FINITE ELEMENT MODELING OF NONLINEAR ELASTIC RESPONSE OF NATURAL AND HIGH DAMPING RUBBER

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It is hereby declared that, except where specific references are made, the work embodied in this thesis is the result of investigation carried out by the author under the supervision of Dr. A.F.M. Saiful Amin, Assistant Professor, Department of Civil Engineering, BUET.

Neither this thesis nor any part of it is concurrently submitted to any other institution in candidature for any degree.

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ABSTRACT

The monotonic response of natural rubber (NR) and high damping rubber (HDR) as revealed through recent experiments are critically examined and reported here in. In understanding the mechanical behavior of rubbers, attention is mostly paid to the compression and shear regime. In studying the mechanical test results in the compression and shear regime, the existence of Mullins’ effect, strain-rate dependency, hysteresis and residual strain effects in all the specimens was noted. In NR, the extent of these effects is found to have an inherent relation with the presence of microstructural voids. The presence of all these effects is also found to be significant in HDR. In addition, the strain-rate dependent high initial stiffness feature at low compressive strain levels was also evident. However, in considering a strain-rate dependent response, equilibrium and instantaneous responses are defined rate-independent responses. In the constitutive model, these responses are usually modelled using a hyperelasticity law. In this context, the current work is carried out using finite element modeling of the rate independent elastic behavior of NR and HDR subjected to compression, shear and their combinations.

To this end, several constitutive models based on phenomenological motivation were thoroughly studied. An improved hyperelasticity model was utilized to formulate the finite element coding for representing the rate-independent nonlinear elastic responses including high initial stiffness characteristics of rubber materials. In doing this an explicit analytical expressions for the second Piola-Kirchhoff stress tensor and the Cauchy stress tensor have been formulated for the distortional part of the hyperelasticity model. The Lagrangian elasticity tensor has also been formulated for the distortional part of the hyperelasticity model to implement in a general-purpose finite element code. The elastic equilibrium and instantaneous responses of NR and HDR under compression and shear have been simulated using material parameters identified from the available experimental observations. The numerical simulation results have been compared with the available experimental observations to discuss the adequacy of the developed finite element procedure in simulating quasi-incompressible response of NR and HDR under uniaxial compression and simple shear deformation. To this end, the simulation results have been verified with available analytical solution for the elastomeric rubber bearing with different shape factors. Numerical experiments have been performed for simultaneous action of
compression and shear. Finally, the possibility of modeling and analyzing the full-scale bridge seats and base isolation bearings have been investigated at different deformation modes by utilizing the developed finite element procedure. In this process, the numerical results have been compared with the available analytical results.
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1.1 GENERAL

Rubber is obtained in the form of latex from the tree Hevea Braziliensis. A high degree of deformability under the action of comparatively small stresses makes rubber unique among other materials. The typical extensibility falls within the range of 500 -1000% while it can sustain large compressive strain (even up to 90%) without undergoing any failure. The cause of such high deformability lies in the polymeric structure of rubber, interpreted from the molecular point of view (Trealor 1975).

As a polymer compound, natural rubber latex contains long molecular chains. The molecular weights of such polymers fall in the range of 100,000 – 1,000,000, with a typical value of 350,000. The molecules of such dimensions possess the flexibility associated with internal vibrations and rotations. Figure 1.1a presents a schematic representation of molecular chain structure of latex rubber. In the bulk material, the long chain molecules, however, do not stay in isolation but form a coherent network like structure with freely rotating links. In addition, weak secondary forces and interlocking at few places also do occur between the molecular chains to form a three dimensional network to keep the material in solid phase. Yet, such a material has little application in engineering application unless it is vulcanized. During vulcanization process, cross-linkages are introduced between the molecular chains of rubber through sulfur molecules in a chemical process (Fig. 1.1b). Furthermore, carbon-black particles are also added as solid fillers and act as reinforcing agent. These fillers are extremely fine particles with typical mean radii of the order of 100-4700Å (Govindjee 1991). These additives improve stiffness, toughness, hysteresis and rate dependence properties of the natural rubber (NR) system. Therefore, the extent of improvement of these properties is closely related with the filler concentration. In addition, rubbers are also believed to fall in the class of incompressible materials. However, all these characteristic behaviors in the material generally originate from its chemical composition and microstructure. In this context an experimental investigation was carried out by Amin (2001) for understanding the mechanical behavior and microstructure of rubber. The scheme comprised
observations of rubber microstructure through Scanning Electron Microscope (SEM) and mechanical tests in compression as well as in tension regimes. The microstructure of rubber contained voids and solid phage as shown in Fig. 1.2

Figure 1.1 Schematic representation of molecular structure of rubber
(a) Latex rubber, (b) vulcanized rubber
Figure 1.2 SEM images describing the void and solid phases:
(a) NR-I (b) NR-II (c) HDR
1.2 ENGINEERING APPLICATIONS OF RUBBER

Rubber has been a versatile and adaptable material in engineering applications for over 100 years. The use of rubber expanded greatly with the advent of vulcanization process, pioneered by Charles Goodyear in 1839. The very high extensibility and compressive strength, and also fatigue, abrasion and corrosion resistant characteristics of the material made it attractive to the engineers for its application in tires, bearings, seals, shock absorbing bushes, tunnel linings, wind shoes and in a lot others.

In addition, high damping rubber (HDR) was developed for specific applications in base isolation bearings to protect structures from earthquakes (Fujita et al 1990; Kelly 1991 and 1997) and vibrations (Castellani et al 1998). The rubber of this kind contains a high proportion of filler (around 31%) and therefore provides better energy absorption property. Figure 1.3 schematically presents the details of a HDR bearing with layers of steel plates. As a requirement of base isolation philosophy, the horizontal layers of steel plates impart a large vertical stiffness in the bearing, while the HDR layers provide a very low horizontal stiffness in the structure subjected to earthquakes or other vibration induced excitations.

Thus these devices are subjected to compression. However, when the structure is subjected to lateral loads like earthquake and wind, they suffer combined action of
compression and shear. Fig 1.4 shows the probable loading mechanisms. In these
loading mechanisms, it is clear that the rubber as material must have the capacity to withstand compression and tension. Hence rubber with less microstructure voids is more suitable than one with more voids (Fig. 1.2). In this context, different analytical models were proposed to predict the stability and stiffness of laminated NR bearings at different deformation modes (Lim and Hermann 1987; Hermann et al 1988 and 1989; D'Ambrosio et al 1995). Hwang and Ku (1997) also tried to simulate the responses obtained from HDR bearings using an analytical model. However, these approaches of studying bearing responses were concerned with developing an equivalent homogeneous model for predicting average response of bearings and were not capable of considering the behaviors of rubber and steel as different materials. Hence, the capabilities of these models in predicting local stress and strain conditions that exist between the layers were very limited. To this end, Ali and Abdel-Ghaffar (1995) proposed a finite element model that considered the constitutive behaviors of individual materials through respective constitutive relations and predicted rate-independent response in shear. In a recent study, Dorfmann and Burtscher (2000) studied the development of cavitations damage in HDR bearings through finite-element models in shear regime. However, these two efforts were concerned with the prediction of rate-independent response in shear regime whereas the applications of these models in compression-shear regimes are unavailable.
1.3 DEVELOPING CONSTITUTIVE MODEL FOR SIMULATION OF LARGE STRAIN RESPONSE: PRIMARY CONSIDERATIONS

In the past decades, there had been considerable efforts to model, analyze and design structures composed of rubber-like materials using a numerical approach such as finite element methods. Yet, the core of a reliable numerical analysis lies in an adequate constitutive model.

For structural applications of rubbery materials, two methods are available at present: testing the prototypes and analyzing numerically on computers. Although both the methods have their own difficulties in design and evaluation, the latter one is more suitable.

The mechanical behavior of rubber materials under monotonic loading is dominated by a nonlinear response (Aklonis et al 1972). Due to the presence of polymeric network chains together with filler particles in the vulcanized rubber microstructure, the nonlinear monotonic response is characterized by high initial stiffness feature at low strain level, large deformability at moderate strain level followed by a hardening feature at large strain level (Fig.5).

In addition, such a response significantly depends on applied strain-rate. However, under cyclic loading, other inelastic behaviors such as Mullin's' effect (Mullin's 1969) and hysteresis (Gent 1962a,b) are also observed. All these effects are found to be
more prominent in HDR (Amin 2001). So, a general constitutive model for a rubberlike material is required to be capable of representing all these effects.

Historically, the large extensibility feature of rubbery materials has had ever motivated the researchers to express the large strain nonlinear elastic behavior through hyperelastic models (Charlton *et al.* 1993). These formulations are based on finite strain theories of nonlinear continuum mechanics capable of considering both the material and geometric nonlinearity. Naturally, while formulating these relations, attention was given mostly on modeling the responses in the large strain tension regime. Of these conventional hyperelasticity models, strain invariant-based Mooney-Rivlin model (Mooney 1940; Rivlin 1948) is the oldest one. However, this model cannot represent the hardening feature observed at large strain level. Subsequently, Hart-Smith (1966), Yeoh (1990), Yamashita and Kawabata (1992), Arruda and Boyce (1993) proposed other improved models to include such a feature. Furthermore, stretch based models (Ogden 1984, Peng and Landel 1972) were also formulated. Recently, Amin (2001) investigated the performances of these conventional models and showed their incapability in representing the high initial stiffness feature present in the equilibrium and instantaneous response of NR and HDR in compression. This led to the development (Amin *et al.* 2002) of an improved and versatile hyperelasticity model that can depict the high initial stiffness feature. Thus it was possible to have an adequate representation of equilibrium and instantaneous responses of NR and HDR in both compression and tension (Amin 2001). The very recent experimental observations (Wiraguna 2003) have shown the applicability of this improved model in representing the responses also in shear regime. Yet, the Finite Element Method (FEM) is the ultimate beneficiary of the development of any successful constitutive model.

1.4 BACKGROUND OF THE STUDY

In the preceding sections, the physics, engineering, general mechanical behavior and primary considerations in developing a constitutive model for large strain response simulation of rubbers have been discussed in a summary. From the discussion it appears that:

1. The addition of filler and other elements in vulcanization process creates a
composite system in the microstructure of rubber. As a result, the rubbery materials provide better energy absorption property. Furthermore, there are possibilities of the presence of voids in the microstructure.

2. In engineering applications, rubbers are used to carry tensile as well as compressive stresses. Hence, the study of rubber mechanics in compression is equally important as that in tension or shear. Furthermore, a general constitutive model is required to be capable of representing the mechanical behaviors such as Mullin's effects, nonlinear rate-independent response, strain-rate dependency, hysteresis, residual strain and recovery effects.

3. The constitutive model for rubbers must be developed under the framework of finite deformation kinematics in order to account for the large strain behavior.

4. In a physically motivated constitutive model developed to represent viscoelastic behavior, there must be an adequate hyperelasticity model in the finite deformation model structure in order to describe the elastic boundary states, namely instantaneous and equilibrium responses. Furthermore, the parameters involved in expressing these responses should be determined from direct experimental observations to retain the physical significance of the model.

5. Formulation and computer coding of relevant finite strain theories are required to incorporate the improved hyperelastic model (Amin 2001, Amin et al 2002 and Wiraguna et al 2003) in a general-purpose finite element code.

1.5 SCOPE AND OBJECTIVES

Based on the background summarized in Section 1.4, the present research is carried out to meet the following objectives:

1. To formulate the expressions for the second Piola-Kirchhoff stress tensor and the Cauchy stress tensor for the distortional part of the hyperelasticity model.
2. To formulate the Lagrangian elasticity tensor for the distortional part of the hyperelasticity model.

3. To implement the stress tensor and the elasticity tensor formulations in a general-purpose finite element code.

4. To simulate the equilibrium response and instantaneous responses of NR and HDR under compression and shear using material parameters identified from the available experimental observations.

5. To compare the numerical simulation results with experimental observations to discuss the adequacy of the developed finite element procedure in simulating quasi-incompressible response of NR and HDR under uniaxial compression and simple shear deformation.

6. To investigate the possibilities of modeling and analyzing the full-scale bridge seats and base isolation bearings at different deformation modes by utilizing the developed finite element procedure.

All over the world, natural rubber (NR) is being extensively used in constructing steel plate laminated bridge seats. Furthermore, steel plate laminated lead-plugged NR bearings and high damping rubber (HDR) bearings are becoming popular and being extensively used as base isolation devices in earthquake resistant design. At present, the rubber and construction industries have to depend on expensive and time-consuming prototype test data to design and evaluate these devices. In this context, the outcome of the proposed work is expected to have a wide application both in the design office and in the rubber industry. It will thus widen the scope of having a reliable analysis, design and performance evaluation of the bridge seats and base isolation bearings in numerical approach.
Chapter 2
General Mechanical Behavior of Rubber

2.1 GENERAL

The mechanical behavior of rubbers is dominated by nonlinear rate dependent elastic response (Aklonis et al. 1972) and includes other characteristic behaviors like Mullins' effect (Mullins 1969) and hysteresis (Gent 1962a,b). Furthermore, incompressibility is an important characteristic feature of this material. The current chapter presents the recent experimental results of rubber materials obtained in compression (Amin 2001) and shear (Wiraguna 2003). These results are utilized in the subsequent chapters of this work in developing a general finite element code based on an improved hyperelasticity model.

2.2 SPECIMENS

Amin (2001) performed experiments in compression on two types of natural rubber (NR-I and NR-II) and high damping rubber (HDR) whereas Wiraguna (2003) investigated mechanical tests in shear on natural rubber (NR-II) and high damping rubber (HDR). To present a general understanding of the mechanical behavior of rubbers; the following subsections highlight the significant observations of these experiments (Amin 2001 and Wiraguna 2003). In the subsequent chapters, experimental data obtained from two types of rubber namely NR (NR-II) and HDR are used for simulation purposes. The details of the specimens are presented in Table 2.1.

2.3 EXPERIMENTS IN COMPRESSION

Uniaxial compression tests were carried out by Amin (2001) to observe the mechanical behavior of NR-I, NR-II and HDR. The mechanical behaviors of these rubbers are briefly discussed in the subsequent sub-sections.
2.3.1 Mullin's effect

The first period of stress-strain curves obtained from cyclic loading test on a virgin specimen differs significantly from the shape of subsequent cycles due to a strain-induced stress softening effect. Mullins (1969) was the first to point out this phenomenon and therefore frequently referred to as 'Mullin's effect'.

Table: 2.1 Details of the specimens

<table>
<thead>
<tr>
<th>SPECIMEN DESIGNATION</th>
<th>NR-I</th>
<th>NR-II</th>
<th>HDR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
<td>NR</td>
<td>NR</td>
<td>HDR</td>
</tr>
<tr>
<td>Application</td>
<td>General purpose</td>
<td>Bridge bearing</td>
<td>Bridge bearing</td>
</tr>
<tr>
<td>Manufacturer</td>
<td>Shinoda Rubber Co.</td>
<td>Yokohama Rubber Co.</td>
<td>Yokohama Rubber Co.</td>
</tr>
<tr>
<td>Strength</td>
<td>4.0 MPa*</td>
<td>0.98 MPa**</td>
<td>0.78 MPa**</td>
</tr>
</tbody>
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MECHANICAL TESTS IN COMPRESSION

<table>
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<tr>
<th>Shape</th>
<th>Cubic</th>
<th>Cylindrical</th>
<th>Cylindrical</th>
</tr>
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<tbody>
<tr>
<td>Size</td>
<td>H:50mm, L:50mm, W:50mm</td>
<td>H:41 mm, D:49 mm</td>
<td>H:41 mm, D:49 mm</td>
</tr>
</tbody>
</table>

MECHANICAL TESTS IN SHEAR

<table>
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<tr>
<th>Shape</th>
<th>Flat Strip</th>
<th>Flat Strip</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>L: 25mm, W: 25mm, H: 5mm</td>
<td>L: 25mm, W: 25mm, H: 5mm</td>
</tr>
</tbody>
</table>

The softening has been attributed to breakdown or slippage of weak linkages between the filler and rubber, filler-filler aggregates and breakdown of molecular network chains. The effect is much more pronounced in the vulcanizates containing high proportion of reinforcing fillers.

Figure 2.1 illustrates an example of typical Mullin's effect exhibited by virgin rubber subjected to a cyclic compression test. In this demonstration, stretches (i.e. \(1+\Delta L/L\), where \(L\) is the undeformed length) were applied in three cycles in each stretch level (Fig. 2.1a). Three maximum stretch levels (namely S1, S2 and S3) were considered.
The specimen is initially assumed to be in a reference stress-free virgin state $V$ and loaded in compression to a stretch state $S_1$ along stress-stretch path $P_1$ (Fig. 2.1b).

At this point, when the specimen is unloaded, it follows path $P_2$ and returns to $V$. In the second period, when the material is loaded once again up to $S_1$, the stress-stretch response follows $P_3$ path and behaves like a softer material than the virgin one. However, the unloading path remains the same as $P_2$. In the third period, the loading path remains at $P_3$ and unloading path at $P_2$ provided the maximum stretch in loading is kept constant at $S_1$. In a fourth period, when loading is applied up to $S_2$ stretch, the path $P_3$ is followed to reach $S_1$ state, not the $P_1$ path. If additional loading is applied beyond $S_1$, the path $P_1$ is followed again to reach $S_2$ and during unloading, $P_4$ path is followed. In subsequent loading phases up to $S_2$ stretch, $P_5$ path is followed, while in unloading sequence $P_4$ path is maintained indicating a greater loss in material stiffness.

