RADIATION EFFECT ON MHD FREE CONVECTION BOUNDARY LAYER FLOW ALONG A VERTICAL POROUS PLATE WITH VARIABLE PLATE TEMPERATURE

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The thesis entitled

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I hereby certify that the work which is being presented in this dissertation entitled "Radiation effect on MHD free convection boundary layer flow along a vertical porous plate with variable plate temperature" submitted in partial fulfillment of the requirements for the award of the degree of Master of Philosophy in Mathematics in the Department of Mathematics, Bangladesh University of Engineering and Technology (BUET) , Dhaka-1000 is an authentic record of my own work.

The matter presented in this thesis has not been submitted else where (Universities and Institutions) for the award of any other degree.

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Abstract

In this thesis the effect of radiation on magnetohydrodynamic (MHD) free convection boundary layer flow along a vertical porous plate with variable plate temperature will be investigated. By using the appropriate transformations, the basic equations are transformed to non-dimensional boundary layer equations, which are solved numerically using finite –difference method. Here we have focused our attention on the evolution of the surface shear stress interms of local Nusselt number , velocity distribution as well as temperature distribution for a selection of parameter sets consisting of Radiation parameter (Rd), Magnetic parameter (M), exponent parameter (m), the Prandtl number (Pr) and Delta (Δ). The results of wall shear stress and the rate of heat transfer interms of the local skin friction coefficient and local Nusselt number, velocity distribution as well as temperature distribution have been shown graphically by using the software LAHEY FORTRAN 90 and TECHPLOT .

Nomenclature

a	: Rosseland mean absorption coefficient , m^{-1}
A	: constant
B	: Transverse magnetic field component, T
B_o	: Constant of proportionality of B
f	: Dimensionless stream function, m^2/s
g	: Acceleration dne to gravity, m/s^2
k	: Coefficient of thermal conductivity, W/mK
m	: Exponent parameter
M	: Magnetic field parameter
Pr	: Prandtl number
q_r	: Component of radiative flnx, W/m^2
Q	: Rate of heat transfer, W
R_d	: Radiation parameter, ms K^4 /W
T	: Temperature of the fluid, K
T_∞	: Temperature of the ambient fluid, K
T_w	: Surface temperature/plate temperature, K
u	: Velocity component in the x direction, m/s
v	: Velocity component in the y direction, m/s
V	: Represents the suction of velocity of fluid through the Surface of the plate/Uniform transpiration velocity, m/s
x	: Coordinate measuring distance along plate, m
y	: Coordinate measuring distance normal to plate, m
ΔT	: Temperature difference, K

Greek symbols

σ	: Electrical conductivity, s/m
σ_s	: scattering coefficient, m^{-1}
ψ	: Stream function, m^2/s
μ	: Viscosity of the fluid, kg/ms
ν	: Kinematic viscosity, m^2/s
τ	: Skin friction, kg/ms^2
β	: Coefficient of cubical / volume expansion, K^{-1}
η	: Similarity variable
α	: Coefficient of thermal diffusivity, m^2/s
θ	: Dimensionless temperature function
ρ	: Density of the ambient fluid, kg/m^3
ξ	: Scaled streamwise coordinate
Δ	: Surface temperature parameter, K

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Chapter 1

Introduction

The study of heat transfer is of great interest in many branches of science and engineering. In designing heat exchangers such as boilers, condensers and radiators etc, heat transfer analysis is essential for designing such equipment. For example, in the design of nuclear-reactor cores, a thorough heat transfer analysis of fuel elements is important for proper sizing of fuel element to prevent burnout. In aerospace technology, heat transfer problems are crucial because of weight limitations and safety considerations. In heating and air conditioning applications for buildings a proper heat transfer analysis is necessary to estimate the amount of insnlation needed to prevent excessive heat loses or gains.

The three distinct modes of heat transfer, namely conduction, convection and radiation must be considered. In reality, the combined effects of these three modes of heat transfer control temperatnre distribntion in a medium. Conduction occurs if energy exchange takes place from the region of high temperature to that of low temperature by the kinetic motion or direct impact of molecules, as in the case of fluid at rest, and by the drift of electrons, as in the case of metals. The radiation energy emitted by a body is transmitted in the space in the form of electromagnetic waves. Energy is emitted from a material due to its temperature level, being larger for a larger temperature, and is then transmitted to another surface which may be vacnum or a medium which may absorb, reflect or transmit the radiation depending on the nature and extent of the medium. Considerable effort has been directed at the convective mode of heat transfer. In this mode, relative motion of the fluid provides an additional mechanism for energy transfer. A study of convective heat transfer involves the mechanisms of conduction and, sometimes, those of radiation processes as well. This makes the study of convective mode a very complicated one.

Considerable attention has been given to magneto hydrodynamic (MHD) flows since the beginning of this centnry. The branch of science which incorporates with the motion of a highly condncting fluid in presence of a magnetic flied is called magneto hydrodynamics. The motion of the conducting fluid across the magnetic field generates electric currents which change the magnetic field and the action of the magnetic field on these currents give rise to mechanical forces, which modify the fluid. It is possible to attain equilibrium in a conducting fluid if the current is parallel to the magnetic field. For then the magnetic forces vanish and the equilibrium of the gas is the same as in the absence of magnetic fields, such magnetic fields are called force-free. But most liquids and gases are very poor electrically conductors. In the case when the conductor is either a liquid or a gas, electromagnetic forces will be generated which may be of the same order of magnitude as the hydro dynamical and inertial forces. Thus the equations of motion will have to take these electromagnetic forces into account as well as the other forces.

However, it is possible to make some gases very highly conducting by ionizing them. For ionization to take effect, the gas must be very hot at temperature upwards of 5000 K or so. Such ionized gases are called plasmas. The material within a star is plasma of very high conductivity and it exists within a strong magnetic field.

Solid matter is generally excluded from Magneto hydrodynamics effect, but it should be realized that the same principles apply in Electro hydrodynamics. Electrical conduction in metals and the Hall Effect are two examples. In an electric motor, the magnetic field produced by the armature current affects the operation of the motor in an important way, so that the mechanical and electrical analyses are coupled, just as in MHD. Electromagnetic forces are an essential part of motors and generators, though they generally do not produce significant elastic deformations, and the motions occur with the help of rotating and sliding contacts. Homo polar generators (ones that produce DC currents) are, indeed, closely related to Magneto- hydrodynamics analogous.

Magneto- hydrodynamic was originally applied to astrophysical and geophysical problems, where it is still very important, but more recently to the problem of fusion power, where the application is the creation and containment of hot plasmas by electromagnetic forces, since material walls would be destroyed. Astrophysical problems include solar structure, especially in the outer layers, the solar wind bathing the earth and other planets, and interstellar magnetic fields. The primary geophysical problem is planetary magnetism, produced by currents deep in the planet, a problem that has not been solved to any degree of satisfaction.

The inclusion of radiation terms is complicated and the resulting equation is very difficult to solve. Grief and Habib (1971) have shown that, in the optically thin limit, the physical situation can be simplified and they derived and exact solution of the problem of fully-developed radiating laminar convection flow in an infinite vertical heated channel. Their analysis was based on the work by Cogley et al. (1968). In the optically thin limit the fluid does not absorb its own emitted radiation but the fluid does absorb radiation emitted by the boundaries. It was shown by Cogley et al. (1968) that in optically thin limit for a gray –gas near equilibrium, the following relation holds:

$$\frac{\partial q_r}{\partial y} = 4(T - T_w)\int_0^\infty \kappa_{\lambda w}\left(\frac{\partial e_{b\lambda}}{\partial T}\right)_w d\lambda = 4(T - T_w)I,$$

where $I = \int_0^\infty \kappa_{\lambda w}\left(\frac{\partial e_{b\lambda}}{\partial T}\right)_w d\lambda$

Here q_r is the radiative flux , κ_λ is the absorption coefficient , $e_{b\lambda}$ is the Planck function and the subscript w represents the value of a qnantity at the wall. Fnrther simplification may be made concerning the spectral properties of radiating gases, but this is not essential for the present analysis. It should be mentioned that Soundalgekar and Takhar (1993) considered the radiative free convective flow of an optically thin grey-gas past a semi- infinite vertical plate.

But the Rosseland model is valid for isotropic local intensity and high optical density of the medium and the radiative heat flux is given by

$$q_r = -\frac{16\sigma \nabla T^3}{3(a + \sigma_s)}\nabla T$$

where T denotes the temperature, a is the Rosseland absorption coefficient , σ_s is the scattering coefficient and σ is the Stefan- Boltzman constant. The thermal boundary – layer equation can be written as

$$\rho C_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\frac{\partial}{\partial y}\left[\left(\frac{16\sigma T^3}{3a}+\kappa\right)\frac{\partial T}{\partial y}\right]$$

The problems of radiation effect on free or forced or mixed convection boundary layer flow over or on bodies of various shapes were discussed by many mathematicians, versed engineers and researchers. Amongst them the name of Rees et al. (1999), Hossain and Pop(2001), Arpaci(1968), Chamka(1997), Ganesan and Loganathan(2002), Yih(1999), Bankston(1977) are noteworthy.

Rees et al. (1999) considered the effect of radiation on free convection from a porous vertical plate. Radiation effects on free convection over a vertical flat plate embedded in a porous medium with high porosity investigated by Hossain and Pop(2001). Arpaci (1968) studied the effect of thermal radiation on the laminar free convection from a heated vertical plate. Solar radiation assisted natural convection in uniform porous medium supported by a vertical flat plate discussed by Chamka (1997). Ganesan and Loganathan (2002) investigated radiation and mass transfer effects on flow of an incompressible viscous fluid past a moving vertical cylinder. Radiation effect on natural convection over a vertical cylinder embedded in porous media studied by Yih (1999).Bankston et al. (1977) radiation convection interaction in an absorbing – emitting liquid in natural convection boundary layer flow. Hossain and Alim (1997) investigated natural convection - radiation interaction on boundary layer flow along a thin vertical cylinder. Interaction of thermal radiation with free convection heat transfer was investigated by Cess (1966). Hossain and Takhar (1996) studied radiation effect on mixed convection along a vertical plate with uniform surface temperature. Natural convection-radiation interaction boundary layer flow along a thin vertical cylinder investigated by Hossain and Alim (1997). Takhar et al.(1999) discnssed radiation interaction on forced and free convection across a horizontal cylinder. Effect of conduction-radiation interaction on the mixed convection flow from a horizontal cylinder studied by Pop et al. (1999). Hossain et al. (1998) investigated radiation-conduction interaction on mixed convection flow along a slender vertical cylinder. Radiative free convection flow of gas past a semi –infinite vertical plate was discussed by Soundalgekar and Takhar (1993). All the cases mentioned above, the radiation effect on MHD free convection in presence of transverse magnetic field has not been studied yet.

The effect of a magnetic field on free convection heat transfer was studied by Sparrow and Cess (1961) . Hossain et al. (1997) investigated MHD forced and free convection boundary layer flow along a vertical porous plate. Natural convection between heated vertical plates in horizontal magnetic fields was studied by Ostrle and Yound (1961). Wilks (1976) studied Magneto Hydrodynamics free convection about a semi-infinite vertical plate in a strong cross field. Free convection with mass transfer under the influence of a magnetic field was investigated by Cobble (1979). Poots (1961) considered laminar natural convection flow in magneto hydrodynamics. Steady and transient free convection of an electrically conducting fluid from a vertical plate in the presence of magnetic field was studied by Gupta (1961). Hossain and Ahmed (1990) investigated MHD forced and free convection boundary layer flow near the leading edge. All the cases mentioned above, radiation term was totally absent there.

Free convection induced by a vertical or horizontal wavy surface in a porous medium with uniform heat flux or without heat flux was studied by Rees and Pop (1994) . Non-Darcy natural convection from a vertical wavy surface in a porous medium was also studied by them (1995). In (1997) they also considered the effect of longitudinal surface waves on free convection from vertical surfaces in porous media. Rees (1999) showed the effect of steady stream wise surface temperature variation on free convection. Mixed convection boundary layer flow along a vertical cylinder was investigated by Hossain et al.(1998). Sparrow and Gregg (1956) studied laminar free convection from a vertical flat plate with uniform Surface Heat Flux. Combined forced and free convection in a boundary layer flow was investigated by Sparrow et al.(1959). Pozzi and Lupo (1988) discussed the coupling of conduction with laminar natural convection along a flat plate. Hydro magnetic free convection for high and low Prandtl numbers was investigated by Nanda and Mohanty (1970). Miyamoto et al. (1980) studied the effect of axial heat conduction in a vertical flat plate on free convection heat transfer. Natural convection of an electrically conducting fluid in the presence of a magnetic field was investigated by Lykoudis (1962). Cheng (1977) studied combined free and forced convection flow about inclined surfaces in porous media. Mixed convection in boundary layer flow on a horizontal plate was discussed by Chen et al. (1977). Lin (1976) has shown laminar free convection from a vertical cone with uniform surface heat flux. Deviation from classical free convection boundary layer theory at low Prandtl numbers was studied by Sparrow and Guinle (1968). Keller and Cebeci (1971) have discussed accurate numerical methods for boundary layer two dimensional flows. Mixed convection along a wavy surface was studied by Moulic and Yao (1989).They also studied natural convection along a vertical wavy surface with uniform heat flux. Merkin (1997) investigated mixed convection from a horizontal circular cylinder. In all the cases mentioned above, radiation term and magnetic term were totally absent there.

In chapter-1 reported results regarding radiation effect on MHD free convection have been discussed from both analytical and numerical point of view. In chapter-2, a steady laminar free convective boundary layer flow with variable plate temperature and magnetic parameter including radiation parameter effects is considered. In chapter-3, a two dimensional laminar MHD free convection boundary layer flow with radiation from a vertical porous plate with presence Prandtl and magnetic parameters including joule heating parameter effects is considered. The above two problems have been solved numerically using a most practical, an efficient and accurate solution technique, known as implicit finite difference method together with Keller-box elimination technique which is well documented and widely used by Keller and Cebeci (1971) and recently by Hossain (1992).The effects of various parameters i.e. the radiation parameter (Rd), the magnetic parameter (M), Prandtl number (Pr), the exponent parameter (m), the surface temperature parameter (Δ) entering into the problems are discussed with the help of graphs. We have presented a general conclusion in Chapter-4 of the model studied. At the end all references extracted in the thesis can be found at the last part of the thesis.

Chapter-2

Radiation effect on magneto hydrodynamic free convection flow along a vertical porous plate with variable plate temperature.

2.1 Introduction

In this chapter radiation effect on free convection flow along a vertical porous plate in a steady two – dimensional viscous incompressible fluid in presence of magnetic field with variable plate temperature is considered. Using the appropriate transformations, the governing boundary layer equations are transformed into a non- dimensional form, which are solved numerically using a very efficient implicit finite difference method. We have represented the effect of radiation parameter *(Rd)*, magnetic parameter *(M)*, Prandtl number *(Pr)*, the exponent parameter *(m)*, the surface temperature parameter *(Δ)* on the velocity and temperature including the skin friction coefficient and the rate of heat transfer. The Prandtl number Pr is to be taken 0.76, 0.88, 1.0 which correspond to CO_2, NH_3 and H_2O vapor.

2.2 Governing equations of the flow

The mathematical statement of the basic conservation laws of mass , momentum and energy for the steady viscous and incompressible and electrically conducting are given by

$$\nabla . q = 0 \tag{2.1}$$

$$\rho (q.\nabla) q = -\nabla p + \mu \nabla^2 q + F + J \times B \tag{2.2}$$

$$\rho c_p (q.\nabla) T = k \nabla^2 T \tag{2.3}$$

where $q = (u, v)$, u and v are the velocity components along the x and y axes respectively, F is the body force per unit volume which is defined as $-\rho g$, the terms $\boldsymbol{J}$ and $\boldsymbol{B}$ are respectively the current density and magnetic induction vector and the term $\boldsymbol{J} \times \boldsymbol{B}$ is the force on the fluid per unit volume produced by the interaction of current and magnetic field in the absence of excess charges . T is the temperature of the fluid in the boundary layer, g is the acceleration due to gravity, k is the thermal conductivity, c_p is the specific heat at constant pressure and μ is the viscosity of the fluid. For the energy equation we neglect the viscous dissipation and the Joule heating term. Here $\boldsymbol{B} = \mu_e H_0$, μ_e being the magnetic permeability of the fluid, H_o is the applied magnetic field and ∇ is the vector differential operator and is defined by

$$\nabla = \hat{i}_x \frac{\partial}{\partial x} + \hat{i}_y \frac{\partial}{\partial y}$$

where $\hat{i}_x$ and the $\hat{i}_y$ are the unit vector along x and y axes respectively.

When the external electric field is zero and the induced electric field is negligible, the current density is related to the velocity by Ohm's law as follows

$$J = \sigma (q \times B) \tag{2.4}$$

where σ denotes the electric conductivity of the fluid . Next under the condnction that the magnetic Renold's number is small , the induced magnetic field is negligible compared with applied field . This condition is usually well satisfied in terrestrial applications , especially so in (low velocity) free convection flows . So we can write

$$B = \hat{i}_y H_0 \tag{2.5}$$

Bringing together equations (2.4) and (2.5) the force per unit volume **J** × **B** acting along the x-axis takes the form:

$$J \times B = -\sigma H_0^2 u \tag{2.6}$$

Under the Boussincsq approximation, the variation of ρ is taken into only in the term **F** in equation (2.1) and the variation of ρ in the inertia term is neglected . We then can write:

$$\rho = \rho_\infty \left[1 - \beta(T - T_\infty)\right] \tag{2.7}$$

where ρ_∞ and T_∞ are the density and temperature respectively out side boundary layer, β is the coefficient of thermal expansion.

We consider the steady two dimensional laminar free convection boundary layer flow of an optically dense viscous and incompressible and electrically conducting fluid along a vertical plate in a porous medium as shown in fig-2.1. We assume that the plate is heated to a constant temperature T_w, which is higher than that the temperature T_∞ of the ambient fluid. The flow configuration and the coordinate system are shown on figure 2.1.

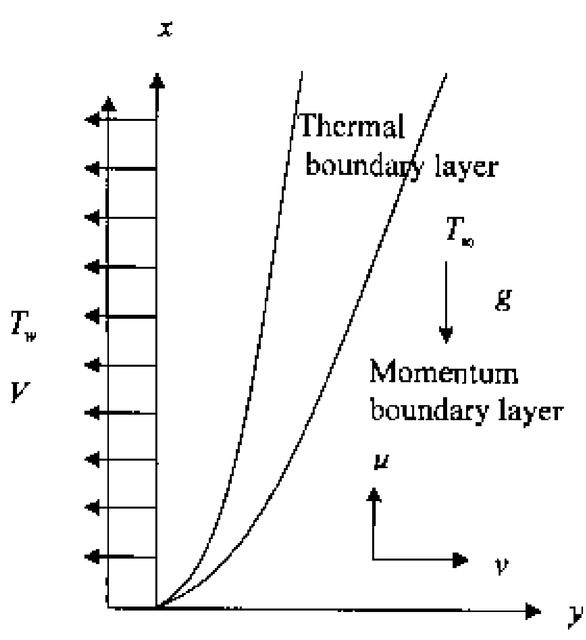

Fig 2.1: Physical model and coordinate systems.

