PREDICTION OF TEMPERATURE PROFILE IN ENTRY REGION OF A TUBE BY ORTHOGONAL COLLOCATION METHOD

A THESIS

SUBMITTED TO THE DEPARTMENT OF CHEMICAL ENGINEERING

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE

DEGREE OF

MASTER OF SCIENCE IN ENGINEERING (CHEMICAL),

BANGLADESH UNIVERSITY OF ENGINEERING & TECHNOLOGY.

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MARCH, 1990

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CERTIFICATION OF THESIS WORK

We, the undersigned, certify that HARENDRA NATH MONDAL, candidate for the degree of Master of Science in Engineering (Chemical) has presented his thesis on the subject PREDICTION OF TEMPERATURE PROFILE IN ENTRY REGION OF A TUBE BY ORTHOGONAL COLLOCATION METHOD, that the thesis is acceptable in form and content, and that the student demonstrated a satisfactory knowledge of the field covered by this thesis in an oral examination held on April 9, 1990.

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ABSTRACT

The energy equation has been solved in the entrance section of a tube for both fully developed velocity profile and developing velocity profile conditions with two different boundary conditions, e. g. constant wall temperature and constant wall heat flux. The parabolic profile is used for fully developed velocity profile and for developing conditions Langhaar[2] velocity is used as axial component of velocity and radial component of velocity is obtained from equation of continuity and Langhaar velocity profile. The solution is obtained using integrator named **BDFSH**, which is capable of solving systems of parabolic, hyperbolic and stiff differential equations coupled with algebraic equations.

The governing partial differential equation is converted into a set of ordinary differential equations by using three different techniques e. g., orthogonal collocation method, finite difference method, and Galerkin's method. Results show that Galerkin's method gives somewhat better results than the orthogonal collocation method in terms of accuracy for lower values of N. As N increases, both methods give identical results. However, Galerkin's method is difficult to use compared to the orthogonal collocation method. On the other hand, the finite difference method is not capable of yielding the same accurate results as obtained by Galerkin's method or by the orthogonal collocation method. Finite difference method requires much more computing time compared to other two methods because a large number of grid points is required to ensure reasonable accuracy.

Numerical solutions have been obtained in the range of $10^{-5} < \zeta < .1$ and show good agreement with analytical solutions where they are avaible.

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ACKNOWLEDGEMENT

The author wishes to express his deepest gratitude and profound indebtedness to Dr. Nazmul Haq, Associate Professor of Chemical Engineering for his interest in this work. He acknowledges the advice and helpful co-operation rendered by Dr. Haq during the course of the work.

He is grateful to Professor Khaliqur Rahman for his valuable advice and suggestions in different phases of the work. It would have been difficult for the author to complete this work without his co-operation.

Thanks are due to Dr. Rafiqul Gani, Associate Professor of Instituttet for Kemiteknik, Denmark, who supplied the **BDFSH** math routine and the FORTRAN compiler and gave valuable suggestions regarding the use of the routine.

Thanks are also due to Professor K. I. Omar, Head of the Department of Chemical Engineering, for kindly allowing the author to use the departmental micro computers in working days as well as in holidays.

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NOMENCLATURE

a_i	—	Coefficients of trial function
a_{i0}	—	Coefficients of trial function at z=0
$\overline{\overline{A}}$	—	Two dimensional matrix
$\overline{C}$	—	One dimensional array
C	—	Concentration, $kmol/m^3$
C_w	—	Concentration at pipe wall, $kmol/m^3$
C_∞	—	Initial mainstream fluid Concentration, $kmol/m^3$
c_p	—	Specific heat, $KJ/kg.^\circ C$
d	—	Diameter of the pipe, m
h	—	Fluid heat transfer coefficient, $W/m.^\circ C$
$(h)_{local}$	—	Local Fluid heat transfer coefficient, $W/m.^\circ C$
$\langle h \rangle$	—	Average heat transfer coefficient, $W/m.^\circ C$
I_0, I_1, I_2	—	Modified Bessels function of order 0, 1 and 2 respectively
k	—	Thermal conductivity of the fluid, $W/m.^\circ C$
m	—	Mass flow rate, kg/s
N_{Re}	—	Reynolds number
N_{Pr}, Pr	—	Prandtl number
Nu	—	Nusselt number
$\langle Nu \rangle$, $(Nu)_m$	—	Mean or average Nusselt number
Pe	—	Peclet number
q	—	Wall heat flux, W/m^2
r	—	Radial distance, cm
r^*	—	Dimensionless radial distance, r/R
R	—	Radius of the tube, cm
Re	—	Reynolds number
Re_r	—	Reynolds number based on tube radius
T	—	Temperature, $^\circ C$

T^* — Dimensionless temperature

— $\frac{T-T_w}{T_\infty - T_w}$, for constant wall temperature

— $\frac{T-T_\infty}{qR/k}$, for constant wall heat flux

T_w — Wall temperature, $°C$

T_∞ — Initial mainstream fluid temperature, $°C$

T_b^* — Normalized bulk temperature

u_a — Average velocity of fluid, cm/sec

u_∞ — Initial mainstream velocity, cm/sec

v_r — Radial component of fluid velocity, cm/sec

v_r^* — $\frac{v_r}{u_a}$

v_z — Axial component of fluid velocity, cm/sec

v_z^* — $\frac{v_z}{u_a}$

z — Absolute axial distance, m

z^* — $\frac{z}{R}$

GREEK LETTERS AND MISCELLANEOUS SYMBOLS

α — Thermal diffusivity, cm^2/sec

β — Argument of Bessels function

ζ — Dimensionless axial distance, either $(x/R)/(Re.Pr)$ or $4(x/d)/(Re.Pr)$

λ — Eigenvalues

μ — Fluid viscosity, kg/m.sec

π — A constant, 3.1415 ⋯

ρ — Fluid density, kg/m^3

∇ — Laplacian

Θ — Dimensionless concentration, $\frac{C-C_w}{C_\infty - C_w}$

CHAPTER 1
INTRODUCTION

Investigation of fluid flow in the entrance region of a pipe or a duct is of considerable practical significance and, not surprisingly, there exist a large number of references in the literature on this topic, especially for incompressible laminar flow. The development of a parabolic Poiseuille profile downstream of entry into a plane channel is one of the standard problems in laminar-flow theory. It has attracted more attention than is warranted by its intrinsic practical importance, because it exemplifies certain general features of viscous flow. It therefore appears in textbooks, and is continually re-examined as new phenomena are introduced.

Forced-convection heat transfer in the thermal entrance region of tubes and ducts has been a subject of analytical study since the pioneering work of Graetz[6]. In most instances consideration has been given to the entrance of the fluid into the heated (or cooled) region of the duct with an already fully developed velocity profile which is unchanging along the duct length. This simplification is not realistic when the fluid enters the heated channel directly from a reservoir or large header. In such a situation the velocity is more nearly uniform across the entrance section, and the velocity and thermal boundary layers develop simultaneously as the fluid moves through the tube and if the velocity and temperature profiles develop simultaneously, the resulting Nusselt numbers are always higher than in the preceding case. This is the case that is more frequently met in technical applications, but the differences are generally significant for a fluid with Prandtl number less than about 10. Since in the case of a low Prandtl number fluid, the velocity profile is established much more slowly along the length of the tube in comparison to the temperature profile, assumption of a fully developed velocity profile at the tube entrance leads to significant error.

The solution of Graetz problem assumes constant physical properties and hence its application is limited to small temperature difference. For physically more realistic problem it is essential to consider property changes with temperature.

Of the physical properties, the temperature dependent viscosity plays a dominant role in influencing the velocity and temperature distribution in flow. The qualitative effect of viscosity is known in literature but knowledge of its quantitative effect is not satisfactory. The difficulty in this regards lies in the nonlinear and stiff character of the governing differential equations. In some problems the largest time step size is governed by the largest eigenvalue and the final time step is usually governed by the smallest eigenvalue. So one must use a very small time step (because of large eigenvalue) for a long time (because of small value). This characteristic of the system of equations causes stiffness. Very often, problems in fluid flow, heat and mass transfer encounter with widely varying time constants which give rise to both long term and short term effects. The resulting differential equations have widely varying eigenvalues. Differential equations of this type are known as stiff equations. In the entry length problem, the system is very stiff due to the development of velocity and temperature profile in a relatively short distance particularly when the Prandtl number is high (since in case of a high Prandtl number fluid, the velocity profile is established much more rapidly than the temperature profile)

If non-stiff numerical methods are applied to solve stiff problem, a very small integration step must be used to ensure stability of the solution. This means large computation time. Stiff problem like entry length problem still becomes unstable with very small integration step. Gear[44] defined "stiff-stability" and developed a program named BDF using backward differentiation to solve system of stiff ordinary differential equations. This method uses large step size which changes to control integration error. The variant of BDF developed by Gani et al[48], called BDFSH which was used to solve the entry length problem in a circular tube in this work. This method is essentially similar to BDF but several modifications have been made (see ref. 49). The modified version has improved the performance of the original code and provides the user with many more options during the integration phase. These include dense or sparse matrix techniques, local error control options, the use of Jacobian copies and restart facilities for discontinuities.

BDFSH method is implicit type linear multivalue or multistep method. This method requires several pieces of information about the dependent variable at time, $t = t_{n-1}$ in order to compute the equivalent pieces of information at $t = t_n$. Often this method uses the values of the dependent variable and its derivative. The original system of differential equations is transformed into a system of nonlinear algebraic equations which is solved by Newton like iteration (see ref. 49).

In recent years, many numerical methods have been applied to the solution of non-linear partial differential equations. Some of the technique apply for solution where the solution does not contain stiff gradient. To determine the ability of the BDFSH package, the Graetz problem is selected because extreme gradients are present in the entrance region in the case of constant wall temperature. A complete and accurate analytical solution from the entrance to a length where the temperature profile is fully developed requires the putting together the Leveque, extended Leveque and series solution containing large number of terms. The versatility of the package is shown by treating heat transfer with constant wall heat flux boundary condition.

Problems also considered in this work are the simultaneous development of velocity and temperature profile for Newtonian fluid in laminar flow in circular tube. The axial velocity component for entrance region given by Langhaar[2] and a radial component obtained from Langhaar's profile and continuity equation, for investigating the effect of radial velocity component on heat transfer.

Numerical solution for problems is worthwhile for problems whose analytical solution is available if such analytical solutions are in the form of multiple series and require either the evaluation of an extremely large number of eigenvalues, especially near the location of step change, or patching up of different solution (such as asymptotic approximation) which are valid for limited range of axial distance. The present work uses orthogonal collocation method, Galerkin's method and finite difference method seperately to solve the differential equations and compares the results.

CHAPTER 2
LITERATURE REVIEW

Historically, the first person to analyze the entrance flow through a smooth pipe was Schiller[58], who used integral analysis of a parabolic velocity profile in the boundary layer. The velocity profile chosen was a modification of the Poiseuille solution in the sense that the pipe radius was replaced by the boundary layer thickness. In other words, when the boundary layer thickness δ becomes equal to the pipe radius R, the analysis predicted automatic establishment of fully developed flow. This apparently gave rise to the idea that the attainment of a fully developed profile is synonymous with δ being equal to R. Schiller's integral solution on the basis of a parabolic profile is inherently questionable, as such a profile does not ensure attainment of free-stream condition at the edge of the boundary layer by not permitting the second derivative of the velocity to be zero, which is an essential boundary condition for flow with a pressure gradient. Later studies, such as that of Schlichting[32], have been primarily oriented towards the investigation of the development of the velocity profile. Schlichting's procedure for a rectangular duct consists of matching a downstream boundary-layer solution with a velocity profile which deviates increasingly from the Poiseuille profile in the upstream direction. This method of perturbing a Poiseuille profile was first adopted by Boussinesq (see Van Dyke[50]). The matching is supposed to take place where the boundary-layer and the deformed-profile solutions are valid simultaneously. This procedure of Schlichting has been re-examined by Van Dyke and Wilson[51] whose major corrections to Schlichting's method consists of proposing second-order stream functions to account for the displacement thickness in the boundary layer region.

A classical example of convective heat or mass transfer is the Graetz problem — transfer from the wall of a tube of radius R to a fluid in fully developed, laminar flow, with an insulating wall upstream at a point $z = 0$ and a wall maintained at a constant temperature or concentration downstream of the point $z = 0$. In terms of heat transfer, the partial differential equation is

$$\rho c_p u_z \frac{\partial T}{\partial z} = 2u_a \left[1 - \left(\frac{r}{R}\right)^2\right] \frac{\partial T}{\partial z} = k \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] \qquad (2\text{-}1)$$

with the following boundary conditions :

$$\left.\begin{array}{l} At\ r = R,\quad z > 0 \quad : T = T_w \\ At\ r = R,\quad z < 0 \quad : T = T_\infty \\ At\ r = 0,\quad \dfrac{\partial T_i}{\partial r} = 0 \\ As\ z \longrightarrow -\infty \quad T \longrightarrow T_\infty \\ As\ z \longrightarrow +\infty \quad T \longrightarrow T_w \end{array}\right\} \qquad (2\text{-}2)$$

In terms of the following dimensionless variables

$$r^* = \frac{r}{R}, \quad T^* = \frac{T - T_w}{T_\infty - T_w}, \qquad z^* = \frac{z}{R}, \quad \zeta = \frac{z^*}{\frac{d u_a \rho c_p}{k}}$$

the equation becomes

$$(1 - r^{*2})\frac{\partial T^*}{\partial \zeta} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) + \frac{\partial^2 T^*}{\partial z^{*2}} \qquad (2\text{-}3)$$

In terms of mass transfer, the partial differential equation, eq. (2-1), becomes

$$u_z \frac{\partial C}{\partial z} = 2u_a \left[1 - \left(\frac{r}{R}\right)^2\right] \frac{\partial C_i}{\partial z} = D \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) + \frac{\partial^2 C}{\partial z^2}\right] \qquad (2\text{-}4)$$

$$\left.\begin{array}{l} At\ r = R,\quad z > 0 \quad : C = C_w \\ At\ r = R,\quad z < 0 \quad : C = C_\infty \\ At\ r = 0,\quad \dfrac{\partial C_i}{\partial r} = 0 \\ As\ z \longrightarrow -\infty \quad C \longrightarrow C_\infty \\ As\ z \longrightarrow +\infty \quad C \longrightarrow C_w \end{array}\right\} \qquad (2\text{-}5)$$

In terms of the following dimensionless variables

$$r^* = \frac{r}{R}, \quad \Theta = \frac{C - C_w}{C_\infty - C_w}, \quad \zeta = \frac{zD}{2u_a r^2}$$

the equation becomes

$$(1 - r^{*2})\frac{\partial \Theta}{\partial \zeta} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial \Theta}{\partial r^*}\right) + \frac{1}{Pe^2}\frac{\partial^2 \Theta}{\partial \zeta^2} \tag{2-6}$$

where $Pe = \frac{2Ru_a}{D}$ is the Peclet number, with boundary conditions

$$\left.\begin{array}{l} At\ r^* = 1, \quad \zeta > 0, \quad \Theta = 0 \\ At\ r^* = 1, \quad \zeta < 0, \quad \dfrac{\partial \Theta}{\partial \zeta} = 0 \\ At\ r^* = 0, \quad \dfrac{\partial \Theta}{\partial r^*} = 0 \\ As\ \zeta \longrightarrow -\infty, \quad \Theta \longrightarrow 1; \text{ and} \\ As\ \zeta \longrightarrow +\infty, \quad \Theta \longrightarrow 0 \end{array}\right\} \tag{2-7}$$

Graetz[6] treated this problem by the method of separation of variables under the condition that the last term in Eq. (2-6) could be neglected. This term, representing axial diffusion, will be small when the Peclet number is large, which holds for most practical applications. In this manner Graetz obtained a series solution involving functions of the radial distance, $R_k(r^*)$, which are defined by a Sturm-Liouville system where the parameter is restricted to discrete eigenvalues λ_k^2. Other workers have refined the Graetz solution. Asymptotic forms of the eigenvalues and coefficients have been worked out for large eigen values, notably by Lauwerier[10] and by Sellers, Tribus, and Klein[24]. Numerical solutions of the Graetz problem, with neglect of axial diffusion, have been given by Kays[9] and by Longwell[15].

Near the entrance i.e., near $z = 0$, the transfer rate becomes infinite, and many terms in the Graetz series are required for an reasonably accurate solution in this region. A similarity solution is developed by Leveque[13] and an extended solution is developed by Newman[17]. The axial diffusion term, neglected in the work already mentioned, becomes important at high Peclet numbers only in a small region near $z = 0$ and at $r = R$, a region which becomes smaller as the Peclet number increases. With the neglect of axial diffusion, the originally elliptic problem becomes parabolic. This region near $z = 0$ is the only place where the elliptic nature of the problem persists; thus, it is a small elliptic region embedded in an otherwise parabolic domain. Axial diffusion has been treated by Singh[26]. Schenk and Dumore[20] have treated the effect of non-zero transfer resistance of a tube wall, which serves to eliminate the infinite transfer rate at $z = 0$. Problems involving a catalytic reaction at the tube wall have been treated by Katz[8] and by Solomon and Hudson[27], and transfer to non-Newtonian fluids has been treated by Schenk and Van Laar[21].

Graetz[6] included axial diffusion in his treatment of plugflow in a tube. Singh[26] did the same thing for a parabolic velocity profile in a tube. However, both of these authors took $\Theta = 1$ in the cross section at the beginning of the transfer section, without apparently recognizing the physically absurd consequences of this condition. The boundary conditions of Eq. (2-4) or Eq. (2-6), on the otherhand, represents a reasonable situation corresponding to an insulated wall upstream of the transfer section. If axial diffusion is important, then the fluid must become depleted, to some extent, upstream of the transfer section. It is only in the limit of an infinite Peclet number that axial diffusion can be neglected and the condition $\Theta = 1$ applied in the cross section $z = 0$. Furthermore, to have a surface with $\Theta = 1$ immediately adjoining a surface with $\Theta = 0$, here the tube wall for $z > 0$, will result in an infinite transfer rate near $z = 0$, a rate which cannot be integrated. In otherwords, with the boundary conditions used by Graetz[6] and Singh[26], the total amount of material transferred in any non-zero length z will be infinite. Bodnarescu[2] treated axial diffusion with an upstream wall maintained at one temperature and a downstream wall at another temperature. Wilson[28] used the same boundary conditions with plug flow. As Drew[5] has pointed out, the total transfer will again be infinite. Schneider[22] treated the same situation as Wilson[28], but with the addition of a transfer resistance at the wall (as proved later, half of Schneider's solutions without axial conduction are wrong).

Siegel and Sparrow[57] analyzed laminar forced convection heat transfer in a parallel plate channel with uniform heat flux at the walls. They assumed both the temperature and velocity to be uniform at the entrance section and used the velocity profile obtained by Schiller[58], who gives the velocity profile as

$$v_z = u_\infty \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \right], \text{ for } 0 \le y \le \delta \qquad (2\text{-}8)$$

and

$$v_z = u_\infty, \text{ for } \delta \le y \le a \qquad (2\text{-}9)$$

where

$a =$ half width of the parallel plate channel,

$y =$ transverse coordinate measured from channel wall and

$\delta =$ velocity boundary layer thickness.

They obtained heat transfer results for Prandtl number range of 0.01 to 50. The approximate nature of their mathematical formulation does not permit precise calculations for thermal entry length. Heaton and Reynolds[59] analyzed the problem of laminar flow heat transfer in an annulus with simultaneously developing velocity and temperature distributions for constant wall heat flux. They used Langhaar[2] velocity distribution for annulus and solved the resulting equation by an integral method similar to that used by Han[54] for the parallel plates.

Kays[9] also studied the same thing for both constant wall heat flux and constant wall temperature conditions. The numerical work that has been done so far in this regard, Kays work is better in comparison to any other workers. Kays obtained the solution by finite difference technique using axial velocity distribution as obtained by Langhaar.

CHAPTER 3
FORMULATION OF EQUATIONS AND SOLUTION PROCEDURE

3.1.1 FORMULATION OF THE DIFFERENTIAL EQUATION FOR CONSTANT WALL TEMPERATURE AND CONSTANT WALL HEAT FLUX

Let us consider an incompressible Newtonian fluid with constant viscosity and thermal conductivity flows through a tube of radius R.

For $z < 0$, the wall of the tube is insulated

At $r = R$, for $z < 0$: $\frac{\partial T}{\partial r} = 0$

For $z > 0$, the wall temperature is maintained constant at T_w :

At $r = R$, for $z > 0$: $T = T_w$

Very far upstream from the entrance to this heated section, the fluid is at a uniform temperature T_∞ :

As $z \longrightarrow -\infty$, for $r \leq R$: $T = T_\infty$

We wish to determine the temperature distribution of the fluid in the heated portion of the tube. The differential energy balance for this situation is

$$\rho c_p v_z \frac{\partial T}{\partial z} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] \tag{3-1}$$

The above differential equation is normalized by using the following normalizing variables :

$r^* = \frac{r}{R}$, $T^* = \frac{T - T_\infty}{T_w - T_\infty}$, $z^* = \frac{z}{R}$, and $v_z^* = \frac{v_z}{u_a}$

Then we have

$$\rho c_p u_a v_z^*(T_w - T_\infty)\frac{1}{R}\frac{\partial T^*}{\partial z^*} = k(T_w - T_\infty)\left[\frac{1}{R^2}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) + \frac{1}{R^2}\frac{\partial^2 T^*}{\partial z^{*2}}\right] \tag{3-2}$$

$\Longrightarrow$

$$\frac{\rho c_p u_a}{kR} v_z^* \frac{\partial T^*}{\partial z^*} = \frac{1}{R^2}\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) + \frac{\partial^2 T^*}{\partial z^{*2}}\right] \tag{3-3}$$

$\Longrightarrow$

$$\frac{\rho c_p u_a}{k} v_z^* \frac{\partial T^*}{\partial z^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) + \frac{\partial^2 T^*}{\partial z^{*2}} \tag{3-4}$$

$\Longrightarrow$

$$N_{Pe} v_z^* \frac{\partial T^*}{\partial z^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) + \frac{\partial^2 T^*}{\partial z^{*2}} \tag{3-5}$$

where $N_{Pe} =$ Peclet number based on tube radius

$$= \frac{\rho c_p R u_a}{k}$$

$$= \frac{\rho R u_a}{\mu}\frac{c_p \mu}{k}$$

$$= N_{Re} N_{Pr}$$

$\Longrightarrow$

$$v_z^* \frac{\partial T^*}{\partial z^*} = \frac{1}{N_{Pe}}\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) + \frac{\partial^2 T^*}{\partial z^{*2}}\right] \tag{3-6}$$

Let $\zeta = \frac{z^*}{N_{Pe}}$

Then we have

$$v_z^* \frac{\partial T^*}{\partial \zeta} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) + \frac{\partial^2 T^*}{\partial z^{*2}} \tag{3-7}$$

The term, $\frac{\partial^2 T^*}{\partial z^{*2}}$, in eq. (3-7) represents the axial conduction. For reasonably high Peclet number, the contribution of axial conduction can be neglected. Then we get

$$v_z^* \frac{\partial T^*}{\partial \zeta} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) \tag{3-8}$$

The boundary conditions are :

At $r^* = 1$, for $\zeta < 0$: $\frac{\partial T^*}{\partial r^*} = 0$

At $r^* = 1$, for $\zeta > 0$: $T^* = 1$

As $\zeta \longrightarrow -\infty$, for $r^* \leq 1$: $T^* = 0$.

If r-component (radial component) of the velocity , v_r is taken into account, then the above equation becomes

$$v_z^* \frac{\partial T^*}{\partial \zeta} + v_r^* \frac{\partial T^*}{\partial r^*} = \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial T^*}{\partial r^*} \right) \tag{3-9}$$

with the same boundary conditions.

For constant wall heat flux the differential equation is as before

$$v_z^* \frac{\partial T^*}{\partial \zeta} = \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial T^*}{\partial r^*} \right) \tag{3-10}$$

where

$$T^* = \frac{T - T_\infty}{qR/k}, \qquad r^* = \frac{r}{R}, \qquad \zeta = \frac{2c_p \rho R u_a}{k}$$

with boundary conditions :

$$\left.\begin{array}{l} At\ r^* = 1,\ for\ \zeta < 0\ :\ \dfrac{\partial T^*}{\partial r^*} = 0 \\ At\ r^* = 1,\ for\ \zeta > 0\ :\ \dfrac{\partial T^*}{\partial r^*} = -1 \\ At\ \zeta = 0,\ for\ r \le 1\ :\ T^* = 0 \end{array}\right\} \tag{3-11}$$

If r-component (radial component) of the velocity , v_r is taken into account, then the above equation becomes

$$v_z^* \frac{\partial T^*}{\partial \zeta} + v_r^* \frac{\partial T^*}{\partial r^*} = \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial T^*}{\partial r^*} \right) \tag{3-12a}$$

with the same boundary conditions.

But when physical property variations are taken into consideration, c_p, ρ and k can no longer be included in dimensionless axial distance, ζ. Then eq. (3-8) or eq. (3-10) is written in the following two forms (i. e. eq. (3-12b) and eq. (3-12c)) for constant wall temperature and for constant wall heat flux respectively :

$$v_z^* \frac{\partial T^*}{\partial z^*} = \frac{k(T^*)}{\rho(T^*) c_p(T^*) u_a R} \left[\frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial T^*}{\partial r^*} \right) + \left.\frac{\partial k}{\partial T}\right|_{T_j} \left(\frac{T_\infty - T_w}{k(T_j)} \right) \left(\frac{\partial T^*}{\partial r^*} \right)^2 \right] \tag{3-12b}$$

$$v_z^* \frac{\partial T^*}{\partial z^*} = \frac{k(T^*)}{\rho(T^*) c_p(T^*) u_a R} \left[\frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial T^*}{\partial r^*} \right) + \left. \frac{\partial k}{\partial T} \right|_{T_j} \left(\frac{qR}{k^2(T_j)} \right) \left(\frac{\partial T^*}{\partial r^*} \right)^2 \right] \quad (3\text{-}12c)$$

The following expressions for variations of c_p, k and ρ with temperature are used for water :

$$c_p = 4.21705 - 3.07139 \times 10^{-3} T + 8.42643 \times 10^{-5} T^2 - 9.31925 \times 10^{7} T^3 + 3.95487 \times 10^{-9} T^4, \quad \text{KJ/KG}^\circ\text{C}$$

$$k = 5.63063 \times 10^{-4} + 1.96429 \times 10^{-6} T - 8.28078 \times 10^{-9} T^2, \quad \text{KW/m}^\circ\text{C}$$

$$\rho = 1000.15 - 6.57555 \times 10^{-3} T - 42.5321 \times 10^{-4} T^2, \quad \text{KG/m}^3$$

3.1.2 EVALUATION OF BULK TEMPERATURE FOR CONSTANT WALL TEMPERATURE AND CONSTANT WALL HEAT FLUX

Since for constant wall heat flux

$$T^* = \frac{T - T_\infty}{qR/k}$$

Then

$$T_b^* = \frac{T_b - T_\infty}{qR/k} \quad \text{and} \quad T = T^*\left(\frac{qR}{k}\right) + T_\infty$$

Now

$$\begin{aligned} T_b &= \frac{\int_0^R T u r \, dr}{\int_0^R u r \, dr} \\ &= \frac{\int_0^R \left[T^*\left(\frac{qR}{k}\right) + T_\infty\right] u r \, dr}{\int_0^R u r \, dr} \\ &= \frac{qR}{k} \frac{\int_0^R T^* u r \, dr}{\int_0^R u r \, dr} + T_\infty \end{aligned}$$

Hence

$$\frac{T_b - T_\infty}{\frac{qR}{k}} = T_b^* = \frac{\int_0^R T^* u r \, dr}{\int_0^R u r \, dr}$$

or,

$$T_b^* = \frac{\int_0^1 T^* u^* r^* \, dr^*}{\int_0^1 u^* r^* \, dr^*} \tag{3-13}$$

For constant wall temperature the normalized temperature is

$$T^* = \frac{T - T_w}{T_\infty - T_w}$$

Using similar procedure as above, we get

$$\frac{T_b - T_w}{T_\infty - T_w} = T_b^* = \frac{\int_0^R T^* u r \, dr}{\int_0^R u r \, dr}$$

or,

$$T_b^* = \frac{\int_0^1 T^* u^* r^* \, dr^*}{\int_0^1 u^* r^* \, dr^*} \tag{3-14}$$

That is, the expression for normalized bulk temperature is same for both cases.

3.1.3 CALCULATION OF LOCAL NUSSELT NUMBER FOR CONSTANT WALL TEMPERATURE AND CONSTANT WALL HEAT FLUX

At any cross-section of the pipe one can write the energy balance equation in the following form :

$$(h)_{local}(T_w - T_b) = -k \left.\frac{\partial T}{\partial r}\right|_{r=R} \tag{3-15}$$

Let us normalized the above equation by substituting

$$T^* = \frac{T - T_\infty}{qR/k} \text{ and } r^* = \frac{r}{R} \tag{3-16}$$

Then we have

$$\left.\frac{\partial T}{\partial r}\right|_{r=R} = \frac{qR}{kR}\left.\frac{\partial T^*}{\partial r^*}\right|_{r^*=1} \tag{3-17}$$

Substituting

$$(h)_{local}(T_w - T_b) = -k\frac{qR}{kR}\left.\frac{\partial T^*}{\partial r^*}\right|_{r^*=1} \tag{3-18}$$

$\Longrightarrow$

$$\frac{(h)_{local}2R}{k} = 2\left(\frac{qR}{k}\right)\frac{\left.\frac{\partial T^*}{\partial r^*}\right|_{r^*=1}}{T_w - T_b}$$

$$= -2\frac{qR}{k}\frac{\left.\frac{\partial T^*}{\partial r^*}\right|_{r^*=1}}{\frac{qR}{k}(T_w^* - T_b^*)}$$

$$= -2\frac{qR}{k}\frac{\left.\frac{\partial T^*}{\partial r^*}\right|_{r^*=1}}{\frac{qR}{k}(T^*|_{r^*=1} - T_b^*)} \tag{3-20}$$

Hence

$$(Nu)_{local} = \frac{-2\left.\frac{\partial T^*}{\partial r^*}\right|_{r^*=1}}{T^*|_{r^*=1} - T_b^*} \tag{3-21}$$

Similarly for constant wall heat flux

$$(Nu)_{local} = \frac{2}{T_w^* - T_b^*} \tag{3-22}$$

Since $\left.\frac{\partial T^*}{\partial r^*}\right|_{r^*=1} = 1$ for constant wall heat flux.