![Figure 2.1](image_url)  
**Figure 2.1** A typical example of Mullins' effect observed in virgin rubber (NR-I) under compression (a) Applied stretch history; (b) Stress-stretch response
In the third maximum stretch level of S3, P1 and P7 loading paths are followed and P6 path is maintained for unloading purpose. However, from Fig. 2.1b, the material appears to reach neighborhood of a stable state after the first loading cycle provided the maximum stretch value is maintained constant. Hence in the subsequent subsections, data obtained after removing the Mullin’s effect will be discussed for subsequent modeling.

![Figure 2.2 A typical nonlinear response obtained from natural rubber under monotonic compression and after removing Mullins’ effect](image)

2.3.2 Nonlinearity in monotonic response

Apart from the Mullin’s effect present in the virgin rubbers, the stress-strain behavior of rubber is recognized to be nonlinearly elastic in its main part. Figure 2.2 presents a typical nonlinear response that one can obtain in a monotonic loading test. The stress-strain response contains three features. At the initiation of stretch application, the presence of fillers gives a little bit high stiffness. With increasing applied stretch, the initial stiffness disappears due to the breakdown of rubber-filler bonds. However, at the end, stiffness starts to increase prominently when the free lengths of the molecular network chains (Fig. 1.1) get depleted and the material approaches the ultimate limit of deformability.

2.3.3 Strain-rate dependency

Figure 2.3 presents a schematic representation of typical rate-dependent responses that can be obtained from a viscoelastic solid. When such a solid is loaded at an infinitely slow rate, the stress-strain curve follows the E-E’ path. This behavior is
called the equilibrium response. On the other hand, in the case of an infinitely fast loading rate, the response takes the I-I' path. Such a response is known as the instantaneous response and defines a domain where viscoelastic effects come into play. However, in practical experiments, it is difficult to apply infinitely fast or slow loading rates to reach these boundaries.

**Figure 2.3** A Schematic representation of responses obtained from a viscoelastic solid

Vulcanized rubber is a typical example of highly viscous solid (Ward 1985), wherein the stress response is highly dependent on the rate of loading. Figure 2.4 presents some typical rate-dependent response that can be obtained, when a rubber is

**Figure 2.4** Viscoelastic effect exhibited by natural rubber at different strain rates in monotonic loading under compression

Vulcanized rubber is a typical example of highly viscous solid (Ward 1985), wherein the stress response is highly dependent on the rate of loading. Figure 2.4 presents some typical rate-dependent response that can be obtained, when a rubber is
subjected to monotonic compression loading at varied strain rates. The comparison of the curves portrays the presence of strain-rate effect on the stress-strain response.

![Diagram showing cyclic compression responses from specimens at different strain rates]

Figure 2.5 Cyclic compression responses from specimens at different strain rates; (a) NR-I, (b) NR-II, (c) HDR
2.3.4 Hysteresis and residual strain

Apart from the strain-rate dependent effects, a rubber also exhibits a significant hysteresis phenomenon and residual strain under cyclic loading. Filler concentration plays an important role on these behaviors (Ward 1985). Figure 2.5 illustrates the hysteresis effect and residual strain feature obtained from a rubber specimen subjected to cyclic loading under compression. Here, the term 'residual strain' refers to the 'set' in the specimen at the end of a cyclic test.

2.3.5 Incompressibility

The resistance of rubber against shear deformation is very low compared to the resistance against volumetric deformation. This gives a very high value of bulk modulus compared to its shear modulus and the material is considered to be incompressible. Under this assumption, the deformed cross-section of the specimen subjected to uniaxial or biaxial deformation can be predicted to calculate the Cauchy (true) stress in the material. However, Herrmann et al (1989) indicated the possibility of the existence of voids in the rubber microstructure that might largely affect the bulk modulus.

An experimental setup capable of measuring the deformed cross section of the rubber specimens subjected to a large uniaxial compression was developed by Amin et al 2003. In Figures 2.6 and 2.7, the mechanical test results have been shown for HDR and NR materials. These experimental evidences justify the near incompressibility feature in the specimens. This observation is used in Chapter 5 and 6 for developing an algorithm capable of simulating a large strain, quasi-incompressible response.

2.4 EXPERIMENTS IN SHEAR

To observe the mechanical behavior of rubber under shear deformation, cyclic test of simple shear loading was carried out by Wiraguna (2003). All specimens were subjected to preloading before actual test to remove Mullin's effect. Cyclic Test with different strain rates, simple relaxation test and cyclic relaxation test were carried out by Wiraguna (2003) on preloaded specimens.
Figure 2.6: Mechanical test on HDR subjected to monotonic compression (a) applied stretch history (b) Cauchy stress vs stretch as obtained from Incompressibility assumption and measurement (c) Volume with increasing stretch.
Figure 2.7: Mechanical test on NR-II subjected to monotonic compression (a) applied stretch history (b) Cauchy stress vs stretch as obtained from Incompressibility assumption and measurement (c) Volume with increasing stretch
2.4.1 Mullin's Effect

Section 2.3.1 presents Mullin's effect of rubber in compression regime. For shear regime, in preloading stage, each virgin specimen was subjected to cyclic shear loading for 5 cycles with a 0.05/s strain rate and a maximum strain level of 250%. Figure 2.8 shows the strain history for the preloading test. Figure 2.9 and 2.10 show the stress history and stress-strain relation from the preloading tests for HDR and NR. The softening effect (Mullin's effect) in the first loading cycle is shown from the figure. After 2-3 loading cycles, the same stress-strain behavior was obtained. The removal of Mullin's effect has been confirmed. In HDR, the softening effect was more pronounced than in NR. All the findings conform to those in compression regime.

![Figure 2.8 Strain histories for preloading test](image)

Figure 2.8 Strain histories for preloading test
Figure 2.9 Stress history and stress-strain relation for HDR
From Figure 2.9 and 2.10, it is seen from the comparison of hysteresis loop area that the amount of energy dissipation in HDR is larger than in NR. This indicates that the energy absorption having direct relation with damping property of HDR is larger than that of NR.
2.4.2 Strain-rate dependency

To find out the instantaneous state and to observe the strain-rate dependency of rubber, a series of cyclic loading tests with different strain rates were carried out by Wiraguna et al (2003). These cyclic tests were conducted to find out an instantaneous state. In theory, the instantaneous response is obtained by applying infinite first loading, however, in practice, it is impossible to apply the infinite fast loading due to the limitation of a testing machine.

In actual tests, the strain rates from 0.05/s to 0.5/s were applied to HDR, and from 0.005/s to 0.1/s were applied to NR-II. The temperature, when the cyclic tests were carried out, was 20° C for HDR and 21° C for NR-II. Table 2.2 presents the detailed data of cyclic test.

Figure 2.11 shows the strain history for cyclic test at 0.1/s. Figure 2.12 shows stress responses of HDR in cyclic test with different strain rates, and Figure 2.13 shows stress responses of NR-II in cyclic test with different strain rates.

Table 2.2. Cyclic tests with different strain rates

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Strain rates ( /sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>HDR</td>
<td>0.05, 0.25, 0.40, 0.50</td>
</tr>
<tr>
<td>NR-II</td>
<td>0.005, 0.01, 0.05, 0.10</td>
</tr>
</tbody>
</table>
Chapter 2: General Mechanical Behavior of Rubber

Figure 2.11 Strain history for cyclic test at 0.1/s

Figure 2.12 Stress responses of HDR in cyclic test with different strain rates
From Figures 2.12 and 2.13, for loading part, the stress increases with increasing strain rates due to viscosity effect. It shows the strain-rate dependency of rubber. However, at higher strain rates, the increase in the stress response to different strain rates is vanished, and the stress responses tend to follow the same path. This indicates that the stress responses at higher strain rates have reached the neighborhood of an instantaneous state. For HDR, the stress responses at strain rates equal or larger than 0.25/s, show the neighborhood of an instantaneous state. Therefore, the stress responses at and over 0.25/s-strain rate can be considered as the neighborhood of an instantaneous state. For NR-II, the stress responses at strain rates equal to or larger than 0.05/s, show the neighborhood of an instantaneous state. Hence, the stress responses at and over 0.05/s-strain rate can be considered as the neighborhood of an instantaneous state. Contrary to loading paths, the unloading paths for all specimens show the absence of strain-rate dependency.

2.4.3 Nonlinearity in monotonic response

Apart from the Mullin's effect present in virgin rubbers, the stress-strain behavior of rubber is recognized to be nonlinearly elastic in its main part. Figure 2.2 presents a typical nonlinear response that one can obtain in a monotonic loading test. The
stress-strain response contains three features. At the initiation of strain application, the presence of fillers gives a little bit high stiffness. With increasingly applied strain, the initial stiffness disappears due to the breakdown of rubber-filler bonds. However, at the end, stiffness starts to increase prominently when the free lengths of the molecular network chains (Fig. 1.1) get depleted and the material approaches the ultimate limit of deformability. Figure 2.12 and 2.13 attributed to the presence of large amount of filler particles (Kelly 1997). When comparing HDR with NR-II, the high initial stiffness at a low strain level is most prominent in HDR at higher strain rate.

2.4.4 Hysteresis and residual strain

Apart from the strain-rate dependent effects, rubber also exhibits a significant hysteresis phenomenon and residual strain under cyclic loading. Filler concentration plays an important role on these behaviors (Ward 1985). Besides this nonlinearity response, Figure 2.12 and 2.13 illustrate the hysteresis effect and residual strain feature obtained from a rubber specimen subjected to cyclic loading under shear. Here, the term 'residual strain' refers to the 'set' in the specimen at the end of a cyclic test.
3.1 GENERAL

In Chapter 2, the nonlinear large strain response of rubbers in compression and shear is introduced. The response of rubber is materially nonlinear. Furthermore, due to large deformation, the deformed shapes of rubbers show the marked geometrical nonlinearity. To describe these responses using a general constitutive law, the theory must be written within the framework of nonlinear continuum mechanics employing finite deformation theory. The theory can take both material and geometrical nonlinearity into account. In this context, this chapter presents the basic elements of nonlinear mechanics and their relation in modeling the nonlinear elastic behaviors.

3.2 NONLINEAR CONTINUUM MECHANICS

Two sources of nonlinearity exist in the analysis of solid continua, namely, material nonlinearity and geometric nonlinearity (Bonet and Wood 1997). The former occurs when, irrespective of the reason, the stress strain behavior given by the constitutive relation is nonlinear. Whereas, the latter is important when changes in geometry, whether large or small, have a significant effect on the load deformation behavior. Geometric nonlinearity includes deformation-dependent boundary and loading conditions.

Despite the obvious success of the assumption of linearity in engineering analysis, it is equally obvious that many situations demand consideration of nonlinear behavior. Nonlinear and linear continuum mechanics deal with the same subjects such as kinematics, stress and equilibrium, and constitutive behavior. But in the linear case an assumption is made that the deformation is sufficiently small to enable the effect of changes in the geometrical configuration of the solid to be ignored, whereas in the nonlinear case, the magnitude of the deformation is unrestricted. The study of the numerical analysis of nonlinear continua using a computer is called nonlinear computational mechanics, which, when applied specifically to the investigation of solid continua, comprises nonlinear continuum mechanics together with the
numerical schemes for solving the resulting governing equations. In this context, the goal of this current work is to describe the observed material nonlinearity through experiments overcoming the effects of geometrical nonlinearity and thereby to identify the nonlinear material constitutive parameters. In the earlier experimental works performed by Amin et al (2002) and Wiraguna et al (2003), the effects geometric nonlinearity of rubber specimens were overcome through suitable test technique. Hence the homogeneous deformation property was recovered from experiments. But the geometric nonlinearity is usually considered in the general finite strain theory. So, these parameters and constitutive relations are used in subsequent simulation process that can consider both material and geometrical nonlinearity.

3.3 NONLINEAR STRAIN MEASURES

In general, structural components or continuum bodies exhibit large strains, when undergoing a geometrically nonlinear deformation process. As an introduction to the different ways in which these large strains can be measured firstly one-dimensional structural element undergoing large displacements and large strains is considered. Subsequently, a brief introduction to the difficulties involved in the definition of correct large strain measures in continuum situations has been given.

A beam member of initial length $L$ and the cross sectional area $A$ is stretched to a final length $l$ and area $a$ as shown in Fig. 3.1. The simplest possible quantity that one can use to measure the strain in the bar is the so-called engineering strain $\varepsilon_E$ defined as:

$$\varepsilon_E = \frac{l - L}{L}$$

Figure 3.1 one-dimensional strains

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Yet, a different measure of strain could be used. For instance, the change in length \( \Delta l = l - L \) could be divided by the final length rather than the initial length. Whichever definition is used, if \( l \approx L \) the small strain quantity \( \varepsilon = \Delta l / l \) is recovered.

An alternative large strain measure can be obtained by adding up all the small strain increments that take place, when the rod is continuously stretched from its original length \( L \) to its final length \( l \). This integration process leads to the definition of the natural or logarithmic strain \( \varepsilon_n \) as:

\[
\varepsilon_n = \int \frac{dl}{l} = \ln \frac{l}{L} \quad (3.2)
\]

Although the above strain definitions can in fact be extrapolated to the deformation of a three-dimensional continuum body, this generalization process is complex and computationally costly (Bonet and Wood, 1997). Strain measures that are much more readily generalized to continuum are the Green strain \( \varepsilon_G \) and Almansi strain \( \varepsilon_A \), defined as:

\[
\varepsilon_G = \frac{l^2 - L^2}{2L^2} \quad (3.3a)
\]
\[
\varepsilon_A = \frac{l^2 - L^2}{2l^2} \quad (3.3b)
\]

Irrespective of the definition of strain is used, a simple Taylor series analysis shows that, for the case, where \( l \approx L \), all the above quantities converge to the small strain definition \( \Delta l / l \). For instance, in the Green strain case, one can observe that:

\[
\varepsilon_G(l \approx L) \approx \frac{l^2 + \Delta l^2 - l^2}{2l^2} = \frac{l^2}{2} + \frac{\Delta l^2 + 2l \Delta l - l^2}{l^2} \approx \frac{\Delta l}{l} \quad (3.4)
\]
3.4 KINEMATICS

A proper description of motion is fundamental requirement to finite deformation analysis and such an emphasis is necessary because infinitesimal deformation analysis implies a host of assumptions that are taken for granted and seldom articulated. Thus, the geometric nonlinearity is also taken into account. Kinematics is the study of motion and deformation without reference to the cause (Bonet and Wood 1997). The following sections show the consideration of finite deformation that enables alternative coordinate systems to be employed, namely, material and spatial descriptions associated with the names of Lagrange and Euler, respectively.

3.4.1 Material and spatial descriptions

In finite deformation analysis, a careful distinction has to be made between the coordinate systems that can be chosen to describe the behavior of the body whose motion is under consideration. Generally speaking, relevant quantities, such as density, can be described in terms of where the body was before deformation or where it is during deformation; the former is called a material description, and the latter is called a spatial description. Alternatively, these are often referred to as Lagrangian and Eulerian descriptions respectively. A material description refers to the behavior of a material particle, whereas a spatial description refers to the behavior at a spatial position.

In order to understand the difference between a material and a spatial description, a simple scalar quantity, such as the material density $\rho$ has been considered.

(a) **Material description**: the variation of $\rho$ over the body is described with respect to the original (or initial) coordinate $X$, used to label a material particle in the continuum at time $t = 0$ as:

$$\rho = \rho(X, t)$$  \hfill (3.5a)

(b) **Spatial description**: $\rho$ is described with respect to the position in space, $x$, ...
currently occupied by a material particle in the continuum at time $t$ as:

$$p = \rho(x,t) \quad (3.5b)$$

In Equation (3.5a), a change in time $t$ implies that the same material particle $x$ has a different density $\rho$. Consequently, an interest is focused on the material particle $X$. In Equation (3.5b), however, a change in the time $t$ implies that a different density is observed at the same spatial position $x$, now probably occupied by a different particle. Consequently, an interest is focused on a spatial position $x$.

### 3.4.2 Deformation gradient

A key quantity in finite deformation analysis is the deformation gradient $F$, which is involved in all equations relating to quantities before deformation to corresponding quantities after (or during) deformation. The deformation gradient tensor enables the relative spatial position of two neighboring particles after deformation to be described in terms of their relative material position before deformation; consequently, it is central to the description of deformation phenomena and hence strains.

