M. Sc. Engineering Thesis

Box-Rectangular Drawings of Planar Graphs

by

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Candidate's Declaration

This is to certify that the work presented in this thesis entitled "**Box-Rectangular Drawings of Planar Graphs**" is the outcome of the investigation carried out by me under the supervision of Professor Dr. Md. Saidur Rahman in the Department of Computer Science and Engineering, Bangladesh University of Engineering and Technology (BUET), Dhaka. It is also declared that neither this thesis nor any part thereof has been submitted or is being currently submitted anywhere else for the award of any degree or diploma.

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Graph Drawing and Information Visualization Lab, Dept. of CSE, BUET is an excellent environment which plays a magnificent role in such kind of ingenious work. I am proud to be a regular member of this lab.

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Abstract

A plane graph is a planar graph with a fixed embedding in the plane. In a box- rectangular drawing of a plane graph, every vertex is drawn as a rectangle, called a box, each edge is drawn as either a horizontal line segment or a vertical line segment, and the contour of each face is drawn as a rectangle. A planar graph is said to have a box-rectangular drawing if at least one of its plane embeddings has a box-rectangular drawing. In this thesis we give a linear-time algorithm to examine whether a planar graph G has a box-rectangular drawing or not, and to find a box-rectangular drawing of G if it exists. We first give an algorithm for box-rectangular drawings of planar graphs of the maximum degree three. We then reduce the problem of box-rectangular drawings of planar graphs of the maximum degree four or more to the problem of box-rectangular drawings of planar graphs of the maximum degree three.

Chapter 1

Introduction

A *graph* is a powerful tool to depict real life circumstances mathematically or by visualization where real world objects and entities are represented by small dots named *vertices* and relationships among them are represented by connecting lines called *edges*. Graph theory is widely used in all branches of Computer Science and Engineering. Uses of graphs are found not only in computer science but also in other fields of engineering, genetics, bioinformatics, molecular biology, chemistry, and even in geology and social sciences. One of the established techniques of information visualization is to draw a graph representing the information to be visualized with desired criteria. The field in which the different aesthetic techniques of drawing graphs are vividly studied is known as "Graph Drawing".

Concept of *Graph Drawing* is very ancient, but the use of it in computer science is not old enough. For the last two decades automatic drawings of graphs have created intense interest due to their broad applications, and as a consequence, a number of drawing styles and corresponding drawing algorithms have emerged [DET99]. The graph in Fig. 1.1(a) having nine vertices and twelve edges can easily be visualized in two ways as in Fig. 1.1(a) and in Fig. 1.1(b) which are different drawing techniques generally used in graph drawing. Different drawing styles play a very important role in circuit layouts, database diagrams, entity relationship diagrams etc. [B96, K96, S84, T87, TTV91], in VLSI floorplanning [KK84, L90, RNN98, RNN00, RNN02, TTSS91], and in architectural floorplanning [MKI00, BGPV08].

In the field of graph drawing, the geometric representations of graphs gener-

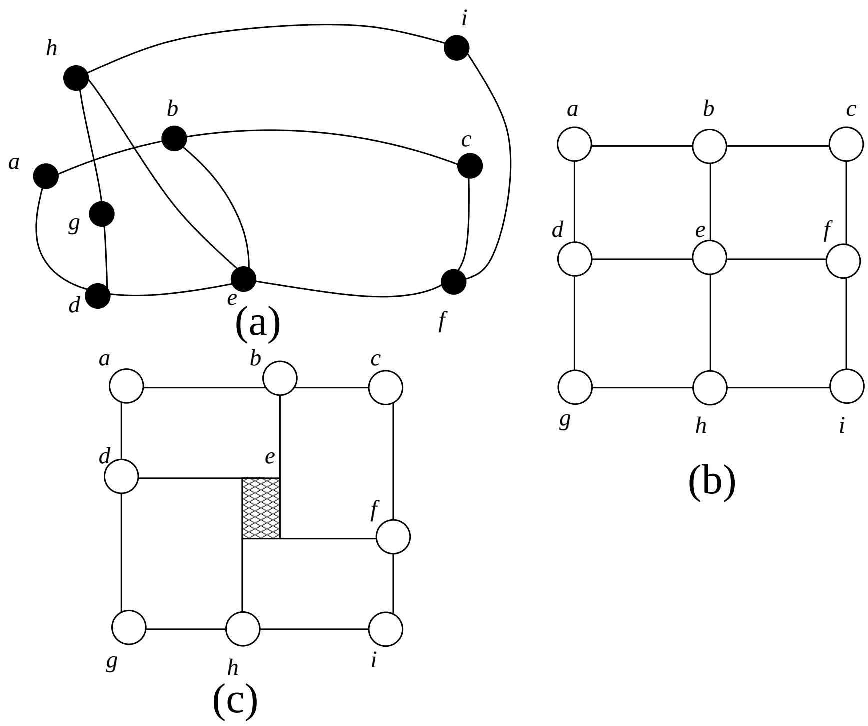


Figure 1.1: Two different drawings (b) and (c) of a same graph (a) in VLSI layout.

ated by graph drawing algorithms are constrained by some predefined geometric or aesthetic properties. The objective of graph drawing is to obtain a nice representation of a graph such that the structure of the graph is easily understandable, moreover the drawing should help to resolve the question arises from the application point of view using predefined properties.

A *plane graph* is a planar graph with a fixed embedding in the plane. A *box-rectangular drawing* of a plane graph G is a drawing of G in which each vertex is drawn as a rectangle, called a box, each edge is drawn as a horizontal line segment or a vertical line segment, and the contour of each face is drawn as a rectangle. A planar graph is said to have a box-rectangular drawing if at least one of its plane embeddings has a box-rectangular drawing. Box-rectangular drawings have many applications in VLSI floorplanning and in architectural floorplanning. Not all graphs have box-rectangular drawings. In this thesis we first address a necessary and sufficient condition for the existence of box-rectangular drawing of a planar graph. Then we establish a linear-time algorithm to find a box-rectangular drawing if it exists .

In the rest of this chapter, we provide with the necessary background and objectives for this thesis. We describe box-rectangular drawings of planar graphs in Section 1.1. Section 1.2 depicts some interesting applications of box-rectangular drawings. Section 1.3 presents the scope of this thesis with a brief overview of the previous results related to the scope. The new results of the thesis are also described in the same Section 1.3. Finally the thesis organization is narrated in Section 1.4.

1.1 Box-Rectangular Drawings of Planar Graphs

In this section we describe the definitions of different types of drawings, and differences between rectangular drawings and box-rectangular drawings of plane graphs.

1.1.1 Rectangular Drawing of a Plane Graph

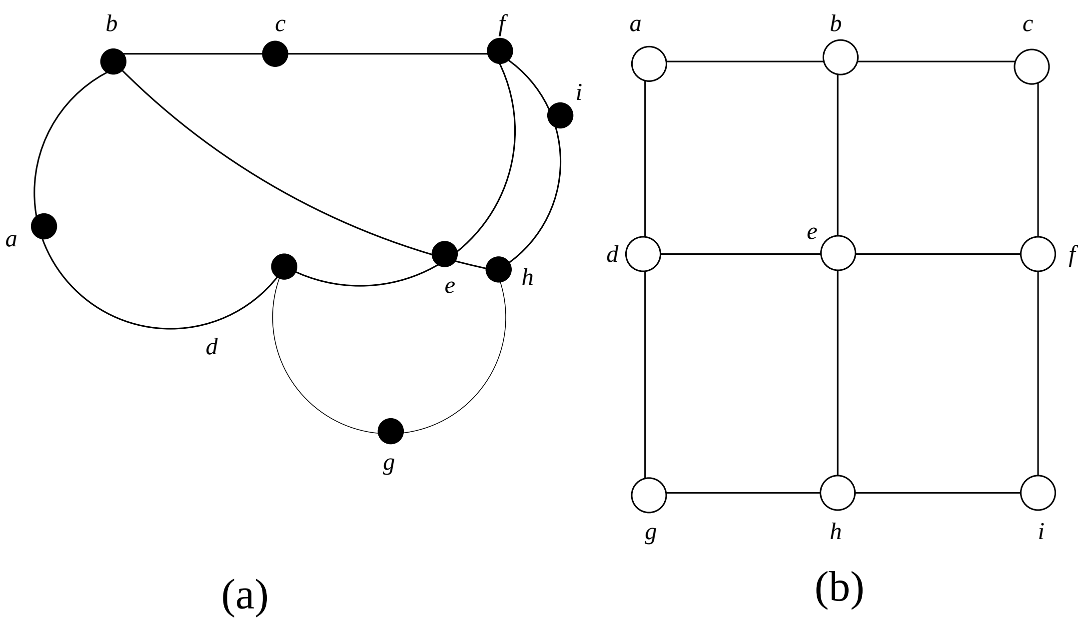


Figure 1.2: (a) A plane graph G, (b) rectangular drawing of the plane graph G.

A *rectangular drawing* of a plane graph G is a drawing of G, where each vertex is drawn as a point, each edge is drawn as a horizontal or vertical line segment, and each face is drawn as a rectangle. Figure 1.2(b) is a rectangular drawing of a planar graph G (Fig. 1.2(a)).

1.1.2 Box-Rectangular Drawing of a Plane graph

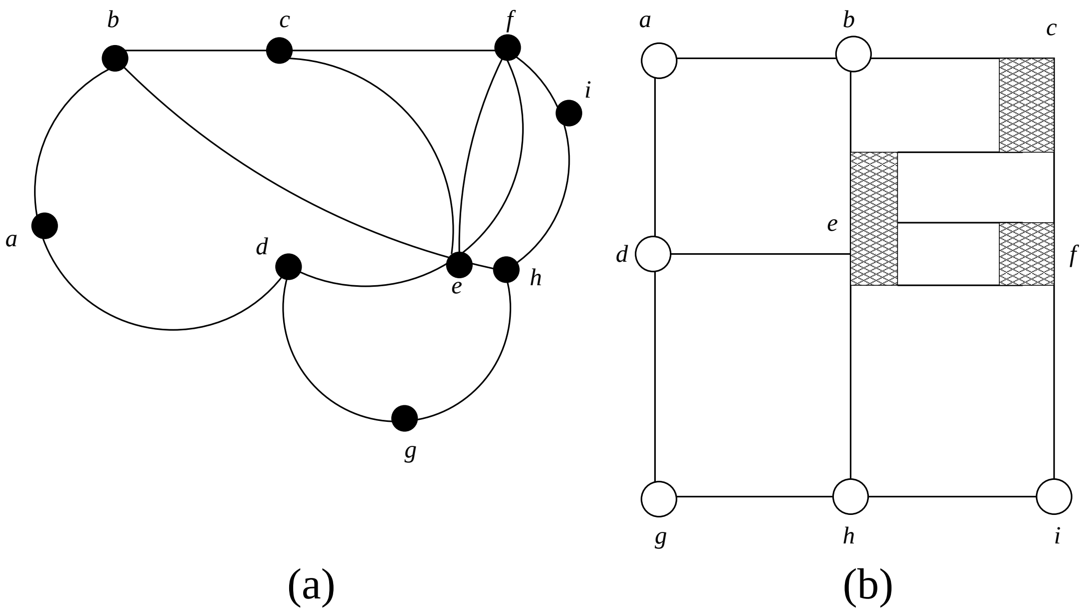


Figure 1.3: (a) A plane graph G, (b) box-rectangular drawing of the plane graph G.

A *box-rectangular drawing* of a plane graph G is a drawing of G, where each vertex is drawn as a rectangle, called a box, each edge is drawn as a horizontal or vertical line segment, and the contour of each face is drawn as a rectangle. Figure 1.3(b) is a box-rectangular drawing of the plane graph G in Fig. 1.3(a).

1.1.3 Differences Between Rectangular Drawings and Box-Rectangular Drawings of Plane Graphs

If a plane graph G with $\Delta \leq 3$ has a rectangular drawing, then the graph G must have a box-rectangular drawing, and the drawings are same for G, as illustrated in Fig. 1.4. But if a plane graph G with $\Delta \leq 3$ has a box-rectangular drawing, then the graph G may not have a rectangular drawing. The plane graph G in Fig. 1.5(b) with $\Delta \leq 3$ has a box-rectangular drawing as in Fig. 1.5(c) but not a rectangular drawing. Similarly if a plane graph G for a planar graph in Fig.

