M. Sc. Engineering Thesis

Rectangular Drawings of Planar Graphs

By

Shubhashis Ghosh

Submitted to

Department of Computer Science and Engineering

in partial fulfillment of the requirements for the Degree of

M. Sc. Engineering (Computer Science and Engineering)

Department of Computer Science and Engineering

Bangladesh University of Engineering and Technology

April 15, 2001

Rectangular Drawings of Planar Graphs

A Thesis submitted by

SHUBHASHIS GHOSH
Student No. 9605019P

For the partial fulfillment of the degree of
M. Sc. Engineering (Computer Science and Engineering)
Examination held on April 15, 2001

Approved as to style and contents by:

DR. MD. SAIDUR RAHMAN Assistant Professor Department of Computer Science and Engineering B.U.E.T., Dhaka-1000, Bangladesh.	Chairman and Supervisor
DR. CHOWDHURY MOFIZUR RAHMAN Associate Professor and Head Department of Computer Science and Engineering B.U.E.T., Dhaka-1000, Bangladesh.	Member (Ex-officio)
DR. M. KAYKOBAD Professor Department of Computer Science and Engineering B.U.E.T., Dhaka-1000, Bangladesh.	Member
DR. MD. ABUL KASHEM MIA Assistant Professor Department of Computer Science and Engineering B.U.E.T., Dhaka-1000, Bangladesh.	Member
DR. MD. LUTFAR RAHMAN Professor Department of Computer Science Dhaka University, Bangladesh.	Member (External)

Contents

List of Figures

List of Tables

Acknowledgements

First of all, I would like to thank my supervisor Dr. Md. Saidur Rahman for introducing me to this interesting and fascinating field of graph drawing, and for teaching me how to carry on a research work. His all out help was the impetus of this work. His profound knowledge in this field, his keen interest and patience and the scholarly guidance and suggestions at all stages of the work have made it possible to complete this thesis. His careful reading of the manuscript led to improvements in both the wording and the technical aspects of the text. I again express my heart-felt and most sincere gratitude to him for his constant supervision, valuable advice and continuous encouragement. Without his patience, concern and efforts this project would have not been possible.

I would like to thank the other members of my graduate committee, Prof. Dr. Md. Lutfar Rahman, Prof. Dr. M. Kaykobad, Associate Prof. Dr. Chowdhury Mofizur Rahman and Assistant Prof. Dr. Md. Abul Kashem Mia, for their valuable suggestions.

I would like to express my special thanks to Prof. Takao Nishizeki for his valuable and worthy suggestions in different stages of this work.

I would like to give my special thanks to my colleagues who always gave me inspiration.

I would like to express my utmost gratitude to my parents for their support, affection and inspiration.

Finally, I would like to express my special thanks to my wife for her endless mental support and inspiration which helped me to carry on my work smoothly.

Abstract

This thesis deals with rectangular drawings of planar graphs. Rectangular drawings have numerous practical applications in different fields of science and technology. A rectangular drawing of a plane graph G, that is a planar graph G with a fixed embedding, is a drawing of G such that each vertex is drawn as a point, each edge is drawn as a horizontal or a vertical line segment, and the contour of each face is drawn as a rectangle. A planar graph G is said to have a rectangular drawing if G has a rectangular drawing for at least one of its planar embeddings. No necessary and sufficient condition is known for a planar graph to have rectangular drawing. In this thesis we establish a necessary and sufficient condition for the existence of a rectangular drawing of a planar graph G for the case where G is a subdivision of a 3-connected cubic planar graph. We also give a linear-time algorithm to determine whether G has a rectangular drawing or not, and to find a rectangular drawing of G if it exists.

Chapter 1

Introduction

The visualization of complex conceptual structures is a key component of support tools for many applications in science and engineering. A graph, which consists of a set of vertices and a set of edges, is used to model conceptual structures containing information. Graphs are used to represent any conceptual structure that can be modeled as objects and relationship between those objects. Thus graph drawing, that is visualization of graphs, finds its applications in many information visualization systems. We will see some of its applications here.

Graphs are used to represent entity-relationship diagram for modeling data of a database sytem; an entity is represented as a vertex of a graph and a relationship between two entities x and y are represented as an edge between x and y. In a graph representing an entity-relationship diagram vertices are drawn as boxes, and edges are drawn as chains of horizontal and vertical line segments. Fig. 1.1 shows an entity-relationship diagram where texts corresponding to entities are written inside boxes. If a diagram is small then one can draw it by hand, but if a diagram is large then it is difficult to draw it by hand. In this case we need algorithms for automatic drawing of that diagram.

Graphs are also used in computer networking to describe hierarchies and interconnec-

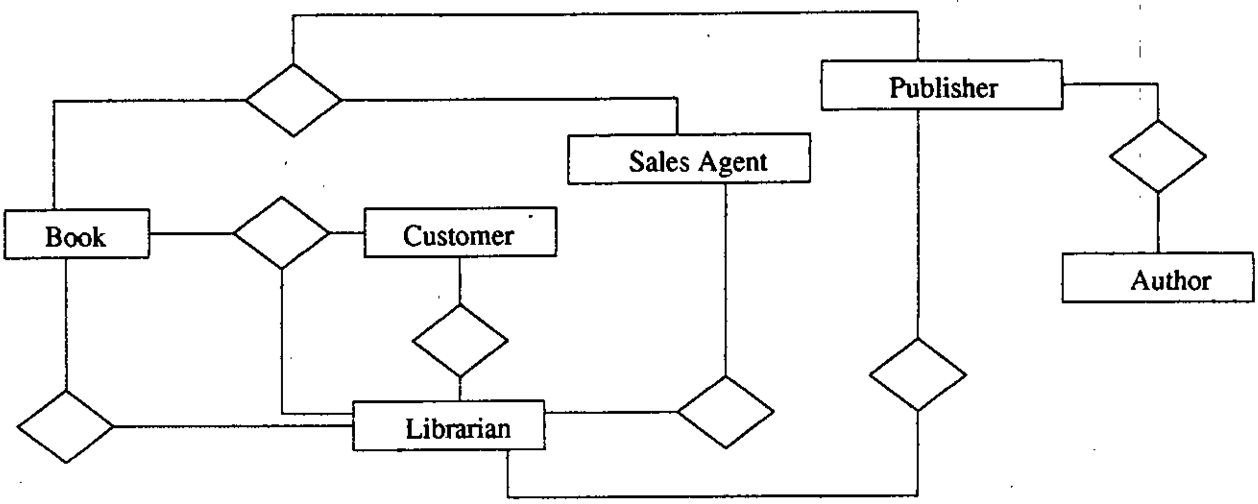


Figure 1.1: An example of an Entity-Relationship Diagram.

tions of components in computer networks; each component is represented as a vertex in a graph and the connection between a component a and a component b is represented by an edge from a and b. These graphs are typically drawn as diagrams with texts at the vertices and the line segment joining the vertices as edges. Fig. 1.2 represents a diagram of a computer network. System administrators use such diagrams for understanding, monitoring and controlling operations of computer networks.

We now consider another example from [R99]. The graph in Fig. 1.3(a) represents the components and connections of an electronic cicuit. In this example, a and b are electronic components and the curved line between them is the connection between the components a and b. The representaion in Fig. 1.3(a) is clumsy and difficult to trace out. Moreover, in this representation one cannot lay the circuit on a PCB because of edge crossings. But the representation in Fig. 1.3(b) looks better and it is easily traceable. The representation in Fig. 1.3(b) can be used for desired PCB layout of the circuit, since there is no edge crossings in this representation. Thus the objective of graph drawing is

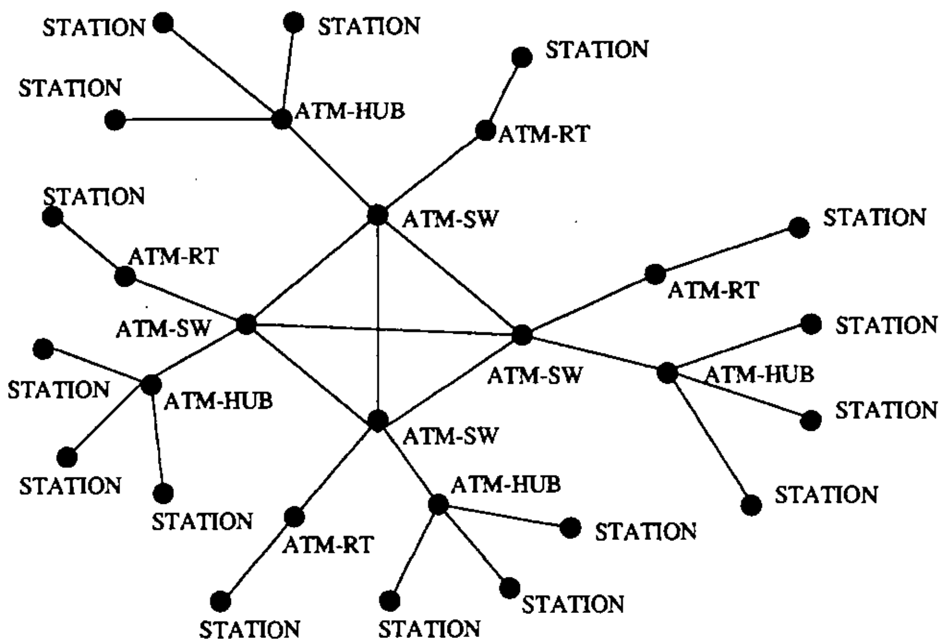


Figure 1.2: A diagram of a computer network.

to obtain a nice representation of graph such that the graph and its contents are easily understandable. Moreover, the drawing should satisfy some criteria that arises from the application point of view.

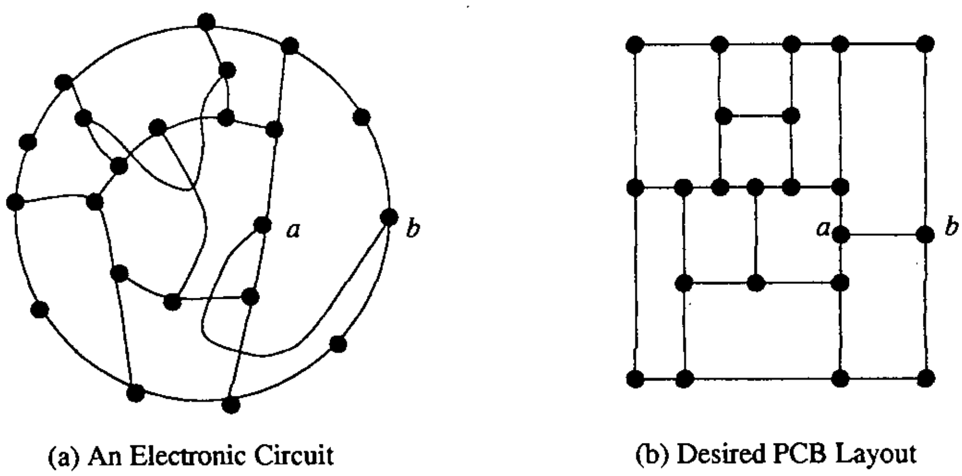


Figure 1.3: An example of a graph drawing in circuit schematics.

Automatic graph drawings have numerous applications not only in database system, computer networks and in PCB layout, but also in VLSI floorplanning, information systems, computer architecture, circuit schematics, architectural floorplanning, etc.

In this chapter we provide the necessary background and objective for this study on graph drawing. In Section 1.1 we give a historical background of the development of the field of graph drawing from [R99]. In Section 1.2 we introduce some conventional drawing styles from [R99]. In Section 1.3 we discuss some drawing aesthetics based on which a drawing is evaluated. In Section 1.4 we illustrate the applications of rectangular drawing. In Section 1.5 we explain the objective of this thesis, and we summarize our results in Section 1.6.

1.1 Historical Background

The first need for graph drawing algorithms arose in late 1960's, when the large number of elements in increasing complex circuit designs made hand-writing too complicated. Algorithms were developed to aid circuit design, an overview can be found in the book of Lengaur [L90]. The field of graph drawing with the objective of producing aesthetically pleasing pictures became of interest in the late 1980's [CON85, TBB88]. The reason for this was the realization that in engineering and in production process, information about the processes is an important resource for management and control. Thus, information managers needed some efficient tools to help them to present information easily. Graph drawing is such a tool.

This field has been flourished intensively in the last two decades. Progress in computational geometry, topological graph theory etc. influenced this field considerably. A comprehensive bibliography in [DETT94] shows that an intensive work is being done in the last two decades.

1.2 Drawing Conventions

In this section we introduce some conventional drawing styles. A drawing convention of a real-life application can be very complex and can involve many details of the drawing. Standard drawing conventions, which are widely used, are given below.

1.2.1 Planar Drawings

A drawing of a graph is *planar* if no two edges intersect in the drawing. Fig. 1.4(a) shows a planar drawing and Fig. 1.4(b) shows a non-planar drawing of the same graph. Unfortunately not all graphs admit planar drawings. It is an interesting problem to find a planar drawing of a graph, if the graph admits such a drawing. The graph, which admits a planar drawing, is called a *planar graph*. A *plane graph* is a planar graph with a fixed planar embedding.

Since all graphs do not admit planar drawings, it is needed to test whether the given graph is planar or not to find a planar drawing of the given graph. If the graph is planar, then one needs to find a planar representation of the graph which is a data structure representing adjacency lists: lists in which the edges incident to a vertex are ordered, all clockwise or counterclockwise, according to the planar representation. Kuratowski [K30] gave the first complete characterization of planar graphs. But this characterization does not lead to an efficient algorithm in order to test planarity. Linear-time algorithms for planarity testing were developed later [HT74, BL76, CNAO85, M92].

1.2.2 Polyline Drawings

A *polyline drawing* is drawing of a graph in which each edge of the graph is represented by a polyline chain. A polyline drawing of a graph is shown in Fig. 1.5(a). The point

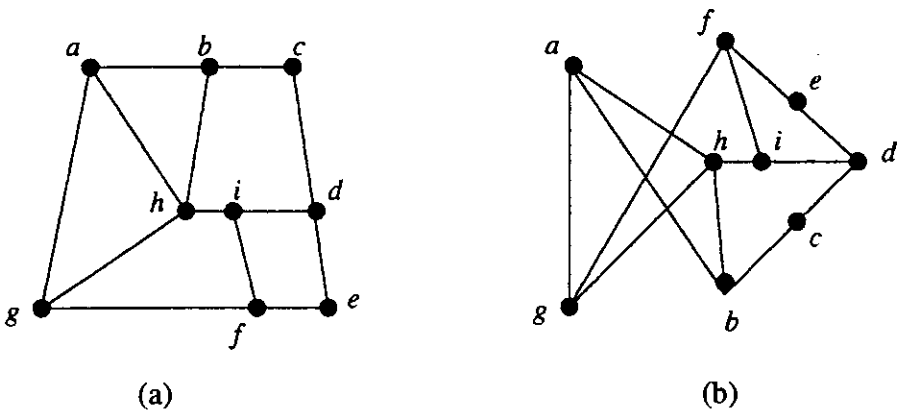


Figure 1.4: (a) A planar drawing and (b) a non-planar drawing of the same graph.

at which an edge changes its direction is called a *bend.* Polyline drawings provide great flexibility since they can approximate drawings with curved edges. However, edges with more than two or three bends may be difficult for the eye to follow.

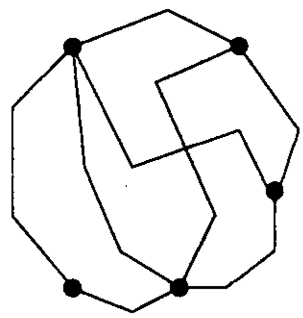
1.2.3 Straight Line Drawings

A *straight line drawing* is a drawing of a graph in which each edge of the graph is represented as a straight-line segment. A straight line drawing of a graph is shown in Fig. 1.5(b). It is a special case of polyline drawing shown in Fig. 1.5(a) where the edges are drawn without bends.

It is proved that every planar graph has a straight line representation [W36, F48, S57]. Many works have been published on straight line drawings of planar graphs [DETT94].

1.2.4 Orthogonal Drawings

An *orthogonal drawing* of a plane graph is a drawing with the given embedding in which each vertex is mapped to a point, each edge is drawn as a sequence of alternate horizontal and vertical line segments. A *bend* is defined to be a point where an edge changes its direction in a drawing. Fig. 1.6(a) is an orthogonal drawing with six bends. To obtain an


(a)

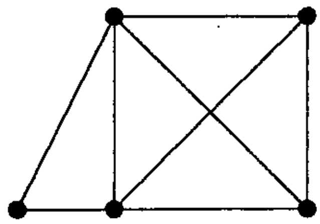

(b)

Figure 1.5: (a) A polyline drawing and (b) a straight-line drawing.

orthogonal drawing of a plane graph with the minimum number of bends is a challenging problem [GT97, T87, RNN99].

