)+\frac{E_c}{R_e}\left[\mu'\left\{2\left(\left(\frac{\partial u'}{\partial x'}\right)^2+\left(\frac{\partial w'}{\partial z'}\right)^2\right)\right.\right.$$
$$\left.\left.+\frac{1}{R_e}\left(R_e\frac{\partial u'}{\partial z'}+\frac{\partial w'}{\partial x'}\right)^2-\frac{2}{3}\left(\frac{\partial u'}{\partial x'}+\frac{\partial w'}{\partial z'}\right)^2\right\}\right]$$

or, $$\rho' c'_p\left(u'\frac{\partial T'}{\partial x'}+w'\frac{\partial T'}{\partial z'}\right)=\frac{1}{P_r}\frac{1}{R_e}\frac{\partial}{\partial x'}\left(\kappa'\frac{\partial T'}{\partial x'}\right)+\frac{1}{P_r}\frac{\partial}{\partial z'}\left(\kappa'\frac{\partial T'}{\partial z'}\right)$$
$$+E_c\left(u'\frac{\partial p'}{\partial x'}+w'\frac{\partial p'}{\partial z'}\right)+\frac{E_c}{R_e}\left[\mu'\left\{2\left(\left(\frac{\partial u'}{\partial x'}\right)^2+\left(\frac{\partial w'}{\partial z'}\right)^2\right)\right.\right.$$
$$\left.\left.+\frac{1}{R_e}\left(R_e\frac{\partial u'}{\partial z'}+\frac{\partial w'}{\partial x'}\right)^2-\frac{2}{3}\left(\frac{\partial u'}{\partial x'}+\frac{\partial w'}{\partial z'}\right)^2\right\}\right]$$

or, $$\rho' c'_p\left(u'\frac{\partial T'}{\partial x'}+w'\frac{\partial T'}{\partial z'}\right)=P_r^{-1}\frac{\partial}{\partial z'}\left(\kappa'\frac{\partial T'}{\partial z'}\right)+O(E_c)+O(\varepsilon^2), \tag{2.14}$$

where $\varepsilon=R_e^{-\frac{1}{2}}$, $\upsilon_r=\frac{\mu_r}{\rho_r}$. Also $F_r\left(=\frac{\tilde{U}^2}{g\ell_r}\right)$, $P_r\left(=\frac{\mu_r c_{p_r}}{\kappa_r}\right)$ and $E_c\left(=\frac{\tilde{U}^2}{c_{p_r}\Delta T}\right)$ are the dimensionless Froude, Prandtl and Eckert numbers of the flow respectively. Now if we consider the limit $\varepsilon\to 0$ with $F_r$ finite, according to first order boundary-layer theory, the *w*-momentum equation asserts that $p'=p'(x')$. However, if we impose the condition $\underset{\varepsilon\to 0}{Lt}\frac{\varepsilon g'_z}{F_r}$ remains finite then the gravity dependent term must be retained in the *w*-momentum equation, resulting in $p'=p'(x',z')$. Iu the present analysis we are concerned with those bonndary-layer flows for which

$$\frac{\varepsilon}{F_r}\left(1-\frac{\rho'_0}{\rho'}\right)g'_z\approx O(1). \tag{2.15}$$

The variation in the buoyancy force normal to the surface is the only means of producing boundary-layer motion on a horizontal surface (i.e., $g_x' = 0$ in the equation (2.12)). The relative importance of the presence of gravity dependent terms in $u$-and $w$-momentum equations depends on the relative magnitude of $g_x'$ and $\varepsilon g_z'$. For horizontal surface since $\varepsilon g_z' >> g_x'$, the equation (2.15) determines the order of the magnitude of characteristic velocity $\tilde{U}$,

$$\text{i.e.,} \quad \tilde{U} \approx O\left( \frac{\rho_s' - \rho_0'}{\rho_s'} g_z (\ell_r \upsilon_r)^{\frac{1}{2}} \right)^{\frac{2}{3}} \tag{2.16}$$

Here suffix 's' denotes the (constant) representative condition at the surface. In natural convection flow the relation (2,16) determines the order of the magnitude of velocity generated by the density differences across the boundary-layer.

In all such situations, inside the first order boundary-layer $p' = p'(x', z')$ provides the mechanism for flow generation. The pressure gradient normal to the surface caused by the density difference $(= \rho_s - \rho_0)$ generates the perturbation pressure field $\tilde{p}(x', z')$ inside the boundary-layer, $x'$-variation is sufficient to cause the motion in the boundary-layer.

Since the derivative of $p'$ occurs in the momentum and energy equations, we may write the general equation of state in the differential form as

$$\rho = \rho(p, T)$$

$$\therefore \quad d\rho = \left( \frac{\partial \rho}{\partial T} \right)_p dT + \left( \frac{\partial \rho}{\partial p} \right)_T dp \text{ and since } \kappa = \frac{1}{\rho}\left( \frac{\partial \rho}{\partial p} \right)_T \text{ and } \beta_T = -\frac{1}{\rho}\left( \frac{\partial \rho}{\partial T} \right)_p,$$

we have -

$$d\rho = -\rho\, \beta_T\, dT + \rho \kappa\, dp$$

$$\text{or,} \quad \frac{d\rho}{\rho} + \beta_T\, dT = \kappa\, d(\tilde{p} + p_0)$$

i.e., $d\tilde{p} = \frac{1}{\kappa}\frac{d\rho}{\rho} + \frac{\beta_T}{\kappa}dT - dp_0$

or, in the above non-dimensional form we get

$$\kappa\rho_r\tilde{U}^2 dp' = \frac{d\rho'}{\rho'} + \beta_T T_r dT' - \kappa\, dp_0 \qquad (2.17)$$

The variations of $p_0$, $\rho_0$, $T_0$ are determined by the hydrostatic relations (2.9) together with some other condition such as, for example, $T_0$ = constant.

If this other condition is stated in rather more general terms as a requirement that any given function of state be constant, it can be shown that (cf. Ackroyd (1974)

$$\kappa\, dp_0,\ d\rho_0' \text{ etc. are all of order } \frac{\ell_0 g\,\beta_T}{c_p}$$

$$\text{i.e., } \kappa\, dp_0 \approx O\left(\frac{\ell_0 g\,\beta_T}{c_p}\right);\ d\rho_0' \approx O\left(\frac{\ell_0 g\,\beta_T}{c_p}\right), \qquad (2.18)$$

where $\ell_0$ represents the vertical scale of the flow field considered and this may be taken to be rather less than $\ell_r$ in most practical situations; ($\ell_0$, for example, can be taken to be the maximum boundary-layer thickness). Typically, $\frac{c_p}{g\,\beta_T}$ represents a length scale, and because of the vary large values associated with this length scale ($10^4$ for air and $10^6$ for water at a atmospheric pressure and temperature), and consequently with the additional provision that $\kappa_r\rho_r\tilde{U}^2 \ll 1$, it follows from equations (2.17) to (2.18) that

$$\rho = \rho(T);\ \rho_0 = \rho_r, \qquad (2.19)$$

so that, variations in $\rho_0$ etc., with altitude, due to hydrostatic relations (2.9) can be ignored.

**Governing boundary-layer equations**

In view of above discussions, the steady laminar boundary-layer equations (i.e., continuity, momentum and energy equations) in dimensional form for a variable properties fluid over a heated horizontal surface, maintained at a temperature different to that of the ambient fluid conditions, are governed by-

$$\frac{\partial}{\partial x}(\rho x u)+\frac{\partial}{\partial z}(\rho x w)=0 \tag{2.20}$$

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) \tag{2.21}$$

$$\frac{\partial \tilde{p}}{\partial z}=(\rho-\rho_r)g_z \tag{2.22}$$

and

$$\rho c_p\left(u\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial z}\right)=\frac{\partial}{\partial z}\left(\kappa\frac{\partial T}{\partial z}\right) \tag{2.23}$$

In the energy equation (2.23) the pressure and viscous dissipation work contributions have been ignored. The Eckert number $E_c$, which governs the significance of these terms, is

$$E_c=\frac{\tilde{U}^2}{c_p\Delta T}=\frac{\tilde{U}^2}{c_p(T_s-T_r)}\approx O\left\{\frac{\beta_T \ell_r g_z}{c_{p_r}}\frac{1}{\beta_r(T_s-T_r)}\frac{\rho_s-\rho_r}{\rho_s}\left(\frac{\tilde{U}\ell_r}{\upsilon_r}\right)^{-\frac{1}{2}}\right\}$$

Now $\dfrac{\frac{\rho_s-\rho_r}{\rho_s}}{\beta_T(T_s-T_r)}$ is of order unity where as $\dfrac{\beta_T \ell_r g_z}{c_{p_r}}$, as seen above, is extremely small compared with unity. However the occurrence of $\left(\dfrac{\tilde{U}\ell_r}{\upsilon_r}\right)^{-\frac{1}{2}}=R_e^{-\frac{1}{2}}$ in the above expression for the Eckert number indicates that terms involving Eckert number should not appear in the first order boundary-layer theory.

Here the independent variables $x$, $z$ denote the co-ordinates measured along the surface from the center of the disc and perpendicular to the plane of the disc respectively, and $u$ and $w$ are velocity components along $x$ and $z$ directions respectively. Also $g_x$ and $g_z$ are the components of the gravitational acceleration along $x$ and $z$ directions respectively.

$\rho$ is the density of the fluid and is defined as the mass per unit volume. It is a thermodynamic property of the fluid and in general is a function of the temperature and pressure, i.e., $\rho = \rho(p,T)$. If density $\rho$ varies with the variation of pressure and temperature, the fluid is then said to be compressible. Otherwise the fluid is said to be incompressible, i.e., for incompressible flow it is assumed that $\rho$ =constant. Again the density differences arising from temperature differences cause buoyant flow. If the density decreases with increasing temperature, bnoyancy forces arise which act as driving forces. This generates the natural convection flows.

The second property of the fluid $\mu$ is called the coefficient of viscosity of the fluid. It is a physical property of the fluid may be defined as the tangential force required per unit area to maintain a unit velocity gradient, i.e., to maintain unit relative velocity between two layers unit distance apart. Thus it relates momentum flux to velocity gradient. Since it establishes the momentum transport perpendicular to the main flow direction, it is also called transport property of the fluid.

The coefficient $\mu$ is in general a function of the temperature and pressure, although the temperature dependence is dominated. So the coefficient of viscosity of a fluid (Newtonian) is directly related to molecular interactions and thus may be considered as a thermodynamic property in the macroscopic sense, varying with temperature and pressure. As the temperature increases, the viscosity of gases generally increases whereas that for liquids decreases. But for gases at ordinary temperature the pressure dependence of viscosity is ignored and only the temperature variations is usually considered. For a perfect or non-viscous fluid, $\mu = 0$.

At higher temperatures, a common approximation for viscosity of dilute gases is the power law: $\frac{\mu}{\mu_0} \simeq \left(\frac{T}{T_0}\right)^n$

where n is of the order of 0.7 and $\mu_0$ is the reference viscosity value at reference temperature $T_0$ . This formula was suggested by Maxwell and later deduced on purely dimensional grounds by Rayleigh.

Another widely used approximation formula resulted from a kinetic theory of gases by Sutherlaud (1893) using an idealized intermolecular-force poteutial is,

$$\frac{\mu}{\mu_0} \simeq \left(\frac{T}{T_0}\right)^{\frac{3}{2}} \frac{T_0 + S}{T + S}$$

where $S$ is an effective temperature, called Sutherland constant, which is characteristic of the gas, i.e., is dependent on the type of gas (e.g., for air $S = 110K$).

For liquids, since the liquid molecules are very closely packed compared to gases and thus dominated by large molecular forces, momentum transport by collisions-so dominate in gases-is small in liquids. If data are available for calibration, the empirical approximation formula for liquid, is given by Bird et al. (1977) and Reid et al. (1977) as $\ln \frac{\mu}{\mu_0} \simeq a + b\left(\frac{T}{T_0}\right) + c\left(\frac{T}{T_0}\right)^2$

where $\mu_0$ , $T_0$ are the reference values and *a, b, c* are dimensionless curve-fit constants (e.g., for water at atmospheric pressure the curve-fit values are $a$ = -2.10, $b$ = - 4.45, $c$ = 6.55, corresponding to $T_0 = 273K$ and $\mu_0 = 0.00179\,\mathrm{kg/m.s.}$). For non polar liquids, $c \simeq 0$, i.e., plot is linear.

The third property of the fluid $c_p$ is the specific heat of the fluid at constant pressure is defined as the amount of heat required to rise the temperature of a unit mass of the fluid by one degree where pressure is assumed to be constant, i.e., $c_p = \left.\frac{\partial Q}{\partial T}\right|_{p=\text{constant}}$, where $\partial Q$ is the amount of heat added to rise the temperature by $\partial T$ at constant pressure. It is also a thermodynamic property of the fluid.

Also $\kappa$ is called the coefficient of thermal conductivity of the fluid, which connects the heat flux with the temperature gradient. It is also a positive physical property so-called heat transport coefficient of the fluid. Since, a fluid is isotropic, i.e., has no directional characteristics, hence $\kappa$ is a thermodynamic property and like viscosity varies with temperature and pressure. By inspection, we see that $\kappa$ should have dimensions of heat per unit time per length per degree, i.e.,

$$\kappa = \frac{\text{Heat flux}}{\text{Temperature Gradient}} = \frac{\text{Btu}}{(\text{h})(\text{ft})(^{\circ}\text{R})}$$ in usual engineering unit.

Also, $\kappa$ has the dimensions of viscosity times specific heats, so that the ratio of these is a fundamental parameter called Prandtl number $= \Pr = \frac{\mu c_p}{\kappa}$. This parameter involves fluid properties only, rather than length and velocity scale of the flow and measure the relative importance of heat conduction and viscosity of fluid.

For routine calculatious with dilute gases, the power law and the Sutherland formula, like viscosity, can also be used for thermal conductivity:

Power law: $\frac{\kappa}{\kappa_0} \simeq \left(\frac{T}{T_0}\right)^n$

Sutherland: $\frac{\kappa}{\kappa_0} \simeq \left(\frac{T}{T_0}\right)^{\frac{3}{2}} \frac{T_0 + S}{T + S}$.

Since for a horizontal surface the component of the buoyancy force parallel to the surface is zero (i.e., $g_x = 0$), so that $g_z$ represents the gravity component normal to the disc surface and in the $z$-direction. We can write $g_z = \pm g$, (2.24)

Also the pressure perturbation $\tilde{p}$, due to motion is related to the absolute pressure $p$ by-

$$p = \tilde{p} + p_r \tag{2.25}$$

Here $p_r$ is the hydrostatic pressure satisfying-

$$\frac{dp_r}{dz} = \rho_r g_z \tag{2.26}$$

Both the hydrostatic density, $\rho_r$, and hydrostatic temperature, $T_r$, can be taken to be constants.

Because of the boundary-layer has its origin at the periphery of the disc, we prefer here to use co-ordinates $(\tilde{x}, z)$ instead of $(x, z)$ and velocity components $(\tilde{u}, w)$ instead of $(u, w)$, where the relations between them are-

$$\left.\begin{aligned} \tilde{x} &= a - x \\ \tilde{u} &= -u \end{aligned}\right\} \tag{2.27}$$

Here, 'a' is the radius of the circular disc, $\tilde{x}$ and $z$ are (non-dimensional) co-ordinates measuring distance from the edge of the disc and normal to it in the upward direction respectively, with $\tilde{u}$ and $w$ be the velocity components in the boundary-layer generated by the buoyancy effect one to density differences almost close to the surface of the disc and in the $\tilde{x}$ and $z$ directions respectively as shown by the Fig. 1.

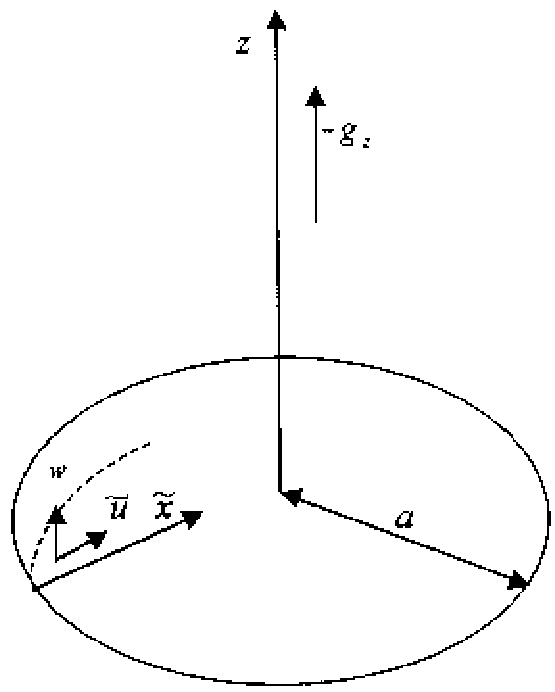


Figure-1: The flow configuration and the co-ordinate system

Using equation (2.27), the governing equations (2.20) to (2.23) for the circular disc become

$$\frac{\partial}{\partial \tilde{x}}\{\rho(a-\tilde{x})\tilde{u}\}+\frac{\partial}{\partial z}\{\rho(a-\tilde{x})w\}=0 \tag{2.28}$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+w\frac{\partial \tilde{u}}{\partial z}=-\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial \tilde{x}}+\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial \tilde{u}}{\partial z}\right) \tag{2.29}$$

$$\frac{\partial \tilde{p}}{\partial z}=(\rho-\rho_r)g_z \tag{2.30}$$

and

$$\rho c_p\left(\tilde{u}\frac{\partial T}{\partial \tilde{x}}+w\frac{\partial T}{\partial z}\right)=\frac{\partial}{\partial z}\left(\kappa\frac{\partial T}{\partial z}\right) \tag{2.31}$$

# Chapter-3

**Similar solutions for the Boussinesq approximation**

In this section we shall discuss the steady free convection laminar boundary-layer equations by simplifying them using Boussinesq approximation. In this approximation density variation other than the variation in the buoyaucy term in momentum equation are ignored. Fluid property variations are completely disregarded in this approximation.

For Boussinesq approximation the forms of the governing boundary-layer equations (2.28) to (2.31) simplify to-

$$\frac{\partial}{\partial \tilde{x}}\{(a-\tilde{x})\tilde{u}\}+\frac{\partial}{\partial z}\{(a-\tilde{x})w\}=0 \tag{3.1}$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+w\frac{\partial \tilde{u}}{\partial z}=-\frac{1}{\rho_r}\frac{\partial \tilde{p}}{\partial \tilde{x}}+\upsilon_r\frac{\partial^2 \tilde{u}}{\partial z^2} \tag{3.2}$$

$$\frac{\partial \tilde{p}}{\partial z}=-\rho_r g_z \beta_T \Delta T\theta \tag{3.3}$$

and $$\tilde{u}\frac{\partial T}{\partial \tilde{x}}+w\frac{\partial T}{\partial z}=\frac{\upsilon_r}{P_r}\frac{\partial^2 T}{\partial z^2} \tag{3.4}$$

Here,

$\upsilon_r=\dfrac{\mu_r}{\rho_r}$ is the kinematic co-efficient of viscosity

and $\rho-\rho_r=-\rho_r\beta_T(T-T_r)$.

Since $\dfrac{T-T_r}{T_s-T_r}=\dfrac{T-T_r}{\Delta T}=\theta$ , $T_s-T_r=\Delta T$ and $\rho\propto\dfrac{1}{T}$ (3.5)

so that, $\rho-\rho_r=-\rho_r\beta_T\Delta T\theta$ (3.6),

(Suffix 's' represents the condition at the surface of disc and suffix 'r' is the constant reference condition in the fluid at rest exterior to the boundary-layer)

We may now introduce the stream function $\psi$, which automatically satisfies the continuity equation (3.1)

$$(a-\tilde{x})\tilde{u}=\frac{\partial\psi}{\partial z}$$

and $$(a-\tilde{x})w=-\frac{\partial\psi}{\partial\tilde{x}}$$

or, $$\tilde{u}=\frac{1}{a-\tilde{x}}\frac{\partial\psi}{\partial z} \tag{3.7}$$

and $$-w=\frac{1}{a-\tilde{x}}\frac{\partial\psi}{\partial\tilde{x}} \tag{3.8}$$

Since for a finite diameter circular disc, the boundary-layer has its origin at the edge of the disc, near the edge of the disc (i.e., $\frac{\tilde{x}}{a}\to 0$ or, $x\to a$) we would expect the boundary-layer to be the same as that obtained on a two-dimensional horizontal flat plate by Stewartson (1958).

Equations (3.1) to (3.4) are non-linear, simultaneous partial differential equations (PDEs) and to obtain solutions for them are extremely difficult. Consequently, we adopt first the method of seeking similarity solutions in order to reduce the system of PDEs (3.2) to (3.4) together with the continuity equation (3.1) into a pair of ordinary differential equations (ODEs). If not, local non-similar solution will be finally achieved. For this purpose we define a new set of variables $(\xi,\tilde{\eta})$, related to $(\tilde{x},z)$ as follows:

$$\left.\begin{aligned}&\xi=\tilde{x}\\ \text{and}\quad&\tilde{\eta}=\frac{z}{\gamma(\tilde{x})}\end{aligned}\right\} \tag{3.9}$$

Here $\gamma(\tilde{x})$ can be thought of being proportional to the local boundary-layer thickness.