3.1.4 CALCULATION OF AVERAGE OR MEAN NUSSELT NUMBER FOR CONSTANT WALL TEMPERATURE

Energy balance for a section of the pipe of length z gives :

$$Q = mc_p(T_b - T_\infty) \quad or, \; dQ = mc_p dT_b \tag{3-23}$$

and

$$Q = 2\pi Rqz \quad or, \; dQ = 2\pi Rqdz \tag{3-24}$$

From (3-23) and (3-24)

$$mc_p dT_b = 2\pi Rqdz \tag{3-25}$$

$\Longrightarrow$

$$\pi R^2 u_a \rho c_p dT_b = 2\pi Rh(T_w - T_b) \tag{3-26}$$

$\Longrightarrow$

$$R^2 \rho u_a c_p \int_{T_\infty}^{T_b} \frac{dT_b}{T_w - T_b} = 2R \int_0^z hdz \tag{3-27}$$

$\Longrightarrow$

$$-R^2 \rho u_a c_p ln\left(\frac{T_w - T_b}{T_w - T_\infty}\right) = 2R(h) \int_0^z dz \tag{3-28}$$

$\Longrightarrow$

$$-R^2 \rho u_a c_p ln\left(\frac{T_b - T_w}{T_\infty - T_w}\right) = 2R(h)z \tag{3-29}$$

$\Longrightarrow$

$$-R^2 \rho u_a c_p ln T_b^* = \left(\frac{2R(h)}{k}\right) kz \tag{3-30}$$

$\Longrightarrow$

$$(Nu) = -\frac{lnT_b^*}{2\zeta} \quad \text{where } \zeta = \frac{zk}{2R^2 \rho c_p u_a} \tag{3-31}$$

3.1.5 CALCULATION OF AVERAGE OR MEAN NUSSELT NUMBER FOR CONSTANT WALL HEAF FLUX

In this case Eq. (3-27) cann't be integrated analytically beacuse T_w is not constant. In this work the following technique is used:

$$
\begin{aligned}
(Nu)_m &= \frac{1}{\zeta}\int_0^{\zeta}(Nu)_{local}\,d\zeta \\
&= \frac{1}{\zeta}\int_0^{\zeta}\frac{2}{T_w - T_b}d\zeta \\
&= \frac{2}{\langle T_w - T_b\rangle} \qquad \text{where } \langle T_w - T_b\rangle = \frac{1}{\zeta}\int_0^{\zeta}(T_w - T_b)d\zeta \\
&= \frac{2\zeta}{\int_0^1 (T_w - T_b)d\zeta} \qquad (3\text{-}32)
\end{aligned}
$$

3.1.6 FLOW DEVELOPMENT IN CIRCULAR TUBES

Langhaar[2] obtained an approximate solution for the axial velocity component in the entrance region of a circular tube by linearizing the boundary-layer equation. The resulting solution is

$$v_z = \frac{I_0(\beta) - I_0(\beta r^*)}{I_2(\beta)} \tag{3-33}$$

where I_0 and I_2 are modified Bessel functions of order zero and two respectively, β is a function of x/Re_r defined by

$$x/Re_r = \int_\beta^\infty g(\beta) f'(\beta)\, d\beta \tag{3-34}$$

where

$$g(\beta) = I_2(2\beta I_1 - \beta^2)$$

$$f(\beta) = \frac{4 I_0 I_2 - (I_0 - 1)^2 - 2I_1^2}{2I_2^2}$$

$$f'(\beta) = \frac{df(\beta)}{d\beta}$$

The argument of the modified Bessel functions is β.

The expression for the radial velocity component v_r is obtained from equation (3-33) and the continuity equation. The continuity equation in cylindrical coordinate is

$$\frac{\partial v_z}{\partial x} + \frac{1}{r}\frac{\partial (r v_r)}{\partial r} = 0 \tag{3-35}$$

Thus

$$v_r = -\frac{1}{r}\int_0^r r \frac{\partial v_z}{\partial x}\, dr = -\frac{1}{r}\frac{\partial}{\partial \beta}\left[\int_0^r r v_z\, dr\right]\frac{\partial \beta}{\partial x} \tag{3-36}$$

From Eq. (3-34)

$$\frac{\partial \beta}{\partial x} = -\frac{1}{Re_r}\left[\frac{1}{g(\beta) f'(\beta)}\right] \tag{3-37}$$

From Eq. (3-33), (3-36) and (3-37) , the expression for radial velocity component is

$$v_r Re_r = \frac{1}{I_2 g(\beta) f'(\beta)}\left[\left\{\frac{r I_0}{2} - \frac{I_1(\beta r^*)}{\beta}\right\}\left(\frac{I_1}{I_2} - \frac{2}{\beta}\right) - \frac{r^*}{2}\left\{I_1 - \frac{2I_2(\beta r^*)}{\beta}\right\}\right] \tag{3-38}$$

where the argument of the modified Bessel functions is β unless otherwise indicated. It might be noted that the computation of v_r from Eq. (3-38) does not constitute a major increase in time and labor since $g(\beta)$, $f'(\beta)$, I_0, I_1 and I_2 are all involved in the computation of v_z.

3.2 METHOD OF SOLUTION

The method of weighted residuals (MWR) is a general method of obtaining solutions for equations of change. The unknown solution is expanded in a set of trial functions, which are specified, but with adjustable constants, which are chosen to give the best solution to the differential equation. Usually the first approximation gives useful qualitative answers, but higher approximations may be used (usually on a computer) to give as precise an answer as desired. The general procedure for using MWR is outlined below :

Let us consider a differential equation of the form :

$$f\left(x, y, \frac{\partial^2 y}{\partial x^2}, \frac{\partial y}{\partial x}\right) = 0 \tag{3-2-1}$$

A solution may be expressed in the form :

$$y_n = y_0 + \sum_{i=1}^{N} a_i y_i \tag{3-2-2}$$

where the functions y_i are specified to satisfy the homogeneous boundary conditions. Then the trial function, eq. (3-2-2), satisfies the boundary conditions for all choices of the constants a_i. This trial function is substituted into the differential equation to form the residual

$$R(a_i, x, y) = f(y_0, a_i, y_i) \tag{3-2-3}$$

If the trial function were exact solution the residual would be zero. In MWR the constants a_i are chosen in such a way that the residual is forced to be zero in an average sense. For this, the weighted integrals of the residual are set to zero :

$$\int W_k R(a_i, x, y)\, dx = o \tag{3-2-4}$$

Finally we choose a criterion or a weighting function, W_k. Each MWR is characterised by a different choice of the sequence of N weighting functions W_k in eq. (3-2-4). For example, Galerkins method uses $\frac{\partial y_n}{\partial a_i}$ as weighting functions whereas least square method uses $\frac{\partial R_N}{\partial a_i}$ and collocation method chooses weighting functions to be *dirac delta* $(W_k = \delta(x - x_k))$.

Whatever may be the weighting functions, eq. (3-2-4) generates either a set of algebraic equations or a set of ordinary differential equations which can be solved numerically using a computer to obtain a_i, which thus gives the approximate solution of the differential equation.

3.2.1 ORTHOGONAL COLLOCATION METHOD

The orthogonal collocation method provides a mechanism for automatically picking the collocation points by making use of orthogonal polynomials. This method chooses the trial function $y(x)$ to be the linear combination

$$y(x) = \sum_{i=1}^{N+2} a_i P_{i-1}(x) \qquad (3\text{-}2\text{-}1a)$$

of a series of orthogonal polynomials $P_m(x)$. The set of polynomials can be written in a condensed form :

$$P_m(x) = \sum_{j=0}^{m} c_{mj} x^j, \qquad m = 0, 1, ..., N = 1 \qquad (3\text{-}2\text{-}1b)$$

The coeffficients c_{mj} are chosen so that the polynomials obey the orthogonality condition

$$\int_a^b w(x) P_k(x) P_m(x)\, dx = 0, \qquad k = 0, 1, ..., (m-1) \qquad (3\text{-}2\text{-}1c)$$

When $P_m(x)$ is chosen to be the Legendre set of orthogonal polynomials, the weighting function $w(x)$ is unity. Similarly, when the weighting function $w(x)$ is either $(1-x)$ or, $(1-x)^{\frac{1}{2}}$, the function $P_m(x)$ becomes either Jacobi or Chebyschef polynomials.

A detailed discussion on orthogonal collocation technique is available in APPENDIX and may also be found in references 29, 30 and 31.

3.2.2 SOLUTION BY GALERKIN'S METHOD

Let us consider eq. (3-8) with $T = T^*$, $r = r^*$ and $z = \zeta$ for simplicity

$$(1-r^2)\frac{\partial T}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{3-2-2a}$$

With boundary conditions:

At $r = 0$, $\frac{\partial T}{\partial r} = 0$; and at $r = 1$, $T = 0$

It is very important to choose an appropriate trial function. Let a trial function of the following form:

$$T_N(r,z) = \sum_{i=1}^{N} a_i(z)T_i(r) \tag{3-2-2b}$$

Where $T_i(r)$ is given by

$$T_i(r) = r^i(1-r)$$

The residual R_N of the equation is

$$R_N(\bar{a},r,z) = (1-r^2)\frac{\partial T_N}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_N}{\partial r}\right)$$

$$= (1-r^2)\sum_{i=1}^{N}\frac{\partial a_i(z)}{\partial z}r^i(1-r) - \sum_{i=1}^{N} a_i(z)[i^2 r^{i-2} - (i+1)^2 r^{i-1}]$$

Then R_N is made orthogonal on the N trial functions $T_j(r)$:

$$\int_0^1 [R_N(a,r,z)]T_j(r)\,dr = 0, \quad j = 1,2,3,...N$$

or,

$$\int_0^1 \left[\sum_{i=1}^{N}\frac{\partial a_i(z)}{\partial z}r^j(1-r) - \sum_{i=1}^{N} a_i(z)[i^2r^{i-2}-(i+1)^2r^{i-1}]\right]T_j(r)\,dr = 0, \; j = 1,2,3...N$$

or,

$$\int_0^1 \left[\sum_{i=1}^{N}(1-r^2)r^i(1-r)\frac{\partial a_i(z)}{\partial z}r^j(1-r) - \sum_{i=0}^{N} a_i(z)r^j(1-r)[i^2r^{i-2} - (i+1)^2r^{i-1}]\right]dr = 0,$$

$$j = 1,2,3...N$$

or in vector form :

$$\overline{\overline{A}}\frac{\partial \overline{a}_i}{\partial z} = \overline{\overline{B}}\overline{a} \tag{3-2-2c}$$

Where

$$A_{ji} = \int_0^1 (1-r^2)r^i(1-r)r^j(1-r)\,dr$$

$$B_{ji} = \int_0^1 \left[i^2 r^{i-2} - (i+1)^2 r^{i-1}\right] r^j(1-r)\,dr$$

After carry out the integration process we get the following results:

$$A_{ji} = \frac{1}{i+j+1} - \frac{2}{i+j+2} + \frac{2}{i+j+4} - \frac{1}{i+j+5}$$

$$B_{ji} = \frac{i^2}{i+j-1} - \frac{(i+1)^2+i^2}{i+j} + \frac{(i+1)^2}{i+j+1}$$

At $z = 0$, all the values of $a_i(z = 0)$ can be determined by using the following technique: [Here T means the normalized temperature T^* which is defined by $T^* = \frac{T-T_w}{T_\infty - T_w}$]. Since at $z = 0$, $T = 1$

$$\int_0^1 \left[1 - \sum_{i=1}^{N} a_{i0} r^i (1-r)\right] \underbrace{(1-r^2)T_j(r)}\,dr = 0$$

Where the underbracketed portion of the above equation is the chosen weighting function. or,

$$\int_0^1 \left[1 - \sum_{i=1}^{N} a_{i0} r^i (1-r)\right] (1-r^2)r^j(1-r)\,dr = 0$$

or,

$$\int_0^1 \sum_{i=1}^{N} a_{i0} r^i (1-r)(1-r^2)r^j(1-r)\,dr = \int_0^1 (1-r^2)r^j(1-r)\,dr$$

or,

$$\overline{\overline{A}}\overline{a}_0 = \overline{C} \tag{3-2-2d}$$

Where A_{ji} is same as before and C_j is given by

$$C_j = \int_0^1 (1 - r^2) r^j (1 - r)\, dr$$
$$= \int_0^1 (r^j - r^{j+1} - r^{j+2} + r^{j+3})\, dr$$

or,

$$C_j = \frac{1}{j+1} - \frac{1}{j+2} - \frac{1}{j+3} + \frac{1}{j+4} \qquad \text{(3-2-2e)}$$

3.2.3 EVALUATION OF BULK TEMPERATURE

For constant physical properties the normalized bulk temperature T_b^* is given by

$$T^* = \frac{\int_0^1 Tur\, dr}{\int_0^1 ur\, dr} \qquad \text{(3-2-3a)}$$

Substituting the expressions for T and u we get,

$$T_b^* = \frac{\int_0^1 \sum_{i=1}^{N} a_i r^i (1 - r)(1 - r^2) r\, dr}{\int_0^1 (1 - r^2) r\, dr} \qquad \text{(3-2-3b)}$$

After carry out the integration we obtain finally the expression for bulk temperature as follows:

$$T_b^* = 4 \sum_{i=1}^{N} a_i \left(\frac{1}{i+2} - \frac{1}{i+4} - \frac{1}{i+3} + \frac{1}{i+5} \right) \qquad \text{(3-2-3c)}$$

3.2.4 FINITE DIFFERENCE METHOD

The finite difference method replaces the derivatives in the differential equations with their finite difference approximations* which is called discretization at each point in the interval of integration, thus converting the equations to a large set of simulteneous nonlinear algebraic equations** which can be solved using Newton's method for simulteneous nonlinear algebraic equations. It should be emphasized, however, that the problem of solving a large set of algebraic equations is not a trivial task. It requires, first, a good initial guess of all the dependent variables, and it involves the evaluation of the $2N \times 2N$ Jacobian matrix for a system which consists of N nonlinear algebraic equations. But if the differential equations are linear, the resulting set of simulteneous algebraic equations will also be linear. In such case, the solution can be obtained by a straightforward application of the Gauss elimination or by the improved Gauss-Jordon procedure. But for stiff systems of differential equations, it is wise not to use finite difference technique.

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*Usually central difference approximation is used in the middle of the interval and forward or backward difference approximation is used at the boundary.

**If the discretization is made in one direction, then the partial differential equation is reduced to a set of ordirary differential equations. In this work, discretization is made in radial direction.

CHAPTER 4

RESULTS AND DISCUSSIONS

The main objective of this work was to determine the temperature profile at the very entrance (say at $\zeta < 10^{-5}$ for Pr > 0.7 and at $\zeta < 10^{-4}$ for Pr ≤ 0.01 where ζ = dimensionless axial distance, $x/R/(Re.Pr)$) section of the tube since solutions in this region is not available in literature. It is very difficult to obtain solutions in these ranges. Kays[9], was able to obtain solution at $\zeta = 0.0002$ (only for $Pr \geq 0.7$) which gave 20-10% too high of the value of local Nusselt number compare to analytical solution obtained by Robert Lipkis[52].

In addition to use of orthogonal collocation, the present work also uses finite difference technique to compare the accuracy and computational time between orthogonal collocation and finite difference technique. It was found that six collocating points is equivalent to forty grids, and nine collocating points is equivalent to eighty grids in finite difference technique with computing times seven to ten times higher. If the number of grid points is increased beyond 100, the accuracy does not increase and it takes excessive computing time. This may be because of numerical truncation error. Actually the way which we used to formulate the solution in orthogonal collocation technique is the method of ordinates (another way is the method of coefficients) and this may be considered as a finite difference method where the grid points are optimally spaced (see ref. 56). Hence in finite difference with large number of grid points requires much more arithmetic calculation than those involving 12, 15 or 18 collocating points resulting in excessive computing time and truncation error.

The present work also uses Galerkin's method to solve the Graetz problem. The results obtained by Galerkin's method is same as that obtained by orthogonal collocation method. But Galerkin's method requires smaller order of trial functions, for example, for $N = 9$ in Galerkin's method gives similar results with $N = 12$ in orthogonal collocation method and obviously requires less computing time. The disadvantage of Galerkin's method is that it is difficult to use, specially, the choice of a trial function (the successfulness depends on it) is very crucial. Once the trial function is chosen successfully, the remaining task is straight forward. The most important thing is, whatever may be the technique or method, the resulting equations should be solved with appropriate integration routine to obtain satisfactory results.

Initially attempt was made to use Gear's routine of adaptive step size control for explicit Runge-Kutta method and then adaptive step size control for polynomial extrapolation (using Richardson's extrapolation) method but both the method failed to give values at $\zeta = 0.001$. In both the cases, very small initial stepsize ($h = 0.0001$) was chosen but it didn't work. Then the package **BDFSH** is tried and proved to be successful. It is quite capable of giving solution at $\zeta = 10^{-5}$ without any numerical instability. This package is used in this work.

The dimensionless temperature profile in the range of $10^{-6} < \zeta < 10^{-1}$ are shown in Figures 1, 2 and 3 — for constant wall temperature and constant wall heat flux respectively. As have been mentioned earlier all the previous solutions in this range of axial distance could be obtained by combination of classical Graetz solution and Leveque approximation requiring large number of terms. The stiffness of the gradient at low ζ provides a good test of numerical techniques. For constant wall temperature problem the local Nusselt number and mean Nusselt number are presented in Table–1 and Table–2 respectively. The value of local Nusselt number at $\zeta = 0.4$ is very close to the asymptotic value of $Nu_{z^* \longrightarrow \infty} = 3.6563$. Both the constant wall temperature profile and constant wall heat flux profile have a slope of zero at the centre of the pipe.

For constant wall heat flux problem the local Nusselt number and mean Nusselt number are presented in Table–3 and Table–4 respectively. The value of local Nusselt number at $\zeta = 0.3$ is in good agreement with analytical asymptotic value of $\frac{48}{11} = 4.36364$. Since the evaluation of flow length mean Nusselt number requires numerical integration of local Nusselt number, the value of mean Nusselt number is less accurate than local Nusselt number. The constant wall heat flux boundary condition gives smooth dimensionless temperature profile and has a slope of one at the wall according to boundary condition at wall i. e., at $r^* = 0$; $\frac{\partial T^*}{\partial r^*} = 1$ for all $z^* \geq 0$.

Table–1 also compares the value of local nusselt number with analytically obtained values obtained by Robert Lipkis[52]. It is quite clear from this table that almost accurate results could be obtained by using 12 collocating points. If collocating points are increased further the accuracy does not increase significantly but requires more computing time. Another important feature is that Nusselt number decreases very quickly in the entrance section and at the very entrance ($\zeta \leq 10^{-5}$), the value of Nusselt number is 4 to 5 times higher than that of at $\zeta = 0.001$). (No analytically or numerically obtained values of Nusselt number are available in literature for $\zeta < 0.0001$.)

Figure–1 shows radial temperature profile for different axial distance for constant wall temperature and fully developed velocity profile. Temperature distribution for

ζ (or z^*) $\leq 10^{-5}$ is not shown on the Figure because plotting (or printing) machine is not capable of distinguishing the difference at this region. But this difference is more important. Due to this difference of stiffness at the wall, the local and mean Nusselt number varies significantly in this region as shown in Figure–2 and Figure–3.

The effect of number of collocating points is shown in Figure–2 and Figure–3 for local and mean Nusselt numbers respectively. From these two Figures it is clear that for normalized axial distance $\zeta \geq 0.001$, the effect of higher collocating points is insignificant i. e., almost accurate results could be obtained using $N = 3$ or 6. But for $\zeta < 0.001$, the effect of collocating points is very significant. For example, at $\zeta = 10^{-5}$, the value of Nusselt number for $N = 6$ is more than 100% higher than that of $N = 3$ and Nusselt number for $N = 9$ is more than 50% higher than that of $N = 6$. But the value of Nusselt number for $N = 12$ is only 1% higher than that of $N = 9$ and further increase in number of collocating points, say $N = 15$ or 18, doesn't have any significance on the value of Nusselt number but it requires 50 minutes (for $N = 15$) to 120 minutes (for $N = 18$) longer computing time compare to $N = 12$.

Figure–4 and Figure–5 show the radial temperature distribution for constant wall heat flux and fully developed velocity profile. Actually Figure–5 shows the upper portion of Figure–4 with an enlarged scale. Figure–6 and Figure–7 show the effect of collocating points on local and mean Nusselt numbers. It can also be noted from these two Figures that for normalized axial distance $\zeta > 0.001$, the value of Nusselt number remains the same as collocating points are increased further. But for $\zeta < 0.001$, specially at $\zeta = 10^{-5}$ or 10^{-6}, the difference is very significant. Increasing the number of collocating points beyond 18, does not produce any significant change in values compared to that obtained using $N = 18$, hence the values obtained using 18 collocating points can be considered quite accurate.

The variation of radial temperature profile for different Prandtl number are shown in Figure–8, 9 and 10. These three Figures show that temperature gradient at wall is higher for lower Prandtl number and the difference decreases with increase of axial distance.

Figure–11, 12 and 13 show the effect of collocating points on local Nusselt number and Figure–14, 15 and 16 show the effect of collocating points on mean Nusselt number for different Prandtl number (both are for constant wall temperature and developing velocity profile). Comparison of local and mean Nusselt number for different Prandtl number is shown in Figure–17 and 18 respectively. But these two Figures are a little bit confusing. Apparently it seems that as Pr number increases Nusselt number decreases for a particular axial distance which is contrary to fact. This is because of presence of Pr number in the dimensionless axial distance. If Pr is omitted from the dimensionless axial distance, Figure–16 looks like Figure–19 which shows clearly the variation of Nu with Pr.

Figure–8 to Figure–18 is similar in nature with Figure–20 to Figure–28 but the later are for constant wall heat flux and developing velocity profile.

Figure–31 shows the variations of local Nusselt number with dimensionless axial distance for constant wall temperature and fully developed velocity profile but the solution is obtained by finite difference technique i. e., eq. (3-8) is discretized in the r direction and then the resulting set of ordinary differential equations are solved using **BDFSH** integrator. This Figure shows that reasonably accurate results can not be obtained in the entrance region using less than 40 grids. Figure–32 shows 15 collocating points give better results than 80 grids and requires much less computing time.

Figure–33 compares the local Nusselt number obtained by collocation method with that obtained by Galerkin's method. This Figure shows that for same values of N, Galerkin's method gives better results. This is because Galerkin's method is inherently "method of coefficients" whereas the method which we used to formulate the solution in orthogonal collocation technique is the "method of ordinates". And the "method of coefficients" is more accurate and its convergence can be more readily assessed by comparing the coefficients at successive values of N (see ref. 56).

Variation of temperature profile for constant and variable property of the fluid for constant wall temperature and constant wall heat flux are shown in Figure–34 and 35 respectively. These Figures show that when physical properties vary with temperature, temperature gradient at wall becomes higher and results higher values of Nusselt number. This is shown in Figure–36.

CHAPTER 5
SUGGESTIONS AND RECOMMENDATIONS

In this work parabolic profile is used for fully developed velocity profile. For developing velocity profile, Langhaar[2] velocity is used as axial component of velocity and radial component is obtained from Langhaar velocity distribution and equation of continuity. This seems to be alright for constant physical properties and incompressible flow, because these assumptions allow an independent solution of the hydrodynamic problem since the momentum and energy equations are not coupled.

But the results that had been obtained considering change in fluid properties with temperature and at the same time using Langhaar velocity is inherently questionable. Because, Langhaar obtained the velocity profile with linearizing the momentum equation and assuming constant physical properties. And since energy equation doesn't contain the viscosity term, which remains implicitly inside the velocity distribution, these results, in fact, ignore the most dominant physical property, viscosity, variation with temperature. But since all other physical properties (heat capacity, thermal conductivity and density) have been taken into consideration, these results may be accepted qualitatively. For example, Figure–34 and Figure–35 show that when the physical properties vary with temperature, the temperature profile becomes stiffer near the wall compared to that obtained using constant physical properties, that means temperature profile developes slowly and thereby resulting higher values of Nusselt number.

Throughout this work, axial conduction term is dropped from the energy equation. This is done because of two reasons : first, it is assumed that at reasonably high Peclet number, its effect is insignificant, second, because of mathematical difficulties to apply orthogonal collocation technique or Galerkin's technique.

Hence, in order to get more accurate results from practical point of view, the momentum and energy equations should be solved simultaneously without neglecting axial conduction term and giving appropriate consideration to physical property variations with temperature. Probably it would be too much complicated to apply orthogonal collocation or Galerkins technique but finite difference method may be tried. In my view, finite element method would be appropriate but in that case computing machine would be an important factor because this method would require, for this particular case, large number of matrices with huge dimension in order to ensure reasonable accuracy. Either a mainframe computer or a PC with a 80386 main processor and having at least 2 megabytes random access memory is recommended.

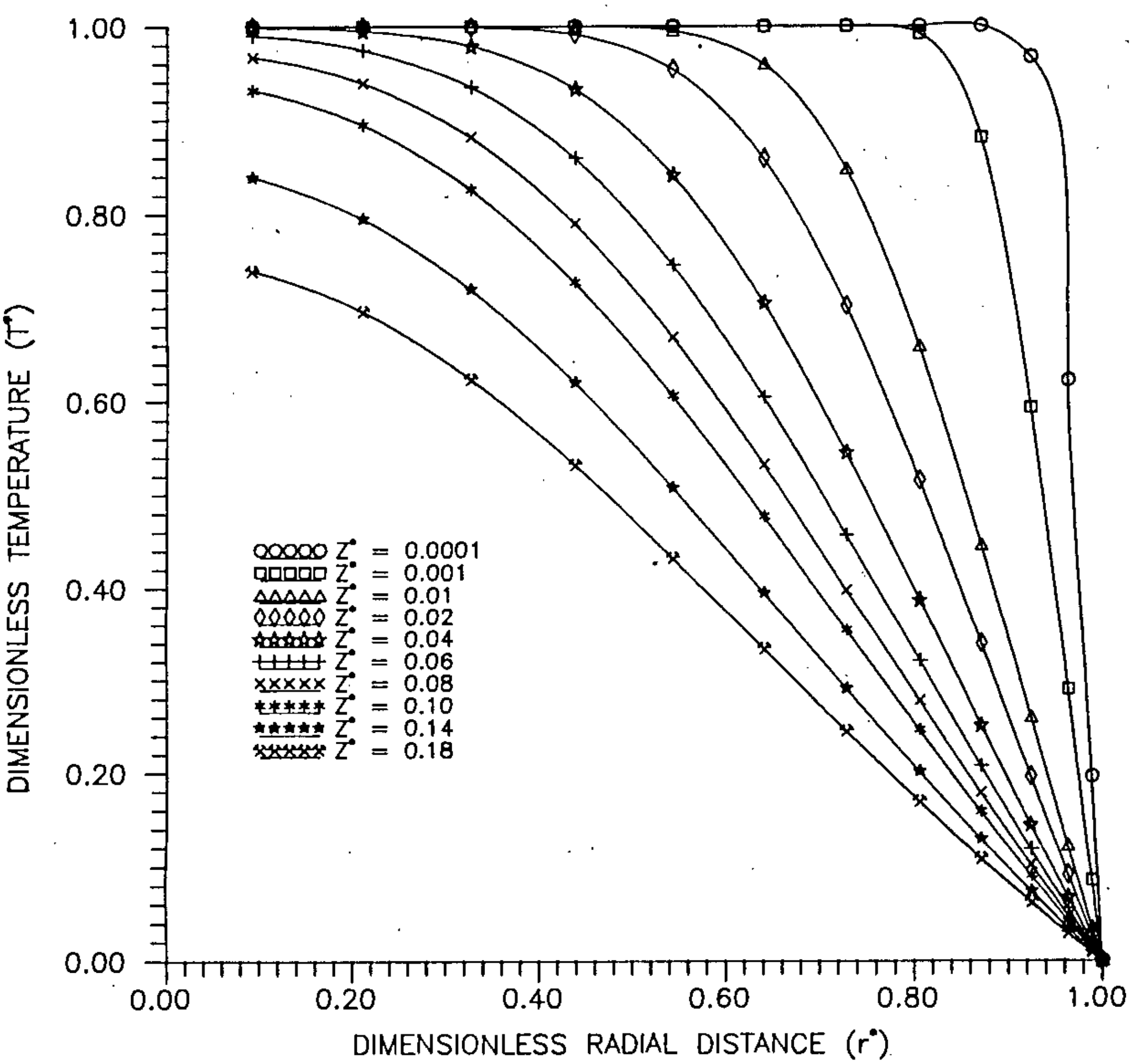

FIG.– 1 RADIAL TEMPERATURE PROFILE FOR DIFFERENT AXIAL DISTANCE, 4(X/D)/(Re.Pr), FOR CONSTANT WALL TEMPERATURE AND FULLY DEVELOPED VELOCITY PROFILE.
(12 COLLOCATING POINTS)