In an orthogonal drawing each vertex is mapped to a point. Therefore, if a graph has vertices of degree more than four then it has no orthogonal drawing as at most four edges can be incident to a vertex in an orthogonal drawing. A *box-orthogonal drawing* of a graph is a drawing such that each vertex is drawn as a rectangle, called a box, and each edge is drawn as a sequence of alternate horizontal or vertical line segments. The drawing in Fig. 1.6(b) is an example of a box-orthogonal drawing. Every plane graph has a box-orthogonal drawing. Several results are known for box-orthogonal drawings [BK94, PT98, R99].

1.2.5 Rectangular Drawings

A *rectangular drawing* of a plane graph is a drawing of graph in which each vertex is drawn as a point, each edge is drawn as a horizontal or vertical line segment without edge crossings and each face is drawn as a rectangle. A rectangular drawing of a plane graph is a special case of orthogonal drawing in which there is no bend and each face of the graph is drawn as a rectangle. The drawing in Fig.1.7(a) is an example of rectangular drawing.

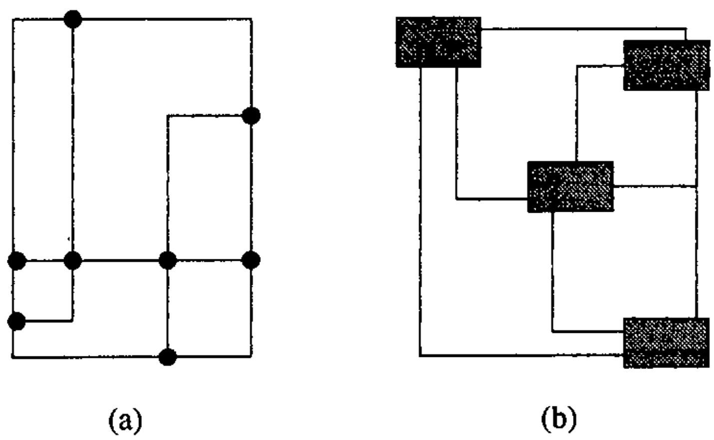


Figure 1.6: (a) An orthogonal drawing and (b) a box-orthogonal drawing.

A planar graph is said to have a rectangular drawing if it has a rectangular drawing for at least one of its planar embeddings.

A *box-rectangular drawing* of a graph is a drawing such that each vertex is drawn as a box, and the contour of each face is drawn as a rectangle. The drawing in fig. 1.7(b) is an example of a box-rectangular drawing. Unfortunately not every plane graph has a box-rectangular drawing. Some works on box-rectangular drawings are done in [R99, RNN00b, H00].

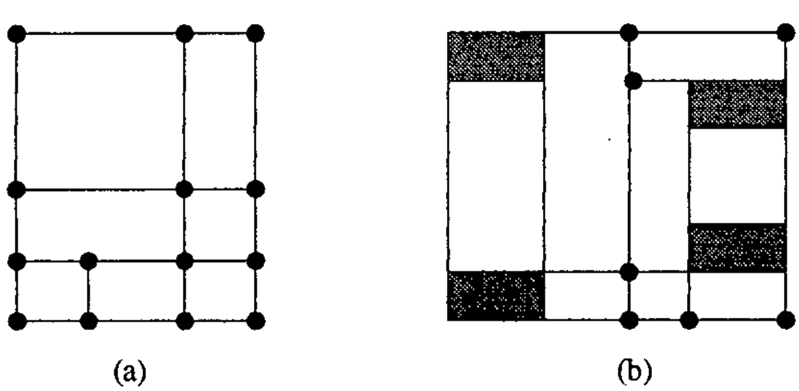
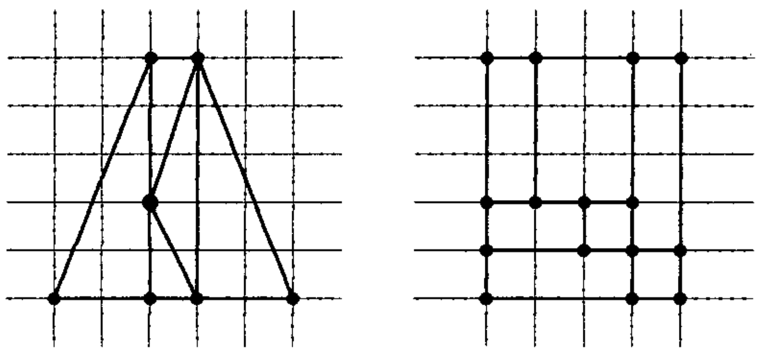


Figure 1.7: (a) A rectangular drawing and (b) a box-rectangular drawing.

1.2.6 Grid drawings

A drawing of a graph in which vertices and bends are located at grid points of an integer grid is called a *grid drawing*. The drawings in Fig. 1.8(a) and in Fig. 1.8(b) are two examples of grid drawings. It is a very challenging problem to draw a plane graph on a grid of the minimum size. In recent years, several works [FPP90, CN98, RNN98] are devoted to this field.



(a) (b)

Figure 1.8: Examples of grid drawings.

1.2.7 Visibility Representation

The *visibility representation* of a plane graph is a representation where each vertex is mapped to a horizontal line and each edge is drawn as a vertical line segment. Fig. 1.9(b) illustrates visibility representation of a plane graph in Fig. 1.9(a).

1.3 Drawing aesthetics

Aesthetics specify graphic properties of the drawing which are used to achieve readability. Actually the degree of readability of a drawing can be evaluated based on the value of these properties. We now mention some common aesthetics.

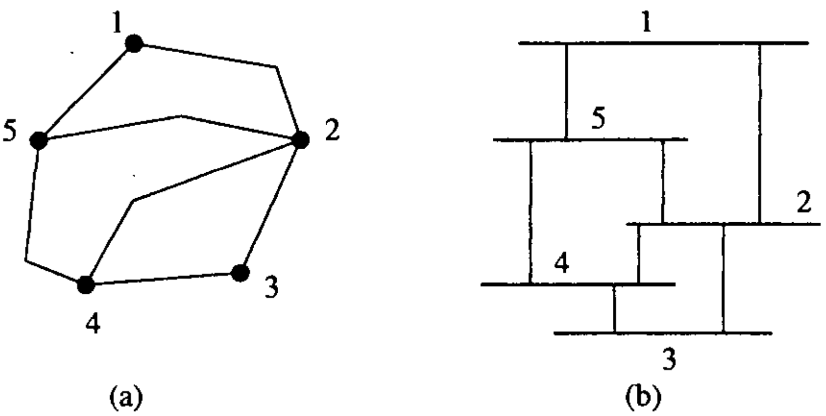


Figure 1.9: Illustration of a visibility representation.

Crossings: Every crossing between edges bears the potential of confusion. Therefore, the total number of edge crossings should be kept small.

Bends: A bend is a point where an edge changes its direction. Bends cause difficulties in implementations of VLSI circuits or PCB layouts of electronic circuits. Therefore, the minimization of both the total number of bends and the number of bends per edge is important.

Area: If a drawing takes a large area, it may be out of viewing area. Then we must decrease resolution or use more pages that may make the drawing unreadable. Therefore, drawings should take small area.

But it is difficult to achieve the optimum value of the above drawing properties. Garey and Johnson showed that the problem of minimizing the number of crossings of a graph is NP-complete [GJ83]. To determine whether a graph can be embedded in a grid of a given size is also NP-complete [KL84]. Garg and Tamassia proved the NP-completeness of the problem of determining the minimum number of bends for orthogonal drawings [GT95].

1.4 Applications of Rectangular Drawings

Major applications of Rectangular graph drawing are in VLSI floorplanning [KK84, L90, RNN98, RNN00a, RNN00b, TTSS91] and in architectural floorplanning [MKI00]. Here we will illustrate how rectangular drawing can be used in VLSI floorplanning problem [RNN00a]. In a VLSI floorplanning problem, the interconnection among modules is usually represented by a planar embedding of a planar graph, every inner face of which is triangulated. Fig. 1.10(a) illustrates such an interconnection graph of 17 modules. The dual-like graph of an interconnection graph is a cubic graph in which every vertex has degree 3 [L90, RNN00b]. Fig. 1.10(b) illustrates a dual-like graph of the graph in Fig. 1.10(a), that has 17 inner faces. Inserting four vertices a, b, c and d of degree 2 in appropriate edges on the outer face contour of the dual-like graph as illustrated in Fig. 1.10(c), one wishes to find a rectangular drawing of the resulting graph as illustrated in Fig. 1.10(d). If there is a rectangular drawing, it yields a floorplan of the interconnection graph. Each vertex of degree 2 is a corner of the rectangle corresponding to outer rectangle.

1.5 Objective of this Thesis

In this section we introduce the objective of this thesis after mentioning some previous results.

A *rectangular drawing* of a planar graph G is a drawing of G on the plane in which each vertex is drawn as a point, each edge is drawn as a horizontal or a vertical line segment without edge-crossings, and the contour of each face is drawn as a rectangle. Rectangular graph drawing has attracted much attention due to its applications in VLSI floorplanning [KK84, L90, RNN98, RNN00a, RNN00b, TTSS91] and architectural floorplanning [MKI00].

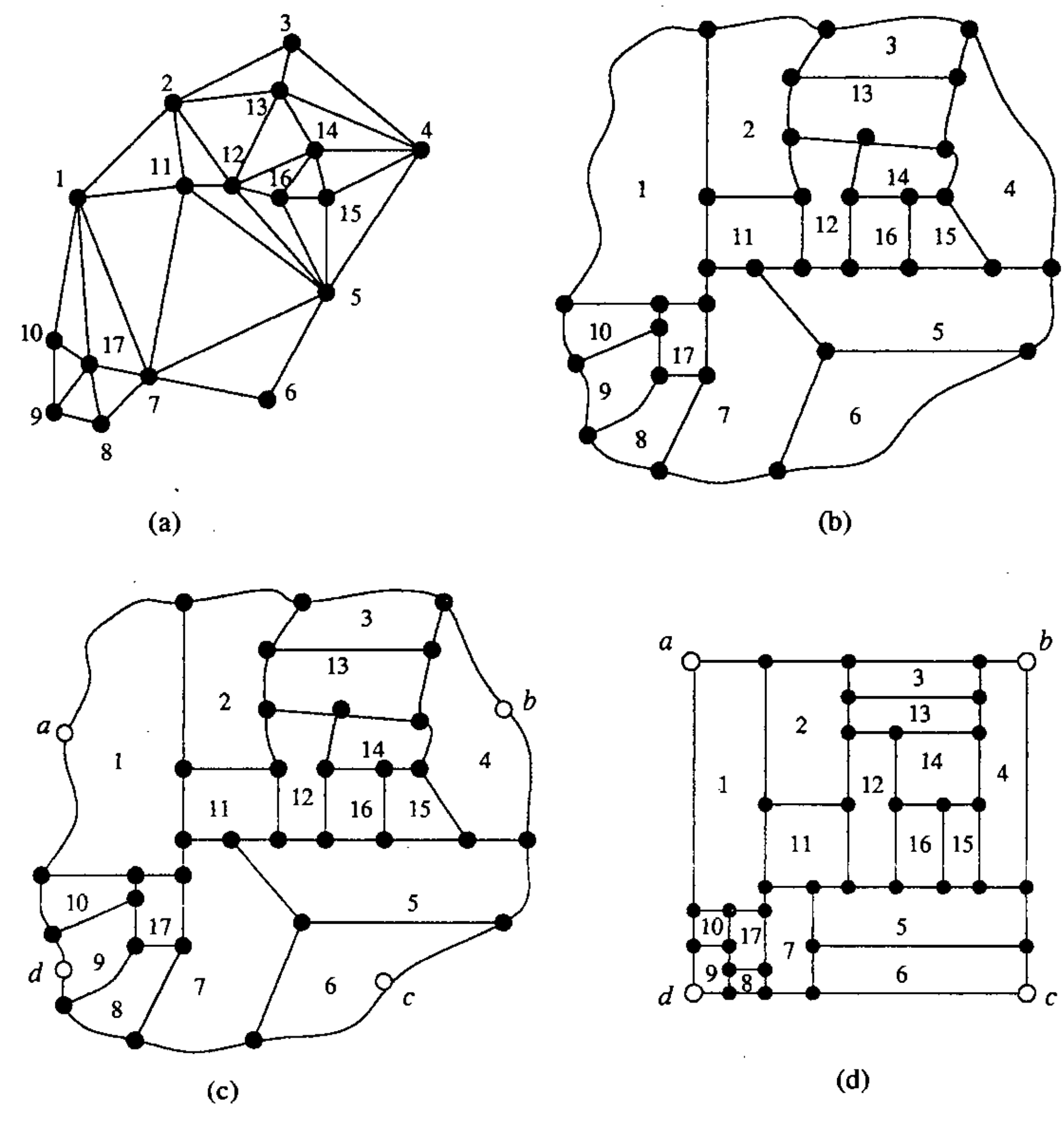


Figure 1.10: (a) Interconnection graph, (b) dual-like graph, (c) Insertion of four corners, and (d) rectangular drawing.

Not every planar graph has a rectangular drawing. Thomassen [T84] obtained a necessary and sufficient condition for a plane graph, i.e, a planar graph with a fixed embedding, to have a rectangular drawing, where four vertices of degree two on the outer face is designated as corners for a rectangular drawing. Linear-time algorithms are given in [BS88, H93, KH97, RNN98] to obtain a rectangular drawing of such a plane graph. Rahman et. al. [RNN00a] gave a necessary and sufficient condition for a plane graph G to have a rectangular drawing, where no vertex is designated as a corner and developed a linear-time algorithm to find a rectangular drawing of G if it exists.

Determining whether a planar graph has a rectangular drawing is not a trivial problem, since a planar graph may have an exponential number of planar embeddings. In Fig. 1.11 four different planar embeddings of the same planar graph are shown. Among the four planar embeddings only the embedding in Fig. 1.11(a) has a rectangular drawing as illustrated in Fig. 1.11(e).

In this thesis we concentrate our attention only on rectangular drawings for planar graphs. Unfortunately no necessary and sufficient condition is known for a planar graph to have a rectangular drawing. Since determining whether a planar graph has a rectangular drawing is a difficult problem, we give our attention on rectangular drawing for a particular class of planar graphs that are subdivisions of 3-connected cubic planar graphs. Thus the objectives of this thesis are as follows:

- To establish a necessary and sufficient condition for a subdivision of a 3-connected cubic planar graph to have a rectangular drawing.
- To give an efficient algorithm to determine whether a subdivision of a 3-connected cubic planar graph satisfies the condition.
- To give an efficient algorithm to find a rectangular drawing of the graph, if it exists.

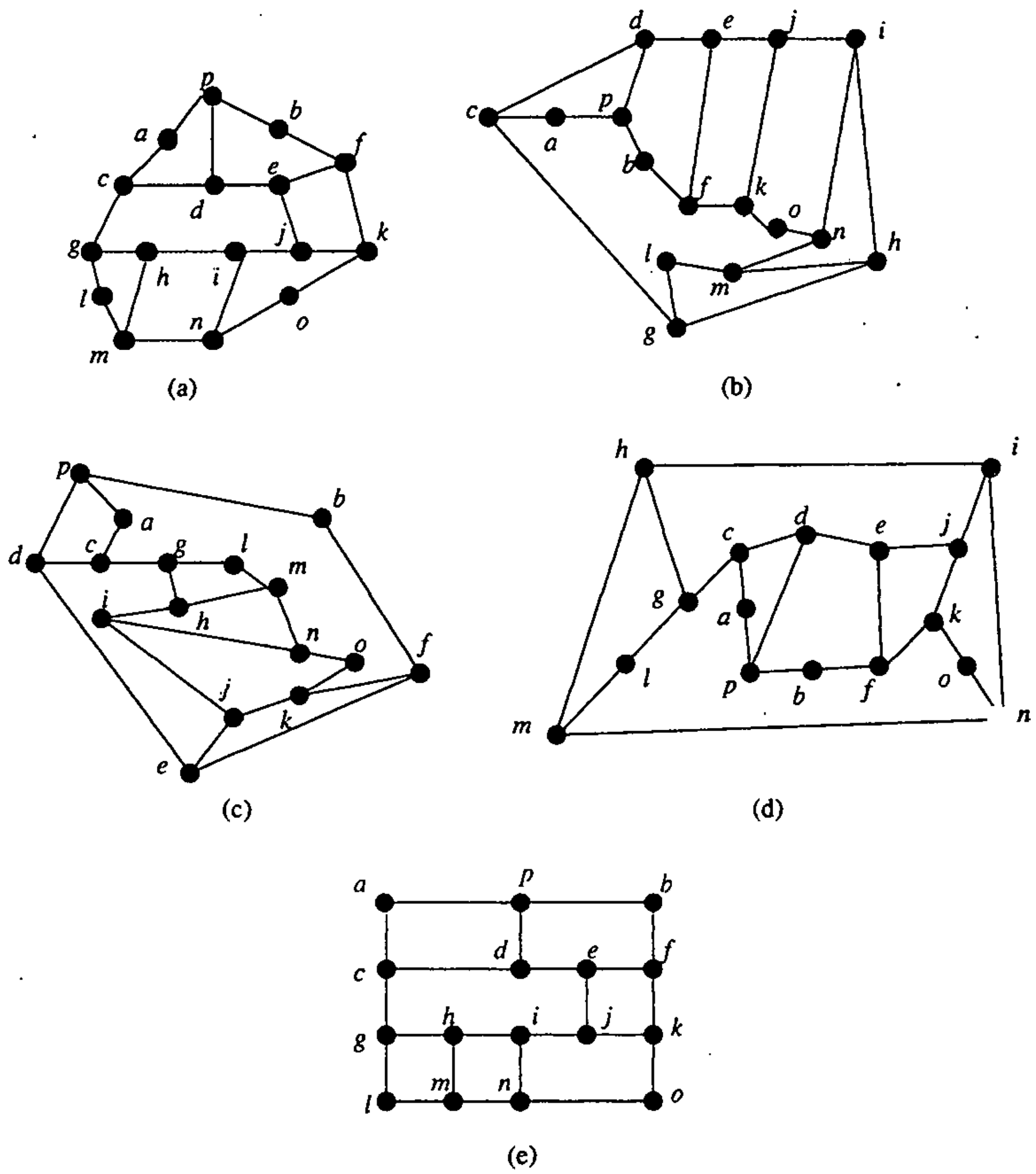


Figure 1.11: Four different planar embeddings of the same graph are shown in (a), (b), (c), and (d); only (a) has a rectangular drawing as shown in (e).