From equation (3.9), we obtain

$$\frac{\partial}{\partial \tilde{x}} = \frac{\partial \xi}{\partial \tilde{x}}\frac{\partial}{\partial \xi} + \frac{\partial \tilde{\eta}}{\partial \tilde{x}}\frac{\partial}{\partial \tilde{\eta}}$$

or, $$\frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial \xi} - \frac{z}{\gamma^2(\xi)}\frac{\partial \gamma(\xi)}{\partial \xi}\frac{\partial}{\partial \tilde{\eta}}$$

or, $$\frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial \xi} - \frac{\gamma_\xi}{\gamma}\tilde{\eta}\frac{\partial}{\partial \tilde{\eta}} \tag{3.10}$$

and similarly,

$$\frac{\partial}{\partial z} = \frac{1}{\gamma}\frac{\partial}{\partial \tilde{\eta}} \tag{3.11}$$

Guided by the idea of the similarity procedure, we may put-

$$\int_0^{\tilde{\eta}} \frac{(a-\xi)\tilde{u}}{\tilde{U}(\xi)} d\tilde{\eta}_1 = \tilde{F}(\xi, \tilde{\eta}) \tag{3.12}$$

where $\tilde{U} = \tilde{U}(\xi)$ is the non-dimensionalising characteristic velocity. Equation (3.9) and (3.12) are the traditional substitutions with a small modification in equation (3.12) for the case of axi-symmetric flow only.

Now substituting equations (3.9) and (3.11) in equation (3.7), we obtain

$$\tilde{u} = \frac{1}{(a-\xi)\gamma(\xi)}\frac{\partial}{\partial \tilde{\eta}}\{\psi(\xi, \tilde{\eta})\}$$

or, $$\frac{(a-\xi)\tilde{u}}{\tilde{U}(\xi)} = \frac{1}{\gamma(\xi)\tilde{U}(\xi)}\frac{\partial}{\partial \tilde{\eta}}\{\psi(\xi, \tilde{\eta})\}$$

Integrating with respect to $\tilde{\eta}$ from 0 to $\tilde{\eta}$ and using equation (3.12), we have

$$\tilde{F}(\xi, \tilde{\eta}) = \frac{1}{\gamma(\xi)\tilde{U}(\xi)}[\psi(\xi, \tilde{\eta}_1)]_0^{\tilde{\eta}}$$

or, $$\tilde{F}(\xi, \tilde{\eta}) = \frac{1}{\gamma(\xi)\tilde{U}(\xi)}[\psi(\xi, \tilde{\eta}) - \psi(\xi, 0)]$$

or, $$\psi(\xi, \tilde{\eta}) = \gamma(\xi)\tilde{U}(\xi)\tilde{F}(\xi, \tilde{\eta}) + \psi(\xi, 0) \tag{3.13}$$

Again using (3.9), (3.10) and (3.13) in equation (3.8), we obtain

$$-w = \frac{1}{a-\xi}\left(\frac{\partial}{\partial\xi} - \frac{\gamma_\xi}{\gamma}\tilde{\eta}\frac{\partial}{\partial\tilde{\eta}}\right)\left\{\gamma(\xi)\tilde{U}(\xi)\tilde{F}(\xi,\tilde{\eta}) + \psi(\xi,0)\right\}$$

$$= \frac{1}{a-\xi}\left\{\left(\gamma\tilde{U}\tilde{F}\right)_\xi - \gamma_\xi\tilde{U}\tilde{\eta}\frac{\partial\tilde{F}(\xi,\tilde{\eta})}{\partial\tilde{\eta}} + \frac{\partial\psi(\xi,0)}{\partial\xi}\right\}$$

$$= \frac{1}{a-\xi}\left\{\left(\gamma\tilde{U}\tilde{F}\right)_\xi - \gamma_\xi\tilde{U}\tilde{\eta}\tilde{F}_{\tilde{\eta}}\right\} + \frac{1}{a-\xi}\psi_\xi(\xi,0)$$

Here suffixes denote the differentiation partially with respect to associated arguments.

or, $$-w = \frac{1}{a-\xi}\left\{\left(\gamma\tilde{U}\tilde{F}\right)_\xi - \gamma_\xi\tilde{U}\tilde{\eta}\tilde{F}_{\tilde{\eta}}\right\} - w_s \qquad (3.14)$$

Here $w_s = -\frac{1}{a-\xi}\frac{\partial\psi(\xi,0)}{\partial\xi} = -\frac{1}{a-\xi}\psi_\xi(\xi,0)$ represents the non-zero wall velocity called the suction or blowing velocity normal to the disc surface, since the surface is taken to be porous, so that fluid will be sucked or blown through it. Physically $w_s < 0$ and $w_s > 0$ represent respectively the suction and blowing velocity through the porous surface. For uniform suction (or blowing) $w_s =$ constant. However $w_s = 0$ implies that the surface is impermeable to the fluid (i.e., the surface is not porous). We consider in our problem that $w_s$ depends on the position of the disc (i.e., on $\xi$ measured from the periphery towards the center of the disc).

Now from equation (3.12), we have

$$\tilde{u} = \frac{\tilde{U}(\xi)}{a-\xi}\frac{\partial\tilde{F}(\xi,\tilde{\eta})}{\partial\tilde{\eta}} = \frac{\tilde{U}(\xi)}{a-\xi}\tilde{F}_{\tilde{\eta}}(\xi,\tilde{\eta}) \qquad (3.15)$$

With the help of (3.10), (3.11), (3.14) and (3.15), the convective operator $\tilde{u}\frac{\partial}{\partial \tilde{x}}+w\frac{\partial}{\partial z}$ becomes

$$\tilde{u}\frac{\partial}{\partial \tilde{x}}+w\frac{\partial}{\partial z}=\frac{\tilde{U}\tilde{F}_{\tilde{\eta}}}{a-\xi}\left(\frac{\partial}{\partial \xi}-\frac{\gamma_\xi}{\gamma}\tilde{\eta}\frac{\partial}{\partial \tilde{\eta}}\right)-\left[\frac{1}{a-\xi}\left\{\left(\gamma\tilde{U}\tilde{F}\right)_\xi-\gamma_\xi\tilde{U}\tilde{\eta}\tilde{F}_{\tilde{\eta}}\right\}-w_s\right]\frac{1}{\gamma}\frac{\partial}{\partial \tilde{\eta}}$$

or, $$\tilde{u}\frac{\partial}{\partial \tilde{x}}+w\frac{\partial}{\partial z}=\frac{\tilde{U}\tilde{F}_{\tilde{\eta}}}{a-\xi}\frac{\partial}{\partial \xi}-\frac{\left(\gamma\tilde{U}\tilde{F}\right)_\xi}{(a-\xi)\gamma}\frac{\partial}{\partial \tilde{\eta}}+\frac{w_s}{\gamma}\frac{\partial}{\partial \tilde{\eta}} \tag{3.16}$$

In attempting separation of variables we assume

$$\tilde{F}(\xi,\tilde{\eta})=\ell(\xi)F(\tilde{\eta}) \tag{3.17}$$

Then from equation (3.15), we have

$$\tilde{u}=\tilde{U}(\xi)\ell(\xi)F_{\tilde{\eta}}(\tilde{\eta}) \tag{3.18}$$

Using (3.17) and (3.18) in equation (3.16), we obtain

$$\tilde{u}\frac{\partial}{\partial \tilde{x}}+w\frac{\partial}{\partial z}=\frac{\tilde{U}\ell}{a-\xi}F_{\tilde{\eta}}\frac{\partial}{\partial \xi}-\frac{\left(\gamma\tilde{U}\ell\right)_\xi}{(a-\xi)\gamma}F\frac{\partial}{\partial \tilde{\eta}}+\frac{w_s}{\gamma}\frac{\partial}{\partial \tilde{\eta}} \tag{3.19}$$

or, $$\left(\tilde{u}\frac{\partial}{\partial \tilde{x}}+w\frac{\partial}{\partial z}\right)\tilde{u}=\left\{\frac{\tilde{U}\ell}{a-\xi}F_{\tilde{\eta}}\frac{\partial}{\partial \xi}-\frac{\left(\gamma\tilde{U}\ell\right)_\xi}{(a-\xi)\gamma}F\frac{\partial}{\partial \tilde{\eta}}+\frac{w_s}{\gamma}\frac{\partial}{\partial \tilde{\eta}}\right\}\left(\frac{\tilde{U}\ell}{a-\xi}F_{\tilde{\eta}}\right)$$

or, $$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+w\frac{\partial \tilde{u}}{\partial z}=\frac{\tilde{U}\ell}{a-\xi}\frac{d}{d\xi}\left(\frac{\tilde{U}\ell}{a-\xi}\right)F_{\tilde{\eta}}^{\,2}-\frac{\tilde{U}\ell\left(\gamma\tilde{U}\ell\right)_\xi}{(a-\xi)^2\gamma}FF_{\tilde{\eta}\tilde{\eta}}+\frac{\tilde{U}\ell w_s}{(a-\xi)\gamma}F_{\tilde{\eta}\tilde{\eta}}$$

or, $$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+w\frac{\partial \tilde{u}}{\partial z}=\left\{\frac{\tilde{U}\ell\left(\tilde{U}\ell\right)_\xi}{(a-\xi)^2}+\frac{\left(\tilde{U}\ell\right)^2}{(a-\xi)^3}\right\}F_{\tilde{\eta}}^{\,2}-\frac{\tilde{U}\ell\left(\gamma\tilde{U}\ell\right)_\xi}{(a-\xi)^2\gamma}FF_{\tilde{\eta}\tilde{\eta}}+\frac{\tilde{U}\ell w_s}{(a-\xi)\gamma}F_{\tilde{\eta}\tilde{\eta}} \tag{3.20}$$

Again we assume

$$\tilde{p}=P(\xi)G(\xi,\tilde{\eta}) \tag{3.21}$$

Using (3.21) in equation (3.10), we have

$$\frac{\partial \tilde{p}}{\partial \tilde{x}}=\left(\frac{\partial}{\partial \xi}-\frac{\gamma_\xi}{\gamma}\tilde{\eta}\frac{\partial}{\partial \tilde{\eta}}\right)\{P(\xi)G(\xi,\tilde{\eta})\}$$

or, $$\frac{\partial \tilde{p}}{\partial \tilde{x}}=\frac{dP(\xi)}{d\xi}G(\xi,\tilde{\eta})+P\frac{\partial G(\xi,\tilde{\eta})}{\partial \xi}-\frac{\gamma_\xi}{\gamma}P\tilde{\eta}\frac{\partial G(\xi,\tilde{\eta})}{\partial \tilde{\eta}}$$

or, $$\frac{\partial\widetilde{p}}{\partial\widetilde{x}} = P_\xi G + P\frac{\partial G}{\partial\xi} - P\frac{\gamma_\xi}{\gamma}\widetilde{\eta}\frac{\partial G}{\partial\widetilde{\eta}} \tag{3.22}$$

Again in view of equation (3.11) and (3.18) we obtain

$$\frac{\partial\widetilde{u}}{\partial z} = \frac{1}{\gamma}\frac{\partial}{\partial\widetilde{\eta}}\left(\frac{\widetilde{U}\ell}{a-\xi}F_{\widetilde{\eta}}\right)$$

or, $$\frac{\partial\widetilde{u}}{\partial z} = \frac{\widetilde{U}\ell}{(a-\xi)\gamma}F_{\widetilde{\eta}\widetilde{\eta}}$$

$$\therefore \quad \frac{\partial^2\widetilde{u}}{\partial z^2} = \frac{\widetilde{U}\ell}{(a-\xi)\gamma^2}F_{\widetilde{\eta}\widetilde{\eta}\widetilde{\eta}} \tag{3.23}$$

Substituting (3.20), (3.22) and (3.23) in equation (3.2), one obtains

$$\left\{\frac{\widetilde{U}\ell(\widetilde{U}\ell)_\xi}{(a-\xi)^2} + \frac{(\widetilde{U}\ell)^2}{(a-\xi)^3}\right\}F_{\widetilde{\eta}}^{\,2} - \frac{\widetilde{U}\ell(\gamma\widetilde{U}\ell)_\xi}{(a-\xi)^2\gamma}FF_{\widetilde{\eta}\widetilde{\eta}} + \frac{\widetilde{U}\ell w_s}{(a-\xi)\gamma}F_{\widetilde{\eta}\widetilde{\eta}}$$

$$= -\frac{1}{\rho_r}\left(P_\xi G + P\frac{\partial G}{\partial\xi} - P\frac{\gamma_\xi}{\gamma}\widetilde{\eta}\frac{\partial G}{\partial\widetilde{\eta}}\right) + \upsilon_r\frac{\widetilde{U}\ell}{(a-\xi)\gamma^2}F_{\widetilde{\eta}\widetilde{\eta}\widetilde{\eta}}$$

Dividing both sides by $\frac{\widetilde{U}\ell}{\gamma^2}$ and multiplying by $(a-\xi)^3$, we have

$$\upsilon_r(a-\xi)^2F_{\widetilde{\eta}\widetilde{\eta}\widetilde{\eta}} + (a-\xi)\gamma(\gamma\widetilde{U}\ell)_\xi FF_{\widetilde{\eta}\widetilde{\eta}} - \gamma w_s(a-\xi)^2F_{\widetilde{\eta}\widetilde{\eta}}$$

$$-\left\{(a-\xi)\gamma^2(\widetilde{U}\ell)_\xi + \gamma^2(\widetilde{U}\ell)\right\}F_{\widetilde{\eta}}^{\,2} = \frac{\gamma^2}{\rho_r\widetilde{U}\ell}(a-\xi)^3\left(P_\xi G + P\frac{\partial G}{\partial\xi} - P\frac{\gamma_\xi}{\gamma}\widetilde{\eta}\frac{\partial G}{\partial\widetilde{\eta}}\right)$$

or, $$\upsilon_r(a-\xi)^2F_{\widetilde{\eta}\widetilde{\eta}\widetilde{\eta}} + (a-\xi)\left\{\gamma(\gamma\widetilde{U})_\xi F - (a-\xi)\gamma w_s\right\}F_{\widetilde{\eta}\widetilde{\eta}} - \left\{(a-\xi)\gamma^2(\widetilde{U}\ell)_\xi\right.$$

$$\left.+\gamma^2(\widetilde{U}\ell)\right\}F_{\widetilde{\eta}}^{\,2} = \frac{\gamma^2}{\rho_r\widetilde{U}\ell}(a-\xi)^3\left(P_\xi G + P\frac{\partial G}{\partial\xi} - P\frac{\gamma_\xi}{\gamma}\widetilde{\eta}\frac{\partial G}{\partial\widetilde{\eta}}\right) \tag{3.24}$$

Again using equation (3.21) in equation (3.11), we have

$$\frac{\partial\widetilde{p}}{\partial z} = \frac{1}{\gamma}\frac{\partial}{\partial\widetilde{\eta}}\{P(\xi)G(\xi,\widetilde{\eta})\}$$

or, $$\frac{\partial\widetilde{p}}{\partial z} = \frac{P}{\gamma}\frac{\partial G}{\partial\widetilde{\eta}} \tag{3.25}$$

Substituting (3.25) in equation (3.3), we obtain

$$\frac{P}{\gamma}\frac{\partial G}{\partial \tilde{\eta}} = -\rho_r g_z \beta_T \Delta T \theta$$

or, $$\frac{\partial G}{\partial \tilde{\eta}} = -\frac{\rho_r g_z \beta_T \gamma \Delta T}{P}\theta \tag{3.26}$$

Again from equation (3.5)

$$T = T_r + \Delta T(\xi)\theta(\xi, \tilde{\eta})$$

and by the method of separation of variables like

$$\theta(\xi, \tilde{\eta}) = \lambda(\xi)\vartheta(\tilde{\eta}) \tag{3.27}$$

we have,

$$T = T_r + \lambda(\xi)\Delta T(\xi)\vartheta(\tilde{\eta}) \tag{3.28}$$

Then equation (3.26) becomes,

$$\frac{\partial G}{\partial \tilde{\eta}} = -\frac{\rho_r g_z \beta_T \gamma\, \lambda(\xi)\Delta T}{P}\vartheta \tag{3.29}$$

Also from equations (3.19) and (3.28), we have

$$\left(\tilde{u}\frac{\partial}{\partial \tilde{x}} + w\frac{\partial}{\partial z}\right)T = \left\{\frac{\tilde{U}\ell}{a-\xi}F_{\tilde{\eta}}\frac{\partial}{\partial \xi} - \frac{\left(\gamma\tilde{U}\ell\right)_\xi}{(a-\xi)\gamma}F\frac{\partial}{\partial \tilde{\eta}} + \frac{w_s}{\gamma}\frac{\partial}{\partial \tilde{\eta}}\right\}\left(T_r + \lambda(\xi)\Delta T\vartheta\right)$$

or, $$\tilde{u}\frac{\partial T}{\partial \tilde{x}} + w\frac{\partial T}{\partial z} = \frac{\tilde{U}\ell\left(\lambda(\xi)\Delta T\right)_\xi}{a-\xi}F_{\tilde{\eta}}\vartheta - \frac{\left(\gamma\tilde{U}\ell\right)_\xi\left(\lambda(\xi)\Delta T\right)}{(a-\xi)\gamma}F\vartheta_{\tilde{\eta}} + \frac{\left(\lambda(\xi)\Delta T\right)w_s}{\gamma}\vartheta_{\tilde{\eta}} \tag{3.30}$$

Using (3.28) in equation (3.11), we get

$$\frac{\partial T}{\partial z} = \frac{1}{\gamma}\frac{\partial}{\partial \tilde{\eta}}\left(T_r + \lambda(\xi)\Delta T\vartheta\right)$$

or, $$\frac{\partial T}{\partial z} = \frac{\lambda(\xi)\Delta T}{\gamma}\vartheta_{\tilde{\eta}}$$

$$\therefore \quad \frac{\partial^2 T}{\partial z^2} = \frac{\lambda(\xi)\Delta T}{\gamma^2}\vartheta_{\tilde{\eta}\tilde{\eta}} \tag{3.31}$$

Substituting (3.30) and (3.31) in equation (3.4), we obtain

$$\frac{\tilde{U}\ell(\lambda(\xi)\Delta T)_\xi}{a-\xi}F_{\bar{\eta}}\vartheta - \frac{(\gamma\tilde{U}\ell)_\xi(\lambda(\xi)\Delta T)}{(a-\xi)\gamma}F\vartheta_{\bar{\eta}} + \frac{(\lambda(\xi)\Delta T)w_s}{\gamma}\vartheta_{\bar{\eta}} = \frac{\upsilon_r}{P_r}\frac{\lambda(\xi)\Delta T}{\gamma^2}\vartheta_{\bar{\eta}\bar{\eta}}$$

Dividing both sides by $\frac{\lambda(\xi)\Delta T}{\gamma^2}$ and multiplying by $(a-\xi)$, we have

or, $$\frac{\upsilon_r}{P_r}(a-\xi)\vartheta_{\bar{\eta}\bar{\eta}} + \{\gamma(\gamma\tilde{U}\ell)_\xi F - (a-\xi)\gamma w_s\}\vartheta_{\bar{\eta}} - (\gamma^2\tilde{U}\ell)\frac{(\lambda(\xi)\Delta T)_\xi}{(\lambda(\xi)\Delta T)}F_{\bar{\eta}}\vartheta = 0$$

or, $$\frac{\upsilon_r}{P_r}(a-\xi)\vartheta_{\bar{\eta}\bar{\eta}} + \{\gamma(\gamma\tilde{U}\ell)_\xi F - (a-\xi)\gamma w_s\}\vartheta_{\bar{\eta}} - (\gamma^2\tilde{U}\ell)(\log\lambda(\xi)\Delta T)_\xi F_{\bar{\eta}}\vartheta = 0$$

(3.32)

There are, however, boundary conditions, which must be imposed in order to determine the solutions of the transformed boundary-layer equations (3.24), (3.29) and (3.32). Boundary conditions will be ascertained from the following physical behaviors of flow configurations.