Under the usual Boussinesq approximation and using the equations (2.4) to (2.6) with respect to the above consideration into the basic equations (2.1) to (2.3), the conservation equations for the steady two dimensional laminar free convection boundary layer flow of a viscous incompressible and electrically conducting fluid with viscosity and also constant thermal conductivity and thermal expansion coefficient past a vertical porous plate take the following form.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.8}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta(T-T_\infty)-\frac{\sigma B_0{}^2 u}{\rho} \tag{2.9}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial y^2}-\frac{1}{k}\frac{\partial}{\partial y}(q_r)\right) \tag{2.10}$$

The appropriate boundary conditions to be satisfied by the above equations are

$$\begin{aligned} &x=0,\quad y>0 \;:\; u=0,\quad T=T_\infty \\ &y=0,\quad x>0 \;:\; u=0,\quad v=-V,\quad T=T_w \\ &y\to\infty, x>0 \;:\; u=0,\quad T=T_\infty \end{aligned} \tag{2.11}$$

In equation (2.11) V represents the suction velocity of the fluid through the surface of the plate. In this chapter we shall consider only the suction case (rather than blowing) and therefore , V is taken as positive throughout, α is the equivalent thermal diffusivity, k is the thermal conductivity and the quantity q_r represents the radiative heat flux in the y direction. In order to rednce the complexity of the problem, we will consider the optically thick radiation limit in the present analysis. Thus, radiative heat flux term is simplified by the Rosseland diffusion approximation proposed by Siegel and Howell (1972) for an optically thick fluid according to

$$q_r=\frac{-4\sigma}{3(a+\sigma_s)}\frac{\partial T^4}{\partial y} \tag{2.12}$$

where σ is the Stefan-Boltzman constant and σ_s is the scattering coefficient and a is the Rosseland mean absorption coefficient. This approximation is valid at points optically far from the boundary surface and is good only for intensive absorption , that is , for an optically thick boundary layer. We observed that the equations (2.8-2.10) together with the boundary condition (2.11) and equation (2.12) are non-linear, partial differential equations. In the following sections the solutions methods of these equations are discussed in details.

2.3 Transformations of the governing equations

In equation (2.11) V represents the suction velocity of the fluid through the surface of the plate. In this chapter we shall consider only the suction case (rather than blowing) and therefore , V is taken as positive throughout. Near the leading edge, the boundary layer is much like that of the free convection boundary layer in the absence of suction, although much further downstream suction it will be found to dominate the flow. Also the magneto hydrodynamic field in the fluid is governed by the boundary layer equations .Therefore to solve the equations (2.8-2.10) subject to the boundary conditions (2.11) the following group of transformations are introduced for the flow starting from up stream to downstream.

$$\eta = \frac{Vy}{\nu\xi}, \qquad \xi = V\left\{\frac{4x}{\upsilon^2 g\beta(T_w - T_\infty)}\right\}^{\frac{1}{4}} \qquad ,\psi = V^{-3}\upsilon^2 g\beta(T_w - T_\infty)\xi^3\left\{f(\xi,\eta) \pm \frac{1}{4}\xi\right\}$$

$$\frac{T - T_\infty}{T_w - T_\infty} = \theta(\xi,\eta), \qquad \Delta = \frac{T_w}{T_\infty} - 1, \qquad Rd = \frac{4\sigma T_\infty^3}{k(a+\sigma_s)}, \qquad T_w - T_\infty = Ax^m \tag{2.13}$$

where η and ξ is the dimensionless similarity variable, ψ is the stream function which satisfies the equation of continuity and $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$, A is constant , Δ is the surface temperature parameter, Rd is the radiation parameter , $\theta(\eta,\xi)$ is the dimensionless temperature, m is the exponent parameter. Substituting equation (2.13) into equations (2.9) and (2.10), we get the following transformed non dimensional equations.

$$f''' + (3+m) f f'' - 2(1+m) f'^2 + \theta - Mf' + \xi f'' = \xi(1-m)\left\{ f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right\} \quad (2.14)$$

$$\Pr^{-1} \theta'' + (3+m) f \theta' + \xi \theta' - 4m\theta f' + \Pr^{-1}\left\{ \frac{4}{3} Rd \, (1+\Delta\theta)^3 \theta' \right\}' = \xi(1-m)\left\{ f' \frac{\partial \theta}{\partial \xi} - \theta' \frac{\partial f}{\partial \xi} \right\}$$

(2.15)

In the above equations the primes denote differentiation with respect to η.

The boundary conditions (2.11) then take the following form:

$$\begin{aligned} & f = 0 \quad , \frac{\partial f}{\partial \eta} = 0 \quad , \theta = 1 \quad at \quad \eta = 0 \\ & \frac{\partial f}{\partial \eta} = 0 \quad , \theta = 0 \quad as \quad \eta \to \infty \end{aligned} \quad (2.16)$$

where Pr is the Prandtl number, $M = \left(\sigma B^2 \Big/ \sqrt{\rho^2 g \beta (T_w - T_0)} \right)$ where $B^2 = 2 B_0{}^2 \sqrt{x}$

is the magnetic parameter.

The solutions of equation (2.14)-(2.15) enable us to calculate the skin friction τ and the rate of heat transfer Q at the surface of the plate from

$$\tau = \left(\frac{V}{g\beta(T_w - T_\infty)} \right) \left(\frac{\partial u}{\partial y} \right)_{y=0} = \xi f''(\xi, 0) \quad (2.17)$$

$$Q = -\left(\frac{v}{V(T_w - T_\infty)} \right) \left(\frac{\partial T}{\partial y} \right)_{y=0} = -\left(1 + \frac{4}{3} Rd \, \Delta^3 \right) \xi^{-1} g'(\xi, 0) \quad (2.18)$$

2.4 The important dimensionless parameters related to problem:

The governing equations of the fluid flow are discussed at the previons section in this chapter. These eqnations contain a number of variables. It is difficult to study the effect of each variable on the process. Moreover these equations are nonlinear. There is no general method to find the solution of these nonlinear equations. In order to bring out the essential features of flow, it is necessary to find important dimensionless parameters, which characterize the flow. These parameters are very useful in the analysis of experimental results. Some non- dimensional parameters related to our problems are discussed below:

Prandtl number P_r

Prandtl gave an important number known as Prandtl number. The Prandtl number is a dimensionless parameter of a convective system that characterizes the regime of convection. It is the ratio of viscous force to the thermal force and is defined follows:

$$P_r = \frac{Viscous\ force}{Thermal\ force}$$

$$= \frac{Kinematic\ viscosity}{Thermal\ diffusivity}$$

$$= \frac{\mu/\rho}{\kappa/\rho C_p} = \frac{\nu}{\alpha}$$

The Prandtl number is larger when thermal conductivity is small and viscosity is large, and small when viscosity is small and thermal conductivity is large .For small value of ν, a thin region will be affected by viscosity, which is known as boundary layer region. For the small value of $\kappa/\rho C_p$, a thin region will be affected by heat conduction which is known as the thermal boundary layer .Prandtl number also gives the relative importance of viscous dissipation to the thermal dissipation. Thus it represents the relative importance of momentum and energy by the diffusion process is comparable. For oils, Pr >>1, hence the momentum diffusion is much greater than the energy diffusion; but for liquid metals, Pr<<1 and the solution is reversed.

Magnetic parameter M

The magnetic parameter is obtained from the ratio of the magnetic force to the inertia force and is define as

$$M = \frac{\sigma B_0^2 L}{\rho \Omega}$$

If this of the order one then the magnetic force is important and the flow is to be considered as hydromagnetic flow .If it is very much less than one, then the flow cal be taken as hydrodynamic. For small value of M, the motion is hardly affected by the magnetic field and for large value of M, the motion is largely controlled by the magnetic field.

Radiation parameter Rd

The radiation parameter is obtained from the radiative heat flux term which is simplified by the Rosseland diffusion approximation proposed by Siegel and Howell (1972) for an optically thick fluid according to

$$q_r = \frac{-4\sigma}{3(a+\sigma_s)} \frac{\partial T^4}{\partial y}$$

From this one can get radiation parameter which is defined as

$$Rd = \frac{4\sigma T_0^3}{k(a+\sigma_s)}$$

where σ is the Stefan-Boltzman constant and σ_s is the scattering coefficient and a is the Rosseland mean absorption coefficient and k is the thermal conductivity. This approximation is valid at points optically far from the boundary surface and is good only for intensive absorption, that is, for an optically thick boundary layer. For small or large value of Rd, the motion is always affected and controlled by the radiation field.

2.5 Results and discussion:

The effect of radiation on magnetohydrodynamic natural convection flow along a vertical porous plate with variable plate temperature has been investigated. The results are obtained in terms of the local skin friction and the local rate of heat transfer for different values of the aforementioned physical parameters and these are shown graphically in Figs.2.4-2.11.The velocity and temperature distributions obtained by the finite difference method for various values of the governing parameters are displayed graphically in Figs.2.12-2.19.

First of all, to verify the proper treatment for the problem, the present solution for $M = 0.0$ and $m = 0.0$ has been compared with that of M.A. Hossain (1999). It can be seen from the Fig.2.2 and Fig. 2.3 that the present results are in excellent agreement with M.A. Hossain.

The effects of varying the radiation parameter Rd on both the skin- friction coefficient $f''(\xi,0)$ and the local rate of heat transfer $\theta'(\xi,0)$ are shown in Fig. 2.4 and Fig.2.5. for the fluid having Prandtl number Pr =0.73, the magnetic parameter $M = 1.0$, the exponent parameter m =0.2 and the surface temperature parameter $\Delta = 0.1$.The figures show that the changes in Rd lead to changes in the asymptotic value of the skin-friction but not in the asymptotic rate of heat transfer. In general, however both the skin- friction and the local rate of heat transfer increase as Rd increases.

The skin- friction coefficient $f''(\xi,0)$ and the local rate of heat transfer $\theta'(\xi,0)$ for different values of the Prandtl number Pr (= 0.76 , 0.88 , 1.0) when the value of the radiation parameter Rd =1.0, the magnetic parameter M =0.5, the exponent parameter m =0.1 and the surface temperature parameter Δ= 0.1 are depicted in Fig.2.6 and Fig.2.7.The values of the Prandtl number Pr are taken to 0.76 that corresponds physically for CO_2 in the temperature range 100-650°F, 0.88 that corresponds for NH_3 vapor in the temperature range 120-400°F and 1.0 corresponding to water vapor in the temperature range 220-900°F. In Fig. 2.6 and Fig. 2.7 it can be seen that as Prandtl number Pr increases, the skin friction coefficient $f''(\xi,0)$ decreases but the rate of heat transfer $\theta'(\xi,0)$ increases.

Numerical values of the skin friction coefficient $f''(\xi,0)$ and the rate of heat transfer $\theta'(\xi,0)$are depicted graphically in Fig.2.8 and Fig.2.9 respectively for different values of the magnetic M (=0.1, 0.5, 0.8. 1.0) for the fluid having Prandtl number Pr = 0.73, the radiation parameter Rd =1.0, the exponent parameter m =0.25 and the surface temperature parameter Δ = 0.1. It is observed that as the magnetic parameter M increases, both the skin friction coefficient $f''(\xi,0)$ and the rate of heat transfer $\theta'(\xi,0)$decrease.

In Fig 2.10 and 2.11, the skin friction coefficient $f''(\xi,0)$ and the rate of heat transfer $\theta'(\xi,0)$ are shown graphically for different values of exponent parameter m (=0 .1, 0.15, 0.25) when Prandtl number Pr =0.73, the radiation parameter Rd = 2.0, the magnetic parameter M =1.0, the surface temperature parameter Δ =0.1. It is observed that the skin friction coefficient $f''(\xi,0)$ decreases with the increase of the exponent parameter m (=0.1, 0.15, 0.25) and as the exponent parameter m increases, the rate of heat transfer $\theta'(\xi,0)$ increases also.

Fig. 2.12 and Fig.2.13 deal with the effect of the radiation parameter Rd (= 1.0, 2.0, 3.0, 4.0, 5.0) for Prandtl number Pr = 0.73, the magnetic parameter M =1.0, the exponent parameter m =0.2 and the surface temperature parameter Δ= 0.1 on the velocity profile $f'(\xi,\eta)$ and the temperatnre profile $\theta(\xi,\eta)$.From Fig. 2.12 it is revealed that the velocity profile $f'(\xi,\eta)$ increases as the values of radiation parameter Rd increases. Near the surface of the plate velocity increases significantly along η and becomes maximum and then decreases slowly and finally approaches to zero, the asymptotic value. From Fig.2.13 it is seen that when the values of the radiation parameter Rd increase, the temperature profile $\theta(\xi,\eta)$ also increase.

Fig. 2.14 depicts that the velocity profile $f'(\xi,\eta)$ for different values of the Prandtl number Pr (=0.76, 0.88, 1.0) while the radiation parameter Rd = 1.0, the magnetic parameter M=0.5, the exponent parameter m =0.1 and the surface temperature parameter Δ= 0.1. Corresponding distribution of the temperature profile $\theta(\xi,\eta)$ in the fluids is shown in Fig. 2.15. From Fig. 2.14, it can be seen that if the Prandtl number increases, the velocity of the fluid decreases. On the other hand, from Fig. 2.15 we observe that the temperature profile decreases within the boundary layer due to increase of the Prandtl number Pr.

Figs. 2.16 and 2.17 display results for the velocity and temperature profiles for different values of magnetic parameter M (=0.1, 0.5, 0.8, 1.0) against η for the fluid having Prandtl number Pr = 0.73, the radiation parameter Rd =1.0, the exponent parameter m =0.25 and the surface temperature parameter Δ = 0.1. It is observed that, as the magnetic parameter increases, the velocity profile decreases between $0 \le \eta \le 3.5$ and then increases with very small difference and finally approaches to zero along η direction. From Fig.2.17 it is seen that when the values of the magnetic parameter M increase, the temperature profile $\theta(\xi,\eta)$ also slightly increase.

Again in Fig. 2.18 and Fig. 2.19, the velocity profile $f'(\xi,\eta)$ and the temperature profile $\theta(\xi,\eta)$ are shown graphically for different values of the exponent m (=0.1, 0.15, 0.25) when the radiation parameter Rd=2.0, the Prandtl number Pr = 0.73, the magnetic parameter M=1.0 and the surface temperature parameter Δ = 0.1. From Fig. 2.18 it is observed that if the exponent m increases the velocity of the fluid decreases and from Fig.2.19 it can be seen that the temperature profile decreases due to increase of the exponent parameter m.

2.6 Conclusion

The effect of radiation on magneto hydrodynamic MHD free convection along a vertical porous plate with variable plate temperature has been investigated introducing a new class of transformations. Numerical solutions of the equations governing the flow are obtained by using the very efficient implicit finite difference method together with the Keller Box scheme. From the present investigation, the following conclusions may be drawn:

1. The skin friction coefficient and the rate of heat transfer increase for increasing value of the radiation parameter *Rd*.
2. Increased value of the radiation parameter *Rd* leads to increase the velocity distribution as well as the temperature distribution.
3. An increasing value of Prandtl number *Pr* leads to decrease the skin friction coefficient but increase the rate of heat transfer.
4. As Prandtl number *Pr* increases, both the velocity and the temperature distributions decrease significantly.
5. With increased of magnetic parameter *M*, the skin friction coefficient and the rate of heat transfer decrease.
6. An increase in values of *M* leads to decrease the velocity distribution but slightly increase the temperature distribution.
7. Increased value of the exponent parameter *m* leads to decrease the skin friction coefficient but increase the rate of heat transfer and as the exponent parameter *m* increases, both the velocity and the temperature distributions decrease significantly.

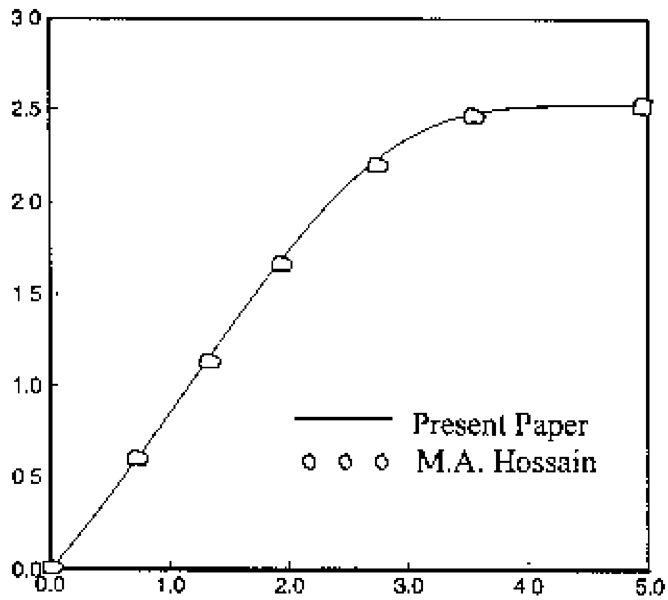

Fig.2.2 : Comparision of skin frictions when Pr = 1.0, Rd = 1.0 ,Δ = 0.1 and M =0.0 and m = 0.0 .

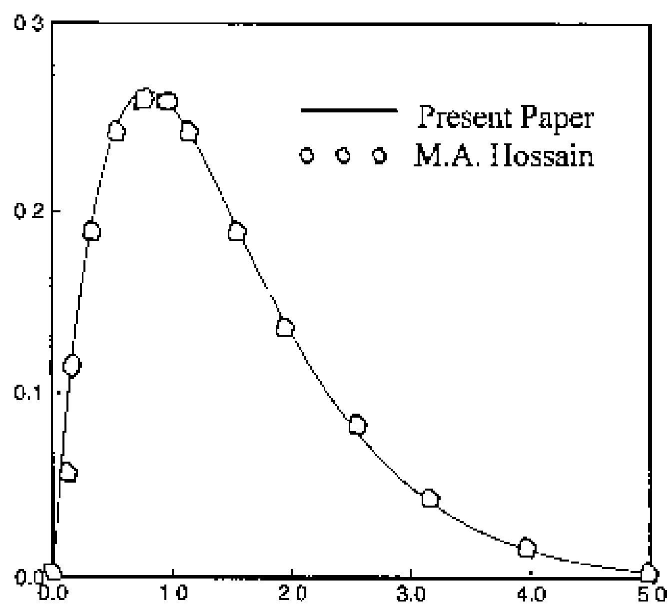

Fig.2.3 : Comparision of velocity profiles when Pr =1.0, Rd =1.0, Δ = 0.1 and M =0.0 and m=0.0 .