Type of drawing	Characterization	Algorithms for checking the condition	
		Time	Reference
Rectangular drawings of plane graphs with designated corners	Thomassen [T84]	$O(n)$	[RNN98]
Rectangular drawings of plane graphs without designated corners	Rahman, Nakano & Nishizeki [RNN00a]	$O(n)$	[RNN00a]
Rectangular drawings of subdivisions of 3-connected cubic planar graphs	**Ours**	$O(n)$	**Ours**

Table 1.1: Characterization for rectangular drawings

Classes of graphs	Time	Reference
Plane graph with designated corners	$O(n)$	[RNN98]
Plane graph without designated corners	$O(n)$	[RNN00a]
Subdivisions of 3-connected cubic planar graphs	$O(n)$	**Ours**

Table 1.2: Algorithms for rectangular drawings

1.6 Summary

In this thesis we establish a necessary and sufficient condition for a subdivision of a 3-connected cubic planar graph to have a rectangular drawing. We also develop a linear-time algorithm to determine whether a graph satisfies the condition or not. Our results together with some previous results are listed in Table 1.1.

We also develop a linear-time algorithm to obtain a rectangular drawing of a subdivision of a 3-connected cubic planar graph, if it exists. Our results together with some previous results are listed in Table 1.2.

The rest of this thesis is organised as follows. Chapter 2 gives preliminaries. Chapter 3 deals with rectangular drawings of subdivisions of 3-connected cubic planar graphs.

Finally we give our conclusion in Chapter 4.

Chapter 2

Preliminaries

In this chapter we give definitions of some basic terms used in graph theory and algorithm theory. Definitions that are not included in this chapter will be introduced later on as they are needed. In Section 2.1 we give some definitions and graph-theoretical terms used throughout the thesis. In Section 2.2 we define some terms related to planar graphs and plane graphs, and in Section 2.3 we introduce the notion of time complexity of an algorithm.

2.1 Basic Terminology

In this section we give some definitions and graph-theoretical terms used throughout the remainder of this thesis. For readers interested in graph theory we refer to [W96].

2.1.1 Graphs and Subgraphs

A *graph* can be defined as a structure (V, E) which consists of a finite set of vertices V and a finite set of edges E; each edge is an unordered pair of distinct vertices. A graph G is said to be a *simple graph* if G has no "multiple edges" or "loops". *Multiple edges* join the same pair of vertices, while a *loop* joins a vertex to itself. Let G be connected simple graph with n vertices and m edges. We denote the set of vertices of G by $V(G)$ and the set of edges by $E(G)$. The graph G in Fig. 2.1 has ten vertices and thirteen edges. Here for graph G, $V(G) = \{v_1, v_2, ..., v_{10}\}$ and $E(G) = \{e_1, e_2, ..., e_{13}\}$.

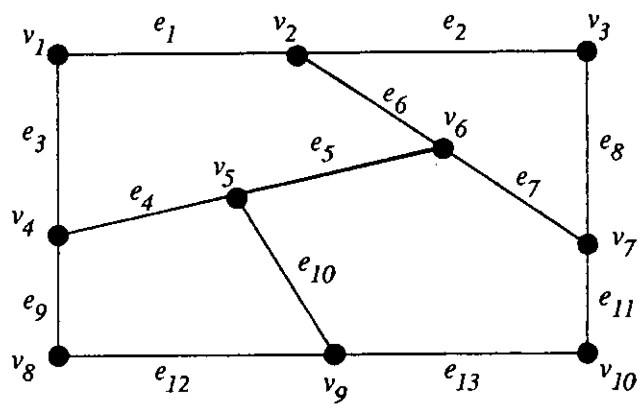


Figure 2.1: A graph with ten vertices and thirteen edges.

We denote an edge between two vertices u and v of G by (u, v) or simply by uv. If $uv \in E$, then we call two vertices u and v are *adjacent*, and edge uv is *incident* to vertices u and v. The *degree* of a vertex v is the number of vertices adjacent to v in G and is denoted by $d(v)$. In Fig. 2.1, $d(v_4) = 3$, since three edges e_3, e_4, e_9 are incident to v_4 and $d(v_3) = 2$ as two edges e_2, e_8 are incident to vertex v_3. We denote the maximum degree of graph G by $\Delta(G)$ or simply Δ. In Fig. 2.1, $\Delta(G) = 3$ since some vertices of G have degree three and no vertex of G has degree greater than three.

A *subgraph* of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$; we then write $G' \subseteq G$. Fig. 2.2 depicts a subgraph of G in Fig. 2.1.

2.1.2 Connectivity

A graph G is *connected* if for every pair $\{u, v\}$ of distinct vertices there is a path from u to v. A graph which is not connected is called *disconnected* graph. A connected component

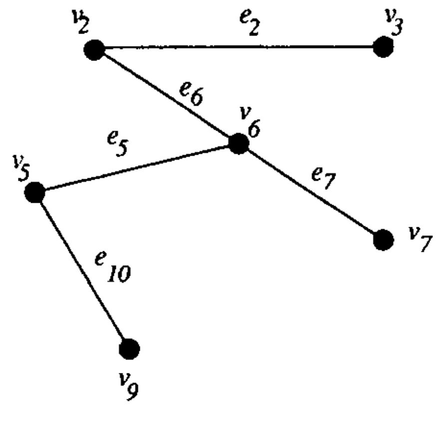


Figure 2.2: An example of a subgraph G' of graph G.

of a graph is a maximal connected subgraph. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal together with all edges adjacent to them results in a disconnected graph or a single vertex graph. We say that G is k-connected if $\kappa(G) \geq k$. The graph in Fig. 2.3(a) is an example of a connected graph. The graph in Fig. 2.3(b) is a biconnected graph, since removal of at least two vertices from the graph results in a disconnected graph. The graph in Fig. 2.3(c) is a triconnected graph. But the graph in Fig. 2.3(d) is a disconnected graph as there is no path from v_1 to v_6. This graph has two connected components H_1 and H_2.

A pair $\{x, y\}$ of vertices of a biconnected graph $G = (V, E)$ is a *separation pair* if there exist two subgraphs $G'_1 = (V_1, E'_1)$ and $G'_2 = (V_2, E'_2)$ satisfying the following conditions (a) and (b):

(a) $V = V_1 \bigcup V_2$, $V_1 \bigcap V_2 = \{x, y\}$;

(b) $E = E'_1 \bigcup E'_2$, $E'_1 \bigcap E'_2 = \Phi$, $|E'_1| \geq 2$, $|E'_2| \geq 2$.

For a separation pair $\{x, y\}$, $G_1 = (V_1, E'_1 + (x, y))$ and $G_2 = (V_2, E'_2 + (x, y))$ are called *split graphs*. The new edges (x, y) added to G_1 and G_2 are called *virtual edges*. The graph in Fig. 2.4(a) is a biconnected graph with separation pair $\{x, y\}$ and Fig. 2.4(b) shows the split graphs due to this separation pair.

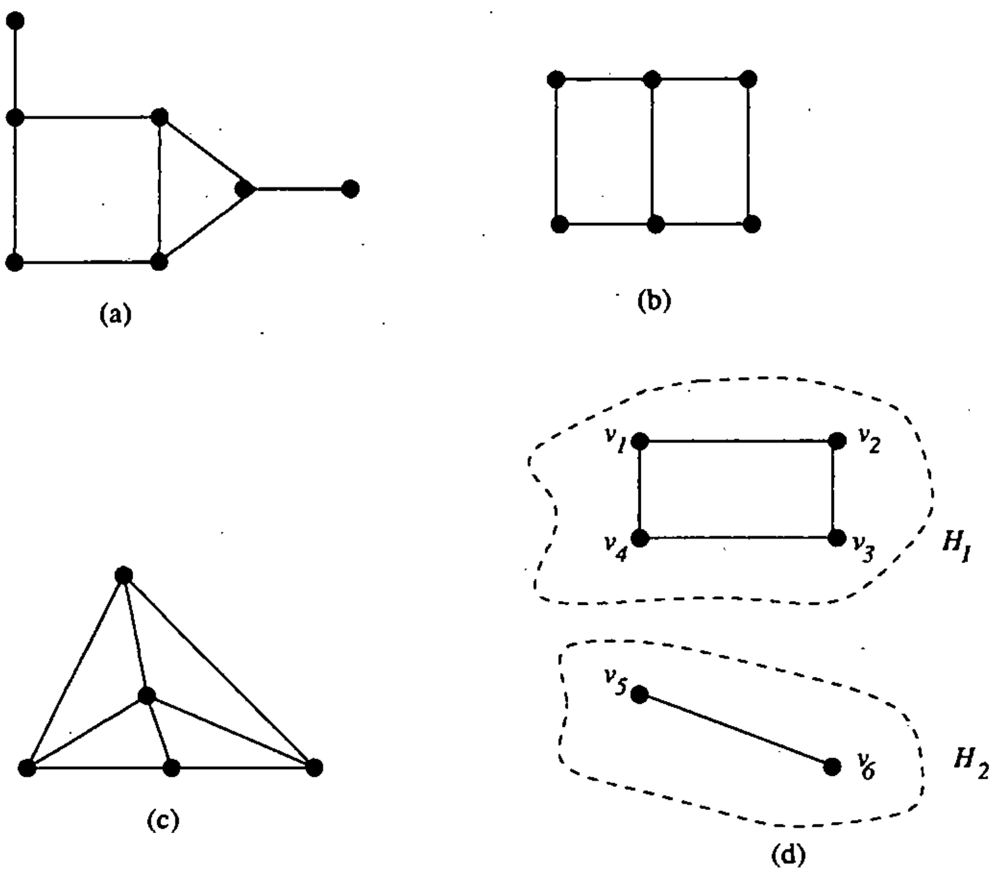


Figure 2.3: (a) A connected graph, (b) a biconnected graph, (c) a triconnected graph and (d) a disconnected graph.

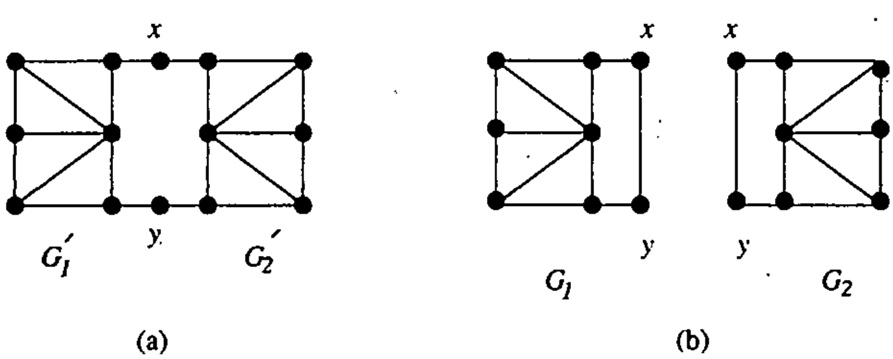


Figure 2.4: (a) A biconnected graph with separation pair $\{x, y\}$ and (b) split graphs G_1 and G_2 for the separation pair $\{x, y\}$.

2.1.3 Cycles and Trees

In a graph G an alternating sequence of vertices and edges, which has the beginning and ending with the same vertex, and in which each edge is incident to two vertices immediately preceding and following it, is called a *cycle*.

A *tree* is a connected graph without any cycle. Fig. 2.5 is an example of a tree. The vertices in a tree are usually called *nodes*. A *rooted tree* is a tree in which one of the nodes is distinguished from the others. The distinguished node is called the *root* of the tree. The root of a tree is generally drawn at the top. In Fig. 2.5 the root is v_1. Every node v other than the root is connected by an edge to some other node u. u is called the *parent* of v. We also called v is the *child* of u. A *leaf* is a node which has no children. In Fig. 2.5 node v_2 is a child of node v_1, and nodes v_5, v_6, v_7, v_8 and v_9 are leaf nodes.

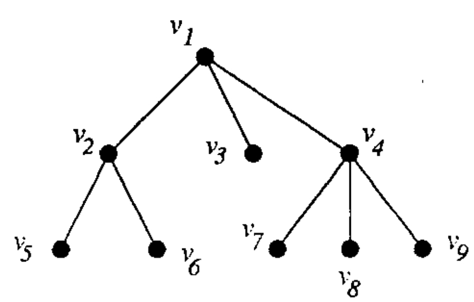


Figure 2.5: A tree.

2.2 Planar Graphs

In this section we give definitions of some terms related to planar graphs and plane graphs. For readers interested in planar graphs we refer to [NC88].

2.2.1 Planar Graphs and Plane Graphs

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A planar graph may have an exponential number of embeddings. Fig. 2.6 shows three different planar embeddings of the same planar graph.

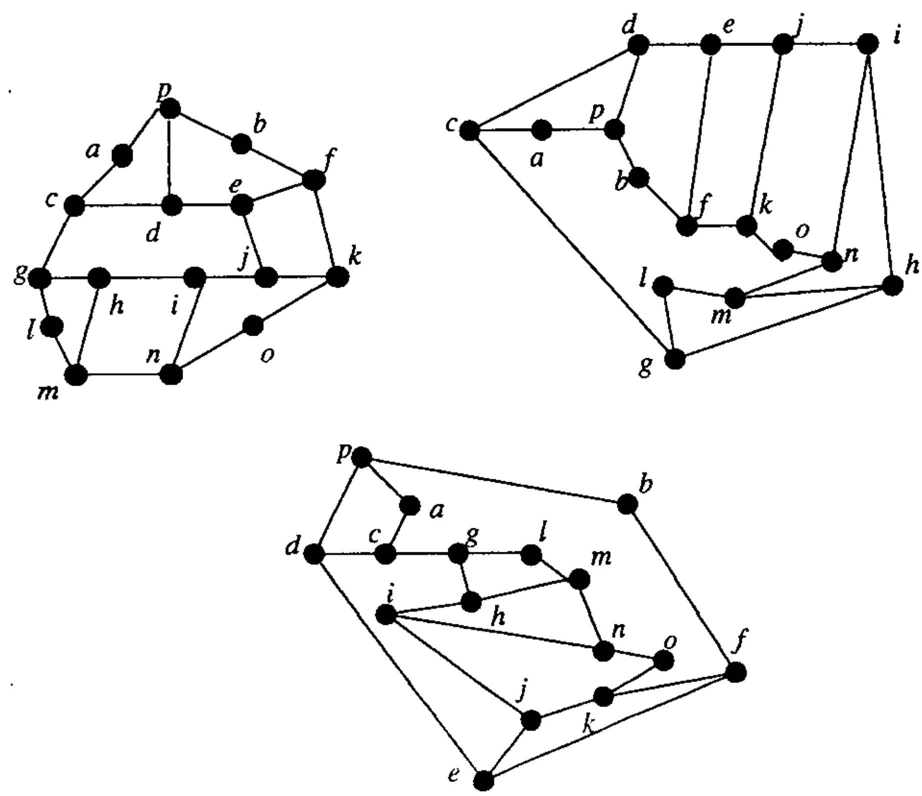


Figure 2.6: Three different planar embeddings of the same graph.

A *plane graph* is a planar graph with a fixed planar embedding. A plane graph divides the plane into connected regions called *faces*. We refer the contour of a face as a clockwise cycle formed by the edges on the boundary of the face. We denote the contour of the outer face of graph G by $C_o(G)$. A cycle of a plane graph is called a *facial cycle* if it is the boundary of a face f and is denoted by C_f. Faces of a plane graph G are shown in Fig. 2.7.

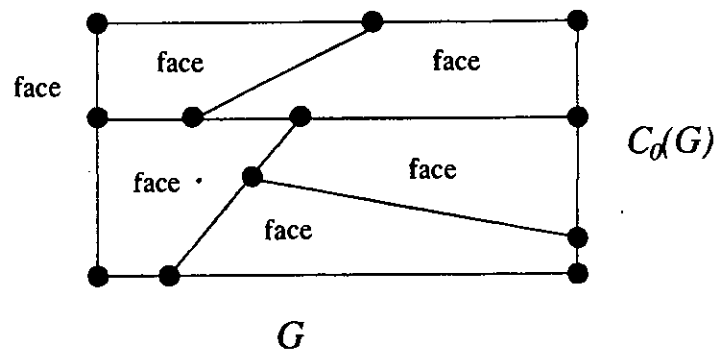


Figure 2.7: A plane graph G and its faces.