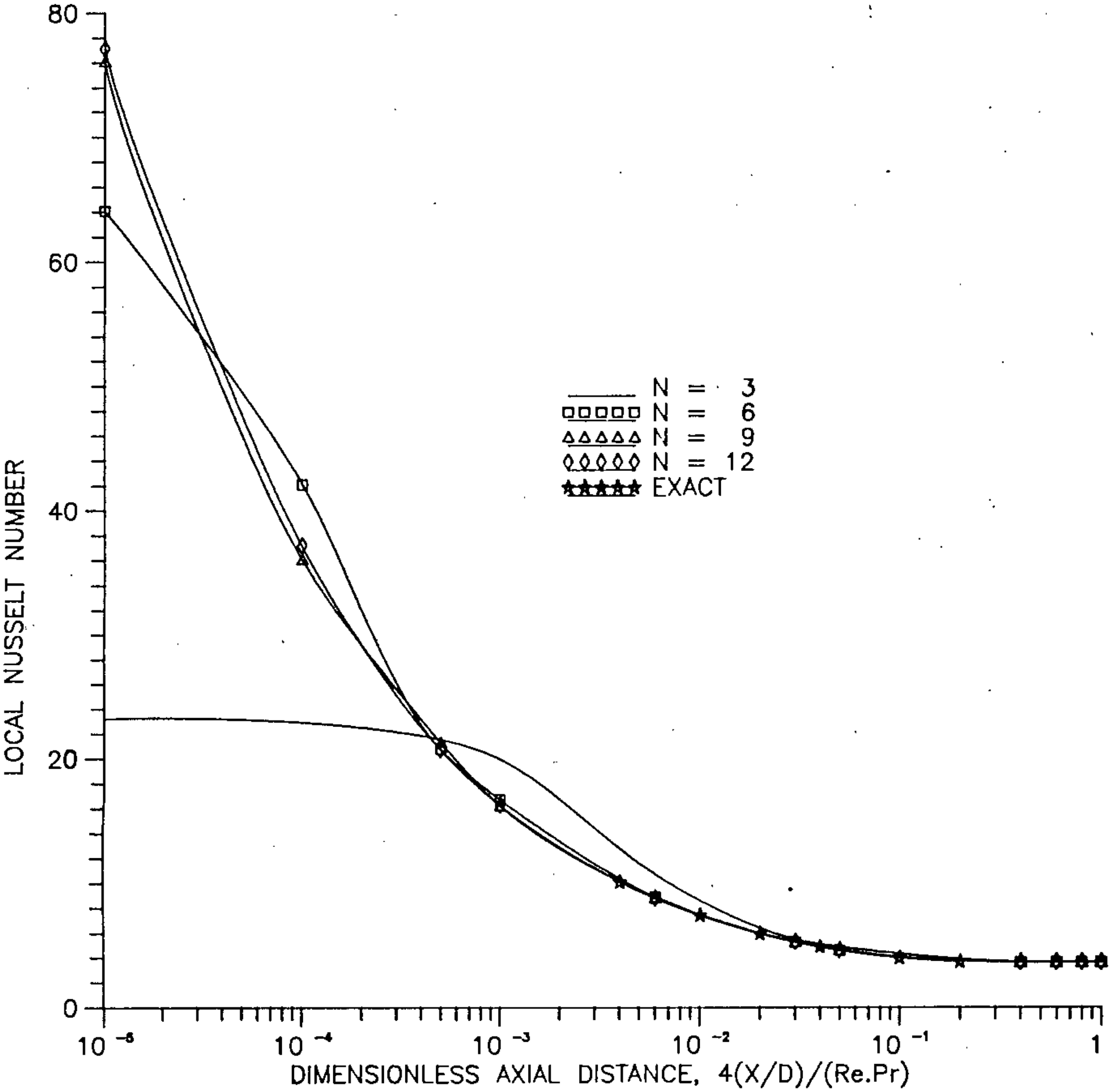

FIG.— 2 VARIATION OF LOCAL NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE FOR CONTANT WALL TEMPERATURE AND FULLY DEVELOPED VELOCITY PROFILE. (N = Number of collocating point)

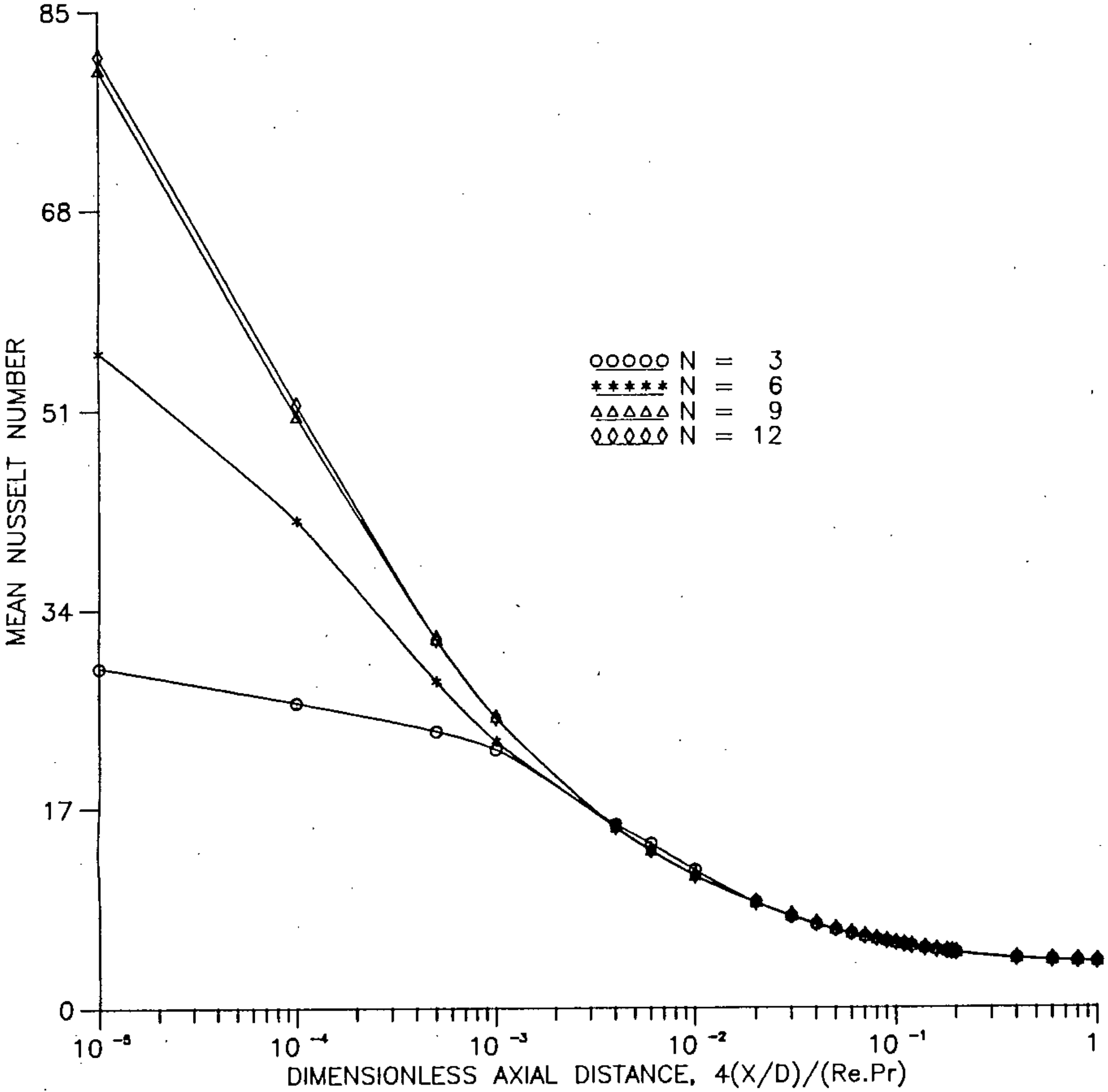

FIG.– 3 VARIATION OF MEAN NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE FOR CONSTANT WALL TEMPERATURE AND FULLY DEVELOPED VELOCITY PROFILE. (N = Number of collocating point)

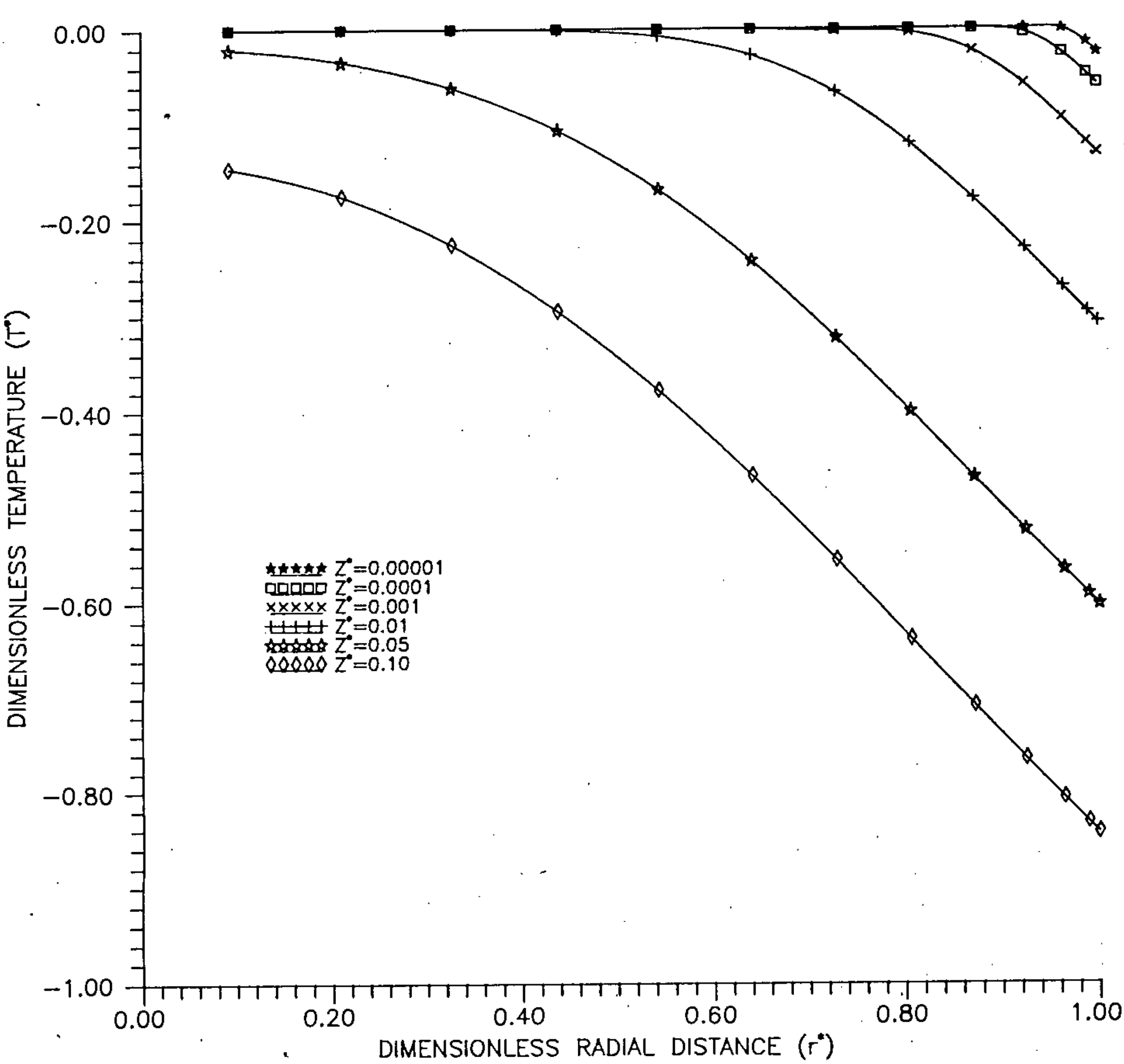

FIG.− 4 RADIAL TEMPERATUE PROFILE FOR DIFFERENT AXIAL DISTANCE, X/R/(Re.Pr), FOR CONSTANT WALL HEAT FLUX AND FULLY DEVELOPED VELOCITY PROFILE. (12 COLLOCATING POINTS)

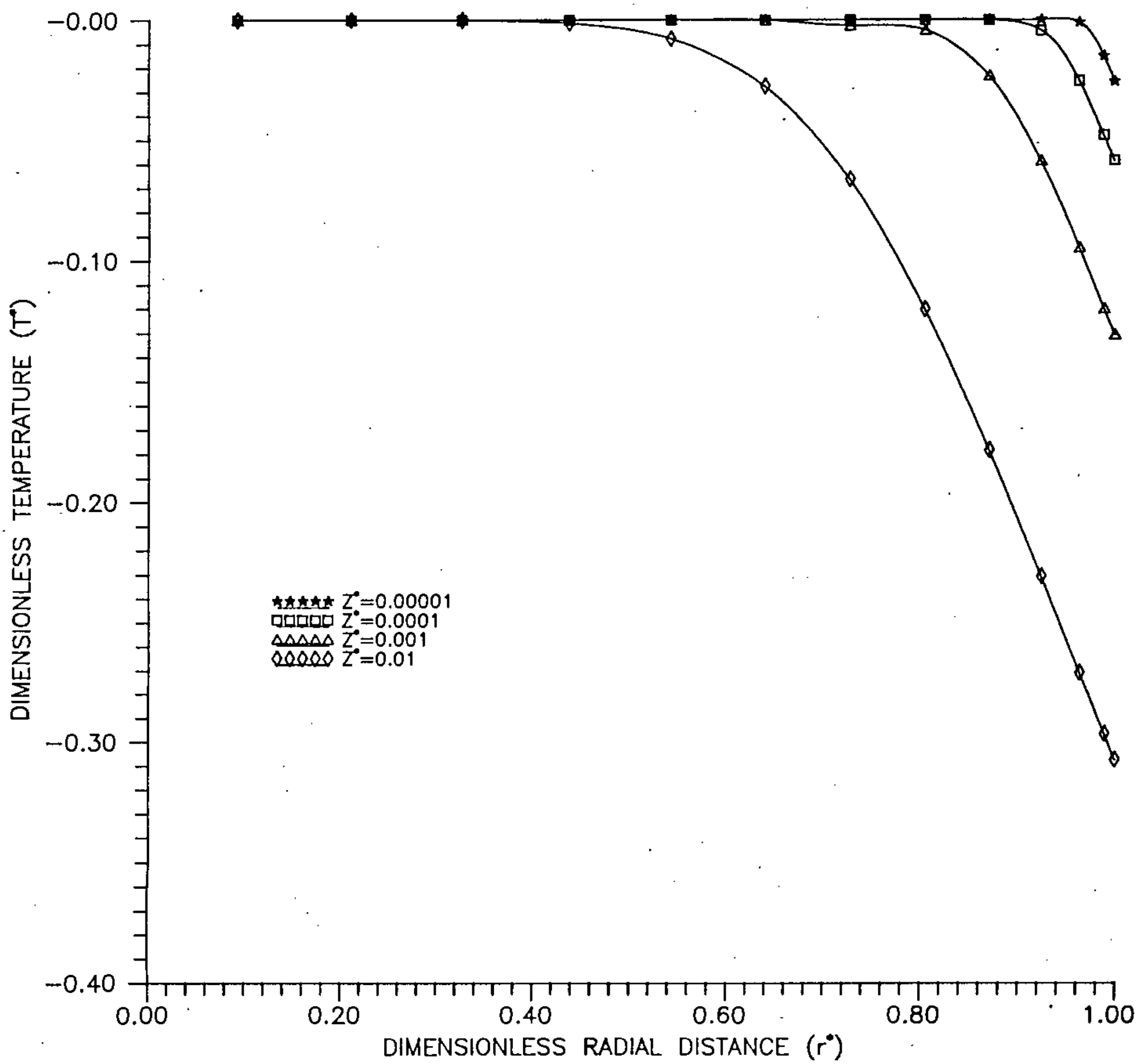

FIG.— 5 RADIAL TEMPERATUE PROFILE FOR DIFFERENT AXIAL DISTANCE, X/R/(Re.Pr), FOR CONSTANT WALL HEAT FLUX AND FULLY DEVELOPED VELOCITY PROFILE. (12 COLLOCATING POINTS) ($Z^* = (X/R)/(Re.Pr)$)

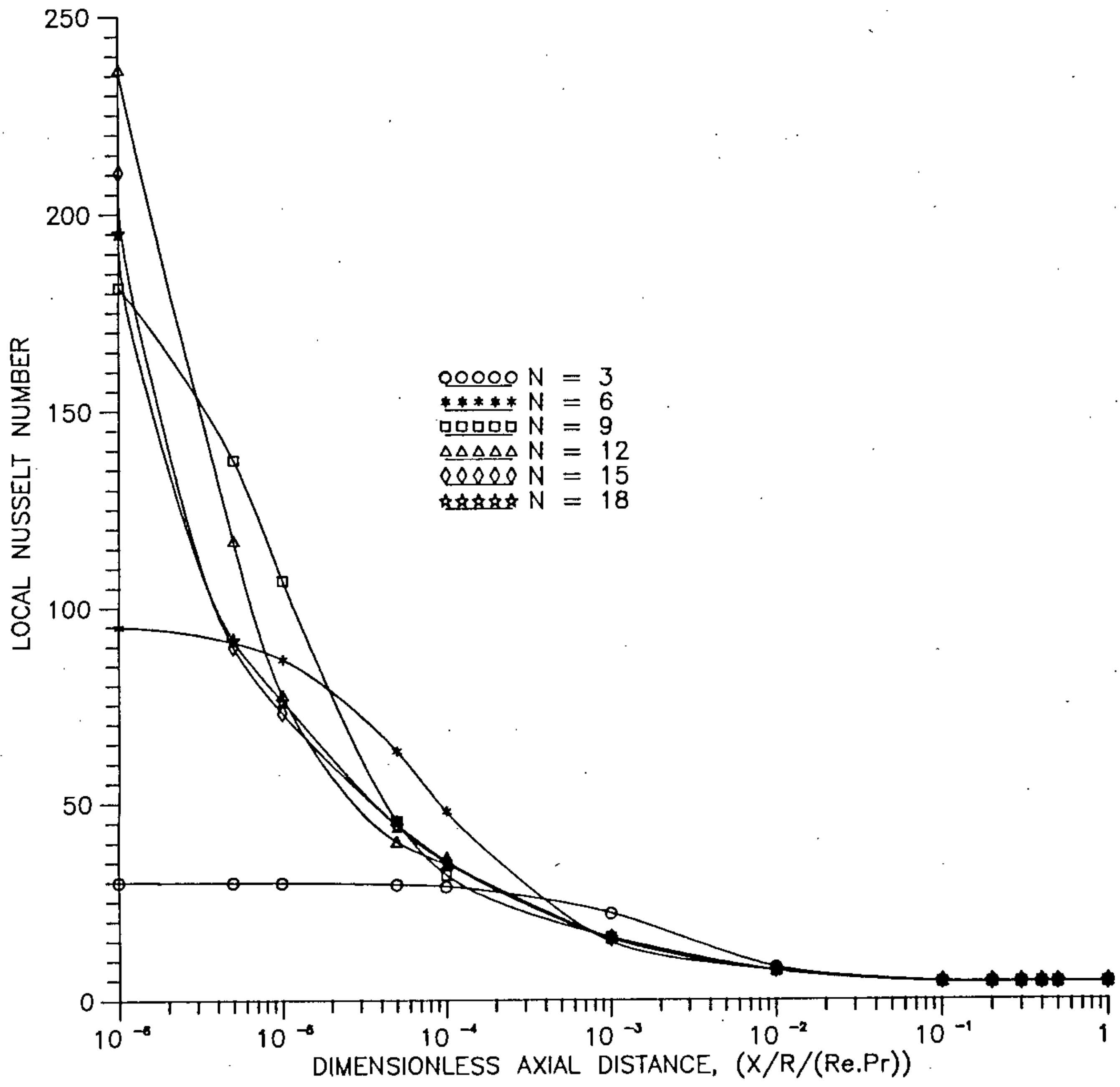

FIG.– 6 VARIATION OF LOCAL NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE FOR DIFFERENT COLLOCATING POINTS. (FOR CONSTANT WALL HEAT FLUX AND FULLY DEVELOPED VELOCITY PROFILE)

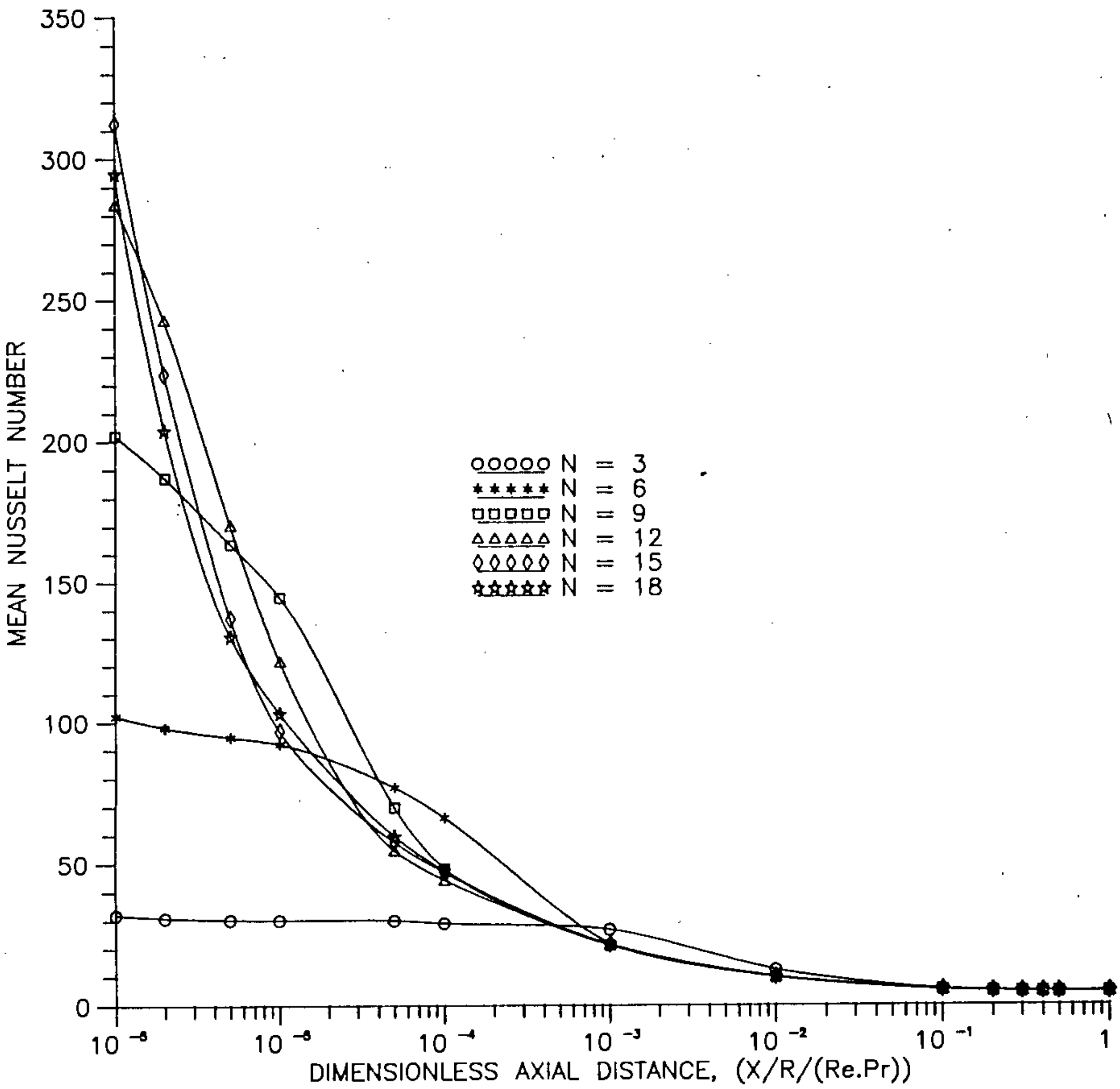

FIG.— 7 VARIATION OF MEAN NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE FOR DIFFERENT COLLOCATING POINTS. (FOR CONSTANT WALL HEAT FLUX AND FULLY DEVELOPED VELOCITY PROFILE)

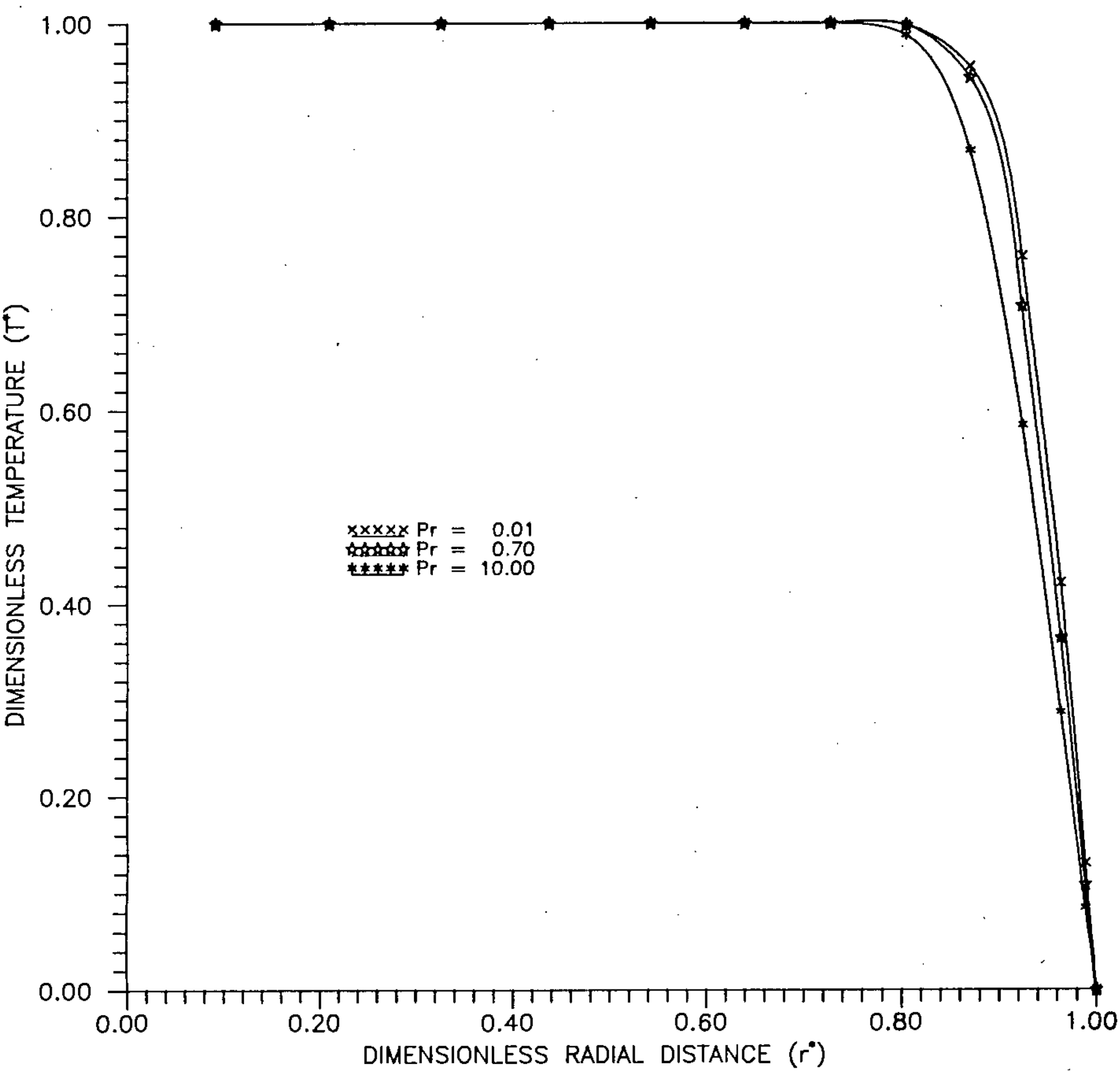

FIG.— 8 RADIAL TEMPERATURE PROFILE FOR SAME AXIAL DISTANCE (X/R/(Re.Pr)=0.001) AND COLLOCATING POINTS (N=12) WITH DIFFERENT Pr.
(FOR CONSTANT WALL TEMPERATURE AND DEVELOPING VELOCITY PROFILE)

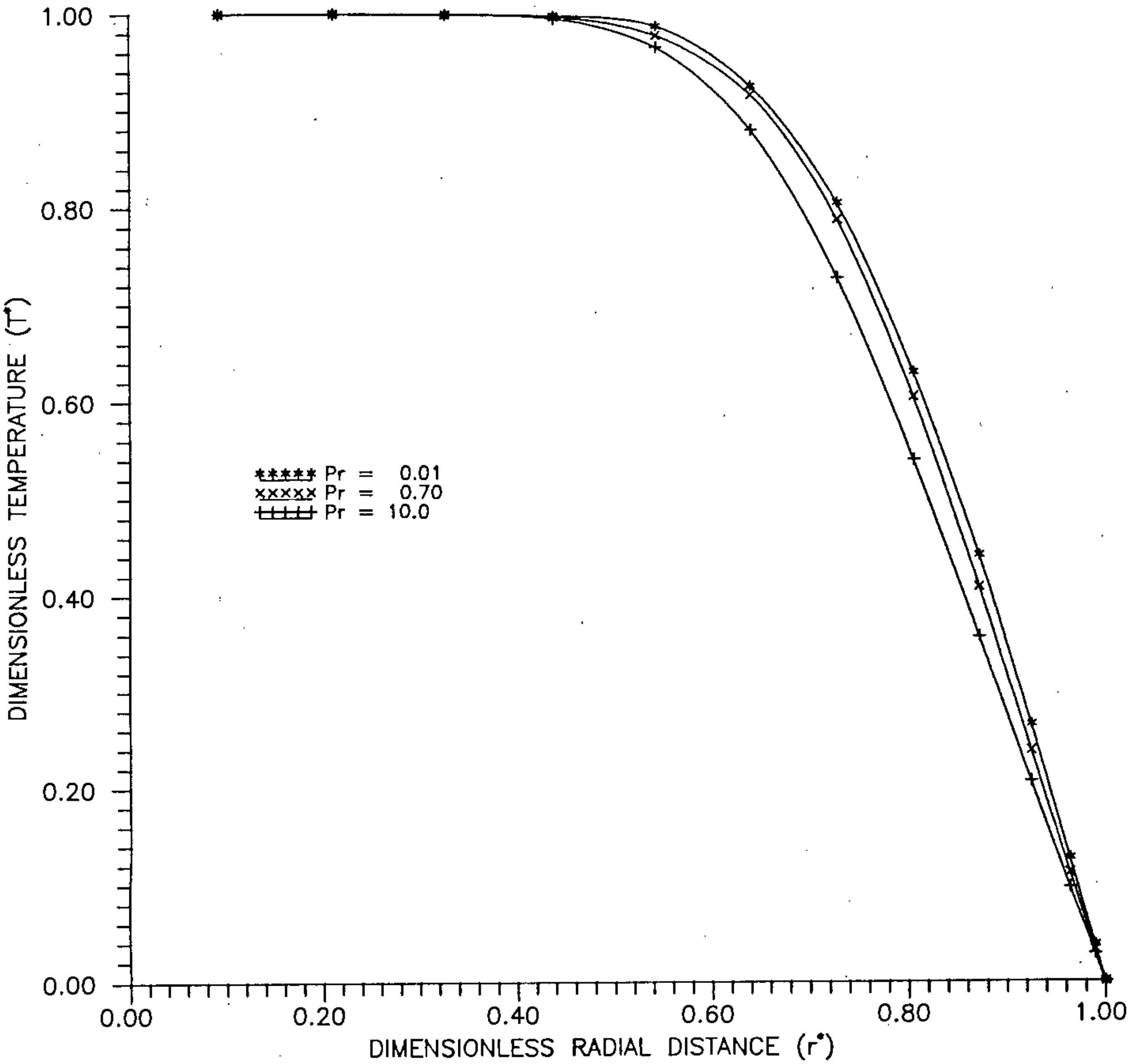

FIG.— 9 RADIAL TEMPERATURE PROFILE FOR DIFFERENT Pr FOR AXIAL DISTANCE ,X/R/(Re.Pr)=0.01 AND 12 COLLOCATING POINTS.
(FOR CONSTANT WALL TEMPERATURE AND DEVELOPING VELOCITY PROFILE)