If $Q_i$ and $Q_j$ are two material particles in the neighborhood as shown in Fig. (3.2) of a material particle $P$ before deformation, the elemental vectors $dX_i$ and $dX_j$ be obtained as follows:

$$dX_i = X_{Q1} - X_P \quad \text{and} \quad dX_j = X_{Q2} - X_P \quad (3.6)$$

after deformation, the material particles $P, Q_i$ and $Q_j$ have deformed to current spatial position given by:

$$x_p = \phi(X_{P,t}) \quad x_{Q1} = \phi(X_{Q1,t}) \quad x_{Q2} = \phi(X_{Q2,t}) \quad (3.7)$$
and the corresponding elemental vectors:

\[
\begin{align*}
\Delta x_1 &= x_{q1} - x_p = \phi(X_{\rho} + dX_1, t) - \phi(X_{\rho}, t) \\
\Delta x_2 &= x_{q2} - x_p = \phi(X_{\rho} + dX_2, t) - \phi(X_{\rho}, t)
\end{align*}
\] (3.8)

Now the deformation gradient can be defined as:

\[
\mathbf{F} = \frac{\partial \phi}{\partial \mathbf{X}} = \nabla_{\mathbf{X}} \phi
\] (3.9)

3.4.3 Distortional component of the deformation gradient tensor

When dealing with incompressible and nearly incompressible materials, it is necessary to separate the volumetric component from the distortional (or isochoric) components of the deformation. Such a separation must ensure that the distortional component, namely, \( \hat{\mathbf{F}} \), does not imply any change in volume. Noting that the determinant of the deformation gradient gives, the volume ratio, the determinant of \( \hat{\mathbf{F}} \) must therefore satisfy
\[ \det \hat{F} = 1 \]  

(3.10)

The deformation gradient \( F \) can now be expressed in terms of the volumetric and distortional components, \( J \) and \( \hat{F} \), respectively, as

\[ F = J^{-1/3} \hat{F} \]  

(3.11)

3.4.4 Rate of deformation

Again the initial elemental vector \( dX_1 \) and \( dX_2 \) introduced in the earlier section and their corresponding pushed-forward spatial counterpart \( dx_1 \) and \( dx_2 \) are considered as shown in Fig.3.3:

\[ dx_1 = FdX_1 ; \quad dx_2 = FdX_2 \]  

(3.12a,b)

In the earlier section, strain was defined and measured as the change in the scalar product of two arbitrary vectors. Similarly, strain rate can now be defined as the rate of change of the scalar product of any pair of vectors. For the purpose of measuring this rate of change, the current scalar product could be expressed in terms of the material vectors \( dX_1 \) and \( dX_2 \) (which are not functions of time) and the time-dependent right Cauchy-Green tensor \( C \) as:

\[ dx_1 \cdot dx_2 = dX_1 \cdot C dX_2 \]  

(3.13)

Differentiating this expression with respect to time and recalling the relationship between the Lagrangian strain tensor \( E \) and the right Cauchy-Green tensor as \( 2E = (C - I) \), it gives the current rate of change of the scalar product in terms of the initial elemental vectors as:

\[ \frac{d}{dt} (dx_1 \cdot dx_2) = dX_1 \cdot CdX_2 = 2dX_1 \cdot \dot{E}dX_2 \]  

(3.14)

where, \( \dot{E} \), the derivative with respect to time of the Lagrangian strain tensor is
known as the material strain rate tensor and can be easily obtained in terms of $\dot{F}$ as:

$$\dot{E} = \frac{1}{2} \dot{C} = \frac{1}{2} (\dot{F}^T F + F^T \dot{F})$$  \hspace{1cm} (3.15)$$

The material strain rate tensor, $\dot{E}$, gives the current rate of change of the scalar product in terms of the initial elemental vectors. Alternatively, it is often convenient to express the same rate of change in terms of the current spatial vectors. For this purpose, Equations (3.12a,b) can be inverted as:

$$dX_1 = F^{-1} dx_1; \quad dX_2 = F^{-1} dx_2$$  \hspace{1cm} (3.16a,b)$$

Introducing these expressions into Eqn. (3.14), it gives the rate of change of the scalar product in terms of $dx_1$ and $dx_2$ as:

$$\frac{1}{2} \frac{d}{dt} (dx_1 \cdot dx_2) = dx_1 \cdot (F^{-T} E F^{-1}) dx_2$$  \hspace{1cm} (3.17)$$

The tensor in the expression on the right-hand side is simply the pushed forward
spatial counterpart of \( \dot{E} \) and is known as the rate of deformation tensor \( d \) given as:

\[
d = \varphi \left[ \dot{E} \right] = F^{-T} \dot{E} F^{-1}; \quad \dot{E} = \varphi^{-1}[d] = F^T dF
\]  

(3.18 a, b)

3.3.5 Strain tensors

As a general measure of deformation, let us consider the change in the scalar product of the two elemental vector \( dX_1 \) and \( dX_2 \) as they deform to \( dx_1 \) and \( dx_2 \). This change will involve both change in length and changes in the enclosed angle between the two vectors. The spatial scalar product \( dx_1 \cdot dx_2 \) can be found in terms of the material vectors \( dX_1 \) and \( dX_2 \) as:

\[
dx_1 \cdot dx_2 = dX_1 \cdot CdX_2
\]  

(3.19)

where, \( C \) is the right Cauchy-Green deformation tensor, which is given in terms of the deformation gradient \( F \) as

\[
C = F^T F
\]  

(3.20)

It can be noted that the tensor \( C \) operates on the material vectors \( dX_1 \) and \( dX_2 \) and consequently \( C \) is called a material tensor quantity. Alternatively, the initial material scalar product \( dX_1 \cdot dX_2 \) can be obtained in terms of the spatial vectors \( dx_1 \) and \( dx_2 \) via the left Cauchy-Green or Finger tensor \( B \) as:

\[
dx_1 \cdot dx_2 = dx_1 \cdot B^{-1} dx_2
\]  

(3.21)

where, \( B \) is:

\[
B = FF^T
\]  

(3.22)

In (3.21), \( B^{-1} \) operates on the spatial vectors \( dx_1 \) and \( dx_2 \), and consequently \( B^{-1} \) or \( B \) itself is a spatial tensor quantity.
The change in scalar product can now be found in terms of the material vectors \( dX_1 \) and \( dX_2 \) and the Lagrangian or Green strain tensor \( E \) as:

\[
\frac{1}{2} (dX_1 \cdot dX_2 - dX_1 \cdot dX_2^2) = dX_1 \cdot EdX_2
\]  

(3.23)

where, the material tensor \( E \) is given by

\[
E = \frac{1}{2} (C - I)
\]

(3.24)

where \( I \) is the identity matrix.

Alternatively, the same change in scalar product can be expressed with reference to the spatial elemental vectors \( dx_1 \) and \( dx_2 \) and the Eulerian or Almansi strain tensor \( e \) as:

\[
\frac{1}{2} (dx_1 \cdot dx_2 - dx_1 \cdot dx_2^2) = dx_1 \cdot edx_2
\]

(3.25)

where, the spatial tensor \( e \) is:

\[
e = \frac{1}{2} (I - B^{-1})
\]

(3.26)

### 3.5 STRESS TENSOR

Stress is first defined in the current configuration in the standard way as force per unit area. This leads to the well-known Cauchy stress tensor as used in linear analysis. In contrast to linear small displacement analysis, stress quantities that refer back to the initial body configuration can also be defined. This will be achieved using work conjugacy concepts that will lead to the Piola-Kirchhoff stress tensors. Finally, the objectivity of several stress rate tensors is considered.

#### 3.5.1 Cauchy stress tensor

For deriving the Cauchy stress tensor, a general deformable body at its current
position, as shown in Fig. 3.4, is considered. In order to develop the concept of stress, it is necessary to study the action of the forces applied by one region $R_1$ of the body on the remaining part $R_2$ with which it is in contact. For this purpose let us consider the element of area $\Delta a$ normal to $n$ in the neighborhood of spatial point $p$ shown in Fig. 3.4. If the resultant force on this area is $\Delta F$, the traction vector $t$ corresponding to the normal $n$ at $p$ is defined as:

$$t(n) = \lim_{\Delta a \to 0} \frac{\Delta F}{\Delta a}$$  \hspace{1cm} (3.27)

where the relationship between $t$ and $n$ must be such that it satisfies Newton's third law of action and reaction

$$t(-n) = -t(n)$$  \hspace{1cm} (3.28)

and hence $t(n) = \sigma n$, where $\sigma$ represents the Cauchy stress tensor and can be represented as below:

$$\sigma = \sum_{i,j=1}^{3} \sigma_{ij} e_i \otimes e_j$$  \hspace{1cm} (3.29)

### 3.5.2 First Piola-Kirchhoff stress tensor

The internal virtual work done by the stresses can be expressed as,

$$\delta W_{\text{int}} = \int \sigma : \varepsilon d\mathbf{v}$$  \hspace{1cm} (3.30)

Pairs such as $\sigma$ and $\varepsilon$ in this equation are said to be work conjugate with respect to the current deformed volume in the sense that their product gives work per unit current volume. Expressing the virtual work equation in the material coordinate system, alternative work conjugate pairs of stresses and strain rates will emerge. To achieve this objective, the spatial virtual work is first expressed with respect to the initial volume and area by transforming the integrals for $d\mathbf{v}$ to give
Chapter 3: Introduction to Nonlinear Continuum Mechanics

Figure 3.4: Description of stress tensor

\[
\int J \sigma : \delta dV = \int f_0 \cdot \delta v \, dV + \oint_{\partial \Omega} t_0 \cdot \delta v \, dA
\]  
(3.31)

where, \( f_0 = Jf \) is the body force per unit undeformed volume and \( t_0 = t\left(\frac{da}{dA}\right) \) is the traction vector per unit initial area, where the area ratio can be obtained after some algebra:

\[
\frac{da}{dA} = \frac{J}{\sqrt{n \cdot Bn}}
\]  
(3.32)

The internal virtual work given by the left-hand side of Eqn. (3.31) can be expressed in terms of the Kirchhoff stress tensor \( \tau \) as:

\[
\delta W_{\text{int}} = \int \tau : \delta dV, \quad \tau = J \sigma
\]  
(3.33 a, b)

This equation reveals that the Kirchhoff stress tensor \( \tau \) is work conjugate to the rate of deformation tensor with respect to the initial volume. The crude transformation
\[ \delta W_{\text{int}} = \int \mathbf{\sigma} : \delta \mathbf{\varepsilon} \, dV \]
\[ = \int \mathbf{\sigma} : (\delta \mathbf{F} F^{-1}) \, dV \]
\[ = \int \text{tr} \left( \mathbf{JF}^{-1} \mathbf{\sigma} \delta \mathbf{F} \right) \, dV \]
\[ = \int \left( \mathbf{JF} F^{-T} \right) : \delta \mathbf{F} \, dV \]  
(3.34)

We observe from this equality that the stress tensor work conjugate to the rate of the deformation gradient \( \dot{\mathbf{F}} \) is the so-called first Piola-Kirchhoff stress tensor given by:

\[ \mathbf{P} = \mathbf{JF} F^{-T} \]  
(3.35)

### 3.5.3 Second Piola-Kirchhoff stress tensor

The first Piola-Kirchhoff stress tensor \( \mathbf{P} \) is an unsymmetric two-point tensor and, as such, is not completely related to the material configuration. It is possible to contrive a totally material symmetric stress tensor, known as the second Piola-Kirchhoff stress \( \mathbf{S} \), by pulling back the spatial element of force \( d\mathbf{p} \) to give a material force vector \( d\mathbf{P} \) as

\[ d\mathbf{P} = \mathbf{\phi}^{-1}[d\mathbf{p}] = \mathbf{F}^{-1} d\mathbf{p} \], where, \( d\mathbf{p} = \mathbf{J} \mathbf{\sigma} \mathbf{F}^{-T} d\mathbf{A} = \mathbf{P} d\mathbf{A} \]  
(3.36)

This gives the transformed force in terms of the second Piola-Kirchhoff stress tensor \( \mathbf{S} \) and the material element of area \( d\mathbf{A} \) as

\[ d\mathbf{P} = \mathbf{S} d\mathbf{A} ; \quad \mathbf{S} = \mathbf{JF}^{-1} \mathbf{\sigma} \mathbf{F}^{-T} \]  
(3.37 a,b)

### 3.5.4 Deviatoric and pressure components of stress

In modeling constitutive behavior of incompressible materials, it is physically relevant to isolate the hydrostatic pressure component \( p \) from the deviatoric component \( \mathbf{\sigma} \) of
the Cauchy stress tensor as:

\[ \sigma = \sigma' + pI; \quad p = \frac{1}{3} \text{tr} \sigma = \frac{1}{3} \sigma : I \]  

\[(3.38)\]

where, the deviatoric Cauchy stress tensor \( \sigma' \) satisfies \( \text{tr} \sigma' = 0 \).

Similar decompositions can be established in terms of the first and second Piola-Kirchhoff stress tensors. For this purpose, we simply substitute the above decomposition into Eqns. (3.35) for \( P \) and (3.37) for \( S \) to give:

\[ P = P' + pJF^{-T}; \quad \text{and} \quad P' = J\sigma' F^{-T} \]

\[ S = S' + pJC^{-1}; \quad S' = JF^{-1} \sigma' F^{-T} \]

\[(3.39a)\] \[ (3.39b)\]

### 3.5.5 Principle of virtual work

In order to describing the principle of virtual work, let \( \delta v \), an arbitrary virtual velocity from the current position of the body as shown in Fig. 3.5. The virtual work, \( \delta w \), per unit volume and time, done by the residual force \( \mathbf{r} \) during this virtual motion is \( \mathbf{r} \cdot \delta v \), and equilibrium implies:

\[ \delta w = \mathbf{r} \cdot \delta v = 0 \]  

\[(3.40)\]

the above scalar equation is fully equivalent to the vector equation \( \mathbf{r} = 0 \). Now a weak statement of the static equilibrium of the body can be obtained by integrating over the volume of the body to give:

\[ \delta W = \int (\text{div} \mathbf{u} + \mathbf{f}) \cdot \delta v \, dv = 0 \]  

\[(3.41)\]

A more common and useful expression can be derived as follows:

\[ \text{div} (\sigma \delta v) = (\text{div} \sigma) \cdot \delta v + \mathbf{\sigma} : \nabla \delta v \]  

\[(3.42)\]

Using this equation together with the Gauss theorem enables Eqn. (3.41) to be
rewritten as
\[ \int_\Omega n \cdot \sigma \delta v \, da - \int \sigma : \nabla \delta v \, dv + \int f \cdot \delta v \, dv = 0 \]  \quad (3.43)

and consequently Eqn. (3.41) becomes,
\[ \int \sigma : \delta \delta v \, dv = \int f \cdot \delta v \, dv + \int t \cdot \delta v \, da \]  \quad (3.44)

Finally, expressing the virtual velocity gradient in terms of the symmetric virtual rate of deformation \( \varepsilon \delta t \) and the antisymmetric virtual spin tensor \( \delta \gamma \) and taking into account again the symmetry of \( \sigma \) gives the spatial virtual work equation as
\[ \delta W = \int \sigma : \varepsilon \delta t \, dv - \int f \cdot \delta v \, dv - \int t \cdot \delta v \, da = 0 \]  \quad (3.45)

This fundamental scalar equation states the equilibrium of a deformable body and will become the basis for the finite element discretization.

Figure 3.5: Description of virtual work principle
3.6 HYPERELASTICITY

The equilibrium equations derived are expressed in terms of the stresses inside the body. These stresses result from the deformation of the material. It is now necessary to express them in terms of some measure of this deformation, such as, the strain. These relationships, known as constitutive equations, obviously depend on the type of material under consideration and may or may not be dependent on time.

Generally, constitutive equations must satisfy certain physical principles. The equations must obviously be frame-invariant. In this section, the constitutive equations will be established in the context of a hyperelastic material and stresses are derived by the partial derivative of potential energy function with respect to strain tensor. So hyperelasticity constitutes the basis for more complex material models, such as elastoplasticity, viscoplasticity, and viscoelasticity.

3.6.1 Definition

Hyperelasticity may be defined as a material that has elastic potential function $W$. Materials for which the constitutive behavior is only a function of the current state of deformation are generally known as elastic. Under such conditions, any stress measured at a particle $X$ is a function of the current deformation gradient $F$ associated with that particle. Instead of using any of the alternative strain measures given in the earlier section, the deformation gradient $F$, together with its conjugate first Piola-Kirchhoff stress measure $P$, will be retained in order to define the basic material relationships. Consequently, elasticity can be generally expressed as

$$P = P(F(X), X)$$

where the direct dependency upon $X$ allows for the possible inhomogeneity of the material. In the special case, when the work done by the stresses during a deformation process is dependent only on the initial state at time $t_0$ and the final configuration at time $t$, the behavior of the material is said to be path-independent and the material is termed hyperelastic.
3.6.2 Lagrangian elasticity tensor

The relationship between $S$ and $C$ or $E = \frac{1}{2}(C - I)$, will invariably be nonlinear. Within the framework of a potential Newton-Raphson solution process, this relationship will need to be linearized with respect to an increment $u$ in the current configuration. Using the chain rule, a linear relationship between the directional derivative of $S$ and the linearized strain $DE[u]$ can be obtained, initially in a component form, as

$$DS_{IJ}[u] = \frac{d}{d\varepsilon_{i=0}} S_{IJ}(E_{KL}[\varepsilon + u])$$

$$= \sum_{K,L=1}^3 \frac{\partial S_{IJ}}{\partial E_{KL}} \frac{d}{d\varepsilon_{i=0}} K_{KL}[\varepsilon + u]$$

$$= \sum_{K,L=1}^3 \frac{\partial S_{IJ}}{\partial E_{KL}} DE_{KL}[u]$$

This relationship between the directional derivatives of $S$ and $E$ is more concisely expressed as

$$DS[u] = C : DE[u]$$

where the symmetric fourth-order tensor $C$, known as the Lagrangian or material elasticity tensor, is defined by the partial derivatives as

$$C = \sum_{I,J,K,L=1}^3 C_{IJKL} E_I \otimes E_J \otimes E_K \otimes E_L,$$

$$C_{IJKL} = \frac{\partial S_{IJ}}{\partial E_{KL}} = -\frac{4\partial^2 W}{\partial C_{IJ} \partial C_{KL}} = C_{KLI}$$

For convenience these expressions are often abbreviated as

$$C = \frac{\partial S}{\partial E} = 2\frac{\partial S}{\partial C} = \frac{4\partial^2 W}{\partial C \partial C}$$
3.6.3 Isotropic hyperelastic material description

The hyperelastic constitutive equations discussed so far are unrestricted in their application. It is required to restrict these equations to the common and important isotropic case. Isotropy is defined by requiring the constitutive behavior to be identical in any material direction. This implies that the relationship between \( W \) and \( C \) must be independent of the material axes chosen and, consequently, \( W \) must only be a function of the invariant of \( C \) as

\[
W(C(X),X) = W(I_{1c}, I_{2c}, I_{3c}, X) \tag{3.51}
\]

where, the invariants of \( C \) are defined here as

\[
I_{1c} = \text{tr} \ C = C : I \tag{3.52a}
\]

\[
I_{2c} = \text{tr} \ C C = C : C \tag{3.52b}
\]

\[
I_{3c} = \det C = J^2 \tag{3.52c}
\]

