Similarity Solutions of the Boundary Layer Equations for the Unsteady Mixed Convection Flows Over a Vertical Porous Flat Plate by the Method of Group Theory

A dissertation submitted in partial fulfillment of the requirements for the award of the degree

of
Master of Philosophy
in Mathematics

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October, 2005

The thesis Entitled "Similarity Solutions of the Boundary Layer Equations for the Unsteady Mixed Convection Flows over a Vertical Porous Flat Plate by the method of Group Theory" submitted by Mohammed Jashim Uddin, Student No: 040009006 F, Session: April-2000, Academic Year: 96-97-98, a full time student of M. Phil (Mathematics) has been accepted as satisfactory in the partial fulfillment for the degree of Master of Philosophy in MATHEMATICS on 30th October, 2005.

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Dedication

This work is dedicated to my Parents.

Contents

Results and Discussions

Abstract

The similarity solutions of unsteady mixed convection boundary layer equations over a flat porous vertical plate has been investigated by the method of one parameter continuous Group Theory. By repeated applications of this method which reduces the number of independent variables, firstly from three to two and finally from two to one, the governing boundary layer partial differential equations are transformed into a pair of nonlinear ordinary differential equations with appropriate boundary conditions. One set of the coupled nonlinear equations are solved numerically. The results thus obtained are compared with all other relevant works in literature.

The heat transfer and skin friction factors $\{q_w(0), \tau_w(0)\}$ are investigated and shown graphically for some values of controlling parameters by using Programming software FORTRAN 77 and visualizing software TECPLOT It would be shown that both the skin friction and heat transfer coefficient increases with suction and decreases with injection.

Declaration

None of the materials in this thesis is / will be submitted in support of any other degree or diploma at any other University or Institute other than publications.

Jasim 30.10.05

(Mohammed Jashim Uddin)
30th October, 2005.

Acknowledgement

All praises are due to almighty Allah for enabling me to carry out the work of this thesis.

I would like to express my indebtness and deep sense of gratitude to my supervisor Dr. Md. Zakerullah, Professor of Mathematics, Bangladesh University of Engineering and Technology, Dhaka-1000 for his continuous guidance, valuable suggestions, constant encouragement and help during this research work.

I wish to express my gratitude to Kh. Abdul Maleque, Assistant Professor of Mathematics, American International University-Bangladesh, for his kind help in solving the equations numerically.

I am indebted to all my teachers, specially all teachers and staffs of the department of Mathematics, BUET, Dhaka-1000 for their all sorts of cooperations.

Finally, it will be an injustice if I don't express my indebtness to my wife, brothers and sister for their sacrifice, cooperation and inspiration during the preparation of this thesis.

Jashim 30.10.05

(Mohammed Jashim Uddin)

Nomenclature

Dimensional variables

Symbol	Quantity	Unit
x'	Distance along the plate	m
y'	Distance perpendicular to the plate	m
t'	Time	s
u'	Velocity component in the boundary layer along the plate	$\frac{m}{s}$
v'	Velocity component in the boundary layer normal to the plate	$\frac{m}{s}$
u_e'	External velocity	$\frac{m}{s}$
U_F	Characteristic velocity generated by buoyancy effect	$\frac{m}{s}$
T'	Temperature	K
L	Characteristic length	m
g'	Acceleration due to gravity	$\frac{m}{s^2}$
q'	Heat transfer rate	$\frac{J}{m^2 s}$
c_p'	Specific heat at constant pressure	$\frac{J}{kg K}$
p'	Pressure	Pa
h'	Enthalpy	J
s'	Entropy	$\frac{J}{K}$

Dimensionless variables

Symbol	Quantity
Gr	Grashof number
f	Velocity function
Nu	Nusselt number
Pr	Prandtl number
Re	Reynolds number
u	Velocity component in the boundary layer along the plate
v	Velocity component in the boundary layer perpendicular to the plate
u_e	External velocity
t	Time
T	Temperature
x	Distance along the plate
y	Distance perpendicular to the plate

Greek symbols

Symbol	Quantity	Unit
α'	Thermal diffusivity	$\frac{m^2}{s}$
β'	Volumetric thermal expansion coefficient	$\frac{1}{K}$
μ'	Coefficient of dynamic viscosity	$\frac{kg}{m\,s}$
ν'	Coefficient of kinematic viscosity	$\frac{m^2}{s}$
ρ'	Density of the fluid	$\frac{kg}{m^3}$
τ'	Local shear stress	$\frac{Kg}{m\,s^2}$
κ'	Thermal conductivity	$\frac{J}{m\,K\,s}$
θ	Dimensionless temperature function	
ψ	Dimensionless stream function	
$\eta's$	Similarity variables	
$\alpha's$	Real numbers	
$\beta's$	Real numbers	

Subscripts

w	Condition at the wall
∞	Condition at the ambient medium

General Introduction

In this thesis an analysis is carried out to study similarity solutions of the boundary layer equations for unsteady mixed convection flows over a porous vertical flat plate by the method of one parameter continuous Group Theory.

Chapter one contains review of the literature of the problem.

Chapter two deals with the properties of viscous fluid, the basic equations, boundary layer equations and allied governing equations. A brief discussion of Group Theory is also presented in this chapter.

In Chapter three, we explore all possible Group invariant solutions including those already published in the literature using Group Theory. In this chapter it is found that present analysis covers all the existing possible six cases derived by **Yang (1960).**

Chapter four deals with the numerical solution of one of the transformed nonlinear ordinary differential equations by shooting method.

Finally, in Chapter five, we present the concluding remarks on overall investigations in the aforementioned chapters.

Chapter One

Review of literature on mixed convection and Group theory

Heat transfer estimates the rate at which heat is transferred across the system, where the boundaries are subjected to a specific temperature difference and temperature distribution of the system. It occurs on the walls of a room, on the outside of warm and cool pipes and between the surfaces and the fluid of all types of heat exchanger. A heat transfer phenomenon may occur in some fields of engineering.

There are three types of thermal energy transport: (i) conduction (ii) convection and (iii) radiation. In various types of studies related to heat transfer or thermal transport, considerable effort has been directed at the convection mode, in which heat transfer process takes place with the motion of the fluid. Convective heat transfer is divided broadly into two basic process, namely, (i) free convection and (ii) forced convection.

Free (Natural) convection

Natural convection flow arise when buoyancy forces due to density difference occur and these acts as a driving forces. In this case fluid motion caused entirely by buoyancy force that arises due to density changes resulting from the temperature variations or concentration differences of the flow. Buoyancy forces may act in different force fields, the gravitational field being most common, the centrifugal force field, the Coriolis force field and the electromagnetic force field are also found in nature. It can be both laminar and turbulent.

Natural convection flow velocities are generally much smaller than those associated with the forced convection and heat transfer rate is generally smaller. In brief, we can say that free convection flow results from the action of body forces on the fluid, that is, forces which are proportional to the mass or density of the fluid.

A heated body cooling in the ambient air produces free convection flow in the region surrounding it .Considerable effort has been given to the convective heat transfer because of its importance in technical applications in which the relative motion of the fluid provides an additional mechanism for the transfer of energy and of the material, the latter being a more important consideration in cases where mass transfer due to concentration difference occurs.

The mechanism of transfer of heat from one fluid element to another adjacent one is conduction. Thus a study of heat transfer involves (i) the mechanism of conduction and (ii) sometimes radiative process.

There are many practical situations in which these factors influence the flow phenomena. For example, in air craft propulsion system, there are components (such as gas turbines and helicopter ramjets) which rotate at high speeds. Associated with these relative speeds the large centrifugal forces similar to gravitational force are also proportional to the fluid density and hence can generate strong natural convection flows.

Free convection heat transfer is important only when there exist no external flow. It is expected from dimensional analysis that for large Reynolds number (i.e., for large flow velocity) and small Grashof number, the influence of free convection on heat transfer can be neglected. On the other hand, for large Grashof number and small Reynolds number the free convection would be a dominating factor.

Forced convection

If the fluid motion arises due to an external agent, such as the externally imposed flow of a fluid stream by a fan, a blower, the winds or the motion of the heated objects itself. The process is known as forced convection. It occurs in electronic devices which are not classified as heat exchangers, such as furnaces with artificial draft and regenerators.

Buoyancy causes variations in the velocity and temperature fields of forced convection flows leading to the variations in the Nusselt number and the wall shear stress or friction coefficient, parameters that are important for most engineering problems. For case of upward forced convection over a flat plate with surface heated to a temperature higher than the surrounding temperature, the buoyancy forces aid convective motion whereas if the surrounding temperature is greater than surface temperature, buoyancy forces oppose the flow.

In nature, we face some situations where forced and free convection act simultaneously in establishing the flow and temperature field near the heated or cooled plate body. That is, if the relative importance of the forced and free convection is of comparable order; the phenomena may be termed as mixed convection flow. The laminar boundary layer flow due to mixed convection has received considerable attention for both steady and unsteady situation in evaluating flow parameters for technical purposes.

In recent years, the transfer of heat to and from enclosed or partially enclosed regions by means of natural convection or by a combination of natural and forced convection has taken a new significance in the field of Aeronautics, Automatic power, Electronics and Chemical engineering and Electrical engineering.

Prandtl boundary layer

Prandtl (1904) first introduced a theory for the boundary layer. He showed that the fluid motion with the small friction may be divided into two regions (i) a very thin region close to the body over which fluid is flowing known as boundary layer where viscous effect dominate. (ii) the region outside the boundary layer where viscous effect is negligible. The flow outside the boundary layer is known as potential flow.

Transition from laminar to turbulent flow in the boundary layer on a flat plate occurs at $\mathrm{Re} \approx 5\times 10^5$, where $\mathrm{Re} = \dfrac{UL}{\nu}$.

A typical inertia term in the Naiver-Stokes equation is $\rho' u' \dfrac{\partial u'}{\partial x'}$ and a typical viscous term is $\mu' \dfrac{\partial^2 u'}{\partial y'^2}$. In the boundary layer they are comparable order and hence we have $\dfrac{\delta}{L} \sim \dfrac{1}{\sqrt{\mathrm{Re}}}$ (Schlichting 1978)

This type of argument is called **scaling analysis** and is a valuable tool in dealing with transport problem.

Porous plate

By porous plate we mean that the plate possesses very fine holes distributed uniformly over the entire surface of the plate through which fluid can flow freely.

Plate with suction and injection

The plate from which the fluid enters into the flow region is known as plate with injection and the plate from which the fluid leaves from the flow region is known as plate with suction.

Some times it is necessary to control the boundary layer flows by injecting or withdrawing fluid through a heated boundary layer wall. Since this can enhance heating (or cooling) of the system and can help delay the transition from laminar to turbulent flow. Boundary layer suction is used to control laminar and turbulent separation, by removing flow of low momentum. The technique is used in aircraft wings, some wind tunnels to remove boundary layer. Blowing (injection) a boundary layer on high temperature components can maintain a thin layer of colder flow that allows the system to function with very high fluid velocity.

Skin friction and separated flows over external surface penalize the performance and economics of the airplanes, ships, cars etc. Generally they can affect the performance of any manufacturing process that employs long piping runs or fluid flows that become unstable. It has been estimated that air craft fuel cost per mile could drop up to 40% if the flow around aircraft could be smoothed out. Hence our purpose is to find economic ways by which we can reduce skin friction and control boundary layer thickness. A variety of active and passive ways had been explored, including external plasma fields, variations, acuators, and flow additives. Among other approaches, one can consider flow blowing or suction in the boundary layer, injection of gas with different viscosity or different temperature.

Mechanics of nonlinear fluids presents a special challenge to the Engineers, Physicists and Mathematicians. All physical phenomena are, in general, governed by partial differential equations with appropriate initial and boundary conditions. It is often difficult and even impossible to find their solution using classical separation of variables, free parameter and dimensional analysis methods. Applied Mathematicians, Engineers and Scientists try to find the ways and means to reduce the partial differential equations into ordinary differential equations to serve their necessary purposes. A vast literature of similarity solutions has appeared in the arena of fluid mechanics, heat transfer, mass transfer, aerodynamics etc. Most existing solutions, in the technical arenas, are similarity solutions in the sense that the pertinent boundary layer equations with boundary conditions under suitable transformations are reduced to a set of ordinary differential equations in terms of similarity variables which are functions of original independent variables. They may be derived by dimensional arguments, by sophisticated group theoretic method, by method of free parameter or by separation of variables. Among them, the group theoretic method which includes the dimensional analysis as special case is the most systematic and sophisticated in generating similarity solutions.

The results obtained from similarity solutions are usable in various technical applications

Similarity solution first introduced by **Blasius (1908)** is one of the pioneer work to reduce the number of independent variables as well as dependent variables. Group theoretic method, first used by **Birkhoff (1955)**, which includes the dimensional analysis, is the most systematic and sophisticated in generating similarity solutions. For comprehensive review one is referred to the text by **Hansen (1964)**. This technique has been applied intensively by **Abd-el-Malek et al (1990, 1990 and 1991), Ames (1985)** and many others. Different types of perturbation techniques are used to solve the nonlinear partial differential equations primarily based on similarity solutions.

Ostrach (1953) analyzed the aspect of natural convection heat transfer. He studied flow between two parallel infinite plates orientated to the direction of generating body force. He **(1954)** latter worked on combined natural and forced convection laminar flows.

Yang (1960) studied analytically the unsteady laminar boundary layer equations for free convection on vertical plates and cylinders. He established some necessary and sufficient conditions for which similarity solutions are possible. He derived the similarity solutions of the governing equations when temperature variations are proportional to $x'^m, e^{mx'}, \frac{1}{ax'+bt'}, \frac{x'}{t'^2}, t'^m$ and $e^{mt'}$.

Most of the works on effect of suction and blowing on free convection boundary layer had been confined to the cases where prescribed wall temperature was considered. The power law variations of the plate temperature and transpiration velocity considered by **Eichhorn (1960)** are those for which similarity solutions exist.

Merkin (1969) considered the boundary layer flow over a semi-infinite vertical plate, heated to a constant temperature in a uniform free stream. He discussed two cases when the buoyancy forces aid and oppose the development of the boundary layer. In the former case, two series solutions were obtained, one of which was valid near the leading edge and other was valid asymptotically. In the latter case, series valid near the leading edge was obtained and it was extended by a numerical method to the point where boundary layer was shown to separate.

Ping Cheng (1977) investigated the combined free and forced convection boundary layer flow along inclined surfaces embedded in porous medium. It was found that when both wall temperature distribution of the plate and velocity parallel to the plate outside the boundary layer vary according to some power function of the distance, then similarity solutions exist.

Self–similar solutions have been studied by **Merkin and Ingham (1987)** for wall temperature prescribed as an inverse square root of the distance from the leading edge.
Unsteady laminar free convection flow of a viscous incompressible and electrically conducting fluid past an accelerated vertical infinite porous plate subject to a suction velocity proportional to $(\text{time})^{-\frac{1}{2}}$ was investigated by **Hossain and Mandal (1985)**.

Aldoss (1994) investigated mixed convection flow over non-isothermal horizontal surface in a porous medium. In his study he considered two conditions of surface heating (i) variable wall temperature (VWT) in the form $T'_w(x') - T'_\infty = a\,x'^n$ and (ii) a variable surface heat flux (VHF) in the form $q'_w = b\,x'^m$.

Soundalgeker (1972) analyzed viscous dissipation effects on unsteady free convective flow past an infinite, vertical porous plate with constant suction. He derived the solutions of the governing coupled nonlinear equations for velocity and temperature field.

Effect of blowing and suction on free convection boundary layer was studied by **Merkin (1972)**. He considered the cases of uniform suction and blowing with the plate held at constant temperature (T'_W) greater than the ambient temperature T'_∞.

Soundalgeker (1977) studied effects on mass transfer and suction on unsteady free convection flow neglecting Soret –Dufour effects on the energy equation. In his study the plate temperature was assumed to be oscillatory.

Pop and Takhar (1993) investigated the free convection flow over a non-isothermal two dimensional body of arbitrary geometric configuration. He discussed in detailed the effects of geometric shape parameter and Prandtl number on velocity and temperature field as well as heat transfer coefficient.

Conjugate free convection on a vertical surface had been discussed in some detailed by **Merkin and Pop (1996).**

Chaudhary, Merkin and Pop (1995) studied in detail, the similarity solutions for free convection boundary layer over a permeable wall in the saturated porous media. It was shown that the system depends on the two parameters m (exponent) and γ (dimensionless surface mass temperature rate).

Williams et al (1987) studied unsteady free convection flow over a vertical plate under the assumption of the variations of wall temperature with time and distance. They found possible semi-similar solutions for a variety of classes of wall temperature distribution.

Possible similarity solutions of three dimensional laminar incompressible boundary layer equations were investigated by **Hansen (1958)**. He exhibited different possible cases in tabular form for $\Delta T'$ variations in addition to those of exterior velocity components.

In order to apply the method of group theory in our problem a general idea of the group is discussed below:

Group: A group G is a set of elements a,b,c together with binary operation (mapping, transformations, rotation etc) $\circ$ defined on G satisfy the following properties:

(i) Closure property: A set is closed under the given operation. If a,b are the elements of G, then $a \circ b = c$ is also a unique element of G.

(ii) Associative property: The given operation is associative, that is, if $a,b,c \in G$ then $(a \circ b) \circ c = a \circ (b \circ c)$.

(iii) Identity exists: There exist a unique element I in G (called identity element) such that $a \circ I = I \circ a$, for all a in G.

(iv) Inverse exist: For every element a in G , there exist a unique element a^{-1} (called inverse of a) such that $a \circ a^{-1} = a^{-1} \circ a = I$.

A continuous r parameter group transformation $(r \geq 1)$ is sometimes known as simply a group.

Definition:

A set of functions $F_\delta(x_1, x_2, x_3, \cdots, x_n)$ is said to be "conformally invariant" under a group transformations $G: x^*{}_i \to x_i$, with a numerical parameter a, if

$$F_\delta(x_i) = f_\delta(x_i^*; a) F_\delta(x_i^*), i = 1,2,3,\cdots,n, \ \delta = 1,2,3,\cdots,m$$

where $F_\delta(x_i^*)$ are exactly the same functions of the x_i^* as $F_\delta(x_i)$ are of the x_i.