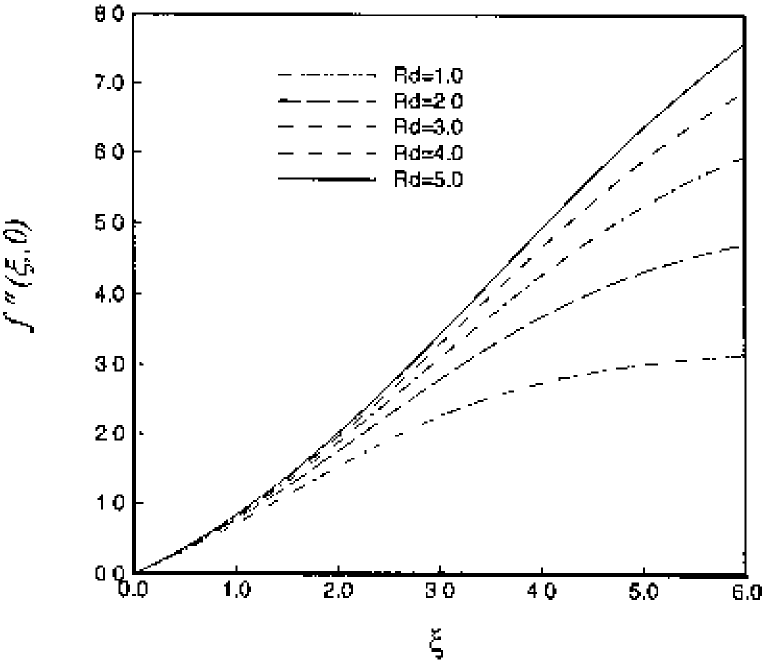

Fig.2.4: Skin friction for different values of Rd when Pr=0.73, M=1.0, m= 0.2 and Δ=0.1.

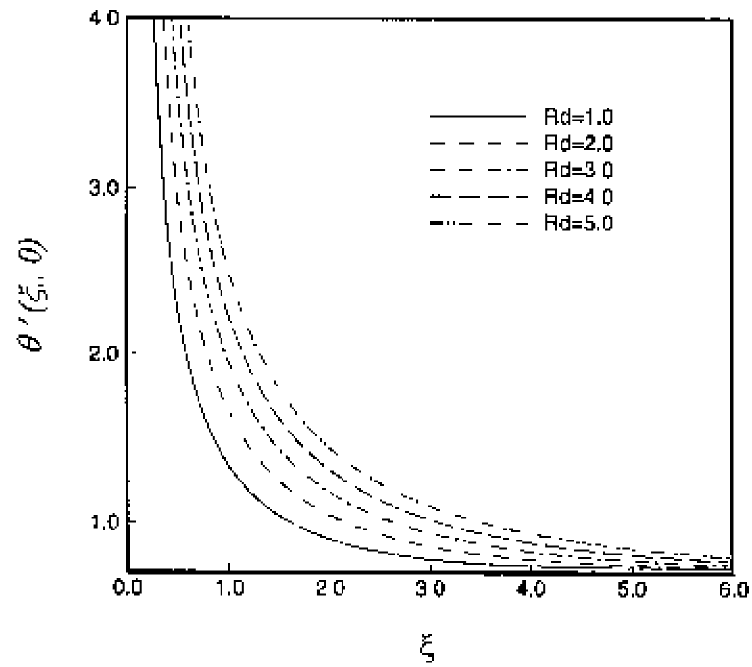

Fig.2.5: Rate of heat transfer for different values of Rd when Pr=0.73, M=0.5, m=0.1 and Δ=0.1.

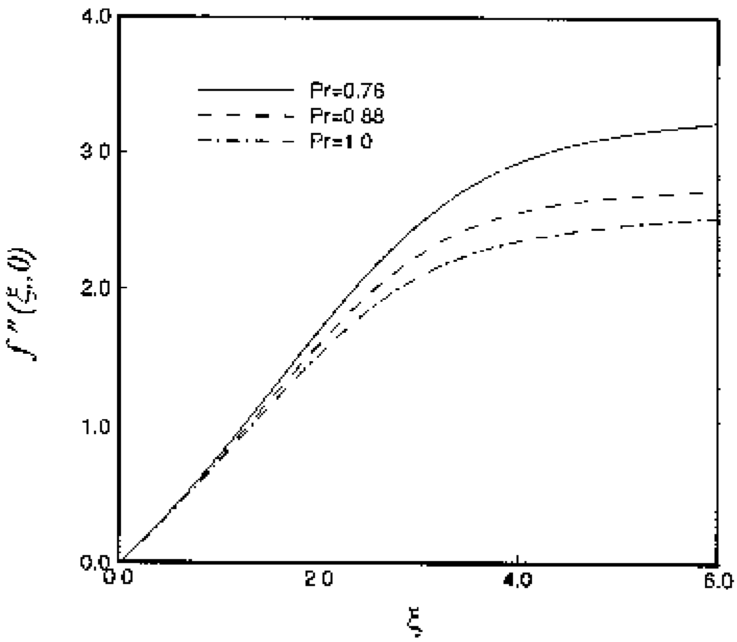

Fig.2.6: Skin friction for different values of Pr when Rd =1.0, M=0.5, m=0.1 and Δ=0.1.

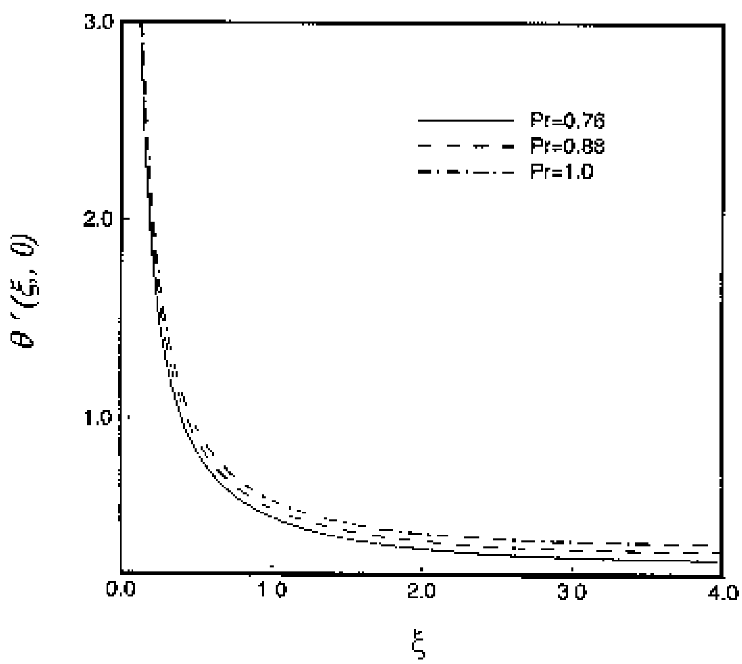

Fig.2.7: Rate of heat transfer for different values of Pr when Rd =1.0, M=0.5, m=0.1 and Δ=0.1.

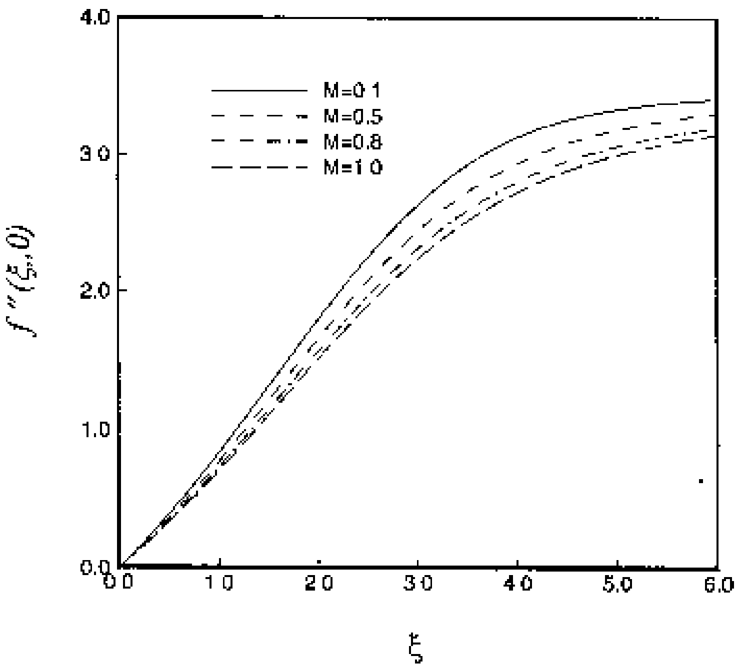

Fig.2.8: Skin friction for different values of M when Pr= 0.73, Rd =1.0, m=0.25 and Δ=0.1.

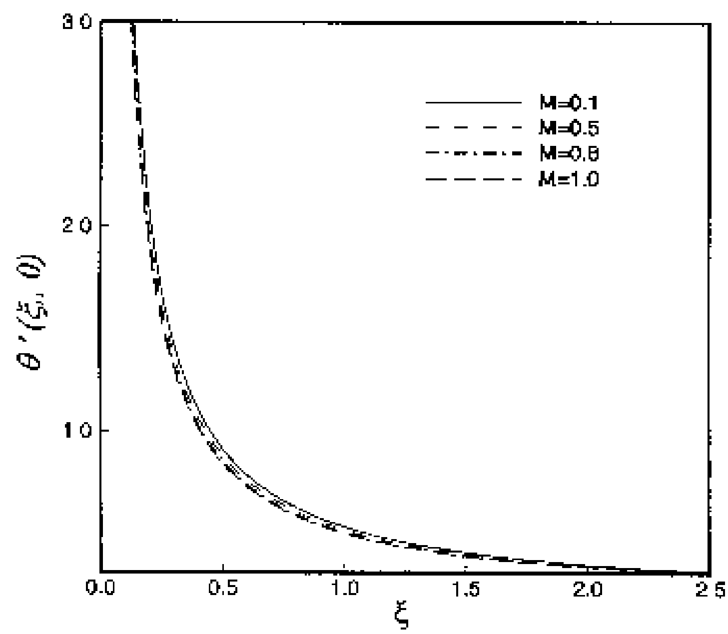

Fig.2.9: Rate of heat transfer for different values of M when Pr= 0.73, Rd =1.0, m=0.25 and Δ=0.1.

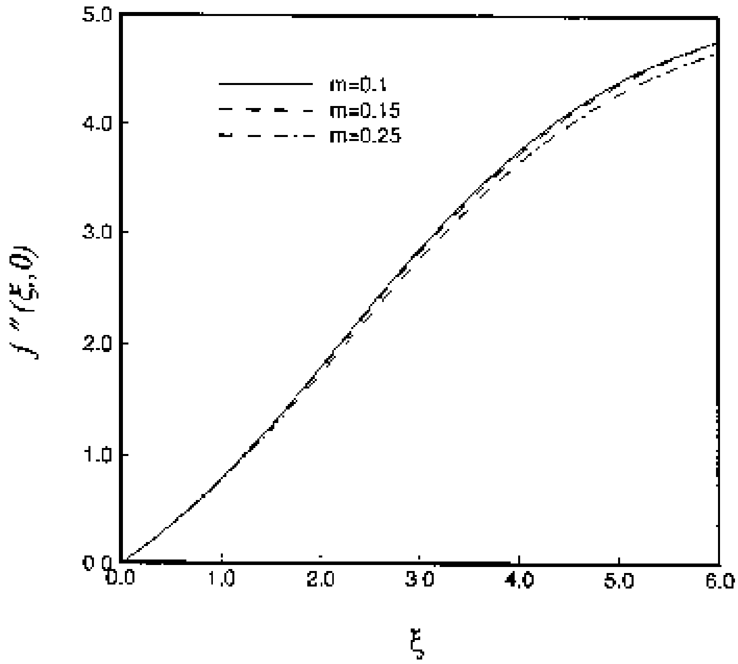

Fig.2.10: Skin friction for different values of m when Pr = 0.73, Rd =2.0, M=1.0 and Δ=0.1.

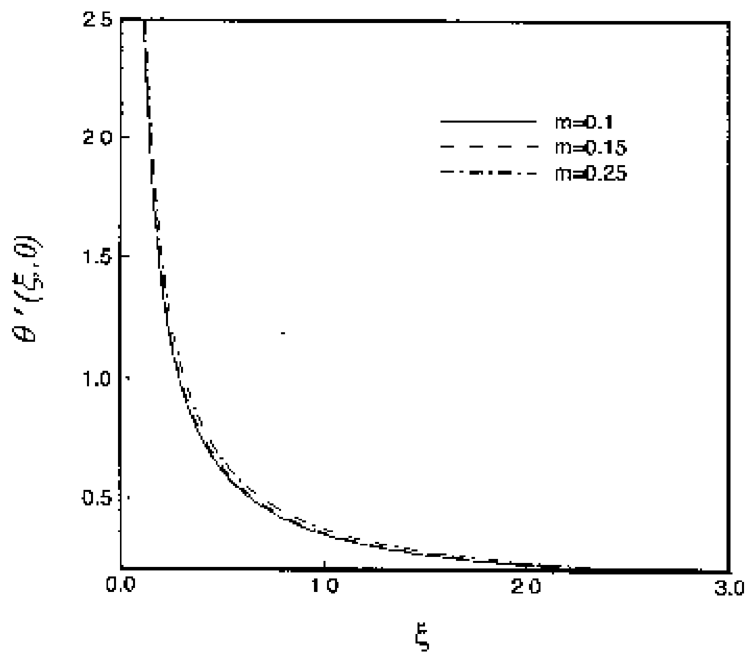

Fig.2.11: Rate of heat transfer for different values of m when Pr = 0.73, Rd =2.0, M=1.0 and Δ=0.1.

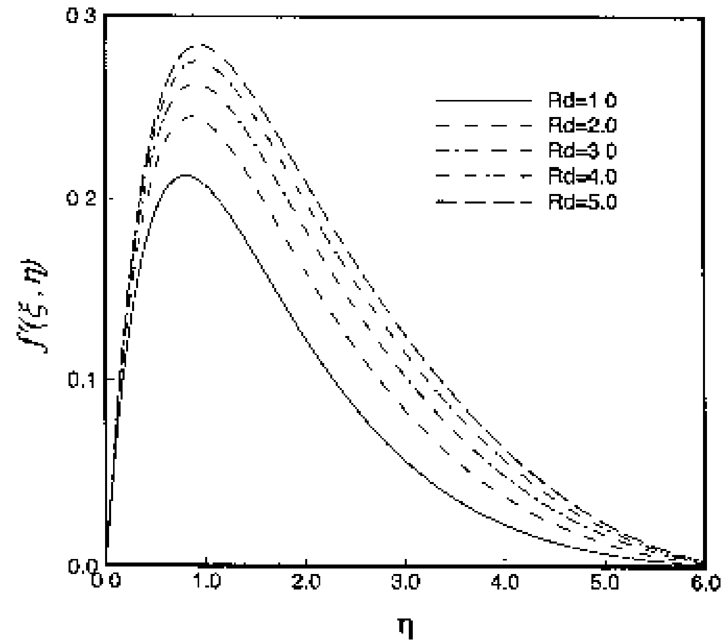

Fig.2.12: Velocity profile for different values of Rd when Pr=0.73, M=1.0, m=0.2 and Δ=0.1.

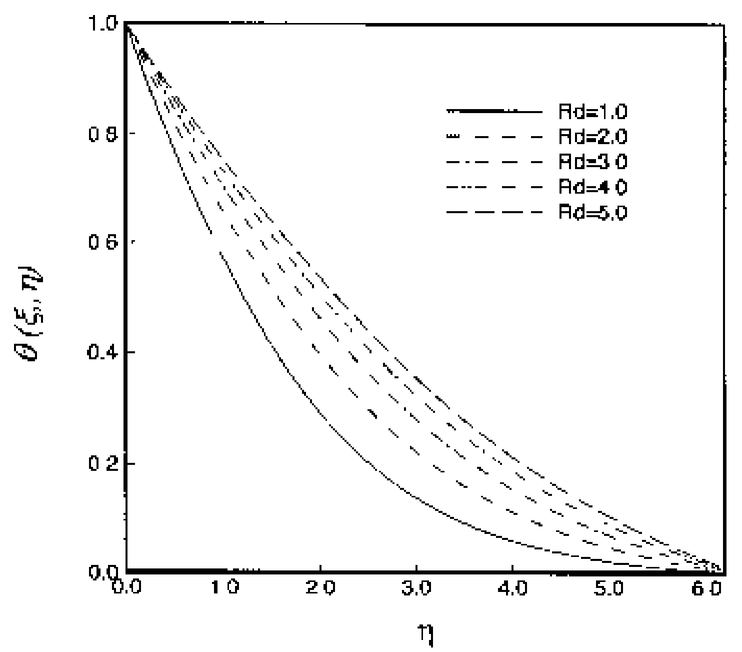

Fig.2.13: Temperature profile for different values of Rd when Pr=0.73, M=1.0, m=0.2 and Δ=0.1.

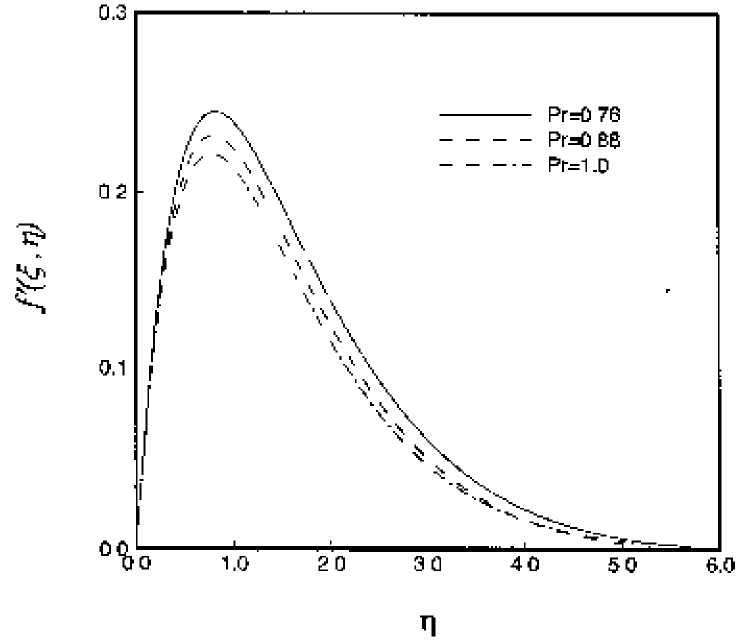

Fig.2.14: Velocity profile for different values of Pr when Rd =1.0, M=0.5, m=0.1 and Δ=0.1.