I. The velocity component $\tilde{u}$ tangential to the surface of the disc vanishes at the surface (no-slip condition). However, since the surface is porous, the velocity component normal to the surface must be equal to the suction (or blowing) velocity, i.e., mathematically:

   $\tilde{u} = 0$ and $w = w_s$ at $z = 0$, implies

   $F(0) = F_{\bar{\eta}}(0) = 0$

II. The velocity of the fluid at a large distance from the surface of the disc must be zero, i.e., mathematically:

   $\tilde{u} = 0$ when $z \to \infty$, implies

   $F_{\bar{\eta}}(\infty) = 0$

III. The temperature of the fluid at the surface of the disc must be equal to the disc temperature, i.e., mathematically:

$T = T_s$ at $z = 0$

or, $\theta = \dfrac{T - T_r}{T_s - T_r} = 1$ at $z = 0$, implies

$\lambda(\xi)\vartheta(0) = 1$

IV. The temperature of the fluid at a large distance from the surface of the disc must be equal to the undisturbed fluid temperature, i.e., mathematically:

$T = T_r$ when $z \to \infty$

or, $\theta = \dfrac{T - T_r}{T_s - T_r} = 0$ when $z \to \infty$, implies

$\vartheta(\infty) = 0$

Without loss of generality we may choose $\tilde{U}\ell = U(\xi)$ and $\lambda(\xi) = 1$. Also for similarity solutions $G$ is assumed to be the function of $\tilde{\eta}$ only. Then equations (3.24), (3.29) and (3.32) become

$$\upsilon_r (a-\xi)^2 F_{\tilde{\eta}\tilde{\eta}\tilde{\eta}} + (a-\xi)\{\gamma(\gamma U)_\xi F - (a-\xi)\gamma w_s\} F_{\tilde{\eta}\tilde{\eta}}$$

$$- \{(a-\xi)\gamma^2 (U)_\xi + \gamma^2 U\} F_{\tilde{\eta}}^{\,2} = \frac{\gamma^2}{\rho_r U}(a-\xi)^3 \left( P_\xi G - P\frac{\gamma_\xi}{\gamma}\tilde{\eta} G_{\tilde{\eta}} \right)$$

$$G_{\tilde{\eta}} = -\frac{\rho_r g_z \beta_T \gamma \Delta T}{P}\vartheta$$

and

$$\frac{\upsilon_r}{P_r}(a-\xi)\vartheta_{\tilde{\eta}\tilde{\eta}} + \{\gamma(\gamma U)_\xi F - (a-\xi)\gamma w_s\}\vartheta_{\tilde{\eta}} - (\gamma^2 U)(\log \Delta T)_\xi F_{\tilde{\eta}}\vartheta = 0$$

Since $\gamma(\gamma U)_\xi = \gamma\gamma_\xi U + \gamma^2 (U)_\xi = \frac{1}{2}\left\{2\gamma\gamma_\xi U + 2\gamma^2 (U)_\xi\right\}$

$$= \frac{1}{2}\left\{\gamma^2 (U)_\xi + 2\gamma\gamma_\xi U + \gamma^2 (U)_\xi\right\}$$

$$= \frac{1}{2}\left\{(\gamma^2 U)_\xi + \gamma^2 (U)_\xi\right\}$$

Hence we have the following system of equations:

$$\upsilon_r (a-\xi)^2 F_{\tilde\eta\tilde\eta\tilde\eta} + (a-\xi)\left[\frac{1}{2}\left\{(\gamma^2 U)_\xi + \gamma^2 (U)_\xi\right\}F - (a-\xi)\gamma w_s\right]F_{\tilde\eta\tilde\eta}$$

$$-\left\{(a-\xi)\gamma^2 (U)_\xi + \gamma^2 U\right\}F_{\tilde\eta}^{\,2} = (a-\xi)^3\left(\frac{\gamma^2}{\rho_r U}P_\xi G - \frac{\gamma\gamma_\xi}{\rho_r U}P\tilde\eta G_{\tilde\eta}\right) \qquad (3.33)$$

$$G_{\tilde\eta} = -\frac{\rho_r g_z \beta_f \gamma \Delta T}{P}\vartheta \qquad (3.34)$$

and

$$\frac{\upsilon_r}{P_r}(a-\xi)\vartheta_{\tilde\eta\tilde\eta} + \left[\frac{1}{2}\left\{(\gamma^2 U)_\xi + \gamma^2 (U)_\xi\right\}F - (a-\xi)\gamma w_s\right]\vartheta_{\tilde\eta}$$

$$-(\gamma^2 U)(\log \Delta T)_\xi F_{\tilde\eta}\vartheta = 0 \qquad (3.35)$$

with boundary conditions:

$$F(0) = F_{\tilde\eta}(0) = 0\ ;\ F_{\tilde\eta}(\infty) = 0$$
$$\vartheta(0) = 1\ ;\ \vartheta(\infty) = 0, \qquad (3.36)$$

It is observed from Merkin's (1983, 1985) analysis that a complete similar solution may not be obtained for a natural convection on circular disc problem. So we are interested to have a local non-similar solution for which we have to consider

$$\left.\begin{array}{ll} \text{(i.)} & (\gamma^2 U)_\xi = a_1 \\ \text{(ii.)} & \gamma^2 (U)_\xi = a_2 \\ \text{(iii.)} & \gamma(\gamma U)_\xi = \frac{1}{2}(a_1 + a_2) = a_3 \\ \text{(iv.)} & \gamma w_s = a_4 \\ \text{(v.)} & \dfrac{\gamma^2}{\rho_r U} P_\xi = a_5 \\ \text{(vi.)} & \dfrac{\gamma \gamma_\xi}{\rho_r U} P = a_6 \\ \text{(vii.)} & -\dfrac{\rho_r g_z \beta_T \gamma \Delta T}{P} = a_7 \\ \text{(viii.)} & (\gamma^2 U)(\log \Delta T)_\xi = a_8 \end{array}\right\} \quad (3.37)$$

where $a_1$, $a_2$, $a_3$,......................................, $a_8$ are all in $\xi$ alone.

Local non-similar solutions for equations (3.33) to (3.35) exist only when all a's are finite; that is to say that all a's must be constants. And thus equations (3.33) to (3.35) will be reduced to local non-similar solution and will finally become non-linear ordinary differential equations in the limiting situations for the remaining variable other than the similarity variable. Consequently, the relations giveu (or stated) by equation (3.37) will be treated conditions. These furnish us the equatious for $U(\xi)$ and $\gamma(\xi)$, the scale factors for the velocity component $\tilde{u}$ and the ordinate

$z$. Uniquely these scale factors together with the suction (or blowing) parameter will determine the flow characteristics of the boundary-layer.

We shall now proceed to find $U(\xi)$, $\gamma(\xi)$ and consequently the suction velocity $w_s$ for the possible two types of self-similarity solutions in the case of the Boussinesq fluid:

**Type-I:** (power-law variation)

From condition (i) of equation (3.37), we have

$$(\gamma^2 U)_\xi = a_1$$

Integrating with respect to $\xi$ we get

$$\gamma^2 U = a_1 \xi + A$$

$$\text{or,} \quad \gamma^2 = \frac{a_1 \xi + A}{U} \tag{3I.1}$$

(Here $A$ is the constant of integration and $U \neq 0$)

Substituting (3I.1) in condition (ii) of equation (3.37), we obtain

$$\gamma^2 (U)_\xi = a_2$$

$$\text{or,} \frac{(a_1\xi + A)(U)_\xi}{U} = a_2$$

$$\text{or,} \frac{(U)_\xi}{U} = \frac{a_2}{a_1\xi + A}$$

$$\text{or,} (\log U)_\xi = \frac{a_2}{a_1\xi + A}$$

Integrating with respect to $\xi$ we get

$$\log U(\xi) + \log B = \frac{a_2}{a_1} \log(a_1\xi + A)$$

$$\log U(\xi)=\log\left\{\frac{(a_1\xi+A)^{\frac{a_2}{a_1}}}{B}\right\}$$

or, $$U(\xi)=\frac{1}{B}(a_1\xi+A)^{\frac{a_2}{a_1}} \tag{3I.2}$$

(Here $B$ is also constant of integration)

With the help of (3I.2) equation (3I.1) becomes

$$\gamma^2(\xi)=B(a_1\xi+A)^{1-\frac{a_2}{a_1}} \tag{3I.3}$$

Substituting (3I.2) and (3I.3) in different conditions stated in equation (3.37) we obtain the following relations:

$a_1$, $a_2$ are arbitrary, $a_6=\frac{a_1-a_2}{4a_2}a_5$; $a_4$, $a_5$ and $a_7$ are disposable constants and

$a_8=\frac{5a_2-a_1}{2}$.

Thus the general equations (3.33) to (3.35) are reduced to

$$\upsilon_r(a-\xi)^2F_{\bar\eta\bar\eta\bar\eta}+(a-\xi)\left\{\frac{1}{2}(a_1+a_2)F-(a-\xi)a_4\right\}F_{\bar\eta\bar\eta}-(a-\xi)a_2F_{\bar\eta}^{\,2}$$
$$-(a_1\xi+A)F_{\bar\eta}^{\,2}=(a-\xi)^3a_5\left(G-\frac{a_1-a_2}{4a_2}\bar\eta G_{\bar\eta}\right) \tag{3I.4}$$

$$G_{\bar\eta}=a_7\vartheta \tag{3I.5}$$

and

$$P_r^{-1}\upsilon_r(a-\xi)\vartheta_{\bar\eta\bar\eta}+\left\{\frac{1}{2}(a_1+a_2)F-(a-\xi)a_4\right\}\vartheta_{\bar\eta}-\frac{1}{2}(5a_2-a_1)F_{\bar\eta}\vartheta=0 \tag{3I.6}$$

Subjected to the boundary conditions:

$$\begin{aligned}&F(0)=F_{\bar\eta}(0)=0\;;\;F_{\bar\eta}(\infty)=0\\&\vartheta(0)=1\;;\;\vartheta(\infty)=0\end{aligned} \tag{3I.7}$$

Let us now substitute

$F = \alpha_1 f$, $\tilde{\eta} = \alpha_2 \eta$ and $G = \alpha_3 \tilde{g}$ , so that

$$\frac{\partial}{\partial \tilde{\eta}} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial \tilde{\eta}} = \frac{1}{\alpha_2}\frac{\partial}{\partial \eta}, \frac{\partial^2}{\partial \tilde{\eta}^2} = \frac{1}{\alpha_2^{\,2}}\frac{\partial^2}{\partial \eta^2} \text{ etc.}$$

Here $\alpha$ 's are used to have the convenieut forms of the similarity solutions.

Then we have from equations (3I.4) to (3I.6)

$$\frac{\upsilon_r \alpha_1}{\alpha_2^3}(a-\xi)^2 f_{\eta\eta\eta} + (a-\xi)\left\{\frac{1}{2}(a_1+a_2)\frac{\alpha_1^2}{\alpha_2^2} f - (a-\xi)\frac{a_4\alpha_1}{\alpha_2^2}\right\} f_{\eta\eta}$$

$$-(a-\xi)\frac{a_2\alpha_1^2}{\alpha_2^2} f_\eta^{\,2} - a_1(\xi+\xi_0) f_\eta^{\,2} = (a-\xi)^3 a_5 \alpha_3\left(\tilde{g} - \frac{a_1 - a_2}{4a_2}\eta \tilde{g}_\eta\right)$$

$$\frac{\alpha_3}{\alpha_2}\tilde{g}_\eta = a_7 \vartheta$$

and

$$\frac{P_r^{-1}\upsilon_r}{\alpha_2^2}(a-\xi)\vartheta_{\eta\eta} + \left\{\frac{1}{2}(a_1+a_2)\frac{\alpha_1}{\alpha_2} f - (a-\xi)\frac{a_4}{\alpha_2}\right\}\vartheta_\eta - \frac{(5a_2-a_1)\alpha_1}{2\alpha_2} f_\eta \vartheta = 0$$

or, $$(a-\xi)^2 f_{\eta\eta\eta} + (a-\xi)\left\{(a_1+a_2)\frac{\alpha_1\alpha_2}{2\upsilon_r} f - (a-\xi)\frac{a_4\alpha_2}{\upsilon_r}\right\} f_{\eta\eta}$$

$$-(a-\xi)\frac{a_2\alpha_1\alpha_2}{\upsilon_r} f_\eta^{\,2} - (\xi+\xi_0)\frac{a_1\alpha_1\alpha_2}{\upsilon_r} f_\eta^{\,2} = (a-\xi)^3 \frac{a_5\alpha_2^3\alpha_3}{\alpha_1\upsilon_r}\left(\tilde{g} - \frac{a_1 - a_2}{4a_2}\eta \tilde{g}_\eta\right)$$

$$\tilde{g}_\eta = \frac{a_7\alpha_2}{\alpha_3}\vartheta$$

and

$$P_r^{-1}(a-\xi)\vartheta_{\eta\eta} + \left\{(a_1+a_2)\frac{\alpha_1\alpha_2}{2\upsilon_r} f - (a-\xi)\frac{a_4\alpha_2}{\upsilon_r}\right\}\vartheta_\eta$$

$$- \frac{(5a_2-a_1)\alpha_1\alpha_2}{2\upsilon_r} f_\eta \vartheta = 0$$

or, $$\left(1-\frac{\xi}{a}\right)^2 f_{\eta\eta\eta}+\left(1-\frac{\xi}{a}\right)\left\{(a_1+a_2)\frac{\alpha_1\alpha_2}{2a\upsilon_r}f-\left(1-\frac{\xi}{a}\right)\frac{a_4\alpha_2}{\upsilon_r}\right\}f_{\eta\eta}$$

$$-\left(1-\frac{\xi}{a}\right)\frac{a_2\alpha_1\alpha_2}{a\upsilon_r}f_\eta^{\,2}-\left(\frac{\xi}{a}+\frac{\xi_0}{a}\right)\frac{a_1\alpha_1\alpha_2}{a\upsilon_r}f_\eta^{\,2}$$

$$=\left(1-\frac{\xi}{a}\right)^3\frac{aa_5\alpha_2^3\alpha_3}{\alpha_1\upsilon_r}\left(\tilde{g}-\frac{a_1-a_2}{4a_2}\eta\tilde{g}_\eta\right) \tag{31.8}$$

$$\tilde{g}_\eta=\frac{a_7\alpha_2}{\alpha_3}\vartheta \tag{31.9}$$

and

$$P_r^{-1}\left(1-\frac{\xi}{a}\right)\vartheta_{\eta\eta}+\left\{(a_1+a_2)\frac{\alpha_1\alpha_2}{2a\upsilon_r}f-\left(1-\frac{\xi}{a}\right)\frac{a_4\alpha_2}{\upsilon_r}\right\}\vartheta_\eta$$

$$-\frac{(5a_2-a_1)\alpha_1\alpha_2}{2a\upsilon_r}f_\eta\vartheta=0 \tag{31.10}$$

Choosing $\alpha_1=\alpha_2$, $\frac{(a_1+a_2)\alpha_1^2}{2a\upsilon_r}=1$ and $\frac{2a_2}{a_1+a_2}=\beta$. Also for a purely free convection flow we are to put $\frac{2a^2a_5\alpha_3}{a_1+a_2}=1$. Then

$$\frac{a_7\alpha_1}{\alpha_3}=\frac{2a^2a_5a_7}{a_1+a_2}\sqrt{\frac{2a\upsilon_r}{a_1+a_2}}$$

$$=\frac{8a^2a_2a_6a_7}{a_1-a_2}\times\frac{\sqrt{2a\upsilon_r}}{(a_1+a_2)\sqrt{a_1+a_2}}$$

$$=\frac{8a^3a_2}{a_1-a_2}\times\frac{\sqrt{2a\upsilon_r}}{(a_1+a_2)\sqrt{a_1+a_2}}\times\frac{\gamma\gamma_\xi}{\rho_rU}P\times\left(-\frac{\rho_rg_z\beta_T\gamma\Delta T}{P}\right)$$

$$=\frac{8a^2a_2}{a_1-a_2}\times\frac{\sqrt{2a\upsilon_r}}{(a_1+a_2)\sqrt{a_1+a_2}}\left(-g_z\beta_T\Delta T\right)\frac{\gamma^2\gamma_\xi}{U}$$

Using equation (3I.3), we get

$$\frac{a_7\alpha_1}{\alpha_3} = \frac{8a^2a_2}{a_1 - a_2} \times \frac{\sqrt{2a\upsilon_r}}{(a_1 + a_2)\sqrt{a_1 + a_2}}\left(-g_z\,\beta_T\Delta T\right)\frac{a_1 - a_2}{2U^2}\sqrt{\frac{a_1\xi + A}{U}}$$

$$\left(\because \gamma\gamma_\xi = \frac{a_1 - a_2}{2U}\right)$$

$$= \frac{4a^2a_2}{(a_1 + a_2)\sqrt{a_1 + a_2}} \times \frac{\sqrt{2a\upsilon_r(a_1\xi + A)}}{U^{\frac{5}{2}}}\left(-g_z\,\beta_T\Delta T\right)$$

$$= \frac{4a^2a_2}{a_1 + a_2} \times \sqrt{\frac{2a_1}{a_1 + a_2}}\left(-g_z\,\beta_T\Delta T\right)\frac{\sqrt{\upsilon_r a(\xi + \xi_0)}}{U^{\frac{5}{2}}}$$

$$= \frac{4a^{\frac{5}{2}}a_2}{a_1 + a_2} \times \sqrt{\frac{2a_1}{a_1 + a_2}}\left(-g_z\,\beta_T\Delta T\right)\frac{\sqrt{\upsilon_r(\xi + \xi_0)}}{U^{\frac{5}{2}}}$$

where $\xi + \xi_0$ is termed as the local characteristic length.

Since $\dfrac{2a_2}{a_1 + a_2} = \beta$, then $\dfrac{2a_1}{a_1 + a_2} = 2 - \beta$, we have

$$\frac{a_7\alpha_1}{\alpha_3} = \frac{2a^{\frac{5}{2}}\beta\sqrt{2-\beta}\left(-g_z\,\beta_T\Delta T\right)\sqrt{\upsilon_r(\xi + \xi_0)}}{U^{\frac{5}{2}}}$$

$$= \left(\frac{U_F}{U}\right)^{\frac{5}{2}}$$

where $U_F = a\left[2\beta\sqrt{2-\beta}\left(-g_z\,\beta_T\Delta T\right)\sqrt{\upsilon_r(\xi + \xi_0)}\right]^{\frac{2}{5}}$ (3I.11)

is called the free convection velocity associated with the local characteristic length $\xi + \xi_0$. Since we are concerned with the free convection flows, without loss of generality we may put $U = U_F$.

Thus the above equations (31.8) to (31.10) can be written as

$$\left(1-\frac{\xi}{a}\right)^2 f_{\eta\eta\eta}+\left(1-\frac{\xi}{a}\right)\left\{(a_1+a_2)\frac{\alpha_1^{\;2}}{2a\upsilon_r}f-\left(1-\frac{\xi}{a}\right)\frac{a_4\alpha_1}{\upsilon_r}\right\}f_{\eta\eta}$$

$$-\left(1-\frac{\xi}{a}\right)\frac{a_2\alpha_1^{\;2}}{a\upsilon_r}f_\eta^{\;2}-\left(\frac{\xi}{a}+\frac{\xi_0}{a}\right)\frac{a_1\alpha_1^{\;2}}{a\upsilon_r}f_\eta^{\;2}$$

$$=\left(1-\frac{\xi}{a}\right)^3\frac{aa_5\alpha_1^2\alpha_3}{\upsilon_r}\left(\tilde{g}-\frac{a_1-a_2}{4a_2}\eta\tilde{g}_\eta\right)$$

$$\tilde{g}_\eta=\left(\frac{U_F}{U}\right)^{\frac{5}{2}}\vartheta$$

and

$$P_r^{-1}\left(1-\frac{\xi}{a}\right)\vartheta_{\eta\eta}+\left\{(a_1+a_2)\frac{\alpha_1^{\;2}}{2a\upsilon_r}f-\left(1-\frac{\xi}{a}\right)\frac{a_4\alpha_1}{\upsilon_r}\right\}\vartheta_\eta$$

$$-\frac{(5a_2-a_1)\alpha_1^{\;2}}{2a\upsilon_r}f_\eta\vartheta=0$$

or, $$\left(1-\frac{\xi}{a}\right)^2 f_{\eta\eta\eta}+\left(1-\frac{\xi}{a}\right)\left\{f-\left(1-\frac{\xi}{a}\right)f_w\right\}f_{\eta\eta}-\left(1-\frac{\xi}{a}\right)\beta f_\eta^{\;2}$$

$$-(2-\beta)\left(\frac{\xi}{a}+\frac{\xi_0}{a}\right)f_\eta^{\;2}=\left(1-\frac{\xi}{a}\right)^3\left(\tilde{g}-\frac{1-\beta}{2\beta}\eta\tilde{g}_\eta\right)$$

$$\tilde{g}_\eta=\vartheta$$

and

$$P_r^{-1}\left(1-\frac{\xi}{a}\right)\vartheta_{\eta\eta}+\left\{f-\left(1-\frac{\xi}{a}\right)f_w\right\}\vartheta_\eta-(3\beta-1)f_\eta\vartheta=0$$

or, if we put $\frac{\xi}{a}=\tilde{\xi}$ ; $\tilde{\xi}<<1$, then

$$(1-\tilde{\xi})^2 f_{\eta\eta\eta}+(1-\tilde{\xi})\{f-(1-\tilde{\xi})f_w\}f_{\eta\eta}-(1-\tilde{\xi})\beta f_\eta^2$$

$$-(2-\beta)(\tilde{\xi}+\tilde{\xi}_0)f_\eta^2=(1-\tilde{\xi})^3\left(\tilde{g}-\frac{1-\beta}{2\beta}\eta\tilde{g}_\eta\right) \qquad (3I.12)$$

$$\tilde{g}_\eta=\vartheta \qquad (3I.13)$$

and

$$P_r^{-1}(1-\tilde{\xi})\vartheta_{\eta\eta}+\{f-(1-\tilde{\xi})f_w\}\vartheta_\eta-(3\beta-1)f_\eta\vartheta=0 \qquad (3I.14)$$

where $f_w$ is the non-dimensional suction (or blowing) velocity at the surface of the disc defined by

$$f_w=a\sqrt{(2-\beta)R_F}\ \frac{w_s}{U_F}$$

$$=a\sqrt{\frac{2-\beta}{\upsilon_r}}\ w_s U_F^{-\frac{1}{2}}(\tilde{\xi}+\xi_{\bar{0}})^{\frac{1}{2}} \qquad (3I.15)$$

Here $R_F$ is the Reynolds number based on free convection velocity $U_F$ given by

$$U_F=\left[2a^3\beta\sqrt{2-\beta}(-g_z\,\beta_T\Delta T)\sqrt{\upsilon_r(\tilde{\xi}+\tilde{\xi}_0)}\right]^{\frac{2}{5}} \qquad (3I.15a)$$

and the local characteristic length $\tilde{\xi}+\tilde{\xi}_0$ as $R_F=\dfrac{U_F(\tilde{\xi}+\tilde{\xi}_0)}{\upsilon_r}$ (3I.16)

or, in terms of $\xi$ , $f_w=\sqrt{a(2-\beta)R_F}\ \dfrac{w_s}{U_F}$ (3I.16a)

where $U_F$ is given by the equation (3I.11) and $R_F=\dfrac{U_F(\xi+\xi_0)}{\upsilon_r}$ (3I.16b)

The transformed boundary conditions are:

$$f(0)=f_\eta(0)=0\ ;\ f_\eta(\infty)=0$$
$$\vartheta(0)=1\ ;\ \vartheta(\infty)=0 \qquad (3I.17)$$

The boundary conditions on $\tilde{g}(\eta)$ may be derived from the last boundary conditions described in equation (3.36). That is,

$$\tilde{g}_\eta(0)=1\ ,\ \tilde{g}_\eta(\infty)=0 \qquad (3I.17a)$$

The similarity function $f(\eta)$, the similarity variable $\eta$ and the pressure function $\tilde{g}(\eta)$ are related to the stream function $\psi$, the physical co-ordinate $(\tilde{x}, z)$ and perturbation pressure $\tilde{p}$ by the following equations respectively

$$\psi = \upsilon_r \sqrt{a(2-\beta)R_F}\ f(\eta)+\psi(\tilde{x},0) \qquad (3I.18)$$

$$\eta = \frac{R_F^{\frac{1}{2}}}{\sqrt{a(2-\beta)}}\frac{z}{(\tilde{x}+x_0)} \qquad (3I.19)$$

and $\tilde{p} = \dfrac{\rho_r U_F^{\ 2}}{2a^2\beta}\tilde{g}(\eta)$ (3I.20)

The velocity components $(=\tilde{u}, w)$ along with the skin friction $\tau_s$ and the local heat transfer co-efficient $q_s$ associated with the equations (3I.12) to (3I.14) are as follows:

$$\tilde{u} = \frac{U_F}{a-\tilde{x}} f_\eta \qquad (3I.21)$$

$$-w = \frac{\upsilon_r}{a-\tilde{x}}\sqrt{\frac{a}{2-\beta}}\frac{R_F^{\frac{1}{2}}}{\tilde{x}+x_0}\{f-(1-\beta)\eta f_\eta\}-w_s$$

$$= \frac{\upsilon_r}{a-\tilde{x}}\sqrt{\frac{a}{2-\beta}}\frac{R_F^{\frac{1}{2}}}{\tilde{x}+x_0}\{f-(1-\beta)\eta f_\eta\}-\frac{U_F R_F^{-\frac{1}{2}}}{\sqrt{a(2-\beta)}} f_w \qquad (3I.22)$$

$$\tau_s = \mu \frac{\partial \tilde{u}}{\partial z}\bigg|_{z=0} = \frac{\mu}{a-\tilde{x}} \frac{U_F}{\sqrt{a(2-\beta)}} \frac{R_F^{\frac{1}{2}}}{(\tilde{x}+x_0)} f_{\eta\eta}(0) \tag{3I.23}$$

and $$q_s = -\kappa \frac{\partial T}{\partial z}\bigg|_{z=0} = -\kappa\,\Delta T\left[\frac{\partial \theta}{\partial z}\right]_{z=0} = -\frac{\kappa\,\Delta T}{\sqrt{a(2-\beta)}} \frac{R_F^{\frac{1}{2}}}{(\tilde{x}+x_0)} \vartheta_\eta(0) \tag{3I.24}$$

Here $q_s$ is the heat transfer rate per unit area of the disc.

