M.Sc. Engineering Thesis

Minimum Segment Drawings of Outerplanar Graphs

By
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The thesis titled "**Minimum Segment Drawings of Outerplanar Graphs**", submitted by Muhammad Abdullah Adnan, Roll No. 100605037P, Session October 2006, to the Department of Computer Science and Engineering, Bangladesh University of Eugineering and Technology, has been accepted as satisfactory in partial fulfillment of the requirements for the degree of Master of Science in Computer Science and Engineering and approved as to its style and contents. Examination held on December 31, 2008.

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Candidate's Declaration

This is to certify that the work presented in this thesis entitled "**Minimum Segment Drawings of Outerplanar Graphs**" is the outcome of the investigation carried out by me under the supervision of Professor Dr. Md. Saidur Rahman in the Department of Computer Science and Engineering, Bangladesh University of Engineering and Technology (BUET), Dhaka. It is also declared that neither this thesis nor any part thereof has been submitted or is being currently submitted anywhere else for the award of any degree or diploma.

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Abstract

In a straight line drawing of a planar graph each vertex is drawn as a point and each edge is drawn as a straight line segment. One of the important aesthetic criteria for a straight line drawing is to minimize the number of maximal straight line segments required for the straight line drawing. Finding a minimum segment drawing of a planar graph is analogous to aligning maximum number of objects according to their relations. Hence the problem of obtaining a minimum segment drawing of a given graph has important practical applications in the fields like Optical Fiber Communication, bend minimization in VLSI Layout Planning, aesthetics in Architectural Floorplanning, antenna placement in Sensor Networks, etc. The problem of finding minimum segment drawings has been studied for different classes of planar graphs which include trees, outerplanar graphs, 2-trees and planar 3-trees. Researchers were able to give bounds on the number of segments required for straight line drawing of the classes of graphs mentioned above. Recently, Samee *et al.* gave an algorithm to find minimum segment drawings of a restricted class of series-parallel graphs with the maximum degree three. Other than that no algorithm has been devised so far for finding minimum segment drawings of non trivial classes of planar graphs.

Outerplanar graphs are an important subclass of planar graphs where every vertex of the graph appears on the outerface. Dujmović *et al.* posed an open problem of finding a polynomial time algorithm to compute an outerplanar drawing of a given outerplanar graph with the minimum number of segments. Motivated by this open problem, in this thesis we give a linear-time algorithm for finding a minimum segment drawing of a dual-path outerplane graph. We also give an algorithm for finding a minimum segment drawing of a subdivision of a dual-path outerplane graph.

Chapter 1

Introduction

A graph consists of a set of vertices and a set of edges, each joining two vertices. Graphs are abstract structures that are used to model structural information arising from many fields, such as economics, engineering, social sciences, genetics, mathematics and computer science. Graphs, as models of information, are often required to be visualized or drawn in ways that are easy to read and understand, or they are required to be laid out while satisfying some physical constraint. Graph drawing addresses the problems of developing algorithmic techniques for their automatic generation. Although graph drawing problems are attractive from a purely mathematical standpoint, they also arise in many application areas, including VLSI design, visualization, and DNA mapping.

There are infinitely many drawings of a graph. Producing a good drawing of a graph typically involves the optimization of several application-specific criteria. More often the idea of a good drawing, regardless of its purpose, coincides with aesthetics and edge straightness. Many bends or equivalently many line segments in the drawing increase the difficulty for the eye to follow the course of the edges incident on a vertex. For this reason, the total number of line segments should be kept small when the readability of a drawing is of concern.

In this thesis, we deal with the problem of drawing graphs with the minimum number of segments. As this problem is relatively a new problem in the area of graph drawing, it has not been studied well so far. As a result, neither the counting of number of segments required for a minimum segment drawing nor any algorithm to construct such a drawing for a given graph is known. However while the recent research works have failed to develop significant algorithms for minimum segment drawings, they were able to give bounds on the number of segments required for a straight line drawing of a graph. Hence it remains open to develop algorithmic techniques both for the counting and the drawing problems.

In this chapter, we discuss the applications of drawing graphs with the minimum number of segments. We also review the previous results regarding the bounds on the number of segments and present the objectives of the thesis. We start with Section 1.1 by giving a precise definition of the minimum segment drawing problem. Section 1.2 describes some practical applications of the problem. Section 1.3 reviews the previous works in this field. Section 1.4 addresses the scope of this thesis. In Section 1.5, we present the summary of the thesis.

1.1 Minimum Segment Drawings

A common requirement for an aesthetically pleasing drawing of a planar graph is that all edges are drawn as straight line segments without edge-crossings [DESW07a, DETT99, Far48, NR04, RNN99]. A *straight line drawing* is such a drawing in which each vertex is drawn as a point and each edge is drawn on a straight line segment. A *maximal segment* is a drawing of a maximal set of edges that form a straight line segment. One of the important criteria for the straight line drawing is to minimize the number of maximal segments. A *minimum segment drawing* of a planar graph G is a straight line drawing of G with the minimum number of maximal segments. Figure 1.1(a) depicts a straight line drawing of a planar graph with 34 maximal segments, while Figure 1.1(b) depicts a minimum segment drawing of the same graph with 15 maximal segments.

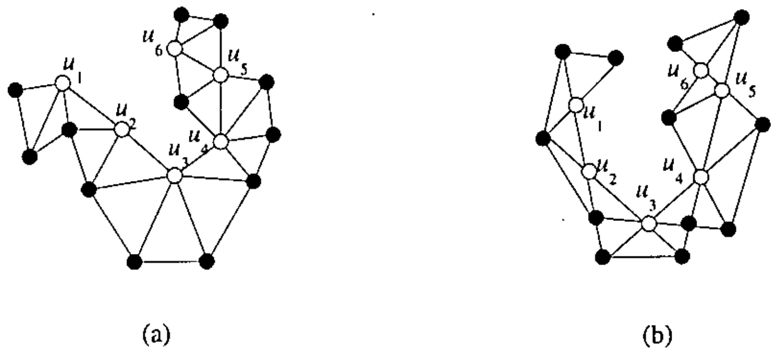


Figure 1.1: (a) A planar graph G, (b) a minimum segment drawing of G.

1.2 Applications of Minimum Segment Drawings

Although planar straight line drawings are considered as the best means for visualizing planar graphs [PCJ96, Pur97], minimization of the number of segments in these drawings can greatly enhance the overall readability [DESW07a]. On the other hand, fewer number of segments in the drawing often implies fewer number of slopes in the drawing [DESW07a]. Both these characteristics have important effects on scan conversion algorithms for lines in raster devices. In raster devices, the grid location of each pixel has to be computed separately. Moreover, this computation is largely dependent on the slope of the segment [FDFH03]. If both the number of segments and the number of slopes in the drawing are few, then these computations can be performed faster yielding a faster rendering of the drawing.

Moreover, finding a minimum segment drawing of a planar graph is analogous to aligning maximum number of objects according to their relations. Hence the problem of obtaining a minimum segment drawing of a given graph has important practical applications in the fields like Optical Fiber Communication, bend minimization in VLSI Layout Planning [KL84, RNN99], aesthetics in Architectural floorplanning [DETT94, DETT99], antenna placement in Sensor Networks [KD05], etc.

1.3 Challenges

In this section, we illustrate the challenges that we face to solve the problem of finding a minimum segment drawing of a graph.

Given a graph first the question arises -"How do we minimize number of segments in a drawing?". One may think of minimizing the number of segments by drawing the faces triangular in the drawing. But this does not lead to a minimum segment drawing because making faces triangular does not always minimizes segments. Figure 1.2(a) shows a straight line drawing of a graph where every face is triangulated. But this drawing is not a minimum segment drawing because this graph has another drawing (see Fig. 1.2(b)) which requires less number of segments.

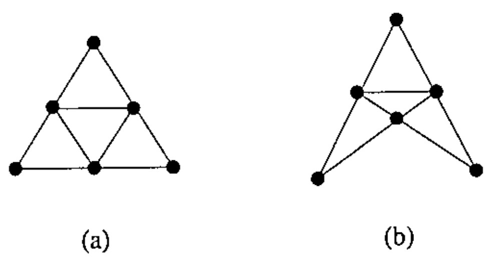


Figure 1.2: (a) A straight line drawing of a graph G, (b) a minimum segment drawing of G.

Another approach to finding minimum segment drawing may be by making edges incident to a vertex pairwise collinear. Then the problem turns into choosing the vertices to which the incident edges will be pairwise collinear. Greedy choice on the vertices with higher degree does not always give optimal solution. Hence the challenge here is to find an optimum choice of vertices. In this thesis we follow this approach of making edges incident to a vertex pairwise collinear and give an algorithm for choosing that set of vertex.

1.4 Previous Results

In this section, we review the previous works regarding drawing planar graphs with few slopes and few segments.

The problem of computing straight line drawings of planar graphs has been studied for long with various application specific objectives in the focus [Far48, FPP90, Pur97, Sch90, Wag36]. Recently, Dujmović *et al.* have studied this problem with the new objective of minimizing the number of segments in a drawing [DESW07a], and the insightful results presented in their work have established a new line of research henceforth. However, as their results suggest, this problem is quite difficult for most of the non-trivial graph classes. For most of the cases, bounds have been given on the number of segments in a drawing, but no algorithm is known for computing a minimum segment drawing. For example, although Dujmović *et al.* have provided an algorithm for computing minimum segment drawings of trees, no such algorithm is known for biconnected and triconnected plane graphs.

Recently, Samee et. al. [SAAR08] studied the problem of finding minimum segment drawing for planar graphs by restricting the maximum degree of that graph. They first gave a lower bound on the number of segments required for straight line drawings of series-parallel graphs with the maximum degree three. Then they presented an algorithm for finding a minimum segment drawing of such a graph. Other than the degree restricted case for series-parallel graphs, to the best of our knowledge no algorithm has yet been devised for computing minimum segment drawings of a general class of graph.

The known results on the minimum segment drawing problem are listed in Table 1.1. Meanings of the notations used in this table are as follows. The symbol η denotes the number of odd degree vertices in a tree. The symbol n denotes the number of vertices in a graph. For a series-parallel graph G with $\Delta(G) = 3$, the symbols P and N denote the number of P-nodes and the number of primitive P-nodes in an SPQ-tree of G, and $k \in \{1, 2\}$ based on a characterization of the SPQ-tree [DETT99] of G.

1.5 Scope of this Thesis

In this thesis, we consider the problem of finding a minimum segment drawing for a subclass of "outerplane graphs." Outerplane graphs comprise an important subclass of plane graphs

Graph class	Bound on segments		Minimum Segment Drawing Algorithm	Reference
	Lower	Upper		
Tree	$\frac{n}{2}$	$\frac{n}{2}$	Yes	[DESW07a]
Plane 2-connected	$\frac{5}{2}n$	–	No	
Planar 2-connected	$2n$	–	No	
Plane 3-connected	$2n$	$\frac{5}{2}n$	No	
Planar 3-connected	$2n$	$\frac{5}{2}n$	No	
Plane 3-connected cubic	–	$n+2$	No	
Series-parallel ($\Delta = 3$)	$P+N+k$	$P+N+k$	Yes	[SAAR08]

Table 1.1: Known results for the minimum segment drawing problem.

where every vertex appear on the outerface. Dujmović *et al.* [DESW07a] posed an open problem of obtaining a polynomial-time algorithm to find a minimum segment drawing of a given outerplane graph. Motivated by this open problem in this thesis, we study the minimum segment drawing problem for subclass of outerplane graphs.

We first study the minimum segment drawing problem for maximal dual-path outerplane graph. To compute a minimum segment drawing, at first the graph is divided into smaller graphs called fan graphs. Then the minimum segment drawings of the fan graphs are computed. Then the drawings of the fan graphs are patched in such a way that after patching the combined drawing becomes a minimum segment drawing of the original graph.

By using the algorithm for finding a minimum segment drawing of a dual-path maximal outerplane graph, we extend our result for dual-path outerplane graphs. To compute a minimum segment drawing of a dual-path outerplane graph, at first the vertices of degree two are removed from the graph. Then the graph is divided into maximal components where each maximal component is a maximal outerplanar graph. Then the minimum segment drawings of the maximal components are computed. After computing the minimum segment drawings of the maximal components, the drawings are patched in such a way that after patching the combined drawing becomes a minimum segment drawing of the original graph. Then the vertices of degree two are added to the drawing.

We then extend our algorithm for subdivision of outerplanar graphs. To compute a minimum segment drawing of a subdivision of an outerplanar graph, the inner vertices of degree two are removed from the graph. Thus the graph transforms into an outerplanar graph and a minimum segment drawing for that graph is computed. Then the vertices of degree two are added to the drawing. Table 1.2 shows the known algorithms for the minimum segment drawing problem for different classes of graphs.

Graph class	Time complexity	Reference
Tree	$O(n)$	[DESW07a]
Series-parallel ($\Delta = 3$)	$O(n)$	[SAAR08]
Dual-path outerplanar	$O(n)$	**[Ours]**
Subdivision of Dual-path outerplanar	$O(n)$	

Table 1.2: Algorithms for the minimum segment drawing problem.

1.6 Summary

In this thesis we develop efficient algorithms for finding minimum segment drawings of subclasses of outerplanar graphs. The main results of this thesis are as follows.

1. We give a linear-time algorithm for computing a minimum segment drawing of a given dual-path maximal outerplanar graph.

2. We present a linear-time algorithm for computing a minimum segment drawing of a given dual-path outerplanar graph.

3. We develop a linear-time algorithm for computing a minimum segment drawing of a subdivision of a dual-path outerplanar graph.

The rest of the thesis is organized as follows. Chapter 2 defines basic terminologies relevant to graphs, graph algorithms and graph drawing problems to understand our research work. Chapter 3 describes the algorithm that computes a minimum segment drawing for a dual-path maximal outerplane graph in linear time. Chapter 4 deals with computing minimum segment drawings of dual-path outerplane graphs. In Chapter 5, we give an algorithm for a subdivision of a dual-path outerplane graph. Finally, Chapter 6 is the conclusion.

Chapter 2

Preliminaries

In this chapter, we define some basic terminology of graph theory and algorithms. Definitions which are not included in this chapter will be introduced as they are needed. We start, in Section 2.1, by giving some definitions of standard graph-theoretical terms used throughout the remainder of this thesis. We devote Section 2.2 to define terms related to plane graphs. The notion of time complexity is introduced in Section 2.3. Finally we give a review of the literature on the minimum segment drawing problem in Section 2.4.