The functions $F_\delta(x_i)$ are called constant "conformally invariant" under the same group transformations if $f_\delta(x_i^*; a)$ are independent of the variables x_i^*.

The functions $F_\delta(x_i)$ are called constant "absolute invariant" under the same group transformations if $f_\delta(x_i^*; a) = 1$.

Literature concerning the group theory is abundant. Representative studies in group theory may be found in review papers by **Abd-el-Malek et al (1990, 91, 97), Morgan (1952), Hansen (1964), Zakerullah (2001) and Birkhoff (1955).**

Similarity solutions of unsteady mixed convection flow about a vertical plate were investigated by **Zubair (1990)** using one parameter continuous group theory method.

Mixed convection flow over a vertical surfaces/plate occurs in many industrial and technical applications which include nuclear reactors cooled by fans during emergency shut down, solar central receivers exposed to wind currents and heat exchangers placed in low velocity environment. **Merkin (1969)** investigated the mixed convection boundary layer flow on semi infinite vertical plate when buoyancy forces aid and oppose the development of boundary layer. Unsteady mixed convection boundary layer flows on vertical surface was studied by **Harris (1999)** and many others.

In this analysis, attention is directed to the situation where forced and free convection act simultaneously in establishing the flow and temperature fields.

Similarity solutions of unsteady mixed convection flow without suction was investigated by **Zubair (1990)** by repeated applications of Group Theory. Similar solutions of mixed convection flows have been studied by many authors, notable amongst them are those by **Wantanable (1991), Rapits, et al (1998), Ramachandran (1998)** and **Cheng (1977).**

However, there have been only a few studies dealing with unsteady mixed convection flow over a flat vertical porous plate by "Group theory method. **Ludlow D.K**

(2000) et al have analyzed new similarity solutions of unsteady incompressible boundary layer equations.

A review of literature shows that very little research has been reported on unsteady mixed convection flow about a porous vertical plate by the method of group theory.

Zakerullah (2001) has derived similarity solutions of some of the possible cases of unsteady mixed convection by Group Theory without suction. We will explore the similarity solutions of the remaining four cases with suction. From the present analysis it is shown that our solutions include some existing solutions as well as many new ones.

The numerical solutions of one of the representative transformed equations for different values of controlling parameters are obtained. Results are therefore compared with the known results in the literature.

The problem of unsteady mixed boundary layer flows has long been a major subject in fluid dynamics because of its importance from both theoretical and practical view point.

Many research papers have been published to date related to unsteady forced convection by many authors. Among them some are **Riley (1975, 1990), Telions (1979, 1981), Harris (2002)** etc.

Harris (1999) has performed an analysis of unsteady mixed convection boundary layer flow from a vertical flat plate embedded in porous medium. He made complete analysis at initial unsteady flow $(t'=0)$ and the steady state flow for large times $(t'\to\infty)$ and a series solution valid for small time obtained using semi-similar coordinates followed by **Smith (1967)**. Very recently, **Roslinda (2004) et al** investigated unsteady mixed convection boundary layer flow near the stagnation point on a vertical surface in a porous medium.

The present analysis is to incorporate the suction effect into the six possible similarity cases derived by **Yang (1960)** and for these similarity cases we introduce the idea of Group Theory to transform the governing equations into ordinary differential equations.

Finally, similarity requirements are exhibited for $\Delta T, u_e$ and v_w variations and we will solve the one of the transformed equations numerically to predict the flow characteristics for different numerical values of controlling parameters involved.

Chapter Two

Fundamental equations of the flow of fluids

Introduction

In this chapter we will discuss the basic properties of the fluid, Continuity equation, Naiver-Stokes equation and Energy equation. We will analyze the order of magnitude of the basic equations so that small order terms can be neglected.

Properties of fluid

There are two types of fluids (i) perfect fluid and (ii) real fluid.

Perfect fluid: It is frictionless and incompressible. In the motion of perfect fluid, two contacting layers experience no tangential forces but act normal forces only.

Real fluid: In the motion of real fluid, two contacting layers experience tangential forces and normal forces.

Viscous fluid possesses the following properties:

i) **Kinematics properties** (linear velocity, angular velocity, vorticity, acceleration and strain rate). These are the properties of the flow field itself rather than of the fluid.

ii) **Transport properties** (viscosity, thermal conductivity, specific heat at constant pressure, the Prandtl number).

iii) **Thermodynamic properties** (Pressure, density, temperature, enthalpy, entropy, bulk modulus, coefficient of thermal expansion).

iv) **Other miscellaneous properties**(Surface tension, vapor pressure, eddy diffusion coefficients, surface accommodation coefficients)

Some properties of (iv) are not the true properties but depend upon the flow conditions, surface conditions and contaminants in the fluid.

Kinematic properties

In fluid mechanics ones first focus is normally with the fluid velocity. If R' is the any property of the fluid and dx', dy', dz' and dt' represent arbitrary changes in x', y', z' and t' then we have

$$\frac{DR'}{Dt'} = \frac{\partial R'}{\partial t'} + (\mathbf{V}' \cdot \nabla')R' \tag{2.1}$$

where $\mathbf{V}' = (u', v', w')$ is the fluid velocity at any point (x', y', z') at any time t' and ∇' is the gradient operator . If R' is $\mathbf{V}'$ itself then

$$\frac{D\mathbf{V}'}{Dt'} = \frac{\partial \mathbf{V}'}{\partial t'} + (\mathbf{V}' \cdot \nabla')\mathbf{V}' = \text{acceleration} = \mathbf{i}\frac{Du'}{Dt'} + \mathbf{j}\frac{Dv'}{Dt'} + \mathbf{k}\frac{Dw'}{Dt'}.$$

Transport properties

The important transport properties of viscous fluid flows are the viscosity, the thermal conductivity, specific heat at constant pressure and the Prandtl number which is the combination of the first three properties.

Viscosity

It is an internal property of a fluid that causes resistance to flow. This property can be thought of as an internal friction. All fluid (liquids or gases) bears the property of viscosity in varying degrees. The dynamic viscosity μ' is defined by **Newton's** law of friction

$$F_S' = \mu' A \frac{du'}{dy'}.$$

Here F_S' is the shearing force or friction of fluids between two parallel layers of fluid, which have equal area A, separated by the distance dy' and one moves parallel to other with velocities u', $u' + du'$ respectively. $\frac{du'}{dy'}$ is the velocity gradient perpendicular to the direction of the flow of the two fluid layers.

If $A = 1$ and $\frac{du'}{dy'} = 1$ then, $F_S' = \mu'$, i.e., μ' is the shearing stress between the two layers of unit area.

Now shear stress per unit area, is

$$\tau' = \mu' \frac{du'}{dy'}. \tag{2.2}$$

(i) If $\tau' = 0$, then $\mu' = 0$, and (2.2) will represent an ideal fluid.

(ii) If $\frac{du'}{dy'} = 0$, then $\mu' \to \infty$ and (2.2) represent an elastic bodies.

(iii) A fluid for which μ' does not change with the rate of deformation (shear strain) is said to be Newtonian fluid.

(iv) If the viscosity varies with the rate of deformation, then it is said to be non-Newtonian fluid. Examples of non-Newtonian fluids are Bingham plastics, Pseudoplastics and Dipotants.

We shall restrict our study to Newtonian fluid.

The coefficient of viscosity of a Newtonian fluid is directly related to the molecular interactions and thus may be considered as a thermodynamic property in the macroscopic sense, varying with temperature and pressure i.e., $\mu' = \mu'(T', p')$. Normally, the viscosity decreases rapidly with temperature for liquid, increases with temperature for low pressure gases (dilute) and always increases with the pressure.

A common approximation for viscosity of dilute gases suggested by **Maxwell** is

$$\frac{\mu'}{\mu_0'} \approx \left(\frac{T'}{T_0'}\right)^n, \; n \text{ is of order } 0.7.$$

If $n = 1$ and surface is the reference condition, then $\mu' = \mu_w' \dfrac{T'}{T_w'}$.

For an isothermal wall, this reduces to $\mu' = (\text{constant}) \;\; T'$.

Experimental measurement of the viscosity of air is related with the temperature by the Southerland equation

$$\frac{\mu'}{\mu_0'} \approx \left(\frac{T'}{T_0'}\right)^{\frac{3}{2}} \frac{T_0' + S}{T' + S}.$$

Here S is an effective temperature called Southerland constant and μ_0', T_0' are reference values.

Thermal conductivity

When a fluid in static equilibrium is heated nonunifomally, heat may be transferred from the region of higher temperature to lower temperature. The basic transport mechanism is conduction which is governed by **Fourier's law** of heat conduction

$$\mathbf{q}' = -\kappa' \nabla' T'.$$

Here $\mathbf{q}'$ is the vector rate of heat flow per unit area (flux). The quantity κ' is called thermal (heat) conductivity; negative sign indicates that the heat flows in the direction of decreasing temperature.

Solid substances often show the anisotropy or directional sensitivity:

$$-\mathbf{q}' = (\kappa'_{x'} \frac{\partial T'}{\partial x'}, \kappa'_{y'} \frac{\partial T'}{\partial y'}, \kappa'_{z'} \frac{\partial T'}{\partial z'}).$$

But a fluid is isotropic, i.e., has no directional characteristics and thus κ' is a thermodynamic property. The value of κ' for a substance depend on the chemical composition, the physical state, temperature and pressure. It varies fluid to fluid. The variation of the thermal conductivity of the gases with temperature is the same as that of dynamic viscosity.

Specific heat at constant pressure ($c_{p'}$)

For air, it is almost constant for a wide range of temperatures.

The Prandtl number

The Prandtl number $\Pr = \dfrac{\mu' c_{p'}}{\kappa'}$ is essentially invariant with temperature.

Therefore, it is assumed that Pr for a gas is constant.

Thermodynamic properties

The most important thermodynamic properties are pressure (p'), density (ρ'), temperature(T'), entropy(s'), enthalpy (h') and internal energy (e'). Consider p' and T' as independent variables and other four variables depend on p' and T'.

First law of thermodynamics

Any thermodynamic system in equilibrium state possesses a state variable called internal energy (E'). Between any two equilibrium state, change in internal energy (dE') is equal to the difference of the heat transfer (dQ') into the system and the work done (dw') by the system.

Mathematically, $dE' = dQ' - dw'$. (2.3)

For a substance at rest with infinitesimal changes

$$d\,w' = -p'\,d\,v' \text{ (for constant volume)}$$
$$dQ' = T'\,d\,s'$$

Here ds' is the change of entropy.

Substituting into (2.3) and expressing the result for unit mass basis, we have

$$de' = T'ds' + \frac{p'}{\rho'^2}d\rho' \tag{2.4}$$

which is one of the forms of the first and second laws combined for infinitesimal process.

From the equation (2.4) we may write

$$e' = e'(s', \rho') = \text{function}(s', \rho').$$

Total derivative of e' is

$$de' = \frac{\partial e'}{\partial s'}ds' + \frac{\partial e'}{\partial \rho'}d\rho' . \tag{2.5}$$

Now comparing (2.4) and (2.5), we have $T' = \left(\frac{\partial e'}{\partial s'}\right)_{\rho'}$ and $p' = \rho'^2\left(\frac{\partial e'}{\partial \rho'}\right)_{s'}$.

From the definition of enthalpy we get,

$$h' = e' + \frac{p'}{\rho'} . \tag{2.6}$$

Thus a single chart of e' versus s' for lines of constant ρ', is sufficient to calculate all thermodynamic properties. Also from (2.4) and (2.5) we have

$$dh' = T'ds' + \frac{1}{\rho'}dp' . \tag{2.7}$$

From (2.7) we may write

$$h' = \text{function}(s', \rho')$$

$$\therefore \quad dh' = \frac{\partial h'}{\partial s'}d\,s' + \frac{\partial h'}{\partial p'}dp' . \tag{2.8}$$

From (2.7) and (2.8) we have

$$T' = \left(\frac{\partial h'}{\partial s'}\right)_{p'}, \quad \frac{1}{\rho'} = \left(\frac{\partial h'}{\partial p'}\right)_{s'}$$

$$e' = h' - \frac{p'}{\rho'} .$$

In this case, a chart of h' versus s for constant pressure p', will define a substance completely.

Coefficient of thermal expansion

For natural convection flow, the flow pattern is due to the buoyant forces caused by temperature differences and buoyant forces are proportional to the coefficient of thermal expansion β', defined as

$$\beta' = -\frac{1}{\rho'}\left(\frac{\partial \rho'}{\partial T'}\right)_{p'}.$$

For perfect gas, $\beta' = \frac{1}{T'}$. For liquid β' is usually smaller than $\frac{1}{T'}$ and may even be negative. For imperfect gas β' may be larger than $\frac{1}{T'}$ at high pressure.

The quantity β' is useful in estimating the dependence of enthalpy on pressure from the thermodynamic relation

$$dh' = c_{p'}\, dT' + \left(1 - \beta' T'\right)\frac{dp'}{\rho'}. \tag{2.9}$$

Fundamental equations of the fluid flows

The fundamental equations of flow of viscous compressible fluids are

(i) Equation of continuity(conservation of mass)

(ii) Equations of motions(conservation of momentum)

(iii) Equation of energy(conservation of energy)

(iv) Equations of species(conservation of species)

(v) Laws of chemical reaction.

For the sake of simplicity we can ignore diffussional chemical reaction, i.e., ignore (iv) and (v). We assume that fluid is uniform and of homogeneous composition.

The three basic equations, namely, continuity equation, Naiver-Stokes equation and energy equation based on conservation of mass, momentum and energy are

$$\frac{\partial \rho'}{\partial t'} + \nabla' \bullet (\rho' \mathbf{v}') = 0 \tag{2.10}$$

$$\rho' \frac{D\mathbf{v}'}{Dt'} = \rho' \mathbf{g}' + \nabla' \bullet \left\{ \mu' \left(\frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i} \right) + \delta'_{ij} \lambda' \nabla' \bullet \mathbf{v}' \right\} - \nabla' p' \tag{2.11}$$

$$\rho' \frac{Dh'}{Dt'} = \frac{Dp'}{Dt'} + \nabla' \bullet (\kappa' \nabla' T') + \left\{ \mu' \left(\frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i} \right) + \delta'_{ij} \lambda' \nabla' \bullet \mathbf{v}' \right\} \frac{\partial u'_i}{\partial x'_j} \tag{2.12}$$

where for a linear Newtonian fluid, the viscous stresses are

$\mu' \left(\frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i} \right) + \delta'_{ij} \lambda' \nabla' \bullet \mathbf{v}'$ (=heat generated due to frictional forces).

Here λ' is the coefficient of second viscosity and $\delta'_{i,j}$ is defined

$$\delta'_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$ is the Kronecker delta.

Assumptions

The above equations are based on the following assumptions.

(i) The fluid is a continuum

(ii) The particles are essentially in thermodynamic equilibrium

(iii) The only effective body forces are due to gravity

(iv) Heat conduction follows Fourier's law

(v) There are no internal heat sources(e.g., radiation, chemical reaction, Joule's heat)

(vi) Stress tensor is symmetric(the fluid has no local torque proportional to the

(vii) volume as would be possible in an electric field)

(viii) The fluid is isotropic(There is no locally preferred direction)

(ix) The fluid is Newtonian

(x) The stokes hypothesis is valid(i.e., $3\lambda' + 2\mu' = 0$).

The last term of right hand side of equation (2.12) is the viscous dissipation term (positive). It is the work done by the viscous stresses. In low speed flow this term will usually be negligible. It is important for gases at extremely low temperature.

There are seven variables involved in equations (2.10)-(2.12) of which three, variables, namely p', v' and T', are assumed to be primary variables. The remaining variables ρ', h', μ' and κ' are assumed to be functions of p' and T'.

The dependencies of the quantities ρ', h', μ' and κ' on pressure are generally very small and may be neglected.

For two dimensional incompressible flows, the above equations become

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0 \tag{2.13}$$

$$\rho'\left(\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}\right)=\rho' g'_{x'}-\frac{\partial p'}{\partial x'}+\mu'\left(\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}\right) \tag{2.14}$$

$$\rho'\left(\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}\right)=\rho' g'_{y'}-\frac{\partial p'}{\partial y'}+\mu'\left(\frac{\partial^2 v'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}\right) \tag{2.15}$$

$$\rho' c_{p'}\left(\frac{\partial T'}{\partial t'}+u'\frac{\partial T'}{\partial x'}+v'\frac{\partial T'}{\partial y'}\right)=\frac{\partial}{\partial x'}\left(\kappa'\frac{\partial T'}{\partial x'}\right)+\frac{\partial}{\partial y'}\left(\kappa'\frac{\partial T'}{\partial y'}\right) \tag{2.16}$$

$$+T'\beta'\left(\frac{\partial p'}{\partial t'}+u'\frac{\partial p'}{\partial x'}+v'\frac{\partial p'}{\partial y'}\right)+\mu'\left[2\left(\frac{\partial u'}{\partial x'}\right)^2+2\left(\frac{\partial v'}{\partial x'}\right)^2+\left\{\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right\}\right]^2.$$

Before proceeding to obtain the solutions of the equations (2.10) - (2.12) we will first find the dimensionless group upon which the solutions must depend. We start by introducing dimensionless quantities into the equation, referring all lengths to some characteristic length L, along the plate considered, velocities with reference to some characteristics velocity U and t' by $\frac{L}{U}$. The density will be made dimensionless with respect to ρ'_0, the pressure will be made dimensionless by $\rho'_0 U^2$ and the temperature by $\Delta T'$. The other transport properties of the fluid $\mu', \kappa', c_{p'}$ and the gravitational components $g'_{x'}$ and $g'_{y'}$ will be made dimensionless by $\mu'_0, \kappa'_0, c_{p'_0}$ and g respectively. Here the suffix 0 refers to some convenient constant reference conditions, undisturbed by the boundary layer. Hence we introduce the following dimensionless quantities.

$$x=\frac{x'}{L},\ y=\frac{y'}{L}, t=\frac{t'}{L}U,\ u=\frac{u'}{U},\ v=\frac{v'}{U},\ p=\frac{p'}{\rho'_0 U^2},$$

$$g_x=\frac{g'_{x'}}{g},\ g_y=\frac{g'_{y'}}{g},\ \rho=\frac{\rho'}{\rho'_0}, c_{p'}=\frac{c'_{p'}}{c_{p'_0}},\ T=\frac{T'-T'_\infty}{T'_w-T'_\infty}$$

$$\mu=\frac{\mu'}{\mu_0}.$$

Introducing the above dimensionless variables in equations (2.10)-(2.12) we obtain the following nondimensional equations.