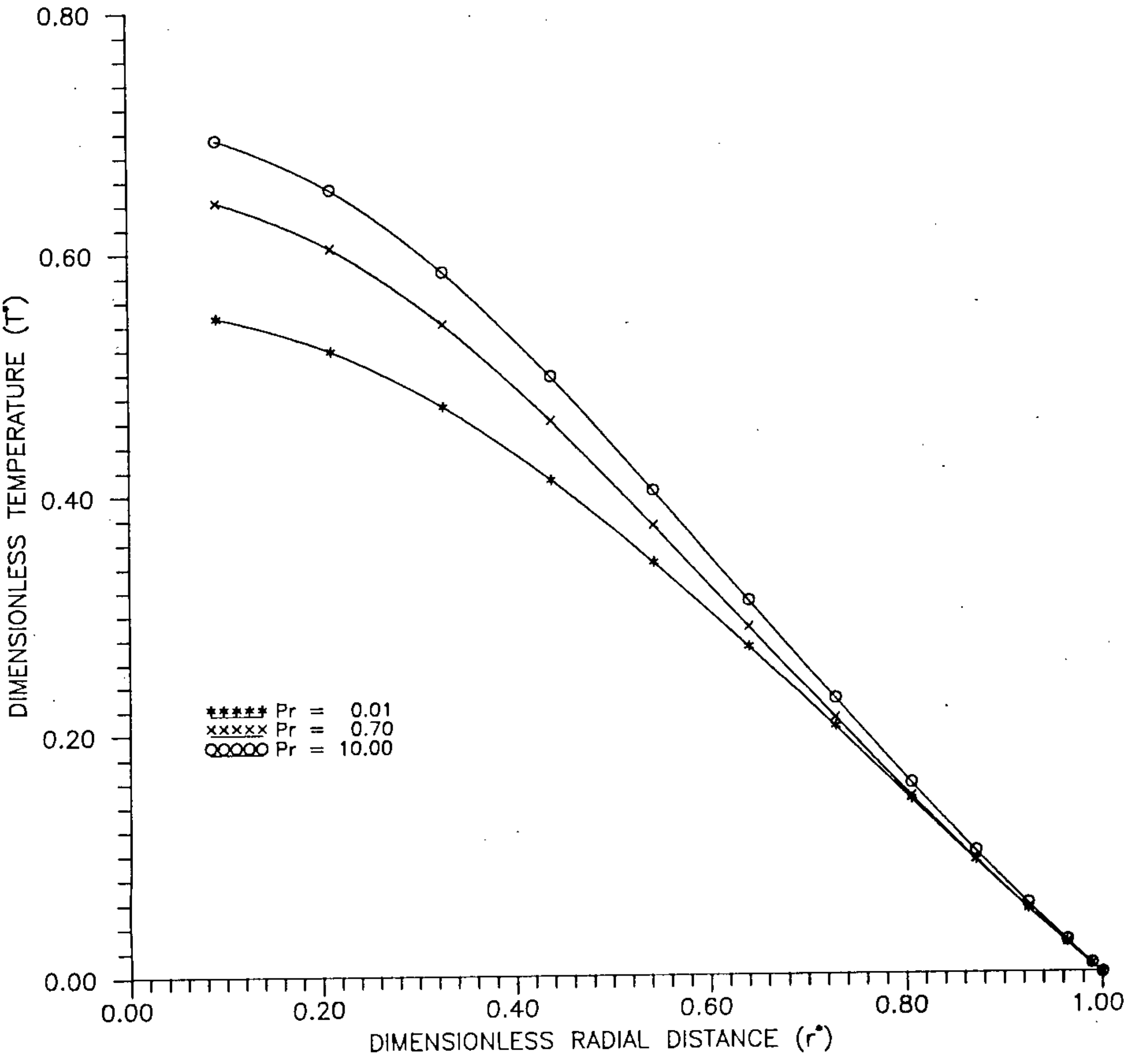

FIG.—10 RADIAL TEMPERATURE PROFILE FOR DIFFERENT Pr FOR AXIAL DISTANCE ,X/R/(Re.Pr)=0.1 AND 12 COLLOCATING POINTS.
(FOR CONSTANT WALL TEMPERATURE AND DEVELOPING VELOCITY PROFILE)

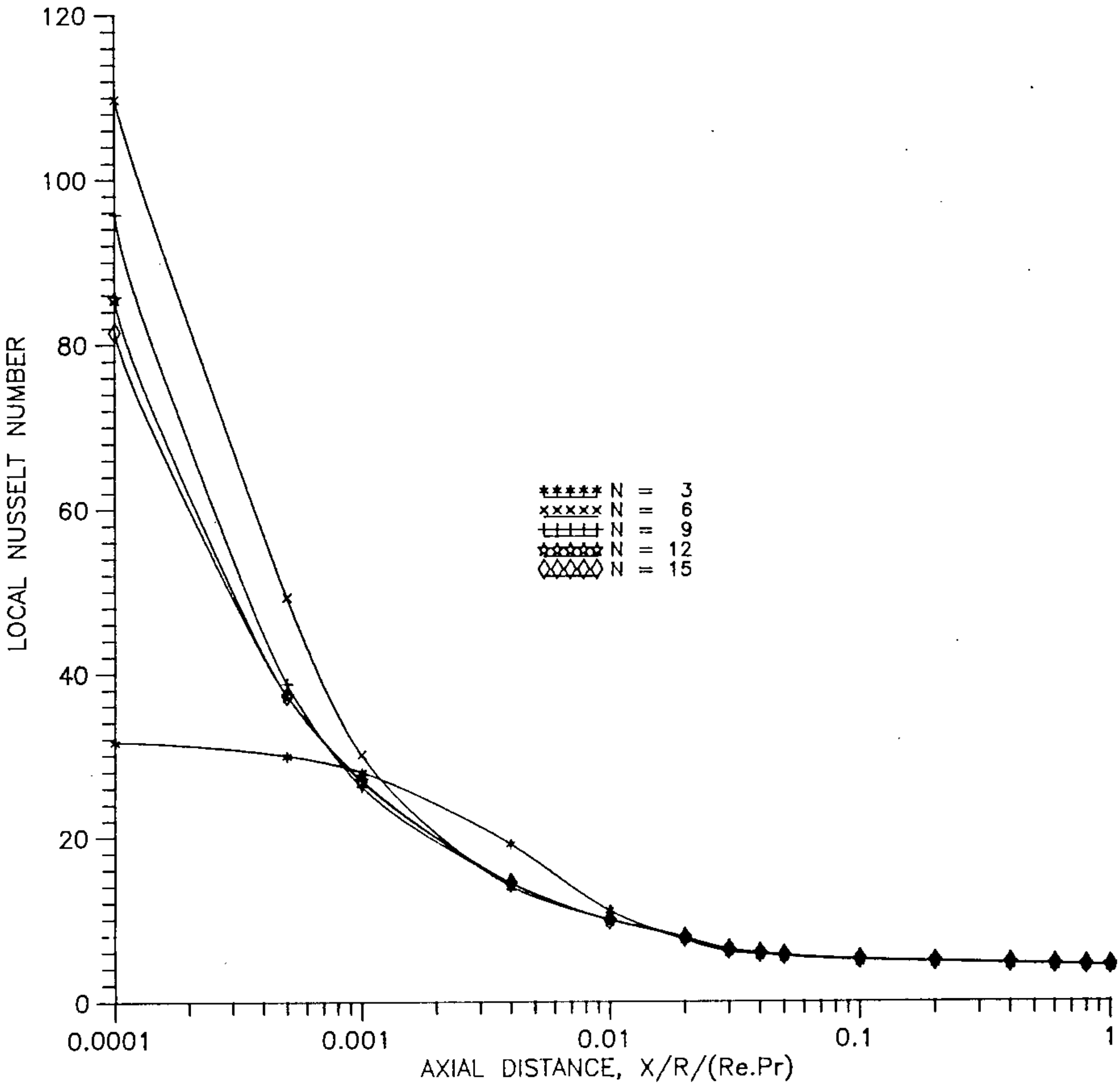

FIG.— **11** VARIATION OF LOCAL NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE (X/R/(Re.Pr)). (FOR CONSTANT WALL TEMPERATURE AND DEVELOPING DEVELOPING VELOCITY PROFILE WITH Pr = 0.01)

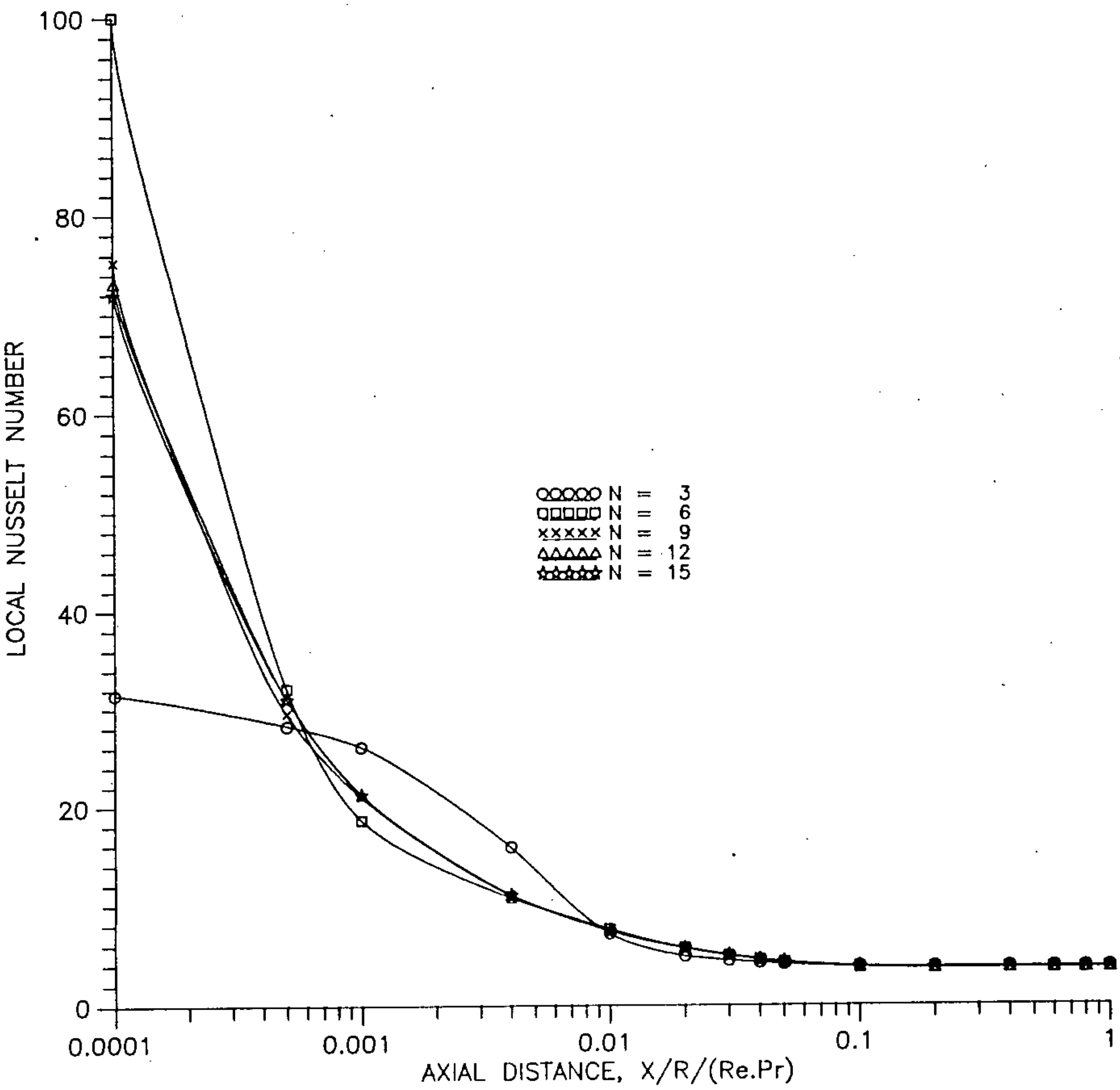

FIG.— 12 VARIATION OF LOCAL NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE (X/R/(Re.Pr)). (FOR CONSTANT WALL TEMPERATURE AND DEVELOPING VELOCITY PROFILE WITH Pr = 0.70)

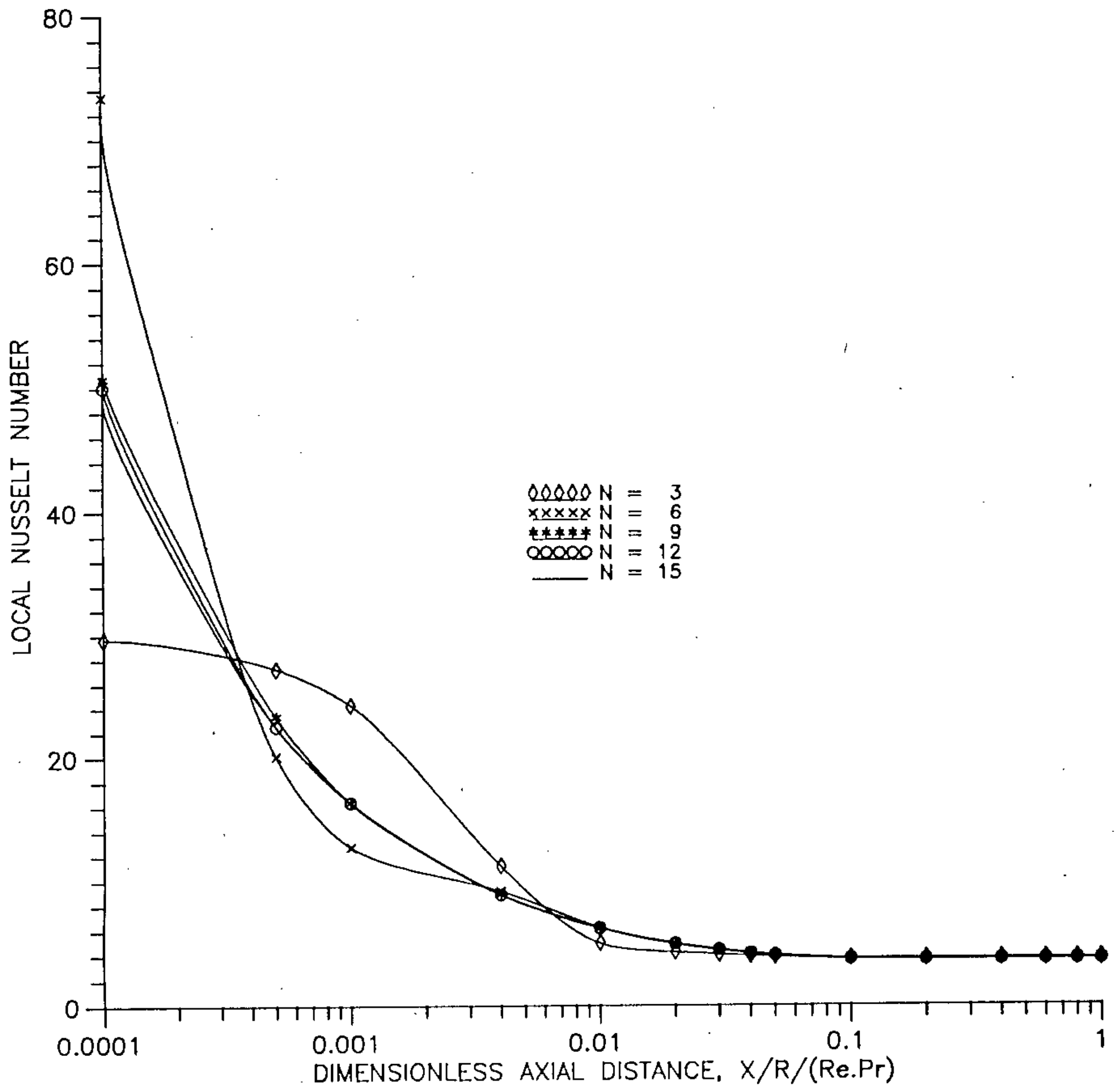

FIG.— 13 VARIATION OF LOCAL NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE (X/R/(Re.Pr)). (FOR CONSTANT WALL TEMPERATURE AND DEVELOPING VELOCITY PROFILE WITH PR = 10)

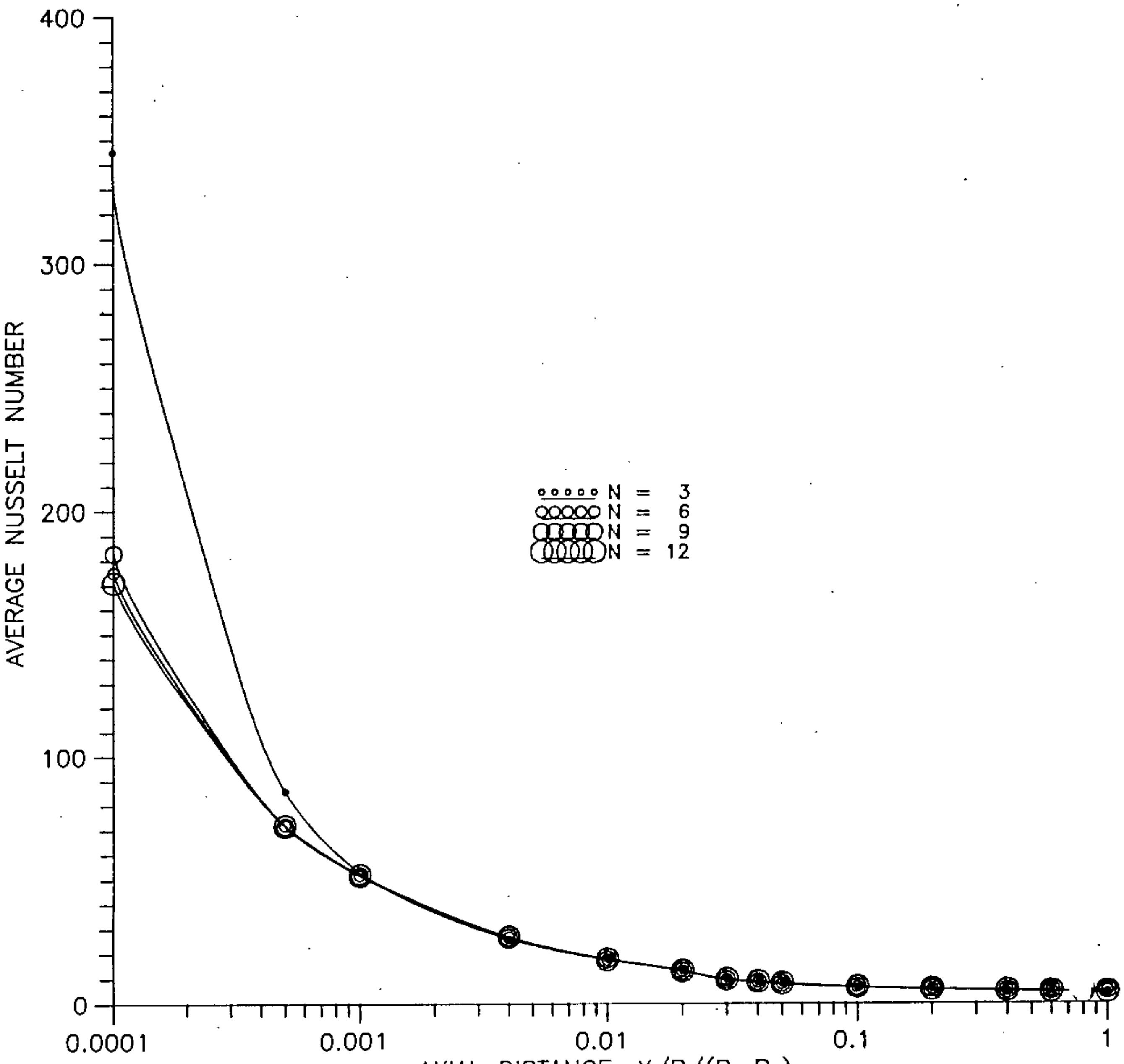

FIG.— **14** VARIATION OF AVERAGE NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE (X/R/(Re.Pr)). (FOR CONSTANT WALL TEMPERATURE AND DEVELOPING VELOCITY PROFILE WITH Pr = 0.01)

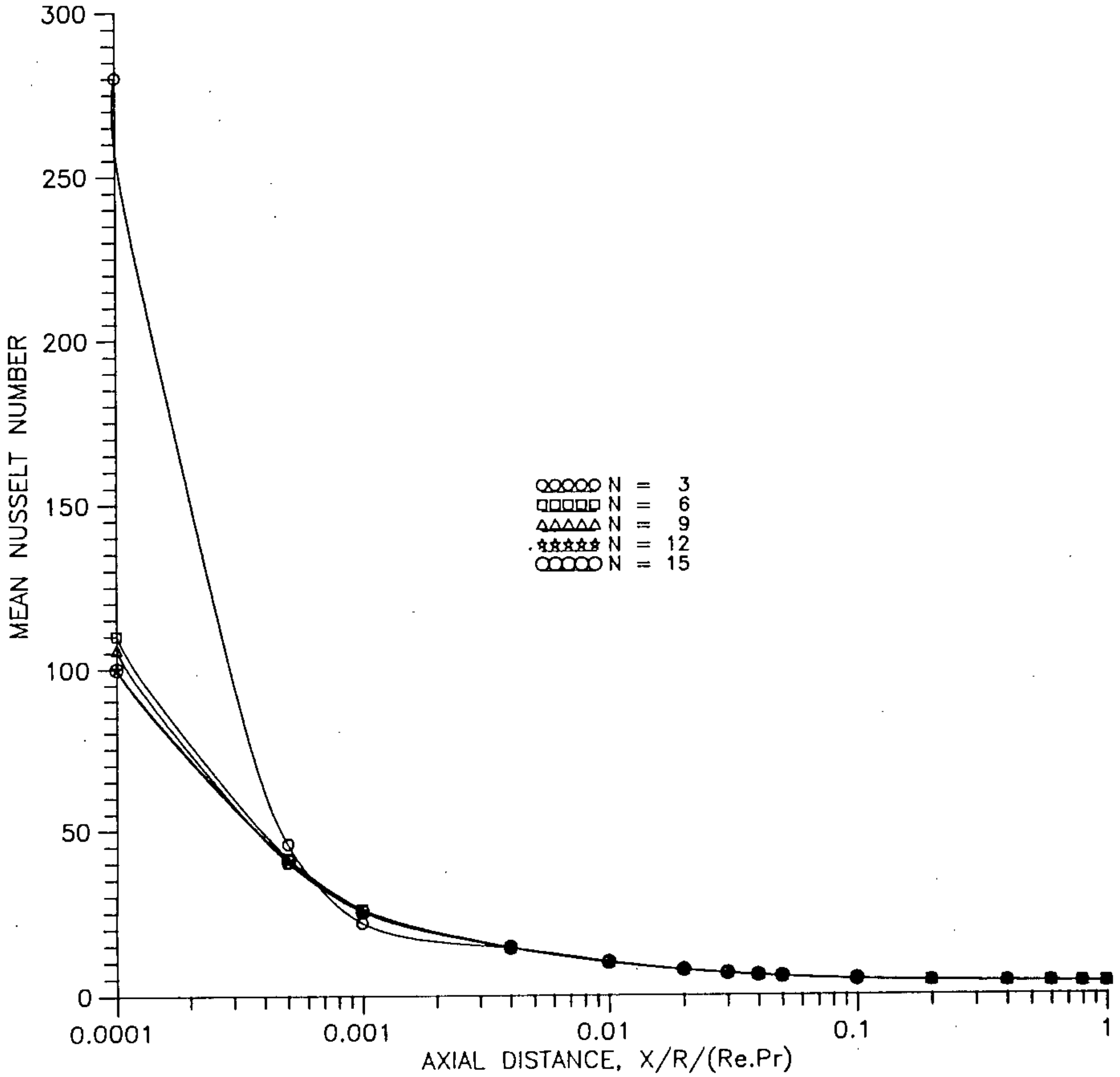

FIG.-15 VARIATION OF MEAN NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE (X/R/(Re.Pr)). (FOR CONSTANT WALL TEMPERATURE AND DEVELOPING VELOCITY PROFILE WITH Pr = 0.70)

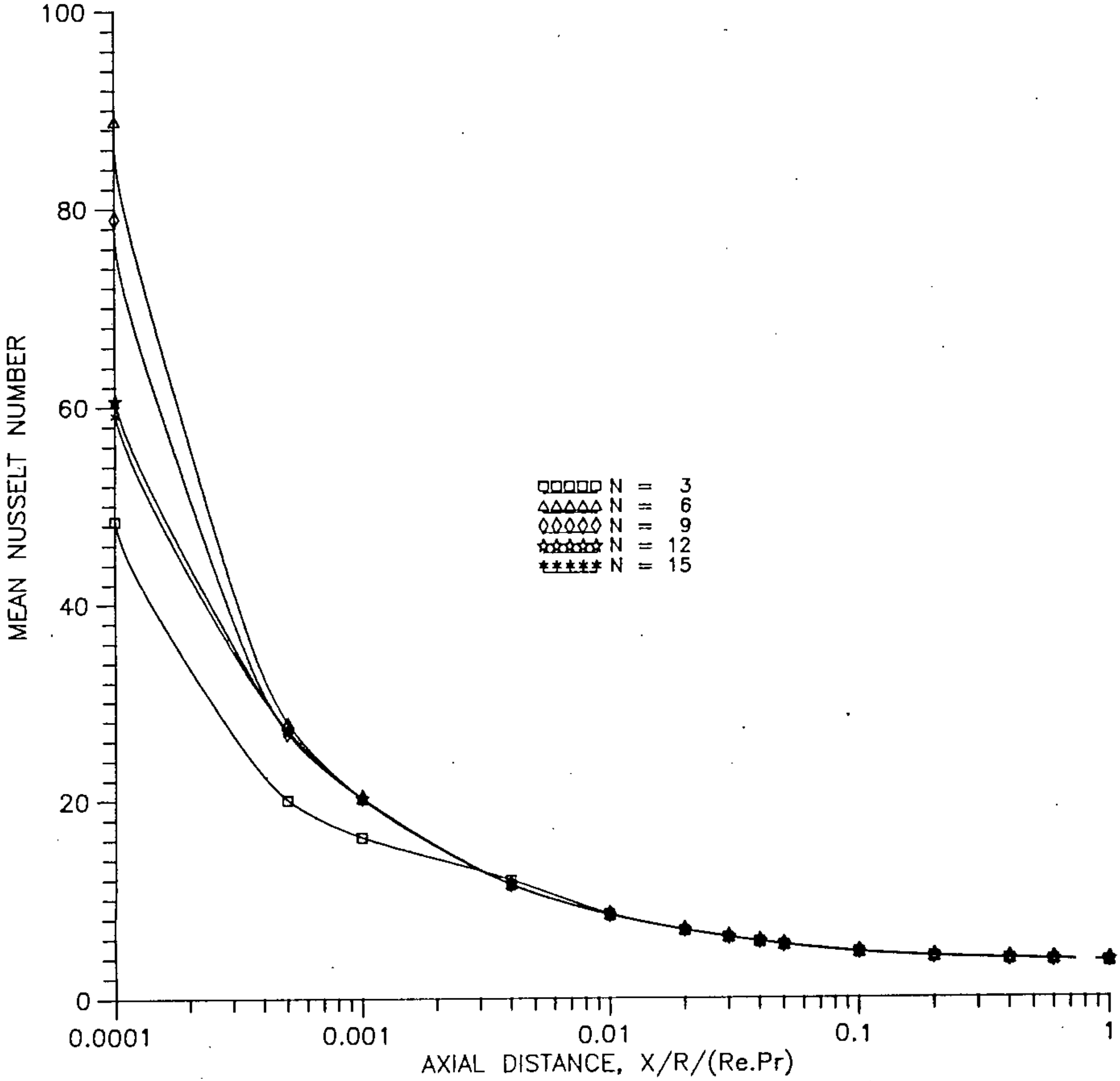

FIG.—16 VARIATION OF MEAN NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE (X/R/(Re.Pr)). (FOR CONSTANT WALL TEMPERATURE AND DEVELOPING VELOCITY PROFILE WITH Pr = 10)

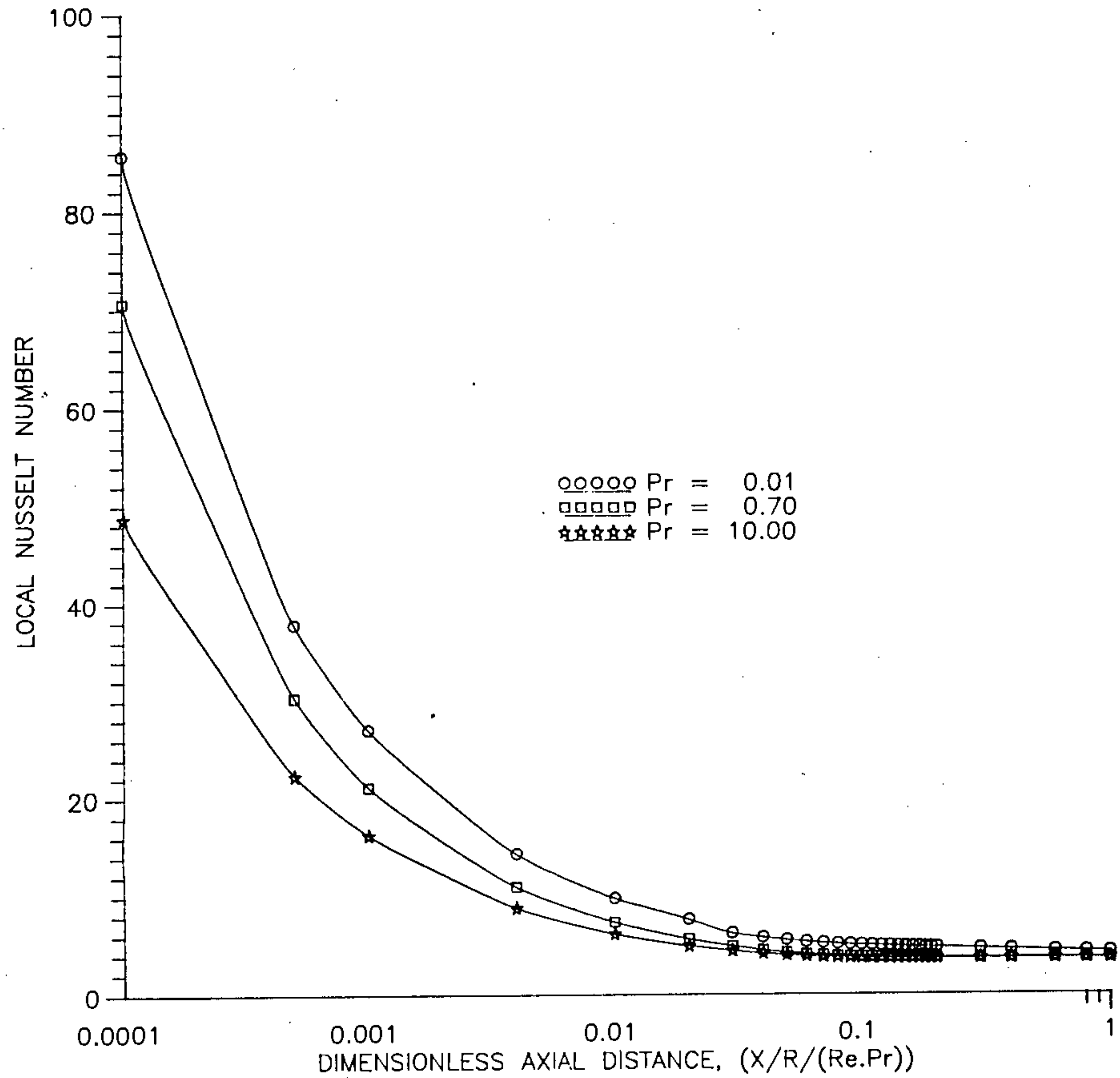

FIG.—17 VARIATION OF LOCAL NUSSELT NUMBER FOR DIFFERENT AXIAL DISTANCE WITH Pr AS A PARAMETER FOR CONSTANT WALL TEMPERATURE AND DEVELOPING VELOCITY PROFILE (15 COLLOCATING POINTS).