For a connected plane graph G with n vertices, m edges and f faces, Euler in 1750 found the following formula:

$$n - m + f = 2$$

This can be shown by an induction over m (see for example [NC88]).

An edge of G, which is incident to exactly one vertex of a simple cycle C and located outside C, is called a leg of the cycle C. The vertex of C to which a leg is incident is called a *leg-vertex* of C. A simple cycle C in G is called a *l-legged* cycle of G if C has exactly l legs in G. In Fig. 2.8 C_1 is a 1-legged cycle, C_2 and C_3 are 2-legged cycles, and C_4, C_5 and C_6 are 3-legged cycles.

As mentioned earlier a planar graph may have an exponential number of planar embeddings. Therefore, one of the major problems is to find the required planar embedding of the graph. Here we will give a brief layout of a linear-time algorithm to find a planar embedding of a planar graph making a specific face as the outer face. For more details of this algorithm we refer to [NC88].

2.2.2 Finding Planar Embedding

We first define some terms and concepts and then present the algorithm for finding the specific planar embedding.

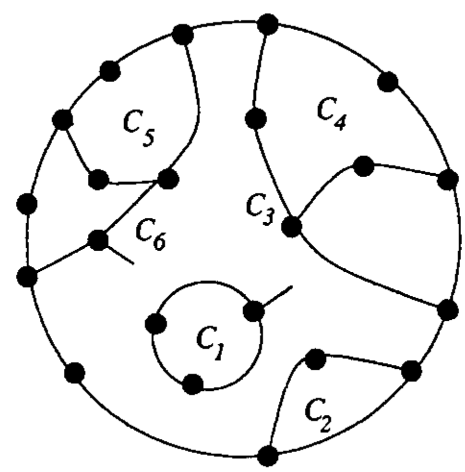


Figure 2.8: Examples of 1-, 2- and 3-legged cycles.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A graph is represented by a set of n lists, called "adjacency lists"; the list Adj(v) for vertex $v \in V$ contains all the neighbours of v. For each $v \in V$ an actual drawing of a planar graph G determines, within a cyclic permutation, the order of v's neighbours embedded around v. *Embedding* a planar graph G means constructing adjacency lists of G such that, in each Adj(v), all the neighbours of v appear in clockwise order with respect to an actual drawing. Such a set Adj of adjacency lists is called *embedding* of G. An example is illustrated in Fig. 2.9(d) which is an embedding of a graph G in Fig. 2.9(a).

The *st-numbering* plays a crucial role in the embedding algorithm. A numbering of the vertices of G by $1, 2, ..., n$ is called an *st-numbering* if the two vertices "1" and "n" are necessarily adjacent and each j of the other vertices is adjacent to two vertices i and k such that $i < j < k$. The vertex "1" is called the *source* and is denoted by s, while the vertex "n" is called a *sink* and is denoted by t. Fig. 2.9(a) illustrates an st-numbering of a graph. Every 2-connected graph G has an st-numbering, and a linear-time algorithm for finding an st-numbering of a graph is given by Even and Tarjan [ET76].

From now on for the embedding algorithm we refer to the vertices of G by their st-

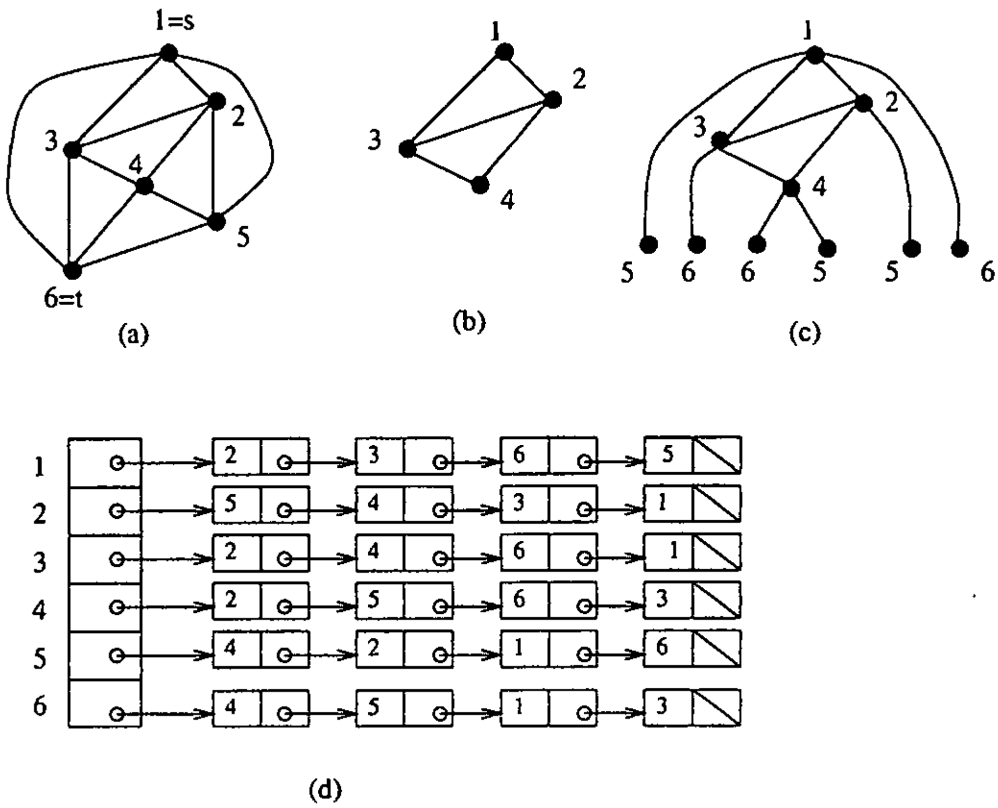


Figure 2.9: Illustration of: (a) st-numbered graph G, (b) G_4, (c) bush form B_4, and (d) adjacency lists of embedded G.

numbers. Let $G_k = (V_k, E_k)$ be the subgraph induced by the vertices $V_k = \{1, 2, ..., k\}$. If $k < n$ then there must exist an edge of G with one end in V_k and the other in $V - V_k$. Let G'_k be the graph formed by adding to G_k all these edges, in which the ends in $V - V_k$ of added edges are kept separate. These edges are called *virtual edges*, and their ends in $V - V_k$ are called *virtual vertices* and labelled as their counterparts in G, but they are kept separate. Thus there may be several virtual vertices with the same label, each with exactly one entering edge. Let B_k be an embedding of G'_k such that all the virtual vertices are placed on the outer face. B_k is called a *bush form* of G'_k. The virtual vertices are usually placed on a horizontal line. G, G_k, and B_k are illustrated in Fig. 2.9. Every planar graph G has a bush form B_k for $1 \leq k \leq n$.

An *upward digraph* D_u is defined to be a digraph obtained from G by assigning a direction to every edge so that it goes from the larger end to the smaller. An *upward embedding* A_u of G is an embedding of the digraph D_u. In an embedding of an undirected graph G, a vertex v appears in list Adj(w) and w appears in list Adj(v) for every edge (v, w). However in an upward embedding A_u of G, the head w appears in adjacency list $A_u(v)$ but the tail v does not appear in $A_u(w)$ for every directed edge (v, w). Fig. 2.10 depicts an upward digraph D_u and an upward embedding A_u for the graph G in Fig. 2.9(a).

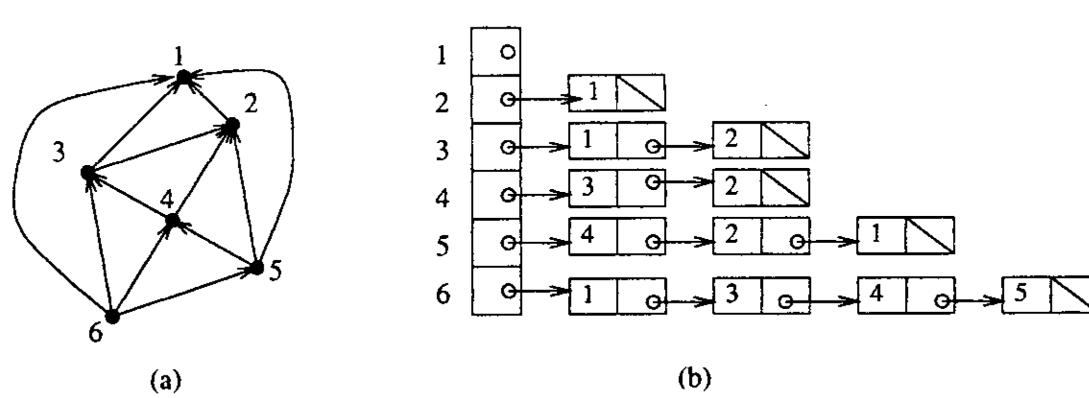
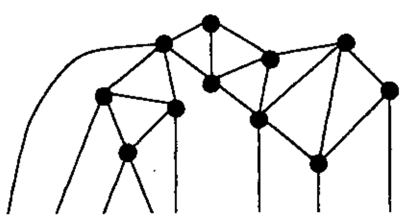
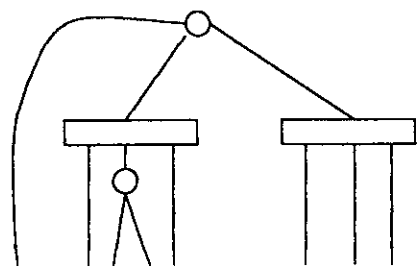


Figure 2.10: Illustration of: (a) upward digraph D_u, and (b) upward embedding A_u for a graph G in Fig. 2.9(a).

We use a special data structure "PQ-tree" to represent B_k. A PQ-tree consists of "P-nodes", "Q-nodes" and "leaves". A P-node represents a cut-vertex of B_k, so the sons of a P-node can be permuted arbitrarily. A Q-node represents a 2-connected component of G_k, and the sons of a Q-node are allowed only to reverse(flip over). A leaf indicates a virtual vertex of B_k. In an illustration of a PQ-tree, a P-node is drawn by a circle and a Q-node by a rectangle. A bush form B_k and a PQ-tree representing B_k are illustrated in Fig. 2.11.



(a)



(b)

Figure 2.11: Illustration of: (a) bush form B_k, and (b) PQ-tree.

For any bush form B_k of subgraph G_k of a planar graph G, there exists a sequence of permutations and reversions to make all the virtual vertices labelled "$k+1$" occupy consecutive positions on the horizontal line. Booth and Lueker [BL76] showed that these permutations and reversions can be found by repeatedly applying the nine transformation rules called the *template matchings* [NC88] to the PQ-tree. A leaf labelled "$k+1$" is said to be *pertinent* in a PQ-tree corresponding to B_k. The *pertinent subtree* is the minimal subtree of a PQ-tree containing all the pertinent leaves. A node of a PQ-tree is said to be *full* if all the leaves among its descendants are pertinent.

Now we present a linear-time embedding algorithm EMBED [CNAO85]. The algorithm EMBED runs in two phases: in the first phase *procedure* **UPWARD-EMBED(G)** determines an upward embedding A_u of a planar graph G; in the second phase *procedure* **ENTIRE-EMBED** constructs an entire embedding Adj of G from A_u.

procedure UPWARD-EMBED(G);

{G is a given planar graph}

begin

Choose an edge (u, v) of the face which is to be embedded as outer face.

Considering u as s and v as t assign st-numbers to all the vertices of G;

Construct a PQ-tree corresponding to G'_1;

for $v := 2$ *to* n

do begin

{reduction step}

apply the template matchings to the PQ-tree, ignoring the direction indicators in it , so that the leaves labelled v occupy consecutive positions;

{vertex addition step}

let $l_1, l_2, ..., l_k$ be the leaves labelled v and direction indicators scanned in this order;

delete $l_1, l_2, ..., l_k$ from the PQ-tree and store them in $A_u(v)$;

if root r of the pertinent subtree is not full

then begin

add in indicator v, directed from l_k to l_1, to the PQ-tree as a son of root r at an arbitrary position among the sons;

replace all the full sons of r by a new P-node

end

else

replace the pertinent subtree by a new P-node;

add to the PQ-tree all the virtual edges adjacent to v as the sons of the P-node

end;

```
      {correction step}
      for v := n down to 1
        do for each element x in A_u(v)
          do if x is a direction indicator
            then begin
              delete x from A_u(v);
              let w be the label of x;
              if the direction of indicator x is
                opposite to that of A_u(v)
              then reverse list A_u(w);
            end
  end;
  procedure ENTIRE-EMBED;
  begin
    copy the upward embedding A_u to the lists Adj;
    mark every vertex "new";
    T := Φ;
    DFS(t);
  end;
procedure DFS(y);
  begin
    mark vertex y "old";
    for each vertex v ∈ A_u(y)
    do begin
      insert vertex y to the top of A_u(v);
```

```
if v is marked "new"
    then begin
        add edge (y, v) to T;
        DFS(v);
    end
  end
end;
```

2.3 Complexity of Algorithms

In this section we introduce some terminologies related to complexity of algorithms. For details we refer [CLR90].

The most widely accepted complexity measure for an algorithm is the *running time* which is expressed by the number of operations it performs before producing the final answer. The number of operations required by an algorithm is not the same for all problem instances. Thus, we consider all inputs of a given size together, and we define the complexity of the algorithm for that input size to be the worst case behavior of the algorithm on any of these inputs. Then running time is a function of size n of input.

Let $f(n)$ and $g(n)$ be the functions from the positive integers to the positive reals, then we write $f(n) = O(g(n))$ if there exists positive constants c_1 and c_2 such that $f(n) \leq c_1 g(n) + c_2$ for all n. Thus the running time of an algorithm may be bounded from above and for $f(n)$ it can be said that "it takes time $O(g(n))$".

An algorithm is said to be polynomial if its complexity is bounded by a polynomial of the size of a problem instance. Examples of such complexities are $O(nlog(n))$, $O(n^5)$, etc. The remaining algorithms are usually referred as *exponential* or *non-polynomial*. $O(2^n)$,

$O(n!)$, etc. are some examples of exponential or non-polynomial algorithms.

When the running time of an algorithm is bounded by $O(n)$, we call it a *linear-time* algorithm.

Chapter 3

Rectangular Drawings of Planar Graphs

In this chapter we consider rectangular drawings of planar graphs. Rectangular drawing of a planar graph G is a planar embedding where each vertex is drawn as a point, each edge is drawn as a horizontal or a vertical line segment without edge-crossings, and the contour of each face is drawn as a rectangle. Determining whether a planar graph has a rectangular drawing is not a trivial problem, since a planar graph may have an exponential number of planar embeddings. In Figure 3.1 three different planar embedings of the same planar graph are shown. Among the three planar embeddings only the embedding in Figure 3.1(a) has a rectangular drawing as illustrated in Figure 3.1(d).

In this chapter we establish a necessary and sufficient condition for the existence of a rectangular drawing of a planar graph G for the case where G is a subdivision of a 3-connected cubic planar graph. We also give a linear-time algorithm to determine whether G has a rectangular drawing or not and to find a rectangular drawing of G if it exists. To the best of our knowledge, no necessary and sufficient condition is known for planar graphs to have a rectangular drawing.

The rest of the chapter is organised as follows. Section 3.1 gives some definitions and presents preliminary results. Section 3.2 presents a characterization for subdivisions of a 3-connected cubic planar graph G to have rectangular drawings. Section 3.3 presents the algorithm to obtain the rectangular drawing of G if it exists. Finally, Section 3.4 concludes with discussions. Partial result of this thesis has been presented in [RGN00].