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{v}) = 0 \tag{2.17}$$

$$\rho \frac{D\mathbf{v}}{Dt} = \frac{1}{Fr} \rho \,\mathbf{g} + \frac{1}{\mathrm{Re}} \nabla \bullet \left\{ \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \delta_{ij} \lambda \nabla \bullet \mathbf{v} \right\} - \nabla p \tag{2.18}$$

$$\rho c_p \frac{DT}{Dt} = Ec \frac{Dp}{Dt} + \frac{1}{\mathrm{Pr}\,\mathrm{Re}} \nabla \bullet \left(\kappa \nabla T \right) + \frac{Ec}{\mathrm{Re}} \varphi \tag{2.19}$$

Here $\varphi = \left\{ \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \delta_{ij} \lambda \nabla \bullet \mathbf{v} \right\} \frac{\partial u_i}{\partial x_j}$ is the viscous dissipation function and $\mathrm{Pr}, \mathrm{Re}, Ec, F_r$ are Prandtl ,Reynolds, Eckert and Froude numbers respectively.

We shall now discuss the physical importance of the nondimensional parameters upon which similarity solutions depend.

Important nondimensional parameters

Prandtl number

The Prandtl number is the ratio of the kinematic viscosity to the thermal diffusivity and is defined as

$$\Pr = \frac{\nu'}{\alpha'} = \frac{\mu' c_{p'}}{\kappa'}.$$

The value of ν shows the effect of viscosity of the fluid. For small value of ν, a thin region in the immediate neighbourhood of the surface will be affected by the viscosity, called thermal boundary layer. The quantity $\alpha' = \dfrac{\kappa'}{\rho' c_{p'}}$ represents thermal diffusivity due to heat conduction. For small value of α', thin regions will be affected by heat conduction which is known as thermal boundary layer.

Thus Prandtl number shows the relative importance of heat conduction and viscosity of the fluid.

It is a material property and thus varies fluid to fluid. Liquid metals have small Prandtl number (e.g. Pr=0.024 for Mercury), gases are slighty less than unity (e.g. Pr=0.70 for Helium), light liquids somewhat higher than unity and oils have very high Pr.

Reynolds number

Reynolds number is the most important parameter of the dynamics of viscous fluid. It represents the ratio of inertia to viscous force and is defined by

$$\mathrm{Re} = \frac{\rho' \dfrac{U^2}{L}}{\dfrac{\mu' U}{L^2}} = \frac{UL}{\nu'}.$$

It is, in fact, a parameter for viscosity, for if Re is small the viscous forces will be predominant and the effect of viscosity will be felt in the whole flow field.

On the other hand, if Re is large the inertia force will be predominant and in such a case effect of viscosity can be considered to be confined in a thin layer know as boundary layer adjacent to the surface. For large Reynolds number the flow ceases to be laminar and becomes turbulent. Reynolds showed that flow in a circular pipe becomes turbulent when Reynolds number of the flow exceeds critical value 2300.

That is $Re = (\frac{\bar{u}}{\nu} d) = 2300 (\text{pipe})$

where $\bar{u}$ is the mean fluid velocity and d is the diameter of the pipe.

Eckert number

For incompressible flow, it determines the relative rise in temperature of the fluid through adiabatic compression. In high speed flow, it is defined by

$$Ec = \frac{U^2}{c_{p'} \Delta T'} = \frac{2(\Delta T')_{ad}}{c_{p'} \Delta T'} = (\gamma - 1) M^2$$

where $M = \frac{U}{c}$ is the Mach number. Here c is the speed of sound.

The work of compression and that of friction become important when the characteristic velocity is comparable with or much greater than the sound or when the prescribed temperature difference is small compared to the absolute temperature of the free stream.

Froude number

It is defined by the ratio of inertial force $(\frac{\rho' U^2}{L})$ to gravity force $(\rho' g')$ and is given by

$$Fr = \frac{U^2}{g' L} = E \frac{\Delta T'}{T_0'} \frac{c_{p'} T_0'}{g' L}.$$

It is important when there is a free surface e.g., in an open channel problem.

For perfect gas

$$\frac{c_{p'} T_0'}{g' L} = \frac{1}{\gamma - 1} \frac{c^2}{g' L}.$$

There are however further conditions which must be imposed in order to have similarity solutions.

(i) $u(x,0,t)=0, v(x,0,t)=0$ for solid plate.(no slip condition)
and $u(x,0,t)=0, v(x,0,t)=v_w(x,t)\neq 0$ for porous plate.

(ii) The velocity at a large distance from the plate must be equal to the undisturbed fluid velocity i.e., $u(x,\infty,t)=u_e(x,t)$.

(iii) The temperature of the plate must be equal to the fluid temperature.
Hence $\theta_w=\theta(x,o,t)=1$. Here $\theta=\dfrac{T'-T'_\infty}{T'_w-T_\infty}$

(iv) The temperature at a large distance from the plate must be equal to the undisturbed fluid temperature i.e., $\theta(x,\infty,t)=0$.

Boussinesq approximation and governing equations

Let us confine our attention to two dimensional incompressible flows past a flat vertical porous plate and the flow configuration is shown in the following figures.

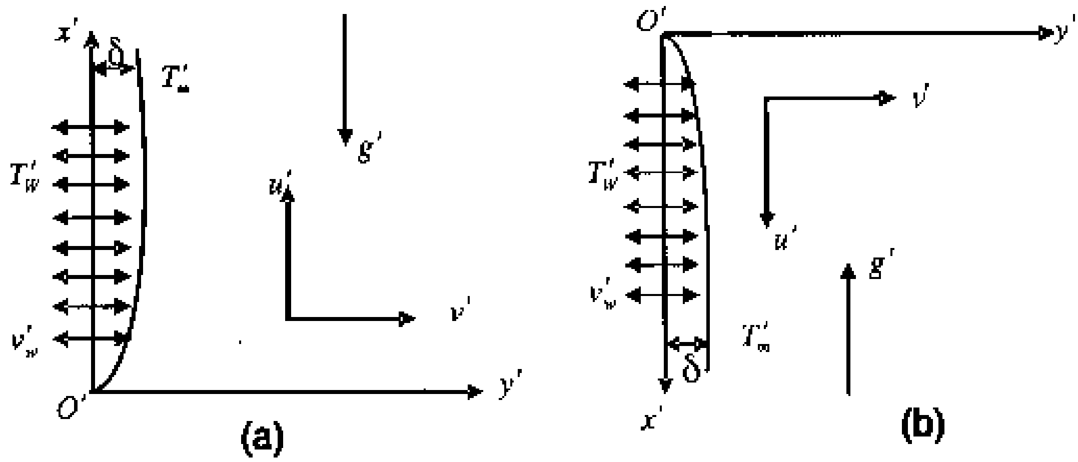

Fig 1 (a) Flow configuration for $T'_w > T'_\infty$
(b) Flow configuration for $T'_w < T'_\infty$

The relevant continuity, Naiver-Stokes and Energy equations in dimensional form are

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0 \tag{2.20}$$

$$\rho'\left(\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}\right)=\rho'\, g'_{x'}\,\beta'(T'-T'_{\infty})-\frac{\partial p'}{\partial x'}+\mu'\left(\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}\right) \tag{2.21}$$

$$\rho'\left(\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+v'\frac{\partial u'}{\partial y'}\right)=\rho'\, g'_{y'}\,\beta'(T'-T'_{\infty})-\frac{\partial p'}{\partial y'}+\mu'\left(\frac{\partial^2 v'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}\right) \tag{2.22}$$

$$\rho' c_{p'}\left(\frac{\partial T'}{\partial t'}+u'\frac{\partial T'}{\partial x'}+v'\frac{\partial T'}{\partial y'}\right)=k'\left(\frac{\partial^2 T'}{\partial x'^2}+\frac{\partial^2 T'}{\partial y'^2}\right) + \mu'\left[2\left(\frac{\partial u'}{\partial x'}\right)^2+2\left(\frac{\partial v'}{\partial y'}\right)^2+\left(\frac{\partial u'}{\partial y'}+\frac{\partial v'}{\partial x'}\right)^2\right] \tag{2.23}$$

The boundary layer equations

The boundary layer equations represent a significant simplification over the full Naiver–Stokes and Energy equations in the boundary layer region. This simplification is done by the order of magnitude analysis i.e., determining which term is small relative to the other terms.

Order of magnitude analysis

We take T', x' and u' as quantities of $o(1)$ and y', and v' of $o(\mathrm{Re}^{\frac{-1}{2}})$.

To make the order of magnitude analysis, we shall nondimensionalize the equations by using the following nondimensional variables .

$$x=\frac{x'}{L},\ y=\frac{y'}{L}\mathrm{Re}^{\frac{1}{2}},\ t=\frac{t'}{L}U,\ T=\frac{T'-T'_{\infty}}{T'_{w}-T'_{\infty}},\ u=\frac{u'}{U},\ v=\frac{v'}{U}\mathrm{Re}^{\frac{1}{2}},\ p=\frac{p'-p'_{0}}{\rho' U^2}$$

Here order of each nondimensional variable is one.

We shall estimate the order of magnitude of each term by taking Reynolds number very large. The order of magnitude of each term is shown in the beneath of each equation.

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$(o) \qquad 1 \qquad 1$$

u momentum equation

$$\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\frac{\Delta T\beta T}{Fr_x}-\frac{\partial p}{\partial x}+\frac{1}{R_e}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

$$(o) \qquad 1 \qquad 1 \qquad 1 \qquad 1 \qquad 1 \qquad 1 \qquad 1$$

v momentum equation

$$\frac{1}{\sqrt{\mathrm{Re}}}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\frac{\Delta T\beta T}{Fr_y}-\frac{\partial p}{\partial y}+\frac{1}{R_e^{\frac{3}{2}}}\frac{\partial^2 v}{\partial x^2}+\frac{1}{\sqrt{\mathrm{Re}}}\frac{\partial^2 v}{\partial y^2}$$

$$(o) \qquad 1 \qquad 1 \qquad 1 \qquad 1 \qquad 1 \qquad \delta^2 \qquad \delta$$

Energy equation

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\kappa}{\rho' c_{p'} UL}\frac{\partial^2 T}{\partial x^2}+\frac{1}{\mathrm{Pr}}\frac{\mathrm{Re}}{UL}\frac{\partial^2 T}{\partial y^2}+\frac{2Ec}{\mathrm{Re}}\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right\}+Ec\left(\frac{\partial u}{\partial y}\right)$$

$$(o) \qquad 1 \qquad 1 \qquad 1 \qquad \delta^2 \qquad \frac{1}{\delta^2} \qquad 1 \qquad \delta^2$$

$$+\frac{1}{\mathrm{Re}}Ec\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+\frac{1}{\mathrm{Re}}Ec\left(\frac{\partial v}{\partial x}\right)^2.$$

$$\delta^2 \qquad \delta^4$$

We retain the terms as $Lt\delta^2 \to 0$.

where $Ec = \dfrac{U^2}{c_{p'}(T'_w - T'_\infty)}$ is the Eckert number and $Fr_i = \dfrac{UL}{g_i L}$ is the Froude number in each direction.Here $i \rightarrow (x, y)$ directions.

There are many things to be noticed

(i) Continuity equation is affected by the consideration of Reynold number.
(ii) The pressure gradient is nearly zero, being affected only by a buoyant term which does not contribute to acceleration in y' direction.

$$\therefore \quad \frac{\partial p'}{\partial y'} = 0 \quad i.e, p' = p'(x', t').$$

We can say that pressure is constant in the direction normal to the boundary layer and may be assumed equal to that at the outer edge of the boundary layer where it is determined by the outer flow (potential flow).

We may therefore write,

$$-\frac{1}{\rho'}\frac{\partial p'}{\partial x'} = \frac{\partial u'_e}{\partial t'} + u'_e \frac{\partial u'_e}{\partial x'}.$$

We can now neglect some terms whose contribution is very small.

In our analysis of the boundary layer equations the following assumptions are taken into accounts.

(i) Reynolds number is very large.

(ii) Eckert number is very small (small velocity, large temperature difference).

(iii) Fluid properties are constant except in the density variations in the body force term (Boussinesq approximation).

Reverting to the dimensional system, we have the following governing boundary layer equations pertinent to our problem viz, the unsteady mixed convection boundary layer equations over a vertical porous flat plate.

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0 \tag{2.24}$$

$$\left(\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}\right)=g'_{x'}\,\beta'(T'-T'_{\infty})+\frac{\partial u'_e}{\partial t'}+u'_e\,\frac{\partial u'_e}{\partial x'}+v'\frac{\partial^2 u'}{\partial y'^2} \tag{2.25}$$

$$\rho' c_{p'}\left(\frac{\partial T'}{\partial t'}+u'\frac{\partial T'}{\partial x'}+v'\frac{\partial T'}{\partial y'}\right)=\frac{v'}{\Pr}\frac{\partial^2 T'}{\partial y'^2}. \tag{2.26}$$

The boundary conditions are

$$\left.\begin{array}{l} u'=0,\ v'=v'_w(x',t'),T'=T'_w \ \text{ at } y'=0 \\ u'=u'_e,\ T'=T'_{\infty} \quad \text{at} \quad y'\to\infty. \end{array}\right\} \tag{2.27}$$

Thus the boundary layer equations provide a significant simplification to the parent Naiver-Stokes equations, in two ways: by allowing invicid solutions to be used outside the boundary layer and by changing equations from elliptic to parabolic inside the boundary layer.

Limitations of the boundary layer equations are

(i) Reynolds number must be large , $\mathrm{Re}>1000$.

(ii) If the outer flow is decelerating $(\frac{\partial u_e}{\partial x'}<0,\frac{\partial p'}{\partial x'}>0)$ a point may be reached where wall shear stress approaches zero, the separation point.

(iii) If $\mathrm{Re}>10^6$, then the laminar solutions become unstable and the transition to turbulent occurs.

Chapter Three

Transformations leading to similarity solutions

Dimensional analysis of the governing equations

In order to have the nondimensional form of the governing equations, we use the following substitutions:

$$x = \frac{x'}{L}, \quad y = \frac{y'}{L}\mathrm{Re}^{\frac{1}{2}}, \quad t = \frac{t'}{L}U$$

$$\Delta T = T'_w(x', t') - T'_\infty, \quad T'(x', t') - T'_\infty = \Delta T\,\theta$$

$$u_e = \frac{u'_e}{U}, \; u = \frac{u'}{U}, \; v = \frac{v'}{U}\mathrm{Re}^{\frac{1}{2}}.$$

where L is the characteristic length, U is the characteristic velocity, $\mathrm{Re} = \frac{UL}{\nu}$ is the Reynolds number.

The nondimensional forms of the equations (2.24) - (2.26) are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{Du}{Dt} = -\frac{g_x\,\beta\,\Delta T L}{U^2}\theta + \frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{D\theta}{Dt} + \theta\left[\frac{\partial}{\partial t}(\ln \Delta T) + u\frac{\partial}{\partial x}(\ln \Delta T)\right] = \mathrm{Pr}^{-1}\frac{\partial^2\theta}{\partial y^2}.$$

The associated boundary conditions are transformed to

$$\left.\begin{array}{l} u = 0, \; v = v_w(x,t), \theta = 1 \text{ at } y = 0 \\ u = u_e, \; \theta = 0 \text{ at } \; y \to \infty. \end{array}\right\}$$

In the case of mixed convection, fluid motion solely depends on temperature difference. Simon **Ostrach (1953)** define a maximum velocity U_F^2 generated by the buoyancy effects for nodimensionalization within the boundary layer in terms of $(T'_w - T'_\infty)$ as

$$U_F^2 = -g_x \beta \, \Delta T \, L$$

where L is the characteristics length.

Thus the above equations are simplified to the following forms

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{Du}{Dt} = \frac{U_F^2}{U^2}\theta + \frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{D\theta}{Dt} + \theta\left[\frac{\partial}{\partial t}(\ln \Delta T) + u\frac{\partial}{\partial x}(\ln \Delta T)\right] = \mathrm{Pr}^{-1}\frac{\partial^2 \theta}{\partial y^2}$$

The associated boundary conditions are transformed to

$$\left.\begin{array}{ll} u = 0,\ v = v_w(x,t), \theta = 1 \text{ at } y = 0 \\ u = u_e,\ \theta = 0 \quad \text{at} \quad y \to \infty. \end{array}\right\}$$

The above set of partial differential equations are simultaneous nonlinear equations and to obtain their solution is extremely difficult. Hence we now proceed to reduce the equations into a pair of ordinary differential equations using one parameter continuous group transformation followed by **Morgan (1952).**

3.1 CASE-1

Unsteady mixed convection with surface temperature varying inversely as a linear combination of x and t, the free stream velocity is constant and the suction velocity varying inversely as a square root of the linear combination of x and t.

Stream function formulation:

An alternative form of the Prandtl boundary layer equations is derived by representing the velocity field in terms of a scalar field ψ, called stream function. The existence of this function is a mere consequence of the incompressibility of the fluid in two dimensional flows. Any solenoidal velocity field in two dimensions can be expressed as $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$. For general unsteady problems $\psi = \psi(x, y, t)$.

The boundary layer equations with the introduction of continuity equation are:

$$\phi_1 = \frac{\partial^2 \psi}{\partial y \partial t} + \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - \frac{U_F^2}{U^2}\theta - \frac{\partial u_e}{\partial t} - u_e\frac{\partial u_e}{\partial x} - \frac{\partial^3 \psi}{\partial y^3} = 0 \tag{3.1.1}$$

$$\phi_2 = \frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} + \theta\left[\frac{\partial}{\partial t}(\ln \Delta T) + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\ln \Delta T)\right] - \Pr^{-1}\frac{\partial^2 \theta}{\partial y^2} = 0 \tag{3.1.2}$$

The boundary conditions are

$$\left.\begin{aligned} &\frac{\partial \psi}{\partial y} = 0,\ -\frac{\partial \psi}{\partial x} = v_w,\ \theta = 1, \text{at } y = 0 \\ &\frac{\partial \psi}{\partial y} \to u_e,\ \theta = 0 \text{ at } y \to \infty. \end{aligned}\right\} \tag{3.1.3}$$

Finding the similarity solutions of the equations (3.1.1) and (3.1.2) ,are equivalent to determe the invariant solutions of these equations under a particular continuous one parameter group.