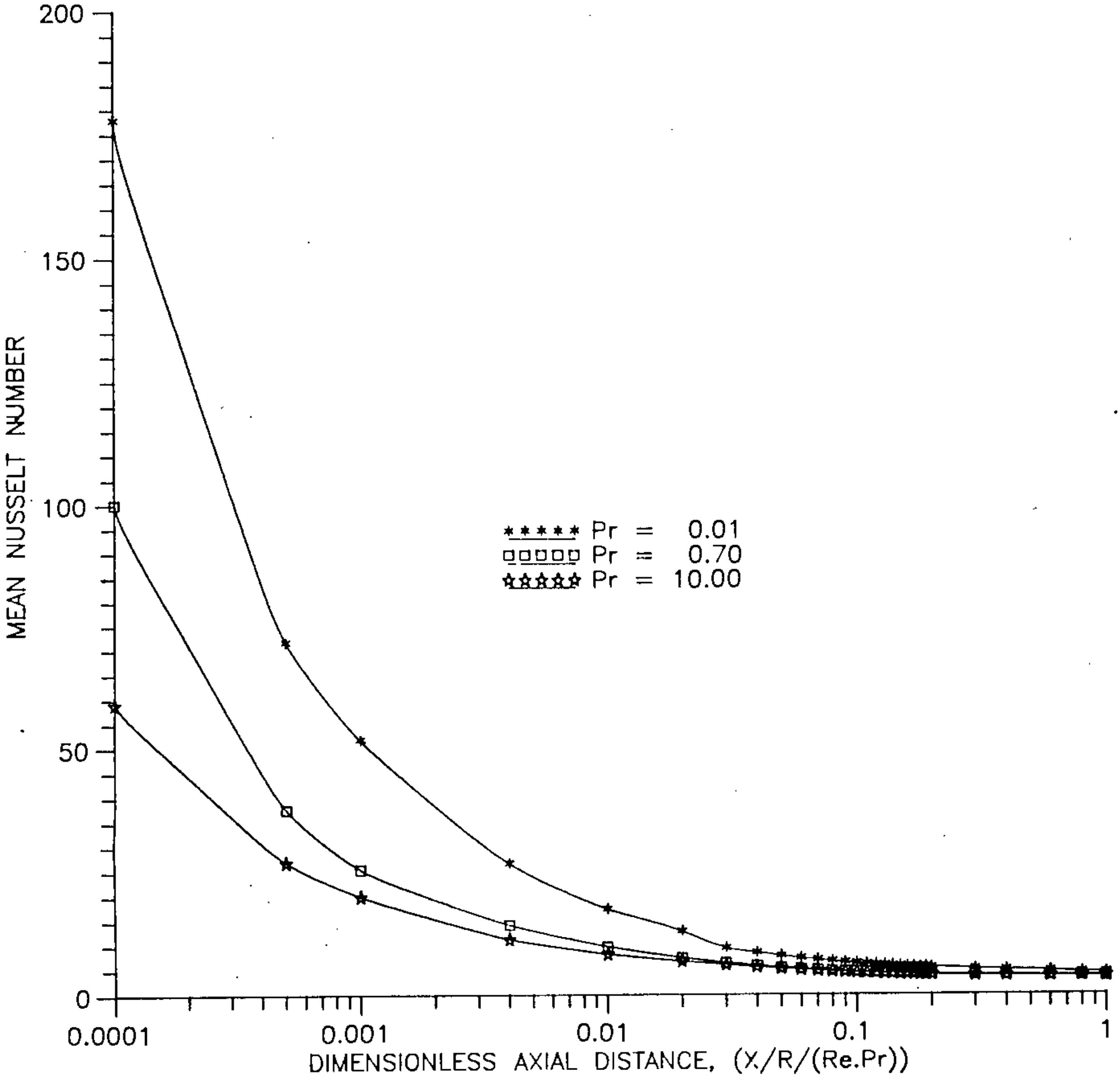

FIG.— 18 VARIATION OF MEAN NUSSELT NUMBER FOR DIFFERENT AXIAL DISTANCE WITH Pr AS A PARAMETER FOR CONSTANT WALL TEMPERATURE AND DEVELOPING VELOCITY PROFILE (15 COLLOCATING POINTS).

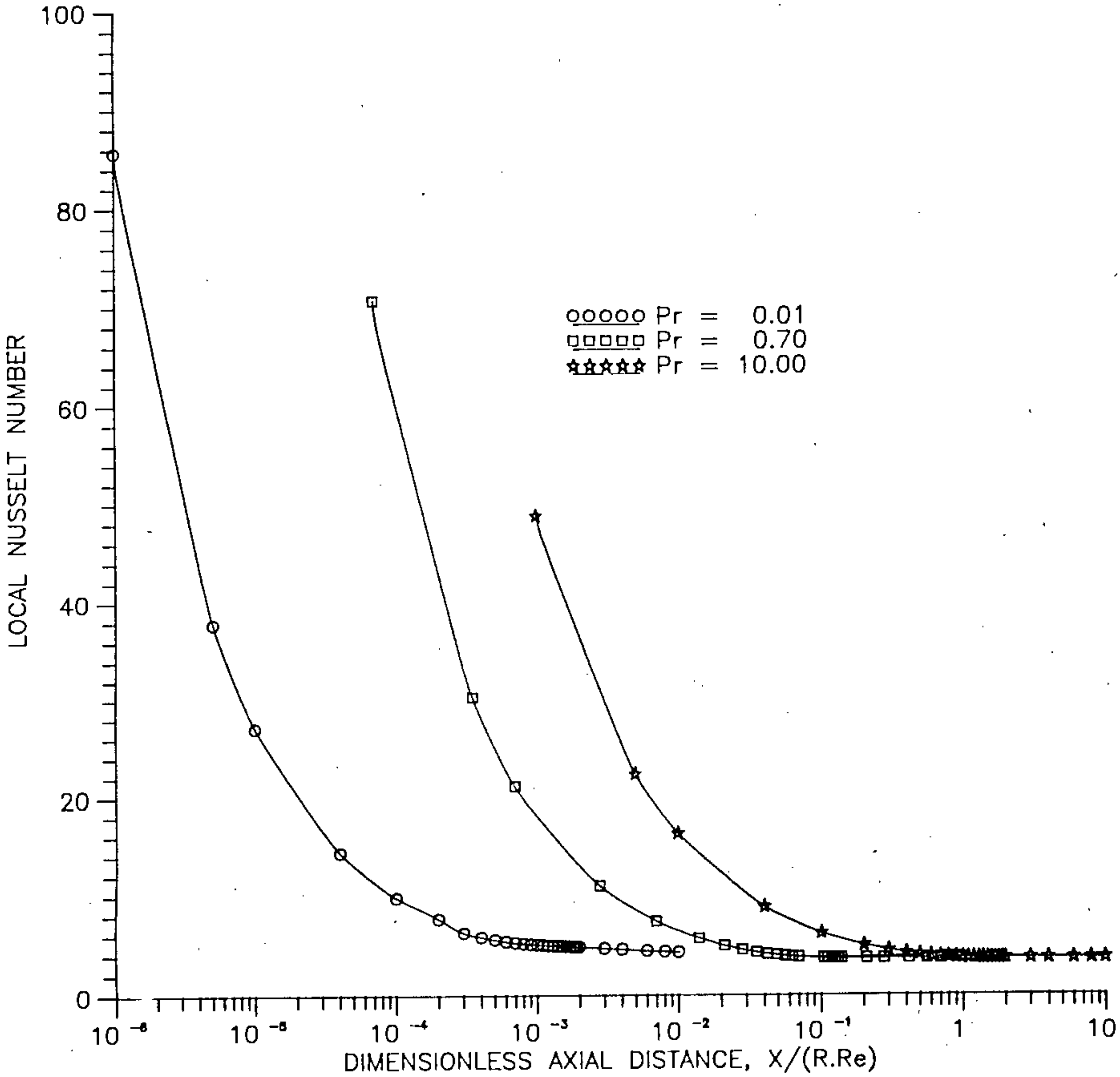

FIG.— 19 VARIATION OF LOCAL NUSSELT NUMBER FOR DIFFERENT AXIAL DISTANCE WITH Pr AS A PARAMETER FOR CONSTANT WALL TEMPERATURE AND DEVELOPING VELOCITY PROFILE (15 COLLOCATING POINTS).

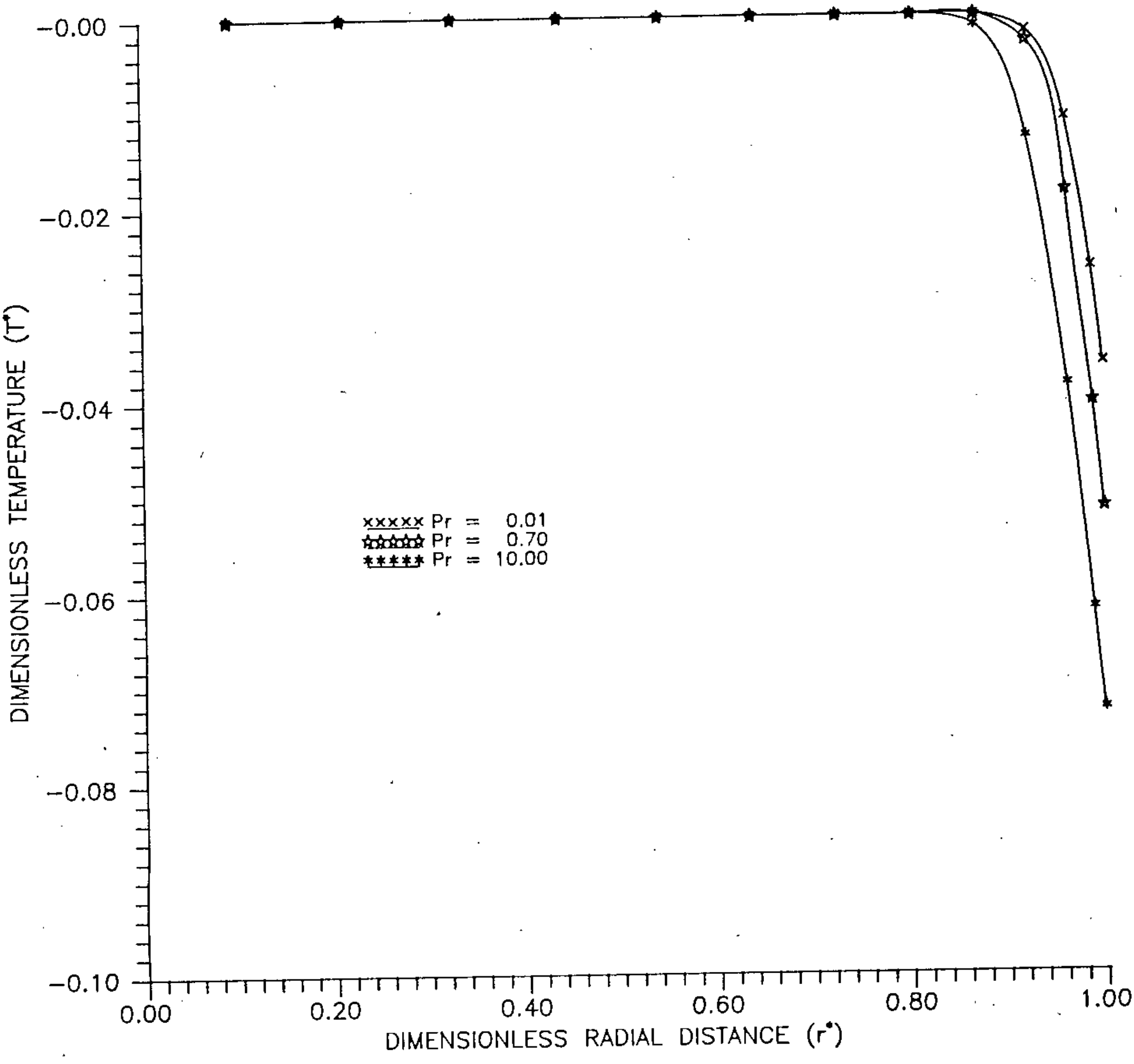

FIG.– 20 RADIAL TEMPERATURE PROFILE FOR SAME AXIAL DISTANCE (X/R/(Re.Pr)=0.001) AND COLLOCATING POINTS (N=12) WITH DIFFERENT Pr.
(FOR CONSTANT WALL HEAT FLUX AND DEVELOPING VELOCITY PROFILE)

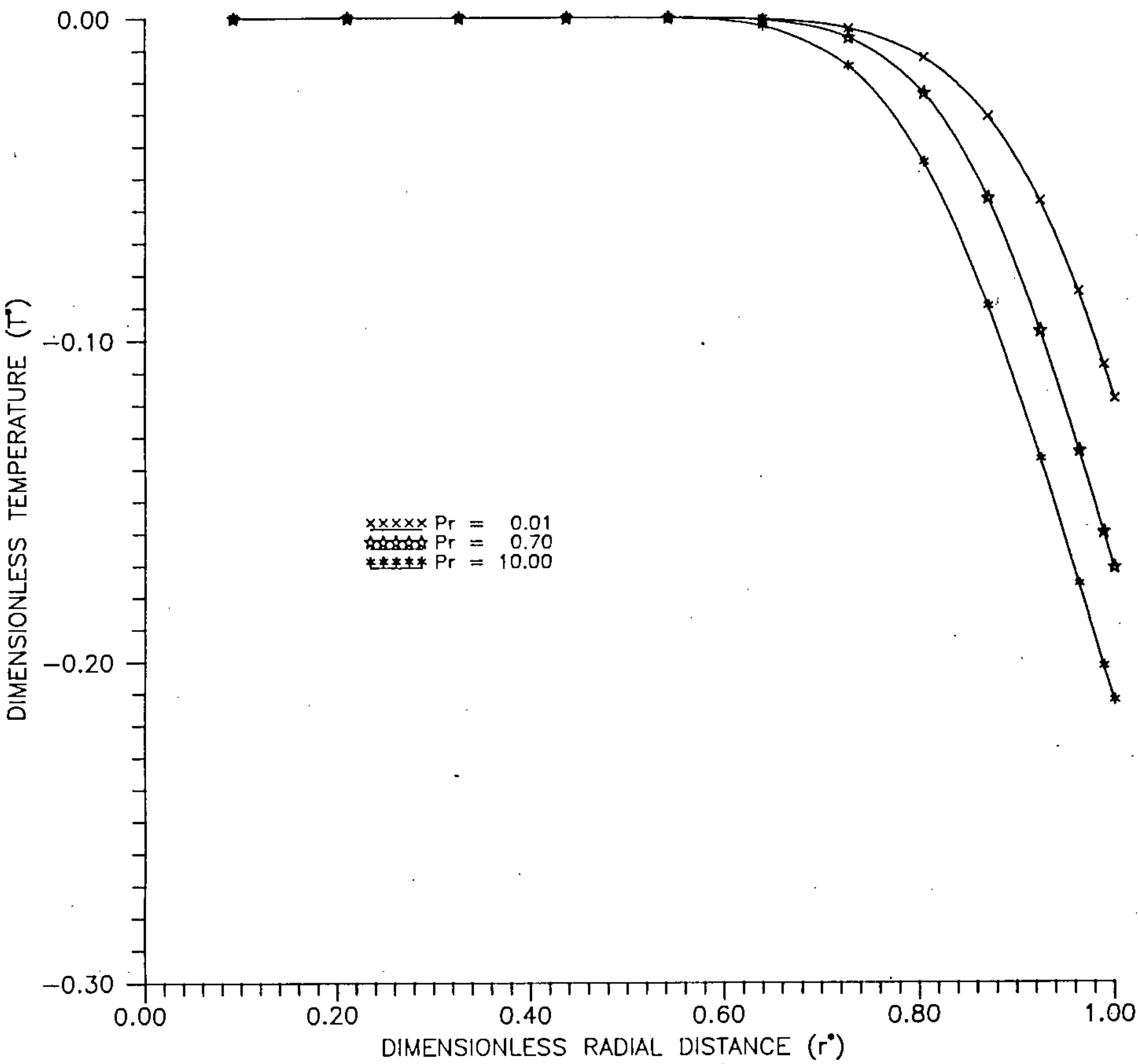

FIG.– 21 RADIAL TEMPERATURE PROFILE FOR SAME AXIAL DISTANCE (X/R/(Re.Pr)=0.01) AND COLLOCATING POINTS (N=12) WITH DIFFERENT Pr. (FOR CONSTANT WALL HEAT FLUX AND DEVELOPING VELOCITY PROFILE)

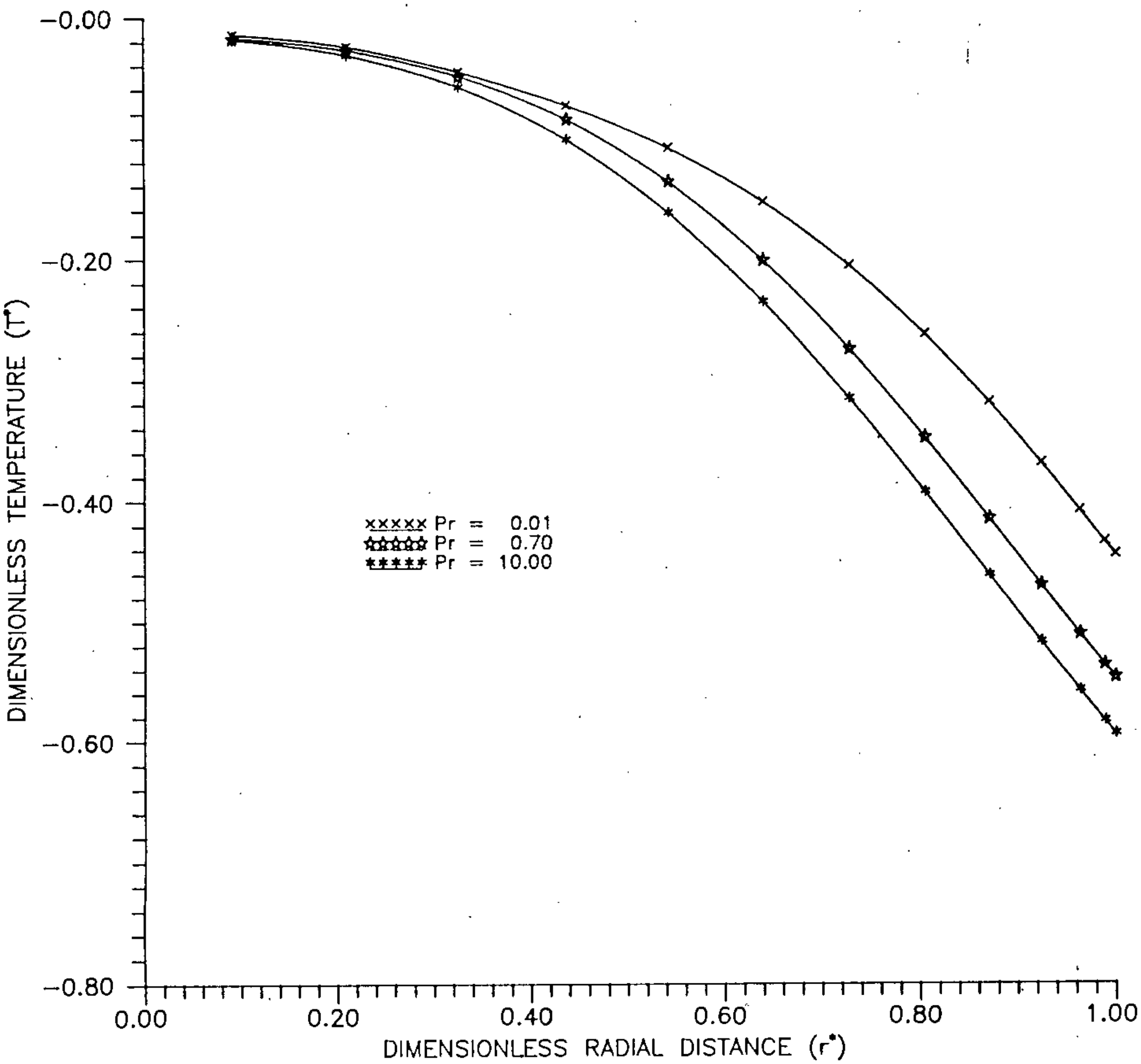

FIG.— 22 RADIAL TEMPERATURE PROFILE FOR SAME AXIAL DISTANCE (X/R/(Re.Pr)=0.1) AND COLLOCATING POINTS (N=12) WITH DIFFERENT Pr.
(FOR CONSTANT WALL HEAT FLUX AND DEVELOPING VELOCITY PROFILE)

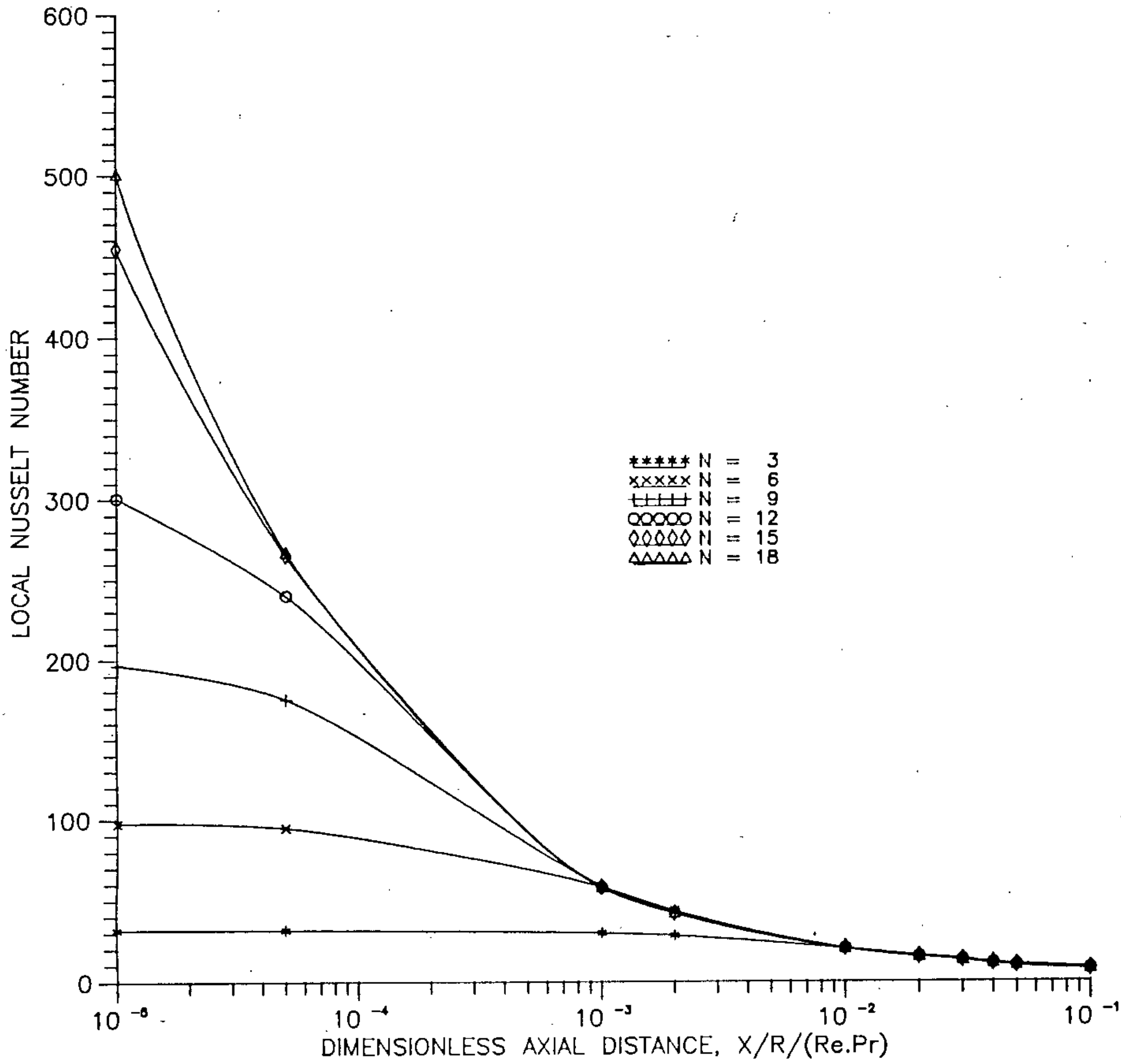

FIG.— 23 VARIATION OF LOCAL NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE (X/R/(Re.Pr)). (FOR CONSTANT WALL HEAT FLUX AND DEVELOPING VELOCITY PROFILE WITH Pr = 0.01)

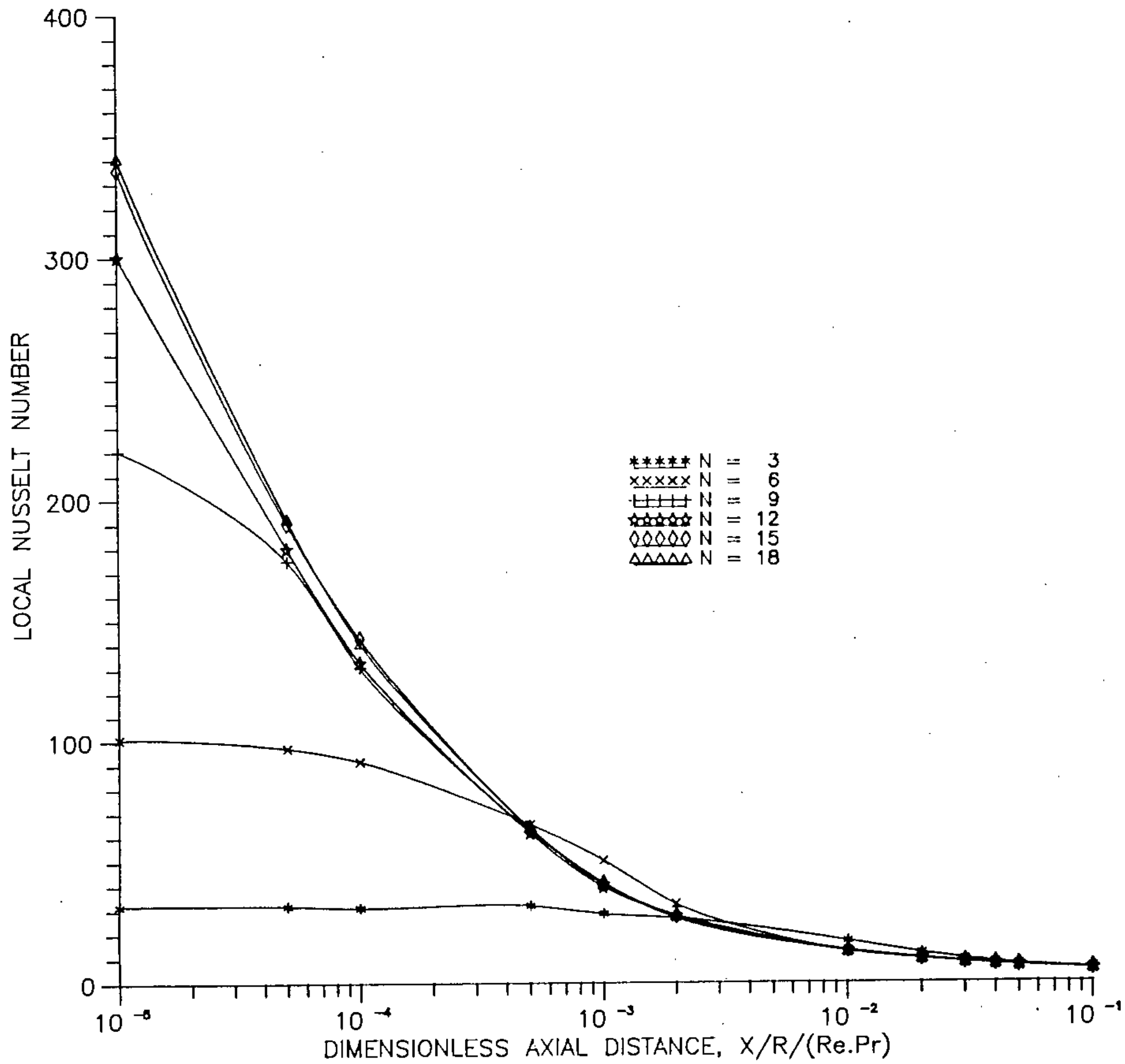

FIG.— 24 VARIATION OF LOCAL NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE (X/R/(Re.Pr)). (FOR CONSTANT WALL HEAT FLUX AND DEVELOPING VELOCITY PROFILE WITH Pr = 0.70)