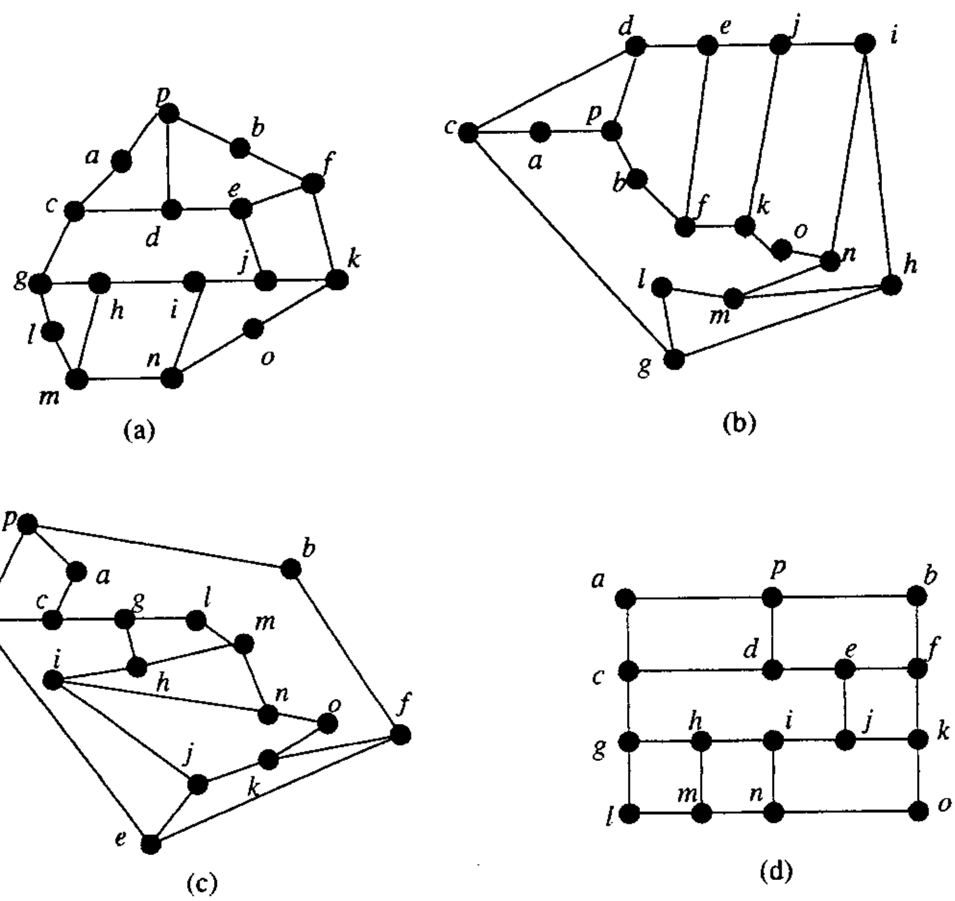


Figure 3.1: Three different planar embeddings of the same graph are shown in (a), (b), and (c); only (a) has a rectangular drawing as shown in (d).

3.1 Preliminaries

In this section we give some definitions and present preliminary results.

Let G be a connected simple graph. We denote the set of vertices of G by $V(G)$, and the set of edges of G by $E(G)$. For a subgraph G' of G we denote by $G - G'$ the subgraph of G induced by the vertices $V(G) - V(G')$.

For a simple cycle C in a plane graph G, we define the plane subgraph of G inside C (including C) as the *inner subgraph* $G_i(C)$ of C and the plane subgraph of G outside C (including C) as the *outer subgraph* $G_o(C)$ of C. An edge of G which is incident to exactly one vertex of a simple cycle C and located inside C is called an *inner adjacent edge* of C. An edge of G which is incident to exactly one vertex of a simple cycle C and located outside C is called a *leg* of the cycle. The vertex of C to which a leg is incident is called a *leg-vertex* of C. A simple cycle C in G is called a *k-legged cycle* of G if C has exactly k legs in G. A k-legged cycle C is a *minimal k-legged cycle* if $G_i(C)$ does not contain any other k-legged cycle of G. Similarly a k-legged cycle C is a *maximal k-legged cycle* if $G - G_i(C)$ contains a cycle and $G_i(C)$ is not contained in any other k-legged cycle C' of G such that $G - G_i(C')$ contains a cycle. We say that cycles C and C' in a plane graph G are independent if $G_i(C)$ and $G_i(C')$ have no common vertex. A set S of cycles is independent if any pair of cycles in S are independent.

Let $\{v_1, v_2, ..., v_{p-1}, v_p\}$, $p \geq 3$, be a set of three or more consecutive vertices on the contour of any cycle C such that the degrees of the first vertex v_1 and the last vertex v_p are exactly three and the degrees of all intermediate vertices $v_2, v_3, ..., v_{p-1}$ are two. Then we call the path induced by $\{v_2, v_3, ..., v_{p-1}\}$ a chain of G, and we call vertices v_1 and v_p the ends of the chain. A vertex of degree 2 in G is contained in exactly one chain.

Subdividing an edge (u, v) of a graph G is the operation of deleting the edge (u, v) and adding a path $u, w_1, w_2, ..., w_k, v$ through new vertices $w_1, w_2, ..., w_k$ of degree 2. A graph G' is said to be a *subdivision* of a graph G if G' is obtained from G by subdividing some of the edges of G.

A *rectangular drawing* of a plane graph G is a drawing of G such that each edge is drawn as a horizontal or a vertical line segment, and each face is drawn as a rectangle. A planar graph is said to have a rectangular drawing if it has a rectangular drawing for at least one of its planar embeddings. The following results on rectangular drawings of plane graphs are known.

Lemma 3.1.1 *[RNN00a] Let G be a plane graph with vertices of degree 2 or 3 such that four or more vertices on $C_o(G)$ have degree 2. Then G has a rectangular drawing if and only if G satisfies the following four conditions:*

(1) G has no 1-legged cycle;

(2) every 2-legged cycle in G contains at least two vertices of degree 2 on $C_o(G)$;

(3) every 3-legged cycle in G contains at least one vertex of degree 2 on $C_o(G)$; and

(4) if an independent set S of cycles in G consists of c_2 2-legged cycles and c_3 3-legged cycles, then $2c_2 + c_3 \leq 4$.

Furthermore one can check in linear time whether G satisfies the condition above, and if G does then one can find a rectangular drawing of G in linear time. ■

Although the results above for plane graphs are known, it is difficult to determine whether a planar graph has a rectangular drawing or not, since a planar graph has an exponential number of planar embeddings. We thus consider a class of planar graphs which are the subdivision of a 3-connected cubic planar graph G. The following properties for such a planar graph are known.

Lemma 3.1.2 *[NC88] Let G be a subdivision of a 3-connected cubic planar graph. Then there is exactly one embedding of G for each face embedded as the outer face. Furthermore,* *for any two planar embeddings Γ and Γ' of G, any facial cycle in Γ is embedded as a facial cycle in Γ' and vice-versa.* ■

Lemma 3.1.3 *[NC88] Let G be a subdivision of a 3-connected cubic planar graph. Then for any separation pair $\{u, v\}$ in G, at least one of the connected components of $G - \{u, v\}$ is a path.* ■

Let G be a subdivision of a 3-connected cubic planar graph. By Lemma 3.1.1 and Lemma 3.1.2 one can determine whether G has a rectangular drawing in $O(n^2)$ time. In the next section we obtain a necessary and sufficient condition for G to have a rectangular drawing. Our characterization leads to a linear-time algorithm to determine whether G has a rectangular drawing and to find a rectangular drawing of G if one exists.

Before going to the next section for giving the characterization, we observe the following properties of subdivisions of 3-connected cubic planar graphs which will be useful in establishing the characterization.

Lemma 3.1.4 *Let G be a subdivision of a 3-connected cubic planar graph. Then G has no 1-legged cycle in any planar embedding of G.*

Proof. Assume that C is a 1-legged cycle in G and that u is the only leg-vertex of C. Then removal of vertex u will make the graph disconneted which contradicts the assumption that G is a subdivision of a 3-connected cubic planar graph. $\mathcal{Q.E.D.}$

Lemma 3.1.5 *Let G be a subdivision of a 3-connected cubic planar graph and let Γ be a planar embedding of G. Then for any 2-legged cycle C in Γ, there is exactly one chain on $C_o(\Gamma)$ which is not in $G_i(C)$ of Γ. Furthermore $G_o(C) - C$ is a chain.*

Proof. If the legs of 2-legged cycle C is not on $C_o(\Gamma)$, then the leg-vertices v_1, v_2 of C is not in $C_o(\Gamma)$. Then $\{v_1, v_2\}$ would be a separation pair in a graph G' obtained from

G by replacing each chain with an edge, a contradiction to the assumption that G is a subdivision of a 3-connected cubic planar graph. Therefore the legs of C is on $C_o(\Gamma)$ and $G - G_i(C)$ is a chain. Clearly only that chain on $C_o(\Gamma)$ is not in $G_i(C)$. $\mathcal{Q.E.D.}$

Lemma 3.1.6 *Let G be a subdivision of a 3-connected cubic planar graph and let Γ be a planar embedding of G. Then for any chain V_c on $C_o(\Gamma)$, there is a 2-legged cycle C in Γ such that $G_i(C)$ contains all vertices of G except the vertices on V_c.*

Proof. Let $v_1, v_2, ..., v_{k-1}, v_k$, $k \geq 3$, be a path on $C_o(\Gamma)$ where $v_2, v_3, ..., v_{k-1}$ is a chain V_c. Then $C_o(G - V_c)$ is a cycle C in Γ where (v_1, v_2) and (v_{k-1}, v_k) are the only 2 legs of C. Therefore C is a 2-legged cycle in Γ and $G_i(C)$ contains all vertices of G except the vertices on V_c. $\mathcal{Q.E.D.}$

Lemma 3.1.7 *Let G be a subdivision of a 3-connected cubic planar graph and let Γ be a planar embedding of G. Then no pair of 2-legged cycles in Γ are independent.*

Proof. Assume that Γ has a pair of 2-legged cycles C_1 and C_2 which are independent. $G_i(C_1)$ and $G_i(C_2)$ has no common vertex. Since C_1 is a 2-legged cycle, the leg-vertices v_1 and v_2 of C_1 are on $C_o(\Gamma)$. Then $\{v_1, v_2\}$ would be a separation pair of a graph G' obtained from G by replacing each chain with an edge, a contradiction to the assumption that G is a subdivision of a 3-connected cubic planar graph. Therefore, no pair of 2-legged cycles in Γ are independent. $\mathcal{Q.E.D.}$

3.2 Characterization of Planar Graphs with Rectangular Drawings

In this section we give a necessary and sufficient condition for a planar graph G to have a rectangular drawing. We will show that the necessary and sufficient condition can be verified in linear time considering any planar embedding of G. To establish our necessary and sufficient condition we need the following lemmas.

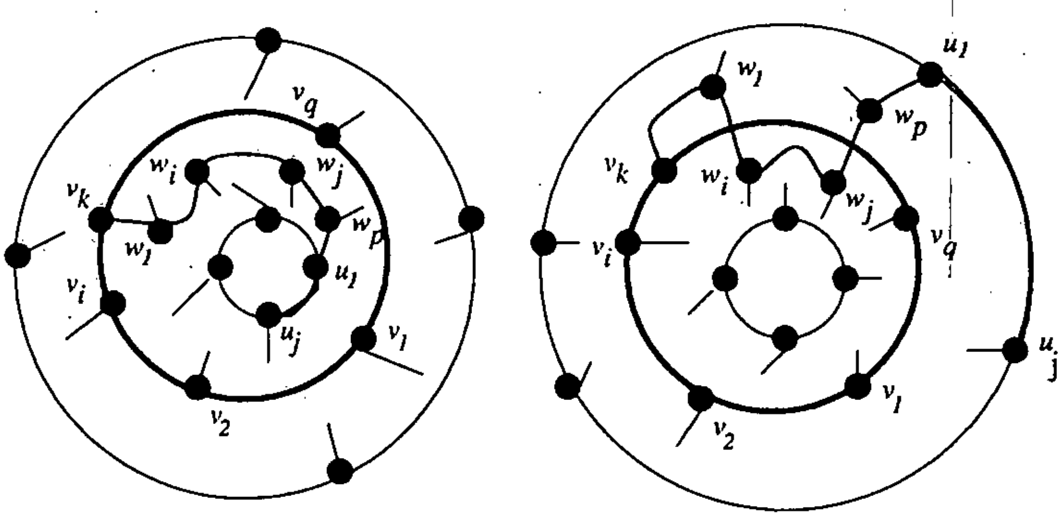
Lemma 3.2.1 *Let G be a subdivision of a 3-connected cubic planar graph and let C be a cycle in G. Assume that in a planar embedding Γ of G, $G_i(C)$ contains a face f and that f is made the outer face of a new planar embedding Γ' of G. Then $G_i(C)$ in Γ will be $G_o(C)$ in Γ' and vice-versa.*

Proof. We only show that $G_i(C)$ in Γ will be $G_o(C)$ in Γ', since the proof for the other claim is similar.

Assume that $u_1, u_2, ..., u_r$ ($r \geq 3$) be the vertices on the contour of face f and $v_1, v_2, ..., v_q$ ($q \geq 3$) be the vertices on the cycle C in a planar embedding Γ of G. There are the following three cases to consider.

Case 1: G has a path $u_l, w_1, w_2, ..., w_p, v_k$, where w_j, $1 \leq j \leq p$, is in $G_i(C)$ in Γ.

Since G is a connected graph, there is a path $(v_k, w_1, ..., w_i, w_j, ..., w_p, u_l)$ where no other verices except v_k and u_l are on the contour of face f or on the cycle C.



(a) (b)

Figure 3.2: Illustration of (a) Γ and (b) Γ' for Case 1 of the proof of Lemma 3.2.1

Let Γ' be a new planar embedding of G where f is embedded as the outer cycle. Since f is the outer cycle in Γ', u_l is in $G_o(C)$ in Γ'. If any of the vertices w_1, w_2, ..., w_p is in $G_i(C)$ in Γ' then the path $(v_k, w_1, ..., w_i, w_j, ..., w_p, u_l)$ will have edge crossing in Γ' as illustrated in Figure 3.2(b) since none of the vertices $w_1, w_2, ..., w_p$ are on C. This is a contradiction to our assumption that Γ' is a planar embedding of G.

Case 2: All the legs of cycle C_f are inner adjacent edges of cycle C in Γ of G.

Let C_f be a k-legged cycle in Γ where each of the legs of C_f is an inner adjacent edge of cycle C. Then there are k faces f_1, f_2, ..., f_k in $G_i(C)$ other than f as illustrated in Figure 3.3(a). The contour of each of these faces contains an edge on the contour of f, two legs of C_f and some edges of C.

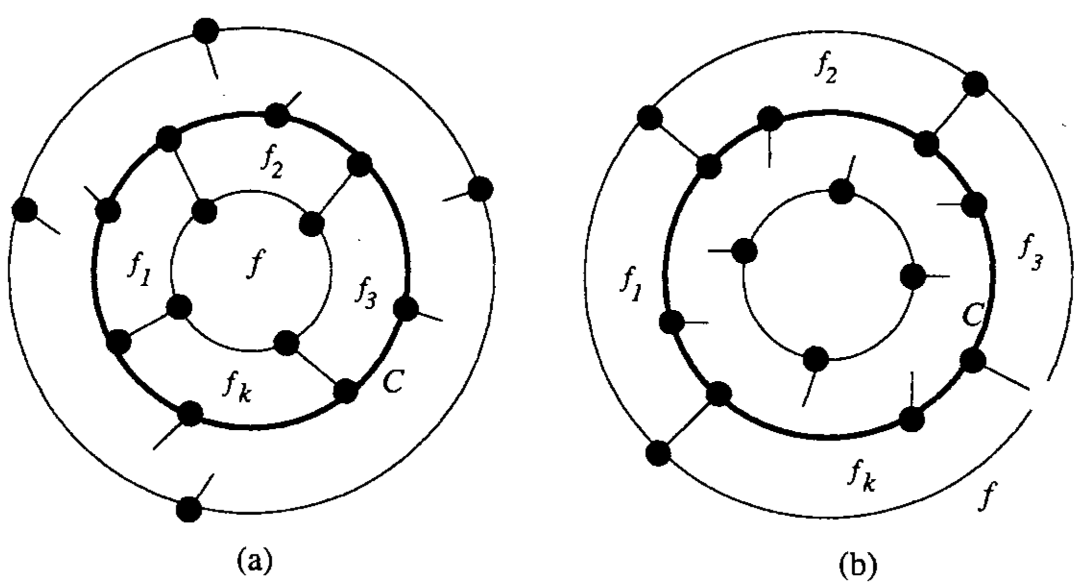


Figure 3.3: Illustration of (a) Γ and (b) Γ' for Case 2 of the proof of Lemma 3.2.1

Let Γ' be a new planar embedding of G where f is embedded as the outer cycle. Since f is the outer face in Γ', the edges on the contour of f is in $G_o(C)$. By Lemma 3.1.2 each face of Γ is a face of Γ' and hence in Γ' the contour of faces f_1, f_2, ..., f_k will be in $G_o(C)$ as illustrated in Figure 3.3(b). Thus $G_i(C)$ in Γ becomes $G_o(C)$ in Γ'.

Case 3: Some of the legs of cycle C_f are legs of cycle C and the rest of the legs of C_f are on C in Γ of G.

In this case $G_i(C)$ contains only one face f' other than f, and the contour of f' contains exactly one edge of the contour of f. This is illustrated in Figure 3.4(a).