One of the simplest methods is to search for a transformation group from the elementary set of one parameter transformation, defined by the following group (G1)

$$\left.\begin{aligned}
x^* &= a^{\alpha_1}\, x \\
y^* &= a^{\alpha_2}\, y \\
t^* &= a^{\alpha_1}\, t \\
\psi^* &= a^{\alpha_3}\, \psi \\
\Delta T^* &= a^{\alpha_4}\, \Delta T \\
U_F^{*2} &= a^{\alpha_4}\, U_F^2 \\
\theta^* &= \theta \\
u_e^* &= u_0 = \text{constant}
\end{aligned}\right\} \tag{3.1.4}$$

Here $a(\neq 0)$ is the parameter of the group and $\alpha's$ are the arbitrary real numbers whose interrelationship will be determine by the subsequent analysis.

We now investigate the relationship between the exponents $\alpha's$ such that

$$\phi_j\left(x^*, y^*, t^*, u^*, v^*, \cdots, \frac{\partial^3\psi^*}{\partial y^{*3}}\right) = H_j\left(x, y, t, u, v, \cdots, \frac{\partial^3\psi}{\partial y^3}; a\right)\phi_j\left((x, y, t, u, v, \cdots, \frac{\partial^3\psi}{\partial y^3}\right) \tag{3.1.5}$$

for this is the requirement that the differential forms ϕ_1, ϕ_2 be conformally invariant under the transformation group (3.1.4).

Substituting the transformations (3.1.4.) in equations (3.1.1) and (3.1.2), we have

$$\begin{aligned}
\phi_1 &= \frac{\partial^2\psi^*}{\partial y^*\partial t^*} + \frac{\partial\psi^*}{\partial y^*}\frac{\partial^2\psi^*}{\partial x^*\partial y^*} - \frac{\partial\psi^*}{\partial x^*}\frac{\partial^2\psi^*}{\partial y^{*2}} - \frac{U_F^{*2}}{U^2}\theta^* - \frac{\partial u_e^*}{\partial t^*} - u_e^*\frac{\partial u_e^*}{\partial x^*} - \frac{\partial^3\psi^*}{\partial y^{*3}} \\
&= a^{\alpha_3-\alpha_1-\alpha_2}\frac{\partial^2\psi}{\partial y\partial t} + a^{2\alpha_3-\alpha_1-2\alpha_2}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - a^{2\alpha_3-\alpha_1-2\alpha_2}\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} \\
&\quad - a^{\alpha_4}\frac{U_F^2}{U^2}\theta - a^{-\alpha_1}\frac{\partial u_e}{\partial t} + a^{-\alpha_1}u_e\frac{\partial u_e}{\partial x} - a^{\alpha_3-3\alpha_2}\frac{\partial^3\psi}{\partial y^3}
\end{aligned} \tag{3.1.6}$$

and

$$\phi_2 = \frac{\partial \theta^*}{\partial t^*} + \frac{\partial \psi^*}{\partial y^*}\frac{\partial \theta^*}{\partial x^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial \theta^*}{\partial y^*} + \theta^*\left[\frac{\partial}{\partial t^*}(\ln \Delta T^*) + \frac{\partial \psi^*}{\partial y^*}\frac{\partial}{\partial x^*}(\ln \Delta T^*)\right] - \frac{1}{\Pr}\frac{\partial^2 \theta^*}{\partial y^{*2}}$$

$$= a^{-\alpha_1}\frac{\partial \theta}{\partial t} + a^{\alpha_3-\alpha_1-\alpha_2}\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - a^{\alpha_3-\alpha_1-\alpha_2}\frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} + a^{-\alpha_1}\theta\frac{\partial}{\partial t}(\ln \Delta T)$$

$$+ a^{\alpha_3-\alpha_1-\alpha_2}\theta\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\ln \Delta T) - a^{-2\alpha_2}\Pr^{-1}\frac{\partial^2 \theta}{\partial y^2}. \qquad (3.1.7)$$

Equating the various exponents of a in equations (3.1.6) and (3.1.7) leads to the following equations

$$\left.\begin{array}{l} \alpha_3 - \alpha_1 - \alpha_2 = 2\alpha_3 - 2\alpha_2 - \alpha_1 = -\alpha_1 = \alpha_3 - 3\alpha_2 = \alpha_4 \\ -\alpha_1 = \alpha_3 - \alpha_2 - \alpha_1 = -2\alpha_2 . \end{array}\right\} \qquad (3.1.8)$$

Solving equations (3.1.8), we have the following relationship between the exponents.

$$\alpha_1 = 2\alpha_2; \quad \alpha_3 = \alpha_2; \quad \alpha_4 = -2\alpha_2 = -\alpha_1; \alpha_2 (\neq 0) \text{ is arbitrary.}$$

It follows that ϕ_1 and ϕ_2 is conformally invariant under the following transformation group.

$$\left.\begin{array}{l} y^* = a^{\alpha_2}\, y = B\, y \\ x^* = a^{\frac{\alpha_1}{\alpha_2}}\, x = B^2 x \\ t^* = a^{\frac{\alpha_1}{\alpha_2}}\, t = B^2 t \\ \psi^* = a^{\frac{\alpha_3}{\alpha_2}}\, \psi = B\psi \\ U_F^{*2} = a^{\frac{\alpha_4}{\alpha_2}} U_F^2 = B^{-2} U_F^2 \\ \Delta T^* = a^{\frac{\alpha_4}{\alpha_2}}\, \Delta T = B^{-2}\Delta T \\ \theta^* = \theta \\ u_e^* = u_0 = \text{constant.} \end{array}\right\} \qquad (3.1.9)$$

Here $a^{\alpha_2} = B$.

We shall now show that ϕ_1, ϕ_2 can be expressed in terms of new independent variable η (similarity variable), F, G, I, U_p^2 and their derivatives with respect to η. The solution of the new system will be a particular set of invariant solutions of the original system in terms of x, y, u, v, etc. The variable η is to be an absolute invariant of the subgroup of the transformation of the independent variables.

In other words, η is to be a function such that

$$\eta(x^*, y^*, t^*) = \eta(x, y, t)$$

where
$$\left.\begin{aligned} x^* &= B^2 x \\ y^* &= B y \\ t^* &= B^2 t. \end{aligned}\right\} \qquad (3.1.10)$$

The way of seeking absolute invariant is not well-defined. From the boundary layer conceptions, it would be a good guess to assume that η might be written in terms of powers of x and t.

Variable Transformation

Independent variable transformation

Assume that

$$\eta = y(ax + bt)^p \qquad (3.1.11)$$

is an absolute invariant of the group (G1)

Now, restriction might be placed on p, in order that η would be invariant under (3.1.9).

So we must have $\eta^* = y^*(a x^* + b t^*)^p = B^{1+2p}\ \eta$.

For absolute invariant put $1 + 2p = 0$

so that $\eta = y(ax + bt)^{-\frac{1}{2}}$ is an absolute invariant.

It doesn't mean that η is the only absolute invariant.

Dependent variable transformation

We now express all dependent variables in terms of η. Since there are five dependent variables, we seek five functions g_i $(i = 1,2,3,4,5)$ which are absolutely invariant under (3.1.4).

Of the countless possible forms which exist, we select

$$\left.\begin{aligned} g_1 &= \psi\,(a x + b t)^q \\ g_2 &= \theta = G(\eta) \\ g_3 &= u_e = u_0 \\ g_4 &= \Delta T\,(a x + b t)^r \\ g_5 &= U_F^2 (ax + b t)^s . \end{aligned}\right\} \quad (3.1.12)$$

One dependent variable has been assigned for each function. The selection of the power forms is in keeping the power form of the transformations (3.1.9).

Employing expression (3.1.9) in g_i gives

$$\left.\begin{aligned} g_1^* &= \psi(a x + b t)^q = B^{-1-2q}\psi^*\left(a x^* + b t^*\right)^q \\ g_2^* &= \theta^* = \theta \\ g_3^* &= u_e^* = u_e \\ g_4^* &= \Delta T(a x + b t)^r = B^{2-2r}\,\Delta T^*\left(a x^* + b t^*\right)^r \\ g_5^* &= U_F^2(a x + b t)^s = B^{2-2s}\,U_F^{*2}\left(a x^* + b t^*\right)^s \end{aligned}\right\} \quad (3.1.13)$$

For constant conformally invariant, we must have

$$\left.\begin{aligned} -1-2q &= 0 \\ 2-2r &= 0 \\ 2-2s &= 0 \end{aligned}\right\} \quad (3.1.14)$$

The invariant solutions of the equations (3.1.1) and (3.1.2) can nowbe expressed in terms of η and the functions F, G and I.

$$\therefore \quad g_1 = \psi\,(ax+bt)^{-\frac{1}{2}}$$

$$\left.\begin{array}{l} \therefore \quad \psi = (ax+bt)^{\frac{1}{2}} F(\eta) \\ \theta = G(\eta) \\ u_e = u_0 \\ \Delta T = (ax+bt)^{-1} I(\eta) \\ U_F^2 = (ax+bt)^{-1} U_f^2(\eta) \end{array}\right\} \qquad (3.1.15)$$

In view of (3.1.1) and (3.1.2), one obtains

$$u = \frac{\partial \psi}{\partial y} = F_\eta \quad , \frac{\partial^2 \psi}{\partial x \partial y} = -\frac{1}{2} a\eta (ax+bt)^{-1} F_{\eta\eta}$$

$$\frac{\partial^2 \psi}{\partial y \partial t} = -\frac{1}{2} b\eta\,(ax+bt)^{-1} F_{\eta\eta}$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = -\frac{1}{2} a\eta\,(ax+bt)^{-1} F_{\eta\eta}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} = -\frac{1}{2} a\eta\,(ax+bt)^{-1} F_\eta F_{\eta\eta}$$

$$-v = \frac{\partial \psi}{\partial x} = \left[\frac{1}{2} a\,F - \frac{1}{2} a\,\eta\,F_\eta\right](ax+bt)^{-\frac{1}{2}} \qquad (3.1.5)'$$

$$\frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \left[\frac{1}{2} a F F_{\eta\eta} - \frac{1}{2} a\eta F_\eta F_{\eta\eta}\right](ax+bt)^{-1}$$

$$\frac{\partial u_e}{\partial x} = 0, \frac{\partial u_e}{\partial t} = 0, \; u_e \frac{\partial u_e}{\partial x} = 0 \text{ and } \frac{\partial^3 \psi}{\partial y^3} = (ax+bt)^{-1} F_{\eta\eta\eta}.$$

Also we have,

$$\theta = G(\eta)$$

$$\frac{\partial\theta}{\partial t} = -\frac{1}{2}b\eta\, G_\eta (ax+bt)^{-1},\quad \frac{\partial\theta}{\partial y} = G_\eta (ax+bt)^{-\frac{1}{2}},\quad \frac{\partial^2\theta}{\partial y^2} = G_{\eta\eta}(ax+bt)^{-1}.$$

$$\frac{\partial\theta}{\partial x} = -\frac{1}{2}a\eta G_\eta (ax+bt)^{-1}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} = -\frac{1}{2}a\eta G_\eta F_\eta (ax+bt)^{-1}$$

$$\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} = \left[\frac{1}{2}aF G_\eta - \frac{1}{2}a\eta F_\eta G_\eta\right](ax+bt)^{-1}$$

$$\theta\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\ln\Delta T) = G\left[-aF_\eta - \frac{a}{2I}\eta F_\eta I_\eta\right](ax+bt)^{-1}$$

$$\theta\frac{\partial}{\partial t}(\ln\Delta T) = \left[-bG - \frac{b}{2I}\eta I_\eta\right](ax+bt)^{-1}$$

Substituting the above values in to the equations (3.1.1) and (3.1.2) we have

$$F_{\eta\eta\eta} + \left(\frac{a}{2}F + \frac{b}{2}\eta\right)F_{\eta\eta} + \frac{U_f^2}{u_e^2}G = 0 \tag{3.1.16}$$

$$\Pr^{-1} G_{\eta\eta} + \left(\frac{b}{2}\eta + \frac{1}{2}aF\right)G_\eta + \left(b + aF_\eta\right)G + \frac{a}{2I}\eta F_\eta I_\eta G + \frac{b}{2I}\eta I_\eta G_\eta = 0. \tag{3.1.17}$$

The boundary conditions becomes

$$\left.\begin{array}{l} F_\eta(0) = 0,\ F(0) = F_w(0) \neq 0,\ G(0) = 1\ at\ \eta = 0 \\ F_\eta(\infty) = 1,\ G(\infty) = 0 \quad at \quad \eta \to \infty. \end{array}\right\} \tag{3.1.18}$$

Here the additional parameter is given in the boundary condition as $F(0) = F_w$ related to the suction v_w.

Here $-v_w = \frac{1}{2}(ax + bt)^{-\frac{1}{2}} F(0)$, $v_w < 0$ signifies suction; $v_w > 0$ injection.

If the function I = constant, we have form (3.1.16-17)

$$F_{\eta\eta\eta} + \left(\frac{a}{2}F + \frac{b}{2}\eta\right)F_{\eta\eta} + \frac{U_f^2}{U^2}G = 0 \quad (3.1.19)$$

$$\Pr^{-1} G_{\eta\eta} + \left(\frac{b}{2}\eta + \frac{a}{2}F\right)G_\eta + \left(b + aF_\eta\right)G = 0 \quad (3.1.20)$$

It should be noted that the similarity solutions is only valid when $(ax + bt)$ is positive.

A minor change in the constants, the equations (3.1.19)-(3.1.20) reduces to that of **Zakerullah (1976).** The minor changes are

$$a = 2\beta, \; b = 2$$

Following the above changes, we have from (3.1.19) – (3.1.20)

$$F_{\eta\eta\eta} + (\eta + \beta F)F_{\eta\eta} + \frac{U_F^2}{u_e^2}G = 0 \quad (3.1.21)$$

$$\Pr^{-1} G_{\eta\eta} + (\eta + \beta F)G_\eta + 2\left(1 + \beta F_\eta\right)G = 0 \quad (3.1.22)$$

The boundary conditions are

$$\left.\begin{aligned} &F_\eta(0) = 0,\; F(0) = F_w(0) \neq 0,\; G(0) = 1 \; at \; \eta = 0 \\ &F_\eta(\infty) = 1,\; G(\infty) = 0 \quad at \quad \eta \to \infty. \end{aligned}\right\} \quad (3.1.23)$$

The controlling parameters are $\beta, \Pr, \frac{U_F^2}{u_e^2}$ and the additional parameter $F_w = F(0)$ related to the suction parameter v_w when $\eta = 0$ for the equation $(3.1.5)'$.

The variations of $\Delta T, u_e$ and suction v_w are proportional to $(\beta x + t)^{-1}, u_0$ and $(\beta x + t)^{-\frac{1}{2}}$ respectively.

3.2 CASE-2

Unsteady mixed convection with surface temperature, free stream velocity and suction velocity varying directly with any power of linear function of x.

For this case consider the group (G_1) is given by the following set of transformations:

$$\left.\begin{aligned} x^* &= a^{\alpha_1} x \\ y^* &= a^{\alpha_2} y \\ t^* &= a^{\alpha_3} t \\ \psi^* &= a^{\alpha_4} \psi \\ \Delta T^* &= a^{\alpha_5} \Delta T \\ u_e^* &= a^{\alpha_6} u_e \\ \theta^* &= a^{\alpha_7} \theta \\ U_F^{*2} &= a^{\alpha_8} U_F^2 \end{aligned}\right\} \quad (3.2.1)$$

where $a\ (\neq 0)$ is the parameter and α's are arbitrary real numbers whose interrelationship will be determine by the subsequent analysis. The functionally independent set of absolute invariants of the group (3.2.1) play a central role in the Group Theory.

Substituting (3.2.1) in equations (3.1.1-2), like the previous case, we have, for constant conformally invariant, the following algebraic equations

$$\left.\begin{aligned} \alpha_4-\alpha_2-\alpha_3 = 2\alpha_4-2\alpha_2-\alpha_1 = \alpha_8+\alpha_7 = \alpha_6-\alpha_3 = 2\alpha_6-\alpha_1 = \alpha_4-3\alpha_2 \\ \alpha_7-\alpha_3 = \alpha_4+\alpha_7-\alpha_1-\alpha_2 = \alpha_7-\alpha_3 = \alpha_7-2\alpha_2. \end{aligned}\right\} \quad (3.2.2)$$

Without loss of generality, we may put $\alpha_7 = 0$, $\alpha_5 = \alpha_8$.

Solving equations (3.2.2), we have the following relationship between the exponents.

$$\frac{\alpha_3}{\alpha_1} = 2n,\ \frac{\alpha_4}{\alpha_1} = 1-n,\ \frac{\alpha_6}{\alpha_1} = 1-2n \quad ,\frac{\alpha_8}{\alpha_1} = 1-4n$$

where $n = \frac{\alpha_2}{\alpha_1}$ is an arbitrary constant.

It follows that ϕ_1, ϕ_2 are conformally invariant under the following transformation group

$$\left.\begin{aligned} x^* &= B\,x \\ y^* &= B^n\,y \\ t^* &= B^{2n}\,t \\ \theta^* &= \theta \\ u_e^* &= B^{1-2n}\,u_e \\ U_F^{*\,2} &= B^{1-4n}\,U_F^2 \\ \psi^* &= B^{1-n}\,\psi \\ \Delta T^* &= B^{1-4n}\,\Delta T \end{aligned}\right\} \qquad (3.2.3)$$

Here $B = a^{\alpha_1}$.

Variable Transformation

Independent variable transformation

We now reduce number of independent variables, like the previous case, from three to two variables η_1 and η_2. Assume that η_1, η_2 might be written in terms of powers of x, y and x, t .Considering x as the common variable for both η_1 and η_2.

Let $\eta_1 = y\,x^s$ and $\eta_2 = t\,x^{s'}$ are two similarity variables.