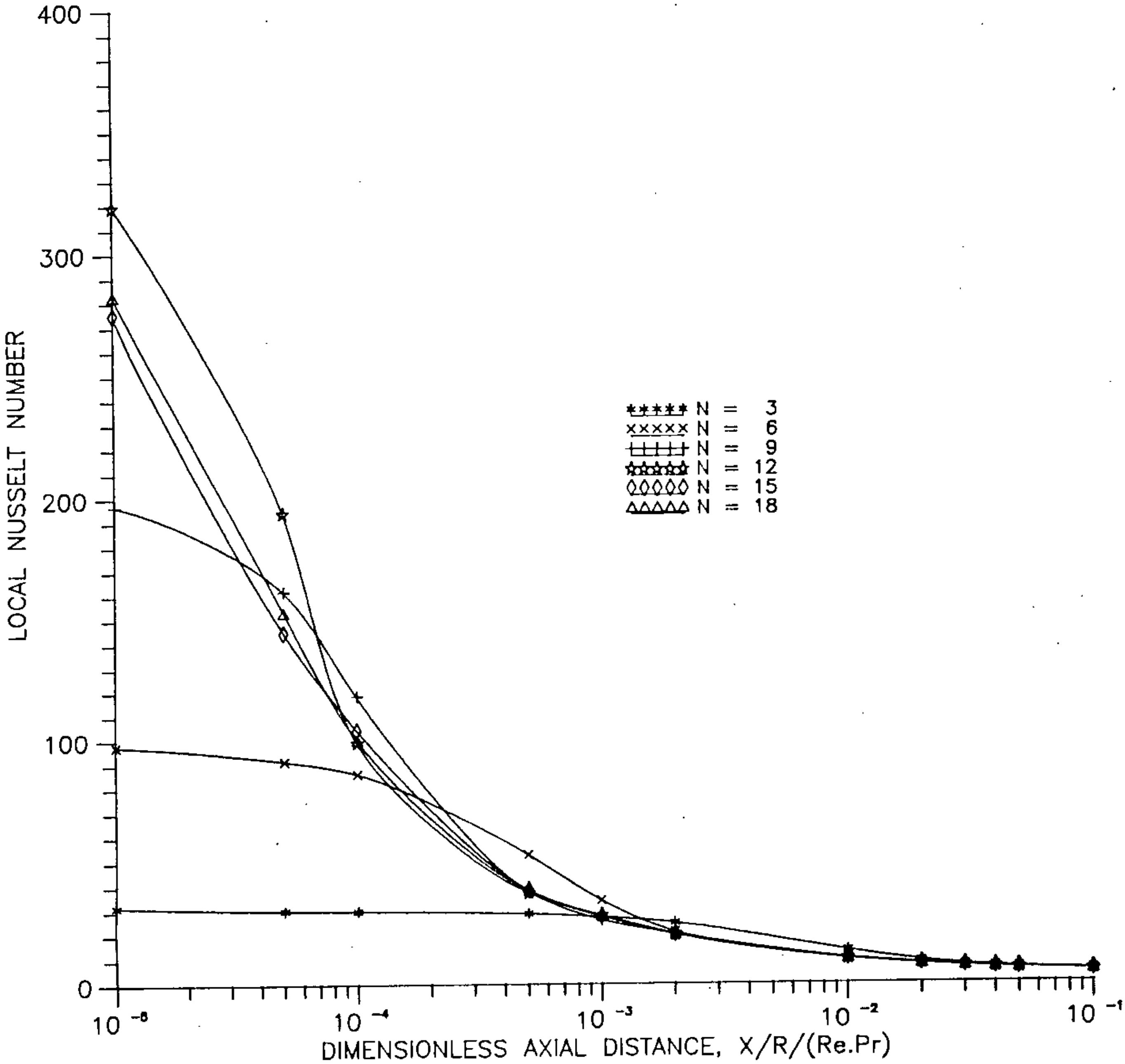

FIG.– 25 VARIATION OF LOCAL NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE (X/R/(Re.Pr)). (FOR CONSTANT WALL HEAT FLUX AND DEVELOPING VELOCITY PROFILE WITH Pr = 10)

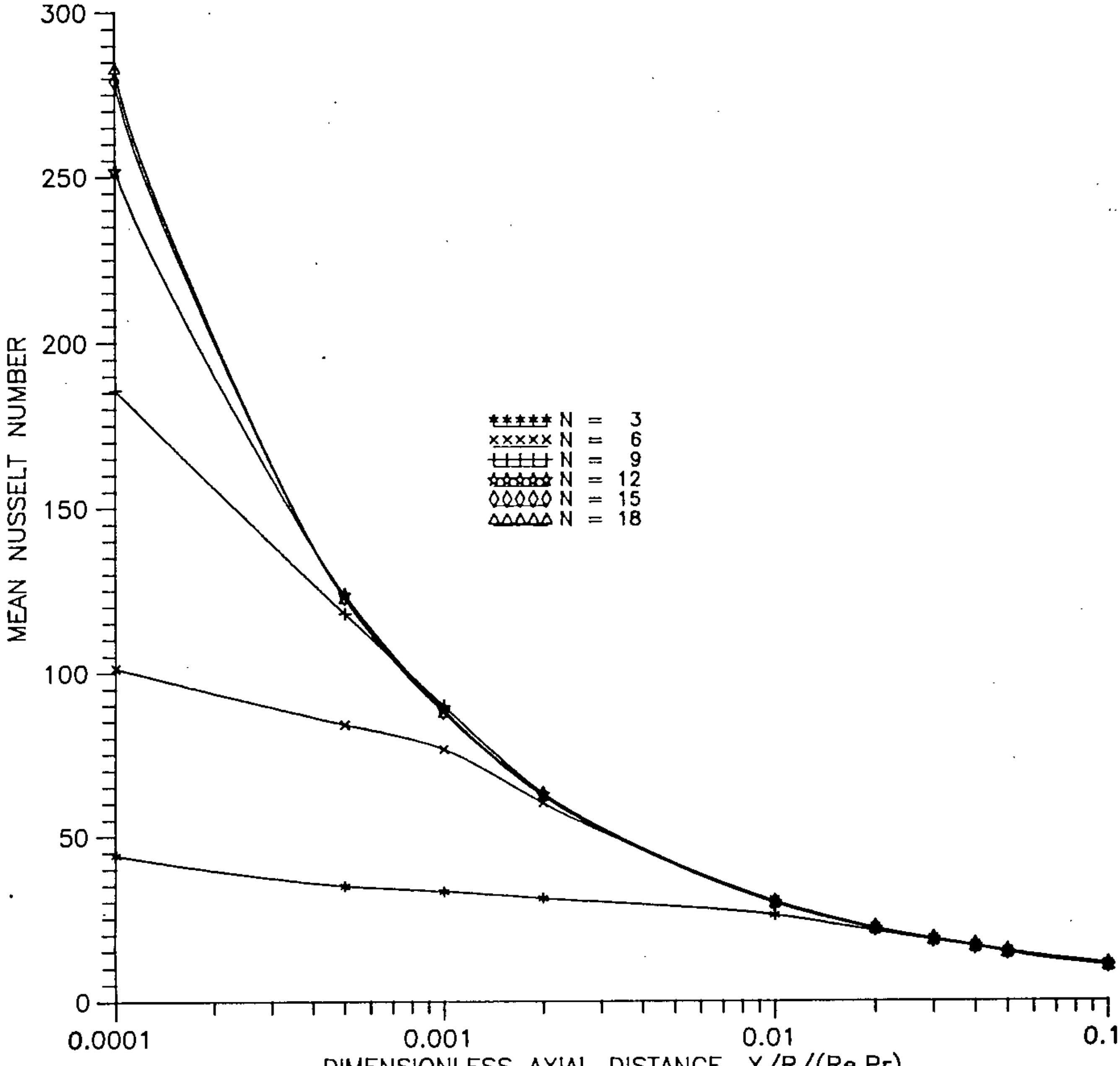

FIG.– 26 VARIATION OF MEAN NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE (X/R/(Re.Pr)). (FOR CONSTANT WALL HEAT FLUX AND DEVELOPING VELOCITY PROFILE WITH Pr = 0.01)

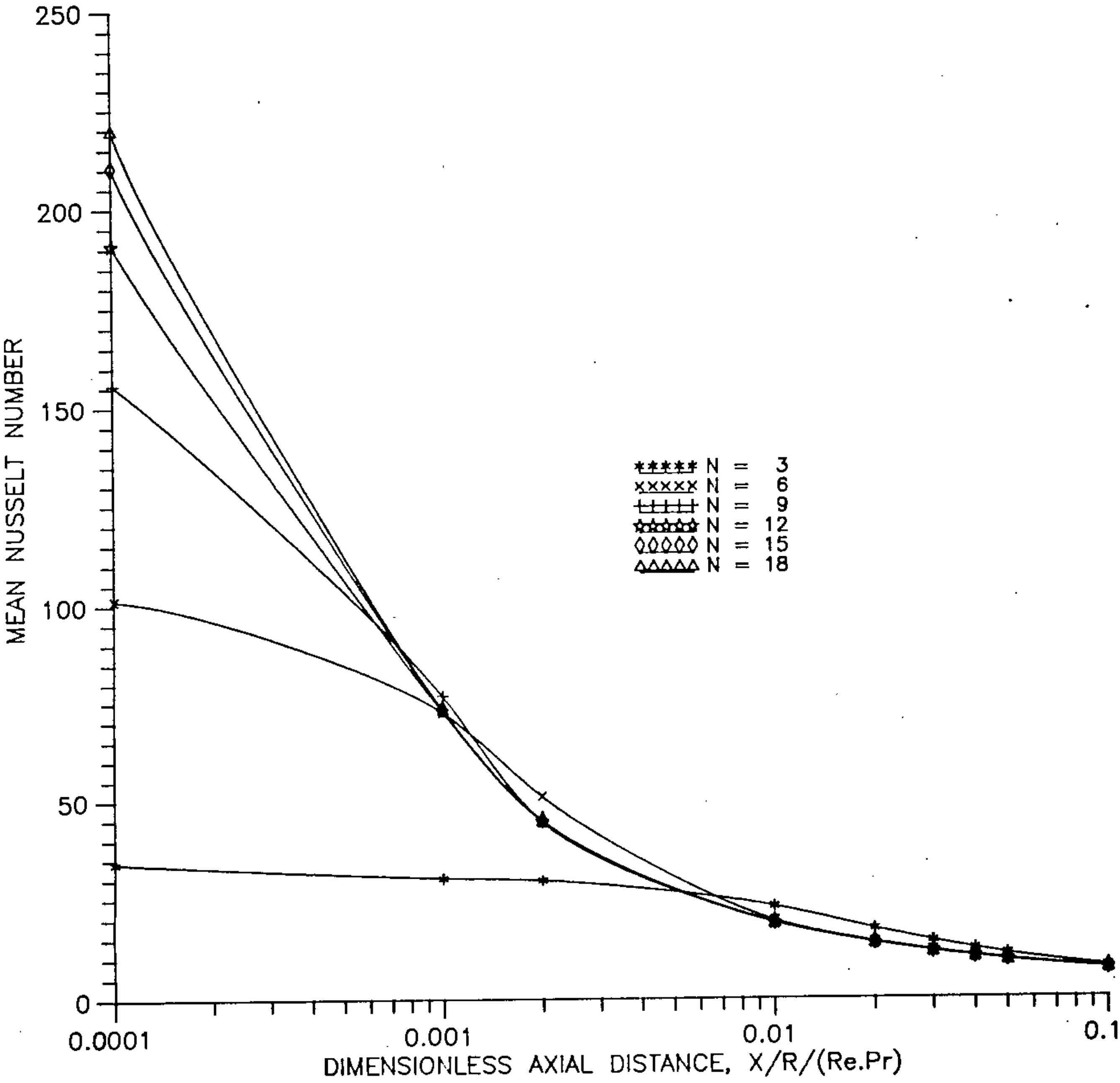

FIG.– **27** VARIATION OF MEAN NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE (X/R/(Re.Pr)). (FOR CONSTANT WALL HEAT FLUX AND DEVELOPING VELOCITY PROFILE WITH Pr = 0.70)

46499

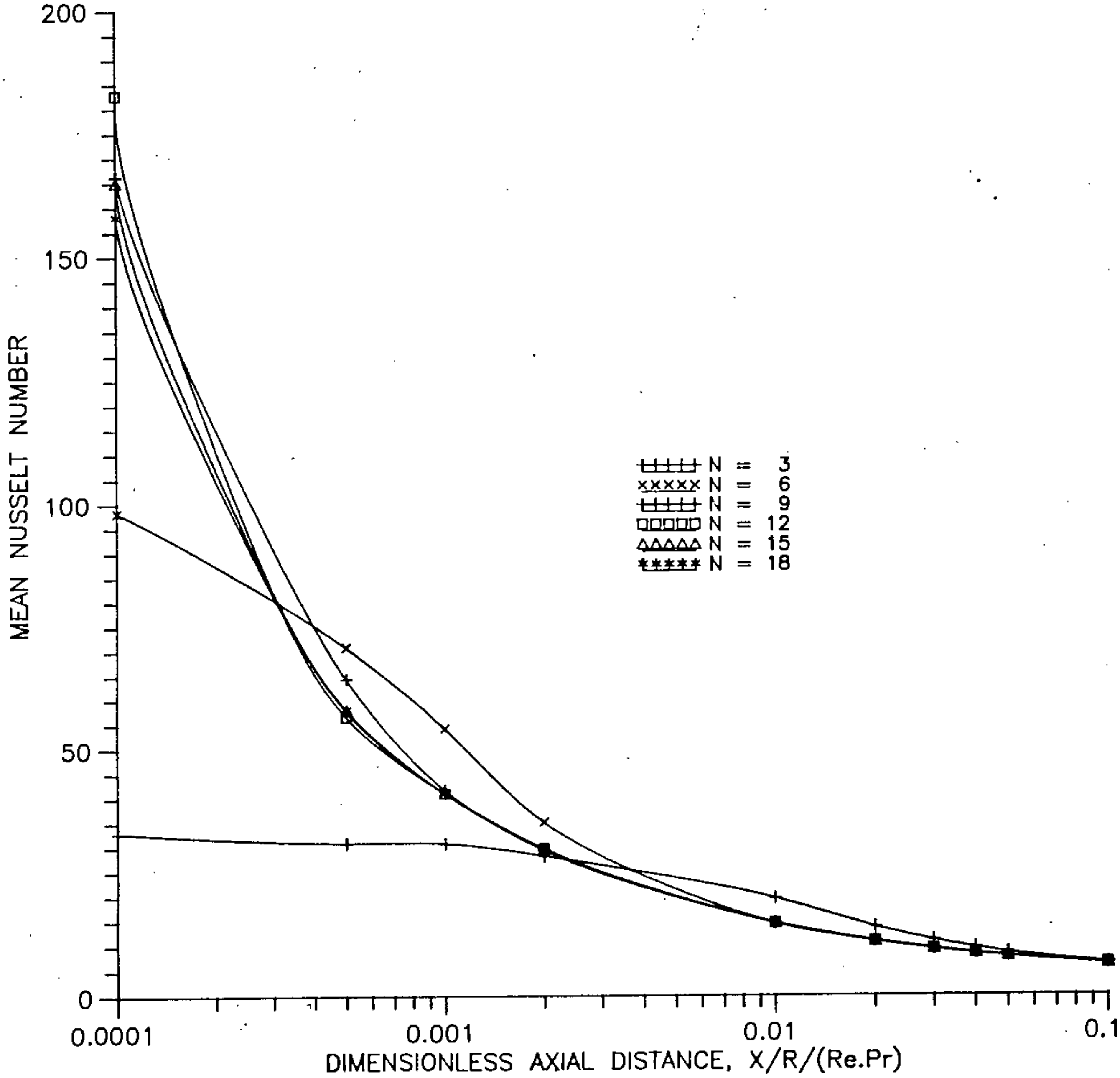

FIG.– 28 VARIATION OF MEAN NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE (X/R/(Re.Pr)). (FOR CONSTANT WALL HEAT FLUX AND DEVELOPING VELOCITY PROFILE WITH Pr = 10)

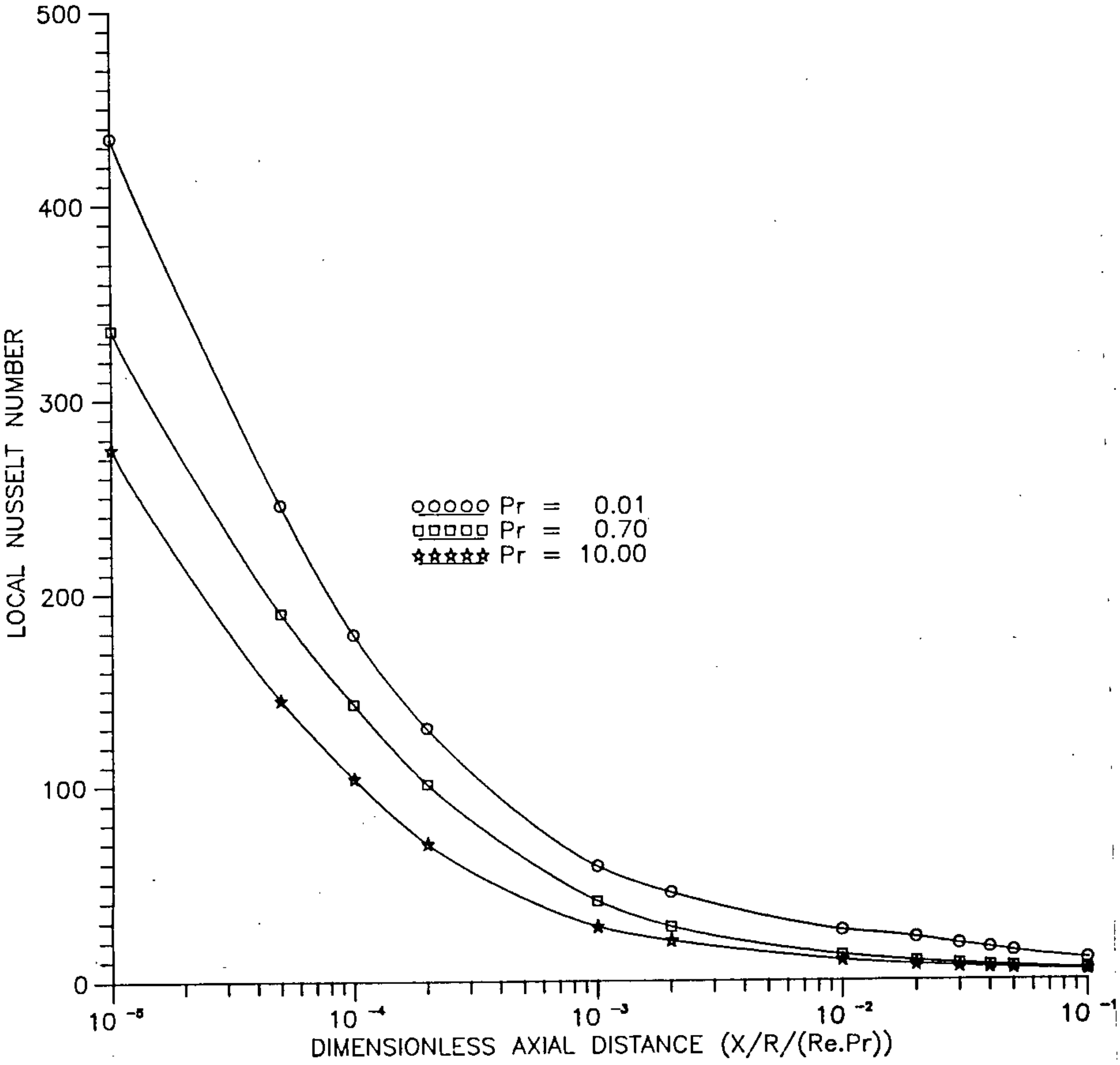

FIG.—29 VARIATION OF LOCAL NUSSELT NUMBER FOR DIFFERENT AXIAL DISTANCE WITH Pr AS A PARAMETER FOR CONSTANT WALL HEAT FLUX AND DEVELOPING VELOCITY PROFILE (15 COLLOCATING POINTS).

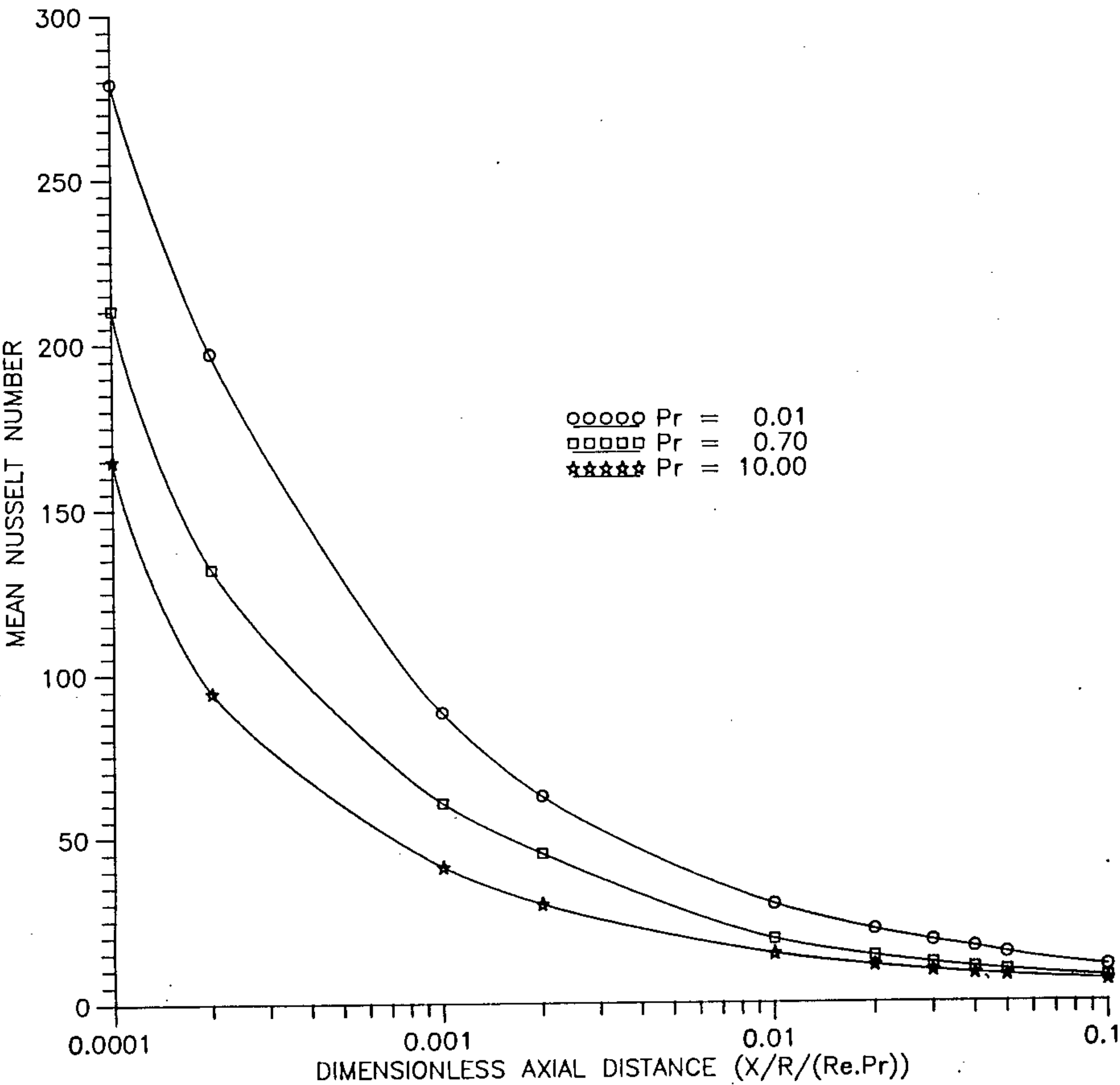

FIG.-30 VARIATION OF MEAN NUSSELT NUMBER FOR DIFFERENT AXIAL DISTANCE WITH Pr AS A PARAMETER FOR CONSTANT WALL HEAT FLUX AND DEVELOPING VELOCITY PROFILE (15 COLLOCATING POINTS).

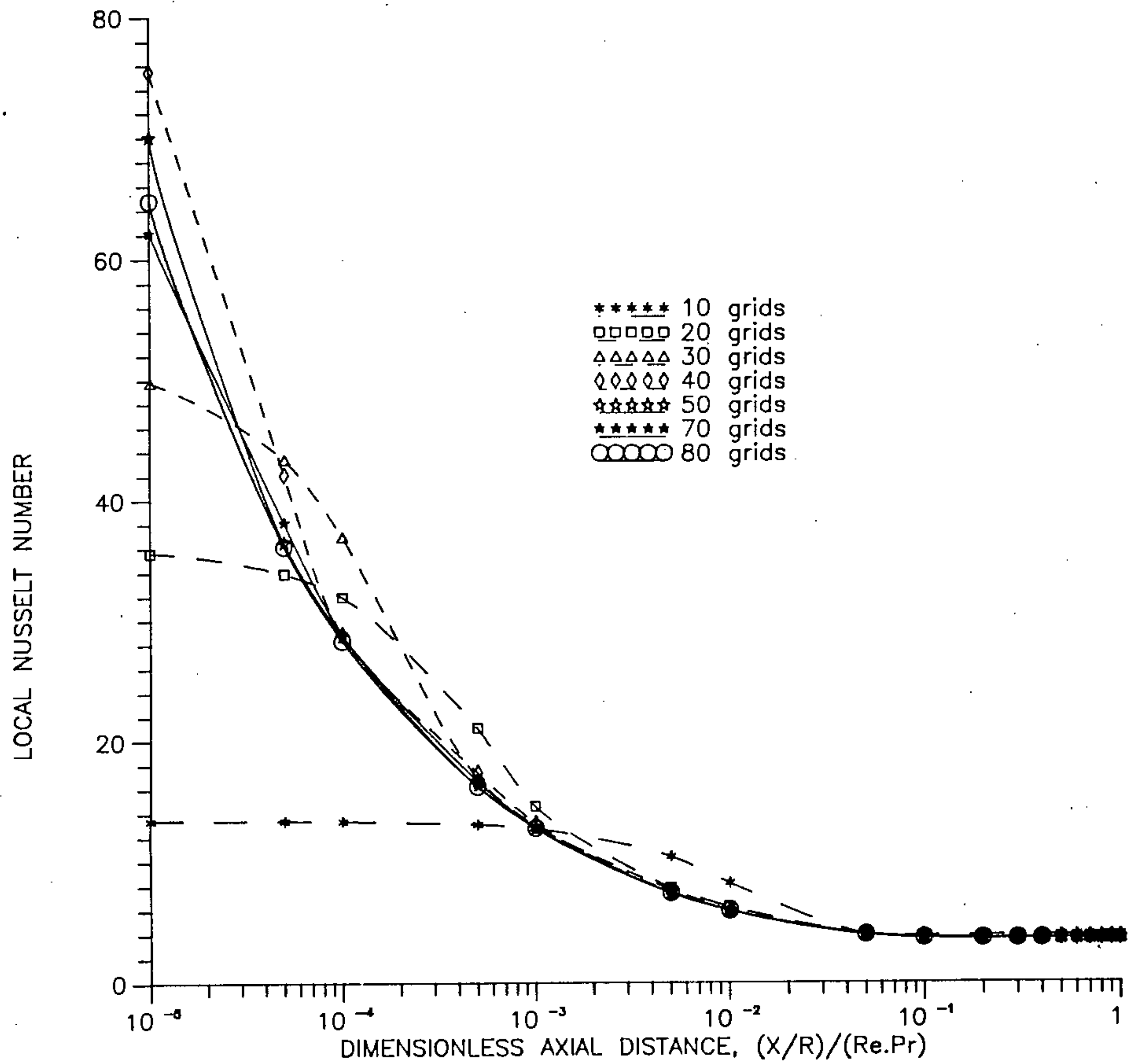

FIG.– 31 VARIATION OF LOCAL NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE FOR CONSTANT WALL TEMPERATURE AND FULLY DEVELOPED VELOCITY PROFILE. (SOLVED BY FINITE DIFERENCE TECHNIQUE)

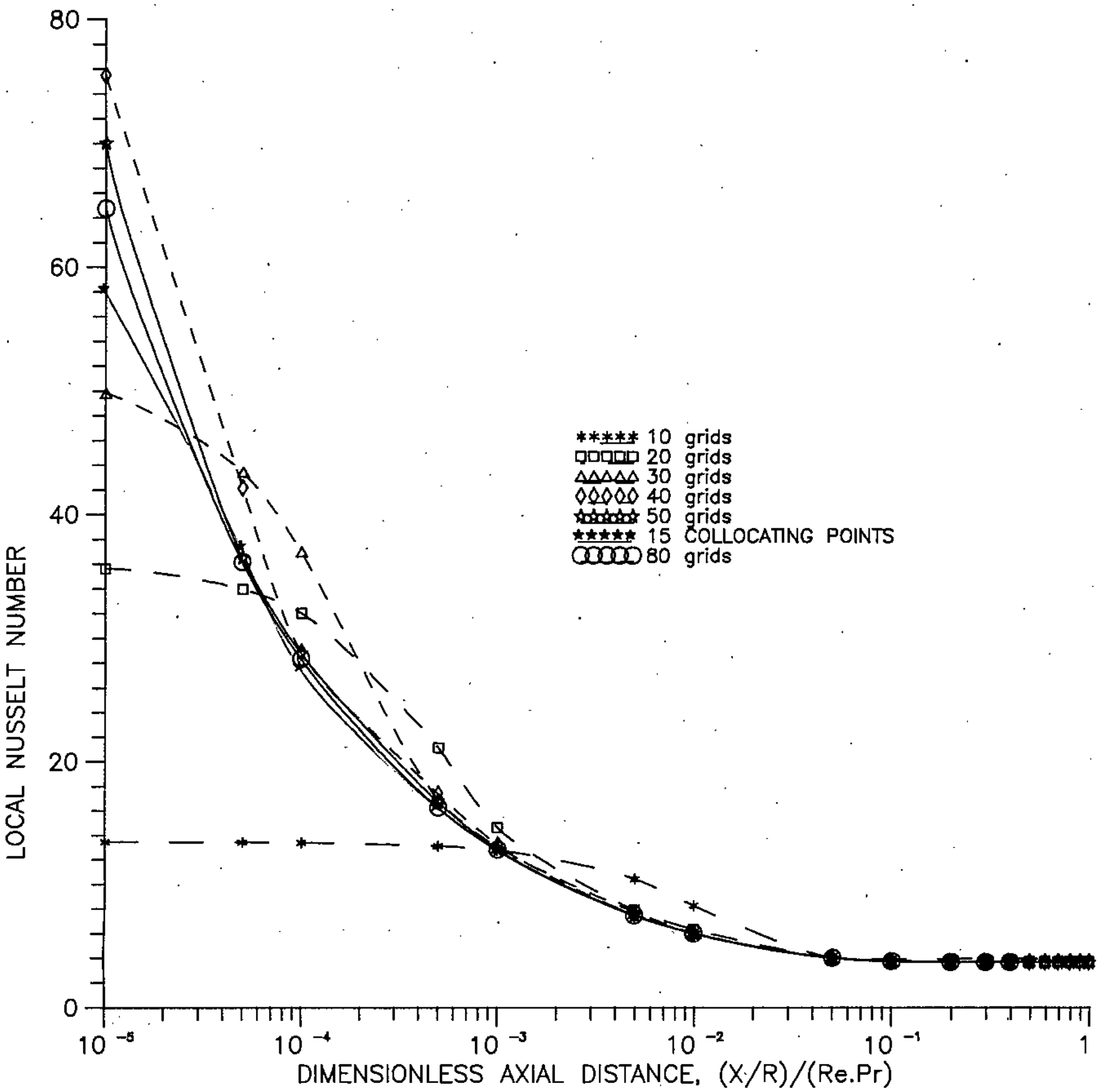

FIG.-32 VARIATION OF LOCAL NUSSELT NUMBER WITH DIMENSIONLESS AXIAL DISTANCE FOR CONSTANT WALL TEMPERATURE AND FULLY DEVELOPED VELOCITY PROFILE. (SOLVED BY FINITE DIFERENCE TECHNIQUE)