2.1 Basic Terminology

In this section we give definitions of some theoretical terms used throughout the remainder of this thesis.

2.1.1 Graphs

A *graph* G is a structure (V, E) which consists of a finite set of *vertices* V and a finite set of *edges* E; each edge is an unordered pair of distinct vertices. We denote the set of vertices of G by $V(G)$ and the set of edges by $E(G)$. Fig. 2.1 depicts a graph G where each vertex in $V(G) = \{v_1, v_2, \ldots, v_6\}$ is drawn as a small dark circle and each edge in $(E(G) = \{e_1, e_2, \ldots, e_9\}$

is drawn by a line segment.

If a graph G has no "multiple edges" or "loops", then G is said to be a *simple graph*. *Multiple edges* join the same pair of vertices, while a *loop* joins a vertex with itself. A graph in which loops and multiple edges are allowed is called a *multigraph*. Often it is clear from the context that the graph is simple. In such cases, a simple graph is called a *graph*. In the remainder of thesis we assume that G has no loop.

We denote an edge between two vertices u and v of G by (u, v) or simply by uv. If $uv \in E$ then two vertices u and v of graph G are said to be *adjacent*; edge uv is then said to be *incident* to vertices u and v; u is a neighbor of v. The *degree* of a vertex v in G, denoted by $d(v)$, is the number of edges incident to v. In the graph shown in Fig. 2.1 vertices v_1 and v_2 are adjacent, and $d(v_1) = 3$, since four of the edges, namely e_1, e_5 and e_6 are incident to v_1. By $\Delta(G)$, we mean the maximum degree of the vertex in a graph.

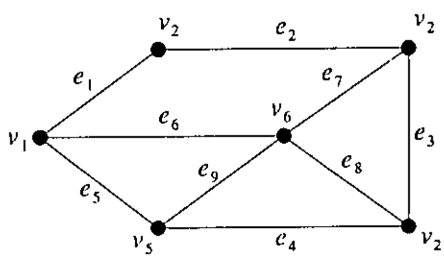


Figure 2.1: Illustration of a graph.

2.1.2 Subgraphs

A *subgraph* of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$; we then write $G' \subseteq G$. If G' contains all the edges of G that join two vertices in V', then G' is said to be the *subgraph induced by* V', and is denoted by $G[V']$. Fig. 2.2 depicts a subgraph of G in Fig. 2.1 induced by $\{v_1, v_2, v_5, v_6\}$.

We often construct new graphs from old ones by deleting some vertices or edges. If v is a vertex of a given graph $G = (V, E)$, then $G - v$ is the subgraph of G obtained by deleting the vertex v and all the edges incident to v. More generally, if V' is a subset of V, then $G - V'$ is

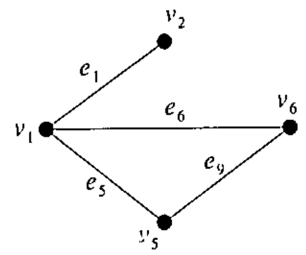


Figure 2.2: A vertex-induced subgraph.

the subgraph of G obtained by deleting the vertices in V' and all the edges incident to them. Then $G - V'$ is a subgraph of G induced by $V - V'$. Similarly, if e is an edge of a G, then $G - e$ is the subgraph of G obtained by deleting the edge e. More generally, if $E' \subseteq E$, then $G - E'$ is the subgraph of G obtained by deleting the edges in E'.

2.1.3 Connectivity

A graph G is a *connected graph* if for every pair $\{u, v\}$ of distinct vertices there is a path between u and v. A graph which is not connected is called a *disconnected graph.* A *connected component* of a graph is a maximal connected subgraph. The graph in Fig. 2.3(a) is a connected graph since there is a path for every pair of distinct vertices of the graph. On the other hand the graph in Fig. 2.3(b) is a disconnected graph since there is no path between v_1 and v_5. The graph in Fig. 2.3(b) has two connected components G_1 and G_2 indicated by dotted lines.

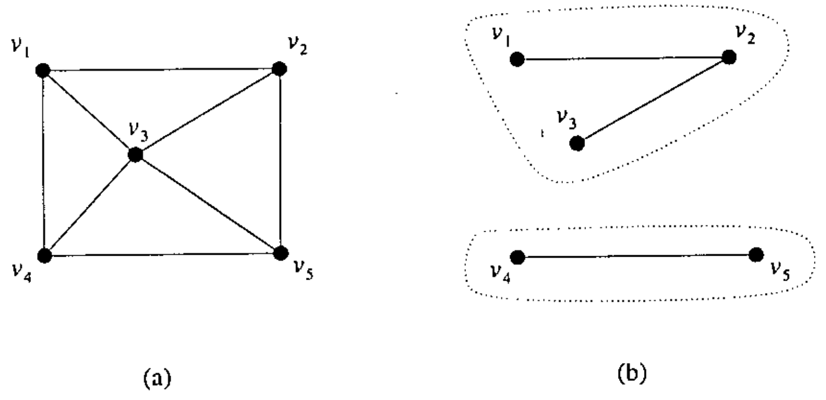


Figure 2.3: (a) A connected graph (b) a disconnected graph with two connected components.

The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph K_1. We say that G is *k-connected* if $\kappa(G) \geq k$. We call a set of vertices in a connected graph G a *separator* or a *vertex cut* if the removal of the vertices in the set results in a disconnected or single-vertex graph. If a vertex-cut contains exactly one vertex then we call the vertex a *cut vertex*. A *block* is a maximal biconnected subgraph of G.

2.1.4 Paths and Cycles

A v_0-v_l *walk*, $v_0, e_1, v_1, \ldots, v_{l-1}, e_l, v_l$, in G is an alternating sequence of vertices and edges of G, beginning and ending with a vertex, in which each edge is incident to two vertices immediately preceding and following it. If the vertices $v_0, v_1, \ldots, v_l$ are distinct (except possibly v_0, v_l), then the walk is called a *path* and usually denoted either by the sequence of vertices $v_0, v_1, \ldots, v_l$ or by the sequence of edges $e_1, e_2, \ldots, e_l$. The length of the path is l, one less than the number of vertices on the path. A path or walk is closed if $v_0 = v_l$. A closed path containing at least one edge is called a *cycle*.

2.1.5 Trees

A *tree* is a connected graph containing no cycle. Figure 2.4 is an example of a tree. The vertices in a tree are usually called *nodes* . A *rooted tree* is a tree in which one of the nodes is distinguished from the others. The distinguished node is called the *root* of the tree. The root of a tree is generally drawn at the top. In Figure 2.4, the root is v_1. Every node u other than the root is connected by an edge to some other node p called the *parent* of u. We also call u a *child* of p. We draw the parent of a node above that node. For example, in Figure 2.4, v_1 is the parent of v_2, v_3 and v_4, while v_2 is the parent of v_5 and v_6; v_2, v_3 and v_4 are children of v_1, while v_5 and v_6 are children of v_2. A *leaf* is a node of a tree that has no children. An *internal node* is a node that has one or more children. Thus every node of a tree is either a leaf or an internal node. In Figure 2.4, the leaves are v_4, v_5, v_6, v_7 and v_8, and the nodes v_1, v_2 and v_3

are internal nodes.

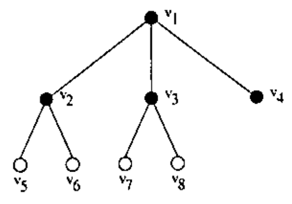


Figure 2.4: Illustration of a tree.

2.2 Planar Graphs and Plane Graphs

In this section we give some definitions related to planar graphs used in the remainder of the thesis. For readers interested in planar graphs we refer to [NC88].

A graph is a *planar graph* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. Note that a planar graph may have an exponential number of embeddings. Fig. 2.5 shows four planar embeddings of the same planar graph.

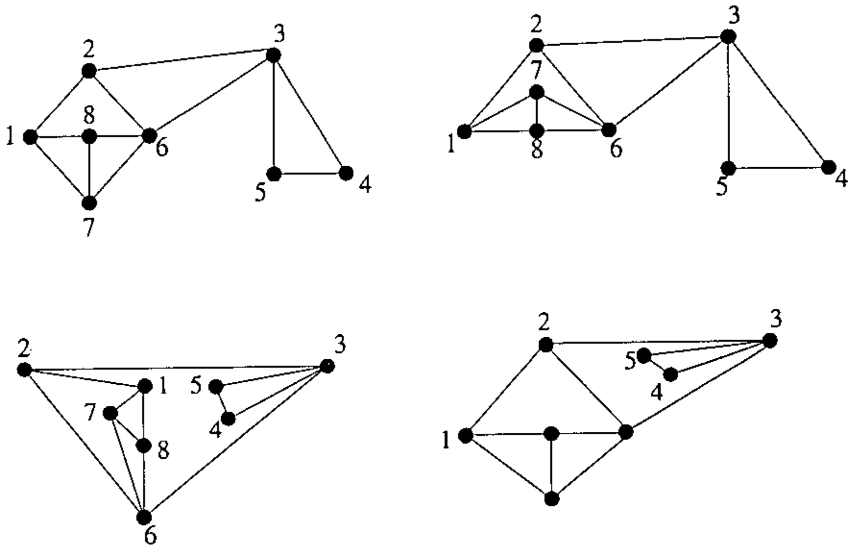
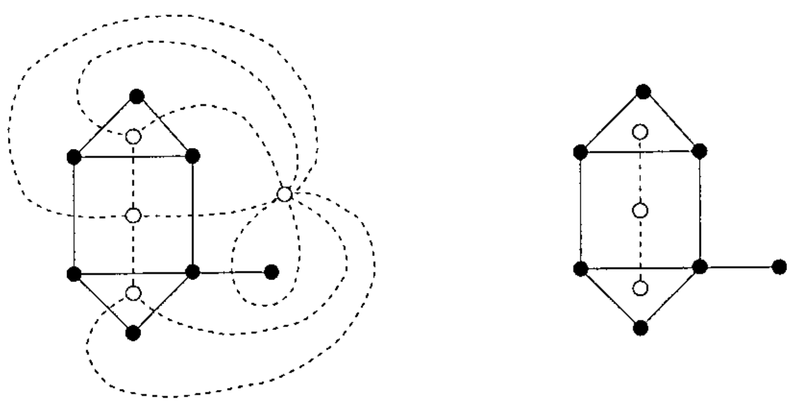


Figure 2.5: Four planar embeddings of the same planar graph.

A *plane graph* is a planar graph with a fixed embedding. A plane graph divides the plane into connected regions called *faces*. We regard the contour of a face as a clockwise cycle formed by the edges on the boundary of the face. We denote the contour of the outer face of graph G by $C_o(G)$. A cycle of a plane graph is called a *facial cycle* if it is the boundary of a face f and denoted by C_f.

2.2.1 Dual Graphs

For a plane graph G, we often construct another graph G^* called the *geometric dual* of G as follows. A vertex v_i^* is placed in each face F_i of G; these are the vertices of G^*. Corresponding to each edge e of G we draw an edge e^* which crosses e (but no other edge of G and joins the vertices v_i^* which lie in the faces F_i adjoining e; these are the edges of G^*. The construction is illustrated in Fig. 2.6; the vertices v_i^* are represented by small white circles, and the edges e^* of G^* by dotted lines. G^* is not necessarily a simple graph even if G is simple. Clearly the dual G^* of a plane graph G is also plane. One can easily observe the following lemma.



(a) (b)

Figure 2.6: A plane graph G and its dual graph G^*.

Lemma 2.2.1 *Let G be a connected plane graph with n vertices, m edges and f faces, and let the dual G^* have n^* vertices, m^* edges and f^* faces; then $n^* = f, m^* = m$, and $f^* = n$.*

Clearly the dual of the dual of the plane graph G is the original graph G. The *weak dual* of a plane graph is the dual of that plane graph disregarding the outerface. Figure 2.6(b) shows the weak dual of the graph.

2.2.2 Outerplane Graphs

Outerplane graphs are an important subclass of plane graph. A plane graph is *outerplane* if all the vertices are on the boundary of the outerface. An outerplane graph G is *maximal* if no edge can be added to G without loosing outerplanarity. Every inner face of a maximal outerplane graph is a triangle. Every outerplane graph has at least two vertices of degree two. In a maximal outerplane graph, the inner face containing a vertex of degree two is called an *ear*. We call the vertex of degree two of an ear an *ear vertex* and the edges of the ear as *ear edges*. The weak dual of an outerplane graph is a tree or a forest. Figure 2.7 represents an outerplane graph G and its weak dual is shown by dotted lines.

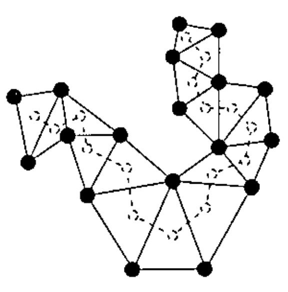


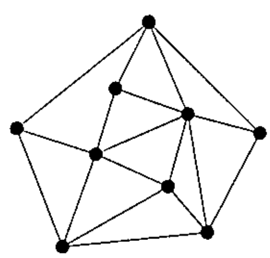
Figure 2.7: An outerplane graph G and its weak dual.

A *dual-path outerplane graph* is defined to be an outerplane graph with one or more inner faces whose weak dual is a path. Thus a dual-path outerplane graph has three or more vertices, is 2-connected, and has at most two ears. The two ears divide the boundary of the outerface into two paths called *outer paths*.

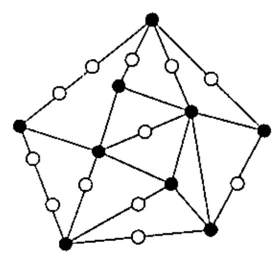
A dual-path maximal outerplane graph is called a *fan graph* if a vertex is adjacent to every other vertex; the vertex is called a *center* of the fan graph. The *fan of a vertex* v in an outerplane graph G is the plane subgraph of G induced by $\{v\} \cup N(v)$.

2.3 Subdivision of a Graph

Subdividing an edge (u, v) of a graph G is the operation of deleting the edge (u, v) and adding a path $u(= w_0), w_1, w_2, \cdots, w_k, v(= w_{k+1})$ through new vertices $w_1, w_2, \cdots, w_k$, $k \geq 1$, of degree two. A graph G' is said to be a *subdivision* of a graph G if G' is obtained from G by subdividing some of the edges of G. Figure 2.8(b) shows a subdivision of the graph in Figure 2.8(a).