As a result of the isotropic restriction, the second Piola-Kirchhoff stress tensor can be rewritten as

\[
\sigma = 2 \frac{\partial W}{\partial C} = 2 \frac{\partial W}{\partial I_{1c}} \frac{\partial I_{1c}}{\partial C} + 2 \frac{\partial W}{\partial I_{2c}} \frac{\partial I_{2c}}{\partial C} + 2 \frac{\partial W}{\partial I_{3c}} \frac{\partial I_{3c}}{\partial C} \tag{3.53}
\]

3.6.4 Isotropic hyperelastic spatial description

In design practice it is obviously the cauchy stresses that are of engineering significance. These can be obtained indirectly from the second Piola-Kirchhoff stresses as follows:

\[
\sigma = J^{-T} F S F^T \tag{3.54}
\]

Substituting \( S \) from Eqn. (3.53) and noting that the left cauchy-green tensor is
\[ B = FF^T \text{ gives,} \]
\[ \sigma = 2J^{-1} W_1 B + 4J^{-1} W_2 B^2 + 2J W_3 I \]  \hspace{1cm} (3.55)

In this equation \( W_1, W_2, \) and \( W_3 \) still involve derivatives with respect to the invariants of the material tensor \( C \). Nevertheless it is easy to show that the invariants of \( B \) are identical to the invariants of \( C \), as the following expressions demonstrate:

\[ I_{1b} = \text{tr}[B] = \text{tr}[FF^T] = \text{tr}[F^TF] = \text{tr}[C] = I_{1c} \]  \hspace{1cm} (3.56a)

\[ I_{2b} = \text{tr}[BB] = \text{tr}[FF^TFF^T] = \text{tr}[F^TF][F^TF] = \text{tr}[CC] = I_{2c} \]  \hspace{1cm} (3.56b)

\[ I_{3b} = \det B = \det[FF^T] = \det[F^TF] = \det[C] = I_{3c} \]  \hspace{1cm} (3.56c)

3.7 FINITE ELEMENT IMPLEMENTATION OF INCOMPRESSIBLE RESPONSE: VARIATIONAL METHODS AND INCOMPRESSIBILITY

It is well known in small strain linear elasticity that finding the stationary position of a total energy potential with respect to displacements can derive the equilibrium equation (Bonet and Wood 1997). This applies equally to finite deformation situations and has the additional advantage that such a treatment provides a unified framework within which such topics as incompressibility contact boundary conditions and finite element technology can be formulated. In particular in the context of incompressibility a variational approach conveniently facilitates the introduction of Lagrangian multipliers or penalty methods of constraint (Simo and Taylor 1982), where the resulting multifield variational principles incorporate variables such as the internal pressure. The use of an independent discretization for these additional variables resolve the well-known locking problem associated with incompressible finite element formulations.

3.7.1 Total potential energy and equilibrium

A total potential energy function whose directional derivative yields the principle of virtual work (Bonet and Wood 1997) is
\[ \Pi(\varphi) = \int W(C) dV - \int f_0 \cdot \varphi dV - \int_{\partial V} t_0 \cdot \varphi dA \] (3.57)

To proceed we assume that the body and traction forces are not functions of the motion. This is usually the case for body forces \( f_0 \), but it is unlikely that traction forces \( t_0 \) will conform to this requirement in a finite deformation context. Under these assumptions the stationary position of the above function obtained by equating to zero its derivative in an arbitrary direction \( \delta \nu \) to give:

\[
\begin{align*}
D \Pi(\delta \nu) &= \int \frac{\partial W}{\partial C} : D C(\delta \nu) dV - \int f_0 \cdot \delta \nu \; dV - \int_{\partial V} t_0 \cdot \delta \nu \; dA \\
&= \int S : D E(\delta \nu) dV - \int f_0 \cdot \delta \nu \; dV - \int_{\partial V} t_0 \cdot \delta \nu \; dA = 0
\end{align*}
\] (3.58)

Here, this equation is identical to the principle of virtual work, that is:

\[
D \Pi(\delta \nu) = \delta W(\varphi, \delta \nu)
\] (3.59)

and consequently the equilibrium configuration \( \varphi \) renders the total potential energy. The stationary condition of Eqn. (3.59) is also known as a variational statement of equilibrium.

### 3.7.2 Penalty methods for incompressibility

The analysis of incompressible nonlinear elastic solids is considered by a penalty function approach. The penalty function method provides a simple and effective procedure of reducing a constrained minimization problem, to one without constraints.

In the context of nonlinear elastostatics, the penalty function method provides a procedure of enforcing the incompressibility constraint, without restraining the configuration space to isochoric deformations. For this type of problems, the application of the method hinges on a suitable extension to the compressible range, of the constitutive model for the given incompressible material. The most widely used
extension assumes, as strain energy in the compressible range is the sum of the strain energy potential of the incompressible material plus a penalty term enforcing the incompressibility constraint. For a certain class of incompressible materials, to which the Mooney-Rivlin model belongs, an alternative form of the strain energy potential in the compressible range is proposed by (Simo and Taylor 1982), which leads to a simpler form of the elasticity tensor. This form is, therefore, particularly useful in the context of a finite element formulation.

3.7.3 Penalty formulation for nonlinear elastic materials

In elastostatics, the variational formulation of the incompressibility constraint by a penalty function (Simo and Taylor 1982) procedure involves the extension of the constitutive model for the incompressible material to the compressible range. In the linearized theory this extension is immediate. The constitutive equation

\[ \sigma = -pI + 2\mu \varepsilon \]  

(3.60)

where, \( \varepsilon \) is the deviatoric part of the infinitesimal strain tensor \( \varepsilon \) and it is extended to the compressible range by considering

\[ \sigma = K \text{div}(\mathbf{u}) + 2\mu \varepsilon \]  

(3.61)

This is the same as adopting for the strain energy \( \bar{W} \) in the compressible range a form:

\[ \bar{W} = \bar{W} + \frac{1}{2}K(\text{div}(\mathbf{u}))^2 \]  

(3.62)

where, \( \bar{W} = \mu \text{tr}(\varepsilon \cdot \varepsilon) \) is the distortional part. A penalty formulation makes use of this compressible model, with the bulk modulus \( K \) as a penalty parameter (Zeinkiewicz and Taylor 1996). In the nonlinear theory, however, the situation is quite different. A first formulation considers a strain energy potential of the form

\[ \tilde{W}(I_1, I_2, I_3) = \hat{W}(I_1, I_2) + U(I_3) \]  

(3.63)
in the compressible range. \( W (I_1, I_2) \) is the strain energy of the given incompressible material, and the term \( U(I_3) \) has the structure of a penalty function enforcing the incompressibility constraint.
Chapter 4
Constitutive Modeling for Nonlinear Elastic Response

4.1 GENERAL

Several mathematical constitutive theories of nonlinear large elastic deformation based on strain-energy-density functions have been developed for rubbers. Generally design engineers can use these theories to analyze and design elastomer products operating under large deformed complex states by a finite element method. This chapter introduces the hyperelasticity modeling approach and presents the conventional hyperelasticity models. In later part of this chapter, a new stain energy density function for NR and HDR subjected to compression and shear has been cited.

4.2 HYPERELASTICITY MODELING OF RUBBERS: AN OVERVIEW

Rubber-like solids are ideal models of elastomers for theoretical convenience. They are isotropic, perfectly elastic for quite large deformations and substantially incompressible (Peng et al. 1997). They are called hyperelastic models. In other words, their mechanical properties can be characterized by means of a strain energy function. A strain energy function is a state function of strain and does not depend on the straining path. The state of the strain is completely specified by the invariants of the deformation tensor or the three principal stretches, denoted by \( \lambda_1, \lambda_2 \) and \( \lambda_3 \), measured from a given configuration, together with their directions. There are two approaches for obtaining a strain energy function: the phenomenological theory and the statistical mechanical theory. The former is based on purely mathematical reasoning and the latter on molecular or structural concepts. The aim of a phenomenological theory is to find out a general way of describing the mechanical properties, but not to pursue the connection between the material's molecular and structural features and its mechanical description, while the statistical mechanical theory aims to derive a description directly, usually through the statistical treatment of elastomeric molecular networks.
However, in both approaches the stress-strain relationship is derived from a strain energy density function ($W$) that depends on the final state of strain and is independent of loading history. Furthermore, the responses predicted by these models are essentially rate-independent. The construction of ($W$) either by statistical mechanics or continuum mechanics forms the basis of the hyperelasticity theory.

In general, the deformation of a body may be resolved into principal strains corresponding to the three principal directions along the three mutually perpendicular axes (Figure 4.1).

For a complete specification of elastic properties of the material, it is sufficient to know the form of the strain energy function, ($W$). For isotropic elastic materials, the strain energy function ($W$) can also be expressed as a function of invariant of a deformation tensor $I_i$ ($i=1,3$).

### 4.2.1 Models based on strain invariant

Among the strain energy functions ($W$) based on statistical molecular theory, Arruda and Boyce (1993) function is the most successful one. A compressible version of the Arruda and Boyce model is also available in Anand (1996). The Arruda and Boyce model is expressed in terms of $I_1$ and it needs only two parameters, namely the $\lambda$ and $\lambda_m$: 
Chapter 4: Constitutive Modeling for Nonlinear Elastic Response

\[ W(I_1) = \mu \left\{ \frac{1}{2} (I_1 - 3) + \frac{1}{20\lambda_m^2} (I_1^2 - 9) + \frac{11}{1050\lambda_m^4} (I_1 - 27)^3 + ... \right\} \] (4.1)

Among the strain invariant-based models, a polynomial form of energy density as proposed by Rivlin (1948a,b) is the earliest and the commonest one. (Equation 4.2) depicts the general polynomial form with \( C_{ij} \) as the material parameters:

\[ W(I_1, I_2) = \sum_{i,j} C_{ij}(I_1 - 3)(I_2 - 3) \] (4.2)

The Rivlin model in its most general form (Eqn. 4.2) is reasonably complicated. This led many researchers to develop variations on the general form to suit their own applications. The most commonly referred Mooney-Rivlin function (Mooney 1940; Rivlin 1948a,b) (Equation 4.3) is derived as the first order polynomial expansion of (Eqn. 4.2) with \( C_{10} \) and \( C_{01} \) as material parameters. Sometimes, the single term Neo-Hookean form (Eqn. 4.4) is also employed.

\[ W(I_1, I_2) = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) \] (4.3)

\[ W(I_1, I_2) = C_{10}(I_1 - 3) \] (4.4)

While, the Mooney-Rivlin form is convenient for its simplicity, it is often found to be inadequate for predicting the stresses associated with other modes of deformations. This failure of Mooney-Rivlin equations to provide adequate multi-axial data predictions was thought to arise not for taking enough terms of the possible expansions of (Eqn.4.2), (Tschoegl 1972). Hence, efforts were made to include higher order of expansion terms to obtain better prediction.

In this course, James and Green (1975) employed various higher order expansions of the general Rivlin function in their attempts to reliably fit and use test data from rubber compounds having high carbon-black loading. The second order and the third order invariant expansion with 5 and 9 terms were also tried there (Eqn.4. 5 and 4.6).

\[ W(I_1, I_2) = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + C_{20}(I_1 - 3)^2 + C_{02}(I_1 - 3)^2 \] (4.5)
In the trial, the third order expansion was found to give acceptable predictions within the range of fitted data ($1 < \lambda < 2$). The extrapolation ability beyond that range was found not to be so good as to the expectation. The large strain-hardening feature of rubbers cannot therefore be modeled. This prompted other contemporary researchers to include different forms of $I_2$. In this course, Rivlin and Saunders (1951) found that $\lambda$ is substantially constant and that $\lambda$ is independent of $I_1$, but varies with $I_2$. They performed experiments on two different types of vulcanized rubbers and proposed a strain energy density function, assuming incompressibility in the following form:

$$W(I_1, I_2) = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + C_{20}(I_1 - 3)^2 + C_{02}(I_1 - 3)^2 + C_{21}(I_1 - 3)^2(I_2 - 3) + C_{12}(I_1 - 3)(I_2 - 3)^2 + C_{30}(I_1 - 3)^3 + C_{03}(I_1 - 3)^3$$

(4.6)

In the trial, the third order expansion was found to give acceptable predictions within the range of fitted data ($1 < \lambda < 2$). The extrapolation ability beyond that range was found not to be so good as to the expectation. The large strain-hardening feature of rubbers cannot therefore be modeled. This prompted other contemporary researchers to include different forms of $I_2$. In this course, Rivlin and Saunders (1951) found that $\frac{\partial W}{\partial I_1}$ is substantially constant and that $\frac{\partial W}{\partial I_2}$ is independent of $I_1$, but varies with $I_2$. They performed experiments on two different types of vulcanized rubbers and proposed a strain energy density function, assuming incompressibility in the following form:

$$W(I_1, I_2) = C_{10}(I_1 - 3) + f(I_2 - 3)$$

(4.7)

where, $C_{10}$ is constant and the function $I_2 - 3$ should be left to be determined by specific experiments. They proposed to include an independent function of $I_1$ in the $(W)$ function. Hart-Smith (1966); Hart-Smith and Crisp (1967) followed this path and proposed a modified form of expression based on their test data with $C_1$, $C_2$ and $C_3$ as material parameters:

$$W(I_1, I_2) = C_1 \int \exp\{G(I_1 - 3)^2\} d I_1 + C_2 \ln \frac{I_2}{3}$$

(4.8)

Subsequently, Alexander (1968) further elaborated the idea conceived by Hart-Smith, where a more complicated five-parameter expression was derived (Eqn. 4.9). In deriving the equation, the idea of Rivlin-Saunders (1951) was blended with those of Hart-Smith (1966). Alexander (1968) improved the Hart-Smith function and included a more complicated form of expression of $I_2$ as

$$W(I_1, I_2) = C_1 \int \exp\{C_3(I_1 - 3)^2\} d I_1 + C_2 \ln \left(\frac{I_2 - 3 + \gamma}{\gamma}\right) + C_3(I_2 - 3)$$

(4.9)
where, $C_1$, $C_2$, $C_3$ and $\gamma$ are material parameters. In this course, the idea of Tschogel (1972) was recollected once again by Yeoh (1990, 1993), and suggested an approximate function of $W$ which expresses as a cubic function of $I_1$:

$$W(I_1) = C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3$$

(4.10)

where, $C_{10}$, $C_{20}$ and $C_{30}$ are material parameters.

Apart from all these approaches, Yamashita and Kawabata (1992) considered the strip-biaxial and bi-axial test results and proposed two alternate forms of $W$:

$$W(I_1, I_2) = C_2(I_1 - 3) + C_3(I_2 - 3) + \frac{C_5}{N+1}(I_1 - 3)^N$$

(4.11)

$$W(I_1, I_2) = C_2(I_1 - 3) + C_5(I_2 - 3) + \frac{C_6}{n+1}(I_1 - 3)^m(I_2 - 3)^{n-m}$$

(4.12)

where, $C_2$, $C_3$, $C_5$, $N$, $n$ and $m$ are the material parameters with $N=n-m$. In the parameter identification procedure, based on Rivlin and Saunders (1951); Kawabata et al (1977, 1981) and also Fukahori and Seki (1992) noted the $W$ to be decomposed into the sum of two independent functions of $I_1$ and $I_2$. This striking idea was further considered in a general form by Lambert-Diani and Rey (1999) to arrive at a general strain energy density building procedure. According to that proposal, one experiment with only one principal stretch ratio greater than one is required to obtain the function of $I_1$. However, to obtain the other part of $W$ through an adequate function of $I_2$, they suggested for another experiment with two principal stretch ratios greater than one. In the present work, this concept was thought to be followed in order to obtain an improved $W$ function for NR and HDR in compression and shear.

4.2.2 Models based on principal stretches

Among the stretch ratio based models, Ogden (1972, 1984, 1986) model is most widely used (Eqn. 4.13). The model requires six parameters to describe the stress-stretch relation and it performs well up to a very large deformation range (above
600% strain).

\[
W(\lambda_1, \lambda_2, \lambda_3) = \sum_{n=1}^{m} \frac{\mu_n}{\alpha_n} (\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3) \tag{4.13}
\]

where, \(\mu_n, \alpha_n\) is the material parameters.

In a parallel work, Peng and Landel (1972) also proposed other forms of strain energy density functions presented in (Eqn. 4.14) with \(c\) as the only material parameter.

\[
W(\lambda_1, \lambda_2, \lambda_3) = \sum_{n=1}^{m} c (\lambda_1 - 1 - \ln \lambda_1 - \frac{1}{6} (\ln \lambda_1)^2 + \frac{1}{18} (\ln \lambda_1)^3 - \frac{1}{216} (\ln \lambda_1)^4) \tag{4.14}
\]

However, because of complete phenomenological nature, these functions need a large number of experiments at different deformation modes for the determination and the calibration of a true set of material parameters (Shariff 2000). The other path was undertaken in the present research to adopt an adequate hyperelasticity function for NR and HDR in compression and shear.

### 4.2.3 Improved strain energy function based on first strain invariant \((I_1)\)

Amin (2001) proposed a strain energy function modifying the Yamashita and Kawabata (1992) model in terms of \(I_1\) for NR and HDR in uniaxial compression. (Eqn. 4.15) presents the proposed strain energy density relation as a function of \(I_1\):

\[
W(I_1) = C_5(I_1 - 3) + \frac{C_2}{N+1}(I_1 - 3)^{N+1} + \frac{C_4}{M+1}(I_1 - 3)^{M+1} \tag{4.15}
\]

where, \(C_5, C_3, C_4, M,\) and \(N\) are material parameters with \(N \geq 1.0\) and \(0.0 \leq M \leq 1.0\).