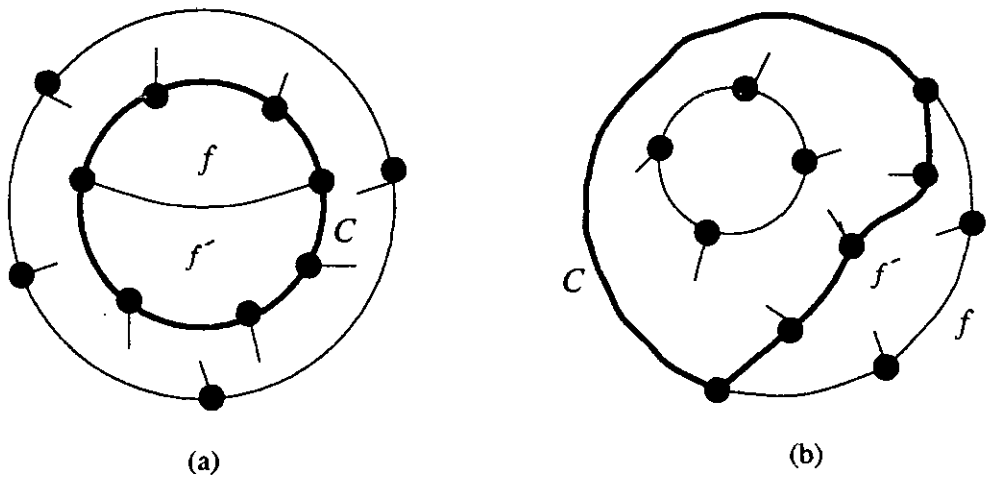


Figure 3.4: Illustration of (a) Γ and (b) Γ' for Case 3 of the proof of Lemma 3.2.1

Let Γ' be a new planar embedding of G where f is embedded as the outer cycle. By Lemma 3.1.2 each face of Γ is a face of Γ' and hence face f' in Γ will be a face f' in Γ' as illustrated in Figure 3.4(b). Clearly f' will be in $G_o(C)$ of Γ' as illustarted in Figure 3.4(b). Thus $G_i(C)$ in Γ becomes $G_o(C)$ in Γ'. $\mathcal{Q.E.D.}$

Lemma 3.2.2 *Let G be a subdivision of a 3-connected cubic planar graph. Let C be k-legged cycle in a planar embedding Γ of G. Assume that $G_i(C)$ in Γ contains a face f and that f is made outer face of a new planar embedding Γ' of G. Then legs of C in Γ will be inner adjacent edges of C in Γ' and inner adjacent edges of C in Γ will be legs of C in Γ'.*

Proof. Let edge (u, v) be a leg of cycle C in Γ where u is on C and v is in $G_o(C)$ of Γ. According to Lemma 3.2.1 v will be in $G_i(C)$ of Γ', but u remains on C. So clearly edge (u, v) will be inner adjacent edge of C in Γ'.

The proof for the other claim is similar. $\mathcal{Q.E.D.}$

Lemma 3.2.3 *Let G be a subdivision of a 3-connected cubic planar graph and let C be a k-legged cycle in a planar embedding Γ of G. Assume that there is a face f in Γ which is in $G_o(C)$ and that f is made outer face of a new planar embedding Γ' of G. Then $G_i(C)$ in Γ will also be $G_i(C)$ in Γ'.*

Proof. Assume that $u_1, u_2, ..., u_r$ ($r \geq 3$) be the vertices on the contour of face f and $v_1, v_2, ..., v_q$ ($q \geq 3$) be the vertices on the cycle C in a planar embedding Γ of G. Since G is connected, there is a path from any vertex on the contour of f to any vertex on C. Let P be a path from u_i to v_j which contains any vertex neither on the contour of face f nor on C except u_i and v_j. Since f is not in $G_i(C)$, P will be in both $G_o(C_f)$ and $G_o(C)$.

Let Γ' be a new planar embedding of G where f is embedded as the outer cycle. Then path P will be in $G_i(C_f)$ of Γ'. If any vertex on P is in $G_i(C)$ of Γ', then Γ would not be a planar embedding of G. Clearly P will be in $G_o(C)$ of Γ'. Therefore, $G_i(C)$ in Γ remains $G_i(C)$ in Γ' and hence legs of C will be unchanged in Γ'. $\mathcal{Q.E.D.}$

One can easily show that there exists a 3-legged cycle in any planar embedding of a subdivision of a 3-connected cubic planar graph [RNN00a]. We now have the following lemmas.

Lemma 3.2.4 *Let G be a subdivision of a 3-connected cubic planar graph and Γ be a planar embedding of G. Assume that $G - G_i(C)$ in Γ is a tree for any 3-legged cycle C. Then $G - G_i(C')$ is a tree for any 3-legged cycle C' in any planar embedding of G.*

Proof. Assume that $G - G_i(C)$ in Γ is a tree for any 3-legged cycle C in Γ. Let Γ' be a new planar embedding of G and $G - G_i(C')$ contains a cycle for a 3-legged cycle C' in Γ'. Then we have the following two cases to consider.

Case 1. $G_i(C')$ of Γ' contains $C_o(\Gamma)$.

C' in Γ' can be one of two types. C' does not contain any edge of $C_o(\Gamma')$ (see Fig. 3.5(a)) and C' contains at least an edge of $C_o(\Gamma')$ (see Fig. 3.5(b)).

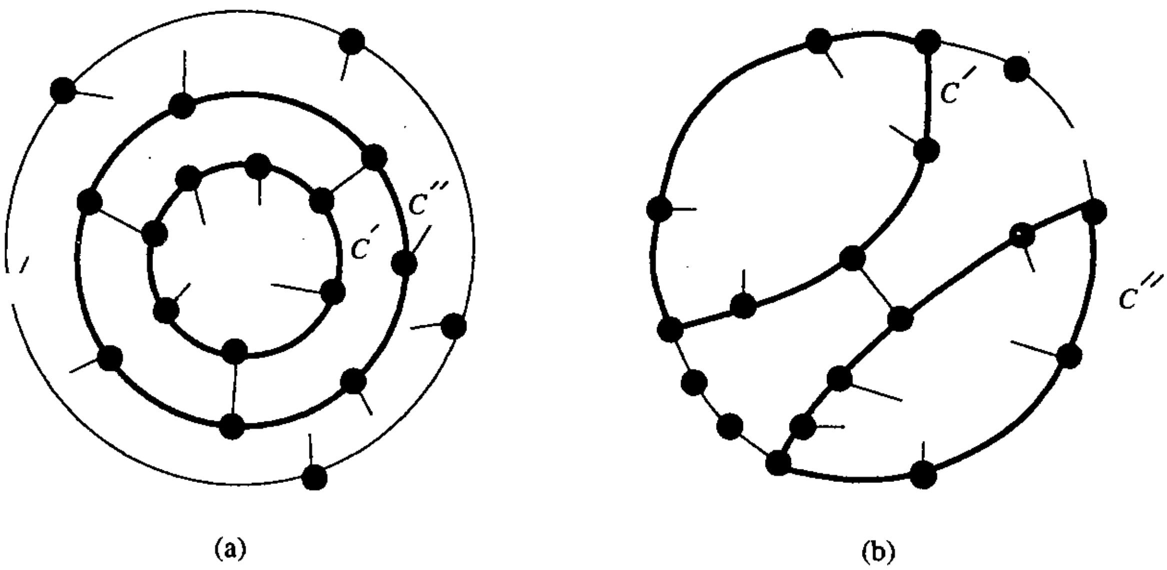


Figure 3.5: Illustration of C' and C'' in Γ' for the proof of Lemma 3.2.4

We first consider the case where C' does not contain any edge of $C_o(\Gamma')$. From the Fig. 3.5(a) one can observe that legs of C' are inner adjacent edges of another cycle C'' in Γ'. Let f be a face in $G_i(C')$ of Γ' which is the outer face of Γ. Then according to Lemma 3.2.1 C'' will be in $G_i(C')$ of Γ. Since f is also in $G_i(C'')$ of Γ', according to Lemma 3.2.2 inner adjacent edges of C'' in Γ' will be legs of C'' in Γ. Since C'' has three inner adjacent edges in Γ', C'' is a 3-legged cycle in Γ, and furthermore the cycle C' will be in $G_o(C'')$ in Γ by Lemma 3.2.1. Hence, $G - G_i(C'')$ is not a tree in Γ which is a contradiction to our assumption that $G - G_i(C)$ in Γ is a tree for any 3-legged cycle C in Γ.

We now consider the case where C' contains at least an edge on $C_o(\Gamma')$. In that case Γ' will have another 3-legged cycle C'' as illustrated in Fig. 3.5(b) and $G - G_i(C'')$ is not a tree in Γ'. Let f be a face in $G_i(C')$ of Γ' which is the outer face of Γ. Then clearly f is in $G_o(C'')$ of Γ'. According to Lemma 3.2.3 $G_i(C'')$ in Γ' will be $G_i(C'')$ in Γ. Since

$G_i(C'')$ is same in both embedding, $G - G_i(C'')$ is also same in both Γ and Γ'. Therefore, $G - G_i(C'')$ is not a tree in Γ which is a contradiction to our assumption that $G - G_i(C)$ in Γ is a tree for any 3-legged cycle C in Γ.

Case 2. $G_o(C')$ in Γ' contains $C_o(\Gamma)$.

Let f be a face in $G_o(C')$ of Γ' which is the outer face of Γ. According to Lemma 3.2.3 $G_i(C')$ in Γ' will be $G_i(C')$ in Γ. Since $G_i(C')$ is same in both embedding, $G - G_i(C')$ is also same in both Γ and Γ'. Therefore, $G - G_i(C')$ is not a tree in Γ which is a contradiction to our assumption that $G - G_i(C)$ in Γ is a tree for any 3-legged cycle C in Γ. $\mathcal{Q.E.D.}$

Theorem 3.2.5 *Let G be a subdivision of a 3-connected cubic planar graph and Γ be a planar embedding of G. Assume that $G - G_i(C)$ in Γ is a tree for any 3-legged cyle C. Then G has a rectangular drawing if and only if there is a face f in Γ such that facial cycle C_f of f satisfies the following conditions:*

(1) C_f conatins at least four vertices of degree 2;

(2) C_f has at least two chains;

(3) if C_f has exactly two chains, then they are non-consecutive and each chain contains at least two vertices of degree two.

Proof: Necessity

Assume that G has a rectangular drawing for its planar embedding λ. Let C be the outer facial cycle of λ.

Since G has a rectangular drawing and the four corner vertices of a rectangular drawing must be of degree 2, C contains at least four vertices of degree 2.

Since C conatins at least four vertices of degree 2, there are chains on C. According to Lemma 3.1.5 for any 2-legged cycle C' in λ, there is exactly one chain on $C_o(\lambda)$ which is not in $G_i(C')$ of λ. Since G has a rectangular drawing, every 2-legged cycle C' in λ contains at least two vertices of degree 2 on $C_o(\lambda)$. Therefore, C has at least two chains.

If C has exactly two chains, they must be non-consecutive; otherwise λ would have a 3-legged cycle containing no vertex of degree 2 on C as drawn by thick lines in Fig. 3.6, contrary to the assumption that G has a rectangular drawing for λ. According to Lemma 3.1.6 for any chain V_c on $C_o(\lambda)$, there is a 2-legged cycle C' in λ such that $G_i(C')$ contains all vertices of G except the vertices on V_c. Since G has a rectangular drawing, every 2-legged cycle C' in λ contains at least two vertices of degree 2 on $C_o(\lambda)$. Therefore, if C has exactly two chains, then each chain contains at least two vertices of degree two.

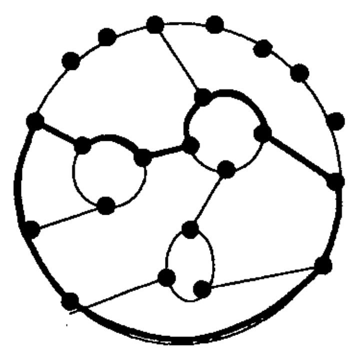


Figure 3.6: Exactly two consecutive chains on outer facial cycle.

Therefore, if G has a rectangular drawing for its planar embedding λ, then C satisfies the conditions of Theorem 3.2.5. According to Lemma 3.1.2, C is a facial cycle in any planar embedding of G. Therefore, if G has a rectangular drawing, then there is a face f such that C_f in Γ satisfies the conditions of Theorem 3.2.5.

Sufficiency

We give a constructive proof for the sufficiency of Theorem 3.2.5.

Assume that Γ has a face f such that C_f satisfies the conditions of Theorem 3.2.5.

We find a new embedding Γ' of G where C_f is embedded as the outer cycle. By Condition(1) in Theorem 3.2.5 C_f in Γ' contains at least four vertices of degree 2. It is sufficient to prove that Γ' satisfies the conditions in Lemma 3.1.1 for a rectangular drawing.

According to Lemma 3.1.4, there is no 1-legged cycle in Γ'. Thus Γ' satisfies the Condition (1) in Lemma 3.1.1.

We now show that Γ' satisfies Condititon (2) in Lemma 3.1.1. By Condition (2) in Theorem 3.2.5 C_f has at least two chains and hence have the following two cases to consider.

Case 1. C_f has exactly two chains.

According to Lemma 3.1.6 for any chain V_c on $C_o(\Gamma')$, there is a 2-legged cycle C in Γ' such that $G_i(C)$ contains all vertices of G except the vertices on V_c. Since C_f has exactly two chains, Γ' has exactly two 2-legged cycles. Again according to Condition (3) of Theorem 3.2.5 if C_f has exactly two chains, each of the chain contains at least two vertices of degree 2. Therefore, each 2-legged cycle contains at least two vertices of degree 2 on C_f.

Case 2. C_f has more than two chains.

According to Lemma 3.1.6 for any chain V_c on $C_o(\Gamma')$, there is a 2-legged cycle C in Γ' such that $G_i(C)$ contains all vertices of G except the vertices on V_c. Therefore, if number of chains on C_f is n ($n \geq 3$), then one can observe that each 2-legged cycle contains $n-1$ chains on C_f that means each 2-legged cycle contains at least two vertices of degree 2.

Thus Γ' satisfies Condition (2) of Lemma 3.1.1.

We next show that Γ' satisfies Condition (3) in Lemma 3.1.1. Since $G - G_i(C)$ in Γ is a tree for any 3-legged cyle C, according to Lemma 3.2.4 $G - G_i(C')$ in Γ' is a tree for any 3-legged cycle C' in Γ'. Let C be a 3-legged cycle in Γ'. One can observe that $G - G_i(C)$ in Γ' contains only one vertex of degree three on C_f and let r be the vertex. Let u, v be the leg vertices of C on C_f. Let P_1 be the path from u to r and P_2 be the path from v to

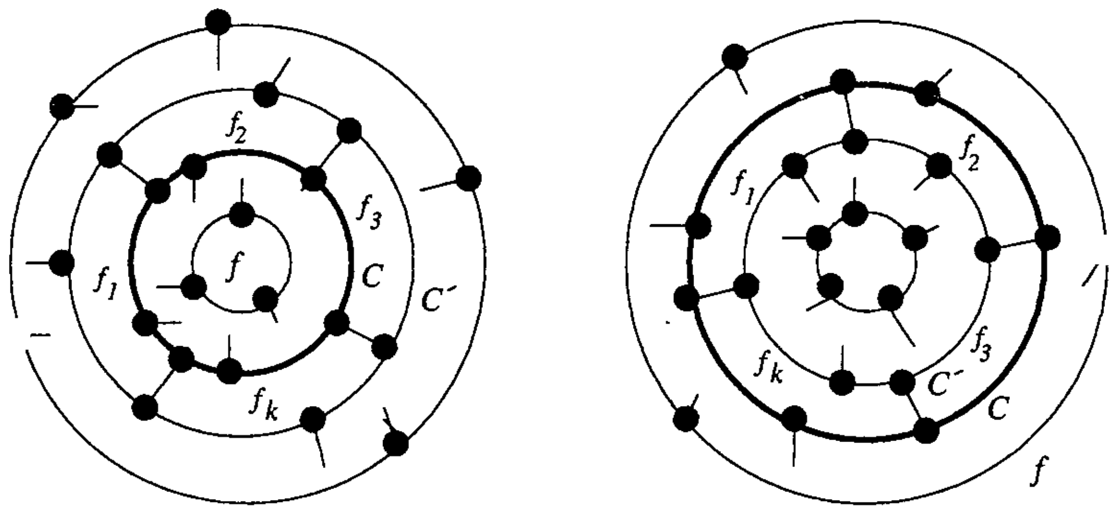
r on C_f. By Condition (2) in Theorem 3.2.5 C_f has at least two chains. If none of P_1 and P_2 contains chain, then C_f has at least two chains which is contained in C. If only one of P_1 and P_2 contains chain, then C_f has at least one chain which is contained in C. If both of P_1 and P_2 contain chain, then these two chains are consective. By Condition (3) in Theorem 3.2.5 if C_f has exactly two chains then they are non-consecutive. In such a case C_f has at least three chains at least one of which is contained in C. Therefore, each 3-legged cycle in Γ' contains at least one vertex of degree 2. Thus Γ' satisfies Condition (3) in Lemma 3.1.1.