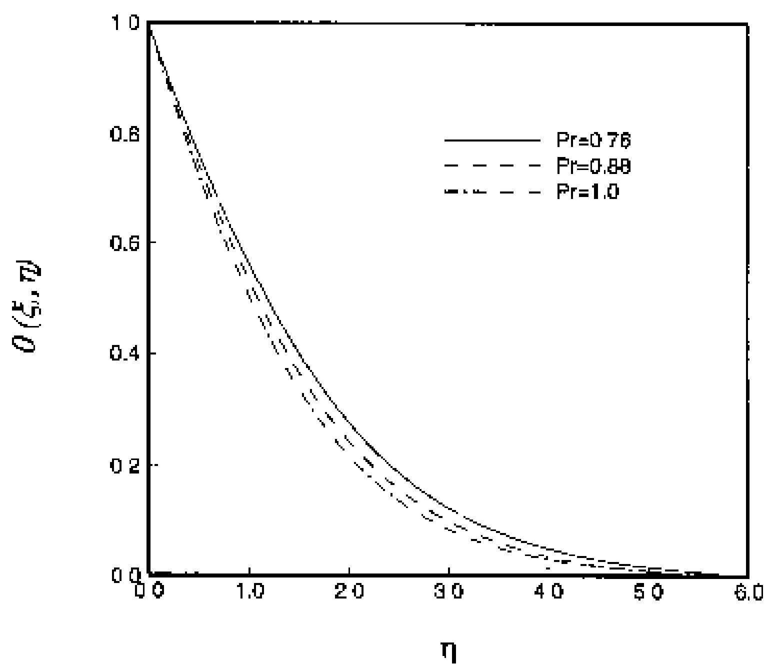

Fig.2.15: Temperature profile for different values of Pr when Rd =1.0, M=0.5, m=0.1 and Δ=0.1.

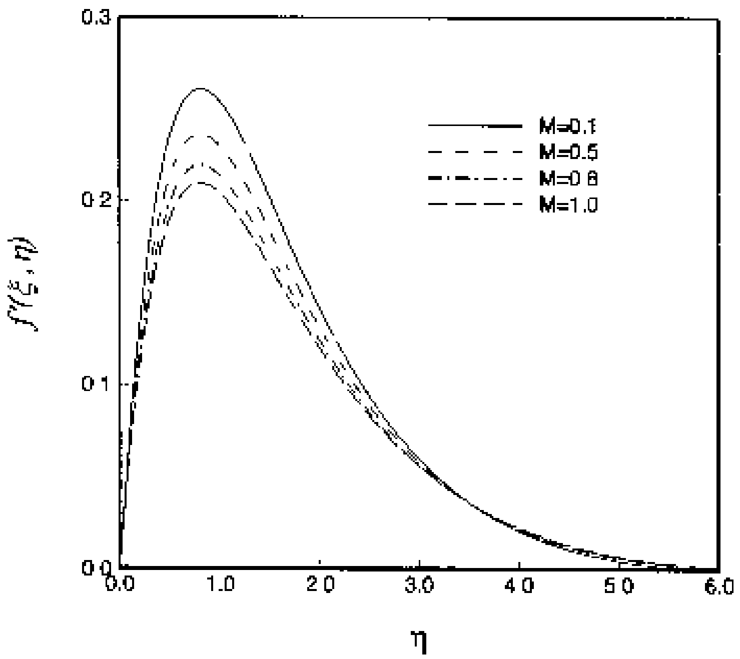

Fig.2.16: Velocity profile for different values of M when Pr= 0.73, Rd =1.0, m=0.25 and Δ=0.1.

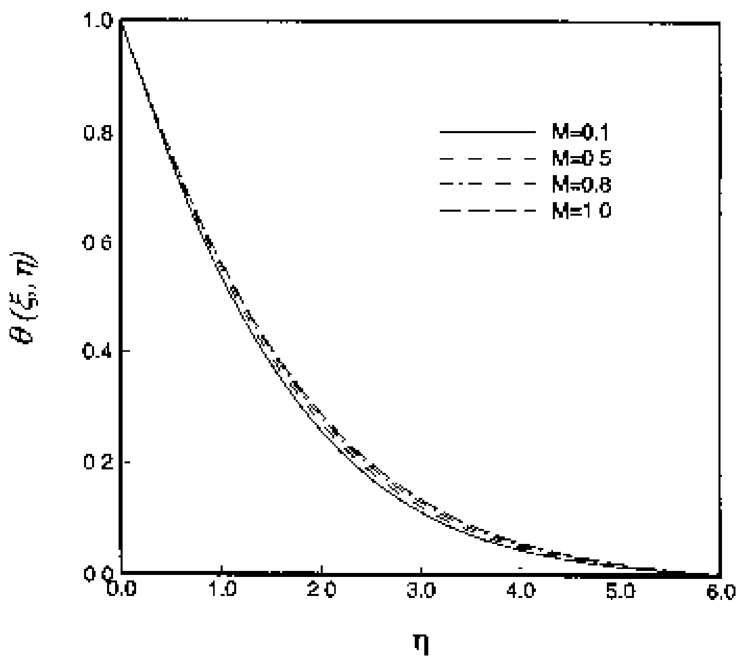

Fig.2.17: Temperature profile for different values of M when Pr= 0.73, Rd =1.0, m=0.25 and Δ=0.1.

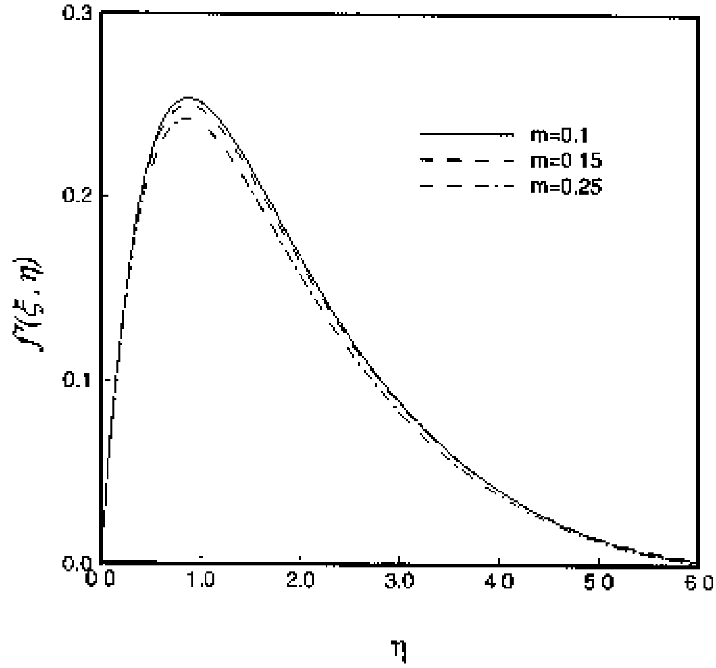

Fig.2.18: Velocity profile for different values of m when Pr = 0.73, Rd =2.0, M=1.0 and Δ=0.1.

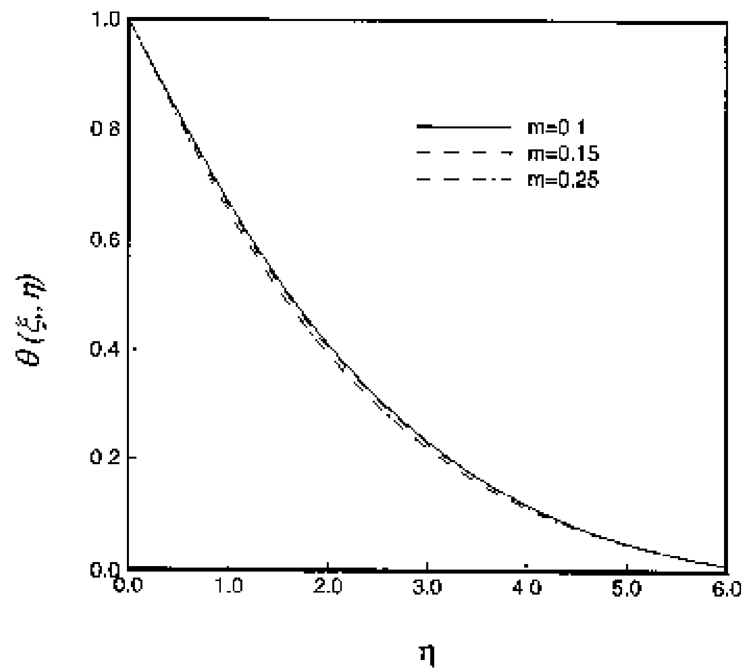

Fig.2.19: Temperature profile for different values of m when Pr = 0.73, Rd =2.0, M=1.0 and Δ=0.1.

To verify the proper treatment for the problem, the present solution for M=0.0 and m = 0.0 has been compared with that of Hossain et al. (1999). It can be seen from the table 2.1 that the present results are in excellent agreement with Hossain et al.

Table 2.1: Comparisons of the present numerical results of skin friction coefficient and the rate of heat transfer against ξ obtained by finite difference method for the values of Prandtl number Pr = 1.0, the radiation parameter Rd = 0.05, the surface temperature parameter Δ = 0.1 with those obtained by Hossain et al. (1999) without the effects of the magnetic parameter M and the exponent parameter m.

Values of ξ	Local skin friction coefficient		Rate of heat transfer	
	Hossain et al. (1999)	Present paper	Hossain et al. (1999)	Present paper
0.10	0.0655	0.0655	6.4627	6.4529
0.20	0.1316	0.1314	3.4928	3.4714
0.40	0.2647	0.2643	2.0229	2.0225
0.60	0.3963	0.3955	1.5439	1.5424
0.80	0.5235	0.5215	1.3247	1.3233
1.00	0.6429	0.6400	1.1995	1.1943
1.50	0.8874	0.8868	1.0574	1.0570
2.00	1.0278	1.0251	1.0120	1.0115
3.00	1.0769	1.0594	1.0001	1.0000

Chapter-3

Joule heating effect on magneto hydrodynamic free convection boundary layer flow along a vertical porous plate in presence of radiation.

3.1 Introduction

Free convection flow is often encountered in cooling of nuclear reactors or in the study of the structure of stars and planets. Along with the free convection flow the phenomenon of the boundary layer flow of an electrically conducting fluid up a vertical flat plate in the presence of a strong magnetic field is also very common because of its application in nuclear engineering in connection with the cooling of reactors. And radiation effects on free convection flow are important in the context of space technology and process involving high temperature, and very little is known about the effects of radiation on the boundary –layer flow of a radiating fluid past a body. Heat transfer from a heated vertical plate provides probably one of the most basic scenarios for natural convection problems. Variations of the problem occur frequently in the literature. Free convective steady hydromagnetic flow about a heated vertical flat plate has been considered by Gupta (1961), Poots (1961), Osterle and Yound (1961), Sparrow and Cess (1961), Lykoudis (1962) and Cramer (1963).The similarity solutions were studied by Gupta (1961) and Lykoudis (1962) considering that the magnetic field differs inversely as the fourth root of the height above the bottom edge of the plate. Afterwards Nanda and Mohanty (1970) made use of the similar technique to solve the hydromagnetic free convection of high and low Prandtl numbers because of realistic applications, as for liquid metals, the Prandtl number is low. Riley (1964) considered a uniform magnetic field and integrated the boundary layer equations over a single boundary layer thickness. Effects of transversely applied magnetic field on free convection of an electrically conducting fluid past a semi-infinite plate were studied by Wilks (1976). Miyamoto et al. (1980) have given an analysis of the relative importance of the parameters of the problem in particular with reference to coaxial heat conduction. Hossain and Ahmed (1990) studied the MHD forced and free convection boundary layer flow near the leading edge. They also investigated the combined forced and free convection of an electrically conducting fluid past a vertical flat plate at which the surface heat flux was uniform and magnetic field was applied parallel to the direction normal to plate. The natural convection boundary layer flow of an electrically conducting fluid up a hot vertical wall in the presence of a strong magnetic field has been studied by several authors because of its application in nuclear engineering in connection with the cooling of reactors.

In all the above studies, the effects of the joule heating and viscous dissipation were neglected because they are of the same order as well as negligibly small. But Gebhart (1962) has shown that the viscous dissipation effect plays an important role in natural convection in various devices which are subjected to large deceleration or which operate at high rotative speeds and also in strong gravitational field processes on large scales (on the planets) and in geological processes. With this understanding Takhar and Soundalgekar (1993) have studied the effects of Joule heating and viscous on the problem proposed by Sparrow and Cess (1961), using the series expansion method of Gebhart (1962). Hossain (1992) have studied the effect of Joule heating and viscous on the flow of an electrically condncting and viscous incompressible fluid past a semi infinite plate of which temperature varies linearly with the distance from the leading edge and in the presence of uniform transverse magnetic field. He has solved the equations numerically governing the flow applying the finite difference method along with Newton's linearization approximation. Alam (1995) has investigated the effects of Joule heating as well as viscous dissipation on the unsteady magneto hydrodynamic free convection and mass transfer flow with Hall current of an electrically conducting and viscous incompressible fluid past an accelerated infinite vertical porous plate with time dependent wall temperature and concentration. In the present study the Jonle heating effects on magnetohydrodynamic boundary layer flow along a vertical porous plate in presence of radiation will be investigated.

The transformed non similar boundary layer equations governing the flow together with the boundary conditions based convection were solved numerically using the

Keller box (implicit finite difference) along with Newton's linearization approximation method in the entire region starting from the lower part of the plate to the down stream for some values of the radiation parameter *Rd*, the magnetic parameter *M*, the joule heating parameter *J* and the Prandtl number *Pr*. The effect of the parameters *Rd*, *J*, *M*, and *Pr* on the velocity and temperature fields as well as on the skin friction coefficient and the rate of heat transfer have been studied. In the following sections detailed derivations of the governing equations for the flow and heat transfer and the method of solutions along with the results and discussions are presented.

3.2 Governing equations of the flow

The steady two dimensional laminar free convection boundary layer flow of an optically dense viscous and incompressible and electrically conducting fluid along a vertical plate in a porous medium as shown in fig-3.1.We assume that the plate is heated to a constant temperature T_w, which is higher than that the temperature T_∞ of the ambient fluid. The flow configuration and the coordinate system are shown on figure 3.1.

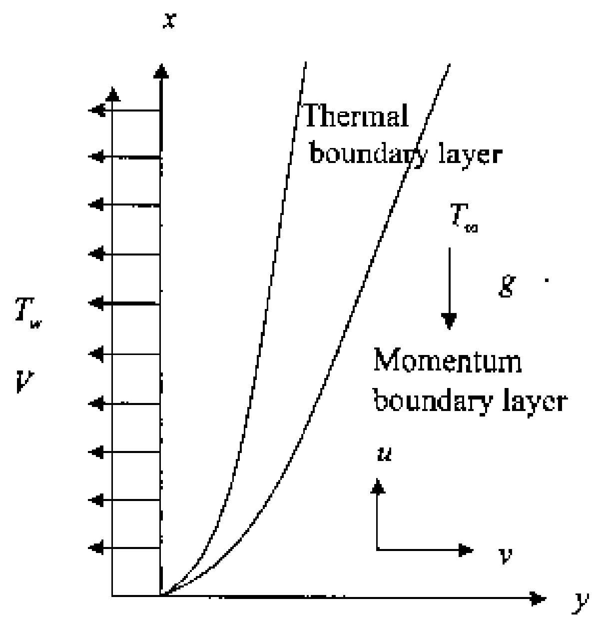

Fig 3.1: Physical model and coordinate systems.

The equations governing the flow are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta(T-T_\infty)-\frac{\sigma B_0^{\,2}u}{\rho} \tag{3.2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial y^2}-\frac{1}{k}\frac{\partial}{\partial y}(q_r)\right)+\frac{\sigma H_0^{\,2}u^2}{\rho c_p} \tag{3.3}$$

The appropriate boundary conditions to be satisfied by the above equations are

$$\begin{aligned}
&x=0\,,\quad y>0 &&: \quad u=0\,,\quad T=T_\infty\\
&y=0\,,\quad x>0 &&: \quad u=0\,,\quad v=-V\,,\quad T=T_w\\
&y\to\infty\,, x>0 &&: \quad u=0\,,\quad T=T_\infty
\end{aligned} \tag{3.4}$$

In equation (3.4) V represents the suction velocity of the fluid through the surface of the plate. In this chapter we shall consider only the suction case (rather than blowing) and therefore , V is taken as positive throughout, α is the equivalent thermal diffusivity, k is the thermal conductivity and the quantity q_r represents the radiative heat flux in the y direction. In order to reduce the complexity of the problem, we will consider the optically thick radiation limit in the present analysis. Thus, radiative heat flux term is simplified by the Rosseland diffusion approximation proposed by Siegel and Howell (1972) for an optically thick fluid according to

$$q_r=\frac{-4\sigma}{3(a+\sigma_s)}\,\frac{\partial T^4}{\partial y} \tag{3.5}$$

where σ is the Stefan-Boltzman constant and σ_s is the scattering coefficient and a is the Rosseland mean absorption coefficient. This approximation is valid at points optically far from the boundary surface and is good only for intensive absorption, that is, for an optically thick boundary layer. We observed that the equations (3.1-3.3) together with the boundary condition (3.4) and equation (3.5) are non-linear, partial differential equations. In the following sections the solutions methods of these equations are discussed in details.

3.3 Transformations of the governing equations

In equation (3.4) V represents the suction velocity of the fluid through the surface of the plate. In this chapter we shall consider only the suction case (rather than blowing) and therefore, V is taken as positive throughout. Near the leading edge, the boundary layer is much like that of the free convection boundary layer in the absence of suction, although much further downstream suction it will be found to dominate the flow. Also the magneto hydrodynamic field in the fluid is governed by the boundary layer equations .Therefore to solve the equations (3.1-3.3) subject to the boundary conditions (3.4) the following group of transformations are introduced for the flow starting from up stream to downstream.

$$\eta = \frac{Vy}{\nu\xi}, \qquad \xi = V\left\{\frac{4x}{\upsilon^2 g\beta(T_w - T_\infty)}\right\}^{\frac{1}{4}} \qquad ,\psi = V^{-3}\upsilon^2 g\beta(T_w - T_\infty)\xi^3\left\{f(\xi,\eta) \pm \frac{1}{4}\xi\right\}$$

$$\frac{T - T_\infty}{T_w - T_\infty} = \theta(\xi,\eta), \qquad \Delta = \frac{T_w}{T_\infty} - 1, \qquad Rd = \frac{4\sigma T_\infty^{\ 3}}{k(a + \sigma_s)} \tag{3.6}$$

where η and ξ is the dimensionless similarity variable, ψ is the stream function which satisfies the equation of continuity and $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$, Δ is the surface temperature parameter, Rd is the radiation parameter , $\theta(\eta,\xi)$ is the dimensionless temperature. Substituting equation (3.6) into equations (3.2) and (3.3), we get the following transformed non dimensional equations.

$$f''' + 3ff'' - 2f'^2 + \theta - Mf' + \xi f'' = \xi\left\{f'\frac{\partial f'}{\partial\xi} - f''\frac{\partial f}{\partial\xi}\right\} \tag{3.7}$$

$$\Pr^{-1}\theta'' + 3f\theta' + \xi\theta' + \Pr^{-1}\left\{\frac{4}{3}Rd\,(1 + \Delta\theta)^3\,\theta'\right\}' + Jf'^2 = \xi\left\{f'\frac{\partial\theta}{\partial\xi} - \theta'\frac{\partial f}{\partial\xi}\right\} \tag{3.8}$$

In the above equations the primes denote differentiation with respect to η.