It should be noted that the first term with coefficient \(C_5\) is a component of original Mooney-Rivlin model (Mooney 1940; Rivlin 1948), while the term with \(C_3\) and \(N\) coefficients was proposed by Yamashita and Kawabata to include the hardening feature observed at higher strain levels. In order to incorporate the initial stiffness part, Amin et al (2002) proposed the incorporation of the third term associated with coefficient \(C_4\) and \(M\).
4.2.4 Improved strain energy function based on first and second strain invariant \((I_1 \text{ and } I_2)\)

After considering the shear and compression regime experiments, Wiraguna et al. (2002) proposed an additional term by modifying the energy function given in (Eq. 4.15)

\[
W(I_1, I_2) = C_1(I_1 - 3) + C_2(I_2 - 3) + \frac{C_3}{N+1}(I_1 - 3)^{N+1} + \frac{C_4}{M+1}(I_1 - 3)^{M+1}
\]  

(4.16)

Amin et al. (2002) reported that the additional term increases accuracy of model result at low stretch level. Lambert –Diani and Rey (1999) suggested the strain energy function with \(I_1\) was enough for uniaxial test. However, the second strain invariant \(I_2\) has greater importance in multi-axial deformation than in uniaxial deformation. In this present work a general finite element code has been developed using this two (Eqns. 4.15 and 4.16) strain energy functions.

4.3 IDENTIFICATION OF ELASTICITY PARAMETERS

To find out the stress-strain relationship in equilibrium and instantaneous states, identification of material parameters in the hyperelasticity model is important. The method of Gauss least squares was used by Wiraguna (2003) to find these material parameters on the basis shear and compression experiments. For simplicity, only positive loading path of simple shear tests was considered by Wiraguna (2003) to determine material parameters. Figures 4.2 and 4.3 show the positive loading paths of the cyclic tests with different strain rates for HDR and NR (NR-II). The equilibrium locus from cyclic relaxation test was also plotted in this figure to illustrate the viscous domain. From Figures 4.2 and 4.3, it is seen that the stresses increase with increasing strain rate due to viscosity effect. At higher strain rates, a diminishing trend in the increase of stress response was observed. This indicates the approach of the instantaneous state. In HDR, this phenomenon is more prominent than that in NR. From this figure, the stress response in HDR at and over 0.25/s can be
considered as neighborhood of the instantaneous state, on the other hand, in NR, the stress response at and over 0.05/s corresponds to it.

Figures 4.4 and 4.5 show the equilibrium locus and the monotonic loading paths from compression tests with different strain rates for HDR and NR (Amin et al. 2002). From Figures 4.4 and 4.5, the stress response at 0.88/s can be considered as the neighborhood of instantaneous state for HDR and the stress response at 0.65/s for NR respectively.

Based on experimental data for shear and compression, identification of parameters for the proposed hyperelasticity model was carried out by (Wiraguna et al. 2003) using the least square method. These parameters except for M and N are determined for the equilibrium and instantaneous states independently. Hence, two ways of the parameter identification schemes are considered. One way is that parameters M and N are identified from experimental data at the equilibrium state, and the other from those at the instantaneous state.

The values of the material parameters used in computer coding of the hyperelasticity models were listed in Table 4.1 to Table 4.4.
Figure 4.2. Monotonic path of simple shear at different strain rates

Figure 4.3. Monotonic path of simple shear at different strain rates
Chapter 4: Constitutive Modeling for Nonlinear Elastic Response

Figure 4.4. Monotonic path of Compression at different strain rates

Figure 4.5. Monotonic path of compression at different strain rates
Table 4.1. Elastic Material Parameters for HDR (M,N from Equilibrium state)

<table>
<thead>
<tr>
<th>Responses</th>
<th>C2 (MPa)</th>
<th>C3 (MPa)</th>
<th>C4 (MPa)</th>
<th>C5 (MPa)</th>
<th>M</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilibrium</td>
<td>0.139</td>
<td>0.120</td>
<td>-0.906</td>
<td>0.940</td>
<td>0.16</td>
<td>0.82</td>
</tr>
<tr>
<td>Instantaneous</td>
<td>0.124</td>
<td>0.236</td>
<td>-2.024</td>
<td>2.347</td>
<td>0.16</td>
<td>0.82</td>
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Table 4.2. Elastic Material Parameters for NR (M,N from Equilibrium state)

<table>
<thead>
<tr>
<th>Responses</th>
<th>C2 (MPa)</th>
<th>C3 (MPa)</th>
<th>C4 (MPa)</th>
<th>C5 (MPa)</th>
<th>M</th>
<th>N</th>
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<tbody>
<tr>
<td>Equilibrium</td>
<td>0.096</td>
<td>0.056</td>
<td>-0.346</td>
<td>0.548</td>
<td>0.32</td>
<td>1.01</td>
</tr>
<tr>
<td>Instantaneous</td>
<td>0.177</td>
<td>0.115</td>
<td>-0.599</td>
<td>0.720</td>
<td>0.32</td>
<td>1.01</td>
</tr>
</tbody>
</table>

Table 4.3. Elastic Material Parameters for NR (M,N from Instantaneous state)

<table>
<thead>
<tr>
<th>Responses</th>
<th>C2 (MPa)</th>
<th>C3 (MPa)</th>
<th>C4 (MPa)</th>
<th>C5 (MPa)</th>
<th>M</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilibrium</td>
<td>0.095</td>
<td>0.019</td>
<td>-0.515</td>
<td>0.754</td>
<td>0.15</td>
<td>1.29</td>
</tr>
<tr>
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<td>0.176</td>
<td>0.043</td>
<td>-0.861</td>
<td>1.056</td>
<td>0.15</td>
<td>1.29</td>
</tr>
</tbody>
</table>

Table 4.4. Elastic Material Parameters for HDR (M,N from Instantaneous state)

<table>
<thead>
<tr>
<th>Responses</th>
<th>C2 (MPa)</th>
<th>C3 (MPa)</th>
<th>C4 (MPa)</th>
<th>C5 (MPa)</th>
<th>M</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilibrium</td>
<td>0.145</td>
<td>1.182</td>
<td>-5.297</td>
<td>4.262</td>
<td>0.06</td>
<td>0.27</td>
</tr>
<tr>
<td>Instantaneous</td>
<td>0.166</td>
<td>2.477</td>
<td>-11.689</td>
<td>9.707</td>
<td>0.06</td>
<td>0.27</td>
</tr>
</tbody>
</table>
Chapter 5
Finite Element Formulation

5.1 GENERAL

The key part in finite element formulation of nonlinear elastic materials is to derive the stress and the material constitutive law from a material model. As long as the material models are given, the stress, the constitutive law and other parameters can easily be derived. In Chapter 3, it has been discussed that strain energy function of hyperelastic materials can be expressed in terms of strain invariants and stretches. In Chapter 4, the different strain energy functions developed until now have been briefly discussed and an improved strain energy function proposed by Amin et al (2002) has also been cited. This chapter is devoted to discuss how the stress components can be derived from strain energy functions. In the subsection, the improved strain energy function (Amin et al 2002, Wiraguna et al 2003) will be utilized as strain energy function for formulation purpose.

5.2 DERIVATION OF CAUCHY STRESS TENSOR

The large strain elastic response of rubber-like solids is described either by using an approach based on statistical thermodynamics or by adopting a phenomenological approach considering the material as a continuum. However, in both approaches the stress-strain relationship is derived from a strain energy density function \( W \) that depends on the final state of strain and is independent of loading history. The construction of \( W \) either by statistical mechanics or continuum mechanics forms the basis of the theory of hyperelasticity.

For a complete specification of elastic properties of the material, it is sufficient to know the form of the strain energy function, \( W \). For isotropic elastic materials, the strain energy function \( W \) can be expressed as a function of invariants of a deformation tensor \( I_i \) \((i=1,3)\):

\[
W = W(I_1, I_2, I_3).
\]  
(5.1)
When the material is incompressible, the third invariant $I_3 = 1$, and $W$ is represented as a function of $I_1$ and $I_2$ only:

$$W = W(I_1, I_2).$$

(5.2)

The deformation invariants can be written in terms of the left Cauchy Green deformation tensor $B$:

$$I_1 = \text{tr} B$$
$$I_2 = \frac{1}{2} \{(\text{tr}B)^2 - \text{tr}(BB)\}$$
$$I_3 = \det B$$

(5.3)

Furthermore, these strain invariants are expressed in terms of the principal stretches $\lambda$ (for uniaxial compression) as:

![Figure 5.1 Definition of stretch, $\lambda = \frac{l}{l_0}$](image)

The stretch $\lambda$ is defined as the ratio of current length (deformed length, $l$) to original length (undeformed length, $l_0$).

$$I_1 = \text{tr} B = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$
$$I_2 = \frac{1}{2} \{(\text{tr} B)^2 - \text{tr}(BB)\} = (\lambda_1 \lambda_2)^2 + (\lambda_2 \lambda_3)^2 + (\lambda_3 \lambda_1)^2,$$
$$I_3 = \det B = (\lambda_1 \lambda_2 \lambda_3)^2,$$

(5.4)
From Truesdell and Noll (1992), it follows that the Cauchy stress $\sigma$ is decomposed into volumetric (pressure) part $(-pI)$ and extra part $(\sigma_E)$ as:

$$\sigma = -pI + \sigma_E,$$  \hspace{1cm} (5.5)

$$\sigma_E = 2\frac{\partial W}{\partial I}B - 2\frac{\partial W}{\partial l_2}B^{-1}$$  \hspace{1cm} (5.6)

where $I$ is the identity tensor, $p$ the hydrostatic pressure, a constitutive indeterminate element and it has to be solved by underlying equilibrium and boundary condition of the particular problem; the subscript ‘$E$’ denotes the deviatoric part.

**For uniaxial compression case**, the cauchy stress tensor $\sigma$ can be expressed as:

$$\sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (5.7)

The deformation gradient tensor $F$ can be written as:

$$F = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$  \hspace{1cm} (5.8)

Due to material isotropy, symmetry of applied deformation and incompressibility condition, we get:

$$\lambda_2^2 = \lambda_3^2 = \lambda_1^{-1}$$  \hspace{1cm} (5.9)

Again, the left Cauchy-Green tensor, $B=FF^T$. Using the equation (5.8), the Cauchy Green tensor can be obtained as:
Chapter 5: Finite Element Formulation

\[
B = \begin{pmatrix}
\lambda_1^2 & 0 & 0 \\
0 & \frac{1}{\lambda_1} & 0 \\
0 & 0 & \frac{1}{\lambda_1}
\end{pmatrix}
\]  
(5.10)

and the inverse of \( B \) matrix

\[
B^{-1} = \begin{pmatrix}
\frac{1}{\lambda_1^2} & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \lambda_1
\end{pmatrix}
\]  
(5.11)

In uniaxial case, for determining \( \sigma_{11} \); we find that \( \sigma_{22} = \sigma_{33} = 0 \); Hence:

\[
\sigma_{22} = 0 = -p + \sigma_{22E}
\]  
(5.12)

Now the volumetric part is obtained as:

\[
p = \sigma_{22E} = \frac{2}{\lambda_1} \frac{\partial W}{\partial l_1} - 2\lambda_1 \frac{\partial W}{\partial l_2}
\]  
(5.13)

and the extra part:

\[
\sigma_{11E} = 2\lambda_1^2 \frac{\partial W}{\partial l_1} - \frac{2}{\lambda_1^2} \frac{\partial W}{\partial l_2}
\]  
(5.14)

Therefore, using Equations (5.3), (5.11) and (5.12) the Cauchy Stress can be obtained

\[
\sigma_{11} = -p + \sigma_{11E}
\]  
(5.15)

\[
\sigma_{11} = -\left(\frac{2 \frac{\partial W}{\partial l_1}}{\lambda_1} - 2\lambda_1 \frac{\partial W}{\partial l_2}\right) + \left(2\lambda_1^2 \frac{\partial W}{\partial l_1} - \frac{2}{\lambda_1^2} \frac{\partial W}{\partial l_2}\right)
\]  
(5.16)
Chapter 5: Finite Element Formulation

For simple Shear:

For simple shear, the shear stress tensor $\tau$ can be expressed as:

\[ \tau = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \]  \hspace{1cm} (5.18)

The deformation gradient tensor $F$ can be written as:

\[ F = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]  \hspace{1cm} (5.19)

Again, the left Cauchy-Green tensor, $B = FF^T$. Using the equation (5.19), the Cauchy Green tensor can be obtained as:
\[ B = \begin{pmatrix} 1 + \gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \] (5.20)

and the inverse of \( B \) matrix

\[ B^{-1} = \begin{pmatrix} 1 & \gamma & 0 \\ -\gamma & 1 + \gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \] (5.21)

For simple shear deformation, these strain invariants are expressed as:

\[ I_1 = \text{tr}B = 3 + \gamma^2 \]

\[ I_2 = \frac{1}{2} (\text{tr}B)^2 - \text{tr}(BB) = 3 + \gamma^2 \] (5.22)

\[ I_3 = \text{det}B = 1 \]

where, \( \gamma \) is shear strain.

Now following similar procedures as applied in the case uniaxial compression the shear stress is obtained as:

\[ \sigma_{12} = 2 \gamma \left( \frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} \right) \] (5.23)

5.2.1 Derivation of Cauchy stress tensor using strain energy density based on first invariant \( (I_1) \)

Amin et al (2002) proposed an improved strain energy function to modify the Yamashita and Kawabata (1992) model in terms of first invariant for NR and HDR in uniaxial compression. The improved strain energy function \( W(I_1) \) is given as stated below:

For uniaxial compression:

\[ W(I_1) = C_5(I_1 - 3) + \frac{C_3}{N+1}(I_1 - 3)^{N+1} + \frac{C_4}{M+1}(I_1 - 3)^{M+1} \] (5.24)

where, \( C_3, C_4, C_5, M, N \) are the material parameters as determined by Amin et al.
(2002) by using least square method (unrestrained fitting) for uniaxial compression data. The first strain invariant can be expressed (Bonet and Wood 1997) as:

\[ I_1 = I_B = \text{tr} \, B = I_C = \text{tr} \, C \]  

(5.25)

The strain energy density function (Amin et al. 2002) can be expressed as:

\[ W(C) = C_3 \left( \text{tr} \, C - 3 \right) + \frac{C_4}{N+1} \left( \text{tr} \, C - 3 \right)^{N+1} + \frac{C_5}{M+1} \left( \text{tr} \, C - 3 \right)^{M+1} \]  

(5.26)

Now the second Piola-Kirchhoff Stress \( S \) can be obtained (Bonet and Wood, 1997)

\[ S = 2 \left( \frac{\partial W(C)}{\partial C} + \text{pressure} \right) \]

\[ = 2 \left[ C_3 \left( \text{tr} \, C - 3 \right) + \frac{C_4}{N+1} \left( \text{tr} \, C - 3 \right)^{N+1} + \frac{C_5}{M+1} \left( \text{tr} \, C - 3 \right)^{M+1} \right] + \text{pressure} \]

(5.27)

\[ S = \left[ S_1 + S_2 + S_3 \right] + \text{pressure} \]

(5.28)

Now, the first, second and third term namely, \( S_1, S_2, S_3 \) can be derived as:

\[ S_1 = 2 \frac{\partial}{\partial C} \left[ C_3 \left( \text{tr} \, C - 3 \right) \right]; \]

\[ = 2C_3 \frac{\partial}{\partial C} \left[ \text{tr} \, C - 3 \right]; \]

\[ = 2C_3 \frac{\partial}{\partial C} \text{tr} \, C; \]

\[ = 2C_3 \frac{\partial}{\partial C} \left[ I_3^{-1/3} C : I \right]; \]

\[ = 2C_3 I_3^{-1/3} \left( I - \frac{1}{3} I \right) \text{tr} \, C^{-1} \]

(5.29)

\[ \left( I - \frac{1}{3} I \right) \text{tr} \, C^{-1} \]

(5.30)
Similarly,

\[ S_2 = 2C_3 \left( \frac{\partial}{\partial C} \left[ \frac{C_3}{N+1} \left( \text{tr} C - 3 \right)^{N+1} \right] \right)^N \left( \text{tr} C \right)^{-\frac{1}{3}} \]

\[ = 2C_3 \left( \frac{\partial}{\partial C} \left( \text{tr} C \right)^{N} \right)^N \left( \text{tr} C \right)^{-\frac{1}{3}} \]

\[ = 2C_3 \left[ J_3^{-\frac{1}{3}} C : I - 3 \right] M J_3^{-\frac{1}{3}} \left( I - \frac{1}{3} l C^{-1} \right) \]

\[ = 2C_3 \left[ J_3^{-\frac{1}{3}} C : I - 3 \right] M J_3^{-\frac{1}{3}} \left( I - \frac{1}{3} l C^{-1} \right) \]

and

\[ S_3 = 2C_4 \left[ J_3^{-\frac{1}{3}} C : I - 3 \right] M J_3^{-\frac{1}{3}} \left( I - \frac{1}{3} l C^{-1} \right) \]

\[ = 2C_4 \left[ J_3^{-\frac{1}{3}} C : I - 3 \right] M J_3^{-\frac{1}{3}} \left( I - \frac{1}{3} l C^{-1} \right) \]

Now, Cauchy stress, \( \sigma \) can be obtained as stated (Bonet and Wood, 1997) below:

\[ \sigma = J'FSF^T \]  

(5.33)