(a)



(b)

Figure 2.8: An outerplane graph G and its weak dual.

2.4 Algorithms and Complexity

In this section we briefly introduce some terminologies related to complexity of algorithms. For interested readers, we refer the book of Garey and Johnson [GJ79].

The most widely accepted complexity measure for an algorithm is the *running time* which is expressed by the number of operations it performs before producing the final answer. The number of operations required by an algorithm is not the same for all problem instances. Thus, we consider all inputs of a given size together, and we define the *complexity of the algorithm for that input size* to be the worst case behavior of the algorithm on any of these inputs. Then the running time is a function of size n of the input.

2.4.1 The Notation $O(n)$

In analyzing the complexity of an algorithm, we are often interested only in the "asymptotic behavior", that is, the behavior of the algorithm when applied to very large inputs. To deal with such a property of functions we shall use the following notations for asymptotic running time. Let $f(n)$ and $g(n)$ are the functions from the positive integers to the positive reals, then we write $f(n) = O(g(n))$ if there exists positive constants c_1 and c_2 such that $f(n) \leq c_1 g(n) + c_2$ for all n. Thus the running time of an algorithm may be bounded from above by phrasing like "takes time $O(n^2)$".

2.4.2 Polynomial Algorithms

An algorithm is said to be *polynomially bounded* (or simply *polynomial*) if its complexity is bounded by a polynomial of the size of a problem instance. Examples of such complexities are $O(n)$, $O(nlogn)$, $O(n^{100})$, etc. The remaining algorithms are usually referred as *exponential* or *nonpolynomial*. Examples of such complexity are $O(2^n)$, $O(n!)$, etc. When the running time of an algorithm is bounded by $O(n)$, we call it a *linear-time* algorithm or simply a *linear* algorithm.

2.4.3 NP-complete

There are a number of interesting computational problems for which it has not been proved whether there is a polynomial time algorithm or not. Most of them are "NP-complete", which we will briefly explain in this section.

The state of algorithms consists of the current values of all the variables and the location of the current instruction to be executed. A *deterministic algorithm* is one for which each state, upon execution of the instruction, uniquely determines at most one of the following state (next state). All computers, which exist now, run deterministically. A problem Q is in the *class P* if there exists a deterministic polynomial-time algorithm which solves Q. In contrast, a *nondeterministic algorithm* is one for which a state may determine many next states simultaneously. We may regard a nondeterministic algorithm as having the capability of branching off into many copies of itself, one for the each next state. Thus, while a deterministic algorithm must explore a set of alternatives one at a time, a nondeterministic algorithm examines all alternatives at the same time. A problem Q is in the *class NP* if there exists a nondeterministic polynomial-time algorithm which solves Q. Clearly $P \subseteq NP$.

Among the problems in *NP* are those that are hardest in the sense that if one can be solved in polynomial-time then so can every problem in *NP*. These are called *NP-complete* problems. The class of *NP*-complete problems has the following interesting properties.

(a) No *NP*-complete problem can be solved by any known polynomial algorithm.

(b) If there is a polynomial algorithm for any *NP*-complete problem, then there are polynomial algorithms for all *NP*-complete problems.

Sometimes we may be able to show that, if problem Q is solvable in polynomial time, all problems in *NP* are so, but we are unable to argue that $Q \in NP$. So Q does not qualify to be called *NP*-complete. Yet, undoubtedly Q is as hard as any problem in *NP*. Such a problem Q is called *NP-hard*.

2.5 Drawing Graphs with Few Segments

A *straight line drawing* of a plane graph is a drawing in which each vertex is drawn as a point and each edge is drawn on a straight line segment. A *maximal segment* is a drawing of a maximal set of edges that form a straight line segment. We call the number of maximal segments in a straight line drawing D of a plane graph the *segment count of* D, and denote it by $sc(D)$. We call the number of maximal segments in a minimum segment drawing of a plane graph G the *segment count of* G, and denote it by $sc(G)$.

The problem of computing minimum segment drawings of planar graphs is a relatively new one, and was originated from the seminal work of Dujmović *et al.* [DESW07a]. In this section we give an overview of some of the most important results presented in [DESW07a]. It is worth mentioning that, although Dujmović *et al.* have given both lower bounds and upper bounds on the number of segments in drawings of several important graph classes, algorithm for computing minimum segment drawings was given only for trees. More interestingly, for some non-trivial graph classes, like plane biconnected and planar biconnected graphs, even no upper bound was given. Similarly, for plane triconnected cubic graphs, no lower bound was given. Nevertheless, each of these results is quite insightful and is necessary for subsequent research on this problem.

2.5.1 Trees

Let T be a tree. Let η denote the number of odd degree vertices of T. It was shown in [DESW07a] that any planar straight-line drawing Γ of T requires at least $\frac{\eta}{2}$ number of segments. The claim holds since each odd degree vertex u of T is an endpoint of some segment in Γ. It is notable that, the number of odd degree vertices in a graph is even and hence, $\frac{\eta}{2}$ is an integer.

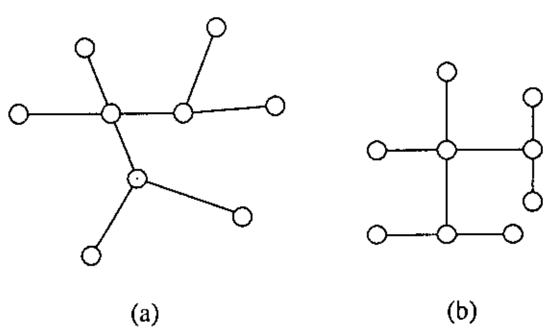
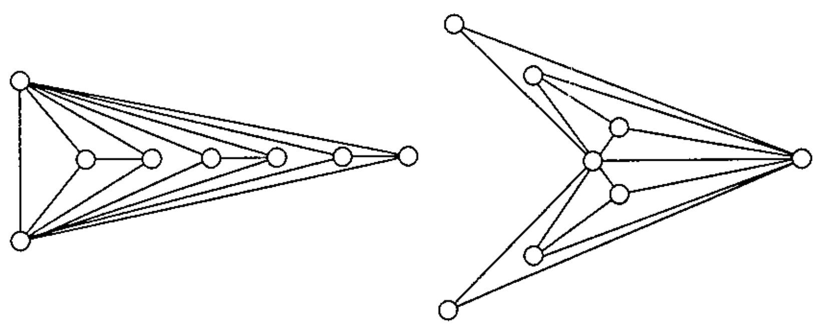


Figure 2.9: (a) A tree T, and (b) a minimum segment drawing of T

It has also been proved in [DESW07a] that T admits a planar straight-line drawing on exactly $\frac{\eta}{2}$ number of segments. The proof of this claim is constructive. To prove this claim, a drawing Γ of T has been computed in [DESW07a] such that every odd degree vertex of T is an endpoint of exactly one segment in Γ and no even degree vertex is an endpoint of a segment in Γ. Such a drawing of a tree T in Fig. 2.9(a) is illustrated in Fig. 2.9(b).

2.5.2 2-Connected Graphs

It was shown in [DESW07a] that there is an n-vertex 2-connected plane graph G with $\frac{5}{2}n - 4$ edges such that any straight line drawing of G requires $\frac{5}{2}n - 4$ number of segments. Such a graph G is shown in Fig. 2.10(a). However, it was also shown in [DESW07a] that the same graph requires at least $2n - 1$ segments in every planar drawing as shown in Fig. 2.10(b). In summary, the known result on minimum segment drawing problem states that there is an n-vertex plane 2-connected graph that can be drawn using at most $\frac{5}{2}n$ number of segments, and an n-vertex planar 2-connected graph that requires at least $2n + O(1)$ number of segments in any planar drawing.



(a) (b)

Figure 2.10: (a) A 2-connected plane graph G that requires $\frac{5}{2}n - 4$ segments in any drawing, and (b) a drawing of G on $2n - 1$ segments.

2.5.3 3-Connected Graphs

Let G be a 3-connected graph. Based on a canonical decomposition [Kan96] of G, it was shown in [DESW07a] that every 3-connected graph G has a plane drawing with at most $\frac{5}{2}n$ line segments. Although it was not shown whether $\frac{5}{2}n$ line segments are necessary for every drawing of G, it was shown that there is a 3-connected plane graph G with $n = 3k$ ($k \in \mathbb{N}$) vertices that requires at least $2n$ number of segments in any planar straight-line drawing. Such a graph G with 12 vertices is shown in Fig. 2.11.

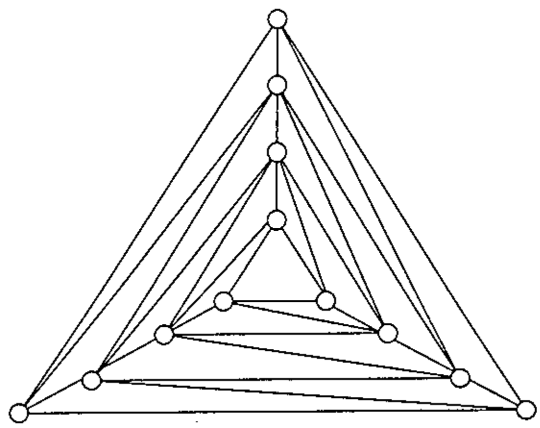


Figure 2.11: A 3-connected graph G that requires at least $2n-6$ segments in any drawing.

2.5.4 3-Connected Cubic Plane Graph

Let G be a 3-connected cubic plane graph. Based on a canonical decomposition of G, it was shown in [DESW07a] that G can always be drawn using at most $n+2$ number of segments. Although this establishes an upper bound of the number of segments required for any drawing of G, no lower bound of the number of segments required for any drawing of G is known as yet. An example of a drawing of a 3-connected cubic plane graph G using exactly $n+2$ segments is shown in Fig. 2.12.

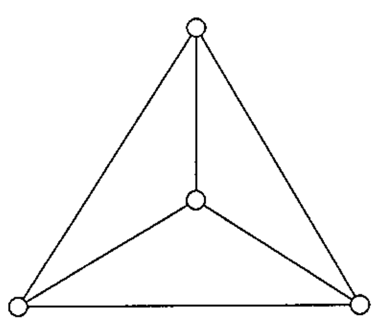


Figure 2.12: Drawing of a 3-connected cubic graph G using $n+2$ segments.

2.5.5 Series-Parallel Graph with the maximum Degree Three

Recently, Samee et. al. [SAAR08] studied the problem of finding minimum segment drawing for planar graphs by restricting the maximum degree of that graph. They first gave a lower bound on the number of segments required for straight line drawings of series-parallel graphs with the maximum degree three. Then they presented a linear-time algorithm for finding a minimum segment drawing of such a graph. Other than the degree restricted case for series-parallel graphs, no algorithm has yet been devised for computing minimum segment drawings of a general class of graph. An example of a minimum segment drawing of a series-parallel graph G with the maximum degree three is shown in Fig. 2.13.

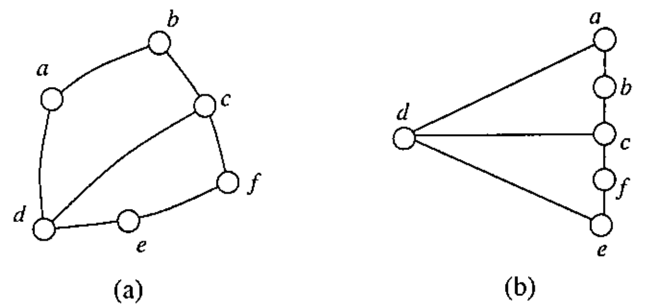


Figure 2.13: (a) A series-parallel graph G and (b) a minimum segment drawing of a series-parallel graph with the maximum degree three.

Chapter 3

Dual-Path Maximal Outerplane Graph

In this chapter we present an algorithm for computing a minimum segment drawing of a dual-path maximal outerplane graph G. Our algorithm is outlined as follows: we first divide G into fan graphs; we then find the minimum segment drawings of the fan graphs; and we finally obtain a minimum segment drawing of G by patching the drawings of the fan graphs. For example, Figure 3.1(b) depicts a decomposition of the graph in Figure 3.1(a) into fan graphs, Figure 3.1(c) illustrates the minimum segment drawings of the fan graphs, and Figure 3.1(d) depicts a minimum segment drawing of the dual-path maximal outerplane graph in Figure 3.1(a).

3.1 Minimum Segment Drawing of a Fan Graph

In this section, we give an algorithm to compute a minimum segment drawing of a fan graph G. We first have the following lemma on the minimum segment drawings of fan graphs.

Lemma 3.1.1 *Let G be a fan graph, and let v be the center of G. Then:*

(a) $sc(G) = 3$ *if* $d(v) = 2$; *and*

(b) $sc(G) = \lfloor d(v)/2 \rfloor + 3$ *if* $d(v) \geq 3$.

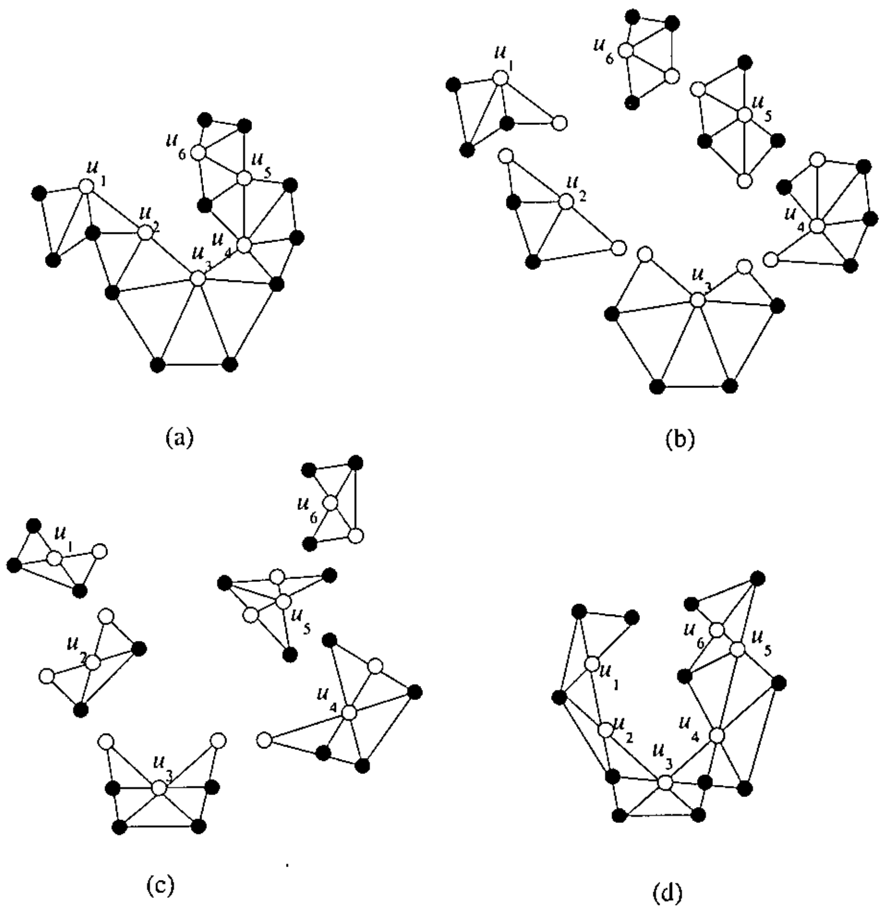


Figure 3.1: (a) A dual-path maximal outerplane graph G, (b) fan graphs of G, (c) minimum segment drawings of the fan graphs, and (d) a minimum segment drawing of G.