The boundary conditions (3.4) then take the following form:

$$f=0 \quad ,\frac{\partial f}{\partial \eta}=0 \quad ,\theta=1 \quad at \quad \eta=0$$
$$\frac{\partial f}{\partial \eta}=0 \quad ,\theta=0 \quad as \quad \eta\to\infty \tag{3.9}$$

where Pr is the Prandtl number, $M=\left(\sigma B^2 \Big/ \sqrt{\rho^2 g\beta(T_w-T_0)}\right)$ where $B^2=2B_0^{\,2}\sqrt{x}$

is the magnetic parameter and $J=\left(g\beta H^2 \Big/ \sqrt{\rho^2 c_p^{\,2} g\beta(T_w-T_\infty)}\right)$ where

$H^2=8\sigma_0 H_0^{\,2}\left(\sqrt{x}\right)^3$ is the joule heating parameter.

The solutions of equation (3.7)-(3.8) enable us to calculate the skin friction τ and the rate of heat transfer Q at the surface of the plate from

$$\tau=\left(\frac{V}{g\beta(T_w-T_\infty)}\right)\left(\frac{\partial u}{\partial y}\right)_{y=0}=\xi f''(\xi.0) \tag{3.10}$$

$$Q=-\left(\frac{v}{V(T_w-T_\infty)}\right)\left(\frac{\partial T}{\partial y}\right)_{y=0}=-\left(1+\frac{4}{3}Rd\ \Delta^3\right)\xi^{-1} g'(\xi,0) \tag{3.11}$$

3.4 Joule heating parameter:

In electronics, and in physics more broadly, **Joule heating** or **ohmic heating** refers to the increase in temperature of a conductor as a result of resistance to an electrical current flowing through it.

At an atomic level, Joule heating is the result of moving electrons colliding with atoms in a conductor, where upon momentum is transferred to the atom, increasing its kinetic energy . Joule heating is named for James Prescott Joule, the first to articulate what is now Joule's law, relating the amount of heat released from an electrical resistor to its resistance and the charge passed through it. In our problem we got a dimensionless parameter $J = \left(g\beta H^2 \Big/ \sqrt{\rho^2 c_p^{\,2} g\beta(T_w - T_\infty)} \right)$ which is Joule heating parameter.

3.5 Method of solution:

To get the solutions of the parabolic differential equations (3.7) and (3.8) along with the boundary condition (3.9), we shall employ implicit finite difference method together with Keller- box elimination technique which is well documented and widely used by Keller and Cebeci (1971) and recently by Hossain (1992). Since a good description of this method has been discussed in details in Chapter-2 i.e. in Appendix, further discussion is disregarded here. The numerical results obtained are presented in the following section.

3.6 Results and discussion:

In this chapter Joule heating effect on magnetohydrodynamic free convection boundary layer flow along a vertical porous plate in presence of radiation has been investigated. Solutions are obtained for the fluid having Prandtl number Pr = (0.76, 0.88, 1.0) and for a wide range of the values of the Joule heating parameter J = (0.1, 0.2, 0.5, 1.0, 2.0, 3.0), the Radiation parameter Rd = (0.5, 1.0, 3.0, 5.0), the magnetic parameter M = (0.1, 0.5, 0.8, 0.9, 1.0) and the surface temperature parameter Δ. If we know the values of the functions $f(\xi,\eta)$, $\theta(\xi,\eta)$ and their derivatives for different values of the Prandtl number Pr, the Radiation parameter Rd , the Joule heating parameter J and the magnetic parameter M, we may calculate the numerical values of the local rate of heat transfer $\theta'(\xi,0)$ and the velocity profile $f'(\xi,\eta)$ at surface that are important from physical point of view.

Numerical values of the skin friction coefficient and the local rate of heat transfer depicted graphically in Fig.3.2 and Fig.3.3 respectively for different values of the Joule heating parameter J = (0.5, 1.0, 2.0, 3.0) for the fluid having Prandtl number Pr = 0.73, the Radiation parameter Rd = 1.0, the magnetic parameter M = 0.5 and the surface temperature parameter Δ=0.1. In Fig.3.4 and Fig.3.5 the skin friction coefficient $f''(\xi,0)$ and the local rate of heat transfer $\theta'(\xi,0)$ are shown graphically for different values of the Prandtl number Pr = (0.76, 0.88, 1.0) when the value of the radiation parameter Rd = 0.5, the magnetic parameter M = 1.0, the Joule heating parameter J = 0.5, and the surface temperature parameter Δ=0.1. The values of the Prandtl number Pr are taken to 0.76 that corresponds physically for CO_2 in the temperature range 100-650°F, 0.88 that corresponds for NH_3 vapor in the temperature range 120-400°F and 1.0 corresponding to water vapor in the temperature range 220-900°F.

In Fig.3.6 and Fig.3.7 the skin friction coefficient $f''(\xi,0)$ and the local rate of heat transfer $\theta'(\xi,0)$ are shown graphically for different values of the radiation parameter Rd = (0.5, 1.0, 3.0, 5.0) when the value of the Prandtl number Pr = 0.73, the magnetic parameter M = 0.8, the Joule heating parameter J = 0.2, and the surface temperature parameter Δ=0.1.In Fig.3.8 and Fig.3.9 the skin friction coefficient $f''(\xi,0)$ and the local rate of heat transfer $\theta'(\xi,0)$ are shown graphically for different values of the magnetic parameter M = (0.1, 0.5, 0.9) when the value of the Prandtl number Pr= 0.73, the radiation parameter Rd = 0.5, the Joule heating parameter J = 0.1, and the surface temperature parameter Δ=0.1.

From Fig.3.2, it is shown that the skin friction coefficient $f''(\xi,0)$ increases with the increase of the Joule heating parameter J (=0.5, 1.0, 2.0, 3.0) and from the Fig.3.3, opposite result is observed on the local rate of heat transfer $\theta'(\xi,0)$ due to increase of the value of the Joule heating parameter J when the value of the Prandtl number is 0.73, the Radiation parameter Rd = 1.0, the magnetic parameter M = 0.5 and the surface temperature parameter Δ=0.1.

From Fig. 3.4, it is observed that the skin friction coefficient $f''(\xi,0)$ decreases monotonically with the increase of the Prandtl number Pr (=0.76, 0.88, 1.0). Again Fig.3.5 shows that the increase of the Prandtl number Pr leads to increase of the local rate of heat transfer $\theta'(\xi,0)$ when the value of joule heating parameter J = 0.5, the Radiation parameter Rd = 0.5, the magnetic parameter M = 1.0 and the surface temperature parameter Δ=0.1.

From Fig. 3.6, it is shown that the skin friction coefficient $f''(\xi,0)$ increases with the increase of the Radiation parameter Rd (=0.5, 1.0, 3.0, 5.0) and from the Fig.3.7, the same result is observed on the local rate of heat transfer $\theta'(\xi,0)$ due to increase of the value of the Radiation parameter Rd when the value of joule heating parameter J = 0.2, the Prandtl number Pr =0.73, the magnetic parameter M = 0.8 and the surface temperature parameter Δ=0.1.

From Fig. 3.8, it is observed that increase in the value of the magnetic parameter M leads to decrease of the value of the skin friction coefficient $f''(\xi,0)$ which is usually expected. Again Fig.3.9 shows that the increase of the magnetic parameter M leads to decrease of the local rate of heat transfer $\theta'(\xi,0)$ when the value of joule heating parameter J = 0.1, the Prandtl number Pr =0.73, the radiation parameter Rd = 0.5 and the surface temperature parameter Δ=0.1.

Fig.3.10 depicts the velocity profile for different values of the Joule heating parameter J (= 0.5, 1.0, 2.0, 3.0) while the Radiation parameter Rd = 1.0, the Prandtl number Pr = 0.73, the magnetic parameter M = 0.5 and the surface temperature parameter Δ=0.1.Corresponding distribution of the temperature profile $\theta(\xi,\eta)$ in the fluids is shown in Fig. 3.11. From Fig. 3.10, it is seen that if Joule heating parameter J increases, the velocity of the fluid increases. Small increment is shown from Fig.3.11 on the temperature profile θ (η, x) for increasing values of J.

In Fig.3.12 and Fig.3.13, the velocity profile $f'(\xi,\eta)$ and the temperature profile $\theta(\xi.\eta)$ are shown graphically for different values of the Prandtl number Pr (= 0.76, 0.88, 1.0) when the value of the radiation parameter Rd = 0.5, the magnetic parameter M = 1.0, the Joule heating parameter J = 0.5, and the surface temperature parameter Δ=0.1. From Fig. 3.12, it is seen that if the Prandtl number increases, the velocity of the fluid decreases. We also observe from Fig.3.13 that the temperature profile decreases within the boundary layer due to increase of the Prandtl number Pr.

Fig.3.14 and Fig.3.15 deal with the effect of the radiation parameter Rd (= 0.5, 1.0, 3.0, 5.0) for Prandtl number Pr = 0.73, the magnetic parameter M = 0.8, the Joule heating parameter J = 0.2, and for the surface temperature parameter Δ=0.1 on the velocity profile $f'(\xi,\eta)$ and the temperature profile $\theta(\xi,\eta)$. From Fig. 3.14, it is revealed that the velocity profile $f'(\xi,\eta)$ increases with the increase of the radiation parameter Rd which indicates that radiation increases the fluid motion. We also observe from Fig.3.15 that the temperature profile increases within the boundary layer due to increase of the radiation parameter Rd.

In Fig.3.16 and Fig.3.17, the velocity profile $f'(\xi,\eta)$ and the temperature profile $\theta(\xi,\eta)$ are shown graphically for different values of the magnetic parameter M (= 0.1, 0.5, 0.9) when value of the Prandtl number Pr is 0.73, the radiation parameter Rd = 0.5, the Joule heating parameter J = 0.1, and the surface temperature parameter

Δ=0.1.From Fig. 3.16, it is seen that if the magnetic parameter increases, the velocity of the fluid decreases. We also observe from Fig.3.17 that the temperature profile increases within the boundary layer due to increase of the magnetic parameter M.

3.7 Conclusion:

The effect of Joule heating parameter J, the radiation parameter Rd and the magnetic parameter M for different values of Prandtl number Pr on the magneto hydrodynamic (MHD) free convection boundary layer flow along a vertical porous plate with radiation has been investigated by introducing a new class of transformations. Numerical solutions of the equations governing the flow are obtained by using the very efficient implicit finite difference method together with the Keller Box scheme. From the present investigation, the following conclusions may be drawn:

1. The skin friction coefficient increases for increasing value of the joule heating parameter J but increased value of the joule heating parameter J leads to decrease the rate of heat transfer.
2. Increased value of the joule heating parameter J leads to increase the velocity distribution as well as the temperature distribution.
3. It has been observed that the skin friction coefficient, the rate of heat transfer, the temperature distribution over the whole bonndary layer and the velocity distribution increase with the increase of the radiation parameter Rd.
4. An increasing value of Prandtl number Pr leads to decrease the skin friction coefficient but increase the rate of heat transfer .As Prandtl number Pr increases, both the velocity and the temperature distributions decrease significantly.
5. The skin friction coefficient, the rate of heat transfer and the velocity profile decrease while the temperature profile slightly increase for increased values of the magnetic parameter M.

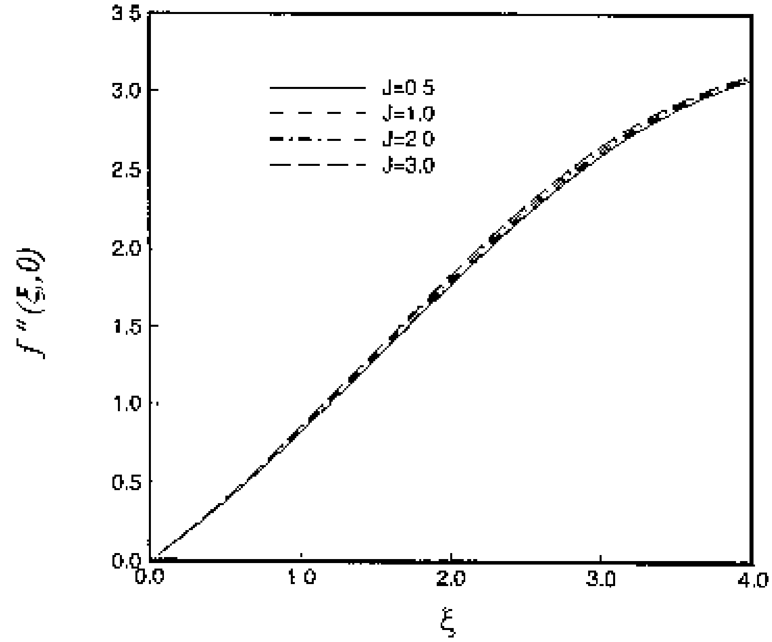

Fig.3.2: Skin friction for different values of J when Pr=0.73, Rd=1.0, M= 0.5 and Δ=0.1.

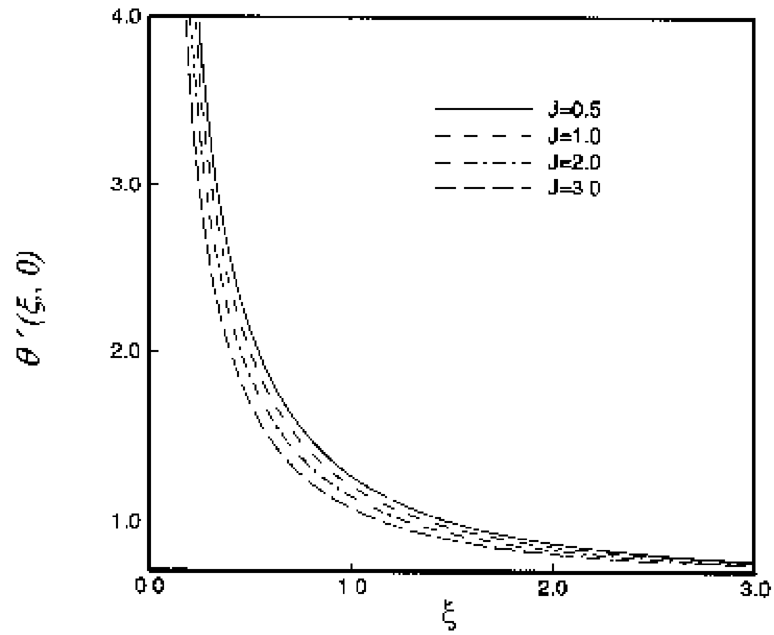

Fig.3.3: Rate of heat transfer for different values of J when Pr=0.73, Rd=1.0, M=0.5 and Δ=0.1.

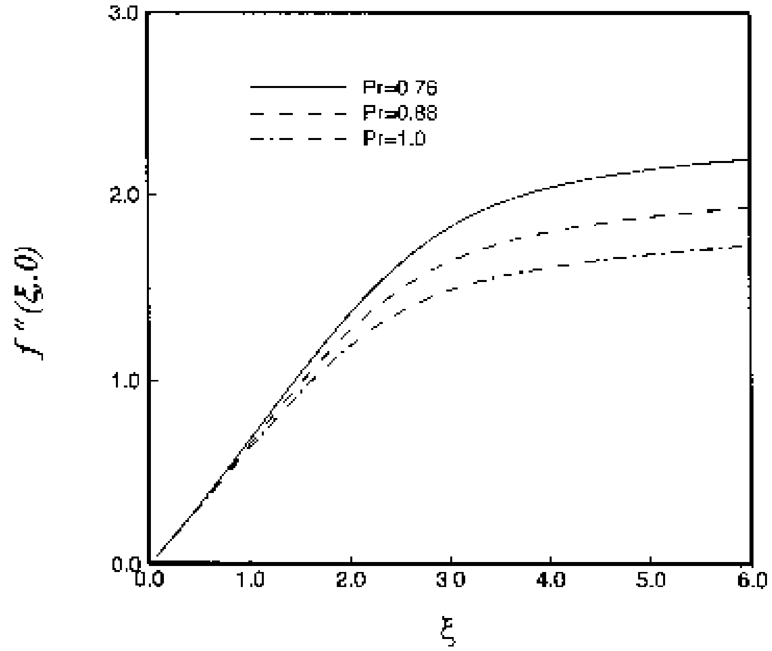

Fig.3.4: Skin friction for different values of Pr when Rd =0.5. J=0.5, M=1.0 and Δ=0.1.

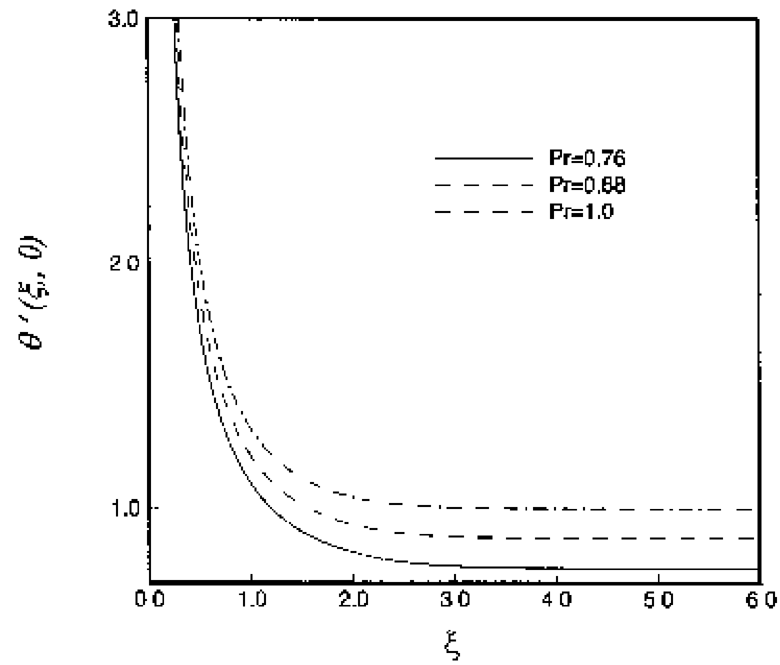

Fig.3.5: Rate of heat transfer for different values of Pr when Rd =0.5, J=0.5, M=1.0 and Δ=0.1.