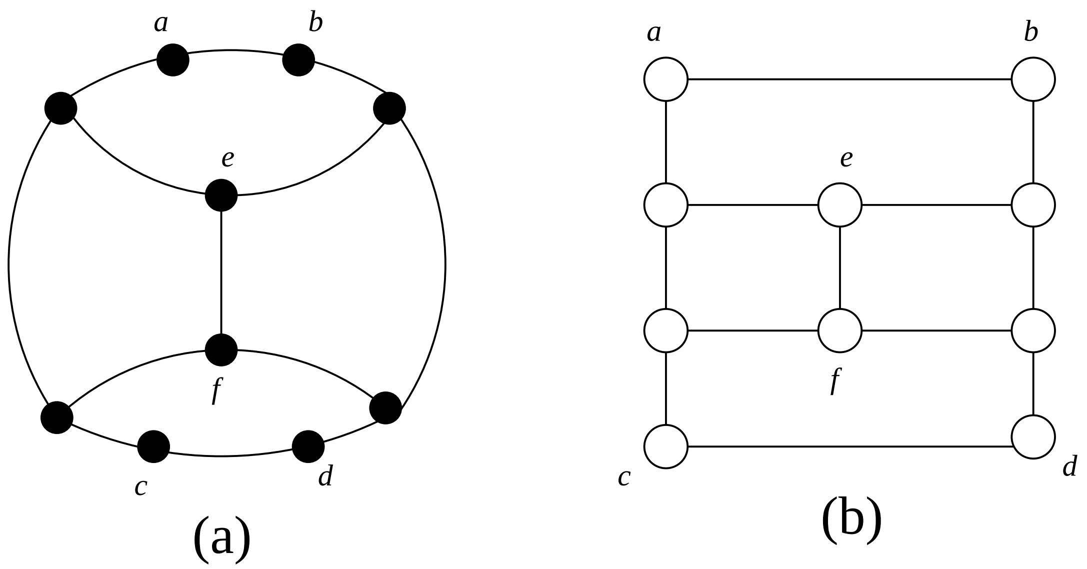


Figure 1.4: (a) A plane graph G with $\Delta \leq 3$, (b) same rectangular drawing and box-rectangular drawing of the plane graph G.

1.1(a) with $\Delta = 4$ has a rectangular drawing, then the graph G must have a box-rectangular drawing, but the drawings are not same for G, as illustrated in Figures 1.1(b) and 1.1(c). Because vertices of degree 4 or more are drawn as real boxes in the box-rectangular of a plane graph with $\Delta \geq 4$. If a plane graph G with $\Delta = 4$ has a box-rectangular drawing, then the graph G may not have a rectangular drawing. A multi graph does not have a rectangular drawing but may have a box-rectangular drawing like Fig. 1.3. For a plane graph G with $\Delta > 4$, there is no rectangular drawing, but box-rectangular drawing may exist. Figure 1.3 describes this case. Lastly a graph with a cut vertex does not have a rectangular drawing but may have a box-rectangular drawing.

1.1.4 Box-Rectangular Drawing of a Planar Graph

A planar graph is said to have a box-rectangular drawing if at least one of its plane embeddings has a box-rectangular drawing. Figure 1.5(a) is a planar graph G. Figure 1.5(b) is a plane embedding of G for which G has a box-rectangular drawing. Finally figure 1.5(c) is a box-rectangular drawing of the planar graph G.

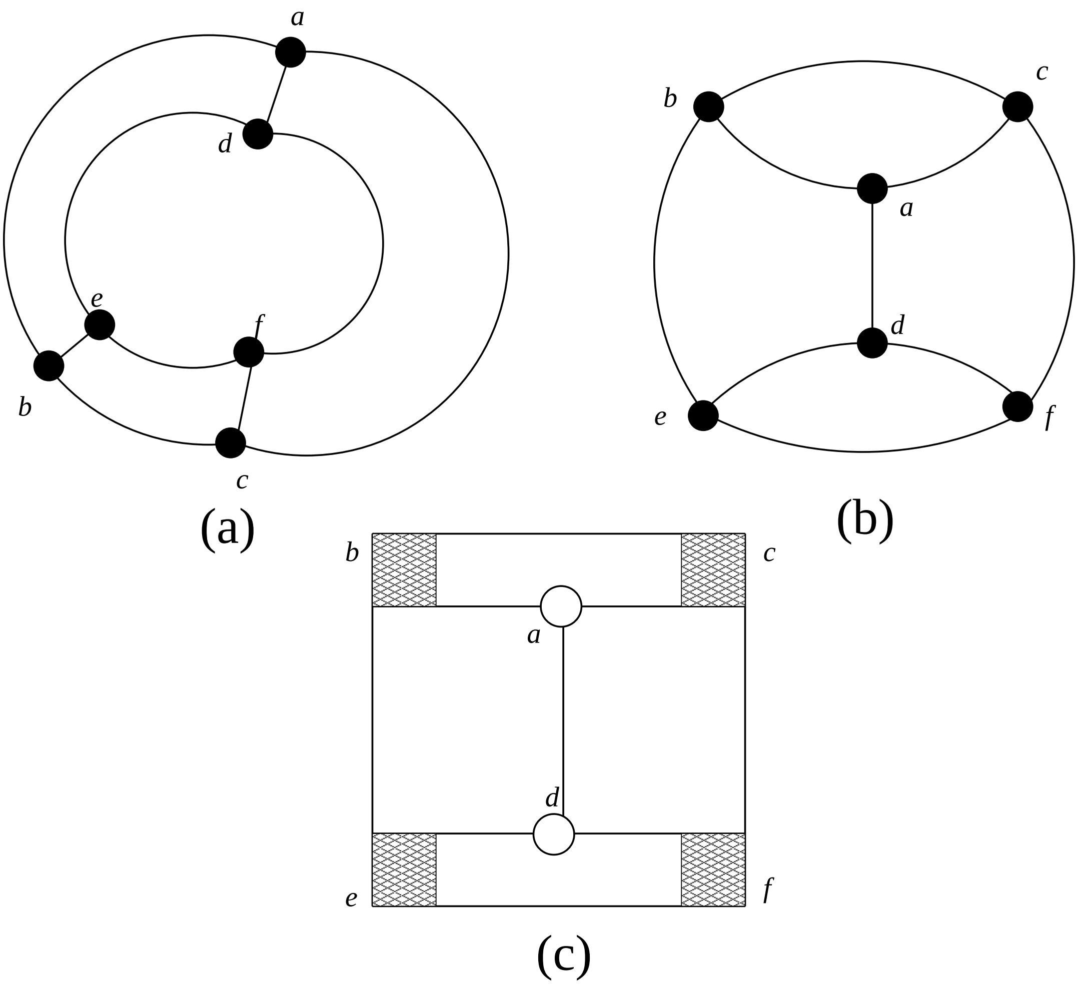


Figure 1.5: (a) A planar graph G, (b) a plane embedding Γ of G for which box-rectangular drawing exists, and (c) box-rectangular drawing of the planar graph G.

1.2 Applications of Box-Rectangular Drawings

Box-rectangular drawings of planar graphs have number of applications in the areas of VLSI Layouts and architectural floorplanning .

1.2.1 VLSI Layout

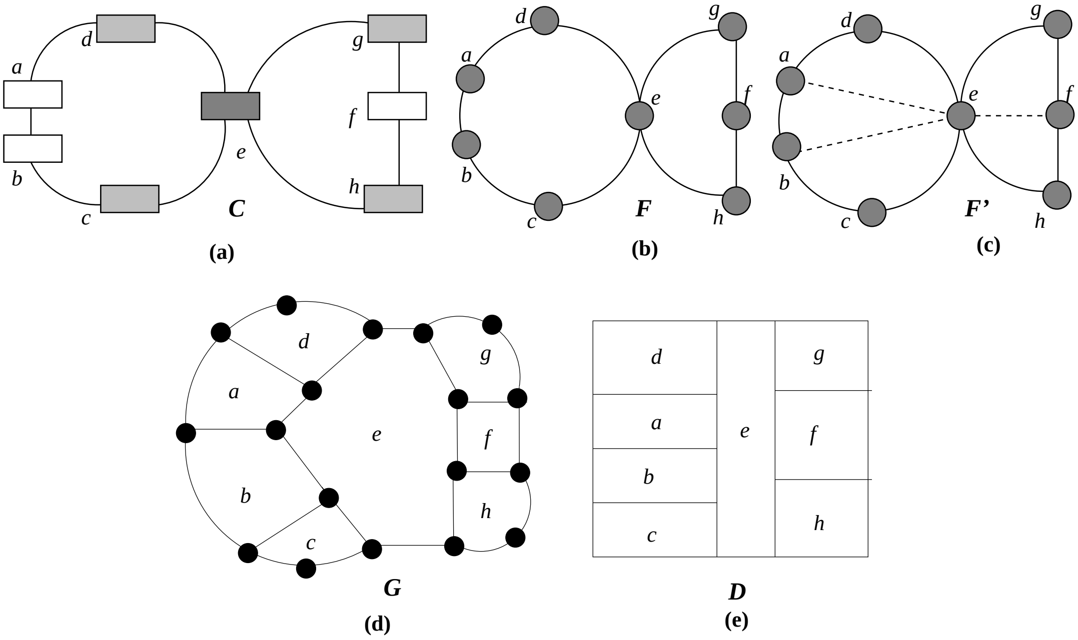


Figure 1.6: Floorplanning by a rectangular drawing.

Rectangular drawings have practical applications in VLSI floorplanning. In a VLSI floorplanning problem, an input is a circuit schematic C as illustrated in Figure 1.6(a) so that it can be transformed to a plane graph F as in Figure 1.6(b); F represents the functional entities of the chip, called *modules*, and interconnections among the modules; each vertex of F represents a module, and an edge between two vertices of F represents the interconnection between the two corresponding modules. An output of the problem for the transformed graph F is a partition of a rectangular chip area into smaller rectangles as illustrated in Figure 1.6(e); each module is assigned to a smaller rectangle, and furthermore, if two modules have interconnections, then their corresponding rectangles must be adjacent, that is, must have common boundary.

A conventional floorplanning algorithm using rectangular drawings is outlined as follows. First, obtain a graph F' by triangulating all inner faces of

F as illustrated in Figure 1.6(c), where dotted lines indicate new edges added to F. Then obtain a dual-like graph G of F' as illustrated in Figure 1.6(d). Finally, by finding a rectangular drawing of G, obtain a possible floorplan for F as illustrated in Figure 1.6(e).

In the conventional floorplan above, two rectangles are always adjacent if the modules corresponding to them have interconnections. However, two rectangles may be adjacent even if the modules corresponding to them have no interconnections. For example, modules a and e have no interconnection in Figure 1.6(a), but their corresponding rectangles are adjacent in the floorplan as in Figure 1.6(e). Such unwanted adjacencies are not desirable in some other floorplanning problems. In floorplanning of an MCM, two chips generating excessive heat should not be adjacent, or two chips operating on high frequency should not be adjacent to avoid malfunctioning due to their interference [S95].

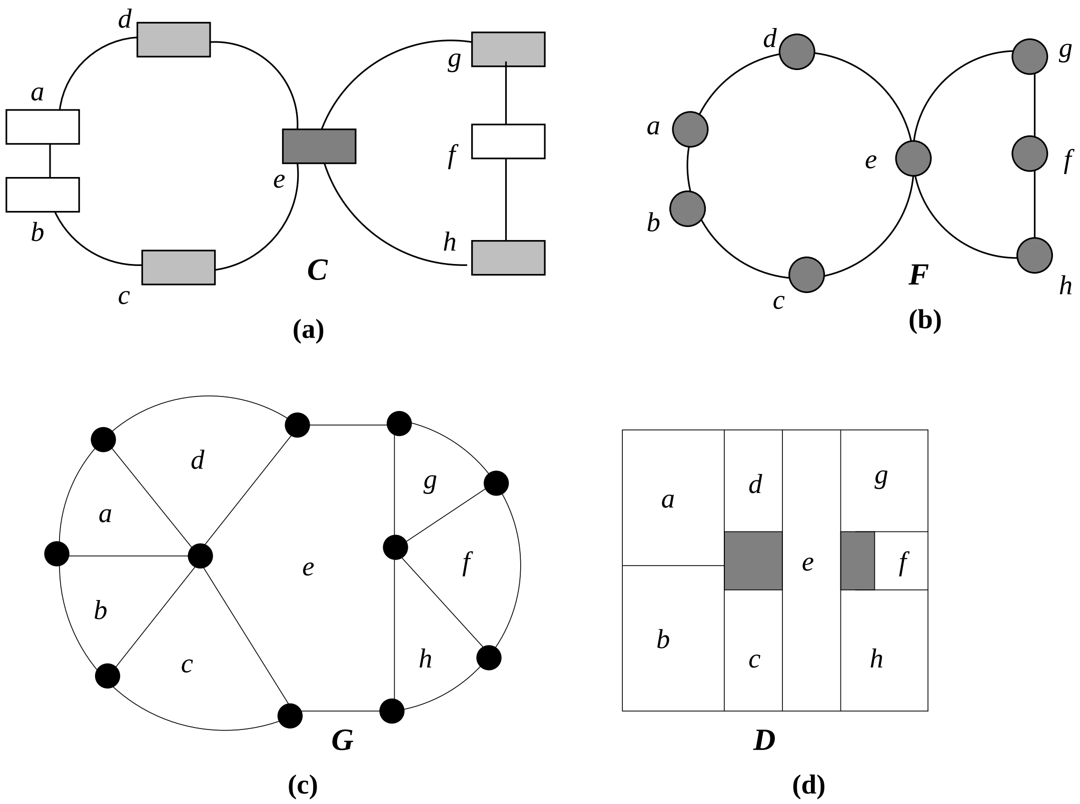


Figure 1.7: Floorplanning by a box-rectangular drawing.