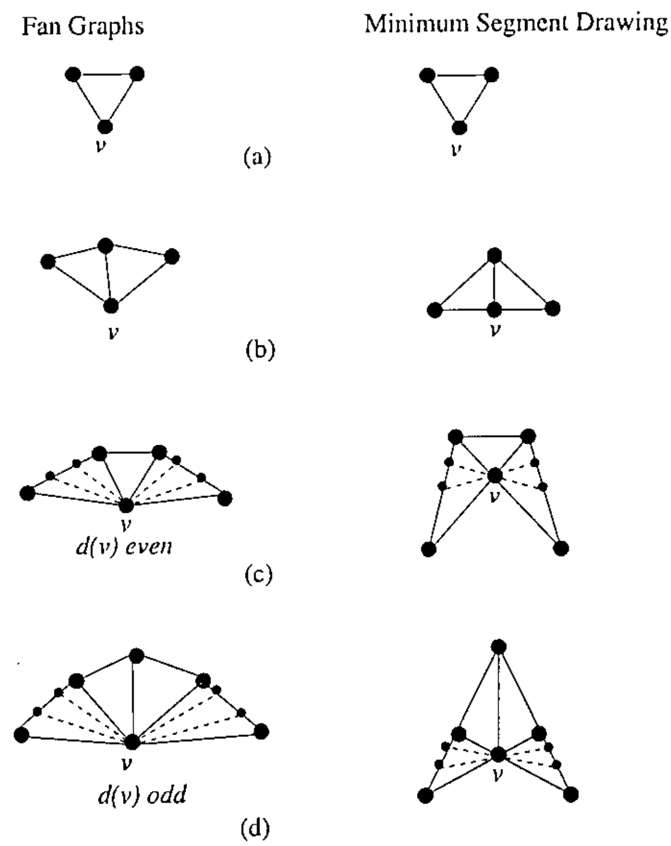


Figure 3.2: Illustration of fan graphs and their minimum segment drawings.

Proof. (a) Since G is a triangle, G has essentially exactly one straight line drawing, which has three maximal segments, as illustrated in Fig. 3.2(a). Thus $sc(G) = 3$.

(b) (i) We first show that $sc(G) \leq \lfloor d(v)/2 \rfloor + 3$. We have the following two cases to consider.

Case 1: $d(v)$ is even and $d(v) \geq 4$.

We take the center v as the intersection point of $d(v)/2$ distinct straight line segments passing through v. The $d(v)$ neighbors of v are placed on the $d(v)$ endpoints of the straight line segments so that each of the two sets of $d(v)/2$ endpoints are collinear, as illustrated in the right drawing of Fig. 3.2(c). Thus one can draw G with $d(v)/2 + 3$ maximal segments, and hence $sc(G) \leq \lfloor d(v)/2 \rfloor + 3$.

Case 2: $d(v)$ is odd and $d(v) \geq 3$.

Assume that $d(v) = 3$. Then G has essentially exactly one straight line drawing with four maximal segments as illustrated in the right drawing of Fig. 3.2(b), and G has no straight line drawing with three or fewer maximal segments. Hence $sc(G) \leq \lfloor d(v)/2 \rfloor + 3$ for G with $d(v) = 3$. We now assume that $d(v) \geq 5$. We take the center v as the intersection point of $\lfloor d(v)/2 \rfloor$ distinct straight line segments passing through v. We draw another straight line segment whose one end is v and the other end is the middle neighbor z of v. Now the $d(v)$ neighbors of v are placed on the $d(v)$ endpoints of all these $\lfloor d(v)/2 \rfloor + 1$ line segments so that each of the two sets of $\lfloor d(v)/2 \rfloor$ endpoints and z are collinear, as illustrated in Fig. 3.2(d). Thus one can draw G with $\lfloor d(v)/2 \rfloor + 3$ maximal segments, and hence $sc(G) \leq \lfloor d(v)/2 \rfloor + 3$.

(ii) We then show that $sc(G) \geq \lfloor d(v)/2 \rfloor + 3$ if $d(v) \geq 3$. We only give a proof for the claim that $sc(G) \geq \lfloor d(v)/2 \rfloor + 3$ if $d(v) \geq 3$ and $d(v)$ is even; the proof for the case where $d(v)$ is odd is similar. We prove the claim by induction on $d(v)$. For the basis of the induction we consider the fan graph G with $d(v) = 4$. G has essentially two distinct drawings with exactly 5 maximal segments as illustrated in Fig. 3.3(a), and G has no drawing with fewer than 5 maximal segments. Thus $sc(G) = 5 \geq \lfloor d(v)/2 \rfloor + 3$, and hence the basis is true.

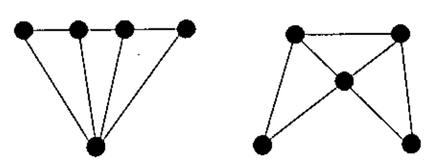


Figure 3.3: Two drawings of the fan graph of $d(v) = 4$ with five maximal segments.

Assume that $sc(G') \geq \lfloor d(v, G')/2 \rfloor + 3$ for the fan graph G' with $d(v, G') = 2k$ for some integer $k \geq 2$. Let G be the fan graph with $d(v, G) = 2k + 2 (\geq 6)$, and let D be a minimum segment drawing of G. Since $sc(D) = sc(G)$, it suffices to prove that $sc(D) \geq \lfloor d(v, G)/2 \rfloor + 3$.

We first consider the case where D has a maximal segment xvy such that each of vertices x and y has degree three in G. Since every inner face is a triangle, x cannot be a neighbor of y in G. Let p and q be the neighbors of x other than v, and let s and t be the neighbors of y other than v. Let G' be a graph obtained from G by deleting x and y and by adding new edges (p, q) and (s, t). Then G' is a dual-path maximal outerplane graph and $d(v, G') = 2k$.

We can obtain a straight line drawing D' of G' by deleting from D the drawings of vertices x and y and their incident edges and by drawing each of the edges (p,q) and (s,t) with a straight line segment. One can observe that the deletion of edges $(p,x),(x,q),(s,y)$ and (y,t) and the addition of the edges (p,q) and (s,t) neither increase the number of maximal segments nor produce any edge crossing. Furthermore, the maximal segment xvy in D disappears in D'. Therefore $sc(D)-1 \geq sc(D')$. Since $sc(D') \geq sc(G') \geq k+3$ by the induction hypothesis, we have $sc(D) \geq sc(D')+1 \geq sc(G')+1 \geq (k+3)+1 = (k+1)+3 = \lfloor d(v,G)/2 \rfloor + 3$. Hence the claim holds.

We then consider the case where D has no maximal segment xvy such that each of x and y has degree 3. Since G has exactly two vertices of degree 2, D has at most two maximal segments passing through v. Each such maximal segment contains exactly two vertices other than v. Since $d(v,G) \geq 6$, v has two neighbors i and j of degree three such that each of the edges (i,v) and (j,v) is a maximal segment in D. We assume that i is not adjacent to j in G. (The proof for the case where i is adjacent to j is similar.) Let p and q be the neighbors of i other than v, and let s and t be the neighbors of j other than v. Let G' be a graph obtained from G by deleting i and j and by adding new edges (p,q) and (s,t). Then G' is a dual-path maximal outerplane graph and $d(v,G') = 2k$. We can obtain a drawing D' of G' by deleting from D the drawings of vertices i and j and their incident edges and by drawing each of the edges (p,q) and (s,t) with a straight line segment. One can observe that the deletion of edges $(p,i),(i,q),(s,j)$ and (j,t) and the addition of the edges (p,q) and (s,t) neither increase the number of maximal segments nor produce any edge-crossing. Furthermore, the two maximal segments iv and jv in D disappear in D'. Therefore $sc(D)-2 \geq sc(D')$. Since $sc(D') \geq sc(G') \geq k+3$ by the inductive hypothesis, we have $sc(G) = sc(D) \geq sc(D')+2 \geq sc(G')+2 \geq (k+3)+2 = (k+1)+4 > (k+1)+3 = \lfloor d(v,G)/2 \rfloor + 3$. Hence the claim holds. $\mathcal{Q.E.D.}$

3.2 Properties of Minimum Segment Drawings of Fan Graphs

In this section we illustrate some properties of the minimum segment drawings of a fan graph. This properties will be used during the patching of minimum segment drawings of the fan graphs.

Let $G = (V, E)$ be a fan graph and v be the center of G. Let D be a straight line drawing of G. We call the edges incident to a vertex $u \in V(G)$ as *pairwise collinear* in D if the edges incident to u are drawn using exactly $\lceil d(u)/2 \rceil$ maximal line segments. If the edges incident to u are not pairwise collinear in D then we call u an *apex* of D. Let n_D be the number of apices in D. The outerface of G is always drawn as a polygon in D which requires at least three convex corners, and hence D has at least three apices, i.e. $n_D \geq 3$. Since G is a fan graph, G contains two ears, each of which always forms an apex in D. Thus among the apices in D two apices are ear vertices. We have the following lemma on the number of apices of D.

Lemma 3.2.1 *Let D be a minimum segment drawing of a fan graph G with center v, and let n_D be the number of apices in D. Then the following (a)-(d) hold:*

(a) $n_D = 3$ if $d(v) \leq 3$;

(b) $3 \leq n_D \leq 4$ if $d(v) = 4$;

(c) $n_D = 4$ if $d(v) > 4$ and $d(v)$ is even; and

(d) $n_D = 3$ if $d(v) > 4$ and $d(v)$ is odd.

Proof. (a) We need at least three convex corners to draw the outerface of G as a polygon. These three convex corners form three apices. Hence $n_D \geq 3$. It is now sufficient to show that $n_D \leq 3$. If $d(v) = 2$ then there are three vertices in G and $n_D = 3$. If $d(v) = 3$ then v has three neighbors. Let v_1, v_2 and w be the neighbors of v with $d(v_1) = d(v_2) = 2$ and $d(w) = 3$. Suppose v, w are both apices. Then the three edges incident to each of v and w are drawn

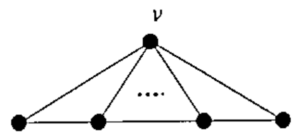


Figure 3.4: Straight line drawing of G with $d(v) > 4$ where v forms a convex corner.

on three different straight line segments. Since (v, w) is an edge, the total number of distinct straight line segment is five. But in this case $sc(G) = 4$ by Lemma 3.1.1(b), a contradiction to the assumption that D is a minimum segment drawing. Hence one of v and w are not apices, and thus $n_D \leq 3$.

(b) We need at least three convex corners to draw the outerface of G as a polygon. These three convex corners form three apices. Hence $n_D \geq 3$. It is now sufficient to show that $n_D \leq 4$. Suppose for a contradiction that $n_D > 4$. Then the edges incident to v and to the neighbors of v are not pairwise collinear, and clearly the number of segments incident to v in D is at least $\lfloor d(v)/2 \rfloor + 1$. Since the three edges on the outerface which are not incident to v are not collinear, $sc(D) \geq (\lfloor d(v)/2 \rfloor + 1) + 3$. By Lemma 3.1.1(b), $sc(G) = \lfloor d(v)/2 \rfloor + 3$. Hence $sc(D) > sc(G)$, a contradiction to the assumption that D is a minimum segment drawing.

(c) (i) We first consider the case where v is an apex in D. In this case we claim that v is not a convex corner in D. Assume for a contradiction that v is a convex corner in D. Then $sc(D) \geq d(v) + 1$, since no two edges incident to v are drawn on the same straight line segment and at least one additional segment is required to complete the drawing of the outerface, as illustrated in Fig. 3.4. By Lemma 3.1.1(b), $sc(G) = \lfloor d(v)/2 \rfloor + 3$. Then $sc(D) > sc(G)$, a contradiction to the assumption that D is a minimum segment drawing. Therefore v cannot be a convex corner in D. We now show that $n_D \geq 4$. Since we need at least three convex corners to draw the outerface as a polygon and v is not a convex corner in D, there are at least three vertices other than v on the outerface of G each of which are convex corners in D. These three vertices along with v form the four apices and hence $n_D \geq 4$.

We now prove that $n_D \leq 4$. Suppose for a contradiction that $n_D > 4$. Then the edges incident to v and at least four neighbors of v are not pairwise collinear. Then the number of segments incident to v is at least $\lfloor d(v)/2 \rfloor + 1$. Also there are three edges on the outerface, other than the edges incident to v, which are not collinear. Hence $sc(D) \geq (\lfloor d(v)/2 \rfloor + 1) + 3$. By Lemma 3.1.1(b), $sc(G) = \lfloor d(v)/2 \rfloor + 3$. Then $sc(D) > sc(G)$, a contradiction to the assumption that D is a minimum segment drawing.