Now,

\[ \Rightarrow \sigma = J^{-1}F ( S_1 + S_2 + S_3 ) F^T; \]

\[ \Rightarrow \sigma = J^{-1}FS_1 F^T + J^{-1}FS_2 F^T + J^{-1}FS_3 F^T; \]

\[ \Rightarrow \sigma = \sigma_1' + \sigma_2' + \sigma_3' + \text{pressure} \]

(5.34)

Now, the first three terms can be derived as stated below:
\[ \sigma_j^* = J^{-1}F \left[ 2C_s I_3^{-1/3} \left( I - \frac{1}{3} I_c C^{-1} \right) \right] F^T; \]

\[ \Rightarrow \sigma_j^* = 2C_s J^{-1/3} \left( I - \frac{1}{3} I_c C^{-1} \right) F^T; \quad \therefore I_3 = \det C = J^2 \]

\[ \Rightarrow \sigma_j^* = 2C_s J^{-1/3} \left( B - \frac{1}{3} I_B I \right) \] \hspace{1cm} (5.35)

and,

\[ \sigma_2^* = J^{-1}F \left[ 2C_3 \left( I - \frac{1}{3} I_c C^{-1} \right) \right] F^T; \]

\[ \Rightarrow \sigma_2^* = 2C_3 \left( I - \frac{1}{3} I_c C^{-1} \right) F^T; \] \hspace{1cm} (5.36)

\[ \Rightarrow \sigma_2^* = 2C_3 J^{-1/3} \left( B - \frac{1}{3} I_B I \right) \]

Similarly,

\[ \sigma_3^* = J^{-1}F \left[ 2C_4 \left( I - \frac{1}{3} I_c C^{-1} \right) \right] F^T; \]

\[ \Rightarrow \sigma_3^* = 2C_4 \left( I - \frac{1}{3} I_c C^{-1} \right) F^T; \] \hspace{1cm} (5.37)

\[ \Rightarrow \sigma_3^* = 2C_4 J^{-1/3} \left( B - \frac{1}{3} I_B I \right) \]

Now the Cauchy stress tensor \( \sigma \) can be obtained (Eqn. 5.34):

\[ \Rightarrow \sigma = \left( B - \frac{1}{3} I_B I \right) \ast \left( 2 \ast J^{-1/3} \right) \ast \left( C_s + C_3 \ast (I_B - 3)^N + C_4 \ast (I_B - 3)^M \right) \] \hspace{1cm} (5.38)

5.2.2 Derivation of Lagrangian elasticity tensor based on first invariant \( (I_1) \)

The deviatoric part of lagrangian elasticity tensor have been evaluated as stated (Bonet and Wood 1997 and Bathe 1996) below:

\[ \mathbf{C} = 2 \frac{\partial S}{\partial \mathbf{C}} = \hat{\mathbf{C}} + \text{ pressure}; \]

and

\[ \hat{\mathbf{C}} = 2 \frac{\partial S}{\partial \mathbf{C}} = 2 \frac{\partial}{\partial \mathbf{C}} (S_1 + \hat{S}_2 + \hat{S}_3) = \hat{\mathbf{C}}_1 + \hat{\mathbf{C}}_2 + \hat{\mathbf{C}}_3. \] \hspace{1cm} (5.39)

Now the three terms can be derived as:
\[
\hat{C}_1 = 2 \frac{\partial \hat{S}_1}{\partial \hat{C}} = 4 C_3 I_3 \left[ \frac{1}{3} I_c I_3 - \frac{1}{3} I \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right] ; \tag{5.40}
\]
\[
\hat{C}_2 = 2 \frac{\partial \hat{S}_2}{\partial \hat{C}} = 4 C_3 (I_B - 3) I_3 \left[ \frac{1}{3} I_c I_3 - \frac{1}{3} I \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right] ; \tag{5.41}
\]
\[
\hat{C}_3 = 2 \frac{\partial \hat{S}_3}{\partial \hat{C}} = 4 C_3 (I_B - 3) M_3 \left[ \frac{1}{3} I_c I_3 - \frac{1}{3} I \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right] ; \tag{5.42}
\]

Now, the Lagrangian elasticity tensor is
\[
\hat{C} = 2 \frac{\partial \hat{S}}{\partial \hat{C}} = 4 J^{-\frac{1}{2}} \left[ \frac{1}{3} I_c I_3 - \frac{1}{3} I \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right] x \left[ C_5 + C_3 (I_B - 3)^N + C_4 (I_B - 3)^M \right] \tag{5.43}
\]

### 5.2.3 Derivation of Cauchy stress tensor using strain energy function based on first and second strain invariants \((I_1, I_2)\)

Yamaashita and Kawabata (1992) considered the strip-biaxial and bi-axial test results and proposed two alternative forms of W as stated below:

\[
W(I_1 I_2) = C_5 (I_1 - 3) + C_2 (I_2 - 3) + \frac{C_3}{N+1} (I_2 - 3)^{N+1} \tag{5.44}
\]
\[
W(I_1, I_2) = C_5 (I_1 - 3) + C_2 (I_2 - 3) + \frac{C_3}{N+1} (I_1 - 3)^{N+1} (I_2 - 3)^{N+1-m} \tag{5.45}
\]

where, \(C_5, C_2, C_3, N, n\) and \(m\) are the material parameters with \(N = n-m\).

Wiraguna et al (2003) proposed the improved strain density energy with two invariants as:

\[
W(I_1, I_2) = C_5 (I_B - 3) + \frac{C_3}{N+1} (I_B - 3)^{N+1} + \frac{C_4}{M+1} (I_B - 3)^{M+1} + C_2 (I_2 - 3) \tag{5.46}
\]

where, \(C_5, C_2, C_3, C_4, N\) and \(M\) are the material parameters to be determined from experimental results by Wiraguna (2003) using least square method. The strain energy density function \(W(I_1, I_2)\) can be represented in terms of \(\text{tr}\hat{C}\) as:
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The second Piola-Kirchhoff stress can be evaluated as stated:

\[ w(C) = C_3 (\text{tr} \hat{C} - 3)^{N+1} + \frac{C_4}{M+1} (\text{tr} \hat{C} - 3)^{M+1} + C_2 \left( \frac{1}{2} \right) (\text{tr} C)^2 - \text{tr} CC \]  

(5.47)

where,

\[ S_1 = 2 \frac{\partial w(C)}{\partial C} \left[ C_3 (\text{tr} \hat{C} - 3) \right] \]

\[ = 2 C_3 \frac{\partial}{\partial C} \left[ (\text{tr} \hat{C} - 3) \right] \]

\[ = 2 C_3 \frac{\partial}{\partial C} \left[ \text{tr} \hat{C} \right] \]

\[ = 2 C_3 \frac{\partial}{\partial C} \left[ J_3^{-1/3} C : I \right] \]

\[ = 2 C_3 J_3^{-1/3} \left( I - \frac{1}{3} I_c C^{-1} \right) \]  

(5.49)

\[ S_2 = 2 \frac{\partial}{\partial C} \left[ \frac{C_3}{N+1} (\text{tr} \hat{C} - 3)^{N+1} \right] \]

\[ = 2 C_3 (\text{tr} \hat{C} - 3)^{N} \frac{\partial}{\partial C} \left( \text{tr} \hat{C} \right) \]

\[ = 2 C_3 \left[ J_3^{-1/3} C : I - 3 \right]^{N} J_3^{-1/3} \left( I - \frac{1}{3} I_c C^{-1} \right) \]

\[ = 2 C_3 \left[ J_3^{-1/3} C : I - 3 \right]^{N} J_3^{-1/3} \left( I - \frac{1}{3} I_c C^{-1} \right) \]  

(5.50)

Similarly,

\[ S_3 = 2 C_4 \left[ J_3^{-1/3} C : I - 3 \right]^{M} J_3^{-1/3} \left( I - \frac{1}{3} I_c C^{-1} \right) \]

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\[ \sigma = 2 C_4 \left[ I_B - 3 \right] J_3 \left( I - \frac{1}{3} I c C^{-1} \right) \] \hspace{1cm} (5.51)

and

\[ s_4 = 2 \frac{\partial}{\partial C} \left[ C_2 \left( I_2 - 3 \right) \right] \]

\[ = 2 C_2 \frac{\partial}{\partial C} \left( \frac{1}{2} \right) \left( \text{tr} \ C \right)^2 \cdot \text{tr} \left( C C \right) - 3 \]

\[ = 2 C_2 \text{tr} \ C \frac{\partial}{\partial C} \left( \text{tr} \ C \right) - 0 \]

\[ = 2 C_2 \text{tr} \ C \frac{\partial}{\partial C} \left[ I_j^{-1/3} C : I \right] \]

\[ = 2 C_2 I_j^{-1/3} \left( I - \frac{1}{3} I c C^{-1} \right) \text{tr} \ C \] \hspace{1cm} (5.52)

Now following the similar steps as made in the case of the strain energy function based on \( I_i \), the Cauchy stress can be evaluated as stated below. So, the Cauchy stress becomes:

\[ \Rightarrow \sigma = 2 C_5 J^{-5/3} \left( B - \frac{1}{3} I_B I \right) + 2 C_3 J^{-5/3} \left( I_B - 3 \right) N \left( B - \frac{1}{3} I_B I \right) + 2 C_4 J^{-5/3} \]

\[ \left( I_B - 3 \right)^M \left( B - \frac{1}{3} I_B I \right) + 2 C_2 J^{-5/3} \left( B * I_B - \frac{1}{3} * I_B * I_B * I \right) \] \hspace{1cm} (5.53)

\[ \Rightarrow \sigma = \left( B - \frac{1}{3} I_B I \right) \left( 2 * J^{-5/3} \right) \left( C_5 + C_3 * (I_B - 3) N + C_4 * (I_B - 3) M \right) + \left( B * I_B - \frac{1}{3} * I_B * I_B * I \right) \] \hspace{1cm} (5.54)

where, \( C_2, C_3, C_4, C_5, N \) and \( M \) are the material parameters to be determined from experimental results by Wiraguna (2003) using least square method.

5.2.4 Derivation of Lagrangian elasticity tensor based on first and second invariants \( (I_1 \) and \( I_2 \))

The deviatoric part of Lagrangian elasticity tensor has been evaluated as stated below
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(Bonet and Wood, 1997 and Bathe 1996). The first three terms of Lagrangian elasticity tensor can be evaluated following the similar steps as followed in (5.39 to 5.42) but the last term can be derived as follows:

\[ \mathbf{\mathcal{C}}_4 = \frac{\partial S_A}{\partial \mathbf{C}} \]

[Putting \( \text{tr} \mathbf{C} = I_5^{-13} \mathbf{C} \)]

\[
\Rightarrow \frac{\partial}{\partial \mathbf{C}} \left[ I_5^{-13} \mathbf{I} - \frac{1}{3} I_5^{-49} I_5 \mathbf{C}^{-1} \left( \mathbf{C} : \mathbf{I} \right) \frac{2 \mathbf{C}_2}{(2 \mathbf{C}_2)} \text{tr} \mathbf{C} \right] \quad (5.55)
\]

Now, following the simple rule of successive differentiation, we have the last term of the Lagrangian elasticity tensor as stated below:

\[ \mathbf{\mathcal{C}}_4 = 2 \frac{\partial S_A}{\partial \mathbf{C}} = 4 \mathbf{C}_2 (\text{tr} \mathbf{C}) \frac{1}{3} \left[ \frac{1}{3} I_5 I_5 - \frac{1}{3} \mathbf{I} \otimes \mathbf{C}^{-1} - \frac{1}{3} \mathbf{C}^{-1} \otimes \mathbf{I} + \frac{1}{9} I_5 \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} \right] \quad (5.56) \]

Now, the final form of Lagrangian elasticity tensor can be obtained as follows (Bonet and Wood 1997):

\[
\mathbf{\mathcal{C}} = 4 J^{1/3} \mathbf{C}_2 \left[ \frac{1}{3} I_5 I_5 - \frac{1}{3} \mathbf{I} \otimes \mathbf{C}^{-1} - \frac{1}{3} \mathbf{C}^{-1} \otimes \mathbf{I} + \frac{1}{9} I_5 \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} \right] \left[ \mathbf{C}_3 + \mathbf{C}_4 (\mathbf{I}_b - 3)^N + \mathbf{C}_5 (\mathbf{I}_b - 3)^M \right] + \\
4 J^{1/3} \mathbf{C}_2 \left[ \frac{1}{3} I_5 I_5 - \frac{1}{3} \mathbf{I} \otimes \mathbf{C}^{-1} - \frac{1}{3} \mathbf{C}^{-1} \otimes \mathbf{I} + \frac{1}{9} I_5 \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} \right] \left[ \text{tr} \mathbf{C} \right] \quad (5.57)
\]

where, \( \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4, \mathbf{C}_5, N \) and \( M \) are the material parameters to be determined from experimental results by (Wiraguna 2003) using a least square method. The formulations for Cauchy stress tensor and Lagrangian elasticity tensor stated in Eqns. (5.24) to (5.57) give fundamental basis for a general FEM coding.
6.1 GENERAL

The experimental findings and the results obtained from the constitutive relationship are presented in Chapter 2, 3, and 4. Several derivations of Cauchy stresses and Lagrangian elasticity tensors have been introduced in Chapter 5. Furthermore, an explicit scheme was performed by Wiraguna (2003) to identify the elastic parameters of the materials in a physically meaningful way. To this end, the identified material parameters were employed in the constitutive model to carry out the numerical simulation of monotonic compression and shear tests. This chapter is devoted to compare the experimental results with finite element simulation results and thereby to illustrate the capability of the developed FEM code. For the purpose of FEM coding, a general-purpose finite element program, FEAP has been used in this work. FEAP has been developed by Taylor (2000) at University of California at Berkeley and partially documented in Zeinkiewicz and Taylor (1996). Two subroutine subprograms have been developed to this end for use in FEAP.

6.2 GEOMETRY OF RUBBER MODEL

Different geometrical configurations are used for the purpose of finite element simulation. Figures 6.1 to 6.6 represent the different simulation models (2D and 3D) used in FEAP program to simulate for compression, shear and combined action of shear and compression. The actual geometries of the test specimens (Chapter 2) were used to outline these models. This eliminates the possibility of the interference in simulation due to the difference between shape (shape factor) of the FE models and the actual test specimens. Figure 6.1 present 2D models using 36 and 625 4-noded quadrilateral elements for compression test simulation. Figure 6.2 presents 3D model using 216 8-noded brick elements for compression test simulation. Figure 6.3 present 2D models using 24 and 216 4-noded quadrilateral elements for shear test simulation. Figure 6.4 presents 3D model using 100 8-noded brick elements for shear test simulation.
Mesh 1

Boundary conditions:
Node A at bottom edge is restrained in 1-direction and 2-direction
All other nodes at bottom edge are free in 1-direction and restrained in 2-direction
Node B at top edge is restrained in 1-direction
All other nodes at top edge are free

Figure 6.1 2D Plane strain simulation models using quadrilateral element for compression test (a) 36 elements (b) 625 elements
Mesh 3

Figure 6.2 3D Simulation model using 216 brick elements for compression test

Mesh 4

Figure 6.3 (a) 2D Plane strain simulation model using 24 quadrilateral elements for shear test
Mesh 5

Figure 6.3 (b) 2D Plane strain simulation model using 216 quadrilateral elements for shear test

Mesh 6

Figure 6.4 3D Simulation model using 100 brick elements for shear test
Mesh 7

Boundary conditions:
- All nodes at the bottom edge are restrained in both 1-direction and 2-direction.
- All nodes at the top edge are restrained in 2-direction.

10 mm

(a)

Mesh 8

Boundary conditions:
- All nodes at the bottom edge are restrained in both 1-direction and 2-direction.
- All nodes at the top edge are restrained in 2-direction.

10 mm

(b)

Figure 6.5 2D Plane strain simulation models using quadrilateral element for combined action of compression and shear (a) 24 elements  (b) 216 elements
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Mesh 9

![3D Simulation model using 100 brick element for combined action of compression and shear](image)

Figure 6.6 3D Simulation model using 100 brick element for combined action of compression and shear

Figure 6.5 presents 2D models using 24 and 216 quadrilateral elements to be used for simulating combined action of compression and shear. Figure 6.6 presents 3D model using 100 brick elements for simulating responses obtained from combined action of compression and shear.

6.3 NUMERICAL SIMULATION USING FIRST STRAIN IN Variant ($I_1$)

The general expression of Cauchy stress tensor (Truesdell and Noll 1992) was developed in the Chapter 5. The short version of Cauchy stress tensor can be be written:

$$\sigma = -pI + \beta_1 B + \beta_2 B^{-1}$$

(6.1)

Where,

$$\beta_1 = 2 \frac{\partial W}{\partial I_1}$$

$$\beta_2 = -2 \frac{\partial W}{\partial I_2}$$

From that explicitly derived relation, the Cauchy stress tensor can be derived using Eqn.5.26:

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\[ \sigma = -p I + 2\left[C_5 + C_3 (I_B - 3)^\nu + C_4 (I_B - 3)^\mu \right] B \]  \hspace{1cm} (6.2)

In this current work, a subroutine has been written following the strain energy function stated in Eqn (5.26) with a view to simulating the strain energy function with experimental and the constitutive results in FE program. In Figure 6.7, the simulation results along with the other results have been shown. Different 2D and 3D simulation models (as shown in Figure 6.1 and 6.2) have been made for numerical simulation. From Figures 6.7 and 6.8, it has been found that 3D model gives a good agreement with those obtained from the experimental and constitutive relation. The 2D model does not, however, represent good agreement with the experimental observations due to plane strain idealization. But in case three-dimensional analysis, no geometric idealization has been made. A good agreement has been obtained there.