Now, restrictions might be placed on s, s' in order that η_1, η_2 would be invariant under (3.2.3) we must obtain $y^* x^{*s} = y x^s$ and $t^* x^{*s'} = t x^{s'}$

so that absolute invariant is satisfied. Hence we obtain the relation between the exponents with arbitrary n.

$$\therefore s = -n \text{ and } s' = -2n$$

$$\therefore \quad \eta_1 = y x^{-n} \text{ and } \eta_2 = t x^{-2n}$$

It doesn't mean that η_1, η_2 are the only two absolute invariants.

We now express all dependent variables in terms of η_1 and η_2. Since there are five dependent variables, we seek five functions g_i $(i = 1,2,3,4,5)$ which are absolute invariant under (3.2.1).

Dependent variable transformation

Of the countless existing possible forms, we select

$$g_1 = \psi x^b, \quad g_2 = \theta x^c, \quad g_3 = u_e x^d$$
$$g_4 = \Delta T x^e \quad , g_5 = U_F^2 x^f.$$

Employing expression (3.2.1) in g_i gives

$$g_1 = B^{b+n-1} \psi^* x^* = \psi x^b$$

For constant conformally invariant, we must have

$$b+n-1=0$$

$$\therefore \quad g_1 = \psi x^{1-n} = F(\eta_1, \eta_2) \text{ (say)}$$

$$\therefore \quad \psi = x^{1-n} F(\eta_1, \eta_2).$$

The dependent variables transformations are

$$\left.\begin{aligned} \psi &= x^{1-n} F(\eta_1, \eta_2) \\ u_e &= x^{1-2n} H(\eta_1, \eta_2) \\ \Delta T &= x^{1-4n} I(\eta_1, \eta_2) \\ U_F^2 &= x^{1-4n} U_f^2(\eta_1, \eta_2) \\ \theta &= G(\eta_1, \eta_2). \end{aligned}\right\} \qquad (3.2.4)$$

Since u_e is the nondimensional external forcing velocity, we are allowed to replace $1-2n$ by m, thus all the dependent variables becomes

$$\left.\begin{array}{l}\psi = x^{\frac{1+m}{2}} F(\eta_1,\eta_2)\\ u_e = x^{m} H(\eta_1,\eta_2)\\ \Delta T = x^{2m-1} I(\eta_1,\eta_2)\\ U_F^2 = x^{2m-1} U_f^2(\eta_1,\eta_2)\\ \theta = G(\eta_1,\eta_2)\end{array}\right\} \qquad (3.2.5)$$

In view of (3.1.1) and (3.1.2), one obtains

$$u = \frac{\partial\psi}{\partial y} = x^m F_{\eta_1}, \quad \frac{\partial^2\psi}{\partial y^2} = x^{\frac{3m-1}{2}} F_{\eta_1\eta_1}, \quad \frac{\partial^2\psi}{\partial y\partial t} = x^{2m-1} F_{\eta_1\eta_2}$$

$$\frac{\partial^2\psi}{\partial x\partial y} = x^{m-1}\left[mF_{\eta_1} + \frac{m-1}{2}\eta_1 F_{\eta_1\eta_1} + (m-1)\eta_2 F_{\eta_1\eta_2}\right]$$

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} = x^{m-1}\left[mF_{\eta_1}^2 + \left(\frac{m-1}{2}\right)\eta_1 F_{\eta_1} F_{\eta_1\eta_1} + (m-1)\eta_2 F_{\eta_1} F_{\eta_1\eta_2}\right]$$

$$-v = \frac{\partial\psi}{\partial x} = x^{\frac{m-1}{2}}\left[\frac{m+1}{2}F + \left(\frac{m-1}{2}\right)\eta_1 F_{\eta_1} + (m-1)\eta_2 F_{\eta_2}\right]$$

$$\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = x^{2m-1}\left[\left(\frac{m+1}{2}\right)F F_{\eta_1\eta_1} + \left(\frac{m-1}{2}\right)\eta_1 F_{\eta_1} F_{\eta_1\eta_1} + (m-1)\eta_2 F_{\eta_2} F_{\eta_1\eta_1}\right]$$

$$\frac{U_F^2}{U^2}\theta = x^{2m-1}\frac{U_f^2}{U^2}G; \quad \frac{\partial u_e}{\partial t} = x^{2m-1} H_{\eta_2}$$

$$\frac{\partial u_e}{\partial x} = x^{m-1}\left[mH + \left(\frac{m-1}{2}\right)\eta_1 H_{\eta_1} + (m-1)\eta_2 H_{\eta_2}\right]$$

$$u_e\frac{\partial u_e}{\partial x} = x^{2m-1}\left[mH^2 + \left(\frac{m-1}{2}\right)\eta_1 H H_{\eta_1} + (m-1)\eta_2 H H_{\eta_2}\right]$$

and $\dfrac{\partial^2\psi}{\partial y^3} = x^{2m-1}F_{\eta_1\eta_1\eta_1}$

Again, $\theta = G(\eta_1, \eta_2)$

$$\frac{\partial\theta}{\partial y} = x^{\frac{m-1}{2}} G_{\eta_1}, \quad \frac{\partial\theta}{\partial t} = x^{m-1} G_{\eta_2}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} = x^{m-1}\left[\left(\frac{m-1}{2}\right)\eta_1 G_{\eta_1} F_{\eta_1} + (m-1)\eta_2 G_{\eta_2} F_{\eta_1}\right]$$

$$\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} = x^{m-1}\left[\left(\frac{m+1}{2}\right)G_{\eta_1} F + \left(\frac{m-1}{2}\right)\eta_1 F_{\eta_1} G_{\eta_1} + (m-1)\eta_2 F_{\eta_2} G_{\eta_1}\right]$$

$$\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\ln\Delta T) = x^{m-1}\left[\left(\frac{m-1}{2I}\right)\eta_1 I_{\eta_1} F_{\eta_1} + (m-1)\eta_2 I_{\eta_2} F_{\eta_1}\frac{1}{I} + (2m-1)F_{\eta_1}\right]$$

$$\theta\frac{\partial}{\partial t}(\ln\Delta T) = x^{m-1} I_{\eta_2}\frac{1}{I}G$$

Substituting the above values into the equations (3.1.1-2), we have after some algebraic manipulations.

$$F_{\eta_1\eta_1\eta_1} + \left(\frac{m+1}{2}\right)F F_{\eta_1\eta_1} + (m-1)\eta_2\left[F_{\eta_2}F_{\eta_1\eta_1} - F_{\eta_1}F_{\eta_1\eta_2}\right] - mF_{\eta_1}^2 - F_{\eta_1\eta_2}G$$

$$+\frac{U_f^2}{U^2} + mH^2 + \left(\frac{m-1}{2}\right)\eta_1 H H_{\eta_1} + (m-1)\eta_2 H H_{\eta_2} + H_{\eta_2} = 0 \qquad (3.2.6)$$

and

$$\left[\left(\frac{m+1}{2}\right)F G_{\eta_1} - (m-1)\eta_2(G_{\eta_2}F_{\eta_1} - F_{\eta_2}G_{\eta_1}) - (2m-1)GF_{\eta_1} - G_{\eta_2}\right]$$

$$-\frac{1}{I}\left[\left(\frac{m-1}{2}\right)\eta_1 G I_{\eta_1} F_{\eta_1} + (m-1)\eta_2 I_{\eta_2} F_{\eta_1} G - G I_{\eta_2}\right] + \Pr^{-1} G_{\eta_1\eta_1} = 0. \qquad (3.2.7)$$

The boundary conditions becomes

$$\left.\begin{array}{l} F_{\eta_1}(0,\eta_2)=0,\ G(0,\eta_2)=1 \\ -v_w(x,t)=x^{\frac{m-1}{2}}\left[\frac{m+1}{2}F(0,\eta_2)+(m-1)\eta_2 F_{\eta_2}(0,\eta_2)\right] \text{at}\ \eta_1=0 \\ F_{\eta_1}(\infty,\eta_2)=H,\quad G(\infty,\eta_2)=0 \qquad \text{at}\quad \eta_1\to\infty. \end{array}\right\} \tag{3.2.8}$$

It is interesting to note several features of equations (3.2.6-7). If the analyses are to be stopped at this stage and assume that H = constan, I = constant and one of the similarity variables (say) $\eta_2 = 0$, equations (3.2.6-7) becomes

$$F_{\eta_1\eta_1\eta_1}+\left(\frac{m+1}{2}\right)F\,F_{\eta_1\eta_1}+m\left(H^2-F_{\eta_1}^2\right)+\frac{U_f^2}{U^2}G=0 \tag{3.2.9}$$

$$G_{\eta_1\eta_1}+\Pr\left[\left(\frac{m+1}{2}\right)F\,G_{\eta_1}-(2m-1)G\,F_{\eta_1}\right]=0. \tag{3.2.10}$$

The boundary conditions are

$$\left.\begin{array}{l} F_{\eta_1}(0)=0,\ G(0)=1 \\ -v_w(x,t)=x^{\frac{m-1}{2}}\left(\frac{m+1}{2}\right)F(0) \quad \text{at}\ \eta_1=0 \\ F_{\eta_1}(\infty)=H,\quad G(\infty)=0 \qquad \text{at}\quad \eta_1\to\infty, \end{array}\right\} \tag{3.2.11}$$

Hence $u_e, \Delta T$ and v_w variations are x^m, x^{2m-1} and $x^{\frac{m-1}{2}}$ respectively.

For constant suction $m=1$.

Comparison with the published results

With following minor change of stream functions and similarity variables

$$F(\eta_1)=\frac{2}{\sqrt{1+m}}F(\eta),\ G(\eta_1)=\theta(\eta),\ \eta=\eta_1\sqrt{1+m}\ \text{and}\ H=2.$$

We have from (3.2.9-10)

$$F_{\eta\eta\eta}+F\,F_{\eta\eta}+\frac{2m}{m+1}\left(1-F_{\eta_1}{}^2\right)+\frac{U_f^2}{U^2}\theta=0 \qquad (3.2.12)$$

$$\theta_{\eta\eta}+\Pr\left[F\,\theta_\eta-\left(\frac{4m-2}{1+m}\right)\theta\,F_\eta\right]=0\ . \qquad (3.2.13)$$

Let $\frac{2m}{m+1}=\beta$, and without loss of generality put $U=u_e$ to have the compabality with the boundary conditions.

Equations (3.2.12-13) becomes

$$F_{\eta\eta\eta}+F\,F_{\eta\eta}+\beta\left(1-F_\eta{}^2\right)+R_F^2\,\theta=0 \qquad (3.2.14)$$

$$\theta_{\eta\eta}+\Pr\left[F\,\theta_\eta-(3\beta-2)\theta\,F_\eta\right]=0. \qquad (3.2.15)$$

The boundary conditions are

$$\left.\begin{array}{ll} F_\eta(0)=0,\ F(0)=F_w(0)\neq 0,\ G(0)=1\ at\ \eta=0 \\ F_\eta(\infty)=1,\ G(\infty)=0 \quad at \quad \eta\to\infty. \end{array}\right\} \qquad (3.2.16)$$

The controlling parameters are β, Pr, $R_F^2=\frac{U_f^2}{U_e^2}$ and the parameter $F_w=F(0)$ related to the suction parameter v_w respectively.

These are purely steady case derived form unsteady mixed convection boundary layer equations.

A flow separation may occur when forced and free convection act in opposite directions. If the buoyancy force has a positive component in the direction of free stream, then the flow is said to be aiding flow, opposing flow is just reverse. For small values of $\frac{U_F^2}{u_e^2}$ forced convection will be predominant and for large value free convection will control.The equations (3.2.14-15) are identical with the equations derived by **Zakerullah (1976)**.

The equations (3.2.9-10) are with the equations of **Zubair (1990)** if we replace $H = \nu, \frac{U_f^2}{u_e^2} = 1$ and $m = \frac{1}{2}$ and the transformed equations are

$$F_{\eta\eta\eta} + \frac{3}{4} F F_{\eta\eta} - \frac{1}{2} F_{\eta_1}^2 + \theta + \frac{1}{2}\nu^2 = 0 \tag{3.2.17}$$

$$\theta_{\eta\eta} + \Pr \frac{3}{4} F \theta_{\eta} = 0. \tag{3.2.18}$$

The boundary conditions are

$$\left.\begin{aligned} &F_\eta(0) = 0,\ F(0) = F_w(0) \neq 0,\ \theta(0) = 1\ at\ \eta = 0 \\ &F'(\infty) = x^{\frac{1}{2}}\nu,\ \theta(\infty) = 0 \quad at \quad \eta \to \infty. \end{aligned}\right\} \tag{3.2.19}$$

Equations (3.2.9-10) are identical with the **Sparrow, Eichorn and Gregg (1959)** following a minor change. Minor changes are

$$F(\eta) = 2F(\eta),\ H = 4, G = \theta \text{ and } U_f^2 = 16 U_f^2$$

and the transformed equations are

$$F_{\eta\eta\eta} + (m+1) F F_{\eta\eta} - 2mF_\eta^2 + 8\frac{U_f^2}{U^2} G + 8m = 0 \tag{3.2.20}$$

$$G_{\eta\eta} + \Pr\left[(m+1) F G_\eta - (4m-2) G F_\eta\right] = 0. \tag{3.2.21}$$

The boundary conditions are

$$\left.\begin{aligned} &F_\eta(0) = 0,\ F(0) = F_w(0) \neq 0,\ G(0) = 1\ at\ \eta = 0 \\ &F_\eta(\infty) = 1,\ G(\infty) = 0 \quad at \quad \eta \to \infty. \end{aligned}\right\} \tag{3.2.22}$$

Here the additional parameter is given in the boundary condition as $F(0) = F_w$ related to the suction v_w.

Results and Discussion

The quantities of physical interest are local skin friction and heat transfer factors. We know local heat transfer q' per unit area from the plate to the fluid may be calculated by Fourier's law, i.e.,

$$q' = -k'\left(\frac{\partial T'}{\partial y'}\right)_{y'=0}$$

It is convenient to express heat transfer results in terms of heat transfer coefficients and Nusselt number according to the following definition

$$h' = \frac{q'}{T'_w - T'_\infty}, \quad Nu = \frac{h' L}{k'} = \theta'(0) \quad .$$

$$\frac{Nu}{\mathrm{Re}^{\frac{1}{2}}} = -\theta'(0).$$

For wall shear stress

$$\tau'_w = \mu'\left(\frac{\partial u'}{\partial y'}\right)_{y'=0}.$$

Defining skin friction coefficient

$$c_f = \frac{2\tau'_w}{\rho' u_e'^2}.$$

and nondimensional skin friction coefficient

$$c_f \, \mathrm{Re}^{\frac{1}{2}} = F_{\eta\eta}(0)$$

3.3 CASE-3

Unsteady mixed convection with surface temperature varying directly with linear function of x inversely with square of the linear function of t, the free stream velocity varying with linear function of x inversely with linear function of t, and the suction velocity varying inversely as a square root of the linear function of t.

It is clear from equation (3.2.6) and (3.2.7) that, they are still partial differential equations with two independent variables η_1 and η_2. To reduce them into single one, we define the following one parameter continuous transformation group (G2) as

$$\left.\begin{aligned} \eta_1^* &= b^{\beta_1}\eta_1 \\ \eta_2^* &= b^{\beta_2}\eta_2 \\ F^* &= b^{\beta_3}F \\ G^* &= b^{\beta_4}G \\ H^* &= b^{\beta_5}H \\ I^* &= b^{\beta_6}I \\ U_F^{*2} &= b^{\beta_7}U_F^2 \end{aligned}\right\} \quad (3.3.1)$$

Here $b(\neq a \neq 0)$ is a parameter of the group and β's are real numbers, determined by previously indicated manner. Substituting (3.3.1) into (3.2.7-8) we have, like previous case, the following algebraic expressions

$$\left.\begin{aligned} &\beta_3-3\beta_1=2\beta_3-2\beta_1=\beta_3-\beta_1-\beta_2=\beta_4+\beta_7=2\beta_5=\beta_5-\beta_2 \\ &\beta_3-2\beta_1=\beta_4+\beta_3-\beta_1=\beta_4-\beta_2 \end{aligned}\right\} \quad (3.3.2)$$

Solving equations (3.3.2), we have the following relationship between the exponents.

$$\frac{\beta_2}{\beta_1}=2,\quad \frac{\beta_3}{\beta_1}=-1,\quad \frac{\beta_5}{\beta_1}=-2,\quad \frac{\beta_6}{\beta_2}=2\beta_1\neq 0.$$

Without loss of generality, we may put $\beta_4=0, \beta_6=\beta_7$.

Proceeding exactly as before, we can find the following invariant transformations.

$$\left.\begin{aligned}
&\eta = \frac{\eta_1}{\sqrt{\eta_2}} \\
&F(\eta_1,\eta_2) = \eta_2^{-\frac{1}{2}}\, f(\eta) \\
&I(\eta_1,\eta_2) = \eta_2^{-2}\, \bar{I}(\eta) \\
&G(\eta_1,\eta_2) = g(\eta) \\
&U_f^2(\eta_1,\eta_2) = \eta_2^{-2} U_f^2(\eta) \\
&H(\eta_1,\eta_2) = \eta_2^{-1}\, h(\eta)
\end{aligned}\right\} \tag{3.3.3}$$

In view of equations (3.2.7-8), we have,

$$F(\eta_1,\eta_2) = \eta_2^{-\frac{1}{2}}\, f(\eta)$$

$$\frac{\partial F}{\partial \eta_1} = \eta\eta_2^{-1}\, f_\eta\,,\quad \frac{\partial^2 F}{\partial \eta_1^{\,2}} = \eta_2^{-\frac{3}{2}}\, f_{\eta\eta}$$

$$\frac{\partial^3 F}{\partial \eta_1^{\,3}} = \eta_2^{-2}\, f_{\eta\eta\eta}$$

$$\frac{\partial^2 F}{\partial \eta_1\eta_2} = \eta_2^{-2}\left[f_\eta - \frac{\eta}{2} f_{\eta\eta}\right]$$

$$\frac{\partial F}{\partial \eta_2} = -\frac{1}{2}\eta_2^{-\frac{3}{2}}\left[f_\eta + \eta\, f_\eta\right]$$

Again,

$$G(\eta_1,\eta_2) = g(\eta)\,,\quad \frac{\partial G}{\partial \eta_1} = \eta_2^{-\frac{1}{2}}\, g_\eta$$

$$\frac{\partial^2 G}{\partial \eta_1\eta_1} = \eta_2^{-1}\, g_{\eta\eta}\,,\quad \frac{\partial G}{\partial \eta_2} = \frac{1}{2}\eta_2^{-1}\left[-\eta\, g_\eta\right]$$

$$H(\eta_1,\eta_2)=\eta_2^{-1} f(\eta),\frac{\partial H}{\partial \eta_1}=\eta_2^{-\frac{3}{2}} h_\eta \text{ and } \frac{\partial H}{\partial \eta_2}=-\eta_2^{-2}\left[h+\frac{\eta}{2}h_\eta\right].$$

Using the transformations (3.3.3) and then putting $\bar{I}=H=1$, we have from (3.2.7-8)

$$f_{\eta\eta\eta}+\left(f+\frac{\eta}{2}\right)f_{\eta\eta}+f_\eta-f_\eta^{\,2}+\frac{U_f^2}{U^2}g=0 \tag{3.3.4}$$

and

$$g_{\eta\eta}+\Pr\left[\left(f+\frac{\eta}{2}\right)g_\eta+\left(2-f_\eta\right)g\right]=0. \tag{3.3.5}$$

The boundary conditions becomes

$$\left.\begin{aligned} &f(0)=f_w, f_\eta(0)=0, g(0)=1\\ &f_\eta(\infty)=1\quad, g(\infty)=0.\end{aligned}\right\} \tag{3.3.6}$$

The variations of ΔT, u_e and v_w are proportional to $\frac{x}{t^2},\frac{x}{t}$ and $\frac{1}{\sqrt{t}}$ respectively.