We can avoid unwanted adjacencies if we obtain a floorplan for F by using a box-rectangular drawing instead of a rectangular drawing, as follows. First, without triangulating the inner faces of F, find a dual-like graph G of F as illustrated in Figure 1.7(c). Then, by finding a box-rectangular drawing of G, obtain a possible floorplan for F as illustrated in Figure 1.7(d). In Figure 1.7(d) modules a and e are not adjacent. They are as separated by a dead space drawn by a rectangular box corresponding to a vertex of G .

1.2.2 Architectural Floorplanning

Unwanted adjacencies may cause a dangerous situation in some architectural floorplanning, too [FW74]. For example, in a chemical industry, a processing unit that deals with poisonous chemicals should not be adjacent to a cafeteria. Such a dead space as in Figure 1.7(d) to separate two rectangles in floorplanning is desirable for ensuring safety in a chemical industry.

1.3 Scope of This Thesis

In this section we first mention the previous results related to box-rectangular drawings of planar graphs. After that we will discuss the results obtained in this thesis.

1.3.1 Previous Results

Thomassen [T84] obtained a necessary and sufficient condition for a plane graph of $\Delta \leq 4$ to have a rectangular drawing when a quadruplet of vertices of degree two on the outer face are designated as convex corners. Linear-time algorithms are given in [BS88, H93, KH97, RNN98] to obtain a rectangular drawing of such a plane graph. We say that *a plane graph G has a rectangular drawing* if there is a rectangular drawing of G for some quadruplet of vertices appropriately chosen as corners. Rahman et al. [RNN02] gave a necessary and sufficient condition for a plane graph of $\Delta \leq 3$ to have a rectangular drawing (for some appropriately chosen quadruplet), and developed a linear-time algorithm to choose such a quadruplet and find a rectangular drawing of a plane graph for the chosen corners if they exist. Rahman et al. [RNN00] gave a necessary and sufficient condition for a plane graph to have a box-rectangular drawing, and developed a linear-time algorithm to draw a box-rectangular drawing of a plane graph if it exists. Xin He [H01] did the same task for proper box- rectangular drawings of plane graphs. Rahman et al. [RNG04] gave a linear-time algorithm to examine whether a planar graph G with $\Delta \leq 3$ has a rectangular drawing and to find a rectangular drawing of G if it exists. Since a planar graph G may have an exponential number of embeddings, determining whether G has a box-rectangular drawing or not using the linear algorithm of Rahman et al. [RNN00] for each embedding of G takes exponential time. Thus to develop an efficient algorithm to examine whether a planar graph has a box-rectangular or not is a non-trivial problem. Rahman et al. mentioned that extending their work in [RNG04], one can obtain a linear-time algorithm which can test whether a planar graph has a box-rectangular drawing or not and find a box-rectangular drawing if it exists. But such algorithm has not been developed till now.

1.3.2 Results in This Thesis

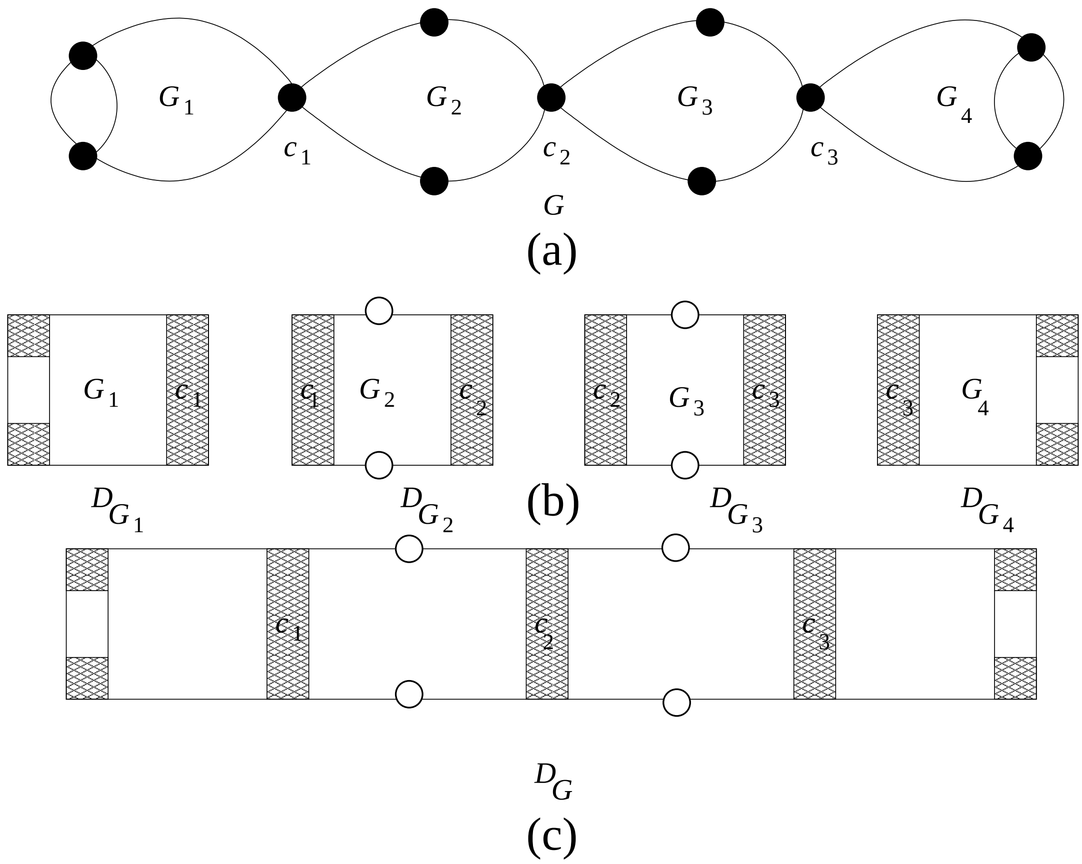


Figure 1.8: Box-rectangular drawing of a planar graph G with cut vertices.

In this thesis we mainly develop two different algorithms. We give linear-time algorithms to examine whether a planar graph G has a box-rectangular drawing and to find a box-rectangular drawing of G if it exists. We can assume in our thesis that, G is connected and has neither a vertex of degree 1 nor a 1-legged cycle; otherwise the planar graph G does not have a box-rectangular drawing as all the faces of the graph can not be drawn as rectangular faces simultaneously. If a planar graph G has neither a vertex of degree 1 nor a 1-legged cycle, and if the graph G is 1-connected, then the cut vertex v must be of degree 4 or more. In Fig. 1.8(a) G is a such kind of planar embedding. c_1, c_2, and c_3 are the cut vertices. If G is not a multi graph, then G_1, G_2, G_3, and G_4 are the different 2-connected components of G with respect to cut vertices. If G has a box-rectangular drawing D_G, then all the cut vertices must reside on the outer face $F_O(D_G)$ of the drawing D_G. All the connected components G_1, G_2, G_3, and G_4 have box-rectangular drawings separately, as illustrated in Fig. 1.8(b). If any of the components G_1, G_2, G_3, and G_4 contains exactly one cut vertex, then that component must have a box-rectangular rectangular drawing with at least one corner box containing two corners. G_1 and G_4 are such kind of components drawn as box-rectangular drawings D_{G_1} and D_{G_4}, respectively as illustrated in Fig 1.8(b). If any of the components G_1, G_2, G_3, and G_4 contains two cut vertices, then that component must have a box-rectangular rectangular drawing with exactly two corner boxes. Each corner box contains exactly two corners. G_2 and G_3 are such kind of components drawn as box-rectangular drawings D_{G_2} and D_{G_3}, respectively as illustrated in Fig 1.8(b). No component contains 3 or more cut vertices; otherwise box-rectangular drawing does not exist for G, as the outer face $F_O(G)$ must be rectangular shape in the drawing. If all the box-rectangular drawings $D_{G_1}, D_{G_2}, D_{G_3}$ and D_{G_4} are merged together, then we will get another box-rectangular drawing D_G of the planar graph G, as illustrated in Fig. 1.8(c). Similar box-rectangular drawing also exists for G, even if the above planar 1-connected graph G is a multigraph. As the case, box-rectangular drawing of a planar 1-connected graph, is a trivial observation, we consider only 2-connected graphs in our thesis.

- We first consider the case for a planar graph G where $\Delta \leq 3$. At first we consider the case where G is a "subdivision" of a planar 3-connected cubic graph. A subdivision of a planar 3-connected cubic graph G has exactly one embedding for each face embedded as the outer face [NC88]. Hence

G has an $O(n)$ number of embeddings, one for each each chosen outer face. Thus, the straightforward algorithm takes $O(n^2)$ time to examine whether the planar graph G has a box-rectangular drawing. We, however, obtain a necessary and sufficient condition for a subdivision of a planar 3-connected cubic graph G to have a box-rectangular drawing, which leads to a linear-time algorithm to examine whether the planar graph G has a box-rectangular drawing. If G is not a subdivision of a planar 3-connected cubic graph, then G may have an exponential number of embeddings, and hence a straightforward algorithm does not run in polynomial time. We, however, develop a linear time algorithm to examine whether G has a box-rectangular drawing or not; we indeed show that it suffices to examine whether only four embeddings of G have box-rectangular drawings or not.

- We secondly consider the case for a planar 2-connected graph G where $\Delta \geq 4$. In this portion we first consider the case where G is a "subdivision" of a planar 3-connected graph. We replace the vertices of degree 4 or more by cycles in an arbitrary plane embedding Γ of G, call the planar graph H, which takes linear-time. Then we develop a linear-time algorithm to examine whether H has a box-rectangular drawing or not. We also show that whether G has a box-rectangular drawing can be tested by whether H has a box-rectangular drawing. Secondly if a graph G with $\Delta \geq 4$ is not a subdivision of a planar 3-connected graph, then the graph is 2-connected. For this case we show here that, we need to check 81 number of graphs, whether any one them has a box-rectangular drawing, to decide whether the planar graph G has a box-rectangular drawing, leads to a linear-time algorithm.

1.4 Thesis Organization

The rest of this thesis is organized as follows. In Chapter 2, we give some basic terminologies of graph theory, and algorithmic theory. Chapter 2 also describes the algorithm for finding box-rectangular drawings of plane graphs, and its related results. In Chapter 3, we describe a necessary and sufficient condition with a linear-time algorithm for a planar graph G with $\Delta \leq 3$ to have a box-rectangular drawing and to do the drawing if drawing exists. Chapter 4 illustrates a necessary and sufficient condition for a planar 2-connected graph G with $\Delta \geq 4$ to have a box-rectangular drawing and describes also the linear-time algorithm for finding out the drawing if drawing exists. Finally Chapter 5 concludes the thesis with a summary of the results and some future works.

Chapter 2

Preliminaries

In this chapter we define some basic terminologies of graph theory, graph drawing, box-rectangular drawing, and algorithm theory, that we will use throughout the rest of this thesis. In Section 2.1, we cover some definitions of standard graph-theoretical terms. We devote Section 2.2 to define terms related to planar graphs. Section 2.3 defines some drawing conventions and Section 2.4 consists of the terms related to a box-rectangular drawing. Description about box-rectangular drawings of plane graphs is told in Section 2.5. Finally we introduce the notion of time complexity of algorithms in Section 2.6.