(ii) We now consider the case where v is not an apex in D. Then the edges incident to v are pairwise collinear. Suppose for a contradiction that $n_D = 3$. Let w be the vertex which forms an apex other than the ear vertices v_1, v_2. Since G is a fan graph, $d(w) = 3$. Let u be a neighbor of v in the embedding from v_1 to w. Since u is not an apex, w, u and v_1 are collinear. Similarly any neighbor of v from w to v_2 is collinear with w and v_2. Hence $sc(D) \leq \lfloor d(v)/2 \rfloor + 2$ where $d(v)$ is even. According to Lemma 3.1.1(b), $sc(G) = \lfloor d(v)/2 \rfloor + 3$. Hence D is not a minimum segment drawing, a contradiction. Thus $n_D \geq 4$.

We now prove that $n_D \leq 4$. Suppose for a contradiction that $n_D > 4$. Then the edges incident to at least five neighbors of v are not pairwise collinear. Let v_1, u_1, u_2, u_3 and v_2 be those apices, where v_1 and v_2 are ear vertices. Then there are at least four outer edges incident to u_1, u_2, u_3 on the outerface which are not collinear. Hence $sc(D) \geq \lfloor d(v)/2 \rfloor + 4$. By Lemma 3.1.1(b), $sc(G) = \lfloor d(v)/2 \rfloor + 3$. Then $sc(D) > sc(G)$, a contradiction to the assumption that D is a minimum segment drawing.

(d) (i) We first consider the case where v is an apex in D. In this case we show that v neither is a convex corner nor forms an apex in D. Assume for a contradiction that v is a convex corner in D. Then $sc(D) \geq d(v) + 1$, since no two edges incident to v are drawn on the same straight line segment and at least one additional segment is required to complete the drawing of the outerface, as illustrated in Fig. 3.4. By Lemma 3.1.1(b), $sc(G) = \lfloor d(v)/2 \rfloor + 3$. Then $sc(D) > sc(G)$, a contradiction to the assumption that D is a minimum segment drawing. Therefore v cannot be a convex corner in D. We now show that v is not an apex in D. Since we need at least three convex corners to draw the outerface as a polygon and v is not a convex corner in D, there are at least three vertices other than v on the outerface of G which are convex corners in D. These three vertices along with v form the four apices and hence $n_D \geq 4$. Since v is an apex, the number of segments incident to v is at least $\lceil d(v)/2 \rceil + 1$. Furthermore, there are at least two edges on the outerface other than the edges incident to v which are not collinear. Hence $sc(D) \geq (\lceil d(v)/2 \rceil + 1) + 2$. By Lemma 3.1.1(b), $sc(G) = \lfloor d(v)/2 \rfloor + 3$. Then $sc(D) > sc(G)$, a contradiction to the assumption that D is a minimum segment drawing. Hence v cannot be an apex in D.

(ii) We now consider the case where v is not an apex in D. Then the edges incident to v are pairwise collinear. Since we need at least three convex corners to draw the outerface as a polygon, $n_D \geq 3$. We now prove that $n_D \leq 3$. Suppose for a contradiction that $n_D > 3$. Then the edges incident to at least four neighbors of v are not pairwise collinear. Let v_1, u_1, u_2 and v_2 be those apices, where v_1 and v_2 are ear vertices. Then there are at least three outer edges incident to u_1 and u_2 on the outerface which are not collinear. Hence $sc(D) \geq \lceil d(v)/2 \rceil + 3$. By Lemma 3.1.1(b), $sc(G) = \lfloor d(v)/2 \rfloor + 3$. Then $sc(D) > sc(G)$, a contradiction to the assumption that D is a minimum segment drawing. $\mathcal{Q.E.D.}$

From the proofs of Lemma 3.2.1(c) and 3.2.1(d), we have the following corollary.

Corollary 3.2.2 *Let G be a fan graph and v be the center of G with $d(v) > 4$. Then the following (a) and (b) hold:*

(a) v is not a convex corner in D.

(b) v is not an apex in D, if $d(v)$ is odd.

Lemma 3.2.3 *Let G be a fan graph and v be the center of G with $d(v) > 4$ and $d(v)$ odd. Then G has a unique minimum segment drawing.*

Proof. Let $d = d(v)$ and $v_1, v_2, \ldots, v_{z-1}, v_z, v_{z+1}, \ldots, v_d$ be the neighbors of v in clockwise order where v_z is the middle neighbor, $z = \lceil d(v)/2 \rceil$. By Corollary 3.2.2(b), v cannot be an apex in any minimum segment drawing of G and by Lemma 3.2.1(c), there are three apices in the minimum segment drawing of G. We know that the two ears always form apices in a drawing of G. Hence it suffices to prove that the third apex is unique. Let D be the minimum segment drawing of G where the middle neighbor v_z of v is an apex in D as illustrated in Fig. 3.2. We will show that there is no minimum segment drawing D' of G other than D. Suppose v_1, v_d

and $v_k(\neq v_z)$ be the apices in D'. Then the vertices from v_1 to v_k are collinear and the vertices from v_k to v_d are collinear. Since v is not an apex, the vertices v_i, v and v_{i+k} are collinear for $1 \leq i < k$. Now if $k < z$ then $2k-1 < d$ and there exists vertices v_j, with $2k-1 < j \leq d$, which is not collinear with v and another neighbor of v. Similarly if $k > z$ then $2k-1 > d$ and there exists vertices v_j, with $1 < j \leq k$, which is not collinear with v and another neighbor of v. Hence v becomes an apex which is a contradiction. Thus $k = z$. Therefore D is the only minimum segment drawing of G. $\mathcal{Q.E.D.}$

Lemma 3.2.4 *Let v be the center of a fan graph G where $d(v) > 4$ and $d(v)$ even. Let $d = d(v)$ and $v_1, v_2, \ldots, v_d$ be the neighbors of v in clockwise order. Let D be a minimum segment drawing of G where v is not an apex. Then the number of neighbors of v between any two consecutive apices v_p and v_q in D is at most $\lceil d(v)/2 \rceil - 2$ where $1 \leq p < q \leq d(v)$.*

Proof. Since v is not an apex the edges incident to v are pairwise collinear. Hence in D, v_i, v and $v_{i+d(v)/2}$ are collinear where $1 \leq i \leq d(v)/2$. Thus v_p is collinear with $v_{p+d(v)/2}$ for $p \leq d(v)/2$. Hence to complete the outerface, q must be less than $p + d(v)/2$. Hence there are at most $d(v)/2 - 2$ neighbors between p and q. $\mathcal{Q.E.D.}$

By Lemma 3.2.3, a fan graph G has a unique minimum segment drawing if the degree of its center v is odd. But if $d(v)$ is even, then one can choose the two apices other than the ear vertices according to Lemma 3.2.4, and find a minimum segment drawing of G. We call the algorithm for finding a minimum segment drawing Algorithm **Draw-Fan**. Clearly Algorithm **Draw-Fan** takes linear time.

3.3 Dual-path Maximal Outerplane Graph

In this section, we give an algorithm to compute a minimum segment drawing of a dual-path maximal outerplane graph. We first decompose the graph into fan components and then we find the minimum segment drawings of the fan component using the algorithm of the previous

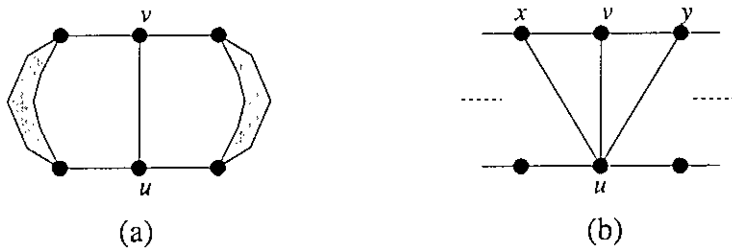


Figure 3.5: Illustration of a non-triangulated face where (a) v is not an ear vertex and (b) u is a vertex with $d(u) > 4$

section. Then we patch the drawings of the fan components to compute the minimum segment drawing of the whole graph.

3.3.1 Decomposition into Fan Components

Let $G = (V, E)$ be a dual-path maximal outerplane graph. We call a subgraph $M = (V_M, E_M)$ of G a *fan component* of G if M is a fan graph. We denote by M_u the fan component with the center vertex u. We have the following lemma.

Lemma 3.3.1 *Let $G = (V, E)$ be a dual-path maximal outerplane graph with $|V| > 4$. Let P_1 and P_2 be the two outerpaths of G, and let v be a vertex on P_1 with $d(v) = 3$. Then there is a vertex $u \in N(v)$ on P_2 such that $d(u) \geq 4$. Moreover $M_v \subset M_u$.*

Proof. Since $d(v) = 3$, v has a neighbor u on P_2. If $d(u) = 3$ then an inner face of G containing v and u would not be a triangle as illustrated in Fig. 3.5(a), a contradiction to the assumption that G is a maximal outerplane graph. Hence $d(u)$ must be greater than three. Let x, y be the two other neighbors of v. Since the faces xuv and yvu are triangles, $\{x, y\} \subset N(u)$ (see Fig. 3.5(b)). Hence $M_v \subset M_u$.

$\mathcal{Q.E.D.}$

We call a fan component M of G a *maximal fan component* if M is not contained in any other fan component of G. The following lemma holds for a maximal fan component of G.

Lemma 3.3.2 *Let $G = (V, E)$ be a dual-path maximal outerplane graph with $|V| > 4$ and let M_v be a fan component with center v. Then M_v is a maximal fan component of G if and only if $d(v) \geq 4$.*

Proof. We first assume that M_v is a maximal fan component. Suppose for a contradiction that $d(v) < 4$. If $d(v) = 2$ then clearly M_v cannot be a a maximal fan component since $|V| > 4$, a contradiction. We thus assume that $d(v) = 3$. Then according to Lemma 3.3.1, v has a neighbor w with $d(w) \geq 4$ and $M_v \subset M_w$. Hence M_v is not a maximal fan component, a contradiction.

We now assume that $d(v) \geq 4$. We will show that there is no vertex $u \in N(v)$ such that $M_v \subset M_u$. Since G is a dual-path outerplane graph, v and u have either one or two common neighbors. Hence, the fan components M_v and M_u have at most two common faces. Since $d(v) \geq 4$, M_v contains at least three inner faces. Thus $M_v \not\subseteq M_u$. $\mathcal{Q.E.D.}$

We now decompose $G = (V, E)$ into maximal fan components. There are several ways of decomposing G into maximal fan components. Here we present a simple algorithm. If $|V| \leq 4$ then G itself is the fan component. Otherwise, G has one or more vertices with degree four or more. We start from an ear vertex and traverse G from that ear to the other. Whenever we get a vertex v with $d(v) \geq 4$, we take the fan of v in G as a fan component. According to Lemma 3.3.2, the fan components obtained in this way are maximal. We order the maximal fan components from one ear to another ear in the embedding of G. Let $M_1, M_2, \ldots, M_q$ be the maximal fan components of G and $u_1, u_2, \ldots, u_q$ be the centers of those fan components respectively. Note that M_1 and M_q are the two fan components containing the ears and u_i, u_{i+1} are adjacent for $1 \leq i < q$. We have the following lemma.

Lemma 3.3.3 *Let $G = (V, E)$, $|V| > 4$ be a dual-path outerplane graph having two consecutive maximal fan components M_i, M_{i+1} with centers u_i and u_{i+1} for $1 \leq i < q$. Then M_i and M_{i-1} have either one or two common faces. If M_i and M_{i+1} have exactly one common face, then u_i and u_{i+1} are on the same outerpath. If M_i and M_{i+1} have two common faces, then u_i and u_{i+1}*

are on different outerpaths.

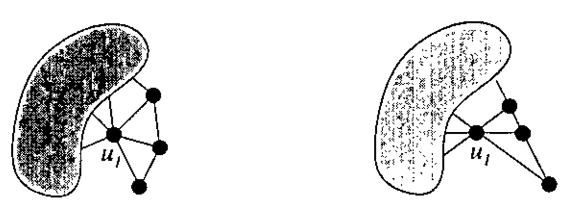
Proof. Since G is a dual-path outerplane graph, u_i and u_{i+1} have either one or two common neighbors. Thus M_i and M_{i-1} have either one or two common faces. We first consider the case where u_i and u_{i+1} have exactly one common neighbor w. Then u_i and u_{i+1} are on the same outerpath and M_i and M_{i+1} have exactly one common face u_iwu_{i+1}. We now consider the case where u_i and u_{i+1} have exactly two common neighbors w_1 and w_2. In this case u_i and u_{i+1} are on different outerpaths forming a quadrilateral $u_iw_1u_{i+1}w_2$ having the edge u_iu_{i+1} as diagonal. Hence $u_iw_1u_{i+1}$ and $u_{i+1}w_2u_i$ are the two common faces of M_i and M_{i+1}. $\mathcal{Q.E.D.}$

3.3.2 Feasible Drawings

After getting the maximal fan components, we compute their minimum segment drawings by the algorithm **Draw-Fan**. Let $M_1, M_2, \ldots, M_q$ be the maximal fan components of G. For each integer i, $1 \leq i \leq q$, we denote by G_i, the graph obtained by the union of the maximal fan components $M_1, M_2, \ldots, M_i$. Then $G_q = G$. Let D be a straight-line drawing of $G_i = M_1 \cup M_2 \cup \ldots \cup M_i$ and let u_i be the fan vertex of M_i. We call D a *feasible drawing* of G_i if the drawing D has the following properties (f1)-(f2):

(f1) D has exactly $sc(G_i)$ number of segments; and

(f2) u_i is not an apex in D.



(a) (b)

Figure 3.6: (a) A dual-path outerplane graph G_i and (b) a feasible drawing D of G_i.

Figure 3.6(a) shows a graph G_i and Figure 3.6(b) illustrates a feasible drawing D of G_i.

3.3.3 Patching of Minimum Segment Drawings

We now patch a minimum segment drawing of a fan graph with a feasible drawing and obtain a feasible drawing of the whole graph. We have the following lemmas.