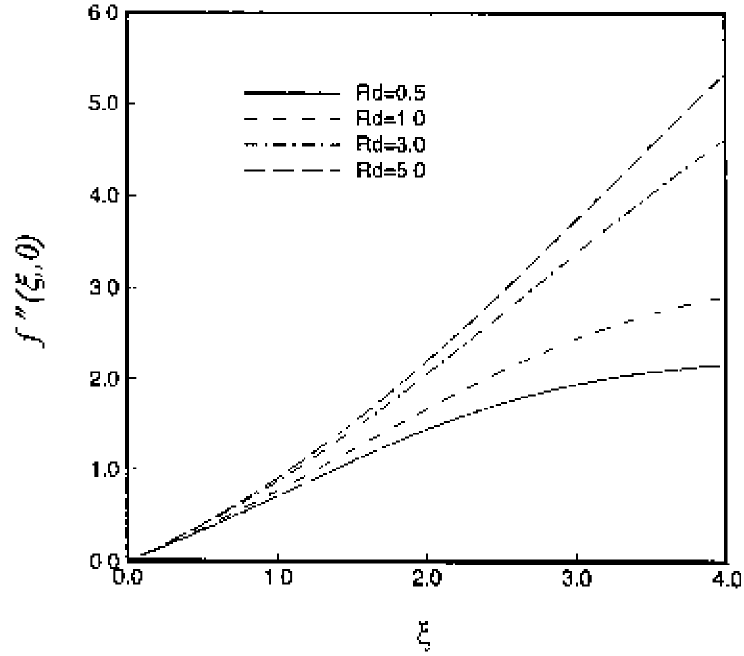

Fig.3.6: Skin friction for different values of Rd when Pr= 0.73, J =0.2, M=0.8 and Δ=0.1.

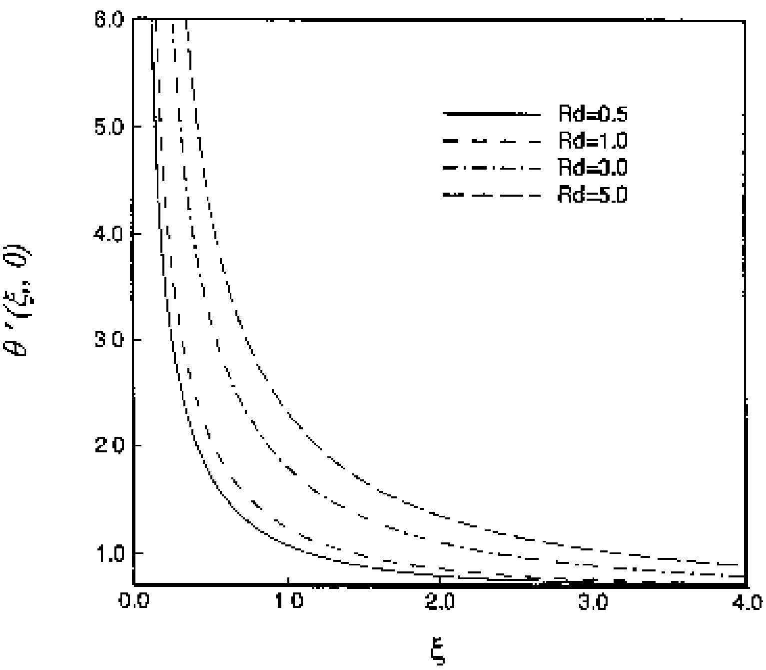

Fig.3.7: Rate of heat transfer for different values of Rd when Pr= 0.73, J =0.2, M=0.8 and Δ=0.1.

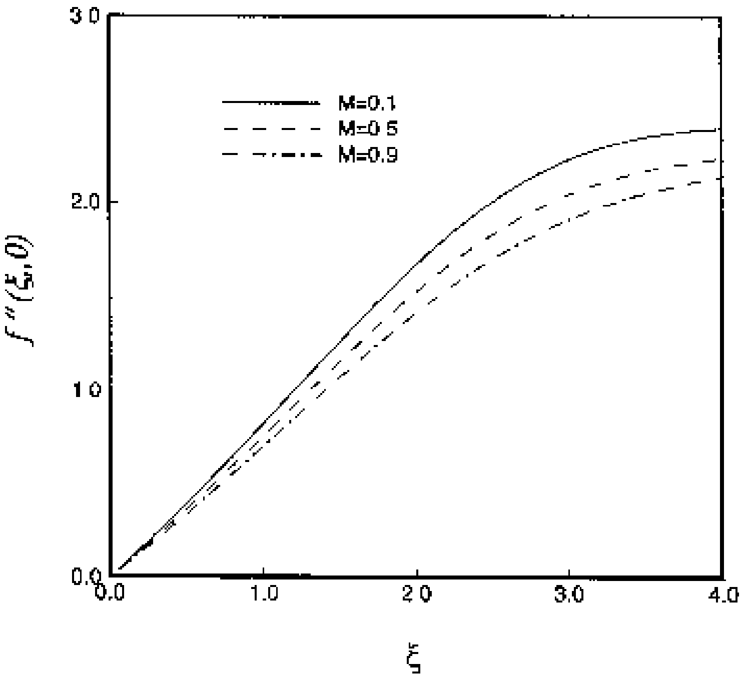

Fig.3.8: Skin friction for different values of M when Pr = 0.73, Rd =0.5, J=0.1 and Δ=0.1.

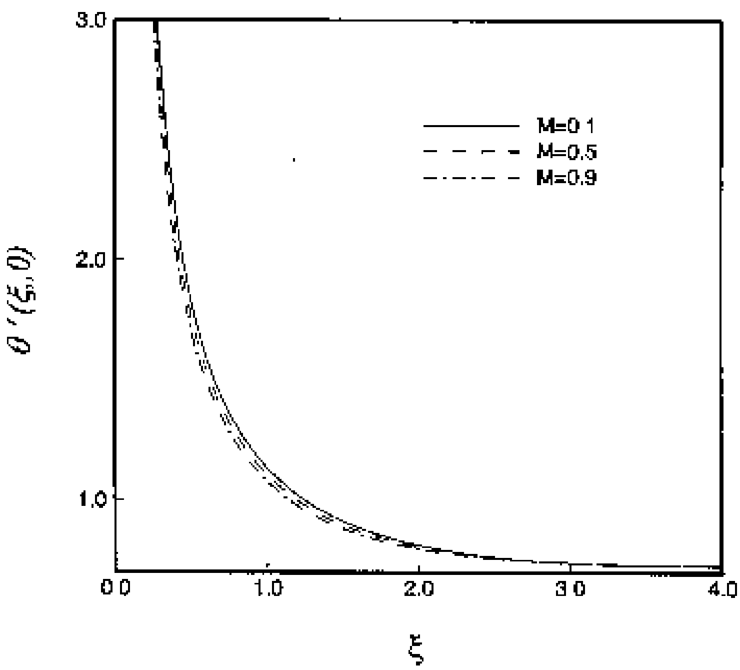

Fig.3.9: Rate of heat transfer for different values of M when Pr = 0.73, Rd =0.5, J=0.1 and Δ=0.1.

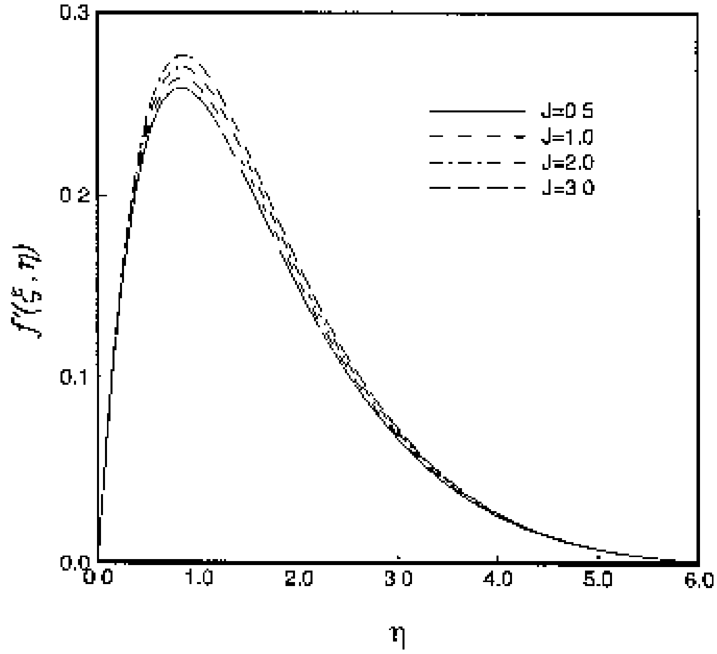

Fig.3.10: Velocity profile for different values of J when Pr=0.73, Rd=1.0, M=0.5 and Δ=0.1.

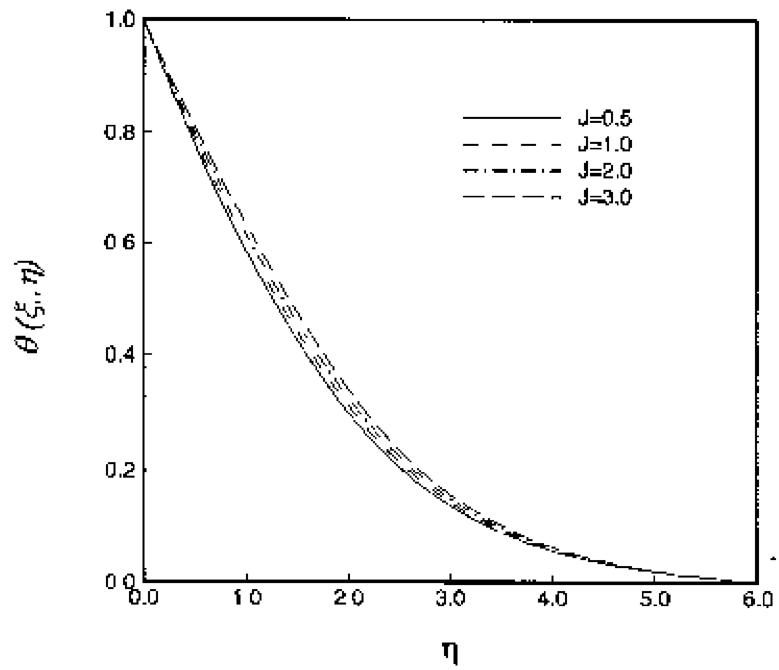

Fig.3.11: Temperature profile for different values of J when Pr=0.73, Rd=1.0, M=0.5 and Δ=0.1.

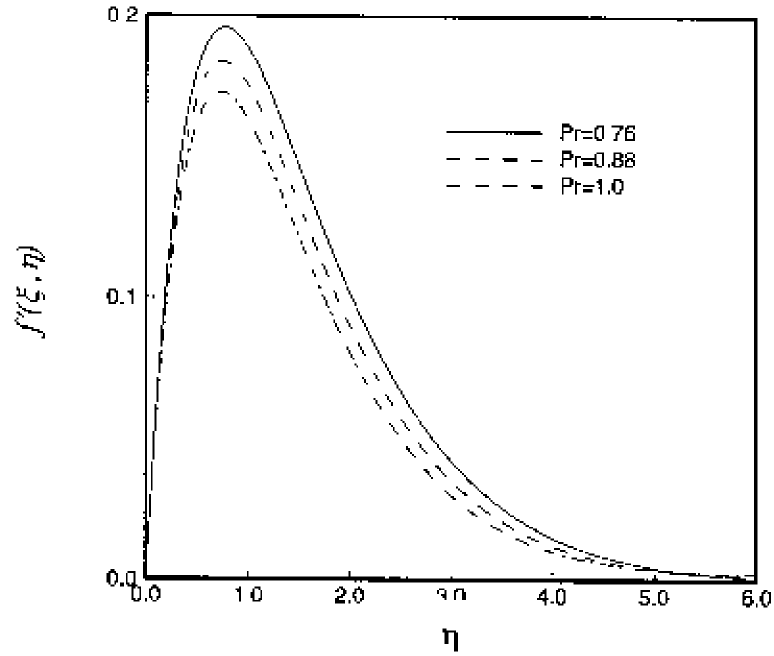

Fig.3.12: Velocity profile for different values of Pr when Rd =0.5, J=0.5, M=1.0 and Δ=0.1.

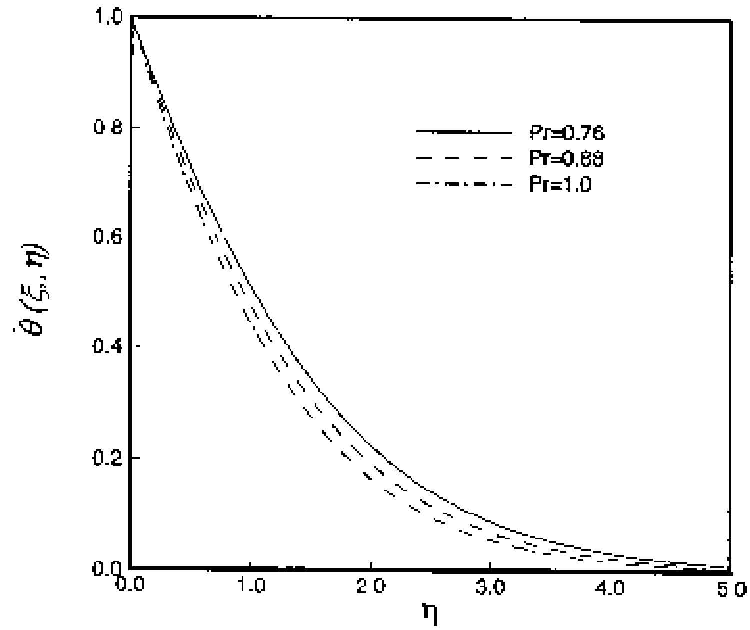

Fig.3.13: Temperature profile for different values of Pr when Rd =0.5, J=0.5, M=1.0 and Δ=0.1.

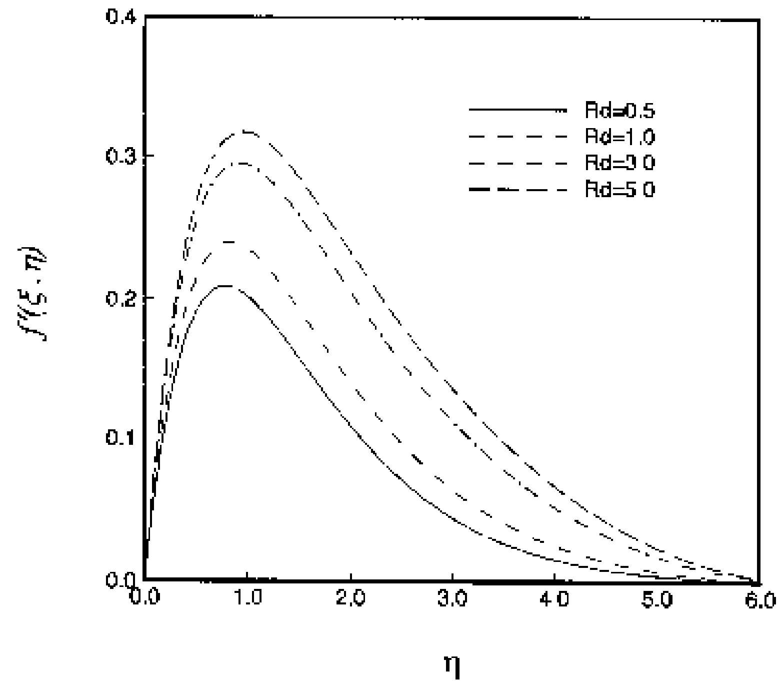

Fig.3.14: Velocity profile for different values of Rd when Pr= 0.73, J =0.2. M=0.8 and Δ=0.1.

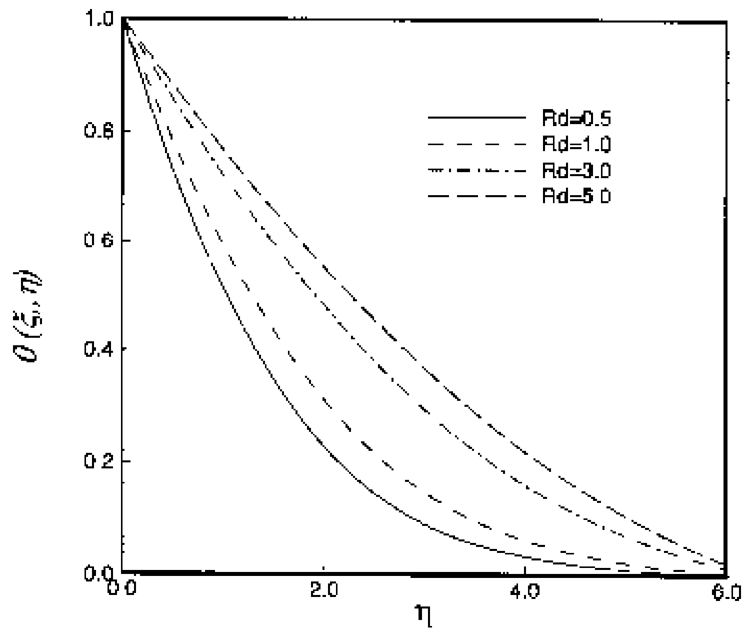

Fig.3.15: Temperature profile for different values of Rd when Pr= 0.73, J =0.2, M=0.8 and Δ=0.1.

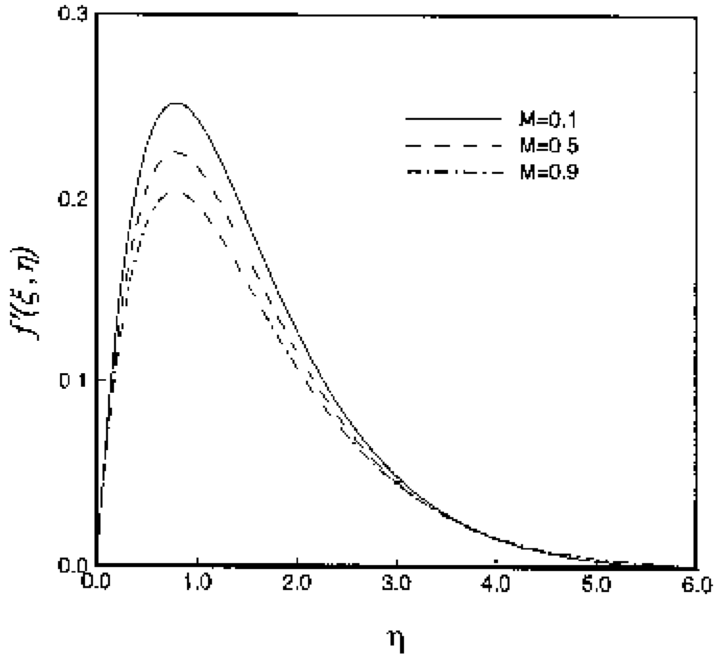

Fig.3.16: Velocity profile for different values of M when Pr = 0.73, Rd =0.5, J=0.1 and Δ=0.1.