2.1 Basic Terminology

In this section we give some definitions of standard graph-theoretical terms used throughout this thesis. For readers interested in more details of graph theory we refer to [NC88, NR04, Wes01].

2.1.1 Graphs and Subgraphs

A *graph* G is a tuple (V, E) which consists of a finite set V of *vertices* and a finite set E of *edges*; each edge being an unordered pair of vertices. Figure 2.1 depicts a graph $G = (V, E)$ where each vertex in $V = \{v_1, v_2, \ldots, v_6\}$ is drawn as a small circle and each edge in $E = \{e_1, e_2, \ldots, e_8\}$ is drawn by a line segment.

We denote an edge joining two vertices u and v of the graph $G = (V, E)$ by (u, v) or simply by uv. If $uv \in E$ then the two vertices u and v of the graph G are said to be *adjacent*; the edge uv is then said to be *incident* to the vertices

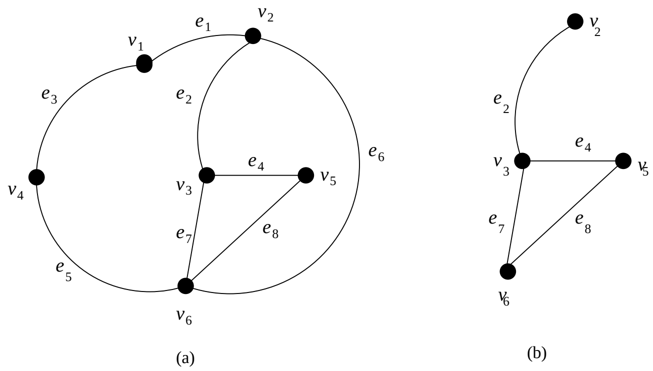


Figure 2.1: (a) A graph G with six vertices and eight edges, (b) a subgraph G' for G in (a).

u and v; also the vertex u is said to be a neighbor of the vertex v (and vice versa). We denote the maximum degree of a graph G by $\Delta(G)$ or simply by Δ. The *degree* of a vertex v in G, denoted by $d(v)$ or $deg(v)$, is the number of edges incident to v in G. In the graph shown in Figure 2.1(a) vertices v_1 and v_2 are adjacent, and $d(v_6) = 4$, since four of the edges, namely e_5, e_6, e_7 and e_8 are incident to v_6. A graph G is called *cubic* if $d(v) = 3$ for every vertex v. A *Subgraph* of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. If G' contains all the edges of G that join vertices in V', then G' is called the *subgraph induced by* V'. Figure 2.1(b) depicts a subgraph of G in Fig. 2.1(a) induced by v_2, v_3, v_5, and v_6. If $V' = V$, then G' is called a *spanning subgraph of* G. For $V' \subseteq V$, $G - V'$ denotes a gaph obtained from G by deleting all vertices in V' together with all edges incident to them. For a subgraph G' of G, we denote by $G - G'$ the graph obtained from G by deleting all vertices in G'.

2.1.2 Simple Graphs and Multigraphs

If a graph G has no "multiple edges" or "loops", then G is said to be a *simple graph. Multiple edges* join the same pair of vertices, while a *loop* joins a vertex with itself. The graph in Figure 2.1(a) is a simple graph.

A graph in which loops and multiple edges are allowed is called a *multigraph.* Multigraphs can arise from various applications. One example is the "call graph" that represents the telephone call history of a network. The graph in Figure 2.2(a) is a call graph that represents the call history among six sub-

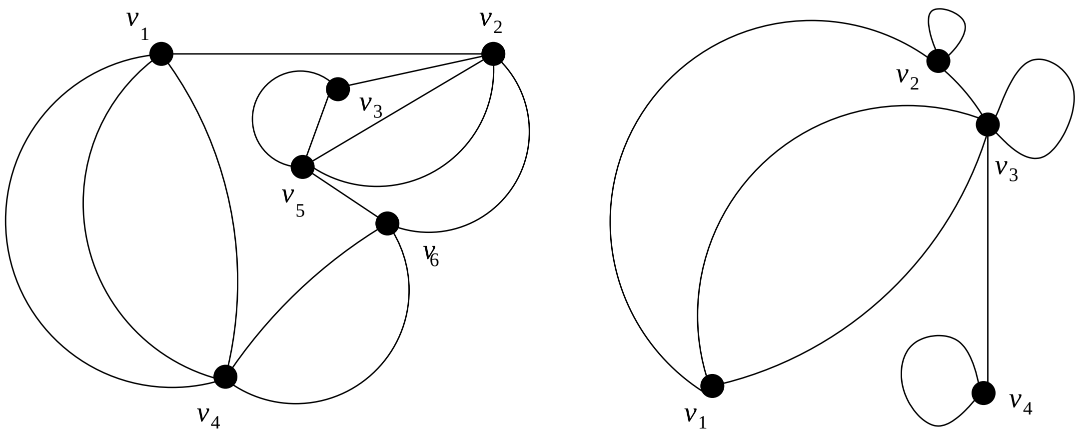


Figure 2.2: Multigraphs.

scribers. Note that there is no loop in this graph. Figure 2.2(b) illustrates another multigraph with multiple edges and loops.

Often it is clear from the context that the graph is simple. In such cases, a simple graph is called a *graph*. In the remainder of thesis we will only be concerned about graphs that may have multiple edges but no loops.

2.1.3 Paths and Cycles

A *walk*, $w = v_0, e_1, v_1, \ldots, v_{l-1}, e_l, v_l$, in a graph G is an alternating sequence of vertices and edges of G, beginning and ending with a vertex, in which each edge is incident to the two vertices immediately preceding and following it. The vertices v_0 and v_l are said to be the end-vertices of the walk w.

If the vertices $v_0, v_1, \ldots, v_l$ are distinct (except possibly v_0 and v_l), then the walk is called a *path* and usually denoted either by the sequence of vertices $v_0, v_1, \ldots, v_l$ or by the sequence of edges $e_1, e_2, \ldots, e_l$. The length of the path is l, one less than the number of vertices on the path. For any two vertices u and v of G, a uv-path in G is a path whose end-vertices are u and v.

A walk or path w is *closed* if the end-vertices of w are the same. A closed path containing at least one edge is called a *cycle*.

2.1.4 Chain and Support

Let $P = w_0, w_1, w_2, \ldots, w_{k+1}$, $k \geq 1$, be a path of G such that $d(w_0) \geq 3$, $d(w_1) = d(w_2) = \ldots d(w_k) = 2$, and $d(w_{k+1}) \geq 3$. Then we call the subpath $P' = w_1, w_2, \ldots, w_k$ of P a *chain* of G and call vertices w_0 and w_{k+1} the *supports* of the chain P'. If G is a subdivision of 3-connected graph, then any vertex of degree 2 in G is contained in exactly one of the chains of G. Two chains of G are *adjacent* if they have a common support.

2.1.5 Graph Subdivision

Subdividing an edge (u, v) of a graph G is the operation of deleting the edge (u, v) and adding a path $u(= w_0), w_1, w_2, \ldots, w_k, v(= w_{k+1})$ passing through new vertices $w_1, w_2, \ldots, w_k$, $k \geq 1$, of degree 2. A graph G is called a *subdivision* of a graph G' if G is obtained from G' by subdividing some of the edges of G'.

2.1.6 Connectivity

A graph G is *connected* if for any two distinct vertices u and v of G, there is a path between u and v. A graph which is not connected is called a *disconnected graph.* A *(connected) component* of a graph is a maximal connected subgraph. The graph in Figure 2.3(a) is a connected graph since there is a path between every pair of distinct vertices of the graph. On the other hand, the graph in Figure 2.3(b) is a disconnected graph since there is no path between, say, v_1 and v_5; v_1 and v_7; v_1 and v_9 . The graph in Figure 2.3(b) has four connected components as indicated by the dotted lines. Note that every connected graph has only one component; the graph itself.

The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph K_1. We say that G is *k-connected* if $\kappa(G) \geq k$. 2-connected and 3- connected graphs are also called biconnected and triconnected graphs, respectively. A *block* is a maximal biconnected subgraph of G. We call a set of vertices in a connected graph G a *separator* or a *vertex cut* if the removal of the vertices in the set results in a disconnected or single-vertex graph. If a vertex-cut contains exactly one vertex then we call the vertex a *cut vertex.* A *separation pair* of 2-connected graph G is a pair of vertices whose deletion disconnects G. A 3-connected graph has no

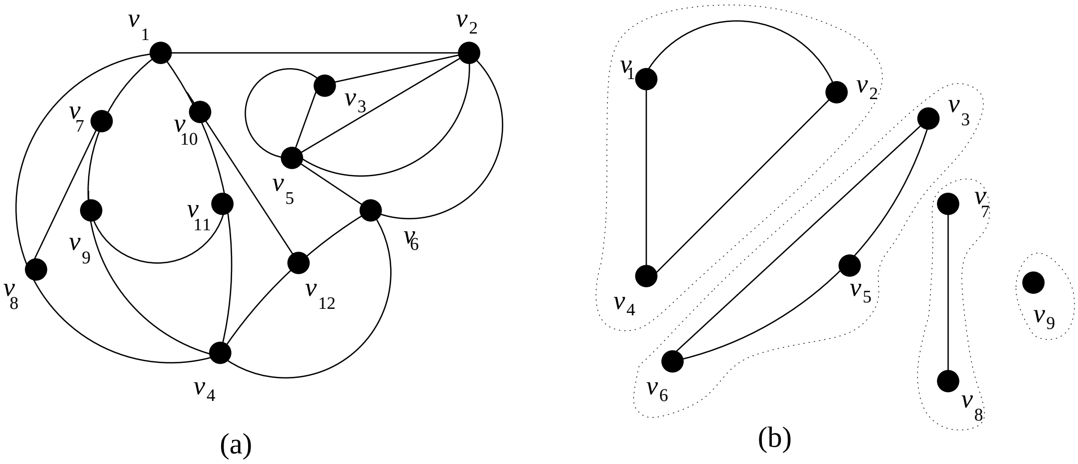


Figure 2.3: (a) A connected graph, (b) a disconnected graph with four connected components.

separation pair. A graph G is called *cyclically* 4-*edge-connected* if the removal of

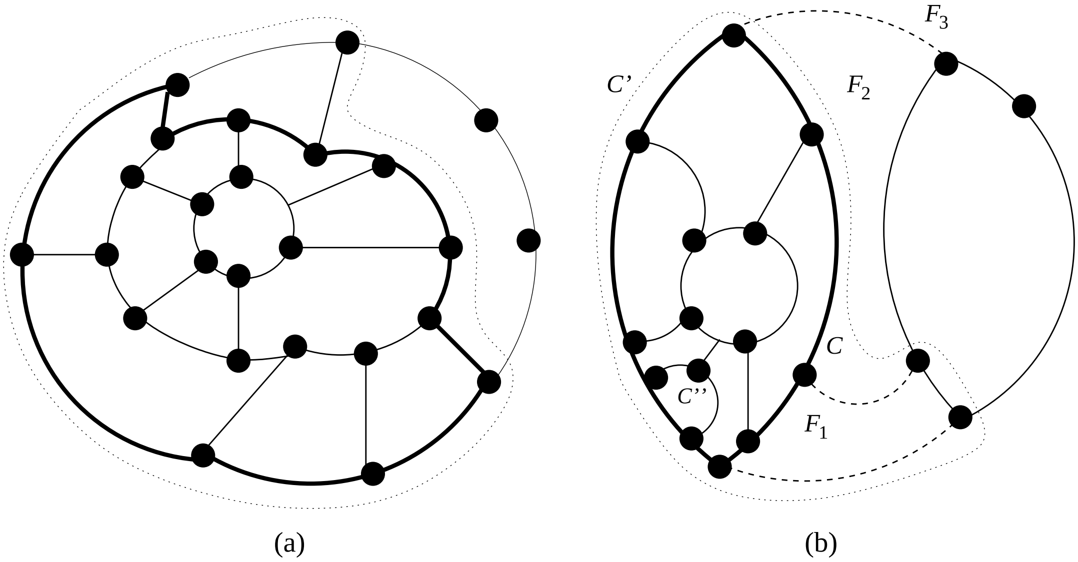


Figure 2.4: (a) A cyclically 4-edge-connected graph, and (b) a graph which is not cyclically 4-edge-connected.

any three or fewer edges leaves a graph such that exactly one of the components has a cycle [T92]. The graph in Fig. 2.4(a) is cyclically 4-edge-connected. On the other hand, the graph in Fig. 2.4(b) is not cyclically 4-edge-connected, since the removal of the three edges drawn by thick dotted lines leaves a graph with two connected components each of which has a cycle.