6.4 NUMERICAL SIMULATION USING FIRST AND SECOND INVARIANT (\(I_1\) and \(I_2\))

Here, the same expression for Cauchy stress tensor as stated in Eqn. (6.1) has been used for calculating the Cauchy stress tensor. The strain energy density function given in Eqn. (5.26) is not sufficient for representing the bi-axial test results. Apart from all these approaches, Yamashita and Kawabata (1992) considered the strip-biaxial and bi-axial test results and proposed two alternate forms of W (Eqn. 5.44 and 5.45).

From that explicitly derived relation, the Cauchy stress tensor can be derived using Eqn.5.46 and 6.1:

\[ \sigma = -p I + 2\left[C_5 + C_3 (I_1 - 3)^\nu + C_4 (I_1 - 3)^\mu \right] B - 2C_2 B^{-1} \]  \hspace{1cm} (6.3)

In this current work, a subroutine has also been written following the strain energy function stated in Eqn (5.46) with a view to simulating the strain energy function with experimental and the constitutive results in FE program. The simulation results obtained by using the strain energy density function Eqn. (5.46) for compressive strain (50% strain) are given in Figure 6.9 and 6.10. For this purpose the different 2D and 3D models have been used as shown in Figures 6.1 and 6.2. From Figures 6.9 and 6.10, it has been seen that the 3D geometric model shows a better agreement than 2D model.
in comparison with those obtained from the experimental and constitutive relation. Here, from the simulation results, it can be noted that the fineness of mesh does not effect on the accuracy of functional results. From Figure 6.9 it has been seen that the HDR at the instantaneous state shows better simulation results than those of HDR at equilibrium state. It may occur due to the strain rate effect. The simulation results obtained by using the strain energy density function as stated in Eqn. (5.46) for shear at 250% strain are given in Figures 6.11 and 6.12. For this purpose, the different 2D and 3D models have been used as shown in Figures 6.3 and 6.4. From Figures 6.11 and 6.12, it has been seen that the simulation results do not vary with the model types. So, it can be said that the either model, i.e. 2D or 3D model shows a good agreement with those obtained from the experimental and constitutive relation. From the simulation results, it can also be noted that the fineness of mesh does not effect on the accuracy of functional result.

6.5 NUMERICAL EXPERIMENTS UNDER SIMULTANEOUS COMPRESSION AND SHEAR

In the last figures, the simulation results for either compression or shear is represented. But in real situations, the structural element is sometimes subjected to the combined actions of compression and shear due to lateral forces. No experimental work for such case has been performed till now. The improved strain energy density function (Amin et al. 2002) of hyperelastic material has been used for the purpose of finite element simulation of such probable experiments. Figures 6.13 to 6.16 illustrate the numerical results for simultaneous action of compression and shear for different shape factors. Shape factor (S.F.) is dimensionless parameter that depends upon the geometry. This parameter shall be discussed in details in the next chapter. In general, higher shape factor represents thinner rubber layers. For the purpose of verification, the numerical simulations have been made for two shape factors (S.F.) such as S.F=1.25 and S.F=24.

Figures 6.13 and 6.14 present the behavior of Cauchy compressive stress \( (\sigma_{22}) \) -strain \( (\lambda_2) \) response and Figures 6.15 and 6.16 present the behavior of shear stress \( (\sigma_{12}) \) - strain \( (\gamma_{12}) \) response for HDR and NR (NR-II) under the combined action of

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compression and shear. From the simulation results (Figure 6.13 and 6.14), it can be said that the hyperelastic behavior for compression has been better represented in HDR than NR for all shape factors. Although, the differences of simulation results due to different shape factors is not significant, but the increasing trends of vertical stiffness continues as the shape factor increases. From the simulation results (Figure 6.15 and 6.16), it can also be said that the hyperelastic behavior for shear has been well represented in HDR and NR for all shape factors. The difference of simulated results due to different shape factors is also insignificant. So, the shape factor does not affect significantly the shear stress of hyperelastic material under the combined action of compression and shear.
Figure 6.7 Cauchy stress -stretch relations and simulation results using \( l_1 \) only (a) HDR (Instantaneous response) (b) HDR (Equilibrium response)
Figure 6.8 Cauchy stress - stretch relations and simulation results using \( I_1 \) only (a) NR (Instantaneous response) (b) NR (Equilibrium response)
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Figure 6.9 Cauchy stress-stretch relations and simulation results using $f_1$ and $f_2$: (a) HDR (Instantaneous response) (b) HDR (Equilibrium response)
Figure 6.10 Cauchy stress-stretch relations and simulation results using $I_1$ and $I_2$: (a) NR (Instantaneous response) (b) NR (Equilibrium response)
Figure 6.11 Shear stress-strain relations and simulation results using $I_1$ and $I_2$ (a) HDR (Instantaneous response) (b) HDR (Equilibrium response)
Figure 6.12 Shear stress-strain relations and simulation results using $i_i$ and $i_j$ (a) NR (Instantaneous response) (b) NR (Equilibrium response)
Figure 6.13 Cauchy stress-stretch relations under simultaneous action of shear and compression using $I_1$ and $I_2$: (a) HDR (Instantaneous response, (compression and 100% shear)) (b) HDR (Equilibrium response, (compression and 80% shear))
Figure 6.14 Cauchy stress-stretch relations under the simultaneous action of shear and compression using $I_1$ and $J_2$ (compression and 80% shear) (a) NR (Instantaneous response) (b) NR (Equilibrium response)
Figure 6.15 Shear stress-strain relations under the simultaneous action of shear and compression using $I_1$ and $I_2$ (a) HDR (instantaneous response, (compression and 100% shear)) (b) HDR (Equilibrium response, (compression and 80% shear))
Figure 6.16 Shear stress-strain relations under the simultaneous action shear and compression (compression and 80% shear) using $l_1$ and $l_2$ (a) NR (Instantaneous response) (b) NR (Equilibrium response)
6.6 STRESS AND DEFORMATION PATTERNS

In Figures 6.7 to 6.12, all the stress–strain responses for uniaxial compression and shear simulation have been presented while the results of FEM simulation for combined action of compression and shear are represented in Figures 6.13 to 6.16. From these simulations, Figures 6.17 to 6.22 present the graphical representation of stress and deformation patterns. From these Figures, the graphical representation of the stress distribution and deformation patterns can be seen in an illustrated way. In Figure 6.17, the stress-22 ($\sigma_{22}$) and deformation-2 ($\lambda_2$) patterns for 3D-compression models have been shown and the uniformity in stress distribution has been attained. In Figure 6.18, the stress-22 ($\sigma_{22}$) and deformation-2 ($\lambda_2$) patterns for 2D-compression models have been shown and the uniformity in stress distribution has also been attained. In Figure 6.19, the stress-12 ($\sigma_{12}$) and deformation-1 ($\lambda_1$) pattern for 3D-shear models have been presented. From this figure, it has been found that the distribution of stress-12 ($\sigma_{12}$) is symmetric about the diagonal line and stress-12 ($\sigma_{12}$) shows the increasing trends along the top-left corner and the right-bottom corner edge of the rubber pad. These facts may be due to the over compression developed due to direct shear at the stated edges. However, the variation of this stress is not significant. Again the homogeneous pattern of deformation-1 ($\lambda_1$) has been attained. In Figure 6.20, the stress-12 ($\sigma_{12}$) and deformation-1 ($\lambda_1$) patterns for 2D-shear models have been well illustrated. From this figure, it has been found that the distribution of stress-12 ($\sigma_{12}$) is symmetric about the diagonal line and almost uniformity in distribution has been attained. However, Figure 6.20 and 6.21 conform the experimental results obtained by Wiraguna (2003). In Figure 6.21, stress-22 ($\sigma_{22}$) and deformation-1 ($\lambda_1$) patterns for combined action of shear and compression have been presented. From this figure, it has been found that the distribution of stress-22 is almost uniform while at some locations the stress concentration is developed. The stress concentration shows the increasing trends along the top-left and bottom-right edges. This fact is happened due to simultaneous action of shear with compression. But the distribution of stress-22 ($\sigma_{22}$) showed the uniform distribution due to homogeneous compression as illustrated in Figure 6.17. In Figure 6.22, stress-12 ($\sigma_{12}$) and deformation-2 ($\lambda_2$) pattern for the combined actions of compression and shear have been presented. From this figure,
it has been found that the distribution of shear stress $\tau_{12}$ is symmetric about the diagonal line with increasing trends at top left and bottom right corners. These facts may be due to the coupling effect and over compression due to direct shear applied at the stated edges. However, Figure 6.20 and 6.21 show the geometric nonlinearity of specimens as the finite deformation model is taking care of.

![DISPLACEMENT 2](image)

![STRESS 2](image)

Figure 6.17 Typical stress and deformation pattern for 3D model (compression) (a) deformation-2 pattern at 0.7 stretch (b) stress -22 pattern at stretch 0.85
Figure 6.18 Stress and deformation pattern for 2D-compression model (a) stress-22 pattern at 0.95 (b) deformation-2 pattern at stretch 0.75
Figure 6.19 Stress and deformation pattern for 3D-shear model (a) deformation-1 at 150% shearing strain (b) stress-12 at 100% strain
Figure 6.20 Stress and deformation pattern for 2D (homogenous shear deformation) model (a) deformation-1 pattern at shearing strain 150% (b) stress-12 pattern at shearing strain 200%
Figure 6.21 Stress and deformation pattern for combined action of shear and compression model (a) deformation-1 pattern at shearing strain 70% and compressive strain 20% (b) Stress-22 pattern at shearing strain 40% and compressive strain 10%
Figure 6.22 Stress and Deformation Pattern for combined action of shear and compression model (a) Stress-12 pattern at shearing strain 40% and compressive strain 10%  (b) Deformation-2 pattern at shearing strain 60% and compressive strain 18%
7.1 GENERAL

The superstructures of bridges and tall buildings undergo motion due to temperature changes, moving loads, earthquakes, concrete shrinkage, creeps and other parameters. Elastomeric bearings are often used to mitigate the effects of these movements. Although, they can be made of plain elastomer, these bearings usually consist of alternative layers of elastomer and steel. However, incorporation of steel provides large vertical stiffness, while the lateral flexibility is provided by the rubber layers (Kelly 1997) (Fig.1.3). Natural rubbers had been used in these devices for a long time and all the theoretical and experimental efforts had been given to analyze only natural rubbers (NR). But another type of rubber like high damping rubber (HDR) shows better performance than NR. In the current work, the simulation for HDR and NR has been made for comparison with experimental results. In the last chapters, the constitutive formulations and subsequent FE implementation along with verification are presented. Thus it was possible to develop an FEM code capable of analyzing rubbers subjected to compression, shear and their combined action.

In the subsequent sections, the behavior of bearings (NR and HDR) has been examined using the FEM code developed in this work. In this process, the available procedures will be introduced and their limitations in modeling these responses will be critically examined.

7.2 SHAPE FACTOR

The vertical frequency of an isolated structure, often an important design parameter, is controlled by the vertical stiffness of the bearing (Kelly 1997). In order to predict this vertical frequency, the designer need only to compute the vertical stiffness of the bearings under the dead load and, for this the linear analysis is traditionally used. But the response of a bearing under the vertical load is very nonlinear and depends on various factors. Hence, any method that can take the feature into account is well needed.
The vertical stiffness of a rubber bearing is given by

\[ K_v = \frac{E_c A}{t_r} \]  \hspace{1cm} (7.1)

where \( A \) is the area of the bearing, \( t \) the total thickness of rubber in the bearing, and \( E_c \) the instantaneous compression modulus of the rubber-steel composite under the specified level of vertical load.

The value of \( E_c \) for a single rubber layer is governed by a shape factor (Kelly 1991). Shape factor \( S \) of a rubber-confined pad is the ratio between the loaded area and the lateral surface free to bulge. Fig. 7.1 shows schematic details of elastomeric pad of bearing, where \( D \) is the diameter of the bearing and \( H \) its total thickness.

![Figure 7.1 Elastomeric bearing](image)

Shape factor is a dimensionless measure of the aspect ratio of the single rubber layer of elastomer. Shape factors of different geometric configuration are stated below.

Shape factor \( S \) for an infinite strip of width \( 2b \) and with a single layer thickness, \( t \), is given by

\[ S = \frac{b}{t_r} \]  \hspace{1cm} (7.2)

For a circular pad of diameter, \( D \), and thickness \( t_r \):

\[ S = \frac{D}{4t_r} \]  \hspace{1cm} (7.3)

and, for a square pad of side dimension, \( a \) and thickness, \( t_r \):

\[ S = \frac{a}{4t_r} \]  \hspace{1cm} (7.4)
7.3 ANALYSIS OF SINGLE RUBBER LAYER USING KELLY’S METHOD

A finite element formulation based on explicit derivation of stress tensor and Lagrangian elasticity tensor developed in the earlier chapter have been used for the purpose of analysis of single rubber layer for both NR and HDR bearings. Results from two and three-dimensional modeling for rubber bearing have been used in this chapter for comparison with analytical results obtained from Kelly (1997). Kelly’s method, the only available method at present for analysis of these bearings under the compression, is an approximate one based on two assumptions stated below:

- points on a vertical line before deformation lie on a parabola after loading
- horizontal planes remain horizontal

The first of the two assumptions is relating to the kinematics of the deformation and the second is related to the stress state. This theory is applicable to bearings with shape factors greater than about five (Kelly 1997).

Here, it is noted that the Kelly’s analysis is based on the linear elastic theory. From the assumptions made by Kelly (1997) mentioned above, the normal stress and shear stress due to pure compression can be evaluated as:

\[
\sigma_{xx,x} = \frac{8Gu_o}{t^2} \quad (7.5)
\]

where, \(G\), \(t\) are Shear modulus and thickness respectively and \(u_o\) can be represented as:

\[
u_o = -\frac{t^2}{8G} \rho_{,x} \quad (7.6)
\]

where, \(\rho_{,x}\) represents the first partial derivation of internal pressure function with respect to the indicated coordinate. For the square shaped bearing, the internal pressure function \(p(x,y)\) can be written as:

\[
p(x,y) = \frac{12G\epsilon_s}{t^2} \left[ \frac{a^2}{2} \left( 1 - \frac{x^2}{a^2} \right) \sum_{m=1}^{\infty} \, \frac{4a^2}{m^2 \Pi^2} \cosh \left( \frac{m\Pi y}{a} \right) \sinh \left( \frac{m\Pi y}{a} \right) \right] \quad (7.7)
\]
From the pressure function mentioned in Eqn. (7.7), the normal stress component \( \sigma_{\text{xx},x} \) can be derived as follows:

\[
\sigma_{\text{xx},x} = -p_x \tag{7.8}
\]

and

\[
\sigma_{11} = \frac{3Ge_c}{t^2} \left[ a^2 \frac{x^2 - y^2}{a^2} \right] \tag{7.9}
\]

Again, the shear stress developed due to the direct compression by the constraint of the rigid layers to which the elastomer material is bonded can be evaluated (Kelly 1997) as:

\[
\tau_{xz,z} = p_x \tag{7.10}
\]

and \( \tau_{xz} \) can also be expressed as:

\[
\tau_{xz} = -8Gu_0 \frac{z}{t^2} \quad \text{and} \quad u_o \text{ can defined as } u_o = -\frac{t^2}{8G} p_x
\]

Now, making a first partial differentiation of the Eqn. (7.7) w.r.t. \( x \), the shear stress is obtained as:

\[
\tau_{12} = \frac{6Ge_c}{t^2} (a - 2x)z \tag{7.11}
\]

The equation of internal pressure distribution with large shape factor can be rewritten (Kelly 1997) as:

\[
p(x, y) = Ke_c \left[ 1 - \frac{\cosh \lambda \left( x - \frac{a}{2} \right)}{\cosh \frac{\lambda a}{2}} - \sum_{m=1}^{\infty} \frac{\lambda^2 a_m}{\alpha^2 m} \cosh \alpha_m y \sin \left( \frac{m \pi x}{a} \right) \right] \tag{7.12}
\]

where, \( K, e_c, \lambda \), represent the modulus of bulk compressibility, compressive strain and second shape factor respectively. Here, it is seen that the contribution of the second part of the Eqn. (7.12) is very insignificant for large shape factor and incompressibility condition. So it has been ignored in the subsequent derivation of this work.

The second shape factor or aspect ratio can be expressed as:
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\[ \lambda = \left( \frac{12G}{Kt^2} \right)^{1/2} \]

where, \( G \) represents the shear modulus of rubber material.

The parameters of Fourier series \( a_m \) and \( \alpha_m \) can be expressed (Kelly 1997) as:

\[ a_m = \frac{4}{m\Pi} \text{ when, } m=1,3,5 \]
\[ = 0 \text{ when, } m=2,4,6 \]

\[ \alpha_m = \left( \lambda^2 + m^2 \frac{\Pi^2}{a^2} \right)^{1/2} \]  

(7.13)  
(7.14)

Now, the shear stress \( \tau_{12} \) and compressive stress \( \sigma_{11} \) distribution function can be evaluated (Kelly 1997) from the Eqns. (7.8), (7.10), (7.12) using the simple differential and integration calculus rules as stated below:

\[ \tau_{12} = -\lambda K\varepsilon_c \frac{\sinh \lambda \left( x - \frac{a}{2} \right)}{\cosh \left( \frac{\lambda a}{2} \right)} \]  

(7.15)

\[ \sigma_{11} = -K\varepsilon_c \frac{\cosh \lambda \left( x - \frac{a}{2} \right)}{\cosh \left( \frac{\lambda a}{2} \right)} \]  

(7.16)

In the following sub-sections the Eqns. (7.15) and (7.16) have been used as the basis for the Kelly’s analytical solution.