Lemma 3.3.4 *Let $G_{i-1} = M_1 \cup M_2 \cup \ldots \cup M_{i-1}$ and $G_i = G_{i-1} \cup M_i$ for $1 < i \leq q$. Assume that M_{i-1} and M_i have exactly one face in common. Then G_i has a feasible drawing with $sc(G_i)$ number of segments where*

$$sc(G_i) = sc(G_{i-1}) + sc(M_i) - 3.$$

Proof. We first obtain a drawing D_i of G_i as follows. Let u_1 and u_2 be the centers of M_{i-1} and M_i and w be the common neighbor of u_1 and u_2. Then M_{i-1} and M_i have the face u_1u_2w in common. (See Fig. 3.7). Let D_{i-1} be a feasible drawing of G_{i-1} and let D' be the minimum segment drawing of M_i, obtained by the Algorithm **Draw-Fan**, as illustrated in Fig. 3.7(a). We obtain D_i by patching the drawing D' with D_{i-1} in such a way that (i) the line segment in D_{i-1} containing the edge (u_1, u_2) and the line segment in D' containing the edge (u_1, u_2) are drawn on the same straight line segment in D_i, (ii) the line segment in D_{i-1} containing the edge (u_1, w) and the line segment in D' containing the edge (u_1, w) are drawn on the same straight line segment in D_i, and (iii) the line segment in D_{i-1} containing the edge (u_2, w) and the line segment in D' containing the edge (u_2, w) are drawn on the same straight line segment in D_i. Furthermore, each line segment containing none of the edges (u_1, u_2), (u_1, w) and (u_2, w) is not affected in the patching above (see Fig. 3.7(b)). Therefore we have $sc(D_i) \leq sc(G_{i-1}) + sc(M_i) - 3$.

We now show that D_i is a feasible drawing of G_i. To show that D_i satisfies (f1) it is sufficient to show that $sc(G_i) \geq sc(G_{i-1}) + sc(M_i) - 3$. Since $d(u_1, M_i)$ is two, the incident edges (u_1, w) and (u_1, u_2) of u_1 in M_i can share at most two segments with a drawing of G_{i-1}. Similarly, since $d(u_2, G_{i-1})$ is two, the incident edges (u_2, w) and (u_2, u_1) of u_2 in G_{i-1}, can share at most two segments with a drawing of M_i. Moreover since $d(w, G_{i-1}) = 3$ and $d(w, M_i) = 3$, no edge other than (u_1, u_2), (u_1, w) and (u_2, w) can share a segment during patching of a drawing of M_i with a drawing of G_{i-1}. Hence at most three segments can be minimized during the patching

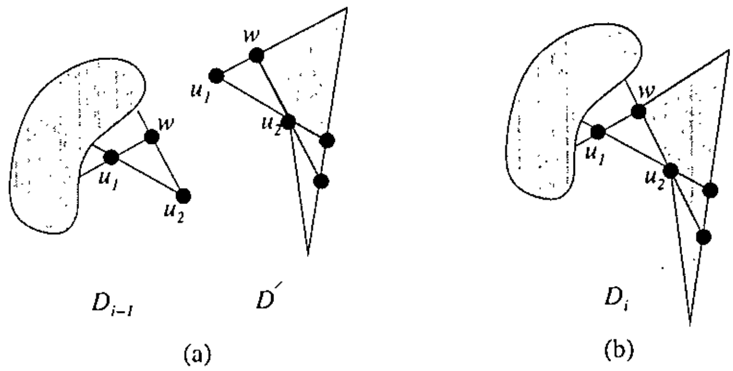


Figure 3.7: Illustration of patching the minimum segment drawing D' of M_i with D_{i-1} where M_{i-1} and M_i have exactly one face in common.

of a drawing of M_i with a drawing of G_{i-1}. Thus $sc(G_i) \geq sc(G_{i-1}) + sc(M_i) - 3$. It is now remaining to show that D_i satisfies (f2). According to Lemma 3.1.1, the edges incident to u_2 are pairwise collinear in D'. Since we have not changed the alignment of the incident edges of u_2 during patching, the incident edges to u_2 remains pairwise collinear in D_i, and hence D_i satisfies (f2). $\mathcal{Q.E.D.}$

Lemma 3.3.5 *Let $G_{i-1} = M_1 \cup M_2 \cup \ldots \cup M_{i-1}$ and $G_i = G_{i-1} \cup M_i$ for $1 < i \leq q$. Let M_{i-1} and M_i have exactly two faces in common and u_1 and u_2 be the centers of M_{i-1} and M_i respectively. Then G_i has a feasible drawing with $sc(G_i)$ number of segments where*

(a) $sc(G_i) = sc(G_{i-1}) + sc(M_i) - 3$ if $d(u_1)$ and $d(u_2)$ is odd; and

(b) $sc(G_i) = sc(G_{i-1}) + sc(M_i) - 4$ otherwise.

Proof. (a) We first consider the case where $d(u_1)$ and $d(u_2)$ are odd. Let v and w be the two common neighbors of u_1 and u_2. Then M_{i-1} and M_i have the faces u_1u_2w and u_1u_2v in common. We obtain a drawing D_i of G_i as follows. Let D_{i-1} be a feasible drawing of G_{i-1} and D' be the minimum segment drawing of M_i obtained by the algorithm **Draw-Fan**, as illustrated in Fig. 3.8(a). We obtain D_i by patching the drawing D' with D_{i-1} in such a way that (i) the line segment in D_{i-1} containing the edge (u_1, u_2) and the line segment in D' containing the edge (u_1, u_2) are drawn on the same straight line segment in D_i, (ii) the line segment in D_{i-1} containing the edge (u_1, w) and the line segment in D' containing the edge (u_1, w) are drawn on the same straight line segment in D_i, and (iii) the line segment in D_{i-1} containing the edge (u_2, v) and the line segment in D' containing the edge (u_2, v) are drawn on the same straight line segment in D_i, (iv) the edge (u_1, v) of M_i is drawn on the line segment in D_{i-1} containing the edge (u_1, v) and it is no longer on the line segment containing (u_1, w) in D', and (v) the edge (u_2, w) of G_{i-1} is drawn on the line segment in D' containing the edge (u_2, w) and it is no longer on the line segment containing (u_2, v) in D'. Furthermore each line segment containing none of the edges (u_1, u_2), (u_1, w), (u_2, v), (u_1, v) and (u_2, w) is not affected in the patching above (see Fig. 3.8(c)). Therefore we have $sc(D_i) \leq sc(G_{i-1}) + sc(M_i) - 3$.

We now show that D_i is a feasible drawing of G_i. To show that D_i satisfies (f1) it is sufficient to show that $sc(G_i) \geq sc(G_{i-1}) + sc(M_i) - 3$. Since $d(u_1, M_i)$ is three, the edges (u_1, w), (u_1, v) and (u_1, u_2) incident to u_1 in M_i can share at most three segments with a drawing of G_{i-1}. Similarly, since $d(u_2, G_{i-1})$ is three, the edges (u_2, v), (u_2, w) and (u_2, u_1) incident to u_2 in G_{i-1}, can share at most three segments with a drawing of M_i. Since $d(v, G_{i-1}) = 3$ and $d(v, M_i) = 3$, no edge other than (u_1, w), (u_1, v), (u_1, u_2) (u_2, w) and (u_2, v) can share a segment during the patching of a drawing of M_i with a drawing of G_{i-1}. Hence at most five segments can be minimized during the patching of a drawing of M_i with a drawing of G_{i-1}.

But, since $d(u_1) > 5$ and $d(u_1, G_{i-1})$ is odd, by Lemma 3.2.3, M_{i-1} has a unique minimum segment drawing where u_2 is not an apex. Since $d(u_2, G_{i-1}) = 3$, the edges (v, u_2) and (u_2, w) are collinear in the minimum segment drawing of G_{i-1} before patching. On the otherhand, since $d(u_2) > 4$ and $d(u_2, M_i)$ is odd, u_2 also is not an apex in the unique minimum segment drawing of M_i. Since $d(u_2, M_i) > 4$, by Lemma 3.2.3, the edges (v, u_2) and (u_2, w) cannot be collinear in the minimum segment drawing for M_i. We need to modify the drawing of M_i such that (v, u_2) and (u_2, w) are on different straight line segment which increases the number of segment of the drawing of M_i by one as illustrated in Fig. 3.8(b). Similarly since $d(u_1, G_{i-1}) > 4$, by Lemma 3.2.3, the edges (v, u_1) and (u_1, w) cannot be collinear in the minimum segment drawing for G_{i-1}. Hence the drawing D_{i-1} needs to be modified in such a way that (v, u_1) and (u_1, w)

are on different straight line segment which increases the number of segment of the drawing of G_{i-1} by one. The modified drawings of D_{i-1} and D' before patching are shown in Fig. 3.8(b). Therefore $sc(G_i) \geq sc(G_{i-1}) + 1 + sc(M_i) + 1 - 5 = sc(G_{i-1}) + sc(M_i) - 3$.

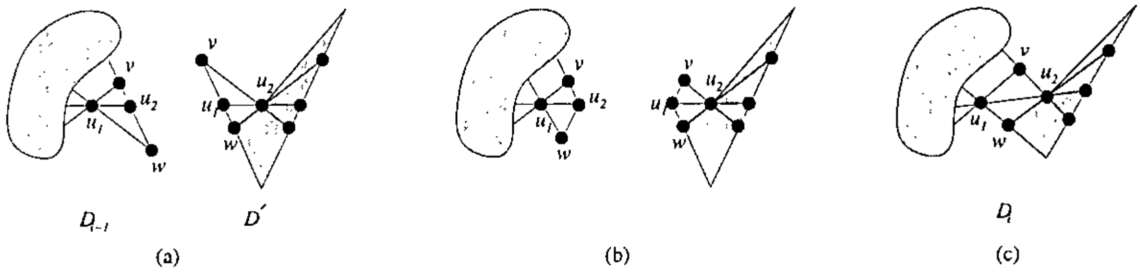


Figure 3.8: Illustration of patching D' with D_{i-1} where $d(u_1)$ and $d(u_2)$ is odd and M_{i-1} and M_i have two faces in common.

It is now remaining to show that D_i satisfies (f2). According to Lemma 3.1.1, the edges incident to u_2 are pairwise collinear in D'. Since we have not changed the alignment of the incident edges of u_2 during patching, the incident edges to u_2 remains pairwise collinear in D_i, and hence D_i satisfies (f2).

(b) We now consider the case where $d(u_1)$ or $d(u_2)$ is even. Let v and w be the two common neighbors of u_1 and u_2. Then M_{i-1} and M_i have the faces u_1u_2w and u_1u_2v in common. We obtain a drawing D_i of G_i as follows. Let D_{i-1} be a feasible drawing of G_{i-1} and D' be the minimum segment drawing of M_i obtained by the algorithm **Draw-Fan**. Here four cases may arise depending on the degree of u_1 and u_2.

Case 1: $d(u_1) = 4$ *and* $d(u_2) = 4$.

Since D_{i-1} is a feasible drawing of G_{i-1}, u_1 is not an apex and u_2, w are apices in D_{i-1}. But since $d(u_2, M_i) = 4$, u_2 may or may not be an apex in D'. First we consider that u_2 is an apex in D'. According to Lemma 3.2.1(b), there are at most three other apices in D'. Since (v, u_1) and (u_1, w) are collinear in D, for patching D' with D_{i-1}, u_1 should not be an apex in D'. Hence by Lemma 3.2.1(b), we can choose that minimum segment drawing of M_i where (v, u_1) and (u_1, w) are collinear. Then one can patch the drawings by sharing at most four maximal segments as illustrated in Fig. 3.9(a) and 3.9(b). Hence $sc(G_i) \geq sc(G_{i-1}) + sc(M_i) - 4$.

We now consider that u_2 is not an apex in D'. Since u_2 is an apex in D_{i-1}, we cannot patch D_{i-1} with D'. We modify the drawing of D_{i-1} such that u_1 becomes an apex and u_2 is not an apex in D_{i-1}. We can do this without increasing the number of segments of D_{i-1}, since M_{i-1} has several minimum segment drawings. Now the edges (v, u_2) and (u_2, w) become collinear. Then one can patch the drawings by sharing at most four maximal segments as illustrated in Fig. 3.9(c) and 3.9(d). Hence $sc(G_i) \geq sc(G_{i-1}) + sc(M_i) - 4$.

In order to obtain a feasible drawing D_i of G_i, we modify the drawing of G_{i-1} such that the vertices v, u_2 and w become collinear. Since D_{i-1} is a feasible drawing, the edges (u_2, v) and (u_2, w) are not collinear in D_{i-1} (see Fig. 3.9(c)). Hence we need to modify the drawing D_{i-1} in such a way that after patching the edges (u_2, v) and (u_2, w) becomes collinear as illustrated in Fig. 3.9(d). Note that during modification of D_{i-1}, the number of segments in D_{i-1} does not increase. Hence four maximal segments is minimized in the combined drawing D_i of G_i. Thus after patching the number of segments in the drawing is $sc(D_i) \leq sc(G_{i-1}) + sc(M_i) - 4$.

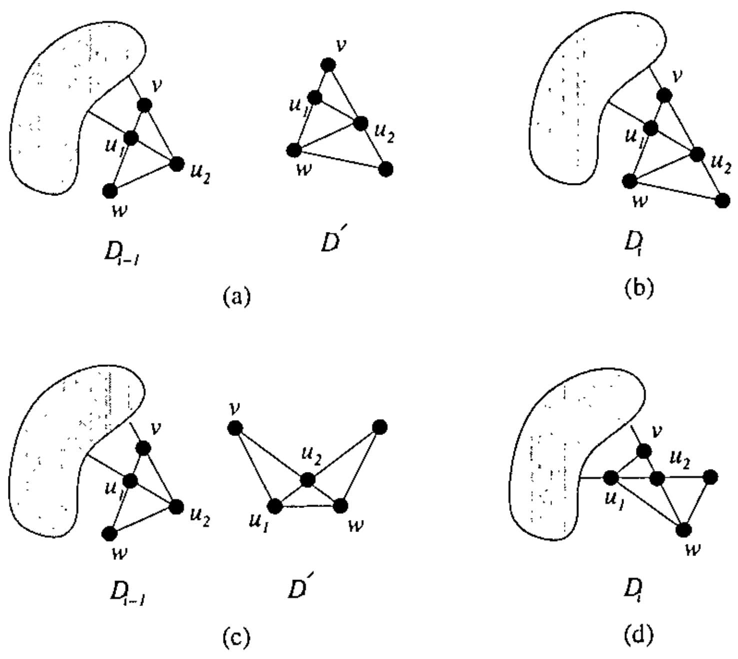


Figure 3.9: Illustration of patching the minimum segment drawing D' of M_i with D_{i-1} where M_{i-1} and M_i have two faces in common with $d(u_1) = d(u_2) = 4$.

Case 2: $d(u_1) > 4$ and $d(u_2) = 4$.