2.1.7 Vertex Replacement

We often use the the following operation on a planar graph G. Let v be a

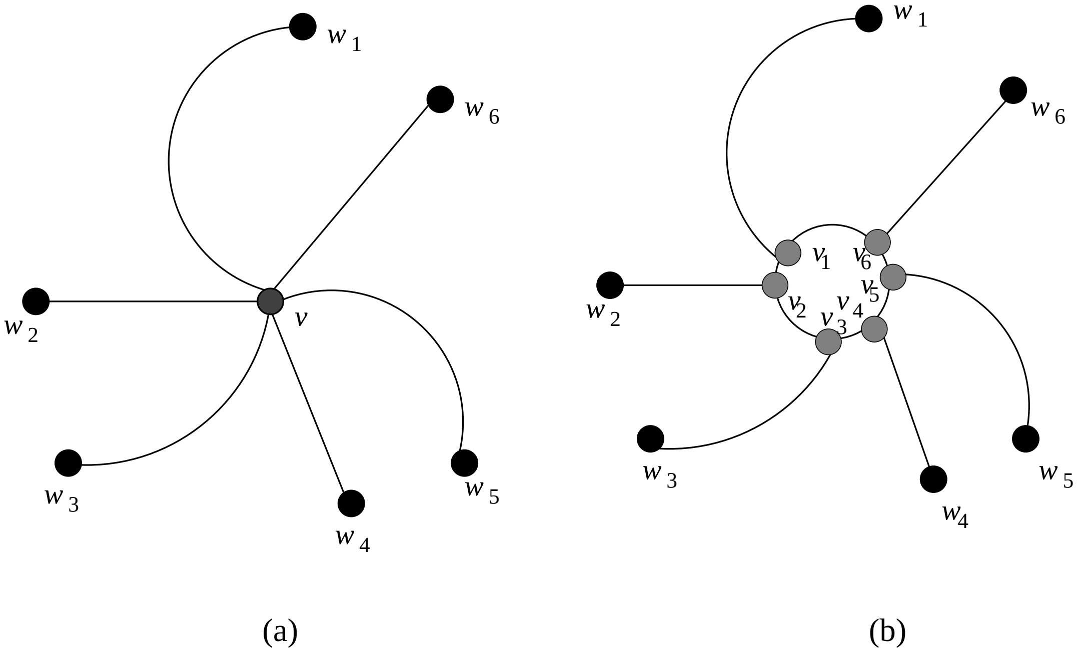


Figure 2.5: Replacement of a vertex v by a cycle.

vertex of degree d in a plane graph Γ of the planar graph G, let $e_1 = vw_1, e_2 = vw_2, \ldots, e_d = vw_d$ be the edges incident to v, and assume that these edges $e_1, e_2, \ldots, e_d$ appear clockwise around v in this order as illustrated in Fig. 2.5(a). Replace v with a cycle $v_1, v_1v_2, v_2, v_2v_3, \ldots, v_dv_1, v_1$, and replace the edges vw_i with v_iw_i for $i = 1, 2, \ldots, d$, as illustrated in Fig. 2.5(b). We call the operation above *replacement of a vertex by a cycle.* The cycle $v_1, v_1v_2, v_2, v_2v_3, \ldots, v_dv_1, v_1$ in the resulting graph is called is called the *replaced cycle* corresponding to the vertex v of Γ.

2.1.8 Removal of a Vertex of Degree 2

We often construct a new graph from a graph as follows. Let v be a vertex of degree 2 in a connected graph G. We replace the two edges u_1v and u_2v incident to v with a single edge u_1u_2 and delete v. We call the operation above *the removal of a vertex of degree* 2 from G. A graph G' is defined to the *homeomorphic* to G if G' is obtained from G by a sequence of removal operations as illustrated in Fig. 2.6 . We call the graph G' the *minimal graph*

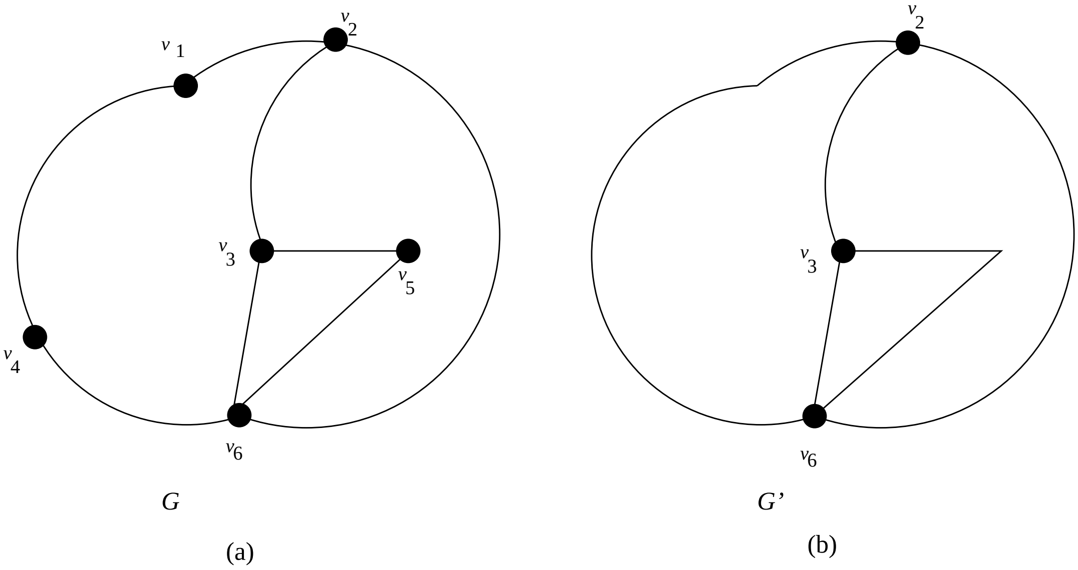


Figure 2.6: Removal of vertices of degree 2.

homeomorphic to G if G' is obtained from G by repeatedly removing vertices of degree 2 until either there is no vertex of degree 2 or the resulting graph has exactly two vertices.

2.2 Planar Graphs

In this section we give some definitions related to planar graphs used in the remainder of the thesis. For readers interested in more details of planar graphs we refer to [NR04].

2.2.1 Planar Graphs and Plane Graphs

A *planar drawing* of a graph G is a two-dimensional drawing of G in which no pair of edges intersect with each other except at their common end-vertex. A planar graph is a graph that has at least one planar drawing. A *planar embedding* of a graph G is a data structure that defines a clockwise (or counter clockwise) ordering of the neighbors of each vertex of G that corresponds to a planar drawing of the graph. Note that a planar graph may have an exponential number of embeddings. Figure 2.7 shows two planar embeddings of the same planar graph. A *plane graph* is a planar graph with a fixed planar embedding.

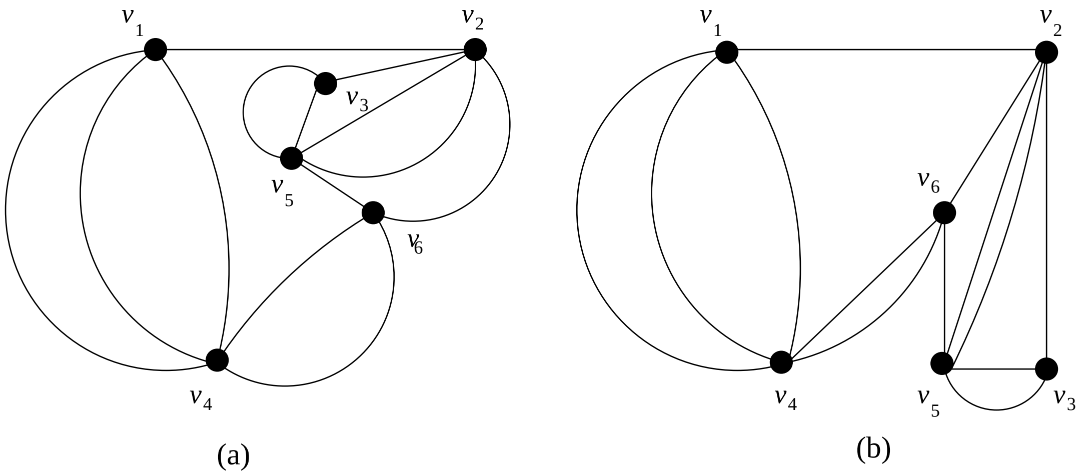


Figure 2.7: Two plane embeddings of the same planar graph.

2.2.2 Face and Facial Cycle

The plane graph G divides the plane into connected regions called *faces*. A finite plane graph G has one unbounded face and it is called the *outer face* of G. The contour of a face is called a *facial cycle*.

2.2.3 Inner Subgraph and Outer Subgraph

Let G be a planar graph, and Γ be an arbitrary plane embedding of G. We denote by $F_O(\Gamma)$ the outer face of Γ. For a cycle C of Γ, we call the plane subgraph of Γ inside C (including C) the *inner subgraph* $\Gamma_1(C)$ for C, and we call the plane subgraph of Γ outside C (including C) the *outer subgraph* $\Gamma_O(C)$ for C.

2.2.4 Leg and k-Legged Cycle

An edge which is incident to exactly one vertex of a cycle C and located outside C is called a *leg* of C. The vertex of C to which a leg is incident is called a *leg-vertex* of C. A cycle C in Γ is called a *k-legged cycle* of Γ if C has exactly k legs and there is no edge which joins two vertices on C and is located outside C. In each of Figs. 2.4(a) and 2.4(b), a 3-legged cycle is drawn by thick solid lines. The set of k legs of a k- legged cycle in Γ corresponds to a "cutset" of k edges.

2.2.5 Peripheral Face

We call a face F of Γ a *peripheral face* for a 3-legged cycle C in Γ if F is in $\Gamma_O(C)$ and the contour of F contains an edge on C. Clearly there are exactly three peripheral faces for any 3-legged cycle in Γ. In Fig. 2.4(b) F_1, F_2, F_3 are the three peripheral faces for the 3-legged cycle C drawn by thick solid lines.

2.2.6 Minimal k-Legged Cycle, Independent Cycles, and Regular 2- or 3-Legged Cycle

A k-legged cycle C is called a *minimal k-legged cycle* if $G_1(C)$ does not contain any other k-legged cycle of G. The 3-legged cycle C drawn by thick lines in Fig. 2.4(b) is not minimal, while the 3-legged facial cycle C'' in Fig. 2.4(b) is minimal. We say that cycles C and C' in Γ are *independent* if $\Gamma_1(C)$ and $\Gamma_1(C')$ have no common vertex. A set S of cycles is *independent* if any pair of cycles in S are independent. The pair of leg-vertices of any 2-legged cycle in Γ is a separation pair. A cycle C in Γ is called *regular* if the plane graph $\Gamma - \Gamma_1(C)$ has a cycle. In the plane graph depicted in Fig. 2.4(b), the cycle C drawn by thick solid lines is a regular 3-legged cycle, while the cycle C' indicated by a thin dotted line in Fig. 2.4(b) is not regular. The 2-legged cycle indicated by thin dotted line in Fig. 2.4(a) is not regular. Clearly a 2-legged cycle C in Γ is not regular if and only if $\Gamma - \Gamma_1(C)$ is a chain of G, while a 3-legged cycle C is not regular if and only if $\Gamma - \Gamma_1(C)$ contains exactly one vertex that has degree 3 in G. Let Γ be a plane embedding of a subdivision G of a planar 3-connected cubic graph, then Γ has no regular 2-legged cycle, but may have a regular 3-legged cycle, and Γ has no regular 3-legged cycle if and only if G is cyclically 4-edge connected.