7.4 COMPARISON FOR COMPRSSIVE STRESS AND SHEAR STRESS
RESULTS WITH KELLY’S SOLUTION DUE TO PURE COMPRESSION

Rubber is modeled as an incompressible hyperelastic material two and three-dimensional (2D and 3D) finite element models incorporating both material as well as geometric (large deformation and large strains) nonlinearities are utilized to analyze elastomeric bearings. Stresses and strains are examined and different patterns for stresses and deformations have been critically analyzed. The penalty approach with Lagrangian multiplier is utilized to enforce the incompressibility condition (Simo et al 1991). In the two and three-dimensional modeling of elastomeric rubber, the model uses quadrilateral and higher–order brick element. Same specimens, geometry with fixed boundary conditions as shown in Fig. 6.2 and Fig. 6.4 for different shape
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Factors have been used for modeling elastomeric rubber pad due to pure compression. In Figures 7.2 to 7.9, the comparisons between the Kelly's analytical results and FEM simulated results for different shape factors have been made. In this subsection, the shear stress developed at the edge of rubber pad due to pure compression is compared between FEM solution and Kelly's solution. The shear stress produced in this manner bears importance for designing rubber pad for the bonding shear between steel and rubber.

Figures 7.2 to 7.5 present the comparisons of Cauchy stress-22 ($\sigma_{22}$) vs strain ($\lambda_2$) between Kelly's solution and FEM solution. It can be said that as the values of shape factor increase, the differences of functional results between FEM and Kelly's solution get reduced. Figures 7.6 to 7.9 illustrate the comparisons of shear stress-12 ($\sigma_{12}$) vs strain ($\lambda_2$) between Kelly's solution and FEM solution. From Figures 7.2 to 7.9, it has been found that as the shape factor increases, the vertical and the horizontal stiffness also increase. However, it must be noted that in seismic isolation applications, there exists a need to have a device with a high vertical stiffness and lower shear stiffness. From the results, it has been seen that the variation of shear stress is inversely proportional to the S.F. and Kelly's method cannot predict the behavior of rubber pads with low S.F. while FEM can predict it. So, it can be said that the value of shape factors in between 5 and 30 may be suitable for better seismic isolation. Here, it can also be seen that the higher the values of shape factors, the lesser the difference of functional result between FEM and Kelly's solution. This large gap between the FEM and Kelly's solution may be explained in that the Kelly's solution is limited to the elastic analysis, but the FEM is independent of geometric idealization of material. From Figures 7.2 to 7.9, it has been seen that the hyperelasticity property of rubber is well experienced for all shape factors in FEM solution. The Kelly's analytical results approach towards the FEM results with increasing shape factors. For lower value of shape factor, the Kelly's solution gives very high functional values compared with those of FEM solution. In Figure 7.10, the variation of maximum Cauchy stress with those obtained from Kelly's analytical method for various shape factors have been presented. It has also been seen that for higher the values of shape factor, the faster the rate of reduction of the gap between FEM and Kelly's Solution. Even though the Kelly's method can predict the maximum stress level in large shape factor, the stress level at intermediate strain levels cannot be attained by the Kelly's method. Figure 7.10 highlights these
revitalizations. In Figure 7.11, the % difference of stress resultant between Kelly’s solution and FEM solution has been shown. The Eqn. of % difference is as:

\[
\text{% Difference} = \frac{\text{FEM Solution} - \text{Kelly's Solution}}{\text{FEM Solution}} \times 100
\]  

(7.17)

![Graph showing variation of Cauchy stress with shape factor](a) For shape factor 0.1 (b) For shape factor 0.25

Figure 7.2 Variation of Cauchy stress with shape factor (a) For shape factor 0.1 (b) For shape factor 0.25
Figure 7.3 Variation of Cauchy stress with shape factor (a) For shape factor 1.25 (b) For shape factor 3.0
Figure 7.4 Variation of Cauchy stress with shape factor (a) For shape factor 6.0 (b) For shape factor 12.0
Figure 7.5 Variation of Cauchy stress with shape factor (a) For shape factor 24.0 (b) For shape factor 42.5
Figure 7.6 Variation of shear stress with shape factor (a) For shape factor 0.1
(b) For shape factor 0.25
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Figure 7.7 Variation of shear stress with shape factor (a) For shape factor 1.25 (b) For shape factor 3.0
Figure 7.8 Variation of shear stress with shape factor (a) For shape factor 6.0  
(b) For shape factor 12.0
Figure 7.9 Variation of shear stress with shape factor (a) For shape factor 24 (b) For shape factor 42.5
Figure 7.10 (a) Variation of maximum Cauchy stress with shape factors (b) Variation of maximum shear stress with shape factors.

Figure 7.10 (a) Variation of maximum Cauchy stress with shape factors (b) Variation of maximum shear stress with shape factors.
Figure 7.11 (a) % Difference of maximum Cauchy stress between FEM and Kelly's solution (b) % Difference of edge maximum shear stress between FEM and Kelly's solution
Figure 7.12 Distribution of Cauchy stress-22 along the 1 Direction (a) shape factor 1.25 (b) shape factor 24
Figure: 7.13 Deformation pattern (for compressive strain 0.32)  (a) Deformation-1 (b) Deformation-2
Figure 7.12 shows the distribution of Cauchy stress-$22 (\sigma_{22})$ along 1 Direction. From Figure 7.12 it has been seen that although the Kelly's solution shows the parabolic variation of stress-$22 (\sigma_{22})$ for lower value of shape factor, but for higher value of shape factor the Kelly's solution shows the almost uniform variation of stress-$22 (\sigma_{22})$ except at the edge as shown for the case of FEM solution. Figure 7.13 shows the pattern of different deformations. However, the nonlinearity of the rubber specimens has also been clearly examined in the Figure 7.13.

### 7.5 ANALYSIS OF FULL SCALE RUBBER PADS WITH MULTIPLE RUBBER LAYERS

A laminated rubber bearing consists of rubber layers bonded and alternated by rigid steel shims. The device, characterized by large stiffness and large horizontal flexibility, is well suited for seismic isolation applications. The evaluation of the collapse conditions is an essential step in designing the elastomeric bearing. The collapse of the device can occur either for global failure, due to buckling or roll-out of the device (Kelly 1997) or for local failure, due to tensile rupture of the rubber, detachment of the rubber from the steel or due to yielding of steel plates.

Therefore, it is necessary to have an accurate knowledge of the global characteristics of the device and of the stress distributions in the inner rubber layers, at the rubber-steel interfaces and the steel shims. Figure 7.14 shows the FEM models. Figure 7.14 (a) presents the FEM model of rubber pad with two steel shims and Figure 7.14 (b and c) show the FEM model of rubber pad with three steel shims. Figures 7.15 (a) and 7.16 (a) show the distribution of Cauchy stress-$22 (\sigma_{22})$ for compression with t strain ($\lambda_t$) for two and three steel shims respectively. Figures 7.15 (b) and 7.16 (b) show the distribution of Cauchy stress-$22 (\sigma_{22})$ with strain ($\lambda_t$) along the coordinate axes 1 for two and three steel shims respectively. From Figures 7.15 and 7.16, it has been seen that the bearing pad with three steel shims experiences lower shear stress than that with two steel shims. So, it can be said that the vertical stiffness of the rubber pad is directly proportional to the number of steel shims used in the rubber pad. Figure 7.17 presents the distribution Cauchy stress-$22 (\sigma_{22})$ with strains ($\lambda_t$) and along the direction-1 for combined action of compression (40 % strain) and shear (90 % shearing strain) for shape factor 20. However, the stress distributions at different interfaces have been observed for different shape factors (SF=10,20,40) in this work. From the observation, it has been seen that the stress
distribution does not vary for different shape factors. For this reason, the stress
distribution only for shape factor 20 has been shown in this section. Figure 7.18
shows the patterns of Cauchy stress-22 ($\sigma_{22}$) of rubber bearing for uniaxial action of
compression for different shape factors (SF=10, 20 and 40). From these Figures, it
has been found that the Cauchy stress-22 ($\sigma_{22}$) distributes uniformly along the
surface except at the edge. This happens due to the presence of the steel shims.
Figure 7.19 shows the patterns of Cauchy stress-22 ($\sigma_{22}$) of rubber bearing for combined actions of compression (40% strain) and shear (90% strain) for different
shape factors (SF=10, 20 and 40). From these Figures, it has been found that, the
Cauchy stress-22 ($\sigma_{22}$) is also uniformly distributed along the surface except at the
edges. Figure 7.20 shows the patterns of shear stress-12 ($\sigma_{12}$) of rubber bearing for combined actions of compression (40% strain) and shear (90% strain) for different
shape factors (SF=10, 20 and 40). From these Figures, it has been observed that the
shear stress-12 ($\sigma_{12}$) is uniformly distributed along the surface except at some
locations. This fact happens due to the combined action of compression and shear
and also due to the presence of steel shims. Figure 7.21 shows the typical patterns
of deformation-2 ($\lambda_2$) for uniaxial compression (40% strain) for the shape factors 20
and 40 respectively. From these Figures, it has been seen the homogeneous
deformation are produced in rubber specimen. Figure 7.22 shows the typical
patterns of deformation-1 ($\lambda_1$) and deformation-2 ($\lambda_2$) for combined actions of compression (40% strain) and shear (90% strain) for the shape factors 10. From
these figures, it has been seen the homogeneity in deformation-2 ($\lambda_2$) is produced in rubber specimen. But, the non-homogeneity in deformation-1 ($\lambda_1$) is clearly identified in different layers of rubber specimen. This happens due to the combined action of compression and shear and also due to the presence of steel shims. Figure 7.23
shows the typical patterns of deformation-1 ($\lambda_1$) and deformation-2 ($\lambda_2$) for combined actions of compression (40% strain) and shear (90% strain) for the shape factors 40.
From these figures, it has been seen the homogeneity in deformation-2 ($\lambda_2$) is produced in rubber specimen. But, the non-homogeneity in deformation-1 ($\lambda_1$) is clearly identified in different layers of rubber specimen. This happens due to the combined action of compression and shear.
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Mesh 10

Boundary condition:
A at bottom edge is restrained in 1-direction and 2-direction.
All other nodes at bottom edge are free in 1-direction and restrained in 2-direction.
Node B at top edge is restrained in 1-direction and all other nodes are free.

Mesh 11

Boundary condition:
A at bottom edge is restrained in 1-direction and 2-direction.
All other nodes at bottom edge are free in 1-direction and restrained in 2-direction.
Node B at top edge is restrained in 1-direction and all other nodes are free.

Mesh 12

Boundary condition:
A at bottom edge is restrained in 1-direction and 2-direction.
All other nodes at bottom edge are free in 1-direction and restrained in 2-direction.
Node B at top edge is restrained in 1-direction and all other nodes are free.

Figure 7.14 FEM models for numerical simulation (a) Rubber pad with 2 steel shims (b) Rubber pad with 3 steel shims (c) Rubber pad with 3 steel shims
Figure 7.15 Stress Distribution for 2 steel shims (a) Distribution of Cauchy stress along the different interface (b) Distribution of Cauchy stress along the different coordinate direction-1
Figure 7.16 Stress Distribution for 3 steel shims (a) Distribution of Cauchy stress-22 along the different interface (b) Distribution of Cauchy stress-22 along the different coordinate direction.
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Figure 7.17 Stress Distribution for 3 steel shims (compression and 90% Shear) (a) Distribution of Cauchy stress along the different interface (b) Distribution of Cauchy stress-22 along the direction-1
Figure 7.18 Typical Cauchy stress-

22 pattern (40% compression strain) (a) shape factor -

10 (b) shape factor -20 (c) shape factor -40
Figure 7.19 Typical Cauchy stress-22 pattern (40% compression strain + 90% shearing strain) (a) shape factor 10 (b) shape factor 20 (c) shape factor 40
Figure 7.20 Stress -12 patterns for (compression+ shear) (a) shape factor 10 (b) shape factor 20 (c) shape factor 40
Figure 7.21 Typical deformation-2 pattern (40% compressive strain) (a) SF=20 (b) SF=40
Figure 7.22 Typical deformation pattern for shape factor 10 (40% compressive and 90% shearing strain) (a) Deformation-1 (b) Deformation-2
Figure 7.23 Typical deformation pattern for shape factor 40 (40% compressive and 90% shearing strain) (a) Deformation-1 (b) Deformation-2
Chapter 8
Conclusions

8.1 FINITE ELEMENT FORMULATIONS

The key part in finite element formulation of nonlinear elastic materials is to derive the stress and the material constitutive law from a material model. In the present work, the stress and the constitutive law have been derived from the improved strain energy functions (Amin et al. 2002 and Wiraguna et al. 2003). The present research work has been performed in two steps: firstly the finite element formulation and secondly verification of the formulation.

The expressions for the second Piola-Kirchhoff stress tensor and the Cauchy stress tensor have been formulated for the distortional part of the hyperelasticity model. The Lagrangian elasticity tensors have also been formulated for the distortional part of the hyperelasticity model to implement in a general-purpose finite element code.

The elastic equilibrium and instantaneous responses of NR (NR-II) and HDR bodies under compression and shear have been simulated using material parameters identified from the available experimental observations (Amin et al. 2002 and Wiraguna et al. 2003).

8.2 VERIFICATION OF FEM FORMULATIONS

The numerical simulation results have been compared with experimental observations to discuss the adequacy of the developed finite element procedure in simulating quasi-incompressible response of NR and HDR under uniaxial compression and simple shear deformation.

For uniaxial compression, 2D and 3D finite element simulations have been carried out to make comparisons with those obtained from the constitutive relation and experiments. From the observations of numerical simulation it can be stated that the 3D model presents better agreement with the results obtained from constitutive relation and experiments than 2D model. The plane strain idealization made in 2D model is accounted for such results.
However, the geometric nonlinearity has been well explained in all these models. For 2D model, numerical simulations were carried out for two types of meshes, i.e. one is fine mesh and the other is coarser mesh. It is very interesting to note that fineness of mesh does not effect on the maximum result of stress function. But the stress distribution at the edge of specimen can clearly be identified with the help of finer mesh. From the numerical simulations it has been seen that stress is uniformly distributed throughout the thickness of the specimen at a level of uniaxial compression. This conforms to the adopted experimental condition.

For simulation of simple shear experiments, 2D and 3D finite element analyses have been carried out to make comparisons with those obtained from the constitutive relation and experiments. From the observations of numerical simulations, it can be concluded that all the models present better agreement with the results obtained from constitutive relation and experiments. This occurs due not to have any idealization effect in 2D and 3D models. Geometric nonlinearity has also been well examined in all these models.

From the numerical simulations, it has also been seen that shear stress is uniformly distributed throughout the thickness except at the edge of the specimen under simple shear. This happens due to the coupling effect of simple shear applied at the upper edge. Homogeneous deformations were critically examined for the case of uniaxial compression and simple shear.

For the case of combined action of compression and shear, 3D finite element simulation has been carried out for different shape factors. From the numerical simulations it has been seen that the vertical stiffness increases as the shape factor increases. It has also been seen that stress function is almost uniformly distributed throughout the thickness except the edge of the specimen for particular strain. Stress concentration was seen to be developed at the edge of the specimen. This happened due to the coupling effect of combined action of compression and shear applied at the upper edge. In this case, non-homogeneous deformation as well as geometric nonlinearity was critically examined. In all cases, HDR material showed stronger nonlinearity property than NR.
8.3 FINITE ELEMENT STUDY ON RUBBER BEARINGS

Comparative studies of rubber bearings were made in two cases, such as one was to study the rubber pad made of single layer and the other made of multiple steel shims with rubber layers. In all cases, Kelly's method was taken as the available analytical method to make comparison with FEM solution.

In the first case, a 3D model of single rubber layer was analyzed by FEM solution and Kelly's method separately for different shape factors and then a comparative study was carried out. From the comparative analysis, the vertical stiffness was found to increase with the shape factors. Again, it was seen that percent difference between the results of FEM and Kelly's solution decreased as the shape factor increased. This difference was due to the idealization made in Kelly's solution method, while in FEM, geometric idealization is not needed. In this case, the edge shear stress (bonding shear stress) was computed from pure compression this is very important in designing the rubber steel interface adhesives.

In the second case, the rubber pad of multiple steel shims was analyzed by FEM for different shape factors due to pure compression and combined action of compression and shear. From the analyses, it can be said that, as the number of steel shim increases (that in turn also increases the shape factor) the vertical stiffness increases. It has also been seen that the zone of stress concentration develops due to the combined action of compression and shear. The phenomenon decreases with increase of shape factor.

8.4 SCOPE OF FUTURE STUDIES

All over the world, natural rubber (NR) is being extensively used in constructing steel plate laminated bridge seats. Furthermore, steel plate laminated lead-plugged NR bearings and high damping rubber bearings are becoming popular and being extensively used as base isolation devices in earthquake resistant design. In the present research, finite element formulations have been made considering only the hyperelasticity property of rubbery materials, but the viscoelasticity property must be considered for the purpose of complete representation of monotonic response rubbery material (Amin et al 2002). So the finite element modeling of rubbery material for the viscoelasticity property has been proposed for the future studies.
Furthermore, the present model is capable of simulating the strain-rate dependency phenomena observed in the monotonic response. In contrast, the experimental observations (Amin et al. 2002 and Wiraguna et al. 2003) indicated the absence of such an effect in the unloading paths of the cyclic tests together with the presence of hysteresis effect. To model such behavior, further formulations are needed to incorporate a damage/plasticity model into the current model structure to increase its capability in simulating the cyclic response as well.

Finally, in the context of Bangladesh, it has been found from the geological studies of the ground that the possibility of occurrence of devastating earthquake increasing rapidly with time. So each and every designer/engineer is required to provide sufficient safety against a jolting of the ground. For this, a comparative assessment for the design of a prototype rubber bearing using the results obtained from the nonlinear finite element simulation and the existing analytical method is required. It is thus proposed for the future studies.
References