Now, if for simplicity we consider $\tilde{\xi}_0 = 0$, then at the periphery of the disc where $\xi \to 0$, or, $\tilde{\xi} \to 0$, (i.e., $\tilde{x} \to 0$, or, $x \to a$), the equations (3I.12) to (3I.14) become-

$$f_{\eta\eta\eta} + (f - f_w) f_{\eta\eta} - \beta f_\eta^2 = \tilde{g} - \frac{1-\beta}{2\beta} \eta \tilde{g}_\eta \tag{3I.25}$$

$$\tilde{g}_\eta = \vartheta \tag{3I.26}$$

and

$$P_r^{-1} \vartheta_{\eta\eta} + (f - f_w)\vartheta_\eta - (3\beta - 1) f_\eta \vartheta = 0 \tag{3I.27}$$

with boundary conditions given in (3I.17) and (3I.17a), and if here, the suction (or blowing) effect is ignored then the equations (3I.25) to (3I.27) reduced to those for the two-dimensional boundary-layer development on a horizontal surface.

Thus, with minor changes in the similarity function and similarity variable (i.e., $F(\eta_R) = 5^{-\frac{1}{4}}\left\{10\beta(2-\beta)^3\right\}^{\frac{1}{5}} f(\eta)$, $\eta_R = 5^{\frac{1}{4}}\left\{\frac{(2-\beta)^2}{10\beta}\right\}^{\frac{1}{5}} \eta$; (where suffix 'R' stands for Rotem and Claassen)) the similarity equations established by Rotem and Claassen (1969a) by the method of group theory may also be obtained.

Again, as $\Delta T$ is fully responsible for the buoyancy flow, therefore, in order to have the existence of the similarity solutions, $\Delta T$ -variation is found to be

$$\Delta T \propto (\tilde{x} + x_0)^{\frac{3\beta-1}{2-\beta}} \tag{3I.28}$$

while the suction velocity $w_s$ varies as

$$w_s \propto (\tilde{x} + x_0)^{\frac{\beta-1}{2-\beta}} \tag{31.29}$$

If it is assumed that $\frac{3\beta-1}{2-\beta} = m$ and $\frac{\beta-1}{2-\beta} = n$, then we can write

$$\Delta T = k_1 \tilde{x}^m \quad \text{(where } k_1 \text{ is a constant)}$$

and $w_s$ must be of the type

$$w_s = k_2 \tilde{x}^n$$

where $k_2$ is a constant of uniform transpiration rate and $n = \frac{1}{5}(m-2)$. That is, there exist a relation between $m$ and $n$ which is $m = 5n + 2$ (31.30)

The constant $k_2$ is negative for the case of suction, and positive for blowing. For an impermeable surface, $k_2 = 0$.

Also it appears that the exponent of $w_s$ and $\Delta T$-variation are finite provided $\beta \neq 2$.

For the case of isothermal surface we have $m = 0$, then $n = -\frac{2}{5}$

i.e., $w_s = k_2 \tilde{x}^{-\frac{2}{5}}$ (31.31)

The special case of $n = 0$ (i.e., $\beta = 1$) defined here as isothermal suction (or blowing).

Since the dependent variables $f$, $\vartheta$ and $\tilde{g}$ mostly depend on the variable $\eta$ (similarity variable), we may expand the dependent variables $f(\tilde{\xi},\eta)$, $\vartheta(\tilde{\xi},\eta)$ and $\tilde{g}(\tilde{\xi},\eta)$ for $|\tilde{\xi}| << 1$, in ascending powers of $\tilde{\xi}$ as follows:

$$f(\tilde{\xi},\eta)=\sum_{j=0}^{\infty}\tilde{\xi}^{j} f_j(\eta)=f_0(\eta)+\tilde{\xi} f_1(\eta)+\tilde{\xi}^2 f_2(\eta)+\ldots\ldots\ldots\ldots \tag{3I.32}$$

$$\vartheta(\tilde{\xi},\eta)=\sum_{j=0}^{\infty}\tilde{\xi}^{j} \vartheta_j(\eta)=\vartheta_0(\eta)+\tilde{\xi} \vartheta_1(\eta)+\tilde{\xi}^2 \vartheta_2(\eta)+\ldots\ldots\ldots\ldots \tag{3I.33}$$

and $$\tilde{g}(\tilde{\xi},\eta)=\sum_{j=0}^{\infty}\tilde{\xi}^{j} \tilde{g}_j(\eta)=\tilde{g}_0(\eta)+\tilde{\xi} \tilde{g}_1(\eta)+\tilde{\xi}^2 \tilde{g}_2(\eta)+\ldots\ldots\ldots\ldots \tag{3I.34}$$

Now, inserting the expansions (3I.32) to (3I.34) into equations (3I.12) to (3I.14) together with the boundary conditions (3I.17), with $\tilde{\xi}_0 = 0$, and equating like powers of $\tilde{\xi}$, we obtain

$\tilde{\xi}^0$:

$$\left.\begin{aligned}
& f_{0\eta\eta\eta}+(f_0-f_w)f_{0\eta\eta}-\beta f_{0\eta}^{\,2}=\tilde{g}_0-\frac{1-\beta}{2\beta}\eta\tilde{g}_{0\eta}\\
& \tilde{g}_{0\eta}=\vartheta_0\\
& P_r^{-1}\vartheta_{0\eta\eta}+(f_0-f_w)\vartheta_{0\eta}-(3\beta-1)f_{0\eta}\vartheta_0=0
\end{aligned}\right\} \tag{3I.35}$$

boundary conditions:

$$\begin{aligned}
& f_0(0)=f_{0\eta}(0)=0\ ;\ f_{0\eta}(\infty)=0\\
& \vartheta_0(0)=1\ ;\ \vartheta_0(\infty)=0\\
& \tilde{g}_{0\eta}(0)=1\ ,\ \tilde{g}_{0\eta}(\infty)=0
\end{aligned} \tag{3I.36}$$

$\tilde{\xi}^1$ :

$$
\left.
\begin{aligned}
& f_{1\eta\eta\eta} - 2f_{0\eta\eta\eta} + f_0 f_{1\eta\eta} + (f_1 - f_0) f_{0\eta\eta} - (f_{1\eta\eta} - 2f_{0\eta\eta}) f_w - 2\beta f_{0\eta} f_{1\eta} \\
& \qquad - 2(1-\beta) f_{0\eta}^{\;2} = \tilde{g}_1 - 3\tilde{g}_0 - \frac{1-\beta}{2\beta}\eta(\tilde{g}_{1\eta} - 3\tilde{g}_{0\eta}) \\
& \tilde{g}_{1\eta} = \vartheta_1 \\
& P_r^{-1}(\vartheta_{1\eta\eta} - \vartheta_{0\eta\eta}) + f_0\vartheta_{1\eta} + f_1\vartheta_{0\eta} - (\vartheta_{1\eta} - \vartheta_{0\eta}) f_w - (3\beta - 1)(f_{0\eta}\vartheta_1 + f_{1\eta}\vartheta_0) = 0
\end{aligned}
\right\}
\tag{31.37}
$$

boundary conditions:

$$
\begin{aligned}
& f_1(0) = f_{1\eta}(0) = 0 \; ; \; f_{1\eta}(\infty) = 0 \\
& \vartheta_1(0) = 0 \; ; \; \vartheta_1(\infty) = 0 \\
& \tilde{g}_{1\eta}(0) = 0 \; , \; \tilde{g}_{1\eta}(\infty) = 0
\end{aligned}
\tag{31.38}
$$

$\tilde{\xi}^2$ :

$$
\left.
\begin{aligned}
& f_{2\eta\eta\eta} - 2f_{1\eta\eta\eta} - f_{0\eta\eta\eta} + f_0 f_{2\eta\eta} + (f_1 - f_0) f_{1\eta\eta} + (f_2 - f_1) f_{0\eta\eta} - (f_{2\eta\eta} - 2f_{1\eta\eta} \\
& \quad + f_{0\eta\eta}) f_w - 2\beta f_{0\eta} f_{2\eta} - 4(1-\beta) f_{0\eta} - 2\beta f_{0\eta} f_{1\eta} - \beta f_{1\eta}^{\;2} = \tilde{g}_2 - 3\tilde{g}_1 + 3\tilde{g}_0 \\
& \qquad - \frac{1-\beta}{2\beta}\eta(\tilde{g}_{2\eta} - 3\tilde{g}_{1\eta} + 3\tilde{g}_{0\eta}) \\
& \tilde{g}_{2\eta} = \vartheta_2 \\
& P_r^{-1}(\vartheta_{2\eta\eta} - \vartheta_{1\eta\eta}) + f_0\vartheta_{2\eta} + f_1\vartheta_{1\eta} + f_2\vartheta_{0\eta} - (\vartheta_{2\eta} - \vartheta_{1\eta}) f_w \\
& \qquad - (3\beta - 1)(f_{0\eta}\vartheta_2 + f_{1\eta}\vartheta_1 + f_{2\eta}\vartheta_0) = 0
\end{aligned}
\right\}
\tag{31.39}
$$

boundary conditions:

$$
\begin{aligned}
& f_2(0) = f_{2\eta}(0) = 0 \; ; \; f_{2\eta}(\infty) = 0 \\
& \vartheta_2(0) = 0 \; ; \; \vartheta_2(\infty) = 0 \\
& \tilde{g}_{2\eta}(0) = 0 \; , \; \tilde{g}_{2\eta}(\infty) = 0
\end{aligned}
\tag{31.40}
$$

and so on.

The equations (3I.35) fully coincide with the equations (3I.25) to (3I.27) and are those for the two-dimensional boundary-layer development on a semi-infinite horizontal surface.

Now using equations (3I.16a) and (3I.15a) in (3I.24) we obtain the local heat transfer rate per unit area of the disc

$$q_s = -\frac{\kappa \Delta T}{a\sqrt{2-\beta}} \frac{\left[2a^3\beta\sqrt{2-\beta}\left(-g_z\beta_T\Delta T\right)\sqrt{\upsilon_r\tilde{\xi}}\right]^{\frac{1}{5}}}{\sqrt{\upsilon_r\tilde{\xi}}} \vartheta_\eta(0)$$

or, $$\frac{q_s}{\kappa\Delta T} = -\{2\beta(-g_z\beta_T\Delta T)\}^{\frac{1}{5}} a^{-\frac{2}{5}}(2-\beta)^{-\frac{2}{5}}\upsilon_r^{-\frac{2}{5}}\tilde{\xi}^{-\frac{2}{5}}\vartheta_\eta(0)$$

or, using (3I.33), we get

$$\frac{q_s}{\kappa\Delta T} = -\{2\beta(-g_z\beta_T\Delta T)\}^{\frac{1}{5}}\{\upsilon_r a(2-\beta)\}^{-\frac{2}{5}}\tilde{\xi}^{-\frac{2}{5}}\{\vartheta_{0_\eta}(0)+\tilde{\xi}\vartheta_{1_\eta}(0)+\tilde{\xi}^2\vartheta_{2_\eta}(0)+\ldots\ldots\}$$

(3I.41)

For elementary ring area of the circular disc (see Fig. 2) of radius $(a-\tilde{x})$ we obtain the overall surface heat transfer rate from the disc $Q$ as

$$Q = -\int_a^0 q_s \times 2\pi(a-\tilde{x})\delta\tilde{x}$$

$$= 2\pi a^2 \int_0^1 q_s\left(1-\tilde{\xi}\right)\delta\tilde{\xi}$$

where $q_s$ is the heat transfer rate per unit area of the disc.

or, using (3I.41), we obtain

$$Q = -2\pi a^2\kappa\Delta T\{2\beta(-g_z\beta_T\Delta T)\}^{\frac{1}{5}}\{\upsilon_r a(2-\beta)\}^{-\frac{2}{5}}$$

$$\int_0^1 \tilde{\xi}^{-\frac{2}{5}}\left(1-\tilde{\xi}\right)\{\vartheta_{0_\eta}(0)+\tilde{\xi}\vartheta_{1_\eta}(0)+\tilde{\xi}^2\vartheta_{2_\eta}(0)+\ldots\ldots\}\delta\tilde{\xi} \qquad (3I.42)$$

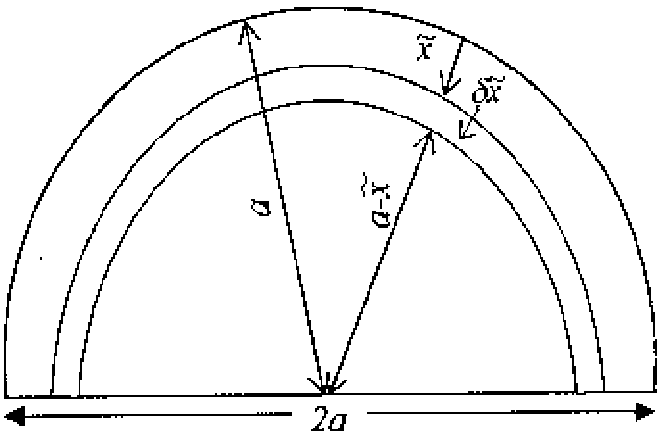


Figure-2: A portion of the elementary circular ring

Now,

$$\int_0^1 \tilde{\xi}^{-\frac{2}{5}}\left(1-\tilde{\xi}\right)\left\{\vartheta_{0\eta}(0)+\tilde{\xi}\vartheta_{1\eta}(0)+\tilde{\xi}^2\vartheta_{2\eta}(0)+\ldots\ldots\right\}\delta\tilde{\xi}$$

$$=\int_0^1\left(\tilde{\xi}^{-\frac{2}{5}}-\tilde{\xi}^{\frac{3}{5}}\right)\left\{\vartheta_{0\eta}(0)+\tilde{\xi}\vartheta_{1\eta}(0)+\tilde{\xi}^2\vartheta_{2\eta}(0)+\ldots\ldots\right\}\delta\tilde{\xi}$$

$$=\int_0^1\left\{\left(\tilde{\xi}^{-\frac{2}{5}}-\tilde{\xi}^{\frac{3}{5}}\right)\vartheta_{0\eta}(0)+\left(\tilde{\xi}^{\frac{3}{5}}-\tilde{\xi}^{\frac{8}{5}}\right)\vartheta_{1\eta}(0)+\left(\tilde{\xi}^{\frac{8}{5}}-\tilde{\xi}^{\frac{13}{5}}\right)\vartheta_{2\eta}(0)+\ldots\ldots\ldots\ldots\ldots\right\}\delta\tilde{\xi}$$

$$=5\left[\left(\frac{1}{3}\tilde{\xi}^{\frac{3}{5}}-\frac{1}{8}\tilde{\xi}^{\frac{8}{5}}\right)\vartheta_{0\eta}(0)+\left(\frac{1}{8}\tilde{\xi}^{\frac{8}{5}}-\frac{1}{13}\tilde{\xi}^{\frac{13}{5}}\right)\vartheta_{1\eta}(0)+\left(\frac{1}{13}\tilde{\xi}^{\frac{13}{5}}-\frac{1}{18}\tilde{\xi}^{\frac{18}{5}}\right)\vartheta_{2\eta}(0)+\ldots\right]_0^1$$

$$=5\left[\left(\frac{1}{3}-\frac{1}{8}\right)\vartheta_{0\eta}(0)+\left(\frac{1}{8}-\frac{1}{13}\right)\vartheta_{1\eta}(0)+\left(\frac{1}{13}-\frac{1}{18}\right)\vartheta_{2\eta}(0)+\ldots\ldots\right]$$

$$=5\left[\frac{5}{24}\vartheta_{0\eta}(0)+\frac{5}{8\times13}\vartheta_{1\eta}(0)+\frac{5}{13\times18}\vartheta_{2\eta}(0)+\ldots\ldots\right]$$

$$=\frac{25}{24}\left[\vartheta_{0\eta}(0)+\frac{24}{8\times13}\vartheta_{1\eta}(0)+\frac{24}{13\times18}\vartheta_{2\eta}(0)+\ldots\ldots\right]$$

$$=\frac{25}{24}\left[\vartheta_{0\eta}(0)+\sum_{j=1}^{\infty}\frac{24}{(5j+3)(5j+8)}\vartheta_{j\eta}(0)\right]$$

Then from equation (3I.42) we obtain

$$Q = -2\pi a^2 \kappa \Delta T\{2\beta(-g_z\beta_T\Delta T)\}^{\frac{1}{5}}\{\upsilon_r a(2-\beta)\}^{-\frac{2}{5}}$$

$$\frac{25}{24}\left[\vartheta_{0\eta}(0)+\sum_{j=1}^{\infty}\frac{24}{(5j+3)(5j+8)}\vartheta_{j\eta}(0)\right] \tag{3I.43}$$

Hence the average heat transfer rate $\tilde{Q}$ is given by

$$\tilde{Q} = \frac{Q}{\pi a^2}$$

or, $$\tilde{Q} = -\kappa\Delta T\{2\beta(-g_z\beta_T\Delta T)\}^{\frac{1}{5}}\{\upsilon_r a(2-\beta)\}^{-\frac{2}{5}}$$

$$\frac{25}{12}\left[\vartheta_{0\eta}(0)+\sum_{j=1}^{\infty}\frac{24}{(5j+3)(5j+8)}\vartheta_{j\eta}(0)\right] \tag{3I.44}$$

Consequently, defining Nusselt and Grashof numbers based on diameter '$2a$' of the disc as:

$$\mathrm{Nu} = \frac{\tilde{Q}.2a}{\kappa\Delta T}$$

$$\mathrm{Gr} = -\frac{(2a)^3 g_z\beta_T\Delta T}{\upsilon_r^2}$$

we may express equation (3I.44) as

$$\mathrm{Nu} = -\frac{25}{12}\{2\beta(-g_z\beta_T\Delta T)\}^{\frac{1}{5}}\{\upsilon_r a(2-\beta)\}^{-\frac{2}{5}}.2a\left[\vartheta_{0\eta}(0)+\sum_{j=1}^{\infty}\frac{24}{(5j+3)(5j+8)}\vartheta_{j\eta}(0)\right]$$

or, $$\mathrm{Nu} = -\frac{25}{12}\left\{-\frac{(2a)^3 g_z\beta_T\Delta T}{\upsilon_r^2}\right\}^{\frac{1}{5}}\left\{\frac{2^3\beta}{(2-\beta)^2}\right\}^{\frac{1}{5}}\left[\vartheta_{0\eta}(0)+\sum_{j=1}^{\infty}\frac{24}{(5j+3)(5j+8)}\vartheta_{j\eta}(0)\right]$$

or, $$\mathrm{Nu} = -\frac{25}{12}\mathrm{Gr}^{\frac{1}{5}}\left\{\frac{8\beta}{(2-\beta)^2}\right\}^{\frac{1}{5}}\left[\vartheta_{0\eta}(0)+\sum_{j=1}^{\infty}\frac{24}{(5j+3)(5j+8)}\vartheta_{j\eta}(0)\right] \tag{3I.45}$$

Equation (3I.45) may be compared with the equation (12b) of Zakerullah and Ackroyd (1979) with $C = \frac{2-\beta}{\sqrt{8\beta}}$. $C \simeq 1$ for $\beta = \frac{1}{3}$, i.e., for a constant temperature difference $\Delta T$. But in this case the suction velocity varies as $\tilde{x}^{-\frac{2}{5}}$.