2.2.7 Hand, Hand-Vertex, 2-Handed Cycle, and Regular 2-Handed Cycle

Similarly an edge of Γ which is incident to exactly one vertex of a cycle C in Γ and located inside C is called a *hand* of C. The vertex of C to which a hand is incident is called a *hand-vertex* of C. A cycle C is called a 2-*handed cycle* if C has exactly two hands in Γ and there is no edge which joins two vertices on C

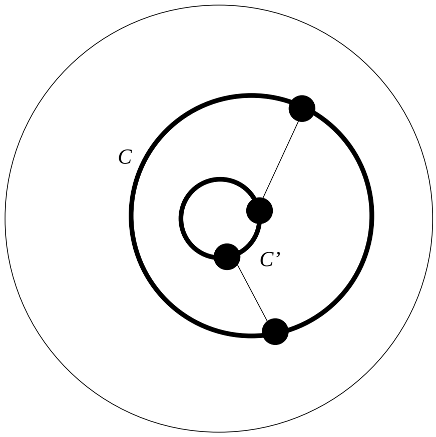


Figure 2.8: Regular 2-handed cycle C and regular 2-legged cycle C'.

and is located inside C. We call a 2-handed cycle C a *regular* 2*-handed cycle* if $\Gamma - \Gamma_O(C)$ contains a cycle. One can observe that any regular 2-handed cycle C corresponds to a regular 2-legged cycle C' of Γ which does not contain any vertex on $F_O(\Gamma)$. In Fig. 2.8 both C and C' are drawn by thick lines.

2.2.8 Some Previous Results on Planar Graph Drawings

Ungar [U53] showed that any plane embedding Γ of a cyclically 4-edge-connected planar cubic graph G has a rectangular drawing if four vertices of degree 2 are inserted on some edges on the outer face $F_O(\Gamma)$. Generalizing the results of Ungar, Thomassen [T84] obtained the following necessary and sufficient conditions stated in Lemma 2.2.1 for a plane graph Γ with $\Delta \leq 3$ to have a rectangular drawing when a quadruplet of vertices of degree 2 on $F_O(\Gamma)$ are designated as corners for a rectangular drawing.

Lemma 2.2.1 *[T84] Let G be a connected plane graph such that all vertices have degree* 3 *except four vertices of degree* 2 *on $C_O(G)$. Then G has a rectangular drawing if and only if G satisfies the following three conditions.*

(r1) G has no 1*-legged cycle.*

(r2) every 2*-legged cycle in G contains at least two vertices of degree* 2

(r3) every 3*-legged cycle in G contains at least one vertex of degree* 2*.*

Generalizing the result of Thomassen, Rahman et al. [RNN02] gave a necessary and sufficient condition for a plane graph Γ with $\Delta \leq 3$ to have a rectangular drawing (for some quadruplet of vertices chosen as corners), and developed a linear-time algorithm to choose a quadruplet and to find a rectangular drawing of Γ for the chosen corners if they exist. A necessary and sufficient condition for a planar graph G with $\Delta \leq 3$ to have a rectangular drawing is given by [RNG04] which runs in linear time. They also developed a linear-time algorithm for finding out the rectangular drawing if drawing exists. On the other hand Rahman at al. [RNN00] gave a necessary and sufficient condition for a plane graph Γ to have a box-rectangular drawing, and developed an algorithm for drawing if it exists, that runs in linear time. Necessary and sufficient conditions for a plane connected graph Γ with $\Delta \leq 3$ to have a box-rectangular drawing are in the following lemma 2.2.2. It is assumed that Γ has a neither a vertex of degree 1 nor a 1-legged cycle; otherwise, Γ has no box-rectangular drawing.

Lemma 2.2.2 *[RNN00] A plane connected graph G with $\Delta \leq 3$ has a box-rectangular drawing if and only if G satisfies the following two conditions:*

(br1) *every 2- or 3- legged cycle in G contains an edge on $C_O(G)$; and*

(br2) $2c_2 + c_3 \leq 4$ *for any independent set ξ of cycles in G, where c_2 and c_3 are the numbers of $2-$ and $3-$ legged cycles in ξ respectively.*

Although the results above for plane embeddings are known, it is difficult to examine whether a planar graph has a box-rectangular drawing or not, since a planar graph may have an exponential number of plane embeddings in general. However, the following fact is known for subdivisions of planar 3-connected cubic graphs.

Fact 2.2.3 *[NC88] Let G be a subdivision of a 3-connected planar graph. Then there is exactly one embedding of G for each face embedded as the outer face. Furthermore, for any two plane embeddings Γ and Γ' of G, any facial cycle in Γ is a facial cycle in Γ'.*

2.3 Drawing Conventions

In this section we introduce some conventional drawing styles, which are found suitable in different application domain. The different drawing styles vary owing to different representations of vertices and edges. Depending on the purpose and objective, the vertices are typically represented with points or boxes and edges are represented with simple Jordan curves [NR04]. A few of the most important drawing styles are introduced below.

2.3.1 Planar Drawing

A drawing Γ of a graph G is planar if no two edges intersect with each other except at their common end-vertices. In Figures 2.9(a) and 2.9(b), we show a planar and a non-planar drawing of the same graph respectively.

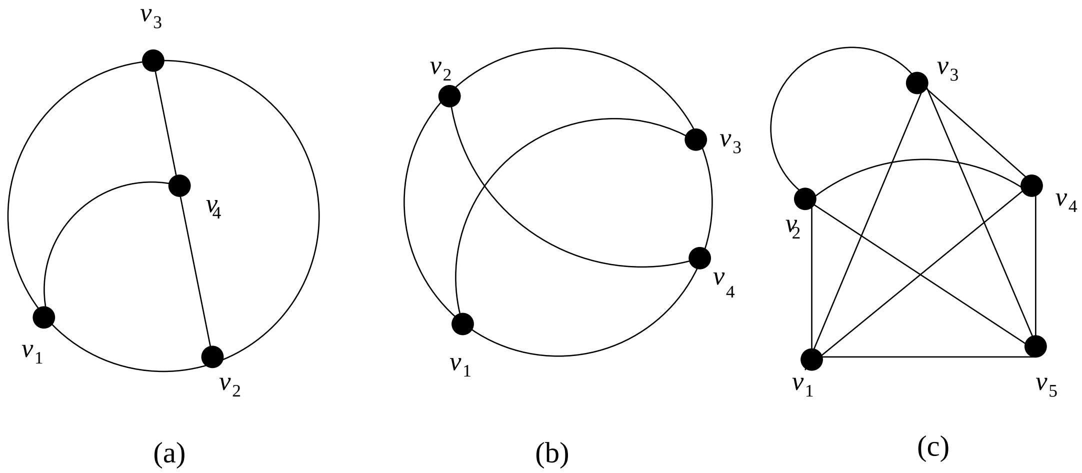


Figure 2.9: (a) A planar drawing, (b) a non-planar drawing of the graph drawn in (a), and (c) a graph which does not have a planar drawing.

Planar drawings of graphs are more convenient than non-planar drawings because, as shown empirically in [Pur97], the presence of edge-crossings in a drawing of a graph makes it more difficult for a person to understand the information being modeled. Unfortunately, not all graphs have a planar drawing. Figure 2.9(c) is an example of one such graph.

2.3.2 Straight Line Drawing

A *straight line drawing* of a plane graph is a drawing in which each edge is drawn as a straight line segment without edge crossings, as illustrated in Fig. 2.10. Wagner [Wag36], Fáry [Far48], and [Ste51] independently proved that every planar graph G has a straight line drawing.

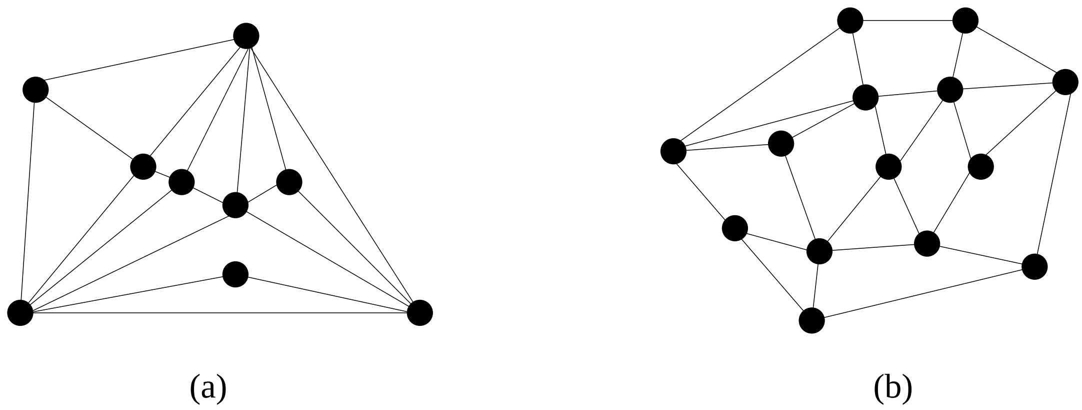


Figure 2.10: (a) A straight line drawing, (b) a convex drawing.

2.3.3 Convex Drawing

Some planar graphs can be drawn in such a way that each edge is drawn as a straight line segment and each face is drawn as a convex polygon, as illustrated in Fig. 2.10(b). Such a drawing is called a *convex drawing*.

2.3.4 Orthogonal Drawing

An *orthogonal drawing* of a planar graph G is a drawing of G, in which each vertex of G is mapped to a point, each edge is drawn as a sequence of alternate horizontal and vertical line segments, and any two edges do not cross except at their common end, as illustrated in Fig. 2.11.

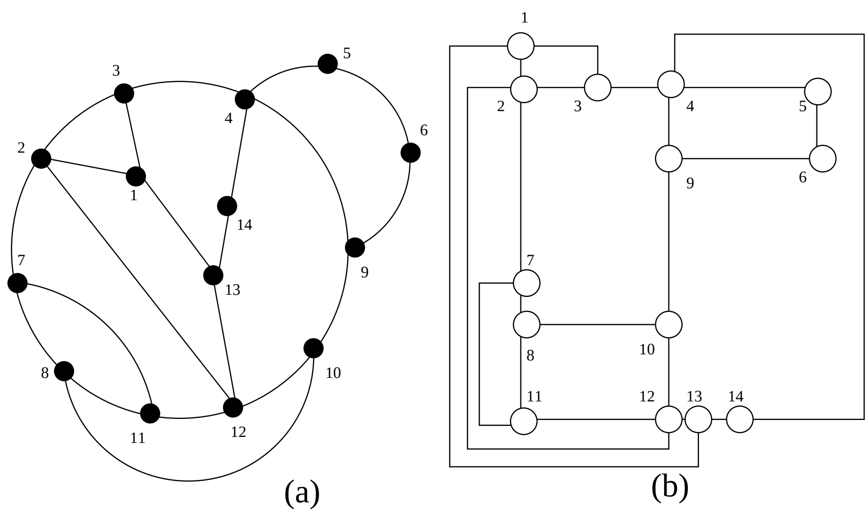


Figure 2.11: (a) A planar graph G, (b) an orthogonal drawing of G.

Clearly the maximum degree Δ of G is at most four if G has an orthogonal drawing. Conversely, every plane graph with $\Delta \leq 4$ has an orthogonal drawing, but may need bends, that is, points where an edge changes its direction in a drawing. However, a plane graph with a vertex of degree 5 or more has no orthogonal drawing.

2.3.5 Box-Orthogonal Drawing

An *box-orthogonal drawing* of a planar graph G is a drawing of G, in which each vertex is drawn as a rectangle, called a box, and each edge is drawn as a sequence of alternate horizontal and vertical line segments along grid lines, as illustrated in Fig. 2.12.

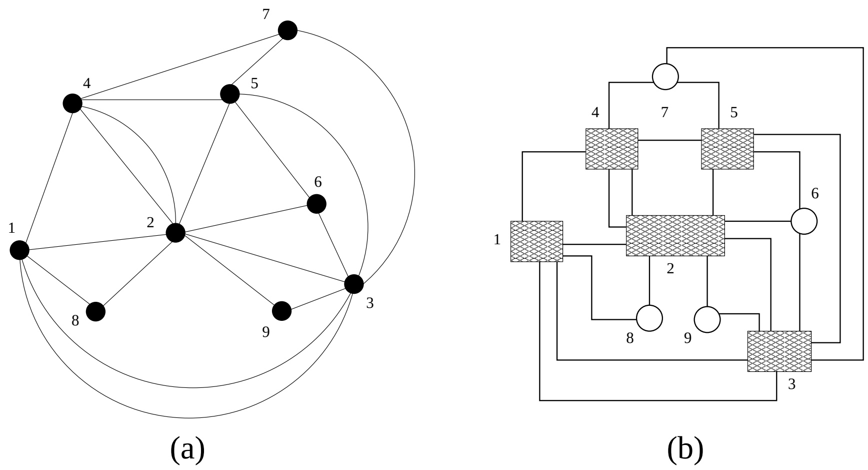


Figure 2.12: (a) A planar graph G, (b) a box-orthogonal drawing of G.