In this case the edges (u_2, v) and (u_2, w) are drawn on the same straight line segment during patching as illustrated in Fig. 3.10. We do not need to modify the drawing of M_i because $d(u_2) = 4$. By an argument similar to that of Case 1, we can prove that at most four maximal segments can be minimized in the combined drawing of G_i. Thus $sc(G_i) = sc(G_{i-1})+sc(M_i)-4$.

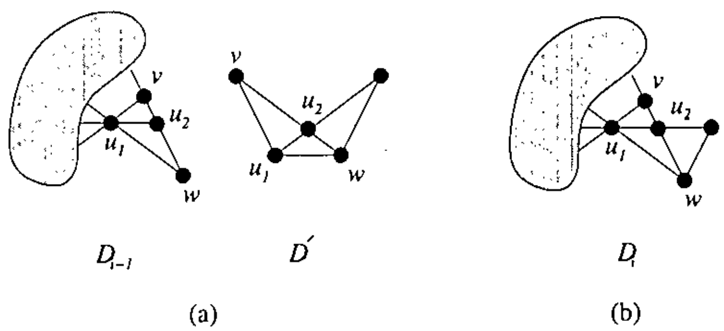


Figure 3.10: Illustration of patching the minimum segment drawing D' of M_i with D_{i-1} where M_{i-1} and M_i have two faces in common with $d(u_1) > 4$ and $d(u_2) = 4$.

Case 3: $d(u_1) = 4$ and $d(u_2) > 4$.

The proof for this case is similar to that of Case 1.

Case 4: $d(u_1) > 4$ and $d(u_2) > 4$.

In this case at least one of u_1 and u_2 have even degree. Suppose without loss of generality that $d(u_1)$ is even. Then by Lemma 3.2.1(c) and Corollary 3.2.2, there are four apices in the minimum segment drawing of M_{i-1} and one can choose the apices according to Lemma 3.2.4. Since by Corollary 3.2.2, u_2 cannot be an apex in any minimum segment drawing of M_i, the edges (v, u_2) and (u_2, w) are not collinear in D'. Hence the minimum segment drawing of M_{i-1} should be chosen such that (v, u_2) and (u_2, w) are not collinear i.e. u_2 is an apex in D_{i-1}. Then we don't need to modify the drawing D_{i-1} as we did in (a). But still we have to modify the drawing of M_i. Because in that case we need to make u_1 an apex in the drawing of M_i which will become collinear with two collinear apices (shown by thick line in Fig. 3.11(b)) in the combined drawing D_i. Then by an argument similar to that in (a), we can prove that $sc(G_i) = sc(G_{i-1}) + sc(M_i) + 1 - 5$.

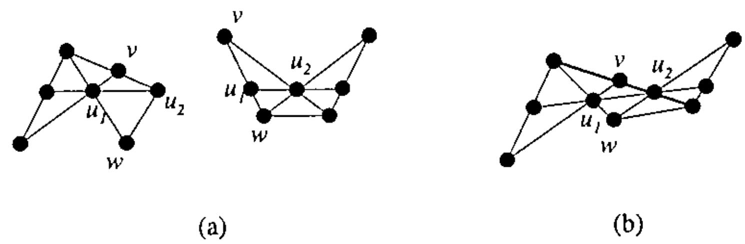


Figure 3.11: Illustration of patching the minimum segment drawing D' of M_i with D_{i-1} where M_{i-1} and M_i have two faces in common and $d(u_1)$ is even.

We now prove that D_i is a feasible drawing of G_i. We have already proved that D_i satisfies (f1). It is remaining to prove that D_i satisfies (f2). According to Lemma 3.1.1, the edges incident to u_2 are pairwise collinear in D'. Since during patching we have not changed the alignment of the incident edges of u_2 in all the cases, the incident edges to u_2 remains pairwise collinear in D_i, and hence D_i satisfies (f2). $\mathcal{Q.E.D.}$

Thus by patching the minimum segment drawings of the maximal fan components we can construct a minimum segment drawing of G. During the patching of the maximal fan components, we compute the co-ordinates of the vertices from the intersection of the line segments passing through it. Note that the relative slopes of the segments are computed during the construction of the minimum segment drawing of the fan components according to the algorithm **Draw-Fan**. By traversing the graph from one ear to another, one vertex at a time, we can compute the minimum segment drawing of the dual-path maximal outerplane graph. Hence the following theorem holds.

Theorem 3.3.6 *A minimum segment drawing of a dual-path maximal outerplane graph can be found in linear time.*

3.4 Upper bound on the Number of Segments

We now give an upper bound on the minimum number of segments required for a minimum segment drawing of a dual-path maximal outerplane graph.

Theorem 3.4.1 *Let G be a dual-path outerplane graph with m edges. Then the minimum number of segments required for a minimum segment drawing of G is at most $5m/7$.*

Proof. The upper bound is obtained if G contains maximum number of maximal fan components. A maximal fan component consists of at least seven edges. Hence there can be at most $m/7$ maximal fan components. By Lemma 3.1.1 the minimum number of segments required to draw a maximal fan component is five. Therefore at most $5m/7$ segments are required for a minimum segment drawing of G. $\mathcal{Q.E.D.}$

3.5 Conclusion

In this chapter, we have established a lower bound on the number of segments in any planar straight line drawing of a dual-path maximal outerplane graph G. We also have presented a linear-time algorithm for computing a minimum segment drawing of G. Then we have given an upper bound on the minimum number of segments required for a minimum segment drawing of a dual-path maximal outerplane graph.

Chapter 4

Dual-path Outerplane Graphs

In this chapter we prsent our algorithm for computing minimum segment drawings of dual-path outerplane graphs. In this case the input graph has vertices of degree two other than the ear vertices. To compute the fan components we first remove all the vertices of degree two except the two ear vertices.

4.1 Decomposition into Maximal Components

Let G be the dual-path outerplane graph. We replace all the chains of vertices of degree two $v_1, v_2, \ldots, v_l$ for $l \geq 3$, by an edge (v_1, v_l) and mark those edges. The graph obtained in this way is denoted by G'. We now divide the graph G' into maximal components. A *maximal component* G_i is a maximal outerplane subgraph of G'. In a maximal component, every face is triangulated. Since every vertex of G' has degree greater than 2, G' cannot contain a face with face length greater than four. Hence to divide the graph G' into maximal components we find out the faces of length four. Let $F = u_1u_2u_3u_4$ be a face of length four where (u_1, u_2) and (u_3, u_4) are outer edges of G'. We delete the outer edges of all such faces of G' and thus we divide G' into maximal components (see Figure 4.1). Since G' is a dual-path outerplane graph, the maximal subgraphs of G' are dual-path maximal outerplane graphs. Hence the following lemma holds.

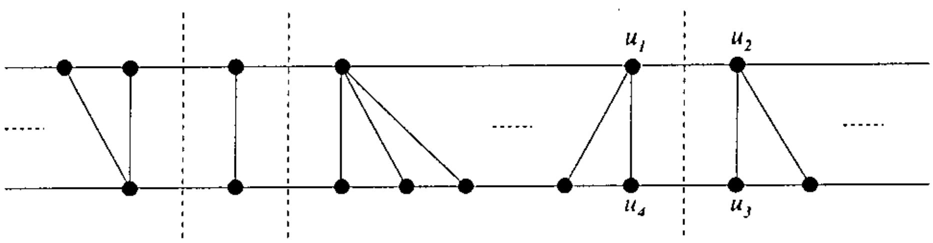


Figure 4.1: Illustration of the method for dividing G' into maximal components.

Lemma 4.1.1 *Each maximal subgraph of G' is a maximal dual-path outerplane graph.*

4.2 Minimum Segment Drawings of Maximal Components

Let $G_1, G_2, \ldots, G_k$ be the k maximal outerplane subgraphs of G'. We now find minimum segment drawings of each G_i for $1 \leq i \leq k$. At first we consider the case that $|V(G_i)| = 2$ i.e. G_i consists of two vertices with an edge between those two vertices. Hence the minimum segment drawing of G_i is the straight line drawing of G_i. We call such a component as an *edge component*. Thus the number of segments required for a minimum segment drawing of an edge component G_i is

$$sc(G_i) = 1 \tag{4.1}$$

We now consider that G_i is not an edge. By Theorem 3.3.6 and Lemma 4.1.1, G_i for $1 \leq i \leq k$ has a minimum segment drawing. We find a minimum segment drawing of G_i by dividing G_i into maximal fan components using the method described in Section 3.1. In each G_i some edges are marked. Let M be a fan component of G_i with fan vertex v and D be the minimum segment drawing of M obtained from the algorithm **Draw-Fan**. We now add chains of vertices of degree two corresponding to each marked edge on D. Note that the total number of segments in D can be reduced if a vertex of degree two is drawn as the intersection point of two distinct straight liue segments incident to that vertex. Thus a vertex of degree two among a chain of degree two vertices can be a convex corner in the drawing of M.

Let $e = (u_1, u_l)$ be a marked edge in M. We replace e by a chain of vertices of degree two $u_1, u_2, \ldots, u_l$ for $l \geq 3$. Either the chain is drawn on the straight line segment of e or one of the chain vertices u_i, $i \in \{2, \ldots, l-1\}$, becomes a convex corner and the other chain vertices are drawn on the two distinct straight line segments incident to u_i. Whether we choose the former or the latter depends on the degree of the fan vertex v of M and the position of marked edge on the embedding of M. Let $d = d(v)$ and $N(v) = \{v_1, v_2, \ldots, v_d\}$. We have two cases here.

Case 1: $d(v)$ is even and edge $(v_{d/2}, v_{d/2-1})$ is marked.

According to Lemma 3.1.1, we need $sc(M) = d(v)/2 + 3$ line segments in any minimum segment drawing of M. It is easy to observe that one segment in the drawing can be reduced by introducing a new convex corner instead of the two convex corners at $v_{d/2}$ and $v_{d/2-1}$. Hence the total number of segments required for a minimum segment drawing of the fan component M is

$$sc(M) = d(v)/2 + 2 \tag{4.2}$$

Thus the number of convex corners required for the drawing is three.

Case 2: otherwise.

In this case the chain of vertices is drawn on the straight line segment of e. Hence the minimum segment drawing for this case is obtained by applying the cases described in Section 3.1.

We now have the minimum segment drawings of the fan components of G_i. By Theorem 3.3.6 each maximal component has a minimum segment drawing. Hence after computing the minimum segment drawings of the fan components we patch them by the method of Chapter 3, to get minimum segment drawings for the maximal components. Then we patch the maximal components to get a minimum segment drawing of the graph G. The detail of which is given below.

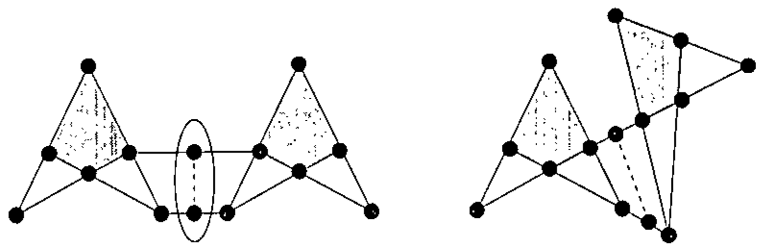


Figure 4.2: Patching an edge with the minimum segment drawing D.

4.3 Patching of the Minimum Segment Drawings of Maximal Components

Let D be the minimum segment drawing of the graph after patching the maximal components $G_1, G_2, \ldots, G_{i-1}$. Let D' be the minimum segment drawing after patching the maximal component G_i with D. We first consider the case where G_i is an edge. Only one segment is required for the minimum segment drawing of G_i. Note that this type of component only occurs as an inner edge. It always increases the number of segments by one as shown in Fig. 4.2.

$$sc(D') = sc(D) + 1 \tag{4.3}$$

We now assume that G_i is a maximal component with two ears. Since G is a dual-path outerplane graph, G_i connects with D by the two edges incident to an ear vertex u and an end point of an ear edge. We find the non-edge maximal component G_k which is nearest to G_i where $1 \leq k \leq i-1$. We omit all the edge components from G_k to G_i. The edge components can easily be patched with the drawing by adding a segment between the two outer paths for each edge component. Four cases arise depending on the four vertices forming the quadrilateral F joining G_i and G_k along which the graph was divided into maximal components. Let v_k and v_i be the fan vertices of G_k and G_i and u_k and u_i be the ear vertices of G_k and G_i respectively. Let w_k and w_i be the other two vertices of the corresponding ears. There are four cases to consider depending on the orientation of the six vertices.

Case 1: F consists of u_k, u_i, w_i and w_k.

In this case G_k and G_i are patched using the outer edges (u_k, u_i) and (w_k, w_i). Since $d(w_k, G_k) = d(w_i, G_i) = 3$, the edge (w_k, w_i) can share segment with the edges (v_k, w_k) and (w_i, v_i). Again since $d(u_k, G_k) = d(u_i, G_i) = 2$, the edge (u_k, u_i) can share segment with the edges (v_k, u_k) and (u_i, v_i). Note that the edges (v_k, w_k), (w_k, w_i), (w_i, v_i), (u_i, v_i), (u_k, u_i) and (v_k, u_k) form a cycle and at least three segments are required to draw a cycle. Hence if the edge (w_k, w_i) share segment with the edges (v_k, w_k) and (w_i, v_i) then the edge (u_k, u_i) cannot share segment with both the edges (v_k, u_k) and (u_i, v_i) and vice versa. Thus at most one segment can be reduced during the patching of G_k and G_i.

Figure 4.3(a) shows the minimum segment drawing for this case where edge (u_k, u_i) share a common segment with both the components and edge (w_k, w_i) share a common segment with G_k. Hence after patching the total number of segments for the drawing D' decreases by one.

$$sc(D') = sc(D) + sc(G_i) - 1 \tag{4.4}$$

Case 2: F consists of u_k, w_i, u_i and w_k.

In this case G_k and G_i are patched using the outer edges (u_k, w_i) and (w_k, u_i). Here also the edges (v_k, w_k), (w_k, u_i), (u_i, v_i), (w_i, v_i), (u_k, w_i) and (v_k, u_k) form a cycle and at most one segment can be reduced during the patching of G_k and G_i.

Figure 4.3(b) shows the minimum segment drawing for this case where edge (u_k, w_i) share a common segment with both the components and edge (w_k, u_i) share a common segment with G_k. Hence after patching the total number of segments for the drawing D' decreases by one.

$$sc(D') = sc(D) + sc(G_i) - 1 \tag{4.5}$$

Case 3: F consists of u_k, v_k, u_i and v_i.