The controlling parameters are $\Pr, \frac{U_f^2}{U^2}$ and $f_w=f(0)$ related to the suction parameter v_w.

3.4 CASE-4(PURELY UNSTEADY CASE)

Unsteady mixed convection with surface temperature and free stream velocity varying any power of linear function of t and the suction velocity varying inversely as a square root of the linear function of t.

In this case in order to seek the invariant solution to the set of governing equations we set the following transformation group($G1$)

$$\left.\begin{aligned} x^* &= a^{\alpha_1} x \\ y^* &= a^{\alpha_2} y \\ t^* &= a^{\alpha_3} t \\ \psi^* &= a^{\alpha_4} \psi \\ \Delta T^* &= a^{\alpha_5} \Delta T \\ u_e^* &= a^{\alpha_6} u_e \\ \theta^* &= a^{\alpha_7} \theta \\ U_F^{*2} &= a^{\alpha_8} U_F^2 \end{aligned}\right\} \quad (3.4.1)$$

Here $a(\neq 0)$ is the parameter of the group and α's are the arbitrary real numbers whose interrelationship will be determined by the subsequent analysis.

For constant conformally invariant we must have,

$$\left.\begin{aligned} &\alpha_4 - \alpha_2 - \alpha_3 = 2\alpha_4 - 2\alpha_2 - \alpha_1 = \alpha_8 + \alpha_7 = \alpha_6 - \alpha_3 = 2\alpha_6 - \alpha_1 = \alpha_4 - 3\alpha_2 \\ &\alpha_7 - \alpha_3 = \alpha_4 + \alpha_7 - \alpha_1 - \alpha_2 = \alpha_7 - \alpha_3 = \alpha_7 - 2\alpha_2. \end{aligned}\right\} \quad (3.4.2)$$

Solving equations (3.4.2) we have the following relationship between the exponents

$$\alpha_3 = 2\alpha_2, \quad \alpha_4 = \alpha_1 - \alpha_2, \quad \alpha_8 + \alpha_7 = \alpha_1 - 4\alpha_2, \quad \alpha_6 = \alpha_1 - \alpha_3$$

Let $\dfrac{\alpha_1}{\alpha_3} = n$ and $\alpha_7 = 0$ and $\alpha_5 = \alpha_7$

$$\therefore \frac{\alpha_2}{\alpha_3} = \frac{1}{2}, \quad \frac{\alpha_4}{\alpha_3} = n - \frac{1}{2}, \quad \frac{\alpha_6}{\alpha_3} = n - 1, \quad \frac{\alpha_8}{\alpha_3} = n - 2 = \frac{\alpha_5}{\alpha_3}$$

It follows that ϕ_1, ϕ_2 are conformally invariant under the following transformation group.

$$\left.\begin{aligned}
t^* &= B\, t \\
x^* &= B^n\, x \\
y^* &= B^{\frac{1}{2}}\, y \\
\psi^* &= B^{n-\frac{1}{2}}\, \psi \\
\theta^* &= \theta \\
\Delta T^* &= B^{n-2} \Delta T \\
U^{*2}_F &= B^{n-2} U_F^2 \\
u_e^* &= B^{n-1}\, u_e
\end{aligned}\right\} \qquad (3.4.3)$$

Here $B = a^{\alpha_3}$.

Variable Transformation

Independent variable Transformation

Let $\eta_1 = y t^s$ and $\eta_2 = x t^{s'}$

For absolute invariant, we have the following relationship between the exponents with arbitrary n.

$$s = -\frac{1}{2} \text{ and } s' = -n.$$

$\therefore \eta_1 = y t^{-\frac{1}{2}}$ and $\eta_2 = x t^n$ are two invariants(out of many possibilities).

We now express all dependent variables in terms of η_1 and η_2. Since there are five dependent variables, we seek five functions g_i $(i = 1,2,3,4,5)$ which are absolutely invariant under (3.4.3).

Dependent variable Transformation

Of the countless possible existing forms, we select

$$\left.\begin{aligned} g_1 &= \psi\, t^b \\ g_2 &= \theta\, t^c \\ g_3 &= u_e\, t^d \\ g_4 &= \Delta T\, t^e \\ g_5 &= U_F^2\, t^f \end{aligned}\right\} \qquad (3.4.4)$$

Employing expression (3.4.3) in g_i gives

$$g_1 = \psi^* t^{*b} = A^{n-\frac{1}{2}+b}\, \psi\, t^b$$

For constant conformally invariant, we must have

$$-b = n - \frac{1}{2}$$

$$\therefore \quad \psi = t^{n-\frac{1}{2}} F(\eta_1, \eta_2)$$

Therefore, we have the following transformations for dependent variables.

$$\left.\begin{aligned} \psi &= t^{n-\frac{1}{2}} F(\eta_1, \eta_2) \\ \theta &= G(\eta_1, \eta_2) \\ u_e &= t^{n-1} H(\eta_1, \eta_2) \\ \Delta T &= t^{n-2} I(\eta_1, \eta_2) \\ U_F^2 &= t^{n-2}\, U_f^2(\eta_1, \eta_2) \end{aligned}\right\} \qquad (3.4.5)$$

Let $n - 1 = m$.

Using the transformations given by (3.4 .4), we have from the equations (3.1.1) and (3.1.2).

$$F_{\eta_1\eta_1\eta_1}+\left(\frac{1}{2}\eta_1+F_{\eta_2}\right)F_{\eta_1\eta_1}+\left[(m+1)\eta_2-1\right]F_{\eta_1\eta_2}-mF_{\eta_1}+\frac{U_f^2}{U^2}G$$
$$+mH+\frac{1}{2}\eta_1H_{\eta_1}+\left[(m+1)\eta_2-1\right]H_{\eta_2}=0 \qquad (3.4.6)$$

and

$$G_{\eta_1\eta_1}+\Pr\left[(m+1)\eta_2-F_{\eta_1}\right]G_{\eta_2}+\left[\frac{\eta_1}{2}+F_{\eta_2}\right]G_{\eta_1}$$
$$+\frac{1}{2I}\left[\eta_1I_{\eta_1}+(m+1)\eta_2I_{\eta_2}-I_{\eta_2}\right]G-(m-1)G=0 \qquad (3.4.7)$$

If analysis is stopped here, we will see some interesting aspect. Putting $H=1$, $I=1$, $\eta=\eta_1$ and $\eta_2=0$, we have

$$F_{\eta\eta\eta}+\frac{1}{2}\eta F_{\eta\eta}-mF_{\eta}+\frac{U_F^2}{U^2}G+m=0 \qquad (3.4.8)$$

and

$$\Pr^{-1}G_{\eta\eta}+\frac{1}{2}\eta G_{\eta}+(1-m)G=0\ . \qquad (3.4.9)$$

The boundary conditions becomes

$$\left.\begin{array}{l} F_\eta(0)=0,\ F(0)=F_w(0)=0,\ G(0)=1\ at\ \eta=0 \\ F_\eta(\infty)=1,\ G(\infty)=0 \qquad at \quad \eta\to\infty. \end{array}\right\} \qquad (3.4.10)$$

Here u_e, ΔT and v_w variations are t^m, t^{m-1} and 0 respectively.

3.5 CASE-5

Unsteady mixed convection with surface temperature and free stream velocity varying with an exponential function of function of t and the suction velocity is zero.

In this case in order to seek the invariant solution of the set of governing equations (3.1.1)-(3.1.2) we set the following spiral group ($G1$)

$$\left.\begin{aligned} t^* &= t + \alpha_1 a \\ x^* &= e^{\alpha_2 a} x \\ y^* &= e^{\alpha_2 a} y \\ \psi^* &= e^{\alpha_4 a} \psi \\ \Delta T^* &= e^{\alpha_5 a} \Delta T \\ u_e^* &= e^{\alpha_6 a} u_e \\ \theta^* &= e^{\alpha_7 a} \theta \\ U_F^{*2} &= e^{\alpha_8 a} U_F^2 \end{aligned}\right\} \qquad (3.5.1)$$

Here $a(\neq 0)$ is the parameter of the group and $\alpha's$ are the arbitrary real numbers.

For constant conformally invariant of the equations (3.1.1)-(3.1.2), we must have,

$$\left.\begin{aligned} &\alpha_4 - \alpha_3 = 2\alpha_4 - \alpha_2 - 2\alpha_3 = \alpha_8 + \alpha_7 = \alpha_6 = 2\alpha_6 - \alpha_2 = \alpha_4 - 3\alpha_2 \\ &\alpha_7 = \alpha_4 + \alpha_7 - \alpha_3 - \alpha_2 = \alpha_7 - 2\alpha_2 \end{aligned}\right\} \qquad (3.5.2)$$

Without loss of generality we may put $\alpha_7 = 0$ and $\alpha_5 = \alpha_8$.

Solving equations (3.5.2), we have the following relationship between the exponents.

$$\alpha_2 = \alpha_4 = \alpha_6 = \alpha_5 = \alpha_8 \text{ and } \alpha_3 = 0.$$

By direct substitution, we can show that ϕ_1, ϕ_2 are conformally invariant under the spiral group transformation (3.5.1).

Variable Transformation

Independent variable Transformation

Let $\eta_1 = y$ and $\eta_2 = xe^{-mt}$ $(m = \frac{\alpha_2}{\alpha_1})$ are two invariants similarity variables (out of many possibilities).

Dependent variable Transformation

Like previously indicated manner we have the following dependent variable transformations.

$$\left.\begin{aligned} \psi &= e^{mt}F(\eta_1,\eta_2) \\ \theta &= G(\eta_1,\eta_2) \\ u_e &= e^{mt}H(\eta_1,\eta_2) \\ \Delta T &= e^{mt}I(\eta_1,\eta_2) \\ U_F^2 &= e^{mt}U_F^2(\eta_1,\eta_2) \end{aligned}\right\} \tag{3.5.3}$$

Using (3.5.3) and considering H = constant , I = constant and $\eta_2 = 0$, we have from (3.2.1) and (3.2.2).

$$F_{\eta_1\eta_1\eta_1} + m(H - F_{\eta_1}) + \frac{U_F^2}{U^2}G = 0 \tag{3.5.4}$$

and

$$G_{\eta_1\eta_1} - mG = 0. \tag{3.5.5}$$

The boundary conditions are

$$\left.\begin{aligned} &F(0) = 0,\ F_{\eta_1}(0) = 0,\ G(0) = 1 \\ &F_{\eta_1}(\infty) = 1,\ G(\infty) = 0. \end{aligned}\right\} \tag{3.5.6}$$

These equations are identical with the case C (II), derived by **Zakerullah (1976)** if we put $m = 1$, and $H = 1$.

The controlling parameters are Pr, m and $\frac{U_F^2}{U^2}$ respectively.

3.6 Case-6

Unsteady mixed convection with surface temperature, free stream velocity, and suction velocity varying with an exponential function of function of x .

In this case in order to seek the invariant solution to the set of governingequations (3.1.1-2) we set the following spiral group ($G1$)

$$\left.\begin{aligned} x^* &= x + \alpha_1 a \\ y^* &= e^{\alpha_2 a} y \\ t^* &= e^{\alpha_3 a} t \\ \psi^* &= e^{\alpha_4 a} \psi \\ \Delta T^* &= e^{\alpha_5 a} \Delta T \\ u_e^* &= e^{\alpha_6 a} u_e \\ \theta^* &= e^{\alpha_7 a} \theta \\ U_F^{*2} &= e^{\alpha_8 a} U_F^2 \end{aligned}\right\} \tag{3.6.1}$$

Here $a(\neq 0)$ is the parameter of the group and $\alpha's$ are the arbitrary real numbers.

For constant conformally invariant of the equation (3.1.1) and (3.1.2) we must have,

$$\left.\begin{aligned} &\alpha_4 - \alpha_2 - \alpha_3 = 2\alpha_4 - 2\alpha_2 = \alpha_8 + \alpha_7 = \alpha_6 - \alpha_3 = 2\alpha_6 = \alpha_4 - 3\alpha_2 \\ &\alpha_7 - \alpha_3 = \alpha_4 + \alpha_7 - \alpha_2 = \alpha_7 - 2\alpha_2 \end{aligned}\right\} \tag{3.6.2}$$

Without loss of generality, put $\alpha_7 = 0$ and $\alpha_5 = \alpha_8$.

Solving (3.6.2) we have the following relations among the exponents.

$\alpha_3 = 2\alpha_2, \alpha_4 = -\alpha_2, \alpha_6 = -2\alpha_2, \alpha_8 = -4\alpha_2$.

By direct substitution we can show that ϕ_1, ϕ_2 are conformally invariant under the spiral group transformation (3.6.1).

Variable Transformation

Independent variable Transformation

Let $\eta_1 = y\,e^{-mx}$ and $\eta_2 = t\,e^{-2mx}$ $\left(m = \dfrac{\alpha_2}{\alpha_1}\right)$ are two invariants similarity variables. (out of many possibilities).

Dependent variable Transformation

In a similar manner applied as before, the following absolute invariants involving dependent variables are found.

$$\left.\begin{aligned}
&\psi = e^{-mx}F(\eta_1,\eta_2)\\
&\theta = G(\eta_1,\eta_2)\\
&u_e = e^{-2mx}H(\eta_1,\eta_2)\\
&\Delta T = e^{-4mx}I(\eta_1,\eta_2)\\
&U_F^2 = e^{-4mx}\,U_F^2(\eta_1,\eta_2)
\end{aligned}\right\} \tag{3.6.3}$$

Using the transformations given by (3.6 .3) we have from the equations (3.2.1) and (3.2.2).

$$F_{\eta_1\eta_1\eta_1} - mFF_{\eta_1\eta_1} + 2m\eta_2\left(F_{\eta_1}F_{\eta_1\eta_2} - F_{\eta_2}F_{\eta_1\eta_1}\right) - F_{\eta_1\eta_2} + 2mF^2\eta_1$$
$$+\frac{U_F^2}{U^2}G + H_{\eta_2} - 2mH^2 - m\eta_1 HH_{\eta_1} - 2m\eta_2 HH_{\eta_2} = 0 \tag{3.6.4}$$

$$\Pr^{-1}G_{\eta_1\eta_1} - mFG_{\eta_1} + 4mF_{\eta_1}G + 2m\eta_2\left(F_{\eta_1}G_{\eta_2} - F_{\eta_2}G_{\eta_1}\right) - G_{\eta_2}$$
$$+\frac{1}{I}\left(m\eta_1 I_{\eta_1}F_{\eta_1}G + 2m\eta_2 I_{\eta_2}F_{\eta_1}G - I_2 G\right) = 0. \tag{3.6.5}$$

The corresponding boundary conditions are

$$
\left.\begin{array}{l}
F_{\eta_1}(0,\eta_2)=0, v_w=e^{-mx}\left[-mF(0,\eta_2)-2m\eta_2 F_{\eta_2}(0,\eta_2)\right].G(0,\eta_2)=1 \\
F_{\eta_1}(\infty,\eta_2)=H, G((\infty,\eta_2)=0.
\end{array}\right\} \quad (3.6.6)
$$

It is interesting to note several features of the equations (3.6.4-5). If the analysis stopped at this stage and by taking H =constant=1, I =constant=1 and one of the similarity variable (say) $\eta_2 = 0$ then we have,

$$
F_{\eta_1\eta_1\eta_1} - mFF_{\eta_1\eta_1} + 2m\left(F_{\eta_1}^2 - 1\right) + \frac{U_F^2}{U^2}G = 0 \quad (3.6.7)
$$

$$
\Pr^{-1} G_{\eta_1\eta_1} - mFG_{\eta_1} + 4m F_{\eta_1} G = 0. \quad (3.6.8)
$$

The corresponding boundary conditions are

$$
\left.\begin{array}{l}
F_\eta(0)=0,\ F(0)=F_w(0)\neq 0,\ G(0)=1 \ at\ \eta=0 \\
F_\eta(\infty)=1,\ G(\infty)=0 \quad at \quad \eta\to\infty.
\end{array}\right\} \quad (3.6.9)
$$

It is clear from equations (3.6.4) and (3.6.5) that, they are still partial differential equations with two independent variables η_1 and η_2. To reduce them into single one, we define the following group (G2) as

$$\left.\begin{aligned}
\eta_1^{\ *} &= b^{\beta_1}\eta_1 \\
\eta_2^{\ *} &= b^{\beta_2}\eta_2 \\
F^* &= b^{\beta_3}F \\
G^* &= b^{\beta_4}G \\
H^* &= b^{\beta_5}H \\
I^* &= b^{\beta_6}I \\
U_F^{*2} &= b^{\beta_7}U_F^2
\end{aligned}\right\} \tag{3.6.10}$$

Here $b(\neq a \neq 0)$ is a parameter of the group and β's are real numbers determined by previously indicated manner. Substituting (3.6.9) into (3.6.4-5) we have the following algebraic expressions

$$\left.\begin{aligned}
&\beta_3 - 3\beta_1 = 2\beta_3 - 2\beta_1 = \beta_3 - \beta_1 - \beta_2 = \beta_4 + \beta_7 = 2\beta_5 = \beta_5 - \beta_2 \\
&\beta_4 - 2\beta_1 = \beta_4 + \beta_3 - \beta_1 = \beta_4 - \beta_2
\end{aligned}\right\} \tag{3.6.11}$$

Solving equations (3.6.11) we have, the following relationship between the exponents.

$$\frac{\beta_2}{\beta_1} = 2, \quad \frac{\beta_3}{\beta_1} = -1, \quad \frac{\beta_5}{\beta_1} = -2, \quad \frac{\beta_7}{\beta_1} = -2, \quad \beta_1 \neq 0. \text{ Set } \beta_4 = 0, \beta_6 = \beta_7.$$

Proceeding exactly as before, we assume the following transformations (out of many possibilities).

$$\left.\begin{aligned}
&\eta = \frac{\eta_1}{\sqrt{\eta_2}} \\
&F(\eta_1,\eta_2) = \eta_2^{-\frac{1}{2}} f(\eta) \\
&I(\eta_1,\eta_2) = \eta_2^{-2}\bar{I}(\eta) \\
&G(\eta_1,\eta_2) = g(\eta) \\
&U_f^2(\eta_1,\eta_2) = \eta_2^{-2}U_f^2(\eta) \\
&H(\eta_1,\eta_2) = \eta_2^{-1}h(\eta)
\end{aligned}\right\} \tag{3.6.12}$$

Using the transformations (3.6.12) and then put $\bar{l} = h = 1$, we have form (3.6.4) and (3.6.5)

$$f_{\eta\eta\eta} + \frac{\eta}{2} f_{\eta\eta} + f_{\eta} - 1 + \frac{U_f^2}{U^2} g = 0 \tag{3.6.13}$$

and

$$\Pr^{-1} g_{\eta\eta} + \frac{\eta}{2} g_{\eta} + 2g = 0 \ . \tag{3.6.14}$$

The boundary conditions becomes

$$\left.\begin{array}{l} f(0) = f_w, f_{\eta}(0) = 0, \ g(0) = 1 \\ f'(\infty) = 1, \ g(\infty) = 0. \end{array}\right\} \tag{3.6.15}$$

The controlling parameters are Pr , $\frac{U_f^2}{U^2}$ and $f_w = f(0)$ related to the suction parameter v_w.