Some of the boxes may be degenerated rectangles i.e., points. A box-orthogonal drawing is a natural generalization of an ordinary orthogonal drawing, and moreover any planar graph has a box-orthogonal drawing even if there is a vertex of degree 5. Several results are known for orthogonal drawings [BK97, FKK96, PT98].

2.3.6 Rectangular Drawing

An orthogonal drawing of a plane graph G is called a *rectangular drawing* of G if each edge of G is drawn as a straight line segment without bends and the contour of each face of G is drawn as a rectangle, as illustrated in Fig. 1.2(b).

Since a rectangular drawing has practical applications in VLSI floorplanning, much attention has been paid to it [KK84, KK88, L90, TTSS91].

Thus a box-orthogonal drawing is a generalization of an orthogonal drawing, while an orthogonal drawing is a generalization of a rectangular drawing. Hence an orthogonal drawing is an intermediate of a box-orthogonal drawing and a rectangular drawing. A box-rectangular drawing is a different style of drawing as intermediate of the two drawing styles.

2.3.7 Box-Rectangular Drawing

A *box-rectangular drawing* of a plane graph G is a drawing of G on an integer grid such that each vertex is drawn as a (possibly degenerated) rectangle, called a *box*, and the contour of each face is drawn as a rectangle, as illustrated in Fig. 1.3(b).

2.4 Box-Rectangular Drawing

We now give some definitions regarding box-rectangular drawings. We say that a vertex of graph G is drawn as a *degenerated box* in a box-rectangular drawing D if the vertex is drawn as a point in D. We often call a degenerated box in D a *point* and call a nondegenerated box a *real box*. We call the rectangle corresponding to $C_O(G)$ the *outer rectangle*, and we call a corner of the outer rectangle simply a *corner*. A box in D containing at least one corner is called a *corner box*. A corner box may be degenerated.

If $n = 1$, that is, G has exactly one vertex, then the box-rectangular drawing is trivial: the drawing is just a degenerated box corresponding to the vertex. Thus in the thesis, we may assume that $n \geq 2$. We now have the following four facts and a lemma.

Fact 2.4.1 *Any box-rectangular drawing has two, three, or four corner boxes.*

Fact 2.4.2 *Any corner box in a box-rectangular drawing contains either one or two corners.*

Fact 2.4.3 *In a box-rectangular drawing D of G, any vertex v of degree* 2 *or* 3 *satisfies* *(i) Vertex v is drawn as a point containing no corner;*

(ii) v is drawn as a corner box containing exactly one corner; and

(iii) v is drawn as a real (corner) box containing exactly two corners.

Fact 2.4.4 *In any box-rectangular drawing D of G, every vertex of degree 5 or more is drawn as a real box.*

In this regard, [RNN00] derived the following lemma which is depicted in figure 1.1(c).

Lemma 2.4.5 *[RNN00] If G has box-rectangular drawing, then G has a box-rectangular drawing in which every vertex of degree 4 or more is drawn as a real box.*

Fact 2.4.6 *In a box-rectangular drawing D of G, any 2-legged cycle of G contains at least two corners, any 3-legged cycle of G contains at least one corner, and any cycle with four or more legs may contain no corner (See Fig. 2.13).*

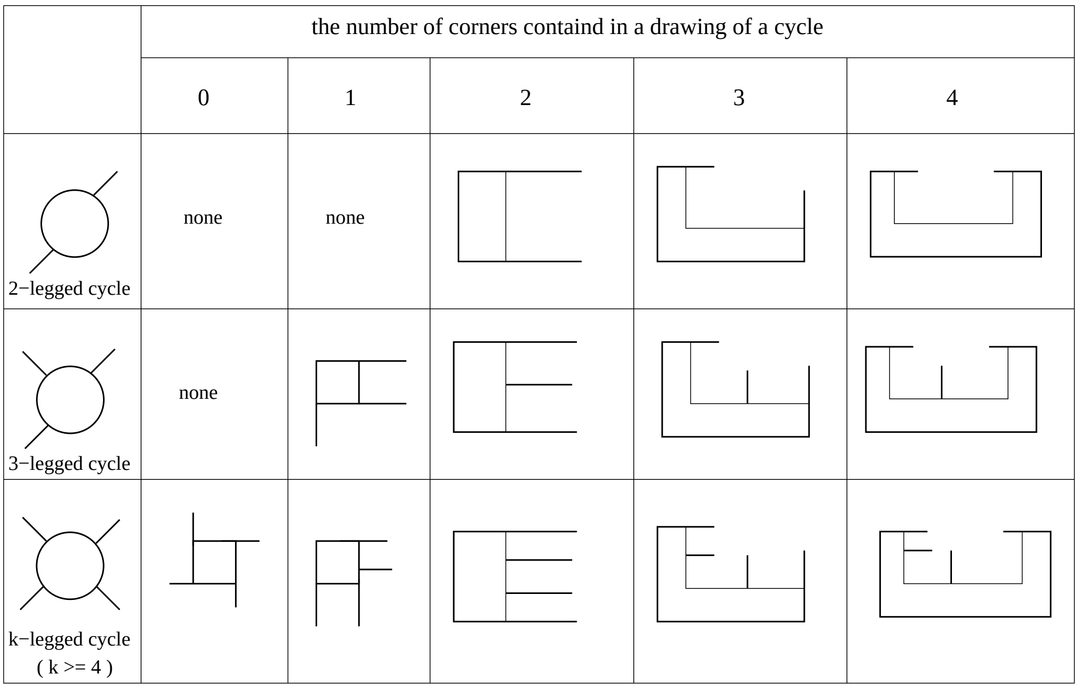
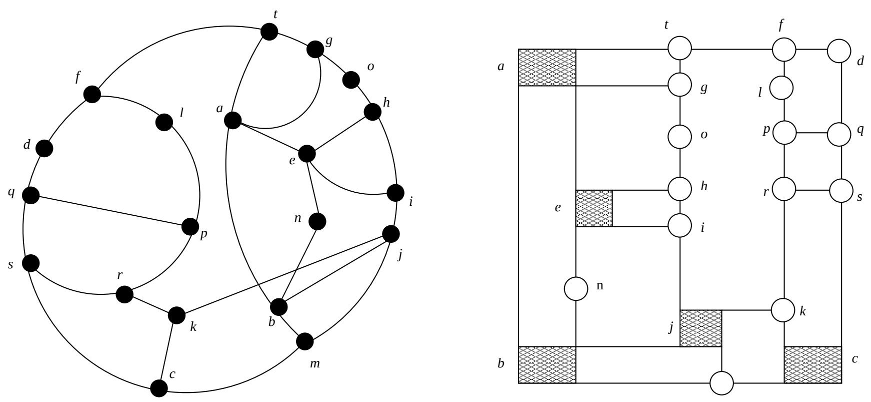


Figure 2.13: Number of corners in drawing of cycles.

The choice of vertices as corner boxes plays an important role in finding a box-rectangular drawing. For example, the graph in Fig. 2.14(a) has a box-



(a) (b)

Figure 2.14: A graph G and its box-rectangular drawing with four corner boxes a, b, c, and d.

rectangular drawing if we choose vertices a, b, c, and d as corner boxes as illustrated in Fig. 2.14(b) however the graph in Fig. 2.14(a) has no box-rectangular drawing if we choose vertices p, q, r, and s as corner boxes. If all vertices corresponding to corner boxes are designated for a drawing, then it is rather easy to determine whether G has a box-rectangular drawing with the designated corner boxes. We deal with this case in Subsection 2.5.1. In Subsections 2.5.2 and 2.5.3 we deal with the general case where no vertex of G is designated corner boxes.

2.5 Box-Rectangular Drawings of Plane Graphs

[RNN00] gave a necessary and sufficient condition for the existence of a box-rectangular drawing of a plane graph G where all vertices of G corresponding to corner boxes are designated and then gave a linear-time algorithm to obtain such a drawing if it exists. Subsection 2.5.1 deals with this case. [RNN00] also gave a necessary and sufficient condition for the existence of a box-rectangular drawing of a plane graph G where no vertex of a plane graph G is designated as a corner box. They first derived a necessary and sufficient condition for a plane graph with maximum degree $\Delta \leq 3$ and gave a linear-time algorithm to obtain such a drawing if it exists. Then, they reduced the box-rectangular problem of a plane graph G with $\Delta \geq 4$ to that of a plane graph Φ with $\Delta \leq 3$. Subsections

2.5.2 and 2.5.3 describes the above cases where no vertex is designated as a corner box, respectively.

2.5.1 Drawing with Exactly Four Designated Corner Boxes

In this subsection we assume that exactly four vertices a, b, c and d in a given plane graph G, on the contour of the outer face are designated as corner boxes. We construct a new graph G'' from G through an intermediate graph G' and reduce the problem of finding a box-rectangular drawing of G with four designated vertices to a problem of finding a rectangular drawing of G''.

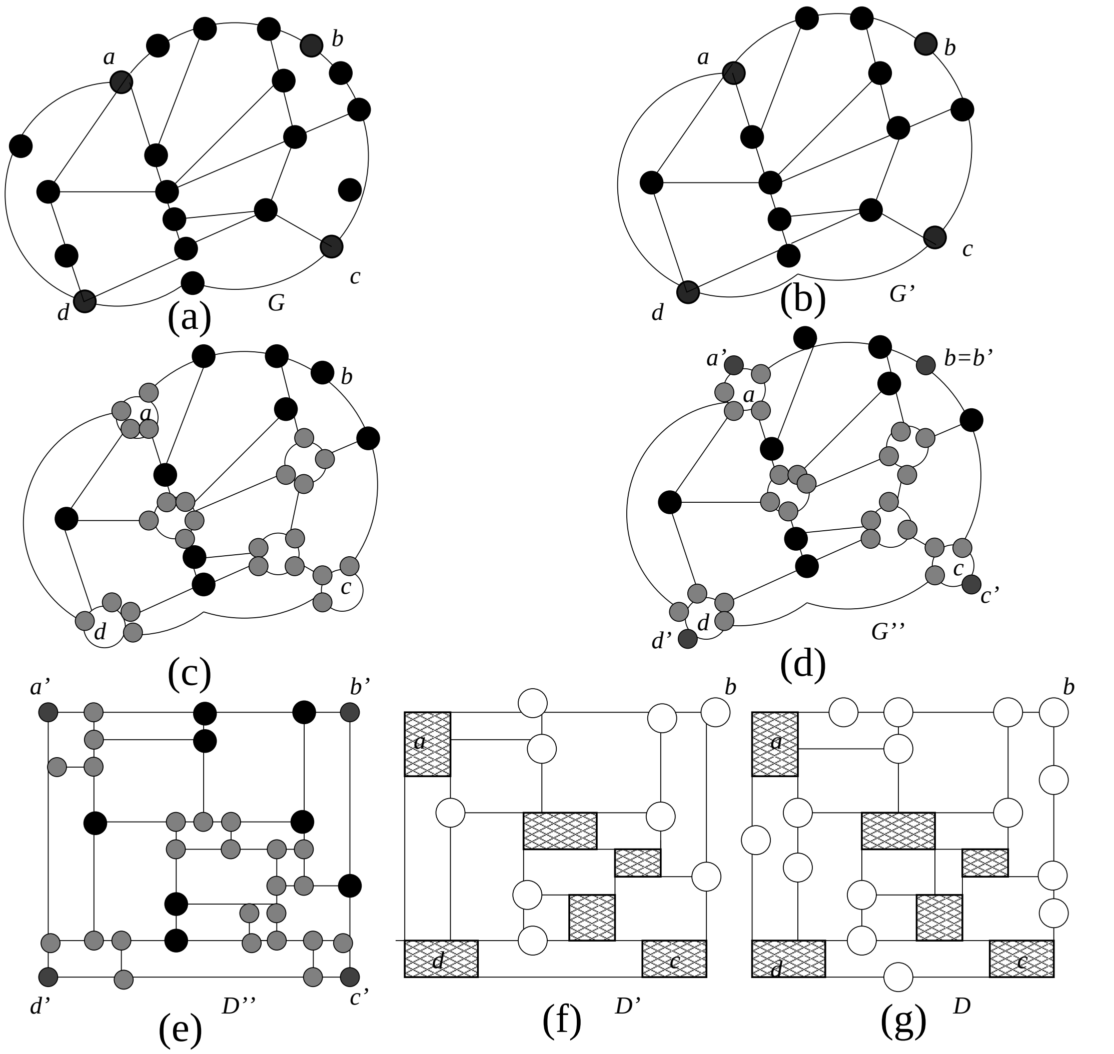


Figure 2.15: Illustration of G, G', G'', D'', D', and D.