**Type-II:** (exponential variation)

From the above type of solution it is observed that the solution is real and finite provided $\beta \neq 2$. So we are now interested to have a solution when $\beta = 2$. We see from the relation $\beta = \dfrac{2a_2}{a_1 + a_2}$ that $\beta = 2$ when $a_1 = 0$,

i.e., $\left(\gamma^2 U\right)_\xi = 0$

or, $\gamma^2 U = k$

or, $\gamma^2(\xi) = \dfrac{k}{U(\xi)}$

where $k$ is constant of integration.

Then from relation (ii) of equation (3.37) we have

$$\frac{k}{U} U_\xi = a_2$$

or, $$\frac{(U)_\xi}{U} = \frac{a_2}{k}$$

or, $$(\log U)_\xi = \frac{a_2}{k}$$

Integrating with respect to $\xi$, we have

or, $$\log U(\xi) = \frac{a_2}{k}\xi + \log C$$

$$\therefore \quad U(\xi) = C e^{\frac{a_2}{k}\xi} \qquad (3\text{II}.1)$$

where $C$ is constant of integration.

Substituting (3II.1) in the above equation, we have

$$\gamma^2(\xi) = \frac{k}{U(\xi)}$$

$$= \frac{k}{Ce^{\frac{a_2}{k}\xi}}$$

$$= \frac{k}{C} e^{-\frac{a_2}{k}\xi}$$

Hence for this exponential case $U(\xi)$ and $\gamma(\xi)$ are given by

$$U(\xi) = Ce^{\frac{a_2}{k}\xi} \tag{3II.2}$$

and $$\gamma^2(\xi) = \frac{k}{C} e^{-\frac{a_2}{k}\xi} \tag{3II.3}$$

By virtne of (3II.2) and (3II.3) with the similarity requirements (3.37), we obtain the following relations between the constants as

$a_1 = 0$, $a_2$ is arbitrary, $a_6 = -\frac{a_5}{4}$; $a_4$, $a_5$ and $a_7$ are disposable constants and $a_8 = \frac{5a_2}{2}$.

Hence the general equations (3.33) to (3.35) take the following form as

$$\upsilon_r (a-\xi)^2 F_{\tilde{\eta}\tilde{\eta}\tilde{\eta}} + (a-\xi)\left\{\frac{1}{2}a_2 F - (a-\xi)a_4\right\}F_{\tilde{\eta}\tilde{\eta}} - (a-\xi)a_2 F_{\tilde{\eta}}^2$$
$$- kF_{\tilde{\eta}}^2 = (a-\xi)^3 a_5 \left(G + \frac{1}{4}\tilde{\eta}G_{\tilde{\eta}}\right) \tag{3II.4}$$

$$G_{\tilde{\eta}} = a_7 \vartheta \tag{3II.5}$$

and

$$P_r^{-1}\upsilon_r (a-\xi)\vartheta_{\tilde{\eta}\tilde{\eta}} + \left\{\frac{1}{2}a_2 F - (a-\xi)a_4\right\}\vartheta_{\tilde{\eta}} - \frac{5}{2}a_2 F_{\tilde{\eta}}\vartheta = 0 \tag{3II.6}$$

with boundary conditions :

$$\begin{aligned} &F(0) = F_{\tilde{\eta}}(0) = 0 \; ; \; F_{\tilde{\eta}}(\infty) = 0 \\ &\vartheta(0) = 1 \; ; \; \vartheta(\infty) = 0 \end{aligned} \tag{3II.7}$$

Now substituting

$$F=\alpha_1 f, \tilde{\eta}=\alpha_2\eta \text{ and } G=\alpha_3\tilde{g} \;,$$

so that

$$\frac{\partial}{\partial\tilde{\eta}}=\frac{\partial}{\partial\eta}\frac{\partial\eta}{\partial\tilde{\eta}}=\frac{1}{\alpha_2}\frac{\partial}{\partial\eta},\frac{\partial^2}{\partial\tilde{\eta}^2}=\frac{1}{\alpha_2{}^2}\frac{\partial^2}{\partial\eta^2} \text{ etc.}$$

Then we have from equations (3II.4) to (3II.6)

$$\frac{\upsilon_r\alpha_1}{\alpha_2^3}(a-\xi)^2 f_{\eta\eta\eta}+(a-\xi)\left\{\frac{a_2\alpha_1^2}{2\alpha_2^2}f-(a-\xi)\frac{a_4\alpha_1}{\alpha_2^2}\right\}f_{\eta\eta}-(a-\xi)\frac{a_2\alpha_1^2}{\alpha_2^2}f_\eta^{\;2}$$
$$-k\frac{\alpha_1^2}{\alpha_2^2}f_\eta^{\;2}=(a-\xi)^3 a_5\alpha_3\left(\tilde{g}+\frac{1}{4}\eta\tilde{g}_\eta\right)$$

$$\frac{\alpha_3}{\alpha_2}\tilde{g}_\eta=a_7\vartheta$$

and

$$\frac{P_r^{-1}\upsilon_r}{\alpha_2^2}(a-\xi)\vartheta_{\eta\eta}+\left\{\frac{a_2\alpha_1}{2\alpha_2}f-(a-\xi)\frac{a_4}{\alpha_2}\right\}\vartheta_\eta-\frac{5a_2\alpha_1}{2\alpha_2}f_\eta\vartheta=0$$

or, $$(a-\xi)^2 f_{\eta\eta\eta}+(a-\xi)\left\{\frac{a_2\alpha_1\alpha_2}{2\upsilon_r}f-(a-\xi)\frac{a_4\alpha_2}{\upsilon_r}\right\}f_{\eta\eta}-(a-\xi)\frac{a_2\alpha_1\alpha_2}{\upsilon_r}f_\eta^{\;2}$$
$$-k\frac{\alpha_1\alpha_2}{\upsilon_r}f_\eta^{\;2}=(a-\xi)^3\frac{a_5\alpha_2^3\alpha_3}{\alpha_1\upsilon_r}\left(\tilde{g}+\frac{1}{4}\eta\tilde{g}_\eta\right)$$

$$\tilde{g}_\eta=\frac{a_7\alpha_2}{\alpha_3}\vartheta$$

and

$$P_r^{-1}(a-\xi)\vartheta_{\eta\eta}+\left\{\frac{a_2\alpha_1\alpha_2}{2\upsilon_r}f-(a-\xi)\frac{a_4\alpha_2}{\upsilon_r}\right\}\vartheta_\eta-\frac{5a_2\alpha_1\alpha_2}{2\upsilon_r}f_\eta\vartheta=0$$

or, $$\left(1-\frac{\xi}{a}\right)^2 f_{\eta\eta\eta}+\left(1-\frac{\xi}{a}\right)\left\{\frac{a_2\alpha_1\alpha_2}{2a\upsilon_r}f-\left(1-\frac{\xi}{a}\right)\frac{a_4\alpha_2}{\upsilon_r}\right\}f_{\eta\eta}-\left(1-\frac{\xi}{a}\right)\frac{a_2\alpha_1\alpha_2}{a\upsilon_r}f_\eta^{\;2}$$
$$-\frac{k\alpha_1\alpha_2}{a^2\upsilon_r}f_\eta^{\;2}=\left(1-\frac{\xi}{a}\right)^3\frac{aa_5\alpha_2^3\alpha_3}{\alpha_1\upsilon_r}\left(\tilde{g}+\frac{1}{4}\eta\tilde{g}_\eta\right) \qquad (3II.8)$$

$$\tilde{g}_\eta=\frac{a_7\alpha_2}{\alpha_3}\vartheta \qquad (3II.9)$$

and

$$P_r^{-1}\left(1-\frac{\xi}{a}\right)\vartheta_{\eta\eta}+\left\{\frac{a_2\alpha_1\alpha_2}{2a\upsilon_r}f-\left(1-\frac{\xi}{a}\right)\frac{a_4\alpha_2}{\upsilon_r}\right\}\vartheta_\eta-\frac{5a_2\alpha_1\alpha_2}{2a\upsilon_r}f_\eta\vartheta=0 \quad (3II.10)$$

Choosing $\alpha_1=\alpha_2$, $\frac{a_2\alpha_1^2}{2a\upsilon_r}=1$, and for a purely free convection flow we are to put

$\frac{2a^2a_5\alpha_3}{a_2}=1$. Then we have from the above equations (3II.8) to (3II.10)

$$\left(1-\frac{\xi}{a}\right)^2 f_{\eta\eta\eta}+\left(1-\frac{\xi}{a}\right)\left\{\frac{a_2\alpha_1^2}{2a\upsilon_r}f-\left(1-\frac{\xi}{a}\right)\frac{a_4\alpha_1}{\upsilon_r}\right\}f_{\eta\eta}-\left(1-\frac{\xi}{a}\right)\frac{a_2\alpha_1^2}{a\upsilon_r}f_\eta^2$$
$$-\frac{k\alpha_1^2}{a^2\upsilon_r}f_\eta^2=\left(1-\frac{\xi}{a}\right)^3\frac{aa_5\alpha_2^2\alpha_3}{\upsilon_r}\left(\tilde{g}+\frac{1}{4}\eta\tilde{g}_\eta\right)$$

$$\tilde{g}_\eta=\left(\frac{U_F}{U}\right)^{\frac{5}{2}}\vartheta$$

and

$$P_r^{-1}\left(1-\frac{\xi}{a}\right)\vartheta_{\eta\eta}+\left\{\frac{a_2\alpha_1^2}{2a\upsilon_r}f-\left(1-\frac{\xi}{a}\right)\frac{a_4\alpha_1}{\upsilon_r}\right\}\vartheta_\eta-\frac{5a_2\alpha_1^2}{2a\upsilon_r}f_\eta\vartheta=0$$

where $U_F=2\left\{a^3\left(-g_z\beta_T\Delta T\sqrt{\upsilon_r\times d}\right)\right\}$ . (3II.11)

is the free convection velocity associated with the fixed characteristic length $d\left(=\frac{k}{aa_2}\right)$. Since we are concerned with the free flows, we may put $U=U_F$.

Hence the above equations become

$$\left(1-\frac{\xi}{a}\right)^2 f_{\eta\eta\eta}+\left(1-\frac{\xi}{a}\right)\left\{f-\left(1-\frac{\xi}{a}\right)f_w\right\}f_{\eta\eta}-2\left(1-\frac{\xi}{a}\right)f_\eta^2-2df_\eta^2$$
$$=\left(1-\frac{\xi}{a}\right)^3\left(\tilde{g}+\frac{1}{4}\eta\tilde{g}_\eta\right)$$

$$\tilde{g}_\eta=\vartheta$$

and

$$P_r^{-1}\left(1-\frac{\xi}{a}\right)\vartheta_{\eta\eta}+\left\{f-\left(1-\frac{\xi}{a}\right)f_w\right\}\vartheta_\eta-5f_\eta\vartheta=0$$

or, if we put $\frac{\xi}{a}=\tilde{\xi}$, then

$$\left(1-\tilde{\xi}\right)^2 f_{\eta\eta\eta}+\left(1-\tilde{\xi}\right)\left\{f-\left(1-\tilde{\xi}\right)f_w\right\}f_{\eta\eta}-2\left(1-\tilde{\xi}\right)f_\eta^{\,2}-2df_\eta^{\,2}$$
$$=\left(1-\tilde{\xi}\right)^3\left(\tilde{g}+\frac{1}{4}\eta\tilde{g}_\eta\right) \qquad (3II.12)$$

$$\tilde{g}_\eta=\vartheta \qquad (3II.13)$$

and

$$P_r^{-1}\left(1-\tilde{\xi}\right)\vartheta_{\eta\eta}+\left\{f-\left(1-\tilde{\xi}\right)f_w\right\}\vartheta_\eta-5f_\eta\vartheta=0 \qquad (3II.14)$$

where $f_w=a\sqrt{2R_{F_d}}\,\frac{w_s}{U_F}$ (3II.15)

Here $R_{F_d}$ is the Reynolds number based on free convection velocity $U_F$ given by (3II.11) and the fixed characteristic length $d$ as

$$R_{F_d}=\frac{U_F d}{\upsilon_r} \qquad (3II.16)$$

The transformed boundary conditions are:

$$f(0)=f_\eta(0)=0\ ;\ f_\eta(\infty)=0$$
$$\vartheta(0)=1\ ;\ \vartheta(\infty)=0 \qquad (3II.17)$$
$$\tilde{g}_\eta(0)=1;\ \tilde{g}_\eta(\infty)=0$$

For this steady exponential case the similarity function $f(\eta)$, the similarity variable $\eta$ and the pressure function $\tilde{g}(\eta)$ are related to the stream function $\psi$, the physical co-ordinate $(\tilde{x},z)$ and perturbation pressure $\tilde{p}$ by the following equations respectively

$$\psi = \upsilon_r a \sqrt{2R_{F_d}}\ f(\eta) + \psi(\tilde{x},0) \tag{3II.18}$$

$$\eta = \frac{1}{a}\sqrt{\frac{R_{F_d}}{2}}\frac{z}{d} \tag{3II.19}$$

and

$$\tilde{p} = \frac{\rho_r U_F^{\ 2}}{4a^2}\tilde{g}(\eta) \tag{3II.20}$$

Here $\Delta T$, which is responsible for the buoyant flow, varies exponentially with $\tilde{x}$ as $\Delta T \propto e^{\frac{5(\tilde{x}+x_0)}{2d}}$ and $w_s$ varies as $w_s \propto e^{\frac{\tilde{x}+x_0}{2d}}$.

As we have seen in the previous case that the exponent of $w_s$ and $\Delta T$ -variation are finite provided $\beta \neq 2$, but as usual in such similarity solutions those are finite when $\beta = 2$, now taken the present exponential form.

The velocity components $(= \tilde{u}, w)$along with the skin friction $\tau_s$, and the local heat trausfer co-efficient $q_s$ are respectively

$$\tilde{u} = \frac{U_F f_\eta}{a-\tilde{x}} \tag{3II.21}$$

$$-w = \frac{\upsilon_r}{(a-\tilde{x})d}\frac{R_{F_d}^{\frac{1}{2}}}{\sqrt{2}}\{f + \eta f_\eta\} - w_s$$

i.e.,
$$-w = \frac{\upsilon_r}{(a-\tilde{x})d}\frac{R_{F_d}^{\frac{1}{2}}}{\sqrt{2}}\{f + \eta f_\eta\} - \frac{U_F R_{F_d}^{-\frac{1}{2}}}{a\sqrt{2}} f_w \tag{3II.22}$$

$$\tau_s = \frac{\mu U_F}{(a-\tilde{x})d}\frac{R_{F_d}^{\frac{1}{2}}}{a\sqrt{2}} f_{\eta\eta}(0) \tag{3II.23}$$

and

$$q_s = -\frac{\kappa\,\Delta T}{a\sqrt{2}}\frac{R_{F_d}^{\frac{1}{2}}}{d}\vartheta_\eta(0) \tag{3II.24}$$

# Chapter-4

## Method of Numerical Solutions

It has been shown previously that as $\xi \to 0$, or, $\tilde{\xi} \to 0$, the equations (3I.12) to (3I.14) with boundary conditions (3I.17) reduce to those for the two-dimensional boundary-layer development on semi-infinite horizontal surface. However it was seen from the analysis of Zakerullah and Ackroyd (1979) that when $\tilde{\xi} > 0$, such a two-dimensional nature is lost immediately and the subsequent boundary-layer development becomes progressively influenced by an axially symmetrical squeezing of the flow as the centre of the disc is approached. Thus close to the disc centre, the boundary-layer theory breaks down and a solution of the full Navier-Stokes equations is necessary in this region. So, in our present investigation we will consider the zeroth-order boundary-layer equations (3I.35) with the boundary conditions (3I.36) for numerical solution at the periphery of the disc.

The set of equation (3I.35) together with the boundary conditions (3I.36) are non-linear and coupled. It is difficult to solve them analytically. Hence we adopt a procedure to obtain the solution numerically. Here we use the standard initial-value solver shooting method namely Nachtsheim-Swigert iteration technique (guessing the missing value) (Nachtsheim & Swigert (1965)) and Runge-Kutta Merson method, in collaboration with Runge-Kutta shooting method.

In a shooting method, the missing (unspecified) initial condition at the initial point of the interval is assumed, and the differential equation is then integrated numerically as an initial value problem to the terminal point. The accuracy of the assumed missing initial condition is then checked by comparing the calculated value of the dependent variable at the terminal point with its given value there. If a difference exists, another value of the missing initial condition must be assumed and the process is repeated. This process in continued until the agreement between the calculated and the given condition at the terminal point is within the specified degree of accuracy. For this type of iterative approach, one naturally inquires whether or not there is a systematic way of finding each succeeding (assumed) value of the missing initial condition.

The Nachtsheim-Swigert iteration technique thus needs to be discussed elaborately. The boundary condition (3I.36) associated with the non-linear ODEs (3I.35) of the boundary-layer type are of the two-point asymptotic class. Two-point boundary conditions have values of the dependent variable specified at two different values of independent variable. Specification of an asymptotic boundary condition implies that the first derivative (and higher derivatives of the boundary-layer equations, if exist) of the dependent variable approaches zero as the outer specified value of the independent variable is approached.

The method of numerically integrating a two-point asymptotic boundary-value problem of the boundary-layer type, the initial-value method, requires that it be recast as an initial-value problem. Thus it is necessary to estimate as many boundary conditions at the surface as were (previously) given at infinity. The governing differential equations are then integrated with these assumed surface boundary conditions. If the required outer boundary condition is satisfied, a solution has been achieved. However, this is not generally the case. Hence, a method must be devised to estimate logically the new surface boundary conditions for the next trial integrations. Asymptotic boundary value problems such as those governing the boundary-layer equations are further complicated by the fact that the outer boundary condition is specified at infinity. In the trial integrations infinity is numerically approximated by some large value of the independent variable. There is no a priori general method of estimating this value. Selecting too small a maximum value for the independent variable may not allow the solution to asymptotically converge to the required accuracy. Selecting large a value may result in divergence of the trial integrations or in slow convergence of surface boundary conditions. Selecting too large a value of the independent variable is expensive in terms of computer time.

Nachtsheim-Swigert developed an iteration method to overcome these difficulties. Extension of the Nachtsheim-Swigert iteration scheme to the system of equation (3I.35) and boundary conditions (3I.36) is straightforward. In equation (3I.36) there are three asymptotic boundary conditions and hence three unknown surface conditions $f''(0)$, $\vartheta'(0)$ and $\tilde{g}(0)$ (dropping the subscript '0').

Within the context of the initial-value method and Nachtsheim-Swigert iteration technique the outer boundary conditions may be functionally represented as

$$f'(\eta_{max}) = f'(f''(0),\ \vartheta'(0), \tilde{g}(0)) = \delta_1 \tag{4.1}$$

$$\vartheta(\eta_{max}) = \vartheta(f''(0),\ \vartheta'(0), \tilde{g}(0)) = \delta_2 \tag{4.2}$$

$$\tilde{g}(\eta_{max}) = \tilde{g}(f''(0),\ \vartheta'(0), \tilde{g}(0)) = \delta_3 \tag{4.3}$$

with the asymptotic convergence criteria given by

$$f''(\eta_{max}) = f''(f''(0),\ \vartheta'(0), \tilde{g}(0)) = \delta_4 \tag{4.4}$$

$$\vartheta'(\eta_{max}) = \vartheta'(f''(0),\ \vartheta'(0), \tilde{g}(0)) = \delta_5 \tag{4.5}$$

$$\tilde{g}'(\eta_{max}) = \tilde{g}'(f''(0),\ \vartheta'(0), \tilde{g}(0)) = \delta_6 \tag{4.6}$$

Choosing $f''(0) = g_1$, $\vartheta'(0) = g_2$ and $\tilde{g}(0) = g_3$ and expanding in a first-order Taylor series after using equations (4.1) to (4.6) yields

$$f'(\eta_{max}) = f'_C(\eta_{max}) + f'_{g_1}\Delta g_1 + f'_{g_2}\Delta g_2 + f'_{g_3}\Delta g_3 = \delta_1 \tag{4.7}$$

$$\vartheta(\eta_{max}) = \vartheta_C(\eta_{max}) + \vartheta_{g_1}\Delta g_1 + \vartheta_{g_2}\Delta g_2 + \vartheta_{g_3}\Delta g_3 = \delta_2 \tag{4.8}$$

$$\tilde{g}(\eta_{max}) = \tilde{g}_C(\eta_{max}) + \tilde{g}_{g_1}\Delta g_1 + \tilde{g}_{g_2}\Delta g_2 + \tilde{g}_{g_3}\Delta g_3 = \delta_3 \tag{4.9}$$

$$f''(\eta_{max}) = f_C''(\eta_{max}) + f_{g1}''\Delta g_1 + f_{g2}''\Delta g_2 + f_{g3}''\Delta g_3 = \delta_4 \qquad (4.10)$$

$$\vartheta'(\eta_{max}) = \vartheta_C'(\eta_{max}) + \vartheta_{g1}'\Delta g_1 + \vartheta_{g2}'\Delta g_2 + \vartheta_{g3}'\Delta g_3 = \delta_5 \qquad (4.11)$$

$$\tilde{g}'(\eta_{max}) = \tilde{g}_C'(\eta_{max}) + \tilde{g}_{g1}'\Delta g_1 + \tilde{g}_{g2}'\Delta g_2 + \tilde{g}'\vartheta_{g3}\Delta g_3 = \delta_6 \qquad (4.12)$$

where subscripts indicate partial differentiation , e. g., $f_{g1}' = \dfrac{\partial f'(\eta_{max})}{\partial g_1}$ etc. and subscript '$C$' indicates the value of the function at $\eta_{max}$ determined from the trial integration.