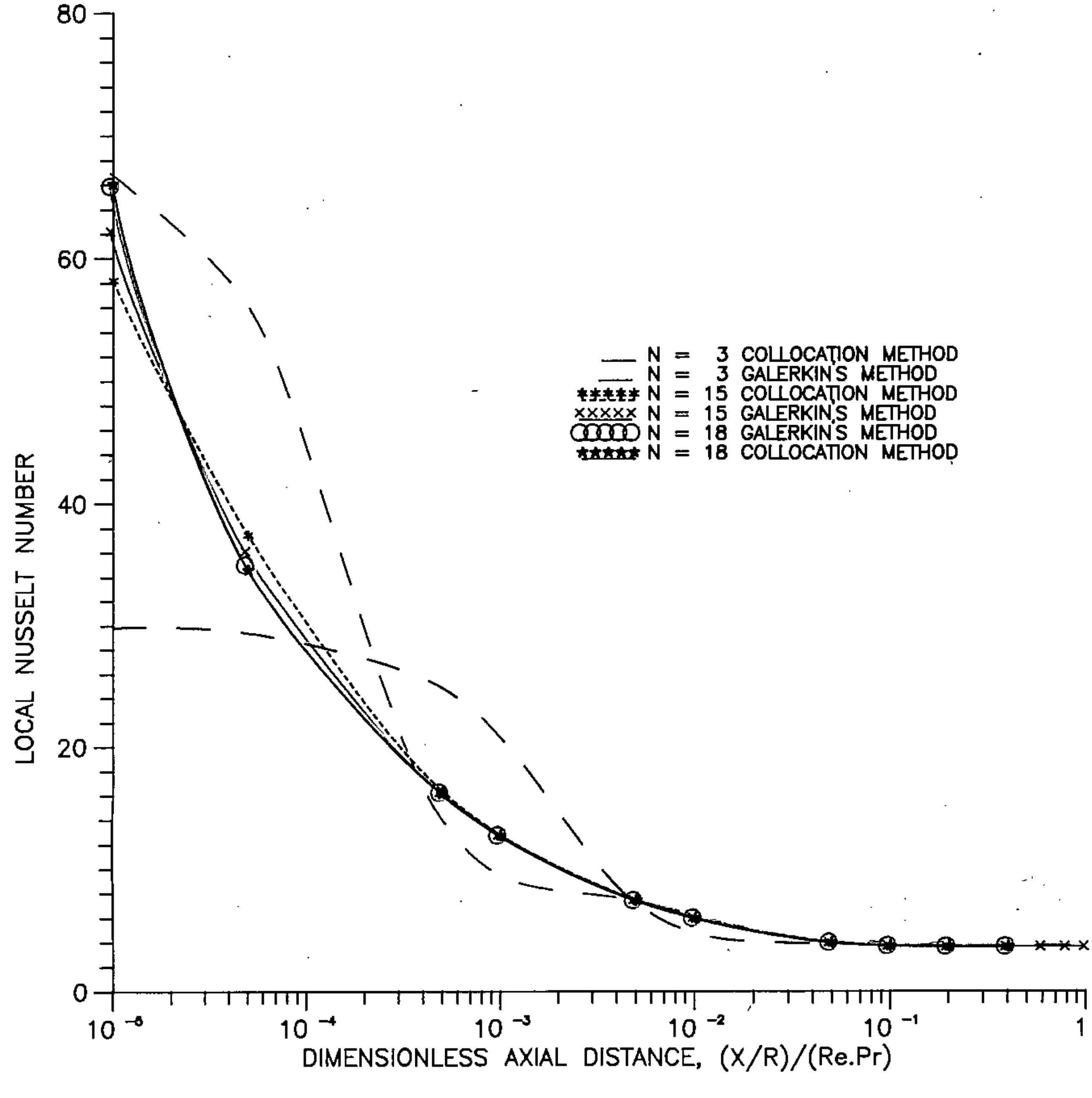

FIG.—33 COMPARISON OF LOCAL NUSSELT NUMBER OBTAINED FROM ORTHOGONAL COLLOCATION TECHNIQUE AND GALERKIN'S TECHNIQUE FOR DIFFERENT N. (FOR CONSTANT WALL TEMPERATURE AND FULLY DEVELOPED VELOCITY PROFILE.)

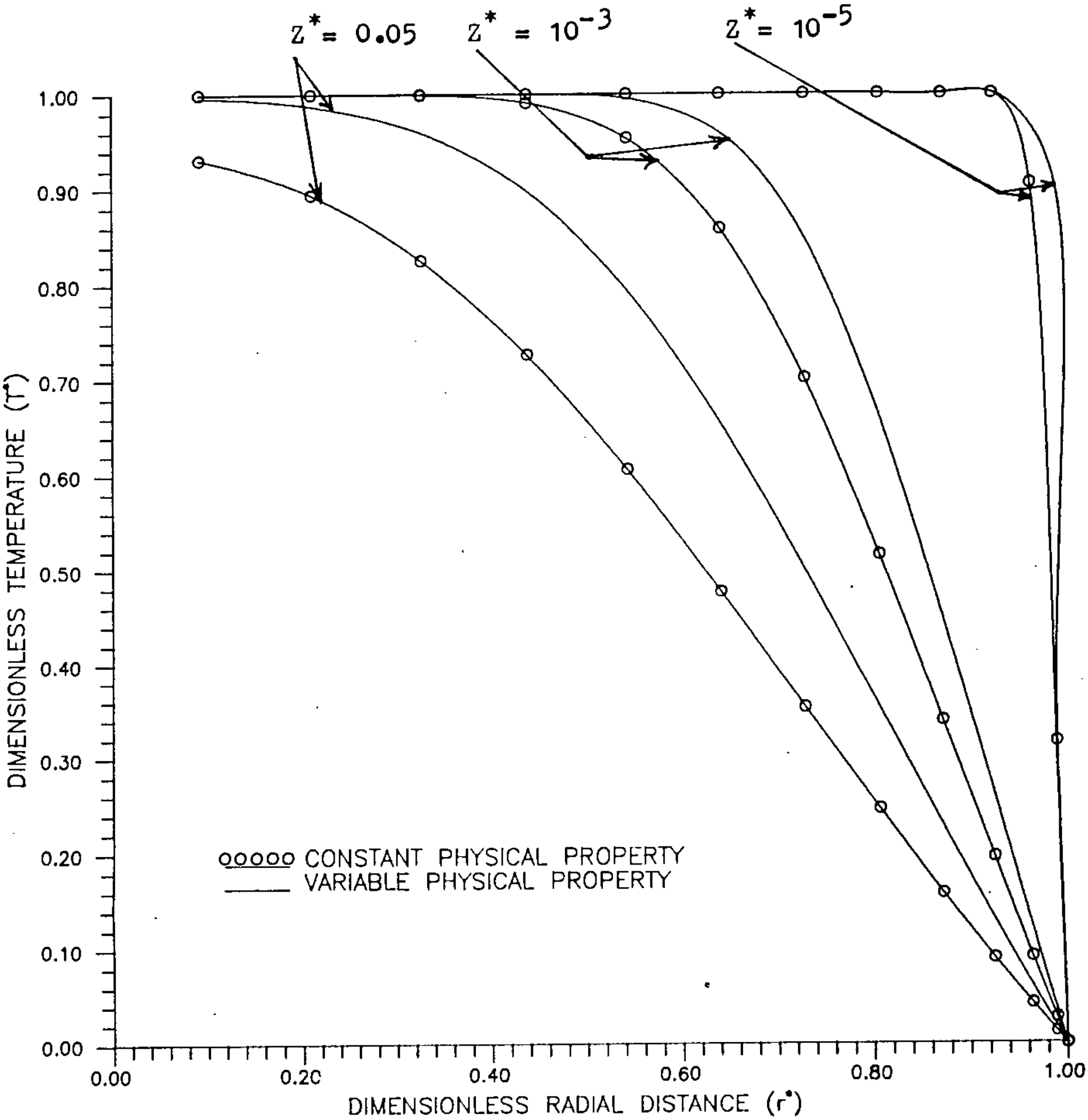

FIG.— 34 COMPARISON OF TEMPERATURE PROFILE FOR CONSTANT WALL TEMPERATURE AND FULLY DEVELOPED VELOCITY PROFILE FOR CONSTANT AND VARIABLE PROPERTY OF THE FLUID. (12 COLLOCATING POINTS) (Z^* = DIMENSIONLESS AXIAL DISTANCE, X/R/(Re.Pr))

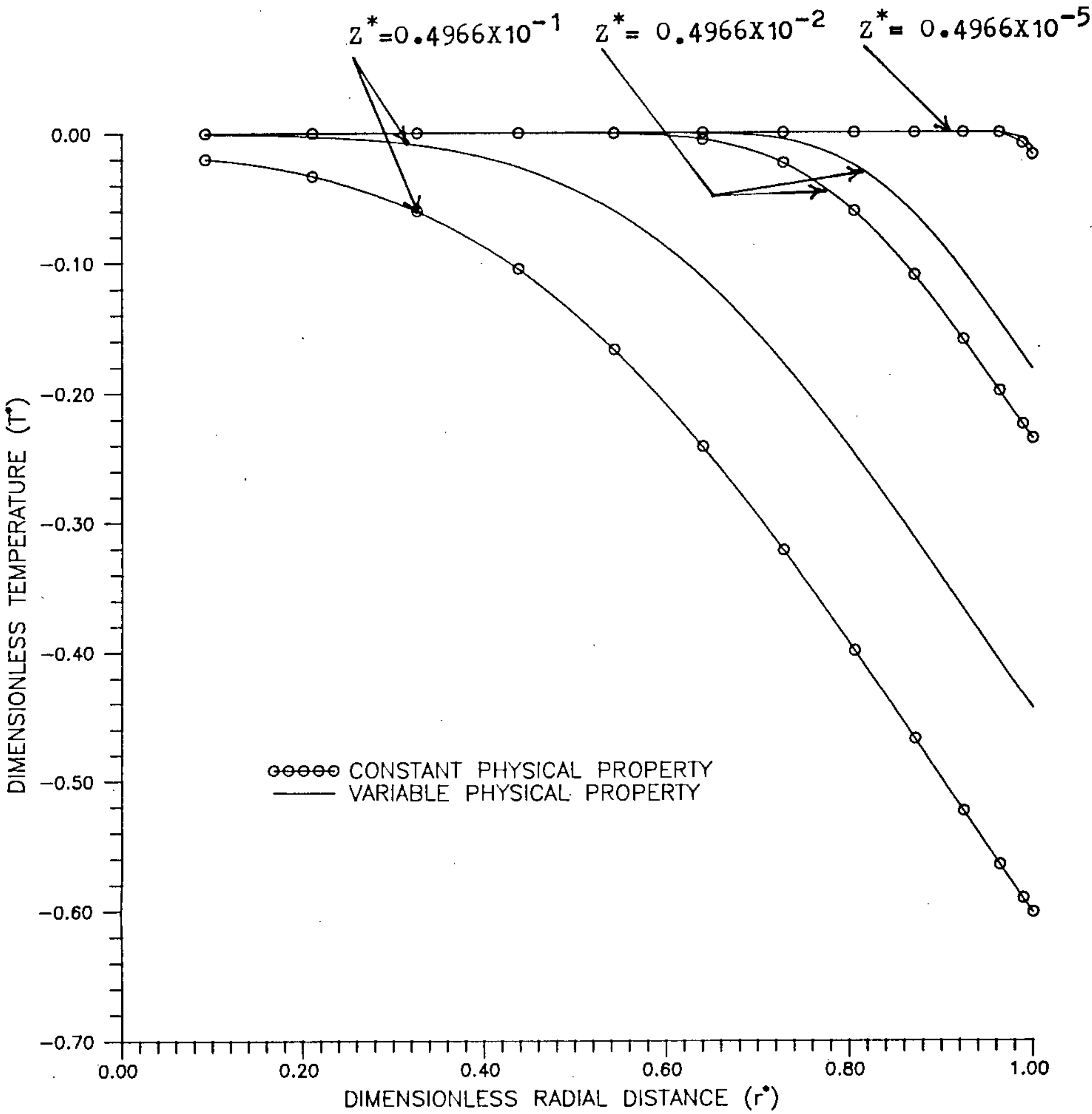

FIG.– 35 COMPARISON OF TEMPERATURE PROFILE FOR CONSTANT WALL HEAT FLUX AND FULLY DEVELOPED VELOCITY PROFILE FOR CONSTANT AND VARIABLE PROPERTY OF THE FLUID). (12 COLLOCATING POINTS) (Z^* = DIMENSIONLESS AXIAL DISTANCE, X/R/(Re.Pr))

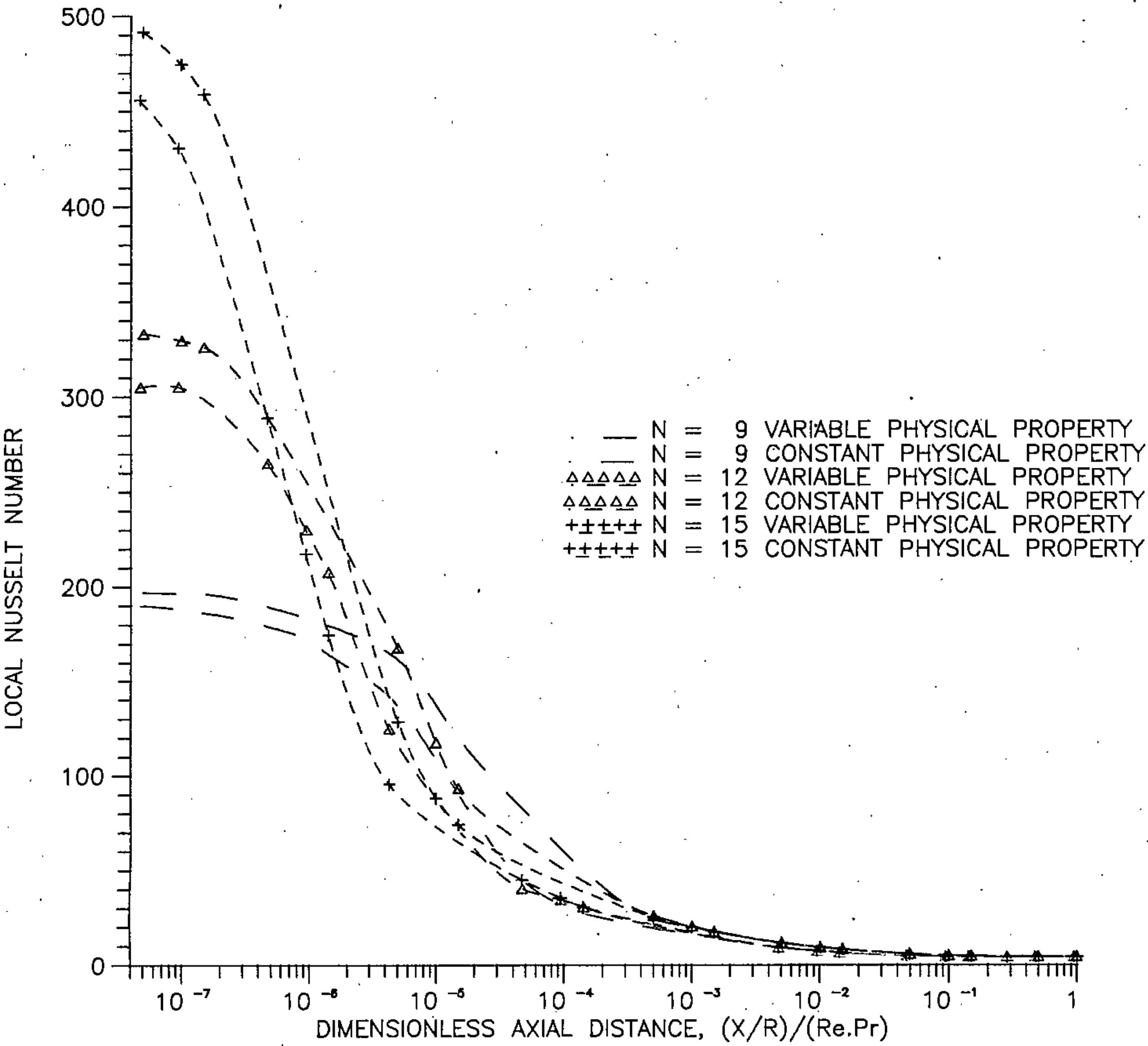

Fig.-36 COMPARISON OF LOCAL NUSSELT NUMBER FOR CONSTANT WALL HEAT FLUX AND FULLY DEVELOPED VELOCITY PROFILE FOR CONSTANT AND VARIABLE PROPERTY OF THE FLUID.

Table 1. Value of local Nusselt number for constant wall temperature and fully developed velocity profile.

N = Number of collocating points.

AXIAL DISTANCE 4(x/d)/(Re.Pr)	NUSSELT NO. N = 3	NUSSELT NO. N = 6	NUSSELT NO. N = 9	NUSSELT NO. N = 12	NUSSELT NO. Exact	NUSSELT NO. N = 18
0.00001	23.254499	64.069702	76.137901	77.126701		77.105645
0.0001	22.961322	42.123402	36.131983	37.204812		35.850536
0.0005	21.566257	20.792566	21.363516	20.751270		20.641351
0.001	19.978306	16.761136	16.277412	16.293546		16.261887
0.004	13.308535	10.499027	10.153810	10.133735	10.1200	10.121364
0.006	10.751418	8.933893	8.853757	8.842843		8.830188
0.01	8.027044	7.486835	7.477353	7.471196	7.4710	7.458778
0.02	6.097226	6.027005	6.002732	6.001462	6.0020	5.988662
0.03	5.561931	5.334010	5.323797	5.323274		5.310280
0.04	5.215073	4.921389	4.915817	4.915734	4.9169	4.902690
0.05	4.936029	4.643690	4.639763	4.639909		4.626873
0.06	4.706024	4.442649	4.439929	4.440203		4.427210
0.07	4.516558	4.290620	4.288959	4.289327		4.276372
0.08	4.360896	4.172406	4.171598	4.172024		4.159123
0.09	4.233341	4.078699	4.078539	4.079005		4.066161
0.10	4.129037	4.003403	4.003706	4.004210	4.0050	3.991410
0.11	4.043893	3.942299	3.942938	3.943465		3.930702
0.12	3.974478	3.892354	3.893248	3.893791		3.881273
0.14	3.871977	3.817510	3.818744	3.819320		3.806642
0.16	3.804246	3.766437	3.767885	3.768477		3.755834
0.18	3.759566	3.731375	3.732973	3.733575		3.720956
0.19	3.743324	3.718189	3.719842	3.720452		3.707838
0.20	3.730148	3.707248	3.708944	3.709560	3.7101	3.696952
0.40	3.674548	3.655097	3.657008	3.657642		3.645070
0.60	3.673716	3.653841	3.655758	3.656388		3.643820
0.80	3.673700	3.653804	3.655724	3.656356		3.643786
1.00	3.673699	3.653803	3.655722	3.656353		3.643784

Table 2. Value of mean Nusselt number for constant wall temperature and fully developed velocity profile.

N = Number of collocating points.

AXIAL DISTANCE 4(x/d)/(Re.Pr)	NUSSELT NO. N = 3	NUSSELT NO. N = 6	NUSSELT NO. N = 9	NUSSELT NO. N = 12
0.00001	28.986900	55.837299	79.987602	81.098701
0.0001	25.993808	41.584987	50.499099	51.475885
0.0005	23.577104	27.837440	31.451824	31.474307
0.001	21.995979	22.746513	24.794202	24.735303
0.004	15.629452	15.411307	15.382027	15.382036
0.006	13.989178	13.418580	13.395754	13.396904
0.01	11.721142	11.265252	11.268306	11.268294
0.02	8.978708	8.942355	8.942957	8.942951
0.03	7.783619	7.839024	7.838410	7.838414
0.04	7.100116	7.155239	7.155091	7.155092
0.05	6.639382	6.678053	6.678104	6.678104
0.06	6.297752	6.320937	6.320999	6.321000
0.07	6.029372	6.041087	6.041118	6.041118
0.08	5.810554	5.814559	5.814567	5.814567
0.09	5.627574	5.626748	5.626745	5.626745
0.10	5.471765	5.468144	5.468138	5.468138
0.11	5.337285	5.332240	5.332235	5.332235
0.12	5.219987	5.214403	5.214400	5.214400
0.14	5.025374	5.020140	5.020139	5.020139
0.16	4.870856	4.866711	4.866711	4.866711
0.18	4.745659	4.742672	4.742672	4.742672
0.19	4.691691	4.689228	4.689228	4.689228
0.20	4.642523	4.640529	4.640529	4.640529
0.40	4.155223	4.155626	4.155626	4.155626
0.60	3.989247	3.989458	3.989459	3.989459
0.80	3.906211	3.906284	3.906286	3.906280
1.00	3.856369	3.856360	3.856357	3.856361

Table 3. Value of local Nusselt number for constant wall heat flux and fully developed velocity profile.

N = Number of collocating points.

AXIAL DISTANCE (x/R)/(Re.Pr)	NUSSELT NO. N = 3	NUSSELT NO. N = 6	NUSSELT NO. N = 9	NUSSELT NO. N = 12	NUSSELT NO. N = 15	NUSSELT NO. N = 18
0.100D-05	29.984213	94.874551	181.368877	236.538502	210.430544	195.206027
0.200D-05	29.972958	93.874058	167.717373	184.902334	147.430578	133.928711
0.300D-05	29.961713	92.896087	156.123183	153.451818	117.197735	110.826974
0.400D-05	29.950477	91.939886	146.154590	132.318394	100.484310	98.981669
0.500D-05	29.939251	91.004737	137.492737	117.176687	90.020037	91.791717
0.100D-04	29.883260	86.623470	106.992930	77.699958	65.984458	76.112103
0.200D-04	29.771969	79.099383	76.252367	54.278578	57.755922	61.697413
0.300D-04	29.661595	72.875325	60.909457	46.502705	52.264036	53.521539
0.400D-04	29.552125	67.641740	51.767876	42.726512	48.139786	48.211696
0.500D-04	29.443549	63.180095	45.758634	40.443516	44.850440	44.440013
0.100D-03	28.913817	48.006950	31.969924	34.512200	35.573785	35.096025
0.200D-03	27.915254	33.700494	25.593243	27.976442	27.813707	27.740476
0.300D-03	26.990704	26.891421	23.100699	24.381409	23.841109	24.217684
0.400D-03	26.132256	22.942211	21.336343	22.000648	21.566965	21.925603
0.500D-03	25.333084	20.402397	20.022506	20.301870	19.988927	20.293785
0.100D-02	22.038476	14.862548	16.031447	15.773302	15.783441	15.819071
0.200D-02	17.703877	12.008102	12.536848	12.531617	12.500157	12.519800
0.300D-02	14.983941	10.750033	10.898261	10.954784	10.946445	10.949168
0.400D-02	13.122088	9.939997	9.924705	9.963073	9.964668	9.966611
0.500D-02	11.772803	9.327739	9.257237	9.273631	9.274678	9.280354
0.100D-01	8.301849	7.541049	7.485977	7.479301	7.479203	7.483254
0.200D-01	6.199059	6.140063	6.138726	6.137203	6.137306	6.142754
0.300D-01	5.441752	5.545886	5.537103	5.537559	5.537604	5.542882
0.400D-01	5.065421	5.198083	5.190106	5.190458	5.189953	5.196926
0.500D-01	4.846249	4.971990	4.963550	4.963863	4.964469	4.971156
0.100D+00	4.461757	4.515830	4.506437	4.512440	4.511078	4.523398
0.200D+00	4.369817	4.376630	4.376211	4.376199	4.376553	4.393032
0.300D+00	4.363726	4.365214	4.365550	4.364834	4.365421	4.388014
0.500D+01	4.363606	4.363606	4.363606	4.363617	4.371039	4.708443

Table 4. Value of mean Nusselt number for constant wall heat flux and fully developed velocity profile.

N = Number of collocating points.

AXIAL DISTANCE (x/R)/(Re.Pr)	NUSSELT NO. N = 3	NUSSELT NO. N = 6	NUSSELT NO. N = 9	NUSSELT NO. N = 12	NUSSELT NO. N = 15	NUSSELT NO. N = 18
0.100D-05	32.13137	102.153655	202.030964	293.476024	312.268946	294.739981
0.200D-05	31.01766	98.108502	187.121253	242.968558	223.701484	204.007030
0.300D-05	31.06947	96.480854	177.803520	211.271529	180.426917	165.574677
0.400D-05	31.04054	95.431316	170.236822	188.305444	154.485501	144.278265
0.500D-05	31.03188	94.611795	163.640673	170.688799	137.209027	130.724101
0.100D-04	30.23180	92.156924	144.744013	121.742975	96.589739	103.279865
0.200D-04	30.20966	89.335865	110.014688	82.788739	75.072985	81.898815
0.300D-04	30.10149	84.338055	90.963204	67.778528	66.841195	71.607816
0.400D-04	30.07278	80.280126	78.622098	59.912696	61.691099	64.897285
0.500D-04	30.11452	76.766287	69.940145	55.031216	57.885860	60.039523
0.100D-03	30.05563	66.345611	48.616919	44.576043	47.202321	47.769710
0.200D-03	29.87512	49.450388	35.396718	36.433986	37.368916	37.420751
0.300D-03	29.01847	40.554297	30.688817	32.159494	32.383577	32.520741
0.400D-03	28.36043	34.953836	27.995041	29.298017	29.218909	29.458420
0.500D-03	27.79123	31.092683	26.133940	27.205219	27.005032	27.288143
0.100D-02	26.87146	22.065459	21.271290	21.578609	21.439365	21.571637
0.200D-02	22.67082	16.460002	16.844538	16.866280	16.810381	16.866971
0.300D-02	20.01345	14.291508	14.646727	14.679542	14.639806	14.674805
0.400D-02	18.06423	13.035809	13.277350	13.317627	13.291701	13.314683
0.500D-02	16.55277	12.171109	12.322499	12.359147	12.341417	12.358850
0.100D-01	12.45582	9.929032	9.953120	9.943339	9.938667	9.943227
0.200D-01	8.948733	7.998831	7.992467	7.988473	7.986874	7.991530
0.300D-01	7.556418	7.104681	7.098509	7.096914	7.096071	7.101167
0.400D-01	6.801204	6.568913	6.561796	6.560613	6.560098	6.565689
0.500D-01	6.326724	6.204169	6.196694	6.196047	6.195745	6.201673
0.100D+00	5.383011	5.402833	5.385650	5.384171	5.382903	5.391002
0.200D+00	4.839265	4.861487	4.852046	4.853134	4.852667	4.864100
0.300D+00	4.670377	4.685471	4.679560	4.680118	4.679995	4.694551
0.500D+01	4.399866	4.398427	4.397497	4.397369	4.400984	4.569754

Table 5. Value of local Nusselt number for constant wall temperature and developing velocity profile.

Pr = 0.01

N = Number of collocating points.