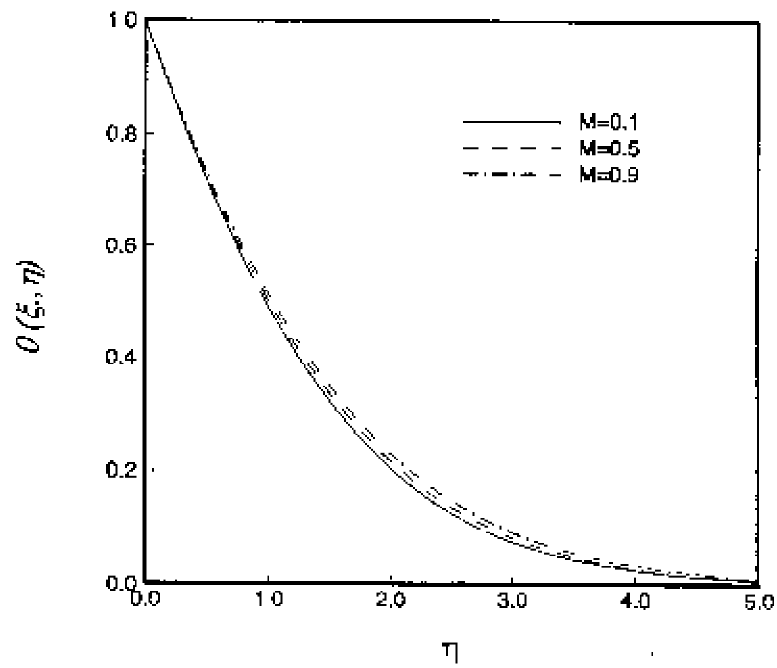

Fig.3.17: Temperature profile for different values of M when Pr = 0.73, Rd =0.5, J=0.1 and Δ=0.1.

Chapter 4

Conclusions

In this dissertation, we have investigated the radiation effect on magneto hydrodynamic free convection flow along a vertical porous plate with variable plate temperature. Using the appropriate transformations the basic equations are transformed to non similar boundary layer equations, which have been solved numerically in the entire region starting from the lower part of the plate to the down stream using a very efficient implicit finite difference method known as Keller box scheme. Here we have focused our attention on the evolution of the skin friction, the surface temperature distribution, velocity distribution as well as temperature distribution for a selection of parameter sets consisting of the Prandtl number *Pr*, the radiation parameter Rd, the magnetic parameter M, and the exponent parameter m and the surface temperature parameter Δ.

From chapter 2

The effect of radiation on magneto hydrodynamic MHD free convection boundary layer flow along a vertical porous plate with variable plate temperature has been investigated introducing a new class of transformations. Numerical solutions of the equations governing the flow are obtained by using the very efficient implicit finite difference method together with the Keller Box scheme. From the present investigation, the following conclusions may be drawn:

1. The skin friction coefficient and the rate of heat transfer increase for increasing value of the radiation parameter *Rd*.
2. Increased value of the radiation parameter *Rd* leads to increase the velocity distribution as well as the temperature distribution.
3. An increasing value of Prandtl number Pr leads to decrease the skin friction coefficient but increase the rate of heat transfer.
4. As Prandtl number Pr increases, both the velocity and the temperature distributions decrease significantly.

5. With effect of magnetic parameter M, the skin friction coefficient and the rate of heat transfer decrease.

6. An increase in values of M leads to decrease the velocity distribution but slightly increase the temperature distribution.

7. Increased value of the exponent parameter m leads to decrease the skin friction coefficient but increase the rate of heat transfer and As the exponent parameter m increases, both the velocity and the temperature distributions decrease significantly.

From chapter 3

The effect of Joule heating parameter J, the radiation parameter Rd and the magnetic parameter M for different values of Prandtl number Pr on the magneto hydrodynamic (MHD) free convection boundary layer flow along a vertical porous plate with radiation has been investigated by introducing a new class of transformations. Numerical solutions of the equations governing the flow are obtained by using the very efficient implicit finite difference method together with the Keller Box scheme. From the present investigation, the following conclusions may be drawn:

1. The skin friction coefficient increases for increasing value of the joule heating parameter J but increased value of the joule heating parameter J leads to decrease the rate of heat transfer.

2. Increased value of the joule heating parameter J leads to increase the velocity distribution as well as the temperature distribution.

3. It has been observed that the skin friction coefficient, the rate of heat transfer, the temperature distribution over the whole bonndary layer and the velocity distribution increase with the increase of the radiation parameter Rd.

4. An increasing value of Prandtl number Pr leads to decrease the skin friction coefficient but increase the rate of heat transfer .As Prandtl number Pr increases, both the velocity and the temperature distributions decrease significantly.

5. The skin friction coefficient, the rate of heat transfer and the velocity profile decrease while the temperature profile slightly increase for increased values of the magnetic parameter M.

Chapter-5

Further Recommendation

The present studies may be extended further,

1. In this work, both the viscosity and thermal conductivity have been considered as a constant but they are functions of temperature. If the viscosity and thermal conductivity are the function of temperature then the problem can be extended.
2. Viscous dissipation is neglected here, so viscous dissipation may be added to the energy equation and hence the problem can be extended.
3. Instead of taking vertical flat plate one can choose vertical cylinder. Thus the problem can also be extended.
4. By using perturbation method or finite volume method to solve the governing equations.

Appendix

Methods of Solution

Implicit finite difference method (IFDM)

To get the solutions of the differential equations (2.14) and (2.15) along with the boundary condition (2.18), we shall employ a most practical, an efficient and accurate solution technique, known as implicit finite difference method together with Keller-box elimination technique which is well documented and widely used by Keller (1978) and Cebeci (1984) and recently by Hossain(1992).

To apply the aforementioned method, we first convert the equations (2.14) and (2.15) into the following system of first order differential equations with dependent variables $u(\xi,\eta)$, $v(\xi,\eta)$ and $p(\xi,\eta)$ along with the boundary condition (2.18) as

$$f' = u \tag{1}$$

$$u' = v \tag{2}$$

$$g' = p \tag{3}$$

Equations (2.14) and (2.15) transform to

$$v' + p_1 fv - p_2 u^2 + p_3 g - p_4 u + p_5 v = p_6 \xi\left(u\frac{\partial u}{\partial \xi} - v\frac{\partial f}{\partial \xi}\right) \tag{4}$$

$$Ap' + p_1 fp + p_5 p - p_7 gu + Bp^2 = p_6 \xi\left(u\frac{\partial g}{\partial \xi} - p\frac{\partial f}{\partial \xi}\right) \tag{5}$$

Where

$$p_1 = 3+m,\ p_2 = 2(1+m),\ p_3 = 1,\ p_4 = M,\ p_5 = \xi,\ p_6 = 1-m,\ p_7 = 4m,$$
$$A = \frac{1}{pr}\left\{1 + (1+\Delta g)^3 .Rdd\right\},\ B = \frac{1}{pr} 3Rdd(1+\Delta g)^2 .\Delta \tag{6}$$

The boundary conditions are:

$$\left.\begin{aligned} f = 0,\quad f' = 0,\quad g = 1 \quad & at\ \eta = 0 \\ f' = 0,\quad g = 0, \quad & at\ \eta \to \infty \end{aligned}\right\} \tag{7}$$

We now consider the net rectangle on the (ξ, η) plane shown in the figure (a) and denote the net points by

$$\left.\begin{aligned} \xi^0 &= 0, \quad \xi^n = \xi^{n-1} + k_n \quad where \quad n = 1,2,\ldots N \\ \eta_0 &= 0, \quad \eta_j = \eta_{j-1} + h_j \quad where \quad j = 1,2,\ldots J \end{aligned}\right\} \tag{8}$$

Here 'n' and 'j' are just sequence of numbers on the (ξ, η) plane, k_n and h_j are the variable mesh widths.

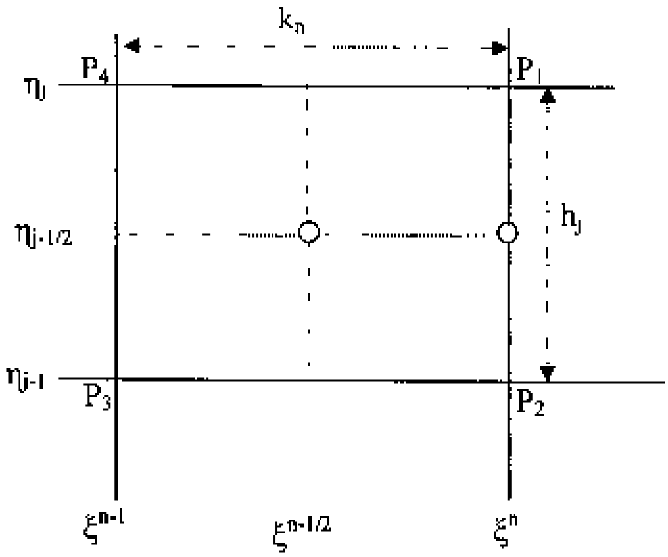

Figure (a): Net rectangle of the difference approximation for the Box scheme.

We approximate the quantities (f, u, v, p) at the points (ξ^n, η_j) of the net by$(f_j^n, u_j^n$ v_j^n, $p_j^n)$ which we call net function . We also employ the notation g_j^n for the quantities midway between net points shown in figure (a) and for any net function as

$$\xi^{n-1/2} = \frac{1}{2}(\xi^n + \xi^{n-1}) \tag{9a}$$

$$\eta_{j-1/2} = \frac{1}{2}(\eta_j - \eta_{j-1}) \tag{9b}$$

$$\theta_j^{n-1/2} = \frac{1}{2}(\theta_j^n + \theta_j^{n-1}) \tag{9c}$$

$$\theta_{j-1/2}^n = \frac{1}{2}(\theta_j^n + \theta_{j-1}^n) \tag{9d}$$

Now we write the difference equations that are to approximate the three first order ordinary differential equations (1)-(3) according to Box method by considering one mesh rectangle. We start by writing the finite difference approximation of the above three equations using central difference quotients and average about the mid-point $(\xi^n, \eta_{j-1/2})$ of the segment P_1P_2 shown in the figure (a) and the finite difference approximations to the two first order differential equations (4)-(5) are written for the mid point $(\xi^{n-1/2}, \eta_{j-1/2})$ of the rectangle $P_1P_2P_3P_4$. This procedure yields.

$$h_j^{-1}(f_j^n - f_{j-1}^n) = u_{j-1/2}^n = \frac{u_{j-1}^n + u_j^n}{2} \tag{10}$$

$$h_j^{-1}(u_j^n - u_{j-1}^n) = v_{j-1/2}^n = \frac{v_{j-1}^n + v_j^n}{2} \tag{11}$$

$$h_j^{-1}(\theta_j^n - \theta_{j-1}^n) = p_{j-1/2}^n = \frac{p_{j-1}^n + p_j^n}{2} \tag{12}$$

$$\begin{aligned}&\frac{1}{2}\left[\frac{v_j^n - v_{j-1}^n}{h_j} + \frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right] + (p_1 fv)_{j-1/2}^{n-1/2} - (p_2 u^2)_{j-1/2}^{n-1/2} + (p_3 g)_{j-1/2}^{n-1/2}\\ &-(p_4 u)_{j-1/2}^{n-1/2} + (p_5 v)_{j-1/2}^{n-1/2}\\ &= (p_6)_{j-1/2}^{n-1/2}\xi_{j-1/2}^{n-1/2}\left[u_{j-1/2}^{n-1/2}\left\{\frac{u_{j-1/2}^n - u_{j-1/2}^{n-1}}{k_n}\right\} - v_{j-1/2}^{n-1/2}\left\{\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right\}\right]\end{aligned} \tag{13}$$

$$\begin{aligned}&\frac{A}{2}\left[\frac{p_j^n - p_{j-1}^n}{h_j} + \frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right] + (p_1 fp)_{j-1/2}^{n-1/2} + (p_5 p)_{j-1/2}^{n-1/2} - (p_7 gu)_{j-1/2}^{n-1/2}\\ &+B(p^2)_{j-1/2}^{n-1/2} = (p_6)_{j-1/2}^{n-1/2}\xi_{j-1/2}^{n-1/2}\left[u_{j-1/2}^{n-1/2}\left\{\frac{g_{j-1/2}^n - g_{j-1/2}^{n-1}}{k_n}\right\} - p_{j-1/2}^{n-1/2}\left\{\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right\}\right]\end{aligned} \tag{14}$$

Now from the equation (13) we get

$$\frac{1}{2}\left(\frac{v_j^n - v_{j-1}^n}{h_j}\right)+\frac{1}{2}\left(\frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right)+\frac{1}{2}\left\{(p_1 fv)_{j-1/2}^n+(p_1 fv)_{j-1/2}^{n-1}\right\}$$

$$-\frac{1}{2}\left\{(p_2 u^2)_{j-1/2}^n + (p_2 u^2)_{j-1/2}^{n-1}\right\}+\frac{1}{2}\left\{(p_3 g)_{j-1/2}^n + (p_3 g)_{j-1/2}^{n-1}\right\}$$

$$-\frac{1}{2}\left\{(p_4 u)_{j-1/2}^n + (p_4 u)_{j-1/2}^{n-1}\right\}+\frac{1}{2}\left\{(p_5 v)_{j-1/2}^n + (p_5 v)_{j-1/2}^{n-1}\right\}$$

$$=\frac{1}{2k_n}(p_6)_{j-1/2}^{n-1/2}\,\xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^n + u_{j-1/2}^{n-1}\right)\left(u_{j-1/2}^n - u_{j-1/2}^{n-1}\right)$$

$$-\frac{1}{2k_n}(p_6)_{j-1/2}^{n-1/2}\,\xi_{j-1/2}^{n-1/2}\left(v_{j-1/2}^n + v_{j-1/2}^{n-1}\right)\left(f_{j-1/2}^n - f_{j-1/2}^{n-1}\right)$$

$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n)+h_j^{-1}(v_j^{n-1} - v_{j-1}^{n-1})+(p_1)_{j-1/2}^n (fv)_{j-1/2}^n + (p_1)_{j-1/2}^{n-1}(fv)_{j-1/2}^{n-1}$$
$$-(p_2)_{j-1/2}^n (u^2)_{j-1/2}^n - (p_2)_{j-1/2}^{n-1}(u^2)_{j-1/2}^{n-1} + (p_3)_{j-1/2}^n (g)_{j-1/2}^n + (p_3)_{j-1/2}^{n-1} g_{j-1/2}^{n-1}$$
$$-(p_4)_{j-1/2}^n (u)_{j-1/2}^n - (p_4)_{j-1/2}^{n-1}(u)_{j-1/2}^{n-1} + (p_5)_{j-1/2}^n (v)_{j-1/2}^n + (p_5)_{j-1/2}^{n-1} v_{j-1/2}^{n-1}$$
$$=\alpha_n\{(u^2)_{j-1/2}^n - (u^2)_{j-1/2}^{n-1} - (fv)_{j-1/2}^n + v_{j-1/2}^n f_{j-1/2}^{n-1} - v_{j-1/2}^{n-1} f_{j-1/2}^n + (fv)_{j-1/2}^{n-1}\}$$

, *Where* $\alpha_n = (p_6)_{j-1/2}^{n-1/2}\dfrac{\xi_{j-1/2}^{n-1/2}}{k_n}$

$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n)+\left\{(p_1)_{j-1/2}^n + \alpha_n\right\}(fv)_{j-1/2}^n - \left\{(p_2)_{j-1/2}^n + \alpha_n\right\}(u^2)_{j-1/2}^n$$
$$+(p_3)_{j-1/2}^n (g)_{j-1/2}^n - (p_4)_{j-1/2}^n u_{j-1/2}^n + (p_5)_{j-1/2}^n (v)_{j-1/2}^n$$
$$=\alpha_n[-(u^2)_{j-1/2}^{n-1} + v_{j-1/2}^n f_{j-1/2}^{n-1} - v_{j-1/2}^{n-1} f_{j-1/2}^n + (fv)_{j-1/2}^{n-1}] - (p_1)_{j-1/2}^{n-1}(fv)_{j-1/2}^{n-1}$$
$$+(p_2)_{j-1/2}^{n-1}(u^2)_{j-1/2}^{n-1} - h_j^{-1}(v_j^{n-1} - v_{j-1}^{n-1}) - (p_3)_{j-1/2}^{n-1} g_{j-1/2}^{n-1} + (p_4)_{j-1/2}^{n-1} u_{j-1/2}^{n-1}$$
$$+(p_5)_{j-1/2}^{n-1} v_{j-1/2}^{n-1}$$

$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n)+\left\{(p_1)_{j-1/2}^n + \alpha_n\right\}(fv)_{j-1/2}^n - \left\{(p_2)_{j-1/2}^n + \alpha_n\right\}(u^2)_{j-1/2}^n$$
$$+(p_3)_{j-1/2}^n (g)_{j-1/2}^n - (p_4)_{j-1/2}^n u_{j-1/2}^n + (p_5)_{j-1/2}^n (v)_{j-1/2}^n + \alpha_n(v_{j-1/2}^{n-1} f_{j-1/2}^n -$$
$$v_{j-1/2}^n f_{j-1/2}^{n-1}) = \alpha_n\left\{(fv)_{j-1/2}^{n-1} - (u^2)_{j-1/2}^{n-1}\right\} - (p_1)_{j-1/2}^{n-1}(fv)_{j-1/2}^{n-1} + (p_2)_{j-1/2}^{n-1}(u^2)_{j-1/2}^{n-1}$$
$$-h_j^{-1}(v_j^{n-1} - v_{j-1}^{n-1}) - (p_3)_{j-1/2}^{n-1}(g)_{j-1/2}^{n-1} + (p_4)_{j-1/2}^{n-1}(u)_{j-1/2}^{n-1} - (p_5)_{j-1/2}^{n-1}(v)_{j-1/2}^{n-1}$$

$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n)+\left\{(p_1)_{j-1/2}^n + \alpha_n\right\}(fv)_{j-1/2}^n - \left\{(p_2)_{j-1/2}^n + \alpha_n\right\}(u^2)_{j-1/2}^n$$
$$+(p_3)_{j-1/2}^n (g)_{j-1/2}^n - (p_4)_{j-1/2}^n u_{j-1/2}^n + (p_5)_{j-1/2}^n (v)_{j-1/2}^n +$$
$$\alpha_n(v_{j-1/2}^{n-1} f_{j-1/2}^n - v_{j-1/2}^n f_{j-1/2}^{n-1}) = -L_{j-1/2}^{n-1} + \alpha_n\left\{(fv)_{j-1/2}^{n-1} - (u^2)_{j-1/2}^{n-1}\right\}$$

where, $L_{j-1/2}^{n-1} = (p_1)_{j-1/2}^{n-1}(fv)_{j-1/2}^{n-1} - (p_2)_{j-1/2}^{n-1}(u^2)_{j-1/2}^{n-1} + h_j^{-1}(v_j^{n-1} - v_{j-1}^{n-1})$
$+(p_3)_{j-1/2}^{n-1}(g)_{j-1/2}^{n-1} - (p_4)_{j-1/2}^{n-1} u_{j-1/2}^{n-1} + (p_5)_{j-1/2}^{n-1}(v)_{j-1/2}^{n-1}$