We finally show that Γ' satisfies Condition (4) in Lemma 3.1.1. Since $G - G_i(C)$ in Γ is a tree for any 3-legged cyle C, according to Lemma 3.2.4 $G - G_i(C')$ in Γ' is a tree for any 3-legged cycle C' in Γ'. Let C and C' are two different 3-legged cycles in Γ'. Then each of $G - G_i(C)$ and $G - G_i(C')$ contains exactly one vertex of degree 3. Since G is a subdivision of 3-connected cubic planar graph, G has at least four vertices of degree 3, and hence $G_i(C)$ and $G_i(C')$ has common vertices. Therefore, there is no independent 3-legged cycle in Γ'. That is $c_3 = 0$. According to Lemma 3.1.7, neither Γ nor Γ' has a pair of independent 2-legged cycles, and hence $c_2 = 0$. Therefore $2c_2 + c_3 = 0$ that means $2c_2 + c_3 \leq 4$ in Γ'. Thus Γ' satisfies the condition (4) of Lemma 3.1.1. $\mathcal{Q.E.D.}$

From now on we assume that G has a 3-legged cycle C such that $G - G_i(C)$ contains a cycle.

Lemma 3.2.6 *Let G be a subdivision of a 3-connected cubic planar graph. Let C be a k-legged cycle in a planar embedding Γ of G such that C does not contain any edge on $C_o(\Gamma)$ and $G - G_i(C)$ contains a cycle. Assume that $G_i(C)$ in Γ contains a face f and that f is made the outer face of a new planar embedding Γ' of G. Then in Γ' the $G_i(C)$ contains a new k-legged cycle C'. Furthermore C' contains any edge neither on C nor on the contour of f.*

Proof. Let C be a k-legged cycle in Γ such that C does not contain any edge on $C_o(\Gamma)$ and $G - G_i(C)$ contains a cycle and let f be a face in $G_i(C)$ of Γ. Then there are k faces $f_1, f_2, ..., f_k$ in $G_o(C)$ containing edges of C. The edges of the face boundaries of $f_1, f_2, ..., f_k$ each of which is neither on C nor a leg of C forms a cycle C'. Clearly f is in $G_i(C')$.



(a) (b)

Figure 3.7: Illustration of (a) Γ and (b) Γ' for the proof of Lemma 3.2.6

Let Γ' be a new planar embedding of G where f is embedded as the outer cycle. By Lemma 3.1.2 each faces of Γ is a face of Γ' and by Lemma 3.2.1 $G_o(C)$ in Γ is $G_i(C)$ in Γ'. Then C' is a k-legged cycle in Γ' as illustrated in Figure 3.7(b). Since f is also in $G_i(C')$ of Γ, according to Lemma 3.2.2 legs of C' in γ are inner adjacent edges of C' in Γ'. So C' is a new k-legged cycle in the sense that legs of C' in Γ' are different from the legs of C' in Γ. One can observe that C' contains any edge neither on C nor on the boundary of f. $\mathcal{Q.E.D.}$

Lemma 3.2.7 *Let G be a subdivision of a 3-connected cubic planar graph. Let C be a k-legged cycle ($k \geq 3$) in a planar embedding Γ of G such that $G - G_i(C)$ contains a cycle and C contains at least an edge on $C_o(\Gamma)$. Assume that $G_i(C)$ in Γ contains a face f and that f is made the outer face of a new planar embedding Γ' of G. Then in Γ' the*

$G_i(C)$ contains a k-legged cycle C' which was also a k-legged cycle in Γ. Furthermore C' contains any edge neither on C nor on the contour of f.

Proof. Let C be a k-legged cycle ($k \geq 3$) in Γ such that $G - G_i(C)$ contains a cycle and C contains at least an edge on $C_o(\Gamma)$ and let f be a face in $G_i(C)$ of Γ. Then there are k faces $f_1, f_2, ..., f_k$ in $G_o(C)$ containing edges of C. Among these faces $k-1$ faces $f_1, f_2, ..., f_{k-1}$ are inner adjacent faces of Γ and the remaining face f_k is the outer face of G in Γ as illustrated in Figure 3.8(a). The edges of the face boundaries of $f_1, f_2, ..., f_k$ each of which is neither on C nor a leg of C forms a k-legged cycle C' as illustrated by shaded cycle in Figure 3.8(a). Clearly C' is in $G_o(C)$.

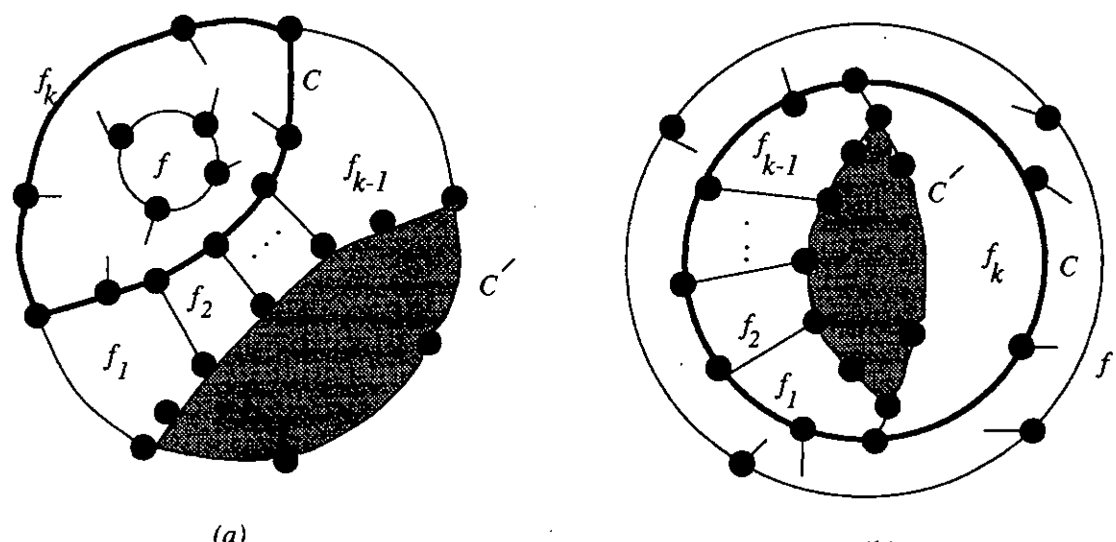


Figure 3.8: Illustration of (a) Γ and (b) Γ' for the proof of Lemma 3.2.7

Let Γ' be a new planar embedding of G where f is embedded as the outer cycle. By Lemma 3.1.2 each faces of Γ is a face of Γ' and by Lemma 3.2.1 $G_o(C)$ in Γ is $G_i(C)$ in Γ'. Then C' is in $G_i(C)$ of Γ'. According to Lemma 3.2.2 legs of C in Γ will be inner adjacent edges of C in Γ'. Clearly these inner adjacent edges are connected with the vertices on C' in Γ'. According to Lemma 3.2.3, $G_i(C')$ in Γ remains $G_i(C')$ in Γ'. Thus the legs of C' in Γ remain the legs of C' in Γ' as illustrated in Figure 3.8(b). So $G_i(C)$ in Γ' contains a k-legged cycle C' of Γ. One can observe that C' contains any edge neither on C nor on the boundary of f. $\mathcal{Q.E.D.}$

We now have the following lemmas on the forbidden structures in a subdivision of a 3-connected cubic planar graph to have rectangular drawing.

Lemma 3.2.8 *Let G be a subdivision of a 3-connected cubic planar graph and let C be a 3-legged cycle in a planar embediing Γ of G such that $G - G_i(C)$ contains a cycle. Assume that $G_i(C)$ in Γ contains a face F and that F is made the outer face of a new planar embedding Γ' of G. Then G has no rectangular drawing for the planar embedding Γ'.*

Proof. Let Γ' be a new planar embedding of the graph G where F is the outer face of the graph G. If C does not contain any edge of $C_o(\Gamma)$, then according to Lemma 3.2.6 $G_i(C)$ contains a new 3-legged cycle in Γ'. If C contains at least an edge of $C_o(\Gamma)$, then according to Lemma 3.2.7 $G_i(C)$ contains a 3-legged cycle of Γ in Γ'. In either case, as the contour of F has no common edge with this 3-legged cycle, rectangular drawing of G is not possible for Γ' according to Lemma 3.1.1. $\mathcal{Q.E.D.}$

Lemma 3.2.9 *Let G be a subdivision of a 3-connected cubic planar graph and let Γ is a planar embedding of G and C be a 3-legged cycle in Γ such that $G - G_i(C)$ contains a cycle. If there is no face in Γ containing at least one edge of each of the maximal 3-legged cycle of G, then G has no rectangular drawing.*

Proof. Let C be a 3-legged cycle in Γ such that $G - G_i(C)$ conatins a cycle and f is a face in $G_i(C)$. Let Γ' be the new planar embedding of G where f is made outer face of G. Then according to Lemma 3.2.8 G has no rectangular drawing for the embedding Γ'. Therefore only the planar embeddings of G where each face outside of all maximal 3-legged cycle is embedded as an outer face may have rectangular drawings.

Assume that f is a face in Γ which is outside of all maximal 3-legged cycle but does not contain an edge from each of the maximal 3-legged cycle of G. If f is made outer face of G then according to Lemma 3.2.3 for any 3-legged cycle C, $G_i(C)$ in Γ will also be $G_i(C)$ in Γ'. Therefore the maximal 3-legged cycle having no edge on the contour of f will not satisfy Condition (3) of Lemma 3.1.1, and hence G will not have a rectangular drawing for this planar embedding. Therefore, if G has no face containing an edge of each maximal 3-legged cycle, G has no rectangular drawing. *Q.E.D.*

We are now ready to prove the following lemma on characterization of subdivisions of 3-connected cubic planar graphs to have rectangular drawing.

Theorem 3.2.10 *Let G be a subdivision of a 3-connected cubic planar graph and Γ be a planar embedding of G. Assume that there exists a 3-legged cycle C' in Γ such that $G - G_i(C')$ contains a cycle. Then G has a rectangular drawing if and only if Γ has a facial cycle C such that C contains at least one edge of each maximal 3-legged cycle in Γ and the following four conditions hold in Γ:*

(1) C contains at least four vertices of degree 2;

(2) for any chain V_c on C, $C - V_c$ must contain at least two vertices of degree 2;

(3) every 3-legged cycle C' for which $G - G_i(C')$ contains a cycle in Γ contains at least one vertex of degree 2 on C;

(4) if Γ has exactly one maximal 3-legged cycle C' then C contains a chain V_c such that V_c is not on C' and does not contain any vertex at which legs of C' are incident; and

(5) if an independent set S of cycles in Γ consists of c_3 3-legged cycles, then $c_3 \leq 4$.

Proof: Necessity

Assume that G has a rectangular drawing for its planar embedding λ. Let C be the outer cycle of λ. Then there are the following two cases to consider.

Case 1: C and $C_o(\Gamma)$ are same.

In this case, by Lemma 3.1.2 λ and Γ are the same planar embedding of G. Since λ has a rectangular drawing, λ satisfies the conditions in Lemma 3.1.1. Therefore C in Γ contains at least an edge of each maximal 3-legged cycle and satisfies the following conditions :

(i) C contains at least four vertices of degree 2. Thus Γ satisfies the condition (1) of Theorem 3.2.10.

(ii) Every 2-legged cycle in Γ contains at least two vertices of degree 2 on C. According to Lemma 3.1.6, for any chain V_c on $C_o(\Gamma)$, there is a 2-legged cycle C' in Γ such that $G_i(C')$ contains all vertices of G except the vertices on V_c. Therefore for any chain V_c, $C - V_c$ must contain at least two vertices of degree 2. Thus Γ satisfies the condition (2) of Theorem 3.2.10.

(iii) Every 3-legged cycle in Γ contains at least one vertex of degree 2 on C, that means every 3-legged cycle C' for which $G - G_i(C')$ contains a cycle in Γ contains at least one vertex of degree 2 on C. Thus Γ satisfies the condition (3) of Theorem 3.2.10.

(iv) Since Γ has a rectangular drawing, Γ contains more than one maximal 3-legged cycle. Thus Γ satisfies the condition (4) of Theorem 3.2.10.

(v) If an independent set S of cycles in Γ consists of c_2 2-legged cycles and c_3 3-legged cycles, then $2c_2 + c_3 \leq 4$. According to Lemma 3.1.7, no pair of 2-legged cycles are independent in Γ. Therfore $c_2 = 0$, and hence $c_3 \leq 4$. Thus Γ satisfies the condition (5) of Theorem 3.2.10.

Case 2: C and $C_o(\Gamma)$ are different.

According to Lemma 3.1.2, C is an inner facial cycle of Γ. Again according to Lemma 3.2.8, if C is inside a 3-legged cycle C' in Γ such that $G - G_i(C')$ contains a cycle in Γ then G has no rectangular darwing for its planar embedding λ. Since λ has a rectangular drawing, C is not inside any 3-legged cycle C' in Γ such that $G - G_i(C')$ contains a cycle in Γ. Since all the 3-legged cycles C' in Γ for which $G - G_i(C')$ contains a cycle in Γ are outside C, according to Lemma 3.2.3 for any 3-legged cycle C' in Γ for which $G - G_i(C')$ contains a cycle in Γ, $G_i(C')$ in λ will also be $G_i(C')$ in Γ. Since λ has a rectangular drawing, according to Lemma 3.1.1 C in λ contains at least an edge of each maximal 3-legged cycles. Therfore C in Γ contains at least an edge of each maximal 3-legged cycles and also satisfies the conditions of Theorem 3.2.10 which can be shown in the following way:

(i) Since C in λ contains at least four vertices of degree 2, C in Γ contains at least four vertices of degree 2. Thus Γ satisfies the condition (1) of Theorem 3.2.10.

(ii) Every 2-legged cycle in λ contains at least two vertices of degree 2 on C. According to Lemma 3.1.6, for any chain V_c on $C_o(\lambda)$, there is a 2-legged cycle C' in λ such that $G_i(C')$ contains all vertices of G except the vertices on V_c. Therefore for any chain V_c on $C_o(\lambda)$, $C - V_c$ must contain at least two vertices of degree 2 in λ. Hence, for any chain V_c on C in Γ, $C - V_c$ must contain at least two vertices of degree 2 in Γ. Thus Γ satisfies the condition (2) of Theorem 3.2.10.

(iii) Every 3-legged cycle in λ contains at least one vertex of degree 2 on C. And according to Lemma 3.2.3 for any 3-legged cycle C' in λ for which $G - G_i(C')$ contains a cycle in λ, $G_i(C')$ in λ will also be $G_i(C')$ in Γ. Therefore, every 3-legged cycle in Γ for which $G - G_i(C')$ contains a cycle in Γ contains at least one vertex of degree 2 on C. Thus Γ satisfies the condition (3) of Theorem 3.2.10.

(iv) We have to consider two cases.

Case 1: If Γ has more than one maximal 3-legged cycle.

Since Γ does not contain exactly one maximal 3-legged cycle, Γ satisfies the condition (4) of Theorem 3.2.10.

Case 2: If Γ has exactly one maximal 3-legged cycle.

Let C' be a 3-legged cycle in λ such that $G - G_i(C')$ contains a cycle in λ. Since C' contains at least an edge on C in λ, λ contains another 3-legged cycle C'' and $G - G_i(C'')$ contains C'. Let Γ be a planar embedding where a face f in $G_i(C')$ of λ is embedded as outer face. According to Lemma 3.2.2, legs of C' in λ will be inner adjacent edges of C' in Γ. Therefore C' will be no longer a 3-legged cycle in Γ. According to Lemma 3.2.3 $G_i(C'')$ in λ will be $G_i(C'')$ in Γ. In such a case Γ has exactly one maximal 3-legged cycle. Since every 3-legged cycle in λ contains at least one vertex of degree 2 on C, C' has a chain V_c on C in λ which exists as a chain on C in Γ also. One can observe that V_c in Γ is not on C'' and does not contain any vertex on which legs of C' are incident. Thus Γ satisfies the condition (4) of Theorem 3.2.10.

(v) If an independent set S of cycles in λ consists of c_2 2-legged cycles and c_3 3-legged cycles, then $2c_2 + c_3 \leq 4$. According to Lemma 3.1.7, no pair of 2-legged cycles are independent neither in λ nor in Γ. Therfore both in λ and Γ, $c_2 = 0$, and hence $c_3 \leq 4$. Thus Γ satisfies the condition (5) of Theorem 3.2.10.

Sufficiency

We give a constructive proof for the sufficiency of Theorem 3.2.10.

Assume that Γ has a facial cycle C such that C contains at least an edge of each maximal 3-legged cycle and Γ satisfies the condition of Theorem 3.2.10.

We find a new embedding Γ' of G where C is embedded as the outer cycle. Then C in Γ' contains at least four vertices of degree 2.

According to Lemma 3.1.4, there is no 1-legged cycle in Γ'. Thus Γ' satisfies the condition (1) of Lemma 3.1.1.

Since in Γ for any chain V_c on C, $C - V_c$ contains at least two vertices of degree 2, in Γ' for those chain V_c on C, $C - V_c$ contains at least two vertices of degree 2. According to Lemma 3.1.5, for any 2-legged cycle C' in Γ', there is exactly one chain on $C_o(\Gamma')$ which is not in $G_i(C')$ of Γ' and $G_o(C') - C$ is a chain. Therfore every 2-legged cycle in Γ' contains at least two vertices of degree 2. Thus Γ' satisfies the condition (2) of Lemma 3.1.1.

Now we show that Γ' satisfies the condition (3) of Lemma 3.1.1. For this we have to consider the following two cases.

Case 1: If Γ has more than one maximal 3-legged cycle.