These equations are special case of the equations of case-4 for $m = -1$.

The variations of ΔT, u_e and v_w are proportional to $\frac{1}{t^2}, \frac{1}{t}$ and $\frac{1}{\sqrt{t}}$ respectively.

3.7 A TABLE FOR SIMILARITY REQUIREMENTS

The nature of $\Delta T, u_e$ and v_w with the similarity variables for which similarity solutions exist are shown in the following table.

CASE	ΔT VARIATIONS	u_e VARIATIONS	v_w VARIATIONS	Similarity Variables
01	$(\beta x + u_0 t)^{-1}$	u_0	$(\beta x + u_0 t)^{-\frac{1}{2}}$	$y(\beta x + u_0 t)^{-\frac{1}{2}}$
02	x^{2m-1}	x^m	$x^{\frac{m-1}{2}}$	$y\, x^{\frac{m-1}{2}}$
03	$\frac{x}{t^2}$	$\frac{x}{t}$	$\frac{1}{\sqrt{t}}$	$\frac{y}{\sqrt{t}}$
04	t^{m-1}	t^m	0	$\frac{y}{\sqrt{t}}$
05	e^{mt}	e^{mt}	0	y
06	e^{-4mx}	e^{-2mx}	e^{-mx}	ye^{-mx}

Chapter Four

Results and Discussions

Numerical results $\{F_{\eta\eta}(0), -\theta_{\eta}(0)\}$ based on equations (3.2.14) and (3.2.15) are presented in Table-1 for $\beta = 1, \Pr = 0.71$ and $R_F^2 = 1$.

Graphs of velocity and temperature profiles are displayed in figures 1, 2, 3 and 4 respectively.

Graphs of effect of suction on skin friction factor and heat transfer factors are shown in figure 5 and 6 respectively.

Table-1 Numerical values of the skin-friction coefficient and heat transfer coefficient for different values of F_w while $\Pr = 0.71$, $\beta = 1$ and $R_F^2 = \dfrac{U_F^2}{u_e^2} = 1$.

F_w	$F_{\eta\eta}(0)$	$-\theta_\eta(0)$
-4	0.4921	0.0409
-3	0.6380	0.0853
-2	0.8689	0.1903
-1	1.2195	0.4078
0	1.7058	0.7669
1	2.3189	1.2542
2	3.0400	1.8243
3	3.8413	2.4443
4	4.6965	3.0910
5	5.5957	3.7712
10	10.3265	7.2137

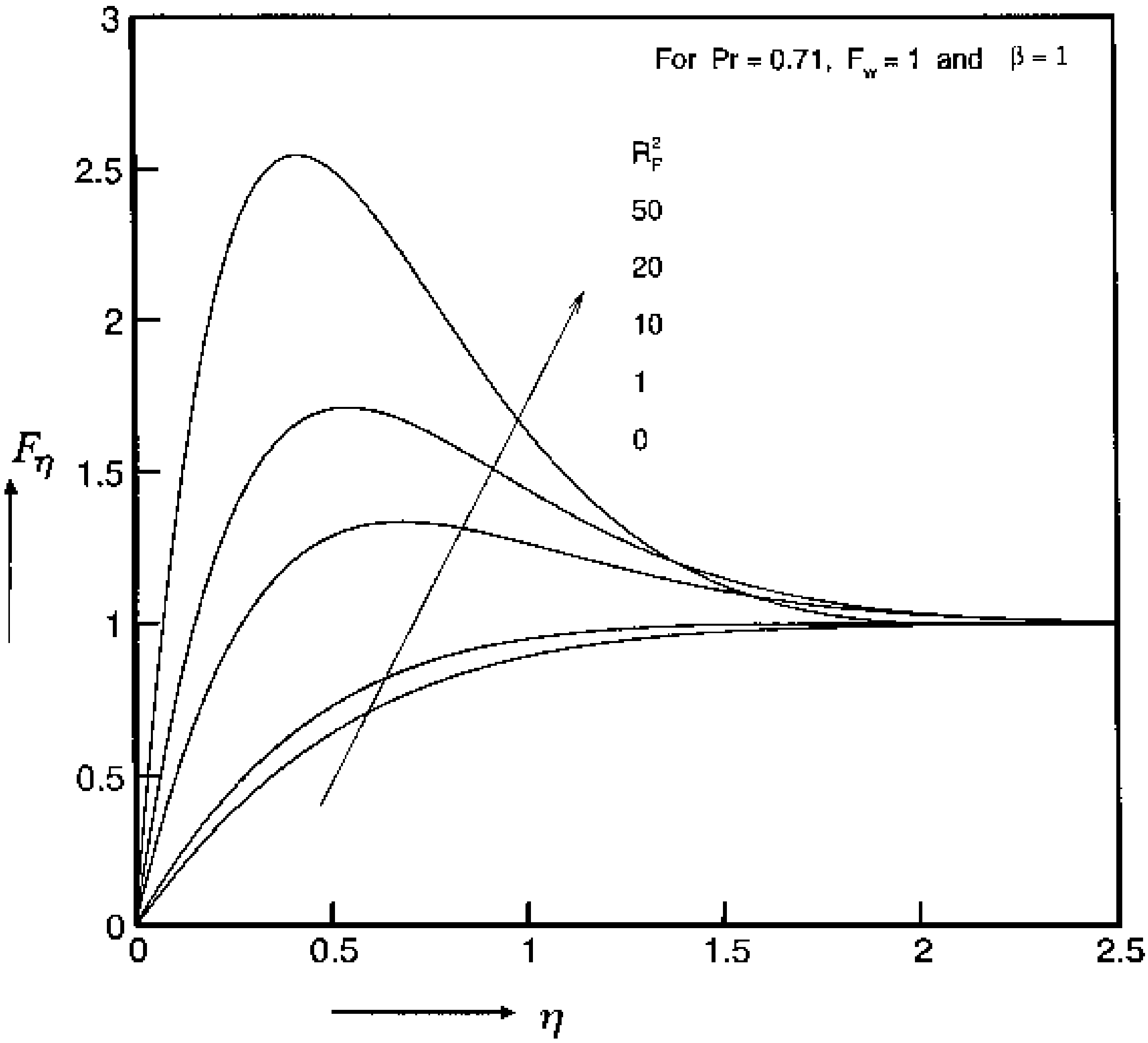

Fig.1 Effect of R_F^2 on velocity profiles.

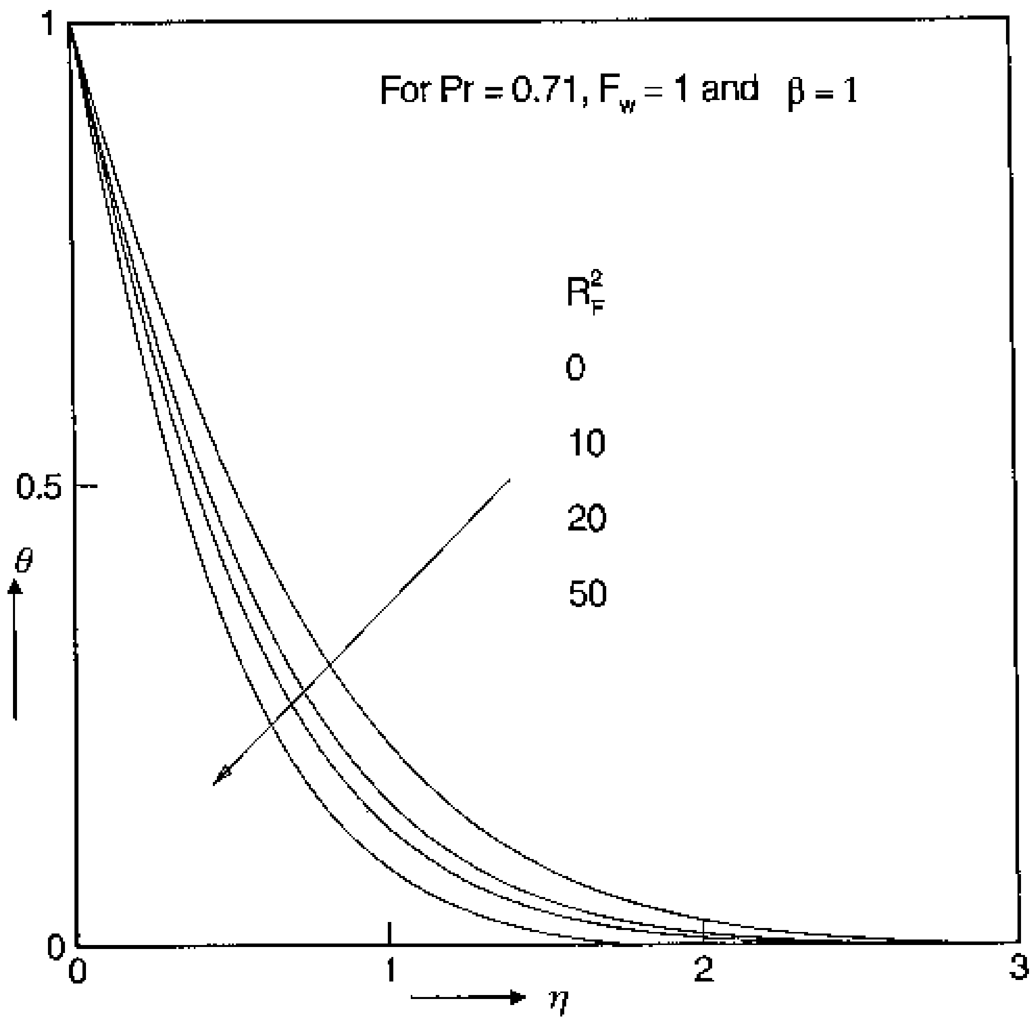

Fig.2 Effect of R_F^2 on temperature profiles.

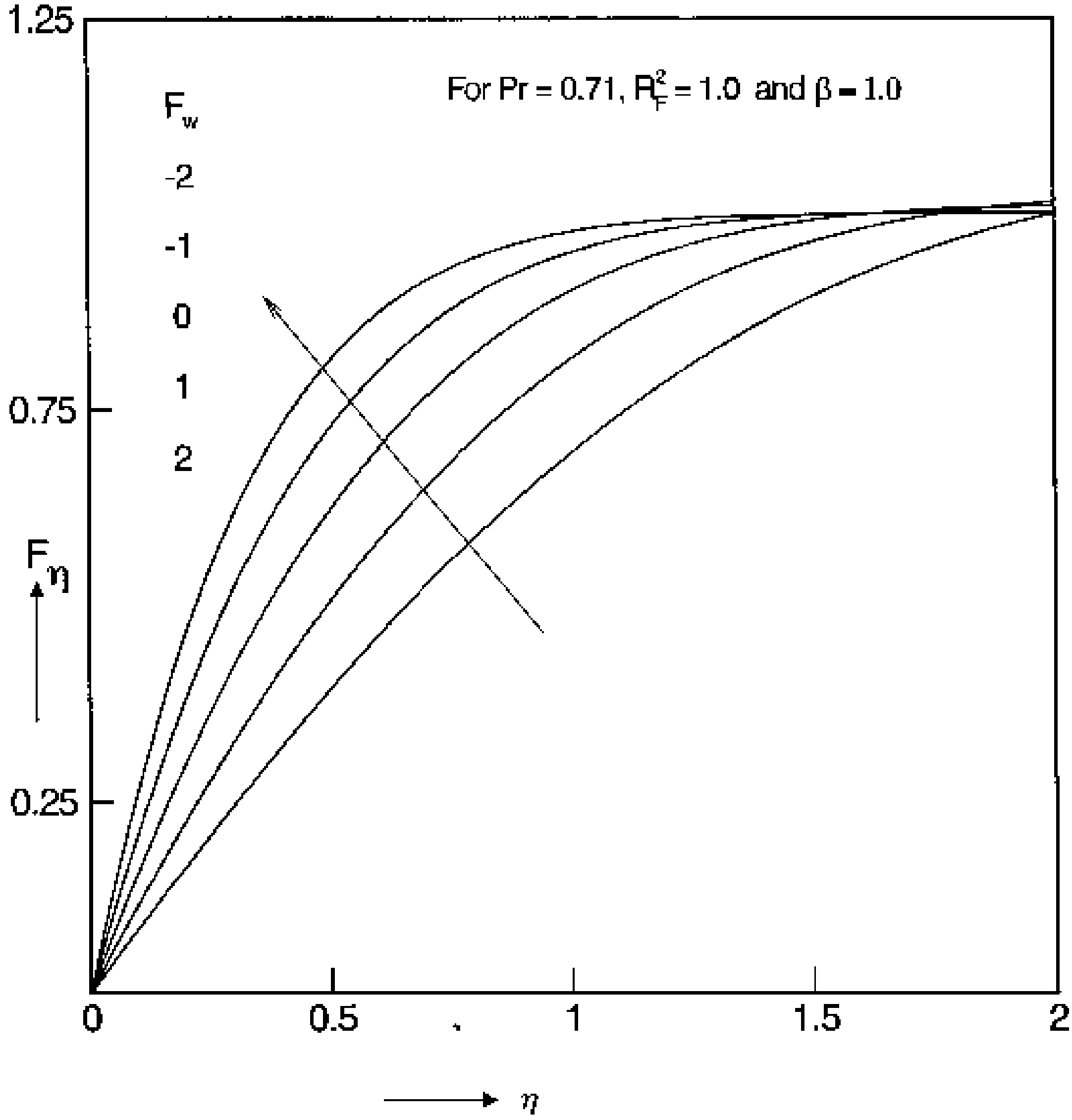

Fig.3 Effect of F_w on velocity profiles

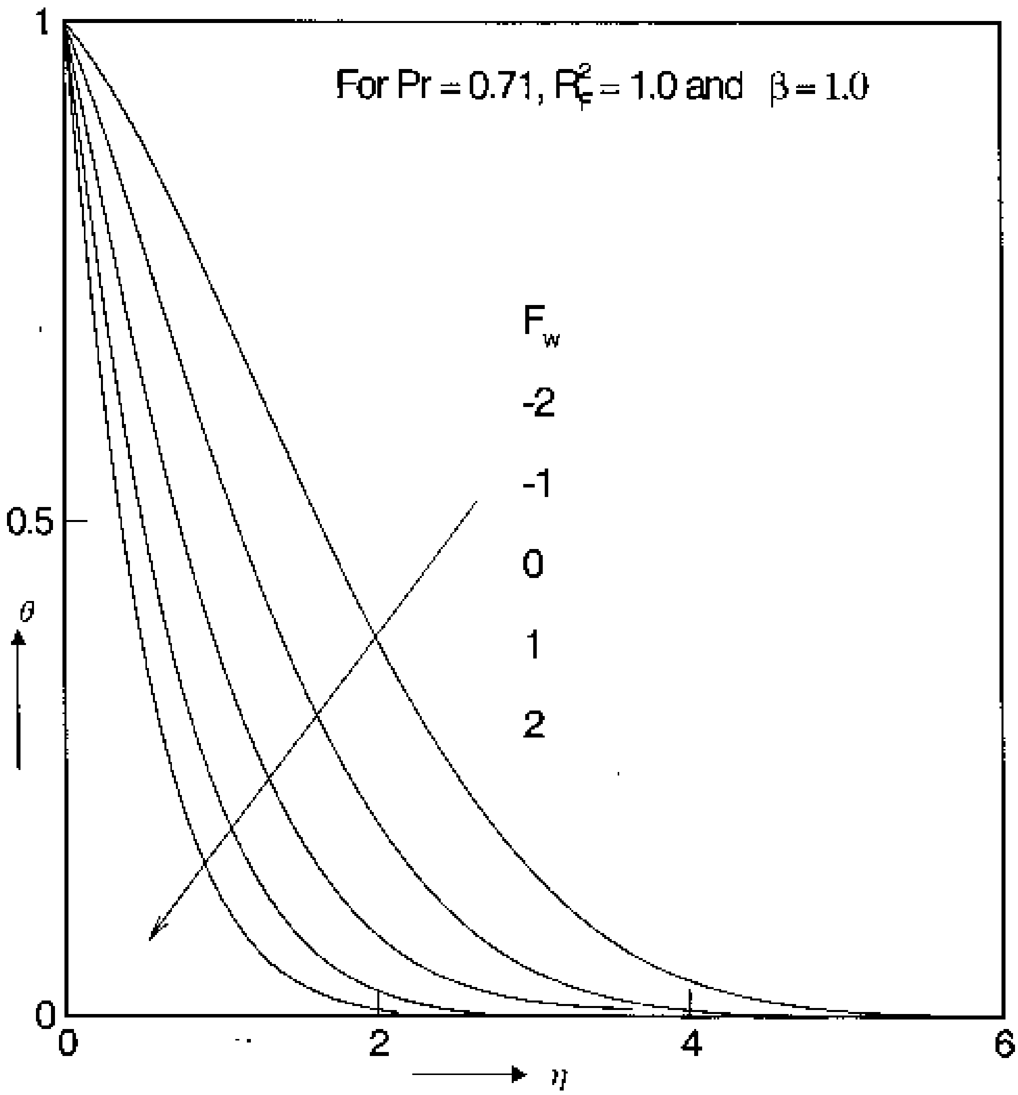

Fig.4 Effect of F_w on temperature profiles.