Solution of these equations in a least-squares sense requires determining the minimum value of

$$E = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 + \delta_5^2 + \delta_6^2 \qquad (4.13)$$

with respect to $g_1$, $g_2$ and $g_3$.

Differentiating $E$ with respect to $g_1$ yields

$$2\delta_1\frac{\partial\delta_1}{\partial g_1} + 2\delta_2\frac{\partial\delta_2}{\partial g_1} + 2\delta_3\frac{\partial\delta_3}{\partial g_1} + 2\delta_4\frac{\partial\delta_4}{\partial g_1} + 2\delta_5\frac{\partial\delta_5}{\partial g_1} + 2\delta_6\frac{\partial\delta_6}{\partial g_1} = 0$$

Using the above equations we get

or, $(f_C' + f_{g1}'\Delta g_1 + f_{g2}'\Delta g_2 + f_{g3}'\Delta g_3)f_{g1}' + (\vartheta_C + \vartheta_{g1}\Delta g_1 + \vartheta_{g2}\Delta g_2 + \vartheta_{g3}\Delta g_3)\vartheta_{g1}$

$+(\tilde{g}_C + \tilde{g}_{g1}\Delta g_1 + \tilde{g}_{g2}\Delta g_2 + \tilde{g}_{g3}\Delta g_3)\tilde{g}_{g1} + (f_C'' + f_{g1}''\Delta g_1 + f_{g2}''\Delta g_2 + f_{g3}''\Delta g_3)f_{g1}''$

$+(\vartheta_C' + \vartheta_{g1}'\Delta g_1 + \vartheta_{g2}'\Delta g_2 + \vartheta_{g3}'\Delta g_3)\vartheta_{g1}' + (\tilde{g}_C' + \tilde{g}_{g1}'\Delta g_1 + \tilde{g}_{g2}'\Delta g_2 + \tilde{g}_{g3}'\Delta g_3)\tilde{g}_{g1}'$

or, $(f_{g1}'^2 + \vartheta_{g1}^2 + \tilde{g}_{g1}^2 + f_{g1}''^2 + \vartheta_{g1}'^2 + \tilde{g}_{g1}'^2)\Delta g_1 + (f_{g1}'f_{g2}' + \vartheta_{g1}\vartheta_{g2} + \tilde{g}_{g1}\tilde{g}_{g2} +$

$f_{g1}''f_{g2}'' + \vartheta_{g1}'\vartheta_{g2}' + \tilde{g}_{g1}'\tilde{g}_{g2}')\Delta g_2 + (f_{g1}'f_{g3}' + \vartheta_{g1}\vartheta_{g3} + \tilde{g}_{g1}\tilde{g}_{g3} + f_{g1}''f_{g3}'' + \vartheta_{g1}'\vartheta_{g3}'$

$+ \tilde{g}_{g1}'\tilde{g}_{g3}')\Delta g_3 = -(f_C'f_{g1}' + \vartheta_C\vartheta_{g1} + \tilde{g}_C\tilde{g}_{g1} + f_C''f_{g1}'' + \vartheta_C'\vartheta_{g1}' + +\tilde{g}_C'\tilde{g}_{g1}')$ (4.14)

Similarly differentiating $E$ with respect to $g_2$ and $g_3$ we obtain

or, $$\left(f'_{g1}f'_{g2}+\vartheta_{g1}\vartheta_{g2}+\widetilde{g}_{g1}\widetilde{g}_{g2}+f''_{g1}f''_{g2}+\vartheta'_{g1}\vartheta'_{g2}+\widetilde{g}'_{g1}\widetilde{g}'_{g2}\right)\Delta g_1+\left(f'^{\,2}_{g2}+\vartheta_{g2}^{\,2}+\widetilde{g}_{g2}^{\,2}\right.$$
$$\left.f''^{\,2}_{g2}+\vartheta'^{\,2}_{g2}+\widetilde{g}'^{\,2}_{g2}\right)\Delta g_2+\left(f'_{g2}f'_{g3}+\vartheta_{g2}\vartheta_{g3}+\widetilde{g}_{g2}\widetilde{g}_{g3}+f''_{g2}f''_{g3}+\vartheta'_{g2}\vartheta'_{g3}\right.$$
$$\left.+\widetilde{g}'_{g2}\widetilde{g}'_{g3}\right)\Delta g_3=-\left(f'_Cf'_{g2}+\vartheta_C\vartheta_{g2}+\widetilde{g}_C\widetilde{g}_{g2}+f''_Cf''_{g2}+\vartheta'_C\vartheta'_{g2}++\widetilde{g}'_C\widetilde{g}'_{g2}\right) \qquad (4.15)$$

and

$$\left(f'_{g1}f'_{g3}+\vartheta_{g1}\vartheta_{g3}+\widetilde{g}_{g1}\widetilde{g}_{g3}+f''_{g1}f''_{g3}+\vartheta'_{g1}\vartheta'_{g3}+\widetilde{g}'_{g1}\widetilde{g}'_{g3}\right)\Delta g_1+\left(f'_{g2}f'_{g3}+\vartheta_{g2}\vartheta_{g3}+\right.$$
$$\left.\widetilde{g}_{g2}\widetilde{g}_{g3}+f''_{g2}f''_{g3}+\vartheta'_{g2}\vartheta'_{g3}+\widetilde{g}'_{g2}\widetilde{g}'_{g3}\right)\Delta g_2+\left(f'^{\,2}_{g3}+\vartheta_{g3}^{\,2}+\widetilde{g}_{g3}^{\,2}+f''^{\,2}_{g3}+\vartheta'^{\,2}_{g3}\right.$$
$$\left.+\widetilde{g}'^{\,2}_{g3}\right)\Delta g_3=-\left(f'_Cf'_{g3}+\vartheta_C\vartheta_{g3}+\widetilde{g}_C\widetilde{g}_{g3}+f''_Cf''_{g3}+\vartheta'_C\vartheta'_{g3}++\widetilde{g}'_C\widetilde{g}'_{g3}\right) \qquad (4.16)$$

We can write equations (4.14) to (4.16) in system of linear equations in the following forms as

$$a_{11}\Delta g_1+a_{12}\Delta g_2+a_{13}\Delta g_3=b_{11} \qquad (4.17)$$
$$a_{21}\Delta g_1+a_{22}\Delta g_2+a_{23}\Delta g_3=b_{22} \qquad (4.18)$$
$$a_{31}\Delta g_1+a_{32}\Delta g_2+a_{33}\Delta g_3=b_{33} \qquad (4.19)$$

where

$$a_{11}=f'^{\,2}_{g1}+\vartheta_{g1}^{\,2}+\widetilde{g}_{g1}^{\,2}+f''^{\,2}_{g1}+\vartheta'^{\,2}_{g1}+\widetilde{g}'^{\,2}_{g1}$$
$$a_{12}=f'_{g1}f'_{g2}+\vartheta_{g1}\vartheta_{g2}+\widetilde{g}_{g1}\widetilde{g}_{g2}+f''_{g1}f''_{g2}+\vartheta'_{g1}\vartheta'_{g2}+\widetilde{g}'_{g1}\widetilde{g}'_{g2}$$
$$a_{13}=f'_{g1}f'_{g3}+\vartheta_{g1}\vartheta_{g3}+\widetilde{g}_{g1}\widetilde{g}_{g3}+f''_{g1}f''_{g3}+\vartheta'_{g1}\vartheta'_{g3}+\widetilde{g}'_{g1}\widetilde{g}'_{g3}$$
$$a_{21}=f'_{g1}f'_{g2}+\vartheta_{g1}\vartheta_{g2}+\widetilde{g}_{g1}\widetilde{g}_{g2}+f''_{g1}f''_{g2}+\vartheta'_{g1}\vartheta'_{g2}+\widetilde{g}'_{g1}\widetilde{g}'_{g2}=a_{12}$$
$$a_{22}=f'^{\,2}_{g2}+\vartheta_{g2}^{\,2}+\widetilde{g}_{g2}^{\,2}+f''^{\,2}_{g2}+\vartheta'^{\,2}_{g2}+\widetilde{g}'^{\,2}_{g2}$$
$$a_{23}=f'_{g2}f'_{g3}+\vartheta_{g2}\vartheta_{g3}+\widetilde{g}_{g2}\widetilde{g}_{g3}+f''_{g2}f''_{g3}+\vartheta'_{g2}\vartheta'_{g3}+\widetilde{g}'_{g2}\widetilde{g}'_{g3}$$
$$a_{31}=a_{13}$$
$$a_{32}=a_{23}$$
$$a_{33}=f'^{\,2}_{g3}+\vartheta_{g3}^{\,2}+\widetilde{g}_{g3}^{\,2}+f''^{\,2}_{g3}+\vartheta'^{\,2}_{g3}+\widetilde{g}'^{\,2}_{g3}$$

$$b_{11} = -\left(f'_C f'_{g1} + \vartheta_C \vartheta_{g1} + \widetilde{g}_C \widetilde{g}_{g1} + f''_C f''_{g1} + \vartheta'_C \vartheta'_{g1} + + \widetilde{g}'_C \widetilde{g}'_{g1}\right)$$

$$b_{22} = -\left(f'_C f'_{g2} + \vartheta_C \vartheta_{g2} + \widetilde{g}_C \widetilde{g}_{g2} + f''_C f''_{g2} + \vartheta'_C \vartheta'_{g2} + + \widetilde{g}'_C \widetilde{g}'_{g2}\right)$$

and $$b_{33} = -\left(f'_C f'_{g3} + \vartheta_C \vartheta_{g3} + \widetilde{g}_C \widetilde{g}_{g3} + f''_C f''_{g3} + \vartheta'_C \vartheta'_{g3} + + \widetilde{g}'_C \widetilde{g}'_{g3}\right)$$

Solving the equations (4.17) to (4.19) we have

$$\Delta g_1 = \frac{\det A_1}{\det A}$$

$$\Delta g_2 = \frac{\det A_2}{\det A}$$

and $$\Delta g_3 = \frac{\det A_3}{\det A}$$

where, $$\det A_1 = \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{22} & a_{22} & a_{23} \\ b_{33} & a_{32} & a_{33} \end{vmatrix} = b_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + b_{22}\left(a_{32}a_{13} - a_{12}a_{33}\right) + b_{33}\left(a_{12}a_{23} - a_{22}a_{13}\right)$$

$$\det A_2 = \begin{vmatrix} a_{11} & b_{11} & a_{13} \\ a_{21} & b_{22} & a_{23} \\ a_{31} & b_{33} & a_{33} \end{vmatrix} = b_{11}\left(a_{31}a_{23} - a_{21}a_{33}\right) + b_{22}\left(a_{11}a_{33} - a_{31}a_{13}\right) + b_{33}\left(a_{21}a_{13} - a_{11}a_{23}\right)$$

$$\det A_3 = \begin{vmatrix} a_{11} & a_{12} & b_{11} \\ a_{21} & a_{22} & b_{22} \\ a_{31} & a_{32} & b_{33} \end{vmatrix} = b_{11}\left(a_{21}a_{32} - a_{31}a_{22}\right) + b_{22}\left(a_{31}a_{12} - a_{11}a_{32}\right) + b_{33}\left(a_{11}a_{22} - a_{21}a_{12}\right)$$

and $$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{21}\left(a_{32}a_{13} - a_{12}a_{33}\right) + a_{31}\left(a_{12}a_{23} - a_{22}a_{13}\right)$$

Then we obtain the missing (unspecified) values as

$$g_1 = g_1 + \Delta g_1$$

$$g_2 = g_2 + \Delta g_2$$

and $$g_3 = g_3 + \Delta g_3$$

Thus adopting the numerical technique aforementioned, the solution of the equation (3I.35) with boundary conditions (3I.36) are obtained together with sixth-order implicit Runge-Kutta initial value solver and determine the velocity, temperature and pressure functions as function of the co-ordinate $\eta$. In the process of integration the skin friction coefficient $f''(0)$ and the heat transfer rate $-\theta'(0)$ are also calculated out and we applied the method for different values of pertinent parameters. Based on the integration done with the aforementioned numerical technique, the results obtained are given in the next chapter with graphs and tables. For more details numerical calculation technique the corresponding FORTRAN program with subroutine is also given in the Appendix 'A' of chapter 5.

# Chapter-5

## Graphs and Tables

## Graphs

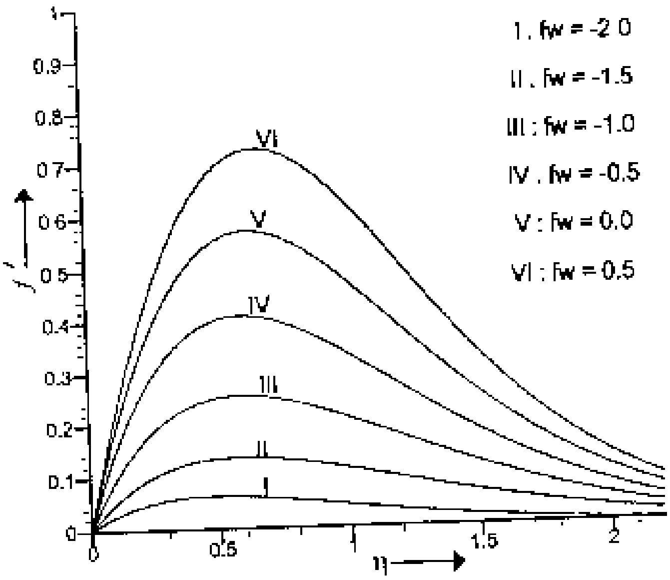


Figure-3: Velocity profiles for different values of $f_w$ at the periphery of the disc with $\beta = 0.33$ and Prandtl No Pr = 0.72.

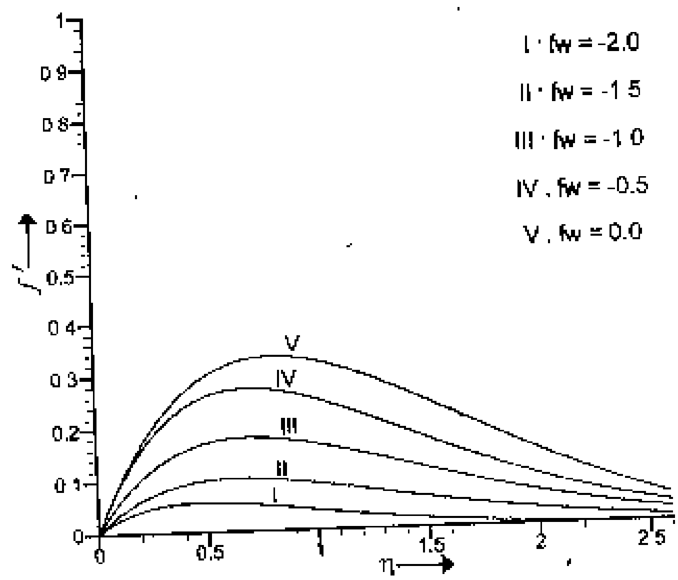


Figure-4: Velocity profiles for different values of $f_w$ at the periphery of the disc with $\beta = 1.0$ and Prandtl No. Pr = 0.72.

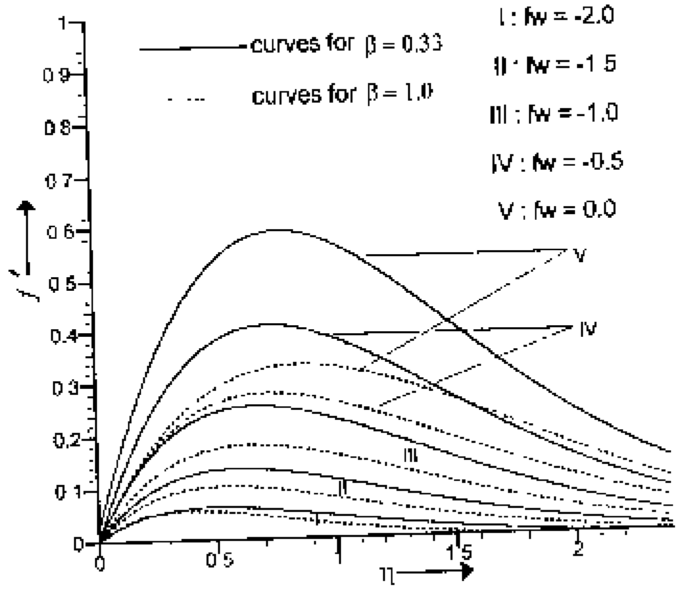


Figure-5: Velocity profiles for different values of $f_w$ at the periphery of the disc with $\beta = 0.33$ & $1.0$ and Prandtl No. Pr = 0.72.

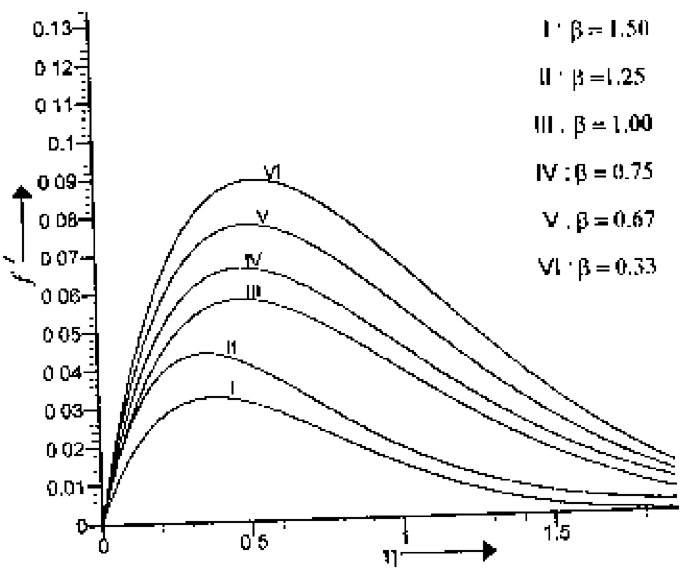


Figure-6: Velocity profiles for different values of $\beta$ at the periphery of the disc with $f_w = -2.0$ and Prandtl No. Pr = 0.72.

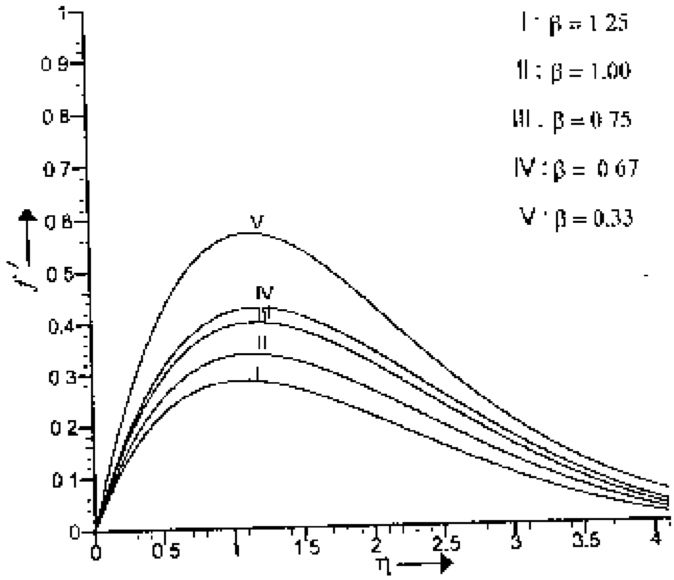


Figure-7: Velocity profiles for different values of $\beta$ at the periphery of the disc with $f_w = 0.0$ and Prandtl No. Pr = 0.72.

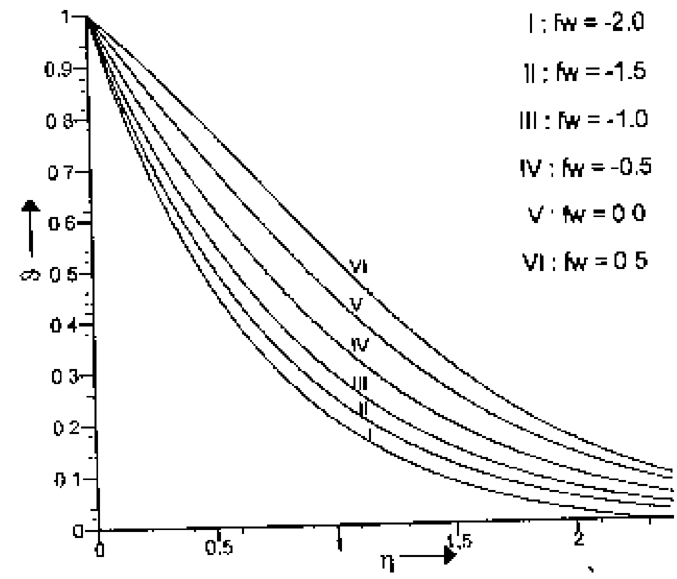


Figure-8: Temperature profiles for different values of $f_w$ at the periphery of the disc with $\beta = 0.33$ and Prandtl No. Pr = 0.72.