$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n) + \{(p_1)_{j-1/2}^n + \alpha_n\}(fv)_{j-1/2}^n - \{(p_2)_{j-1/2}^n + \alpha_n\}(u^2)_{j-1/2}^n$$
$$+(p_3)_{j-1/2}^n (g)_{j-1/2}^n - (p_4)_{j-1/2}^n g_{j-1/2}^n + (p_5)_{j-1/2}^n (v)_{j-1/2}^n$$
$$+\alpha_n (v_{j-1/2}^{n-1} f_{j-1/2}^n - v_{j-1/2}^n f_{j-1/2}^{n-1}) = R_{j-1/2}^{n-1} \quad (15)$$
$$R_{j-1/2}^{n-1} = -L_{j-1/2}^n + \alpha_n \{(fv)_{j-1/2}^{n-1} - (u^2)_{j-1/2}^{n-1}\}$$

Since $p_i = p_i(\xi)$, so we may write the above equation as

$$h_j^{-1}(v_j^n - v_{j-1}^n) + \{(p_1)^n + \alpha_n\}(fv)_{j-1/2}^n - \{(p_2)^n + \alpha_n\}(u^2)_{j-1/2}^n + (p_3)^n (g)_{j-1/2}^n$$
$$-(p_4)^n u_{j-1/2}^n + (p_5)^n v_{j-1/2}^n + \alpha_n (v_{j-1/2}^{n-1} f_{j-1/2}^n - v_{j-1/2}^n f_{j-1/2}^{n-1}) = R_{j-1/2}^{n-1} \quad (16)$$

Again from the equation (14) we get

$$\frac{A}{2}\left[\frac{p_j^n - p_{j-1}^n}{h_j} + \frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right] + (p_1 fp)_{j-1/2}^{n-1/2} + (p_5 p)_{j-1/2}^{n-1/2} - (p_7 gu)_{j-1/2}^{n-1/2}$$
$$+B(p^2)_{j-1/2}^{n-1/2} = (p_6)_{j-1/2}^{n-1/2} \xi_{j-1/2}^{n-1/2}\left[u_{j-1/2}^{n-1/2}\left\{\frac{\theta_{j-1/2}^n - \theta_{j-1/2}^{n-1}}{k_n}\right\} - p_{j-1/2}^{n-1/2}\left\{\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right\}\right]$$

$$\Rightarrow Ah_j^{-1}(p_j^n - p_{j-1}^n) + Ah_j^{-1}(p_j^{n-1} - p_{j-1}^{n-1}) + (p_1)_{j-1/2}^{n-1/2}(fp)_{j-1/2}^n + (p_1)_{j-1/2}^{n-1/2}(fp)_{j-1/2}^{n-1}$$
$$+(p_5)_{j-1/2}^{n-1/2}(p)_{j-1/2}^n + (p_5)_{j-1/2}^{n-1/2}(p)_{j-1/2}^{n-1} - (p_7)_{j-1/2}^{n-1/2}(gu)_{j-1/2}^n - (p_7)_{j-1/2}^{n-1/2}(gu)_{j-1/2}^{n-1}$$
$$+B(p^2)_{j-1/2}^n + B(p^2)_{j-1/2}^{n-1}$$
$$= \alpha_n\{(u_{j-1/2}^n + u_{j-1/2}^{n-1})(g_{j-1/2}^n - g_{j-1/2}^{n-1}) - (p_{j-1/2}^n + p_{j-1/2}^{n-1})(f_{j-1/2}^n - f_{j-1/2}^{n-1})\}$$
$$\Rightarrow Ah_j^{-1}(p_j^n - p_{j-1}^n) + (p_1)_{j-1/2}^{n-1/2}(fp)_{j-1/2}^n + (p_5)_{j-1/2}^{n-1/2}(p)_{j-1/2}^n - (p_7)_{j-1/2}^{n-1/2}(gu)_{j-1/2}^n$$
$$+B(p^2)_{j-1/2}^n = -M_{j-1/2}^{n-1} + \alpha_n\{(ug)_{j-1/2}^n - u_{j-1/2}^n g_{j-1/2}^{n-1} + u_{j-1/2}^{n-1} g_{j-1/2}^n - (pf)_{j-1/2}^n$$
$$+p_{j-1/2}^n f_{j-1/2}^{n-1} - p_{j-1/2}^{n-1} f_{j-1/2}^n\} + \alpha_n\{(fp)_{j-1/2}^{n-1} - (ug)_{j-1/2}^{n-1}\}$$

Where, $M_{j-1/2}^{n-1} = Ah_j^{-1}(p_j^{n-1} - p_{j-1}^{n-1}) + (p_1)_{j-1/2}^{n-1/2}(fp)_{j-1/2}^{n-1} + (p_5)_{j-1/2}^{n-1/2}(p)_{j-1/2}^{n-1}$
$-(p_7)_{j-1/2}^{n-1/2}(gu)_{j-1/2}^{n-1} + B(p^2)_{j-1/2}^{n-1}$

$$\Rightarrow Ah_j^{-1}(p_j^n - p_{j-1}^n) + \{(p_1)_{j-1/2}^{n-1/2} + \alpha_n\}(fp)_{j-1/2}^n - \{(p_7)_{j-1/2}^{n-1/2} + \alpha_n\}(ug)_{j-1/2}^n +$$
$$(p_5)_{j-1/2}^{n-1/2}(p)_{j-1/2}^n + B(p^2)_{j-1/2}^n - \alpha_n(u_{j-1/2}^{n-1} g_{j-1/2}^n - u_{j-1/2}^n g_{j-1/2}^{n-1} + p_{j-1/2}^n f_{j-1/2}^{n-1}$$
$$-p_{j-1/2}^{n-1} f_{j-1/2}^n) = T_{j-1/2}^{n-1} \quad (17)$$
$$T_{j-1/2}^{n-1} = -M_{j-1/2}^{n-1} + \alpha_n\{(fp)_{j-1/2}^{n-1} - (ug)_{j-1/2}^{n-1}\}$$

The boundary conditions become

$$f_0^n = 0 \qquad u_0^n = 0 \qquad g_0^n = 1 \qquad (18)$$
$$u_J^n = 0 \qquad \theta_J^n = 0$$

If we assume $f_j^{n-1}, u_j^{n-1}, v_j^{n-1}, \theta_j^{n-1}, p_j^{n-1}$ to be known for $0 \le j \le J$, equations (10) to (12) and (15) – (18) form a system of 5J + 5 non linear equations for the solutions of the 5J + 5 unknowns ($f_j^n, u_j^n, v_j^n, \theta_j^n, p_j^n$), j = 0,1,2 …J. These non linear system of algebraic equations are to be linearized by Newton's Quassy linearization method . We define the iterates [$f_j^n, u_j^n, v_j^n, \theta_j^n, p_j^n$], i = 0,1,2 ...N with initial values equal those at the previous x-station, which are usually the best initial guess available . For the higher iterates we set:

$$f_j^{(i+1)} = f_j^{(i)} + \delta f_j^{(i)} \qquad (19)$$

$$u_j^{(i+1)} = u_j^{(i)} + \delta u_j^{(i)} \qquad (20)$$

$$v_j^{(i+1)} = v_j^{(i)} + \delta v_j^{(i)} \qquad (21)$$

$$\theta_j^{(i+1)} = \theta_j^{(i)} + \delta \theta_j^{(i)} \qquad (22)$$

$$p_j^{(i+1)} = p_j^{(i)} + \delta p_j^{(i)} \qquad (23)$$

Now we substitute the right hand sides of the above equations in place of f_j^n, u_j^n, v_j^n, θ_j^n and p_j^n in equations (10) to (17) and (18) and omitting the terms that are quadratic in $\delta f_j^i, \delta u_j^i, \delta v_j^i, \delta \theta_j^i$ and δp_j^i we get the equations (10) to (12) in the following form:

$$\delta f_j^{(i)} - \delta f_{j-1}^{(i)} - \frac{h_j}{2}(\delta u_j^{(i)} + \delta u_{j-1}^{(i)}) = (r_1)_j \qquad (24)$$

Where $(r_1)_j = f_{j-1}^{(i)} - f_j^{(i)} + h_j u_{j-1/2}^{(i)}$ (25)

$$\delta u_j^{(i)} - \delta u_{j-1}^{(i)} - \frac{h_j}{2}\left(\delta v_j^{(i)} + \delta v_{j-1}^{(i)}\right) = (r_4)_j \tag{26}$$

$$(r_4)_j = u_{j-1}^{(i)} - u_j^{(i)} + h_j v_{j-1/2}^{(i)} \tag{27}$$

$$\delta \theta_j^{(i)} - \delta \theta_{j-1}^{(i)} - \frac{h_j}{2}\left(\delta p_j^{(i)} + \delta p_{j-1}^{(i)}\right) = (r_5)_j \tag{28}$$

Where $(r_5)_j = \theta_{j-1}^{(i)} - \theta_j^{(i)} + h_j p_{j-1/2}^{(i)}$ (29)

$$\begin{aligned}
&h_j^{-1}\left(v_j^{(i)} + \delta v_j^{(i)} - v_{j-1}^{(i)} - \delta v_{j-1}^{(i)}\right) + \left\{(p_1)_{j-1/2}^{n} + \alpha_n\right\}\left\{(fv)_{j-1/2}^{(i)} + \delta (fv)_{j-1/2}^{(i)}\right\} \\
&- \left\{(p_2)_{j-1/2}^{n} + \alpha_n\right\}\left\{(u^2)_{j-1/2}^{(i)} + \delta (u^2)_{j-1/2}^{(i)}\right\} + (p_3)_{j-1/2}^{n}\left\{(g)_{j-1/2}^{(i)} + \delta (g)_{j-1/2}^{(i)}\right\} \\
&- (p_4)^n\left\{u_{j-1/2}^{(i)} + \delta u_{j-1/2}^{(i)}\right\} + (p_5)^n\left\{v_{j-1/2}^{(i)} + \delta v_{j-1/2}^{(i)}\right\} + \alpha_n\left(f_{j-1/2}^{(i)} + \delta f_{j-1/2}^{(i)}\right)v_{j-1/2}^{n-1} \\
&- \alpha_n\left(v_{j-1/2}^{(i)} + \delta v_{j-1/2}^{(i)}\right)f_{j-1/2}^{n-1} = R_{j-1/2}^{n-1}
\end{aligned}$$

$$\Rightarrow (s_1)_j \delta v_j^{(i)} + (s_2)_j \delta v_{j-1}^{(i)} + (s_3)_j \delta f_j^{(i)} + (s_4)_j \delta f_{j-1}^{(i)} + (s_5)_j \delta u_j^{(i)} + (s_6)_j \delta u_{j-1}^{(i)} + (s_7)_j \delta g_j^{(i)} + (s_8)_j \delta g_{j-1}^{(i)} + (s_9)_j \delta p'_j + (s_{10})_j \delta p'_{j-1} = (r_2)_j \tag{30}$$

Where $(s_1)_j = (h_j^{-1} + \frac{(p_1)_{j-1/2}^n + \alpha_n}{2} f_j^{(i)} + \frac{(p_5)_{j-1/2}^n}{2} - \frac{1}{2}\alpha_n f_{j-1/2}^{n-1})$ (31)

$$(s_2)_j = (-h_j^{-1} + \frac{(p_1)_{j-1/2}^n + \alpha_n}{2} f_{j-1}^{(i)} + \frac{(p_5)_{j-1/2}^n}{2} - \frac{1}{2}\alpha_n f_{j-1/2}^{n-1}) \tag{32}$$

$$(s_3)_j = (\frac{(p_1)_{j-1/2}^n + \alpha_n}{2} v_j^{(i)} + \frac{1}{2}\alpha_n v_{j-1/2}^{n-1} \tag{33}$$

$$(s_4)_j = (\frac{(p_1)_{j-1/2}^n + \alpha_n}{2} v_{j-1}^{(i)} + \frac{1}{2}\alpha_n v_{j-1/2}^{n-1}) \tag{34}$$

$$(s_5)_j = -\{(p_2)_{j-1/2}^n + \alpha_n\} u_j^{(i)} - \frac{(p_4)_{j-1/2}^n}{2} \tag{35}$$

$$(s_6)_j = -\{(p_2)_{j-1/2}^n + \alpha_n\} u_{j-1}^{(i)} - \frac{(p_4)_{j-1/2}^n}{2}) \tag{36}$$

$$(s_7)_j = (p_3)_{j-1/2}^n / 2 \tag{37}$$

$$(s_8)_j = (p_3)_{j-1/2}^n / 2 \tag{38}$$

$$(s_9)_j = 0 \tag{39}$$

$$(s_{10})_j = 0 \tag{40}$$

$$(r_2)_j = R_{j-1/2}^{n-1} - \{h_j^{-1}(v_j^{(i)} - v_{j-1}^{(i)}) + ((p_1)_{j-1/2}^n + \alpha_n)(fv)_{j-1/2}^{(i)}\} + ((p_2)_{j-1/2}^n + \alpha_n)(u^2)_{j-1/2}^{(i)} - \alpha_n\{f_{j-1/2}^{(i)} v_{j-1/2}^{n-1} - f_{j-1/2}^{n-1} v_{j-1/2}^{(i)}\} - \{(p_3)_{j-1/2}^n \theta_{j-1/2}^{(i)} - (p_4)^n u_{j-1/2}^{(i)} + (p_5)^n v_{j-1/2}^{(i)}\} \tag{41}$$

Here the coefficients $(s_9)_j$ and $(s_{10})_j$, which are zero in this case , are included here for the generality.

Similarly by using the equations (19) to (23) in the equation (17) we get the following form:

$$(t_1)_j\,\delta p_j^{(i)} + (t_2)_j\,\delta p_{j-1}^{(i)} + (t_3)_j\,\delta f_j^{(i)} + (t_4)_j\,\delta f_{j-1}^{(i)} + (t_5)_j\,\delta u_j^{(i)} + (t_6)_j\,\delta u_{j-1}^{(i)} + (t_7)_j\,\delta \theta_j^{(i)} + (t_8)_j\,\delta \theta_{j-1}^{(i)} + (t_9)_j\,\delta v_j^{(i)} + (t_{10})_j\,\delta v_{j-1}^{(i)} = (r_3)_j \quad (42)$$

Where $(t_1)_j = Ah_j^{-1} + \dfrac{(p_1)_{j-1/2}^{n-1/2} + \alpha_n}{2} f_j^{(i)} + \dfrac{(p_5)_{j-1/2}^{n-1/2}}{2} + Bp_j^{(i)} - \dfrac{1}{2}\alpha_n f_{j-1/2}^{n-1}$ (43)

$$(t_2)_j = -Ah_j^{-1} + \frac{(p_1)_{j-1/2}^{n-1/2} + \alpha_n}{2} f_{j-1}^{(i)} + \frac{(p_5)_{j-1/2}^{n-1/2}}{2} + Bp_{j-1}^{(i)} - \frac{1}{2}\alpha_n f_{j-1/2}^{n-1} \quad (44)$$

$$(t_3)_j = \frac{(p_1)_{j-1/2}^{n-1/2} + \alpha_n}{2} p_j^{(i)} + \frac{1}{2}\alpha_n p_{j-1/2}^{n-1} \quad (45)$$

$$(t_4)_j = \frac{(p_1)_{j-1/2}^{n-1/2} + \alpha_n}{2} p_{j-1}^{(i)} + \frac{1}{2}\alpha_n p_{j-1/2}^{n-1} \quad (46)$$

$$(t_5)_j = -\frac{(p_7)_{j-1/2}^{n-1/2} + \alpha_n}{2} g_j^{(i)} + \frac{1}{2}\alpha_n g_{j-1/2}^{n-1} \quad (47)$$

$$(t_6)_j = -\frac{(p_7)_{j-1/2}^{n-1/2} + \alpha_n}{2} g_{j-1}^{(i)} + \frac{1}{2}\alpha_n g_{j-1/2}^{n-1} \quad (48)$$

$$(t_7)_j = 0 \quad (49)$$

$$(t_8)_j = 0 \quad (50)$$

$$(t_9)_j = -\frac{(p_7)_{j-1/2}^{n-1/2} + \alpha_n}{2} u_j^{(i)} - \frac{1}{2}\alpha_n u_{j-1/2}^{n-1} \quad (51)$$

$$(t_{10})_j = \frac{(p_7)_{j-1/2}^{n-1/2} + \alpha_n}{2} u_{j-1}^{(i)} - \frac{1}{2}\alpha_n u_{j-1/2}^{n-1} \quad (52)$$

$$(r_3)_j = T_{j-1/2}^{n-1} - Ah_j^{-1}(p_j^{(i)} - p_{j-1}^{(i)}) - \frac{(p_1)_{j-1/2}^{n-1/2} + \alpha_n}{2}(fp)_{j-1/2}^{(i)} + \frac{(p_5)_{j-1/2}^{n-1/2}}{2}(p)_{j-1/2}^{(i)} - \frac{(p_7)_{j-1/2}^{n-1/2} + \alpha_n}{2}(gu)_{j-1/2}^{(i)} + B(p^2)_{j-1/2}^{(i)} + \alpha_n\{(ug)_{j-1/2}^{(i)} + u_{j-1/2}^{n-1} g_{j-1/2}^{(i)} - g_{j-1/2}^{n-1} u'_{j-1/2}\} + \alpha_n\{f_{j-1/2}^{n-1} p_{j-1/2}^{(i)} - p_{j-1/2}^{n-1} f_{j-1/2}^{(i)}\} \quad (53)$$

The boundary conditions (18) become

$$\delta f_0^n = 0,\ \delta u_0^n = 0,\ \delta g_0^n = 0 \qquad \delta u_J^n = 0,\ \delta \theta_J^n = 0 \quad (54)$$

Which just express the requirement for the boundary conditions to remain during the iteration process.

Now the system of linear equations (24) - (30), (41), (42) and (53) together with the boundary conditions (54) can written in a block matrix from a coefficient matrix, which are solved by modified 'Keller Box' methods especially introduced by Keller (1978). Later, this method has been used most efficiently by Cebeci and Bradshow (1984) and recently by Hossain (1992). Hossain et. al. (1994), taking the initial iteration to be given by convergent solution at $\xi = \xi_{j-1}$. Results are shown in graphical form by using the numerical values obtained from the above technique. The solutions of the above equations (2.14) and (2.15) together with the boundary conditions (2.18) enable us to calculate the skin friction τ and the rate of heat transfer Q at the surface in the boundary layer from the following relations:

$$\tau = \left(\frac{V}{g\beta(T_w - T_\infty)} \right) \left(\frac{\partial u}{\partial y} \right)_{y=0} = \xi f''(\xi, 0) \tag{55}$$

$$Q = -\left(\frac{\nu}{V(T_w - T_\infty)} \right) \left(\frac{\partial T}{\partial y} \right)_{y=0} = -\left(1 + \frac{4}{3} Rd\ \Delta^3 \right) \xi^{-1} g'(\xi, 0) \tag{56}$$

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