AXIAL DISTANCE (x/R)/(Re.Pr)	NUSSELT NO. N = 3	NUSSELT NO. N = 6	NUSSELT NO. N = 9	NUSSELT NO. N = 12	NUSSELT NO. N = 15	NUSSELT NO. N = 18
0.100D-03	0.3155538D+02	0.8450072D+02	0.1097042D+03	0.8559784D+02	0.7624016D+02	0.8148976D+02
0.500D-03	0.2988615D+02	0.4929405D+02	0.4929054D+02	0.3734327D+02	0.3740434D+02	0.3731265D+02
0.100D-02	0.2793907D+02	0.3007688D+02	0.2623273D+02	0.2702260D+02	0.2692174D+02	0.2691371D+02
0.400D-02	0.1914240D+02	0.1398669D+02	0.1439481D+02	0.1439944D+02	0.1440056D+02	0.1439222D+02
0.100D-01	0.1103832D+02	0.9940148D+01	0.9901750D+01	0.9899637D+01	0.9897219D+01	0.9886689D+01
0.200D-01	0.7465717D+01	0.7696834D+01	0.7759246D+01	0.7817214D+01	0.7745545D+01	0.7739112D+01
0.300D-01	0.6049463D+01	0.6363501D+01	0.6296548D+01	0.6316580D+01	0.6310547D+01	0.6291997D+01
0.400D-01	0.5687414D+01	0.5858676D+01	0.5871104D+01	0.5872262D+01	0.5871712D+01	0.5856410D+01
0.500D-01	0.5461356D+01	0.5592181D+01	0.5592021D+01	0.5593281D+01	0.5593250D+01	0.5578307D+01
0.600D-01	0.5307270D+01	0.5407187D+01	0.5410707D+01	0.5407787D+01	0.5406992D+01	0.5391951D+01
0.700D-01	0.5201633D+01	0.5280924D+01	0.5283383D+01	0.5281797D+01	0.5280614D+01	0.5266218D+01
0.800D-01	0.5123932D+01	0.5188722D+01	0.5191003D+01	0.5191856D+01	0.5190938D+01	0.5176449D+01
0.900D-01	0.5063069D+01	0.5120664D+01	0.5122162D+01	0.5123359D+01	0.5123280D+01	0.5109319D+01
0.100D+00	0.5013804D+01	0.5067594D+01	0.5069369D+01	0.5069316D+01	0.5070127D+01	0.5056124D+01
0.110D+00	0.4973986D+01	0.5025940D+01	0.5027335D+01	0.5026645D+01	0.5025901D+01	0.5012571D+01
0.120D+00	0.4939571D+01	0.4990813D+01	0.4992099D+01	0.4991029D+01	0.4991125D+01	0.4976480D+01
0.130D+00	0.4910016D+01	0.4959745D+01	0.4961085D+01	0.4961607D+01	0.4961141D+01	0.4946451D+01
0.140D+00	0.4883463D+01	0.4932550D+01	0.4933384D+01	0.4934703D+01	0.4934648D+01	0.4920774D+01
0.150D+00	0.4858924D+01	0.4907496D+01	0.4908707D+01	0.4910380D+01	0.4910689D+01	0.4896350D+01
0.160D+00	0.4836436D+01	0.4886324D+01	0.4887189D+01	0.4888064D+01	0.4888512D+01	0.4874857D+01
0.170D+00	0.4815406D+01	0.4866460D+01	0.4867086D+01	0.4867017D+01	0.4867642D+01	0.4854335D+01
0.180D+00	0.4795857D+01	0.4847850D+01	0.4848458D+01	0.4847489D+01	0.4847636D+01	0.4834543D+01
0.190D+00	0.4777945D+01	0.4830965D+01	0.4831437D+01	0.4829495D+01	0.4830131D+01	0.4815887D+01
0.200D+00	0.4761003D+01	0.4813768D+01	0.4814299D+01	0.4812728D+01	0.4813220D+01	0.4799780D+01
0.300D+00	0.4630273D+01	0.4679814D+01	0.4679669D+01	0.4680317D+01	0.4680781D+01	0.4666917D+01
0.400D+00	0.4537874D+01	0.4585419D+01	0.4584898D+01	0.4584951D+01	0.4585126D+01	0.4571668D+01
0.600D+00	0.4410772D+01	0.4450875D+01	0.4448663D+01	0.4450131D+01	0.4451746D+01	0.4438601D+01
0.800D+00	0.4330194D+01	0.4355723D+01	0.4360510D+01	0.4355243D+01	0.4362187D+01	0.4348133D+01
0.100D+01	0.4252792D+01	0.4190702D+01	0.4196701D+01	0.4317320D+01	0.4303633D+01	0.4291618D+01

Table 6. Value of mean Nusselt number for constant wall temperature and developing velocity profile.

Pr = 0.01

N = Number of collocating points.

AXIAL DISTANCE (x/R)/(Re.Pr)	NUSSELT NO.	NUSSELT NO.	NUSSELT NO.	NUSSELT NO.	NUSSELT NO.	NUSSELT NO.
	N = 3	N = 6	N = 9	N = 12	N = 15	N = 18
0.1000D-03	0.3450305D+03	0.1751550D+03	0.1828560D+03	0.1710196D+03	0.1779877D+03	0.1772299D+03
0.5000D-03	0.8608868D+02	0.7210799D+02	0.7174949D+02	0.7189747D+02	0.7181893D+02	0.7181156D+02
0.1000D-02	0.5323442D+02	0.5204018D+02	0.5178086D+02	0.5187702D+02	0.5188611D+02	0.5188822D+02
0.4000D-02	0.2619160D+02	0.2688788D+02	0.2690398D+02	0.2690168D+02	0.2690169D+02	0.2690387D+02
0.1000D-01	0.1773732D+02	0.1772680D+02	0.1772600D+02	0.1772679D+02	0.1772732D+02	0.1772955D+02
0.2000D-01	0.1317071D+02	0.1314270D+02	0.1316832D+02	0.1315791D+02	0.1316464D+02	0.1317325D+02
0.3000D-01	0.9690171D+01	0.9678678D+01	0.9710840D+01	0.9695378D+01	0.9709813D+01	0.9730165D+01
0.4000D-01	0.8708124D+01	0.8701615D+01	0.8721098D+01	0.8713421D+01	0.8722286D+01	0.8736229D+01
0.5000D-01	0.8048136D+01	0.8041454D+01	0.8056997D+01	0.8050610D+01	0.8057214D+01	0.8068163D+01
0.6000D-01	0.7572069D+01	0.7564648D+01	0.7577858D+01	0.7573039D+01	0.7578786D+01	0.7587925D+01
0.7000D-01	0.7212038D+01	0.7205582D+01	0.7216935D+01	0.7212177D+01	0.7217329D+01	0.7225333D+01
0.8000D-01	0.6931246D+01	0.6925797D+01	0.6935656D+01	0.6930917D+01	0.6935337D+01	0.6942380D+01
0.9000D-01	0.6706658D+01	0.6701668D+01	0.6710330D+01	0.6706102D+01	0.6709804D+01	0.6716099D+01
0.1000D+00	0.6523013D+01	0.6518036D+01	0.6525764D+01	0.6522382D+01	0.6525564D+01	0.6531265D+01
0.1100D+00	0.6369296D+01	0.6364810D+01	0.6371844D+01	0.6368918D+01	0.6372114D+01	0.6377428D+01
0.1200D+00	0.6239256D+01	0.6235209D+01	0.6241626D+01	0.6238990D+01	0.6241570D+01	0.6246781D+01
0.1300D+00	0.6127464D+01	0.6124208D+01	0.6130120D+01	0.6127017D+01	0.6129710D+01	0.6134657D+01
0.1400D+00	0.6030452D+01	0.6027921D+01	0.6033411D+01	0.6030160D+01	0.6032463D+01	0.6036832D+01
0.1500D+00	0.5945601D+01	0.5943347D+01	0.5948424D+01	0.5945236D+01	0.5947122D+01	0.5951431D+01
0.1600D+00	0.5870391D+01	0.5867341D+01	0.5872082D+01	0.5869963D+01	0.5871851D+01	0.5875472D+01
0.1700D+00	0.5803345D+01	0.5799635D+01	0.5804130D+01	0.5802998D+01	0.5804468D+01	0.5808042D+01
0.1800D+00	0.5742931D+01	0.5738528D+01	0.5742694D+01	0.5742732D+01	0.5744100D+01	0.5747314D+01
0.1900D+00	0.5687086D+01	0.5682466D+01	0.5686422D+01	0.5687224D+01	0.5689468D+01	0.5692444D+01
0.2000D+00	0.5636301D+01	0.5632468D+01	0.5636198D+01	0.5636175D+01	0.5638601D+01	0.5642604D+01
0.3000D+00	0.5291439D+01	0.5290018D+01	0.5292289D+01	0.5290129D+01	0.5290766D+01	0.5294128D+01
0.4000D+00	0.5087968D+01	0.5087522D+01	0.5088945D+01	0.5086063D+01	0.5088973D+01	0.5092846D+01
0.6000D+00	0.4823843D+01	0.4811912D+01	0.4811634D+01	D.4821143D+01	0.4824038D+01	0.4830469D+01
0.8000D+00	0.4610570D+01	0.4660365D+01	0.4657944D+01	0.4578776D+01	0.4575286D+01	0.4573537D+01
0.1000D+01	0.4577472D+01	0.4971284D+01	0.4966900D+01	0.4545959D+01	0.4497841D+01	0.4447986D+01

Table 7. Value of local Nusselt number for constant wall temperature and developing velocity profile.

Pr = 0.70

N = Number of collocating points.

AXIAL DISTANCE (x/R)/(Re.Pr)	NUSSELT NO. N = 3	NUSSELT NO. N = 6	NUSSELT NO. N = 9	NUSSELT NO. N = 12	NUSSELT NO. N = 15
0.1000D-03	0.3155321D+02	0.1007042D+03	0.7523912D+02	0.73.1242D+02	0.7198616D+02
0.5000D-03	0.2828381D+02	0.3219786D+02	0.2956905D+02	0.3115445D+02	0.3112184D+02
0.1000D-02	0.2613021D+02	0.1871320D+02	0.2108453D+02	0.2132132D+02	0.2113255D+02
0.4000D-02	0.1586493D+02	0.1079611D+02	0.1106540D+02	0.1106857D+02	0.1106209D+02
0.1000D-01	0.7204341D+01	0.7634669D+01	0.7507104D+01	0.7473482D+01	0.7480252D+01
0.2000D-01	0.4897862D+01	0.5727350D+01	0.5733280D+01	0.5710598D+01	0.5729897D+01
0.3000D-01	0.4476783D+01	0.4971116D+01	0.4980900D+01	0.4983996D+01	0.4982006D+01
0.4000D-01	0.4253012D+01	0.4556568D+01	0.4562436D+01	0.4557441D+01	0.4561471D+01
0.5000D-01	0.4102562D+01	0.4296253D+01	0.4296261D+01	0.4296226D+01	0.4298477D+01
0.6000D-01	0.4001202D+01	0.4119759D+01	0.4120942D+01	0.4120143D+01	0.4120886D+01
0.7000D-01	0.3930102D+01	0.3998682D+01	0.3999869D+01	0.3998147D+01	0.3998491D+01
0.8000D-01	0.3878568D+01	0.3913289D+01	0.3912780D+01	0.3912155D+01	0.3912793D+01
0.9000D-01	0.3841006D+01	0.3851034D+01	0.3849212D+01	0.3850438D+01	0.3850905D+01
0.1000D+00	0.3813583D+01	0.3804773D+01	0.3804063D+01	0.3804750D+01	0.3805108D+01
0.1100D+00	0.3793119D+01	0.3770274D+01	0.3770764D+01	0.3770500D+01	0.3770677D+01
0.1200D+00	0.3777888D+01	0.3745553D+01	0.3745740D+01	0.3744592D+01	0.3744844D+01
0.1300D+00	0.3766324D+01	0.3726679D+01	0.3727025D+01	0.3726125D+01	0.3726306D+01
0.1400D+00	0.3757204D+01	0.3712487D+01	0.3712171D+01	0.3711652D+01	0.3711888D+01
0.1500D+00	0.3749996D+01	0.3701504D+01	0.3700428D+01	0.3701215D+01	0.3701279D+01
0.1600D+00	0.3744363D+01	0.3692462D+01	0.3691261D+01	0.3692521D+01	0.3692642D+01
0.1700D+00	0.3740152D+01	0.3685488D+01	0.3684827D+01	0.3685854D+01	0.3685783D+01
0.1800D+00	0.3736964D+01	0.3679665D+01	0.3679841D+01	0.3680216D+01	0.3680227D+01
0.1900D+00	0.3734175D+01	0.3675041D+01	0.3675661D+01	0.3675659D+01	0.3675569D+01
0.2000D+00	0.3732337D+01	0.3671825D+01	0.3672960D+01	0.3671949D+01	0.3671923D+01
0.3000D+00	0.3727626D+01	0.3664813D+01	0.3663797D+01	0.3665101D+01	0.3665230D+01
0.4000D+00	0.3727553D+01	0.3662957D+01	0.3663603D+01	0.3663218D+01	0.3663208D+01
0.6000D+00	0.3727354D+01	0.3663114D+01	0.3662999D+01	0.3663270D+01	0.3663297D+01
0.8000D+00	0.3727406D+01	0.3663058D+01	0.3663127D+01	0.3662930D+01	0.3662986D+01
0.1000D+01	0.3727356D+01	0.3663051D+01	0.3662984D+01	0.3663186D+01	0.3663217D+01

Table 8. Value of mean Nusselt number for constant wall temperature and developing velocity profile.

Pr = 0.70

N = Number of collocating points.

AXIAL DISTANCE (x/R)/(Re.Pr)	NUSSELT NO. N = 3	NUSSELT NO. N = 6	NUSSELT NO. N = 9	NUSSELT NO. N = 12	NUSSELT NO. N = 15
0.1000D-03	0.2800927D+03	0.1100230D+03	0.1058760D+03	0.1000896D+03	0.100.083D+03
0.5000D-03	0.4594483D+02	0.4158236D+02	0.4016170D+02	0.4087674D+02	0.4087671D+02
0.1000D-02	0.2208119D+02	0.2625830D+02	0.2554179D+02	0.2561743D+02	0.2563737D+02
0.4000D-02	0.1430185D+02	0.1450461D+02	0.1449079D+02	0.1449231D+02	0.1449006D+02
0.1000D-01	0.1023371D+02	0.1000591D+02	0.1000039D+02	0.1000177D+02	0.9999481D+01
0.2000D-01	0.7642554D+01	0.7645929D+01	0.7643573D+01	0.7644814D+01	0.7640797D+01
0.3000D-01	0.6577979D+01	0.6600179D+01	0.6598847D+01	0.6595753D+01	0.6596745D+01
0.4000D-01	0.5977741D+01	0.5989085D+01	0.5988112D+01	0.5985763D+01	0.5986730D+01
0.5000D-01	0.5580512D+01	0.5583765D+01	0.5583356D+01	0.5581130D+01	0.5581793D+01
0.6000D-01	0.5294315D+01	0.5294468D+01	0.5293577D+01	0.5292109D+01	0.5292814D+01
0.7000D-01	0.5078254D+01	0.5077050D+01	0.5076283D+01	0.5075223D+01	0.5075784D+01
0.8000D-01	0.4909913D+01	0.4908169D+01	0.4907835D+01	0.4906574D+01	0.4907064D+01
0.9000D-01	0.4775442D+01	0.4773731D+01	0.4773715D+01	0.4772195D+01	0.4772671D+01
0.1000D+00	0.4665545D+01	0.4664471D+01	0.4664126D+01	0.4662976D+01	0.4663437D+01
0.1100D+00	0.4574511D+01	0.4574042D+01	0.4573441D+01	0.4572663D+01	0.4573105D+01
0.1200D+00	0.4497989D+01	0.4497698D+01	0.4497334D+01	0.4496832D+01	0.4497209D+01
0.1300D+00	0.4432936D+01	0.4432875D+01	0.4432539D+01	0.4431931D+01	0.4432295D+01
0.1400D+00	0.4377181D+01	0.4377082D+01	0.4376965D+01	0.4376293D+01	0.4376606D+01
0.1500D+00	0.4328843D+01	0.4328643D+01	0.4328785D+01	0.4327747D+01	0.4328095D+01
0.1600D+00	0.4286504D+01	0.4286378D+01	0.4286591D+01	0.4285434D+01	0.4285736D+01
0.1700D+00	0.4248854D+01	0.4248997D+01	0.4248915D+01	0.4248019D+01	0.4248372D+01
0.1800D+00	0.4215242D+01	0.4215873D+01	0.4215417D+01	0.4214864D+01	0.4215167D+01
0.1900D+00	0.4185443D+01	0.4186247D+01	0.4185565D+01	0.4185230D+01	0.4185569D+01
0.2000D+00	0.4158141D+01	0.4159132D+01	0.4158336D+01	0.4158559D+01	0.4158856D+01
0.3000D+00	0.3989417D+01	0.3987693D+01	0.3989487D+01	0.3986355D+01	0.3986473D+01
0.4000D+00	0.3897824D+01	0.3898998D+01	0.3897386D+01	0.3898693D+01	0.3898691D+01
0.6000D+00	0.3793997D+01	0.3796218D+01	0.3792673D+01	0.3792622D+01	0.3793425D+01
0.8000D+00	0.3716673D+01	0.3704520D+01	0.3717186D+01	0.3704616D+01	0.3704772D+01
0.1000D+01	0.3607141D+01	0.3647411D+01	0.3602137D+01	0.3634615D+01	0.3632492D+01

Table 9. Value of local Nusselt number for constant wall temperature and developing velocity profile.

Pr = 10

N = Number of collocating points.

AXIAL DISTANCE (x/R)/(Re.Pr)	NUSSELT NO. N = 3	NUSSELT NO. N = 6	NUSSELT NO. N = 9	NUSSELT NO. N = 12	NUSSELT NO. N = 15	NUSSELT NO. N = 18
0.0001	29.68693	73.40760	44.82271	40.82859	50.02529	49.23023
0.0005	27.28655	20.12080	21.83782	22.85393	22.22838	22.42201
0.0010	24.30815	12.81522	17.12442	16.12391	16.38545	16.34424
0.0040	11.25305	9.20954	8.86799	8.92988	8.93899	8.92611
0.0100	5.04002	6.31120	6.25358	6.25396	6.25102	6.24254
0.0200	4.28764	4.96422	5.00046	5.00204	5.00107	4.99038
0.0300	4.09893	4.47581	4.48815	4.48924	4.48899	4.47723
0.0400	3.97339	4.20215	4.20620	4.20560	4.20535	4.19384
0.0500	3.89053	4.03100	4.02877	4.02898	4.02983	4.01714
0.0600	3.83733	3.91612	3.91372	3.91450	3.91333	3.90286
0.0700	3.80135	3.83741	3.83692	3.83689	3.83626	3.82520
0.0800	3.77667	3.78457	3.78415	3.78314	3.78440	3.77153
0.0900	3.76055	3.74874	3.74718	3.74693	3.74841	3.73522
0.1000	3.74995	3.72385	3.72186	3.72238	3.72296	3.71061
0.1100	3.74285	3.70622	3.70481	3.70548	3.70496	3.69384
0.1200	3.73801	3.69362	3.69366	3.69395	3.69306	3.68234
0.1300	3.73467	3.68463	3.68544	3.68536	3.68472	3.67373
0.1400	3.73239	3.67914	3.67944	3.67899	3.67960	3.66739
0.1500	3.73124	3.67495	3.67525	3.67445	3.67547	3.66279
0.1600	3.73036	3.67264	3.67187	3.67188	3.67266	3.66020
0.1700	3.72991	3.67051	3.66940	3.66992	3.67031	3.65832
0.1800	3.72960	3.66924	3.66815	3.66872	3.66844	3.65700
0.1900	3.72921	3.66790	3.66732	3.66801	3.66707	3.65635
0.2000	3.72899	3.66692	3.66645	3.66709	3.66603	3.65541
0.3000	3.72733	3.66375	3.66372	3.66319	3.66461	3.65157
0.4000	3.72753	3.66337	3.66300	3.66336	3.66302	3.65174
0.6000	3.72737	3.66312	3.66298	3.66299	3.66372	3.65136
0.8000	3.72738	3.66296	3.66316	3.66312	3.65890	3.65150
1.0000	3.72739	3.66308	3.66300	3.66303	3.67165	3.65140

Table 10. Value of mean Nusselt number for constant wall temperature and developing velocity profile.

Pr = 10

N = Number of collocating points.

AXIAL DISTANCE (x/R)/(Re.Pr)	NUSSELT NO. N = 3	NUSSELT NO. N = 6	NUSSELT NO. N = 9	NUSSELT NO. N = 12	NUSSELT NO. N = 15	NUSSELT NO. N = 18
0.0001	48.44645	88.81291	78.93982	60.65203	59.08677	59.12140
0.0005	20.10004	27.95827	26.88306	27.00028	27.10271	27.11677
0.0010	16.30047	20.25126	20.27509	20.23733	20.23241	20.23377
0.0040	11.98055	11.55708	11.53318	11.53702	11.53718	11.53918
0.0100	8.53217	8.44873	8.44554	8.44578	8.44620	8.44747
0.0200	6.87507	6.92073	6.91971	6.92071	6.92083	6.92210
0.0300	6.16846	6.18366	6.18348	6.18383	6.18402	6.18515
0.0400	5.72513	5.72039	5.71958	5.72020	5.72020	5.72136
0.0500	5.40715	5.39693	5.39705	5.39726	5.39714	5.39839
0.0600	5.16642	5.15786	5.15780	5.15790	5.15828	5.15893
0.0700	4.98053	4.97374	4.97326	4.97354	4.97365	4.97455
0.0800	4.83313	4.82723	4.82695	4.82743	4.82705	4.82839
0.0900	4.71310	4.70833	4.70846	4.70866	4.70832	4.70962
0.1000	4.61397	4.61031	4.61057	4.61057	4.61049	4.61153
0.1100	4.53108	4.52836	4.52848	4.52847	4.52865	4.52937
0.1200	4.46096	4.45896	4.45873	4.45883	4.45904	4.45971
0.1300	4.40098	4.39950	4.39912	4.39930	4.39943	4.40017
0.1400	4.34914	4.34775	4.34761	4.34785	4.34764	4.34870
0.1500	4.30363	4.30272	4.30260	4.30293	4.30262	4.30379
0.1600	4.26375	4.26287	4.26311	4.26310	4.26297	4.26396
0.1700	4.22830	4.22776	4.22811	4.22791	4.22792	4.22872
0.1800	4.19674	4.19633	4.19653	4.19642	4.19672	4.19727
0.1900	4.16854	4.16829	4.16819	4.16811	4.16868	4.16893
0.2000	4.14302	4.14293	4.14281	4.14276	4.14333	4.14359
0.3000	3.97890	3.97677	3.97913	3.97876	3.97659	3.97958
0.4000	3.88872	3.88865	3.89004	3.88875	3.89205	3.88930
0.6000	3.78572	3.78455	3.78776	3.78430	3.83948	3.78495
0.8000	3.71443	3.70944	3.70661	3.71902	3.81714	3.71837
1.0000	3.59289	3.60159	3.61762	3.59779	4.20596	3.59567

APPENDIX

REVIEW OF ORTHOGONAL COLLCATION METHOD

The orthogonal collocation method has several advantages . Namely, the collocation points are picked up automatically, thus avoiding the arbitrary choice (and posible poor one) by the user, and the error decreases much faster as the number of collocating points increases. In addition, the solution can be derived not in terms of the coefficients in the trial function (method of coefficients) but in terms of the of the value of the solution at the collocation points (method of ordinates). the whole problem is then reduced to a set of matrix equations which can be easily* generated and solved on the computer.

Let us examine the advantage of solving for the solution at the collocation points rather than the coefficients. We expand the solution in the form

$$y(x) = \sum_{i=1}^{N} a_i y_i(x) \tag{A}$$

where $\{y_i(x)\}$ are known functions of position. Usually we express the solution by providing the set $\{a_i\}$. Then we evaluate eq. (A) at a set of N points to give

$$y(x_j) = \sum_{i=1}^{N} a_i y_i(x_j) \tag{B}$$

It is important to note that for all problems the $y_i(x_j)$ are known numbers. Thus using eq. (B) gives $y(x_j)$ if the coefficients $\{a_i\}$ are known. Consequently, rearranging

*But in nonlinear problems the formulation of the equations actually becomes more combersome than solving the equations.

eq. (B) and solving for $\{a_i\}$ we obtain

$$a_i = \sum_{i=1}^{N} \Big[y_i(x_j)\Big]^{-1} \Big[y(x_i)\Big] \tag{C}$$

This means that if the value of the solution is known at N points then the coefficients $\{a_i\}$ are determined. Consequently, we can solve a problem using as unknowns either the coefficients $\{a_i\}$ or the set of solution values at the collocation points $\{y(x_j)\}$. To solve a differential equation that includes higher derivatives of y, we differentiate eq. (A) once or twice, and evaluate the result at the collocation points

$$y'(x_j) = \sum_{i=1}^{N} a_i y_i'(x_j) \tag{D}$$

$$y''(x_j) = \sum_{i=1}^{N} a_i y_i''(x_j) \tag{E}$$

Since the coefficients $\{a_i\}$ can be expressed interms of the solution values at the collocation points $\{y(x_j)\}$, the derivatives can also. We simply substitute eq. (C) in eq. (D) and eq. (E). Then the derivative at a particular collocation point, which is needed for the residual, is expressed in terms of the solution at all the collocation points :

$$y'(x_j) = \sum_{i,k=1}^{N} \Big[y_i(x_k)\Big]^{-1} \Big[y(x_k)\Big] y_i'(x_j) \tag{F}$$

$$y''(x_j) = \sum_{i,k=1}^{N} \Big[y_i(x_k)\Big]^{-1} \Big[y(x_k)\Big] y_i''(x_j) \tag{G}$$

We write the results as

$$y'(x_j) = \sum_{i=1}^{N} A_{jk} y(x_k) \tag{H}$$

$$y''(x_j) = \sum_{i=1}^{N} B_{jk} y(x_k) \tag{I}$$

Thus the derivative at any collocation point is expressed in terms of the value of the function at the collocation points. The next improvement to be introduced into the collocation method is to choose orthogonal polynomials for trial functions. We define the polynomial

$$P_m(x) = \sum_{jo}^{m} c_j x^j \tag{J}$$

and we say that the polynomial has degree m and order $m+1$. The coefficients in eq. (J) are defined by requiring that P_1 be orthogonal to P_0, P_2 be orthogonal to both P_0 and P_1, and P_m be orthogonal to each P_k, where $k \leq m-1$. The orthogonality condition can include a weighting function $W(x) \geq 0$. Thus

$$\int_a^b W(x)P_k(x)P_m(x)\,dx = 0 \qquad k = 0, 1, 2...m-1 \tag{K}$$

The procedure specifies the polynomials to within a multiplicative constant, which are determine by requiring the first coefficient to be one. For example, let us use $W(x) = 1$, $a = 0$ and $b = 1$. The polynomials are

$$P_0 = 1 \qquad P_1 = 1 + bx \qquad P_2 = 1 + cx + dx^2 \tag{L}$$

The first one is already known : $P_0 = 1$. The second one is found by requiring

$$\int_0^1 P_0 P_1\,dx = 0 \qquad or, \int_0^1 (1 + bx)\,dx = 0 \tag{M}$$

which makes $b = -2$. The third one P_2 is found from

$$\int_0^1 P_0 P_2\,dx = 0 \qquad \int_0^1 P_1 P_2\,dx = 0$$

and so forth. The results are

$$P_0 = 1, \; P_1 = 1 - 2x, \; P_2 = 1 - 6x + 6x^2$$

and

$$P_1(x) = 0 \quad \text{at } x = \frac{1}{2} \text{ and}$$
$$P_2(x) = 0 \quad \text{at } x = \frac{1}{2}(1 \pm \sqrt{3}/3)$$

The polynomial $P_m(x)$ has m roots in the interval a to b, and these serve as convenient choices of the collocation points. Thus if the expansion involves P_0 and P_1, such that

$$y = a_1 P_0(x) + a_2 P_1(x) \tag{N}$$

We need two collocation points to evaluate two residuals to determine the two constants a_1 and a_2, and we choose the two roots to $P_2(x) = 0$. We see that the whole procedure is automatic once the weighting function $W(x)$ is chosen. So, one has fewer arbitrary choices as to trial functions and collocating points, although the weighting function must be specified.

The equations which we developed in chapter 3 require that the solution to be symmetric about $r = 0$. Hence it can be expanded in terms of only even powers of r, excluding all the odd powers. In such a case it is our prerogative to include that information in the choice of trial functions. To do this we construct orthogonal polynomials that are functions of r^2. One possible choice is

$$y(r^2) = y(1) + (1 - r^2) \sum_{i=1}^{N} a_i P_{i-1}(r^2) \tag{O}$$

N is the number of interior collocation points. Equivalent choices are

$$y(r^2) = \sum_{i=1}^{N} b_i P_{i-1}(r^2) \tag{P}$$

$$= \sum_{i=1}^{N} d_i r^{2i-2} \tag{Q}$$

We define the polynomial to be orthogonal with the condition

$$\int_0^1 W(r^2)P_k(r^2)P_m(r^2)r^{a-1}\,dr = 0, \qquad k \leq m-1 \tag{R}$$

where a = 1, 2, or 3 for planar, cylindrical, or spherical geometry, respectively. Again we take first coefficient of the polynomial as one, so that the choice of the weighting function $W(r^2)$ completely determines the polynomial, and hence the trial function and the collocation points.

Let us differentiate eq. (Q) and take the first derivative and the laplacian of it, where the laplacian

$$\nabla^2 y = \frac{1}{r^{a-1}}\frac{\partial}{\partial r^{a-1}}\left(r^{a-1}\frac{\partial y}{\partial r^{a-1}}\right) \tag{S}$$

for three geometris. Thus

$$\frac{\partial y}{\partial r} = \sum_{i=1}^{N+1} d_i(2i-2)r^{2i-3} \tag{T}$$

and

$$\nabla^2 y = \sum_{i=1}^{N+1} d_i(2i-2)\Big[(2i-3)+a-1\Big]r^{2i-4} \tag{U}$$

Now the collocating points are N interior points in the interval $0 < r_j < 1$ and one boundary points $r_{n+1} = 1$. The point $r = 0$ is not included because the symmetry condition requires that the first derivative be zero at $r = 0$ and that condition is already built into the trial function. Now the derivatives are evaluated at the collocation points to give

$$y(r_j) = \sum_{i=1}^{N+1} d_i r_j^{2i-2} \tag{V}$$

$$\frac{\partial y(r_j)}{\partial r} = \sum_{i=1}^{N+1} d_i r_j^{2i-3} \tag{V}$$

$$\nabla^2 y(r_j) = \sum_{i=1}^{N+1} d_i \; \nabla^2(r^{2i-2})\Big|_{r_j} \tag{X}$$

And in matrix notation

$$\overline{y} = \overline{\overline{Q}}\,\overline{d} \qquad \overline{\frac{\partial y}{\partial r}} = \overline{\overline{C}}\,\overline{d} \qquad \overline{\nabla}^2\,\overline{y} = \overline{\overline{D}}\,\overline{d}$$

$$Q_{ji} = r_j^{2i-2} \qquad C_{ji} = (2i-2)r_j^{2i-3}$$

$$\begin{aligned} D_{ji} &= \nabla^2\,(r^{2i-2})\big|_{r_j} \\ &= (2i-2)^2 r_j^{2i-4} \qquad \text{for cylindrical geometry.} \end{aligned}$$

Solving for $\overline{d}$

$$\overline{\frac{\partial y}{\partial r}} = \overline{\overline{C}}\,\overline{\overline{Q}}^{-1} = \overline{\overline{A}}\,\overline{y} \tag{Y}$$

$$\overline{\nabla}^2\,\overline{y} = \overline{\overline{D}}\,\overline{\overline{Q}}^{-1}\,\overline{y} = \overline{\overline{B}}\,\overline{y} \tag{Z}$$

The roots of Jacobi polynomials are used throughout this work. Legendre polynomials have zeros well distributed, with some concentrations near the two ends of the interval. Jacobi polynomials emphasize the middle of the approximation interval. Although Legendre polynomials give the smallest approximation errors at both ends, large errors may occur within the approximation inverval. On the other hand Jacobi polynomials give minimum errors in the middle regions.

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