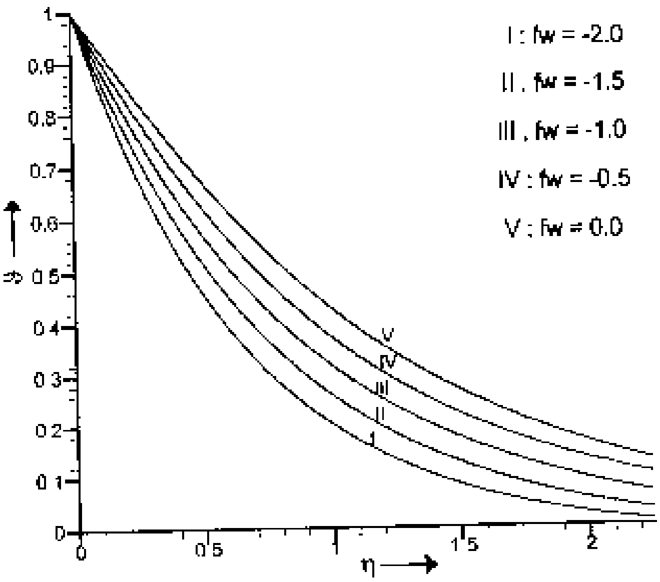


Figure-9: Temperature profiles for different values of $f_w$ at the periphery of the disc with $\beta = 1.0$ and Prandtl No. Pr = 0.72.

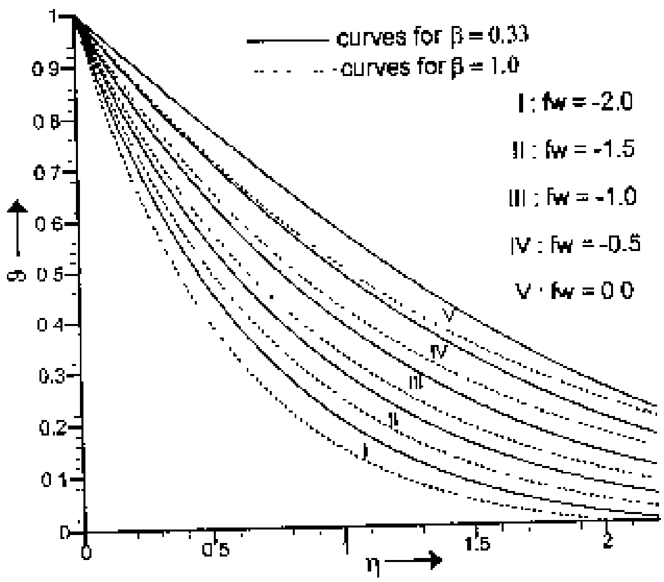


Figure-10: Temperature profiles for different values of $f_w$ at the periphery of the disc with $\beta = 0.33$ & 1.0 and Prandtl No. Pr = 0.72.

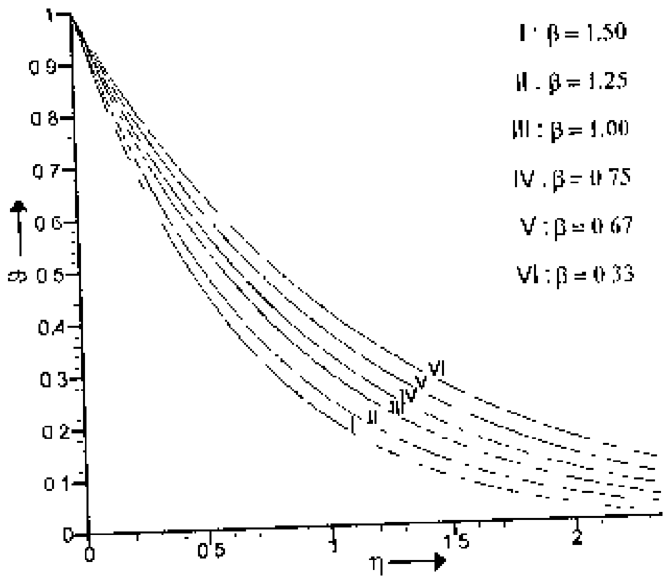


Figure-11: Temperature profiles for different values of $\beta$ at the periphery of the disc with $f_w = -2.0$ and Prandtl No. Pr = 0.72.

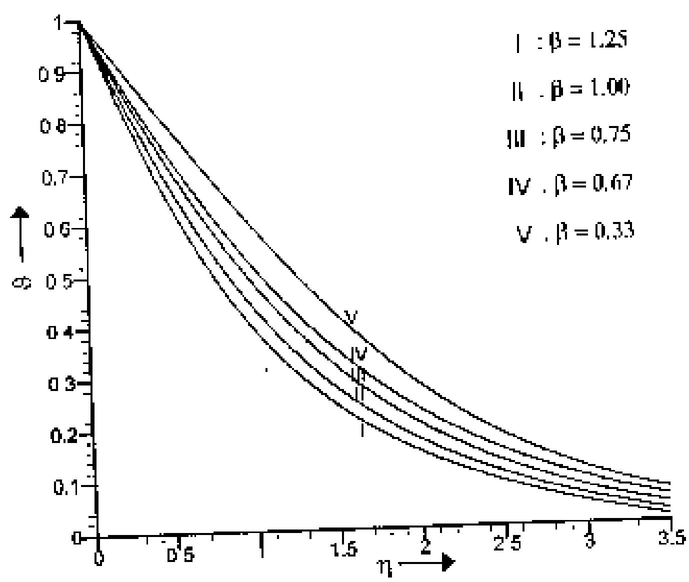


Figure-12: Temperature profiles for different values of $\beta$ at the periphery of the disc with $f_w = 0.0$ and Prandtl No. Pr = 0.72.

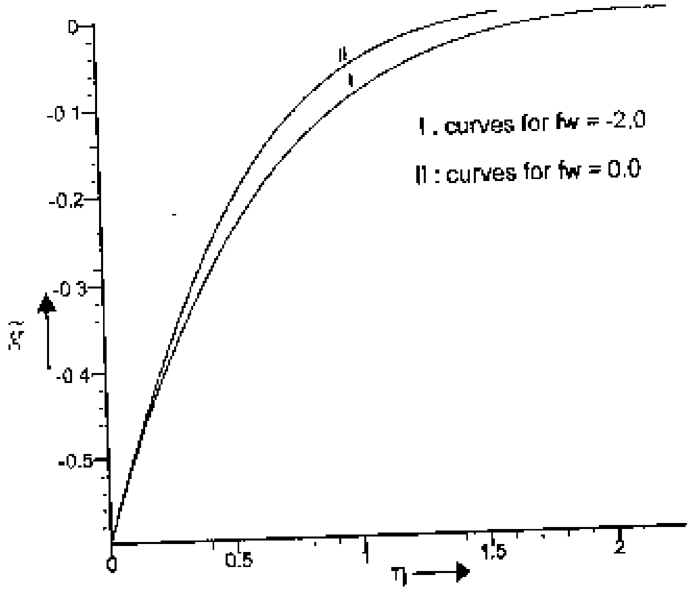


Figure-13: The non-dimensional pressure distributions at the periphery of the disc for $f_w = -2.0$ & 0.0 with $\beta = 0.33$ and Prandtl No. Pr = 0.72.

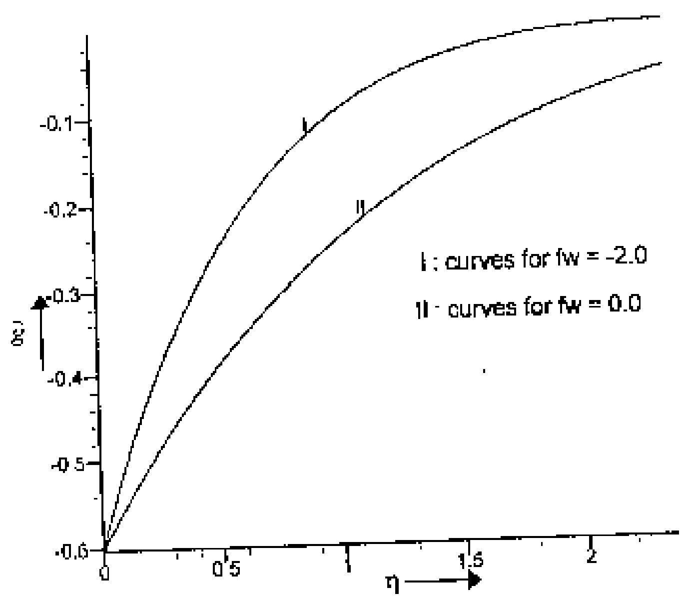


Figure-14: The non-dimensional pressure distributions at the periphery of the disc for $f_w = -2.0$ & 0.0 with $\beta = 1.0$ and Prandtl No. Pr = 0.72.

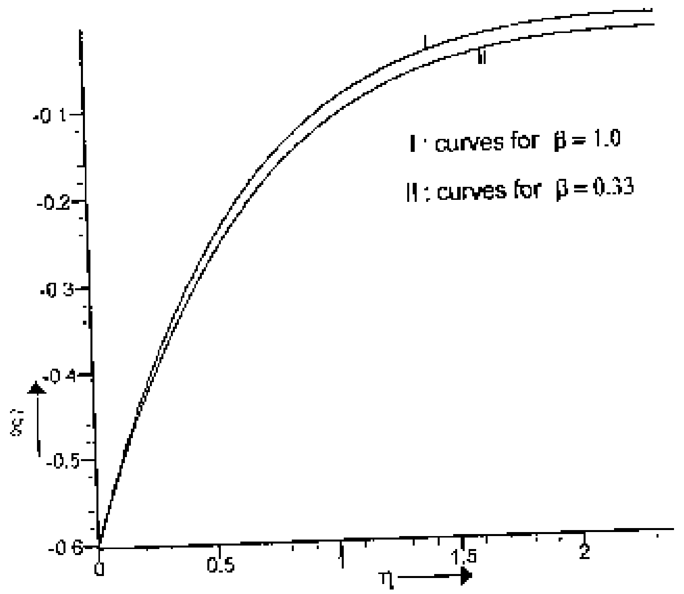


Figure-15: The non-dimensional pressure distributions at the periphery of the disc for $\beta$-variation with $f_w = -2.0$ and Prandtl No. Pr = 0.72.

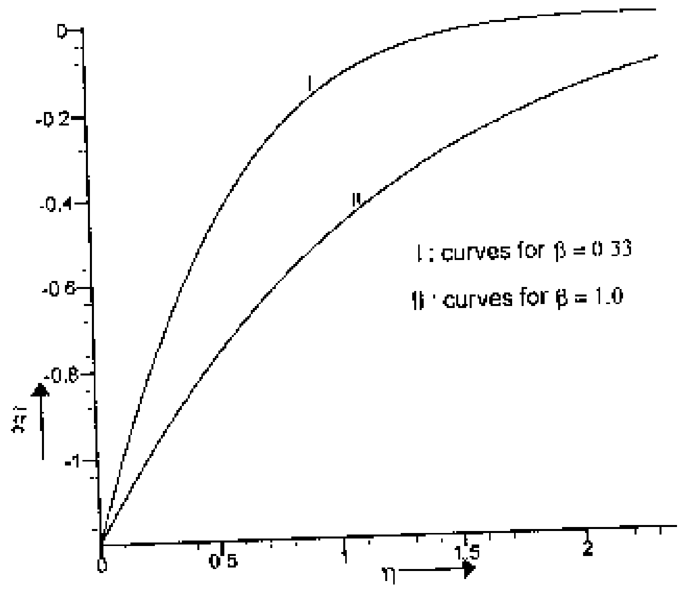


Figure-16: The non-dimensional pressure distributions at the periphery of the disc for $\beta$ -variation with $f_w$ =0.0 and Prandtl No. Pr = 0.72.

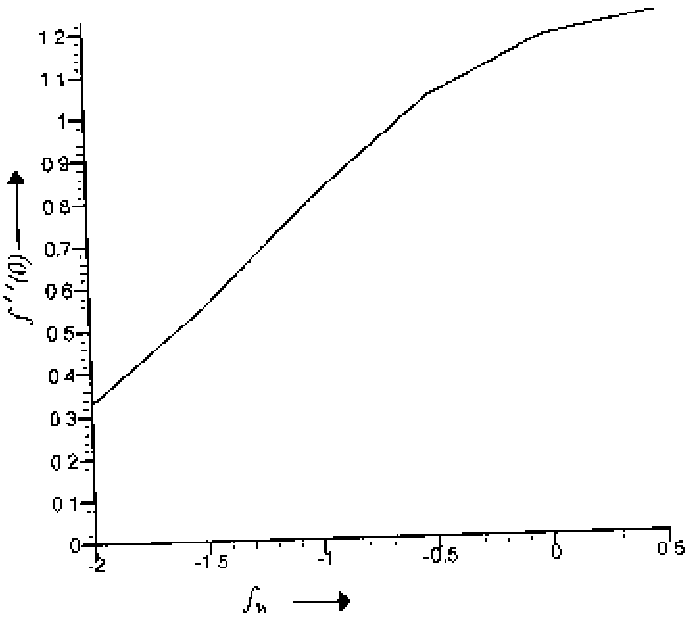


Figure-17: Skin friction factor $(= f''(0))$ for $f_w$ - variation at the periphery of the disc with $\beta = 0.33$ and Prandtl No. Pr = 0.72.

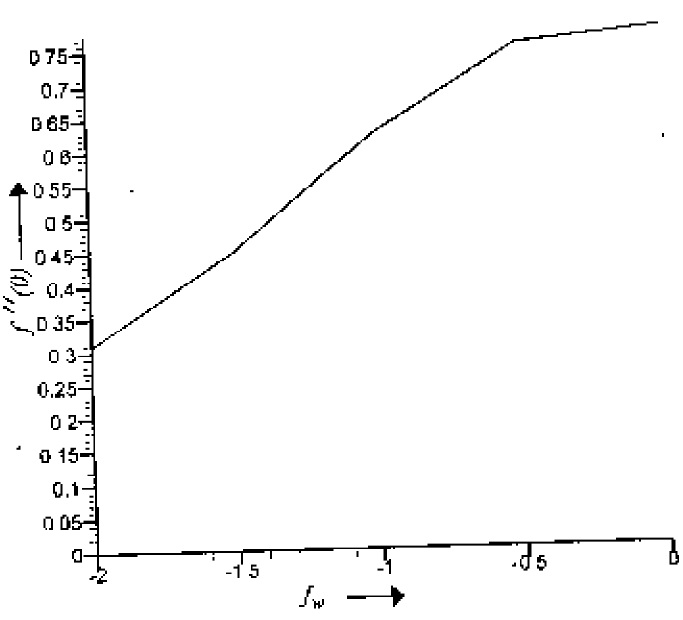


Figure-18: Skin friction factor $(= f''(0))$ for $f_w$ - variation at the periphery of the disc with $\beta = 1.0$ and Prandtl No. Pr = 0.72.

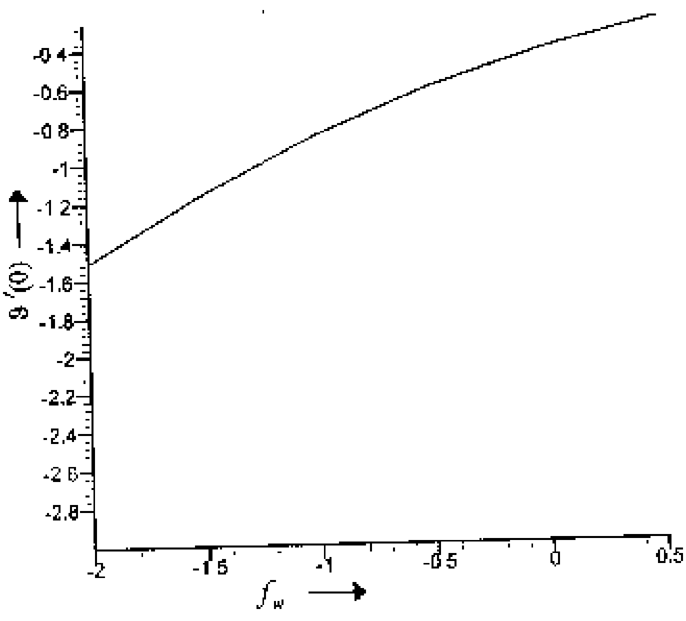


Figure-19: Heat transfer co-efficient ($= -\vartheta'(0)$) for $f_w$ - variation at the periphery of the disc with $\beta = 0.33$ and Prandtl No. Pr = 0.72.

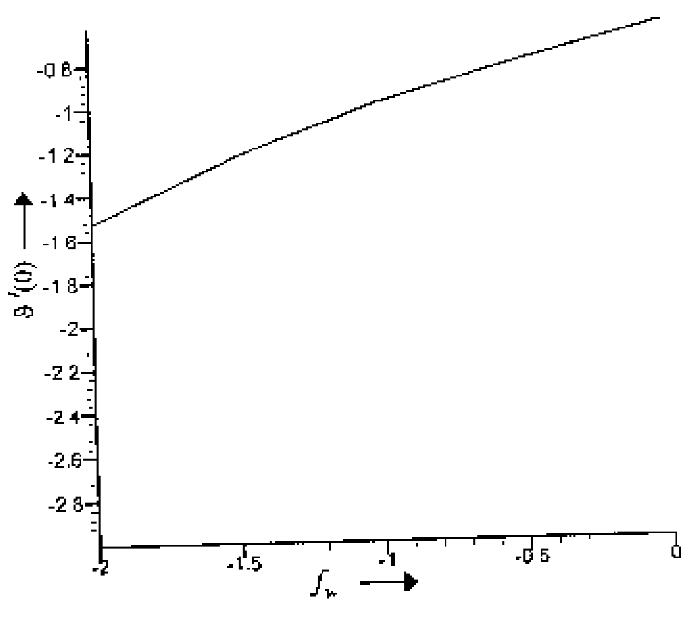


Figure-20: Heat transfer co-efficient ($= -\vartheta'(0)$) for $f_w$ - variation at the periphery of the disc with $\beta = 1.0$ and Prandtl No. Pr = 0.72.

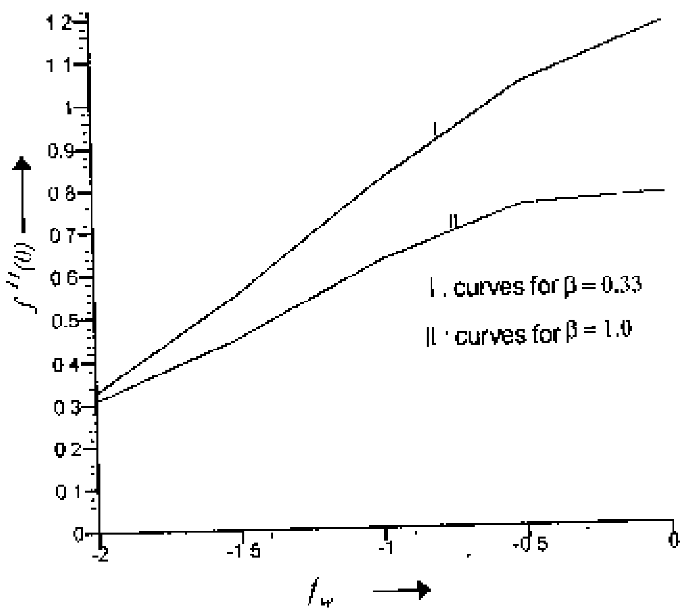


Figure-21: Skin friction factors $(= f''(0))$ for $f_w$ - variation at the periphery of the disc with $\beta = 0.33$ & 1.0 and Prandtl No. Pr = 0.72.

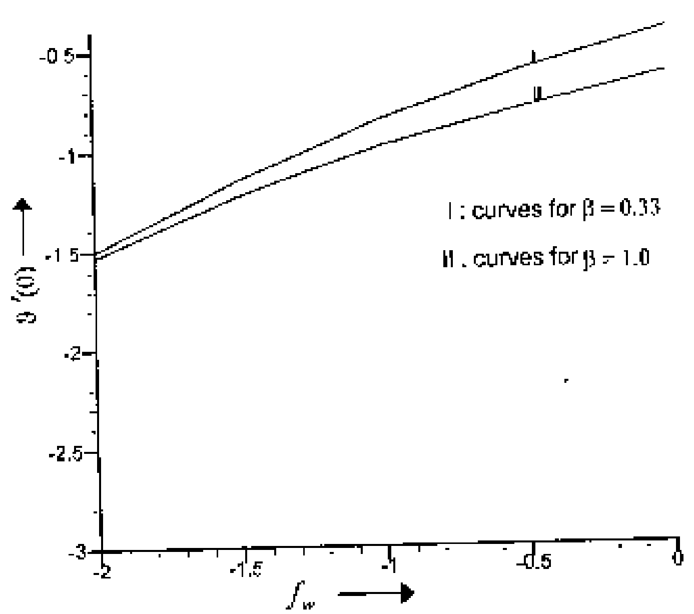


Figure-22: Heat transfer co-efficient $(= -\vartheta'(0))$ for $f_w$ - variation at the periphery of the disc with $\beta = 0.33$ & 1.0 and Prandtl No. Pr = 0.72.