In this case G_k and G_i are patched using the outer edges (u_k, v_i) and (v_k, u_i). Here the edge (u_k, v_i) can share segment with the edge (w_k, u_k) and the edge (v_k, u_i) can share segment with the edge (u_i, w_i). Hence during patching neither any segment can be reduced nor any extra segment is required. But if the degree of one of the fan vertices v_k and v_i is odd then one segment can be minimized by sharing a segment with the unshared edge incident to the fan

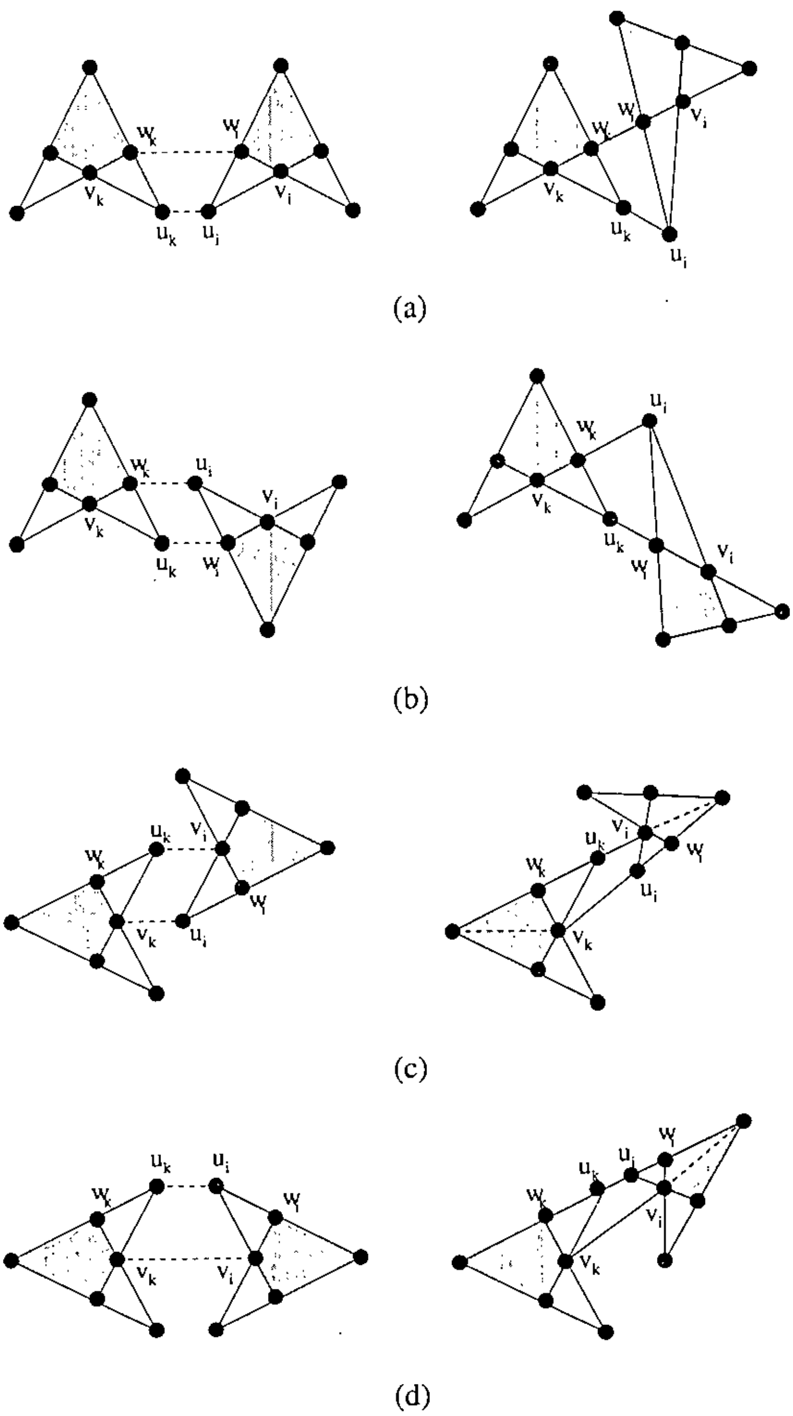


Figure 4.3: Illustration of the method for patching the maximal components to get the minimum segment drawing for G.

vertex with the odd degree.

Figure 4.3(c) shows the minimum segment drawing for this case where edge (u_k, v_i) share a common segment with G_k and edge (v_k, u_i) is drawn as a new segment. If the degree of the fan vertices is odd the number of segments decrease. Hence after patching the total number of segments for the drawing D' may at most decrease by one.

$$sc(D') = \begin{cases} sc(D) + sc(G_i) & \text{if } d(v_i, G_i) \text{ and } d(v_k, G_k) \text{ is even} \\ sc(D) + sc(G_i) - 1 & \text{otherwise} \end{cases} \tag{4.6}$$

Case 4: F consists of u_k, v_k, v_i and u_i.

In this case G_k and G_i are patched using the outer edges (u_k, u_i) and (v_k, v_i). Here the edge (u_k, u_i) can share segment with the edges (w_k, u_k) and (w_i, u_i). Hence during patching neither any segment can be reduced nor any extra segment is required. But if the degree of one of the fan vertices v_k and v_i is odd then one segment can be minimized by sharing a segment with the unshared edge incident to the fan vertex with the odd degree.

Figure 4.3(d) shows the minimum segment drawing for this case where edge (u_k, u_i) share a common segment with both the components and edge (v_k, v_i) is drawn as a new segment. If $d(v_i)$ is odd then the edge (v_k, v_i) share a common segment with G_k. Hence after patching the total number of segments for the drawing D' may at most decrease by one.

$$sc(D') = \begin{cases} sc(D) + sc(G_i) & \text{if } d(u_i, G_i) \text{ and } d(u_k, G_k) \text{ is even} \\ sc(D) + sc(G_i) - 1 & \text{otherwise} \end{cases} \tag{4.7}$$

4.4 Time Complexity

During the patching of the maximal components we compute the co-ordinates of the vertices from the intersection of the line segments passing through it. Note that the relative slopes of the segments are computed during the construction of the minimum segment drawing of the fan components. By traversing the graph from one ear to another one vertex at a time we can compute the minimum segment drawing of the dual-path outerplane graph. Hence we have the following theorem.

Theorem 4.4.1 *A minimum segment drawing of a dual-path outerplane graph can be found in linear time.*

4.5 Conclusion

In this chapter, we have presented an algorithm for finding minimum segment drawing of dual-path outerplane graphs. We first decompose the graph into maximal components and then patch the minimum segment drawings of the maximal components to compute a minimum segment drawing of the outerplanar graph. In the next chapter we are going to extend our algorithm for finding a minimum segment drawing of a subdivision of an outerplanar graph.

Chapter 5

Subdivision of Dual-path Outerplane Graphs

In this chapter we extend the algorithm presented in the previous chapter for computing minimum segment drawings of subdivisions dual-path outerplane graphs. In this case the input graph has vertices of degree two that are not on the outer face. We will prove that those vertices of degree two cannot decrease the number of segments in any minimum segment drawing of the dual-path outerplane graph.

Theorem 5.0.1 *Let G be a dual-path outerplane graph. Assume that G' is obtained from G by subdividing some of the inner edges of G. If G has a minimum segment drawing D with $sc(D)$ number of segments, then G' has a minimum segment drawing D' with $sc(D)$ number of segments.*

Proof. Let $e = (u, v)$ be an inner edge in G and e is subdivided in G' by replacing e with a path u, w, v through a new vertex w. Let G'' be the graph obtained by subdividing edge e of G and D'' be a minimum segment drawing of G''. If the uwv path in G'' is drawn as a straight line then the number of segments in D'' is $sc(D)$. Otherwise if uw and wv edges are drawn on different segments then the number of segments in D'' may decrease. We will show

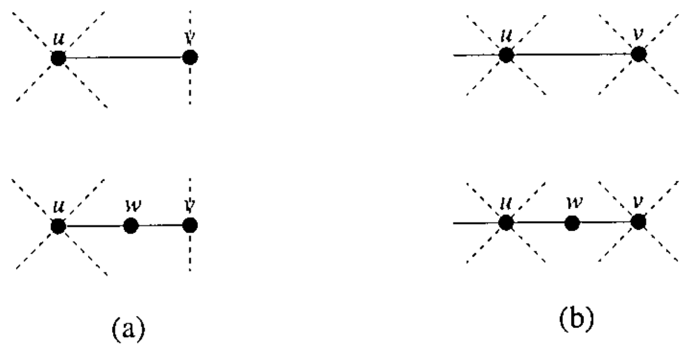


Figure 5.1: (a) e does not share, (b)e shares, a segment with an adjacent edge e' in D .

that subdividing e cannot decrease the number of segments in any drawing of G''. We have two cases:

Case 1: e does not share any segment with the adjacent edges in D.

In this case either e is an ear edge with $d(v) = 3$ or e is an edge with $d(u)$ odd and $d(v) = 3$. In both the cases, one of the incident vertices has $d(v) = 3$. Hence one segment is always required for the edge (u, v) and we insert the new vertex w on that segment (see Figure 5.1(a)).

Case 2: e shares a segment with an adjacent edge e' in D.

Let e' be adjacent to u. Then uw and wv can share the segment with e'. Hence subdividing e cannot decrease the number of segment in the minimum segment drawing of G'' (see Figure 5.1(b)). $\mathcal{Q.E.D.}$

Hence for computing the minimum segment drawing of a subdivision of a dual-path outerplane graph, we first remove the inner vertices of degree two. We then compute a minimum segment drawing of the resulting graph by the method described in the previous section. We then add the inner vertices of degree two on the line segments by subdividing the corresponding line segments. For example Figure 5.2(b) shows a minimum segment drawing of the graph in Figure 5.2(a) and Figure 5.2(c) shows the subdivision of the graph in Figure 5.2(a) and the minimum segment drawing of this graph is shown in Figure 5.2(d). We have following theorem.

Theorem 5.0.2 *A minimum segment drawing of a subdivision of a dual-path outerplane graph can be found in linear time.*

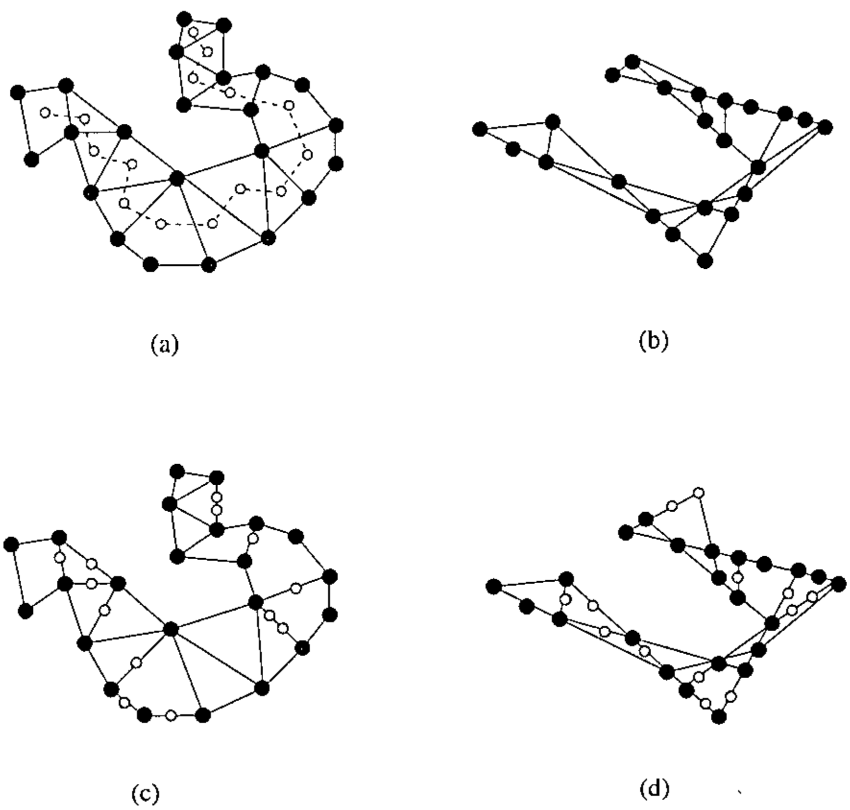


Figure 5.2: (a) A dual-path outerplane graph, (b) a minimum segment drawing of the dual-path outerplanar graph in (a), (c) a subdivision of the graph in (a), (d) a minimum segment drawing of the graph in (c).

In this chapter, we have presented an algorithm for finding a minimum segment drawing of a subdivision of a dual-path outerplane graphs. We have shown that the inner vertices of degree two cannot decrease the number of segments in any drawing of the original dual graph outerplanar graph. Thus a minimum segment drawing can be obtained using the algorithm of chapter 4 by removing the inner vertices of degree two.

Chapter 6

Conclusion

This thesis deals with minimum segment drawings of subclasses of outerplane graphs. We have started with an introductory overview on graph drawing in Chapter 1. In this chapter we have given a precise definition of minimum segment drawing of a graph and discussed several practical applications of this problem. Then we have depicted the challenges that we have faced to solve this problem. We illustrated some previous results on this field and have established our objective in this thesis.

In Chapter 2 we have introduced the preliminary ideas on graph theory and on minimum segment drawings. We have also discussed outerplane graphs and complexity theory in detail in this chapter.

In Chapter 3 we have established a lower bound on the number of segments in any planar straight line drawing of a dual-path maximal outerplane graph G. Finally, we have presented a linear-time algorithm for computing a minimum segment drawing of G.

In Chapter 4 we have extended our algorithm for finding minimum segment drawing of dual-path outerplane graphs.

Finally in Chapter 5 we have presented a linear-time algorithm to compute a minimum segment drawing of a subdivision of a dual-path outerplane graph. To the best of our knowledge our algorithms have been the first such result in the minimum segment drawing problem for an important subclass of outerplanar graphs. However, the following problems remained as future works.

1. To study the minimum segment drawing problem in conjunction with other aesthetic criteria like area requirement and symmetry of the drawing.

2. To obtain minimum segment drawing algorithm for any outerplanar graph.

3. To obtain minimum segment drawing algorithm for outerplanar graphs with imposing restriction on the degree.

4. To obtain minimum segment drawing algorithms for larger subclass of planar graphs.

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