According to Lemma 3.2.3, for any 3-legged cycle C' in Γ for which $G - G_i(C')$ contains a cycle, $G_i(C')$ in Γ will also be $G_i(C')$ in Γ'. Therefore, every 3-legged cycle C' in Γ' for which $G - G_i(C')$ contains a cycle contains at least a vertex of degree 2 on C. It can be easily shown that every 3-legged cycle C'' in Γ' for which $G - G_i(C'')$ is a tree contains at least a vertex of degree 2 on C. Therefore each 3-legged cycle contains at least one vertex of degree 2 on C in Γ'. Thus Γ' satisfies the condition (3) of Lemma 3.1.1.

Case 2: If Γ has exactly one maximal 3-legged cycle.

Let C' be the maximal 3-legged cycle in Γ. According to Lemma 3.2.3, $G_i(C')$ in Γ will also be $G_i(C')$ in Γ'. Γ' also contains another maximal 3-legged cycle C'' as illustrated in Fig 3.9. Since there is a chain V_c on C in Γ such that V_c is not on C' and does not contain any vertex at which legs of C' are incident, C'' contains V_c on C in Γ'. Therefore each 3-legged cycle contains at least one vertex of degree 2 on C in Γ'. Thus Γ' satisfies the condition (3) of Lemma 3.1.1.

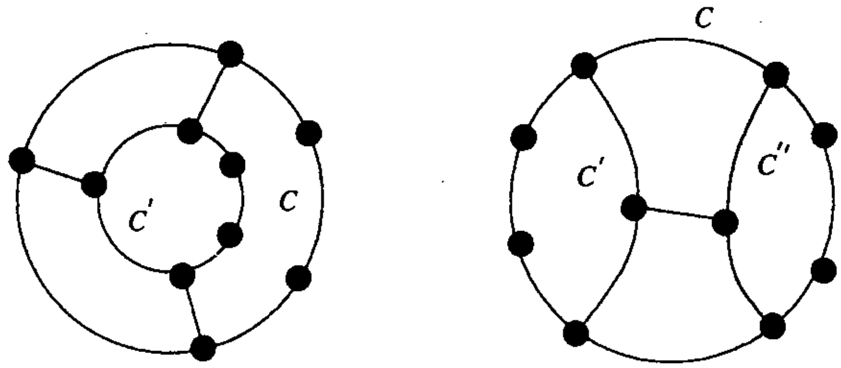


Figure 3.9: Illustration of C'' for the proof of sufficiency of Theorem 3.2.10

If an independent set S of cycles in Γ consists of c_3 3-legged cycles, then $c_3 \leq 4$. According to Lemma 3.1.7, neither Γ nor Γ' has independent 2-legged cycle. Therfore $c_2 = 0$ that means $2c_2 + c_3 \leq 4$ in Γ'. Thus Γ' satisfies the condition (5) of Lemma 3.1.1.

Since Γ' satisfies the conditons of Lemma 3.1.1, Γ' has a rectangular drawing D. Hence D is the rectangular drawing of G.

Lemma 3.2.11 *Let G be a subdivision of a 3-connected cubic planar graph. At most three faces of G can contain an edge of each maximal 3-legged cycle.*

Proof. Let C be a maximal 3-legged cycle. Then there are exactly three faces in $G_o(C)$ which contains an edge of C. Hence, only these three faces may contain an edge of each of the maximal 3-legged cycle, since such a face must contain an edge on C and will be in $G_o(C)$ to contain edges of other maximal 3-legged cycles. $\mathcal{Q.E.D.}$

Corollary 3.2.12 *Let G be a subdivision of a 3-connected cubic planar graph and Γ be a planar embedding of G. Assume that C is a maximal 3-legged cycle in Γ and that f_1, f_2 and f_3 are three faces outside C containing edges on C. Then G has a rectangular* *drawing if and only if any of* Γ_1, Γ_2 *and* Γ_3 *has a rectangular drawing where* Γ_1, Γ_2 *and* Γ_3 *respectively are the planar embeddings of* G *taking* f_1, f_2 *and* f_3 *as the outer face.*

Theorem 3.2.13 *Let* G *be a subdivision of a 3-connected cubic planar graph. One can determine in linear time whether* G *has a rectangular drawing or not.*

Proof. Let Γ be a planar embedding of G. One can identify all the maximal 3-legged cycle in Γ in linear time [RNN00b]. If there is no maximal 3-legged cycle then one can determine in linear time whether there is a face in Γ that satisfies the conditions of Theorem 3.2.5 and G has a rectangular drawing if such a face exists in Γ. Otherwise, according to Lemma 3.2.11 there are only 3 faces f_1, f_2 and f_3 in $G_o(C_m)$ each of which contain edges of a maximal 3-legged cycle C_m. One can check in linear time whether Γ satisfies the conditions of Theorem 3.2.10 considering each facial cycle corresponding to face f_1, f_2 and f_3. If any facial cycle coreesponding to face f_1, f_2 and f_3 satisfies the conditions of Theorem 3.2.10, then G has a rectangular drawing. Therefore, one can determine in linear time whether G has a rectangular drawing or not. $\mathcal{Q.E.D.}$

3.3 Rectangular Drawing Algorithm

In the previous sections we have established a necessary and sufficient condition for a subdivision of a 3-connected graph to have rectangular drawing. We have also shown that this necessary and sufficient condition can be checked in linear time. Now we will describe our algorithm to determine a rectangular drawing of a subdivision of a 3-connected cubic planar graph if it exists. This algorithm also checks whether a subdivision of a 3-connected cubic planar graph has a rectangular drawing or not.

Algorithm Planar-Rectangular-Draw(G)

begin

Let the given planar emebedding of graph G is Γ;

Identify all the maximal 3-legged cycles C in Γ;

if no maximal 3-legged cycle exists *then*

begin

Find a face f in Γ which satisfies Theorem 3.2.5.

if no such face exists *then*

G has no rectangular drawing;

else if a face f in Γ satisfies Theorem 3.2.5 *then*

begin

Determine the planar embedding Γ' of G where f is embedded as the outer face;

Find Γ'' by removing all inner vertices of degree 2 from Γ';

Determine rectangular drawing D' of Γ'';

Determine rectangular drawing D of Γ' by inserting all removed vertices of degree 2 in D'. (D is the rectangular drawing of G.)

end

end

else {maximal 3-legged cycle exists}

begin

let C_m be one of the maximal 3-legged cycles;

Find the three faces f_1, f_2 and f_3 in $G_o(C_m)$ which contains edges of C_m;

Considering each of f_1, f_2, and f_3 in Γ as C check whether C satisfies the condition of Theorem 3.2.10;

if C satisfies the condition of Theorem 3.2.10 *then*

begin

```
14      Determine the planar embedding Γ' of G where C is embedded as outer face;
15      Find Γ'' by removing all inner vertices of degree 2 from Γ';
16      Determine rectangular drawing D' of Γ'';
17      Determine rectangular drawing D of Γ' by inserting all removed vertices
          of degree 2 in D'. ( D is the rectangular drawing of G. )
      end
      else
          G has no rectangular drawing;
    end;
  end.
```

Theorem 3.3.1 *Algorithm Planar-Rectangular-Draw finds a rectangular drawing of a subdivision of a 3-connected cubic planar graph G in linear time if it exists.*

Proof. Using a method similar to the method in [RNN00b], one can find all 3-legged cycles in G in linear time in Step 1 of the algorithm. If there is no maximal 3-legged cycle then the algorithm executes Step 3 to Step 9, otherwise the algorithm executes Step 11 to Step 17. Step 3 takes linear time. Finding a planar embedding Γ' of G where a face f_i is embedded as outer face in Step 6 and in Step 14 takes linear time. Γ'' in Step 7 and in Step 15 can be found in linear time. By Lemma 3.1.1 rectangular drawing D' of Γ'' in Step 8 and in Step 16 can be obtained in linear time. In Step 9 and in Step 17, all the inner vertices of degree 2 can be inserted in linear time. Therefore, the overall time complexity of the algorithm **Planar-Rectangular-Draw** is linear.

3.4 Conclusions

In this chapter we have established a necessary and sufficient condition for the existence of a rectangular drawing of a planar graph G where G is a subdivision of a 3-connected cubic planar graph. We also show that it is possible to determine in linear time whether G has a rectangular drawing and find a rectangular drawing of G if it exists.

Chapter 4

Conclusions

This thesis deals with the characterization of subdivisions of 3-connected cubic planar graphs to have rectangular drawings. We have established a necessary and sufficient condition for subdivisions of a 3-connected cubic planar graph to have a rectangular drawing. We have also presented a linear-time algorithm to determine whether a subdivision of a 3-connected cubic planar graph has a rectangular drawing or not, and obtain such a drawing of that graph, if it exists.

We first summarize each chapter and its contributions. In Chapter 1 we have introduced different drawing conventions, different aspects of graph drawings and also described the objectives of this work.

In Chapter 2 we have introduced some basic terminologies related to graphs and algorithm theory which we have used in our work.

In Chapter 3 we have established a necessary and sufficient condition for a subdivision of a 3-connected cubic planar graph to have a rectangular drawing. We have also given a constructive proof which immediately produces a very simple linear-time algorithm to obtain a rectangular drawing of such a graph, if it exists.

As mentioned earlier, this thesis deals with rectangular drawings of planar graphs.

However, the following problems remained as future works:

1. To establish a necessary and sufficient condition for a biconnected planar graph to have a rectangular drawing.

2. To obtain an efficient algorithm to have a rectangular drawing of a biconnected planar graph, if it exists.

References

[BS88] J. Bhasker and S. Sahni, *A linear algorithm to find a rectangular dual of a planar triangulated graph,* Algorithmica, 3, pp.247-278, 1988.

[BK94] T. C. Biedl and G. Kant, *A better heuristic for orthogonal graph drawings,* Proc. of 2nd European Symposium of Algorithms, Lecture Notes in Computer Science, 885, pp.37-52, 1994.

[BL76] K. S. Booth and G. S. Lueker, *Testing the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms,* J. Comput. Syst. Sci., 13, pp.335-379, 1976.

[CNAO85] N. Chiba, T. Nishizeki, S. Abe and T. Ozawa, *A linear algorithm for embedding planar graphs using PQ-trees,* J. Comput. Syst. Sci., 30, pp.54-76, 1985.

[CON85] N. Chiba, K. Onoguchi and T. Nishizeki, *Drawing planar graphs nicely,* Acta Informatica, 22, pp. 187-201, 1985.

[CN98] M. Chrobak and S. Nakano, *Minimum-width grid drawings of plane graphs,* Comp. Geom. Theory Appl., 11, pp.29-54, 1998.

[CLR90] T. H. Cormen, C. E. Lieserson and R. L. Rivest, *Introduction to Algorithms,* McGraw Hill, 1990.

[DETT94] G. Di Battista, P. Eades, R. Tamassia and I.G. Tollis. *Algorithms for drawing graphs: an annotated bibliography*, Comp Geom. Theoy Appl., 4, pp.235-282, 1994.

[DETT99] G. Di Battista, P. Eades, R. Tamassia and I.G. Tollis. *Graph drawing*, Prentice Hall, Upper Saddle River, NJ, 1999.

[ET76] S. Even and R. E. Tarjan, *Computing an st-numbering*, Theor. Comput. Sci., 2, pp.339-404, 1976.

[F48] I. Fary, *On straight line representation of planar graphs*, Acta. Sci. Math Szeged, 11, pp.229-233, 1948.

[FPP90] H. de. Fraysseix, J. Pach, and R. Polack, *How to draw a planar graph on a grid*, Combinatorica, 10, pp.41-51, 1990.

[GJ83] M. R. Garey and D. S. Johnson, *Crossing number is NP-complete*, SIAM J.Alg. Disc. Methods, 4(3), pp.312-316, 1983.

[GT95] A. Garg and R. Tamassia, *On the computational complexity of upward and rectilinear planarity testing*, Proc. of Graph Drawing'94, Lecture notes in Computer Science, 894, pp.99-110, 1995.

[GT97] A. Garg and R. Tamassia, *A new minimum cost flow algorithm with application to graph drawing*, Proc. of Graph Drawing'96, Lecture notes in Computer Science, 1190, pp.201-226, 1997.

[H93] X. He, *On finding the rectangular duals of planar triangulated graphs*, SIAM J. Comput., 22(6), pp.1218-1226, 1993.

[H95] X. He, *An efficient parallel algorithm for finding rectangular duals of plane triangulated graphs*, Algorithmica, 13, pp.553-572, 1995.

[H00] X. He, *A simple linear time algorithm for proper box rectangular drawings of plane graphs*, Department of Computer Science and Engineering, State University of New York at Buffalo, 2000, (Manuscript).

[HT74] J. E. Hopcroft and R. E. Tarjan, *Efficient planarity testing*, J. Assoc. Comput. Mach., 21, pp.549-568, 1974.

[KH97] G. Kant and X. He, *Regular edge labeling of 4-connected plane graphs and its applications in graph drawing problems*, Theoretical Computer Science, 172, pp.175-193, 1997.

[KK84] K. Kozminski and E. Kinnen, *An algorithm for finding a rectangular dual of a planar graph for use in area planning for VLSI integrated circuits*, Proc.21st DAC, Albuquerque, pp.655-656, 1984.

[KL84] M. R. Kramer and J. van Leeuwen, *The complexity of wire routing and finding minimum area layouts for VLSI circuits*, (Eds.) F. P. Preparata, Advances in Computing Research, vol. 2: VLSI Theory, JAI press, Reading, MA, pp.129-146, 1984.

[K30] C. Kuratowski, *Surle probleme des corbes gauches en topologie*, Fundamental math., 15, pp.271-283, 1930.

[L90] T. Lengauer, *Combinatirial Algorithms for Integrated Circuit Layout*, John Wiley & Sons, Chichester, 1990.

[MKI00] S. Munemoto, N. Katoh and G. Imamura, *Finding an optimal floor layout based on an orthogonal graph drawing algorithm*, J. Archit. Plann. Enviroment Eng. AIJ, No. 524, pp.279-286, 2000.

[M92] P. Mutzel, *A first linear time embedding algorithm based on the Hopcroft-Tarjan planarity test*, Tech. Report, 1992.

[NC88] T. Nishizeki and N. Chiba, *Planar graphs: theory and algorithms*, North-Holland, Amsterdam, 1988.

[PT98] A. Papakostas and I. G. Tollis, *Orthogonal drawings of high degree graphs with small area and few bends*, Proc. of 5th Workshop on Algorithms and Data Structures, Lecture Notes in Computer Science, 1272, pp.354-357, 1998.

[RGN00] M. S. Rahman, S. Ghosh and T. Nishizeki, *Rectangular drawings of subdivisions of 3-connected planar graphs*, Proc. of International Conference on Computer and Information Technology 2000, pp.158-162, 2000.

[RNN98] M. S. Rahman, S. Nakano and T. Nishizeki, *Rectangular grid drawings of plane graphs*, Comp. Geom. Theo. Appl., 10(3), pp.203-220, 1998.

[R99] M. S. Rahman, *Efficient algorithms for drawing planar graphs*, Ph.D. Thesis, Department of System Information Sciences, Tohoku University, Japan, 1999.

[RNN99] M. S. Rahman, S. Nakano and T. Nishizeki, *A linear algorithm for bend-optimal orthogonal drawings of triconnected cubic plane graphs*, Journal of Graph Alg. Appl., 3, No.4, pp.31-62, 1999.

[RNN00a] M. S. Rahman, S. Nakano and T. Nishizeki, *Rectangular drawings of plane graphs without designated corners*, Proc. of COCOON 2000, Lecture Notes in Computer Science, 1858, pp.85-94, 2000. Comp. Geom. Theo. Appl. (to appear).

[RNN00b] M. S. Rahman, S. Nakano and T. Nishizeki, *Box-rectangular drawings of plane graphs*, Journal of Algorithms, 37(2), pp.363-398, 2000.

[S57] K. S. Stein, *Convex maps*, Proc. Amer Math. Soc., 2, pp.464-466, 1957.

[T87] R. Tamassia, *On embedding a graph in the grid with the minimum number of bends*, SIAM J. Computing, 16(3), pp.421-444, 1987.

[TBB88] R. Tamassia, G. Di Battista and C. Batini, *Automatic graph drawing and readability of diagrams*, IEEE Trans. on Syst. Man Cybern, SMC-18, pp.61-79, 1988.

[TTSS91] K. Tani, S. Tsukiyama, S. Shinoda and I. Shirakawa, *On area-efficient drawings of rectangular duals for VLSI floor-plan*, Mathematical Programming 52, pp.29-43, 1991.

[T84] C. Thomassen, *Plane representations of graphs*,(Eds.) J.A. Bondy and U.S.R. Murty, Progress in Graph Theory, Academic Press Canada, pp.43-69, 1984.

[W36] K. Wagner, *Bemerkunjen zum vierfar benproblem*, Jahresber, Dutchsch. Math-Verien., 46, pp.26-32, 1936.

[W96] D. B. West, *Introduction to Graph theory*, Prentice Hall, Upper Saddle River, New Jersy, 1996.

Index