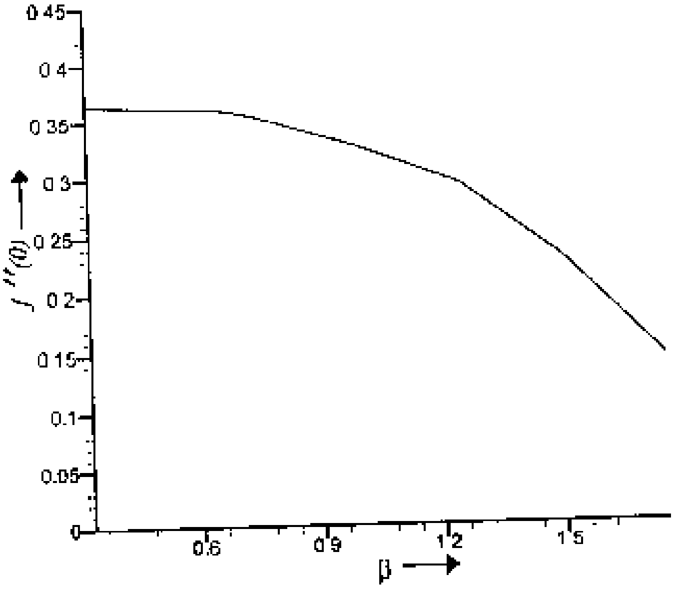


Figure-23: Skin friction factor (= $f''(0)$) for $\beta$ - variation at the periphery of the disc with $f_w = -2.0$ and Prandtl No. Pr = 0.72.

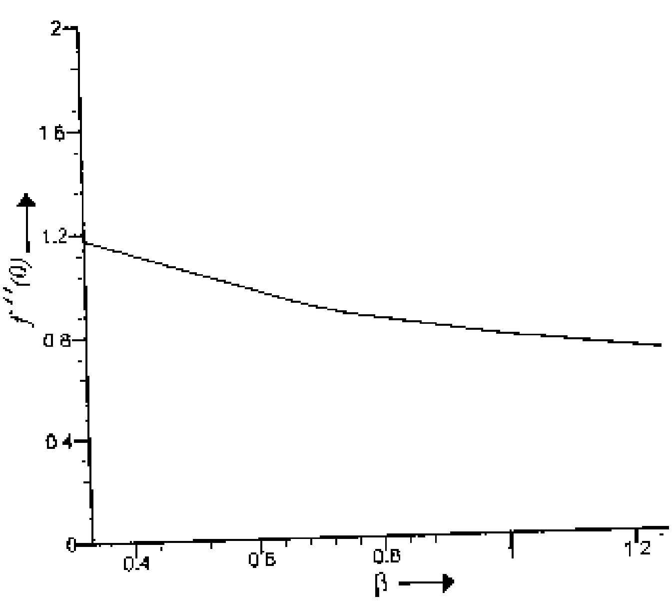


Figure-24: Skin friction factor (= $f''(0)$) for $\beta$ - variation at the periphery of the disc with $f_w = 0.0$ and Prandtl No. Pr = 0.72.

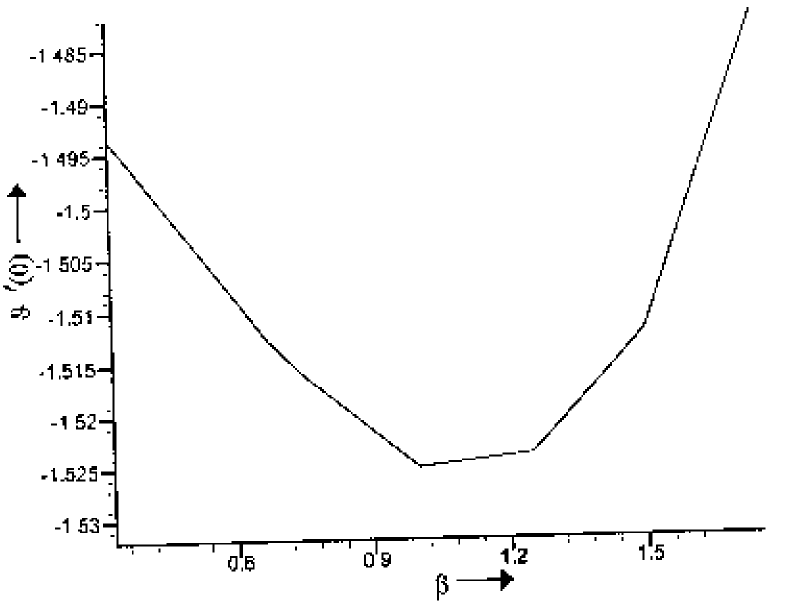


Figure-25: Heat transfer co-efficient $(= -\vartheta'(0))$ for $\beta$ - variation at the periphery of the disc with $f_w = -2.0$ and Prandtl No. Pr = 0.72.

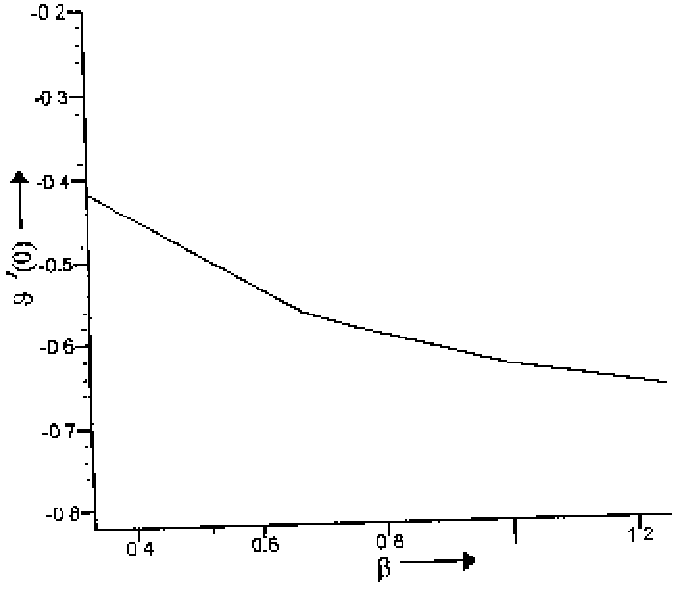


Figure-26: Heat transfer co-efficient $(= -\vartheta'(0))$ for $\beta$ - variation at the periphery of the disc with $f_w = 0.0$ and Prandtl No. Pr = 0.72.

## Tables

| $f_w$ | $f''(0)$ | $-\vartheta'(0)$ | $\beta$ |
|---|---|---|---|
| -2.00 | 0.36397 | 1.49356 | 0.33 |
| -1.50 | 0.55170 | 1.15770 | |
| -1.00 | 0.81350 | 0.86050 | |
| -0.50 | 1.04040 | 0.61530 | |
| 0.0 | 1.17535 | 0.41707 | |
| 0.50 | 1.22752 | 0.26399 | |

Table-1: The skin friction and heat transfer coefficients for different values of $f_w$ with $\beta = 0.33$

| $f_w$ | $f''(0)$ | $-\vartheta'(0)$ | $\beta$ |
|---|---|---|---|
| -2.00 | 0.32577 | 1.52513 | 1.00 |
| -1.50 | 0.44916 | 1.22735 | |
| -1.00 | 0.62381 | 0.98412 | |
| -0.50 | 0.75316 | 0.80067 | |
| 0.0 | 0.77969 | 0.63301 | |

Table-2: The skin friction and heat transfer coefficients for different values of $f_w$ with $\beta = 1.00$

| $\beta$ | $f''(0)$ | $-\theta'(0)$ | $f_w$ |
|---|---|---|---|
| 0.33 | 0.36979 | 1.49356 | |
| 0.67 | 0.35818 | 1.51269 | |
| 0 75 | 0.35230 | 1.51613 | |
| 1.00 | 0.32577 | 1.52513 | -2.00 |
| 1.25 | 0 29243 | 1.52385 | |
| 1.50 | 0.22831 | 1.51216 | |
| 1.75 | 0.14383 | 1.48168 | |

Table-3: The skin friction and heat transfer coefficients for different values of $\beta$ with $f_w = -2.0$

| $\beta$ | $f''(0)$ | $-\theta'(0)$ | $f_w$ |
|---|---|---|---|
| 0.33 | 1.17535 | 0.41707 | |
| 0.50 | 1.05185 | 0.50490 | 0.00 |
| 0.67 | 0.92110 | 0.56475 | |
| 0.75 | 0.87650 | 0.58394 | |
| 1.00 | 0.77969 | 0.63301 | |
| 1.25 | 0.70887 | 0.66254 | |

Table-4: The skin friction and heat transfer coefficients for different values of $\beta$ with $f_w = 0.0$

**Results and discussions**

On the basis of the numerical results of the set of equations (31.35) and (31.36) the dimensionless velocity and temperature profiles along with the pressure distributions are presented in Fig. 3 to Fig. 16, whereas the skin friction factors $(f''(0))$ and the heat transfer co-efficients $(-\vartheta'(0))$ are displayed in Fig. 17 to Fig. 26 for the fixed value of Prandtl number Pr = 0.72 (the typical value of air) with several selected values of established parameters $f_w$ and $\beta$ so far zeroth order boundary-layer is concerned. Figures show that profiles vary as usual with variations of parameters $f_w$ and $\beta$.

The displayed Fig. 3 shows that for isothermal temperature $\beta = 0.33$ the velocity profiles rise sharply with the increasing value of $f_w$ from negative to positive (i.e., with decreasing suction) and the rises are higher than those of $\beta = 1.0$ (i.e., for isothermal suction) [Fig. 4]. Consequently, sharp rise will increase the wall shear stresses.

For $\beta = 0.33$, $f_w = 0.0$ and $\mathrm{Pr} = 0.72$, the numerical results shown in Tab. 1 coincide with those displayed by Rotem and Claassen (1969a) and Zakerullah and Ackroyd (1979) for some particular values of the paramcters concerned.

For a fixed value of suction parameter $(f_w = -2.0)$ the velocity profiles [Fig. 6] exhibit the remarkable behaviors for $\beta$ – variation. Here, the velocity profiles rise usually with the decrease of the disc temperature. The same situation arises if no suction is applied [Fig. 7] but the sharp rise happens in this case.

Fig. 8 and Fig. 9 predict that the temperature profiles are higher near the surface of the disc and away from the disc they decrease asymptotically. Here we also infer that the temperature profiles decrease with the decreasing value of the suction parameter $f_w$ (i.e., with increasing suction). Thus, as $f_w$ increases from negative value to positive ones, the temperature gradient at the wall increases. This corresponds to the physical situation in which heat is transferred from the disc to the fluid.

Fig. 10 exhibits a comparison of the temperature profiles for isothermal disc's surfaces ($\beta = 0.33$) and isothermal suction ($\beta = 1.0$), for different values of $f_w$.

From Fig. 17 and Fig. 18 we observe that for constant wall temperature the skin friction gradually decreases with the increase of suction parameter but the skin friction more decreases with the increasing of wall temperature. The Fig. 19 and Fig. 20 concern that suctions increase the heat transfer rate highly. For suction the fluid at the ambient temperature being brought closer to the surface resulting in an increase in heat transfer. It is evident that the effects of suctions to suck away the warm fluids present on the wall and thus decrease the thermal boundary-layer thickness and thereby increase the heat transfer rate. It is thns confirmly predict that very small suction velocity plays a vital role on the effect of the skin friction and heat transfer.

From Fig. 23 and Fig. 24 it is observed that the skin friction also decreases with the increasing of wall temperature either suction is applied or not, but the rate of decrease with suction ($f_w = -2.0$) are more dominant than those of without suction ($f_w = 0.0$).

The heat transfer coefficients for $\beta$ – variation are shown in Fig. 25 and Fig. 26 with $f_w = -2.0$ & $0.0$. It is anticipated from the figures that if no suction is applied the heat transfer co-efficients increase with the increase of $\beta$. But a different behavior is observed if the suction is considered. Here, the heat transfer co-efficient increases with the increase of $\beta$ and the higher heat transfer occurs when $\beta = 1.0$ (isothermal suction).

In addition to this we find from the numerical solutions [Tab. 4] that for uniform heat flux (i.e., for $\beta = 0.5$) with Pr = 0.72, the results obtained coincide with those deduced by Pera and Gabhart (1973) in absence of suction parameter $f_w$.

Since the flow characteristics associated with heat transfer and skin friction co-efficients of the present problem are of practical interest, so the numerical results for $f''(0)$ and $-\vartheta'(0)$ are presented in tabular forms. Tab. 1 and Tab. 2 display the effects of skin friction and heat transfer co-efficients for the variation of $f_w$ with $\beta = 0.33$ (isothermal surface) and 1.0 (isothermal suction) for Pr = 0.72. Also Tab. 3 and Tab. 4 display the same for $\beta$ – variation with $f_w = -2.0$ & $0.0$. We observe that for isothermal suction, the skin friction and heat transfer co-efficients are less than those of isothermal surface [Fig. 21 and Fig. 22].

## Conclusion

It is desirable to evaluate the other correction terms like $\vartheta_i'(0)$ of equation (3I.44) for $i=1,2$ etc. But $\tilde{\xi}$-dependent terms like $J\left(\frac{f, f'}{\tilde{\xi}, \eta}\right)$ and $J\left(\frac{f, \vartheta}{\tilde{\xi}, \eta}\right)$ from the right-hand side of momentum and energy equations are ignored in the present study embodied with Boussinesq approximation. By the substitutions of the present similarity variable, in the $\tilde{\xi}$-dependent terms $f(\tilde{\xi}, \eta)$, $\vartheta(\tilde{\xi}, \eta)$ and $\tilde{g}(\tilde{\xi}, \eta)$ of the governing equations (3.1) to (3.4), the forms of the first order and second order perturbed equations like (3I.32) to (3I.34) would be affected, although zeroth order remains unchanged. So it needs further study to include more terms for equations (3I.37) to (3I.40) in the calculation of over all heat transfer and drug co-efficients numerically.

# Appendix 'A'

**Program**

```
c=======mainprogram=================================================
c     shooting method
      implicit real*8(a-h,o-z)
      common/p/fw,pr,bt
      common/v/ir,ix
      common/vv/g1,g2,g3
      open(unit=3,file='sss3.dat')
      open(unit=2,file='osss.dat')
      read(3,*)ir,ix,g1,g2,g3,fw,bt,pr
c     it=0
c11   it=it+1
      call drff0
      call comp1
c     if(it.gt.10)stop
c     go to 11
      stop
      end
c=======drff0=======================================================
      subroutine drff0
      implicit real*8(a-h,o-z)
      common/p/fw,pr,bt
      common/v/ir,ix
      common/vv/g1,g2,g3
      dimension xd(60),xk(3,60),f(60),x(60)
      external derf0
      n=24
      itmax=8
      kk=0
 555  kk=kk+1
      if(kk.eq.10)stop
      write(*,*)'ir =',ir
      do 101 iter=1,ir
       t=0.0
       do k=1,n
        x(k)=0.0
       enddo
```

```
      x(3)=g1
      x(4)=1.
      x(5)=g2
      x(6)=g3
      x(9)=1.
      x(17)=1.
      x(24)=1.
      h=.01
c     h=dsinh(1./aa)
      do i=1,ir
      call rksys(derf0,t,h,x,xd,xk,f,n)
       do k=1,n
        x(k)=xd(k)
       enddo
c       h=dsinh(float(i)/aa)-dsinh(float(i-1)/aa)
        t=t+h
      enddo
       a11=x(8)**2+x(9)**2+x(10)**2+x(11)**2+x(12)**2
       a12=x(8)*x(14)+x(9)*x(15)+x(10)*x(16)+x(11)*x(17)+x(12)*x(18)
       a13=x(8)*x(20)+x(9)*x(21)+x(10)*x(22)+x(11)*x(23)+x(12)*x(24)
       a21=a12
       a22=x(14)**2+x(15)**2+x(16)**2+x(17)**2+x(18)**2
       a23=x(14)*x(20)+x(15)*x(21)+x(16)*x(22)+x(17)*x(23)+x(18)*x(24)
       a31=a13
       a32=a23
       a33=x(20)**2+x(21)**2+x(22)**2+x(23)**2+x(24)**2
       b1=-(x(2)*x(8)+x(3)*x(9)+x(4)*x(10)+x(5)*x(11)+x(6)*x(12))
       b2=-(x(2)*x(14)+x(3)*x(15)+x(4)*x(16)+x(5)*x(17)+x(6)*x(18))
       b3=-(x(2)*x(20)+x(3)*x(21)+x(4)*x(22)+x(5)*x(23)+x(6)*x(24))
       err=x(2)**2+x(4)**2+x(6)**2
       write(*,39)err
       write(*,39)g1,g2,g3
       if(err.le. 0.00001)go to 22
       del1=b1*(a22*a33-a32*a23)-b2*(a12*a33-a32*a13)
     1         +b3*(a12*a23-a22*a13)
       del2=-b1*(a21*a33-a31*a23)+b2*(a11*a33-a31*a13)
     1         -b3*(a11*a23-a21*a13)
       del3 =b1*(a21*a32-a31*a22)-b2*(a11*a32-a31*a12)
     1         +b3*(a11*a22-a21*a12)
       delA=a11*(a22*a33-a32*a23)-a21*(a12*a33-a32*a13)
```

```
1        +a31*(a12*a23-a22*a13)
     dg1=del1/delA
     dg2=del2/delA
     dg3=del3/delA
     g1=g1+dg1
     g2=g2+dg2
     g3=g3+dg3
     if(iter.ge.itmax) then
     ir=ir+ix
     go to 555
     endif
101   continue
22    write(*,39)'g1=',g1,g2,g3
      write(2,39)g1,g2,g3
39    format(2x,3f9.5)
      return
      end
c========compl=========================================
      subroutine comp1
      implicit real*8(a-h,o-z)
      common/p/fw,pr,bt
      common/v/ir,ix
      common/vv/g1,g2,g3
      dimension xd(60),xk(3,60),f(60),x(60)
      external derf0
      n=6
c     itmax=8
c     kk=0
c 555   kk=kk+1
c      if(kk.eq.10)stop
c      write(*,*)'ir =',ir
c      do 101 iter=1,ir
       t=0.0
       do k=1,n
        x(k)=0.0
       enddo
       x(3)=g1
       x(4)=1.
       x(5)=g2
       x(6)=g3
```

```
      write(*,39)t,x(2),x(3),x(4),x(5),x(6),fw
      write(2,39)t,x(2),x(3),x(4),x(5),x(6),fw
      h=.01
c      h=dsinh(1./aa)
      do i=1,ir
      call rksys(derf0,t,h,x,xd,xk,f,n)
       do k=1,n
        x(k)=xd(k)
       enddo
c      h=dsinh(float(i)/aa)-dsinh(float(i-1)/aa)
      t=t+h
      write(*,39)t,x(3),x(5),x(6)
 22   write(2,39)t,x(2),x(3),x(4),x(5),x(6),fw
      enddo
 39   format(f5.3,2x,6f9.4)
      return
      end
c========derf0==========================================================
      subroutine derf0(x,t,f,n)
      implicit real*8(a-h,o-z)
      common/p/fw,pr,bt
      dimension x(n),f(n)
       p1=fw
       p2=(1.-bt)/2.*bt
       p3=3.*bt-1.
       f(1)=x(2)
       f(2)=x(3)
       f(3)=-(x(1)*x(3)-p1*x(3)-bt*x(2)**2+p2*t*x(4)-x(6))
       f(4)=x(5)
       f(5)=-pr*(x(1)*x(5)-p1*x(5)-p3*x(2)*x(4))
       f(6)=x(4)
       do j=1,3
         k=6*j
        f(k+1)=x(k+2)
        f(k+2)=x(k+3)
        f(k+3)=-(x(1)*x(k+3)+x(3)*x(k+1)-p1*x(k+3)-bt*2*x(k+2)
     1          +p2*t*x(k+4)-x(k+6))
        f(k+4)=x(k+5)
        f(k+5)=-(pr*(x(1)*x(k+5)+x(k+1)*x(5)-p1*x(k+5)-p3*(x(2)*x(k+4)
     1          +x(k+2)*x(4))))
```

```
      f(k+6)=x(k+4)
      enddo
      return
      end
c=====================================================================
c     Implicit R-K Sixth order method
c=====================================================================
      subroutine rksys(derivs,t,h,x,xd,xk,f,n)
      implicit real*8(a-h,o-z)
      dimension x(n),xd(n),xk(4,n),f(n)
      sqt=sqrt(15.0)
      a1=(5.-sqt)/10.0
      a2=1./2.
      a3=(5.+sqt)/10.0
      b1=5./36.
      b2=(10.-3.*sqt)/45.
      b3=(25.-6.*sqt)/180.
      c1=(10.+3.*sqt)/72.
      c2=2./9.
      c3=(10.-3.*sqt)/72.
      d1=(25.+6.*sqt)/180.
      d2=(10.+3.*sqt)/45.
      d3=5./36.
      call derivs(x,t,f,n)
      do i=1,n
       xk(1,i)=h*f(i)
       xk(2,i)=h*f(i)
       xk(3,i)=h*f(i)
       xd(i)=x(i)+b1*xk(1,i)+b2*xk(2,i)+b3*xk(3,i)
      enddo
      call derivs(xd,t+a1*h,f,n)
      do i=1,n
       xk(1,i)=h*f(i)
       xd(i)=x(i)+c1*xk(1,i)+c2*xk(2,i)+c3*xk(3,i)
      enddo
      call derivs(xd,t+a2*h,f,n)
      do i=1,n
       xk(2,i)=h*f(i)
       xd(i)=x(i)+d1*xk(1,i)+d2*xk(2,i)+d3*xk(3,i)
      enddo
```

```
call derivs(xd,t+a3*h,f,n)
do i=1,n
  xk(3,i)=h*f(i)
  xd(i)=x(i)+(5.*xk(1,i)+8.*xk(2,i)+5.*xk(3,i))/18.0
enddo
retnrn
end
```

# Appendix 'B'

**References**


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