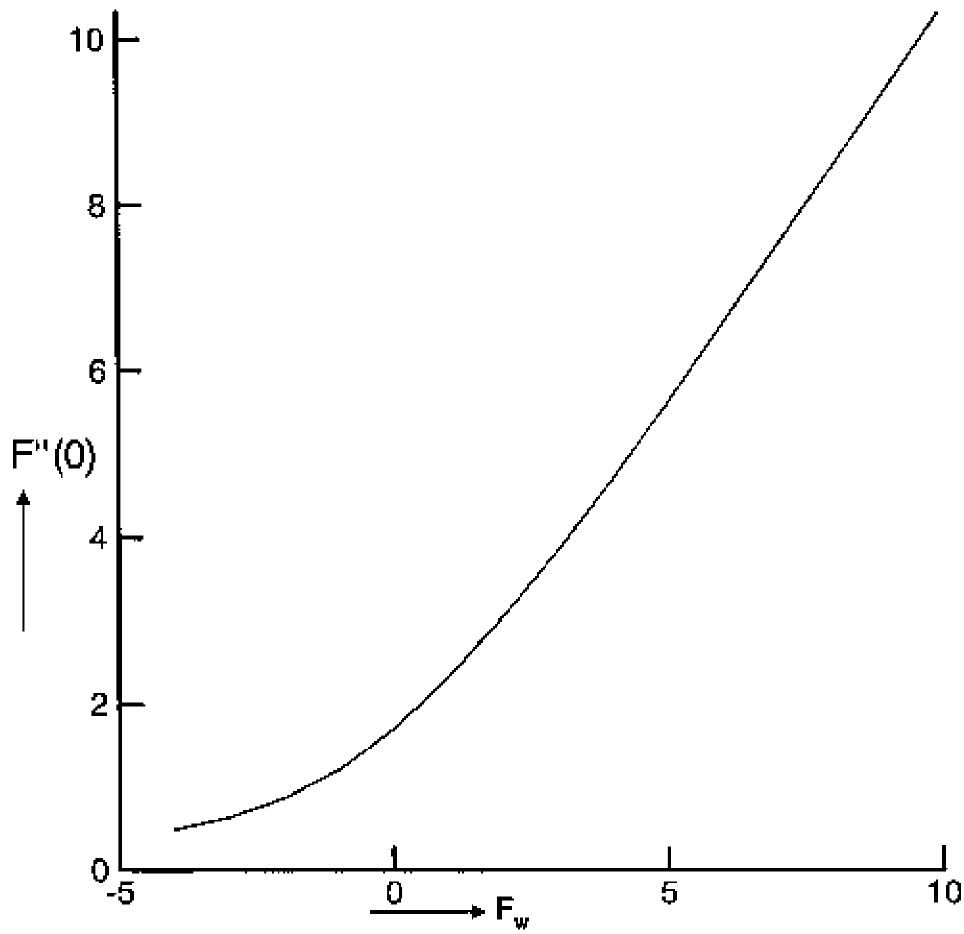

Figure.5 Effect of F_w on skin friction factor

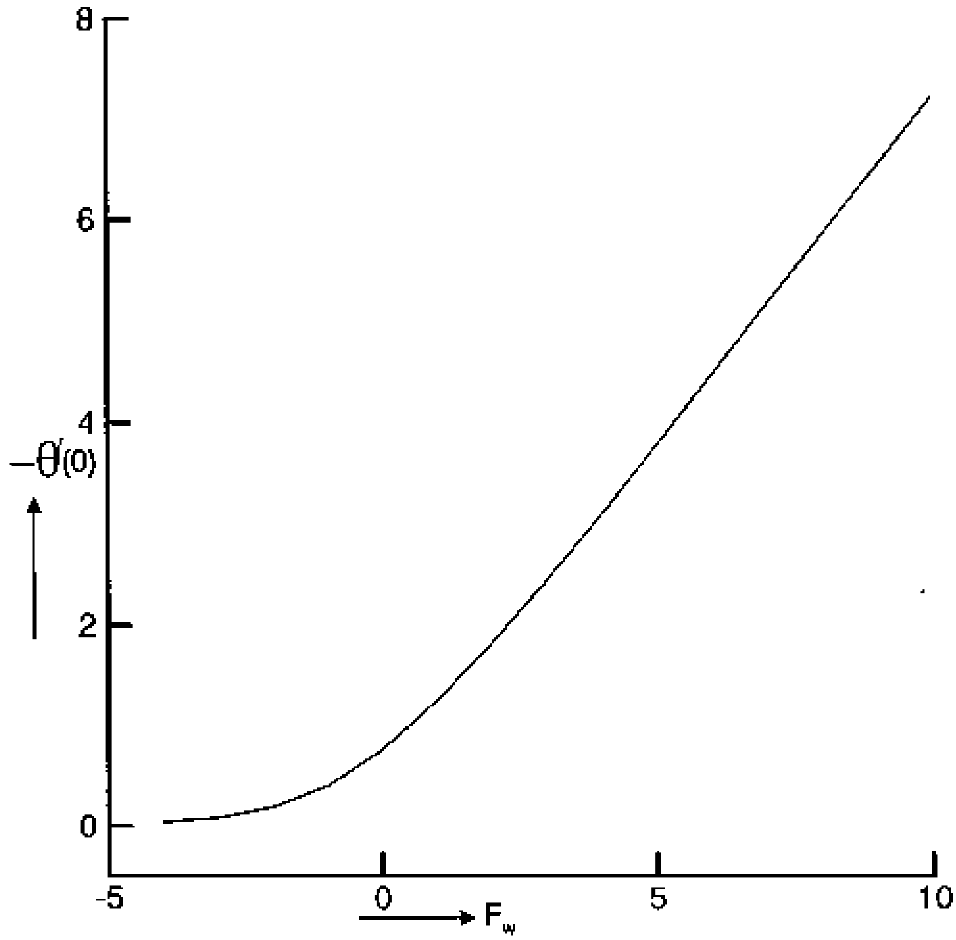

Figure.6 Effect of F_w on heat transfer factor

Discussions

In the present investigation, one set of nonlinear ordinary differential equation (3.2.14) and (3.2.15) is solved numerically by shooting method. The calculations are carried out for several values of the parameter F_w ($F_w > 0$ for suction and $F_w < 0$ for injection) for $\Pr = 0.71$, $R_F^2 = \frac{U_F^2}{u_e^2} = 1$ and $\beta = 1$.

Figure 1 represents velocity profiles to show the effect of the parameter $R_F^2 = \frac{U_F^2}{u_e^2}$ for $\Pr = 0.71$, $F_w = 1$ (suction) and $\beta = 1$. From the figure it can be concluded that velocity profiles increases as $R_F^2 = \frac{U_F^2}{u_e^2}$ increases.

Figure 2 represents temperature profiles to show the effect of the parameter $R_F^2 = \frac{U_F^2}{u_e^2}$ for $\Pr = 0.71$, $F_w = 1$(suction) and $\beta = 1$. From the figure it can be concluded that temperature profiles decreases as $R_F^2 = \frac{U_F^2}{u_e^2}$ increases.

Figure 3 represents velocity profiles to show the effect of the parameter F_w for $\Pr = 0.71$, $R_F^2 = 1$ and $\beta = 1$. From the figure it can be concluded that velocity profiles decreases as F_w increases.

Figure 4 represents temperature profiles to show the effect of the parameter F_w for $\Pr = 0.71$, $R_F^2 = 1$ and $\beta = 1$. From the figure it can be concluded that temperature profiles decrease as F_w increases.

From figure 5 we observe that the skin friction gradually increases with increasing of F_w.

Figure 6 shows that heat transfer rate increases remarkably with increase of F_w.

Chapter Five

Conclusion

The similarity solutions of unsteady mixed convection boundary layer equations over a flat porous vertical plate has been investigated by repeated applications of the method of one parameter continuous Group Theory. By applying group theory we have converted the governing partial differential equations into a pair of ordinary differential equations with appropriate boundary conditions. We have analyzed six possible cases for which similarity solutions exist. Out of six cases two cases were derived by **Zakerullah (2001)** without suction. It is found that our solutions are consistent with some of the published results in the literature. One set of the coupled nonlinear equations are solved numerically. This set is a purely steady one. In most practical purposes steady cases are generally dealt with.

The heat transfer and skin friction factors $\{q_w(0), \tau_w(0)\}$ are displayed and shown graphically for some values of the parameter F_W. It is shown that both the skin friction and heat transfer coefficient increases with suction and the effect of injection is just reverse.

It is desirable to solve certain classes of problem (but not all) by Group Theory method. Each problem has its own special features. So it requires a thorough knowledge of the happenings of the problem. The method of Group Theory may also be applied to certain classes of the boundary value problems for which the governing partial differential equations are expressed in spherical or cylindrical co-ordinates.

If a number of dependent and independent variable are present in the problem, first a group of independent similarity variables $\eta_1, \eta_2, \eta_3, \cdots$ are sought from the original independent variables and are one less in number. The η_i are absolute invariant. For each dependent variable, an absolute invariant g_i is sought which involve the dependent variable.

A good choice is $g_i = u_i h_i(x_1, x_2, \cdots, x_n)$, where u_i is the dependent variable.

The function g_i is then equated to a function

$$F(\eta_1, \eta_2, \cdots, \eta_{m-1}).$$

If $g_i = u_i h_i(x_1, x_2, \cdots, x_n)$ then

$$u_i = \frac{F_i(\eta_1, \eta_2, \cdots, \eta_{m-1})}{h_i(x_1, x_2, \cdots, x_m)}$$

is the dependent variable transformation. Substituting various transformations into the original system of equations, the new system stands with number of independent variable reduced by one.

Thus the reductions of variables in the problem carry more and more restrictions to develop various types of possible cases. It would be quite simple to investigate these possibilities. Finally, we may reach to a position to give the analytical solution of the problem under restricted conditions.

References

Abd-el-Malek, M.B. and Badran, N.A., (1990) "Group method analysis of unsteady free-convective boundary-layer flow on a nonisothermal vertical circular cylinder,"Acta. Mech., Vol. 24, No.4, pp.193-206.

Abd-el-Malek, M.B. and Badran, N.A., (1991) "Group method analysis of steady free-convective boundary-layer flow on a nonisothermal vertical circular cylinder," J. Comput. Appl.Math., Vol.36, No.2, pp.227-238.

Abd-el-Malek, M.B., (1997) "Group method analysis of nonlinear temperature variation across the Lake depth," Proc. XXI International Colloquium on Group Theoretical Methods in PHYSICS group 21(Goslar, Germany.15-20 July, 1996), World Scientific, Singapore, pp.225-260.

Abd-el-Malek, M.B., Boutrons, Y.Z. and Badran, N.A., (1990) "Group method analysis of unsteady free-convective boundary-layer flow on a nonisothermal vertical flat plat," J. Engr. Math., Vol.24, No.4, pp.343-368.

Aldoss, T.k., (1994) "Mixed convection over nonisothermal horizontal surfaces in porous medium; the entire regime," Num. Heat Mass Transfer, Vol. 25, pp.685-701.

Ames, W.F., and Nucci, M.C., (1985) "Analysis of fluid equations by Group methods," J. Engr. Math., Vol.20, pp.181-187.

Birkhoff, G., (1955) "Hydrodynamics; A study of Logic, Fact and Similitude," New York, Dover Publications.

Choaudhary, M.A., Merkin, J.H., and Pop, I., (1995) "Similarity solutions in free convection boundary layer flows adjacent to vertical permeable surface in porous media," Eur. Mech., B/Fluids, Vol.14.No.2 pp.217-237.

Cheng, P., (1977) "Combined free and Forced Convection Flow about inclined surface in porous media," Int. J. Heat Mass Transfer. Vol. 20, pp. 807-814.

Eichhorn, R., (1960) "The effect of mass transfer on free convection," J. Heat Transfer, Vol. 82, pp.260-263.

Hansen, A.G., (1958) "Possible Similarity solutions of laminar incompressible boundary layer equations," Trans A.S.M.E 80, Vol. 80, pp. 1553-1559.

Hansen, A.G., (1964) "Similarity Analysis of Boundary Layer Problems in Engineering,"Prentice-Hall,Inc., Englewood Cliff,New Jersey.

Hossain, M.A., and Mandal, A.C., (1985) "Unsteady hydro magnetic free convection flow past an accelerated infinite porous plate," Astrophysics and Space Science, Vol. 111, No.1, pp. 87-95.

Harris, S.D., Ingham, D.B., Pop, I., (1999) "Unsteady mixed convection boundary layer flow on a vertical surface in a porous medium," Int. J. Heat Mass Transfer, Vol. 42, pp.357-372.

Ludlow, D.k., Clarkson, P.A., and Bassom, A.P., (2000) "New similarity solutions of unsteady incompressible boundary layer equations," Quarterly Journal of Mech. Appl. Math., Vol. 53, pp.175-206.

Merkin, J.H., (1972) "Free Convection with Blowing and Suction," Int. J. Heat Mass Transfer, Vol. 15, pp. 989-999.

Merkin, J.H., (1969) "The effects of buoyancy forces on the boundary layer flow over a semi infinite vertical plate," Journal of Fluid Mech., Vol.35, pp.439-450.

Morgan, A.J.A., (1952) "The reduction of one of the Number of Independent Variables in Some System of Partial Differential equations," Quart. Journal of Math, Oxford (2) pp.250-259.

Merkin, J.H. and Ingham, D.B., (1996) "Conjugate free convection on a vertical surface," Int. J. Heat Mass Transfer, Vol. 39, pp.1527-1534.

Muloani, L and Rahman, M., (2000) "Similarity analysis for natural convection from a vertical plate with distributed wall concentration," Int. Journal of Math and Math Sci., Vol.35, No. 5, pp.319-334.

Merkin, J.H. and Ingham, D.B., (1987) "Mixed convection similarity solutions on a horizontal surface," ZAMP, Vol. 38, pp.102-115

Ostrach, S., (1953) "An analysis of laminar free convection flow and heat transfer about a flat plate parallel to the direction of the generating body force," NACA, Rept.1111, 1953.

Ostrach, S.,(1954) "Combined natural and forced convection laminar flows and heat transfer of fluids with and without heat sources in the channels with linearly varying wall temperature," NACA. Tech. Note. 3141

Pop, I., and Takhar, H.S., (1993) "Free convection from a curved surface," ZAMMP,Z.angew,Math.Mech. Vol. 736, pp.534-539.

Ramachandran, N., Chen, T.S. and Armaly, B.F., (1998) "Mixed convection in stagnation flows adjacent to a vertical surface," Journal of Heat Transfer, Vol.110, pp.373-377.

Rapits, A., and **Pedikis, C.P., (1998)** "Combined Forced and Free Convection flow through a porous medium," Int. Journal of Energy Research," Vol.12, pp.557-560.

Roslinda, N., Norsarahaida, A., and **Pop, I.,** (2004) "Unsteady mixed convection boundary layer flow near stagnation point on a vertical surface in a porous medium," Int. J. Heat Mass Transfer, Vol. 47, pp.2681-2688.

Soundalgeker,V.M and Wavre, P.W., (1977) "Unsteady Free Convection Flow past an infinite vertical plate with suction and mass transfer," Int. J. Heat Mass Transfer. Vol. 20, pp. 1363-1373.

Soundagekar, V.M., (1972) "Viscous dissipation on unsteady free convective flow past an infinite vertical porous plate with constant suction," Int. J. Heat Mass Transfer. Vol. 15, pp. 1253-1261.

Sparrow, E.M., Eichhorn, R., and Gregg J.L., (1959) "Similar boundary layer solutions for combined forced and free convection," Journal of Physics of Fluid, Vol.2, pp.319-328.

Schlicting (1978), Boundary layer theory, sixth edition, MaGraw-Hill, New York.

Wantanable, T., (1991) "Forced and Free Convection boundary layer flow with uniform suction or injection on vertical plate," Acta Mechanica, Vol.89, pp.123-132.

Williams, J.C., Mulligan, J.C., (1987) "Semi-similar solutions for unsteady free convection boundary layer flow on a vertical flat plate," Journal of fluid mechanics, Vol.175, pp. 309-332.

Yang, K.T., (1960) "Possible similarity solutions for laminar free convection on vertical plates and cylinders," J. Appl. Mech. Vol. 27, pp.230-236.

Zakerullah, Md., (2001) "Similarity Analysis," First Edition, Bangladesh University of Engineering and Technology, Dhaka-1000.

Zakerullah, Md., (1976) "Laminar natural and combined convection flows, Ph.D. Thesis," The Victoria University of Manchester, U.K.

Zubair, S.M.,and Kadaba, P.V., (1990) "Similarity Transformations for Boundary Layer Equations in Unsteady Mixed Convection," Int.Comm. Heat Mass Transfer, Vol.17, pp. 215-226.