We first construct G' from G as follows. If a vertex v of degree 2 in G as vertex b in Fig. 2.15(a) is a designated vertex, then v is drawn as a corner point in a box-rectangular drawing of G. Otherwise, the two edges incident to v must be drawn on a straight line segment. We thus remove all nondesignated vertices of degree 2 one by one from G. The resulting graph is G'. Thus all vertices of degree 2 in G' are designated vertices. Clearly, G has a box-rectangular drawing with the four designated corner boxes if and only if G' has a box-rectangular drawing with the four designated corner boxes. Fig. 2.15(a) illustrates a plane graph G with four designated vertices a, b, c, and d, and Fig. 2.15(b) illustrates G'. Fig. 2.15(e) is a rectangular drawing as well as a box-rectangular drawing D'' of G'' in Fig. 2.15(d), and Fig. 2.15(f) illustrates a box-rectangular drawing D' of G'. Fig. 2.15(g) illustrates a box-rectangular drawing D of G.

Since every vertex of degree 2 in G' is a designated vertex, it is drawn as a (corner) point in any box-rectangular drawing of G'. Every designated vertex of degree 3 in G', as vertex c in Fig. 2.15 (b) is drawn as a real box since it is a corner. On the other hand, every nondesignated vertex of degree 3 in G' is drawn as point. These facts together with Lemma 2.4.5 imply that if G' has a box-rectangular drawing then G' has a box-rectangular drawing D' in which all designated vertices of degree 3 and all vertices of degree 4 or more are drawn as real boxes.

We now construct G'' from G'. Replace by a cycle each of the designated vertices of degree 3 and the vertices of degree 4 or more as illustrated in Fig. 2.15(c). The replaced cycle corresponding to a designated vertex x of degree 3 or more contains exactly one edge, say e_x, on the contour of the outer face, where $x = a, b, c$, or d. Put a dummy vertex x' of degree 2 on e_x. The resulting graph is G''. We let $x' = x$ if a designated vertex x has degree 2. (See Fig. 2.15(d)). Now G'' has exactly four vertices a', b', c', and d' of degree 2 on $C_0(G'')$, and all other vertices have degree 3.

[RNN00] states the theorem.

Theorem 2.5.1 *[RNN00] Let G be a connected plane graph with four designated vertices a, b, c, and d on $C_O(G)$, and let G'' be the graph transformed from G' as mentioned above. Then G has a box-rectangular drawing with four corner boxes corresponding to a, b, c, and d if and only if G'' has a rectangular drawing.*

2.5.2 Box-Rectangular Drawings of Plane Graphs with No Designated Corner Boxes for $\Delta \leq 3$

In this subsection we consider the general case where no vertex of a plane graph G with $\Delta \leq 3$ is designated as a corner box. By Fact 2.4.1 there are two, three, or four corner boxes in any box-rectangular drawing of G. Therefore, considering all combinations of two, three, and four vertices on $C_O(G)$ as corner boxes and applying the algorithm in the previous subsection for each of the combinations, one can determine whether G has a box-rectangular drawing. Such a straightforward method requires time $O(n^5)$ since there are $O(n^4)$ combinations and the algorithm in Subsection 2.5.1 can determine in linear-time whether G has a box-rectangular drawing for each of them.

[RNN00] states the following Lemmas.

Lemma 2.5.2 *[RNN00] Let G be a plane cubic connected graph. Assume that G satisfies ($br1$) and ($br2$) in Lemma 2.2.2, that is, G has four or more vertices on $C_O(G)$, and that there is exactly one $C_O(G)$ component. Then*

(a) G has a 3-legged cycle; and

(b) if G has two or more independent 3-legged cycles, then the set of a minimal 3-legged cycles in G is independent.

Lemma 2.5.3 *[RNN00] Let G be a plane cubic graph. Assume that G satisfies Conditions ($br1$) and ($br2$) in Lemma 2.2.2 and that G has four or more vertices of degree 3 on $C_O(G)$. Then G has a box-rectangular drawing.*

If G has a 2-legged cycle C then G has a pair of independent 2-legged cycles. Let v be the end of any leg of C that is not on C. Since G is cubic, v has degree 3. Then G has a 2-legged cycle C' which has v as a leg-vertex. Clearly C and C' are independent.

We can consider the following two cases.

Case 1: *G has no 2-legged cycle.* In this case G has exactly one $C_O(G)$ component; otherwise, G would have a 2- legged cycle. Then by Lemma 2.5.2(a) G has a 3-legged cycle. We choose four vertices on $C_O(G)$ as the four corner boxes for a box-rectangular drawing of G, as follows.

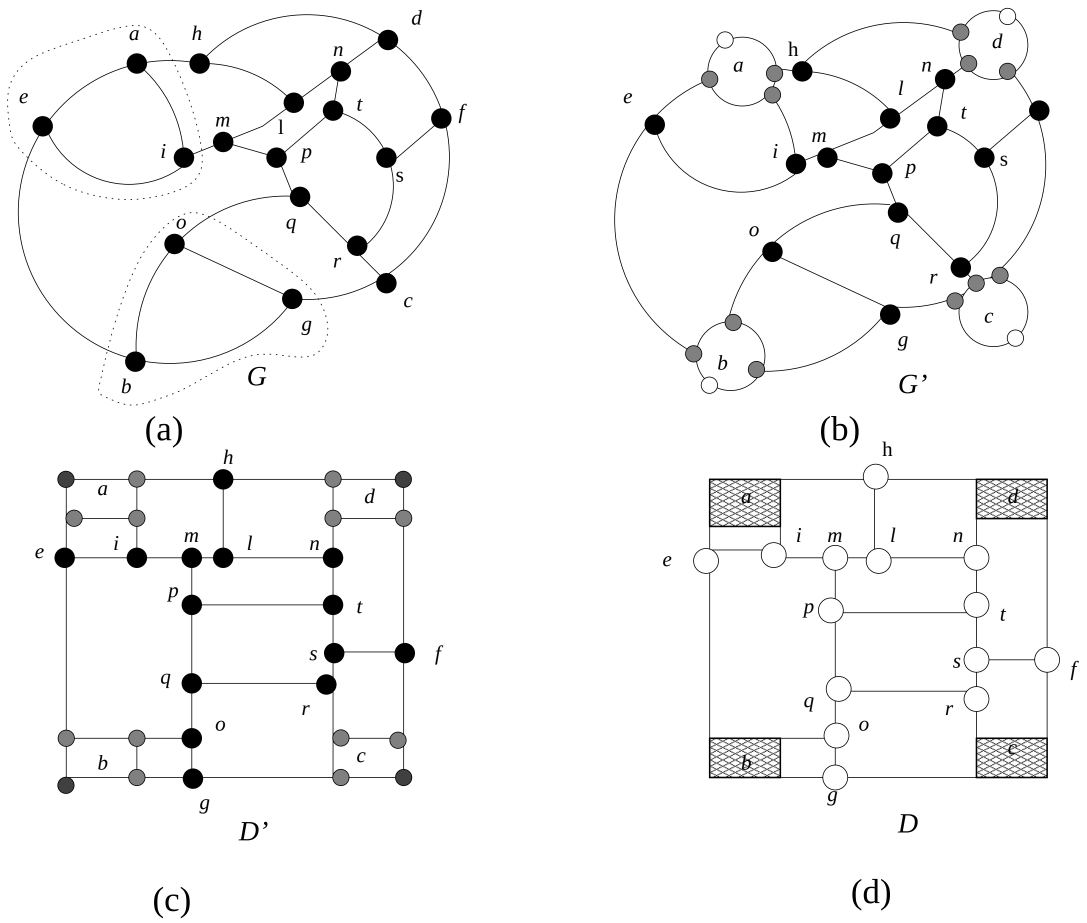


Figure 2.16: Illustration of Case 1.

We first consider the case where G has no pair of independent 3-legged cycles. We arbitrarily choose four vertices on $C_O(G)$ as the four corner boxes for a box-rectangular drawing of G. We now claim that every 3-legged cycle C in G has at least one designated vertex. Since C has an edge on $C_O(G)$, exactly two of the three legs of C lie on $C_O(G)$. Let x and y be the two leg-vertices of the two legs. Let P be the path on C starting at x and ending at y without passing through any edge on C. Then P has exactly one intermediate vertex, say z; otherwise, either G would have more than one $C_O(G)$ component or G would have a pair of of independent 3-legged cycles, a contradiction. Thus one can easily know that all three legs of C are incident to z. Therefore, all the vertices on $C_O(G)$ except z lie on C. Hence regardless of whether z is one of the four designated vertices or not. C contains at least one of the four designated vertices.

We then consider the case where G has a pair of independent 3-legged cycles. Let M be the set of all minimal 3-legged cycles in G. By Lemma 2.5.2(b) M is independent. Let $k = |M|$, then $k \leq 4$ by condition (br2). For each 3-legged cycle C_m in M, we arbitrarily choose a vertex on C_m which is also on $C_O(G)$. If $k < 4$, we arbitrarily choose $4-k$ vertices on $C_O(G)$ which are not chosen so far. Thus we have chosen exactly four vertices on $C_o(G)$, and we regard them as the four designated vertices for a box-rectangular drawing of G. In Fig. 2.16(a) four vertices a, b, c and d are on $C_O(G)$ are chosen as designated vertices. Vertices a and b are chosen on two independent minimal 3-legged cycles indicated by dotted lines, whereas vertices c and d are chosen arbitrarily on $C_O(G)$. We now claim that every 3-legged cycle C of G has at least one designated vertex. Clearly C contains a designated vertex if C is minimal, that is, $C \in M$. Thus one may assume C is not minimal. Then $G(C)$ contains a minimal 3-legged cycle $C_m \in M$, and every vertex of C_m on $C_O(G)$ is also on C. Since C_m has a designated vertex on $C_O(G)$, the vertex is also on C.

Thus we have chosen four designated vertics on $C_O(G)$. We now give a method to find a box-rectangular drawing of G with the four designated vertices. We replace each of the four vertices by a cycle and put a dummy vertex of degree 2 on the edge of the cycle on the contour of the outer face. Let G' be the resulting graph (See Fig. 2.16(b) where the vertices of degree 2 are drawn by white circles). G' has exactly four vertices of degree 2 on $C_O(G')$, and all other vertices of G' have degree 3. Since G has no 1-legged cycle , G' has no 1-legged cycle and hence G' satisfies Condition (r1) in Lemma 2.2.1. Since G has no 2-legged cycle, any 2-legged cycle C in G' contains all vertices on $C_O(G')$ except a vertex of degree 2. Therefore, C contains three vertices of degree 2, and hence G' satisfies Condition (r2) in Lemma 2.2.1. Moreover since any 3-legged cycle in G' contains at least one designated vertex, any 3-legged cycle in G' contains at least one vertex of degree 2 and hence G' satisfies condition (r3). Thus by Lemma 2.2.1 G' has a rectangular drawing D' as illustrated in Fig. 2.16(c). Regarding each face in D' corresponding to a replaced cycle as a box, we immediately obtain a box-rectangular drawing D of G from D' as illustrated in Fig. 2.16(d).

Case 2: *G has a pair of independent 2-legged cycles.* Let C_1 and C_2 be independent 2-legged cycles in G. One may assume that both C_1 and C_2 are minimal 2-legged cycles. By Condition (br2) at most two 2-legged cycles of G are independent. Therefore, for any other 2-legged cycles $C'(\neq C_1, C_2)$, $G(C')$ contains either C_1 or C_2.

Let k_i, $i = 1$ or 2, be the number of all minimal (not always independent) 3-legged cycles in $G'(C_i)$. Then we claim that $k_i \leq 2$. First consider the case where C_i has exactly three vertices on $C_O(G)$. Then $G(C_i)$ has exactly two inner faces; otherwise, $G(C_i)$ would have a cycle which has two or three legs and has no edge on $C_O(G)$, contrary to condition (br1). The contour of the two faces are minimal 3-legged cycles, and there is no other minimal 3-legged cycle in $G(C_i)$. Thus $k_i = 2$. We next consider the case where C_i has four or more vertices on $C_O(G)$. That is, the set of all minimal 3-legged cycles of G in $G(C_i)$ is independent. Furthermore, $k_i \leq 2$; otherwise, Condition (br2) would not hold for the independent set ξ of $k_i + 1$ cycles: the k_i (≥ 3) 3-legged cycles in $G(C_i)$ and the 2-legged cycle C_j, $j = 1$ or 2 and $j \neq i$.

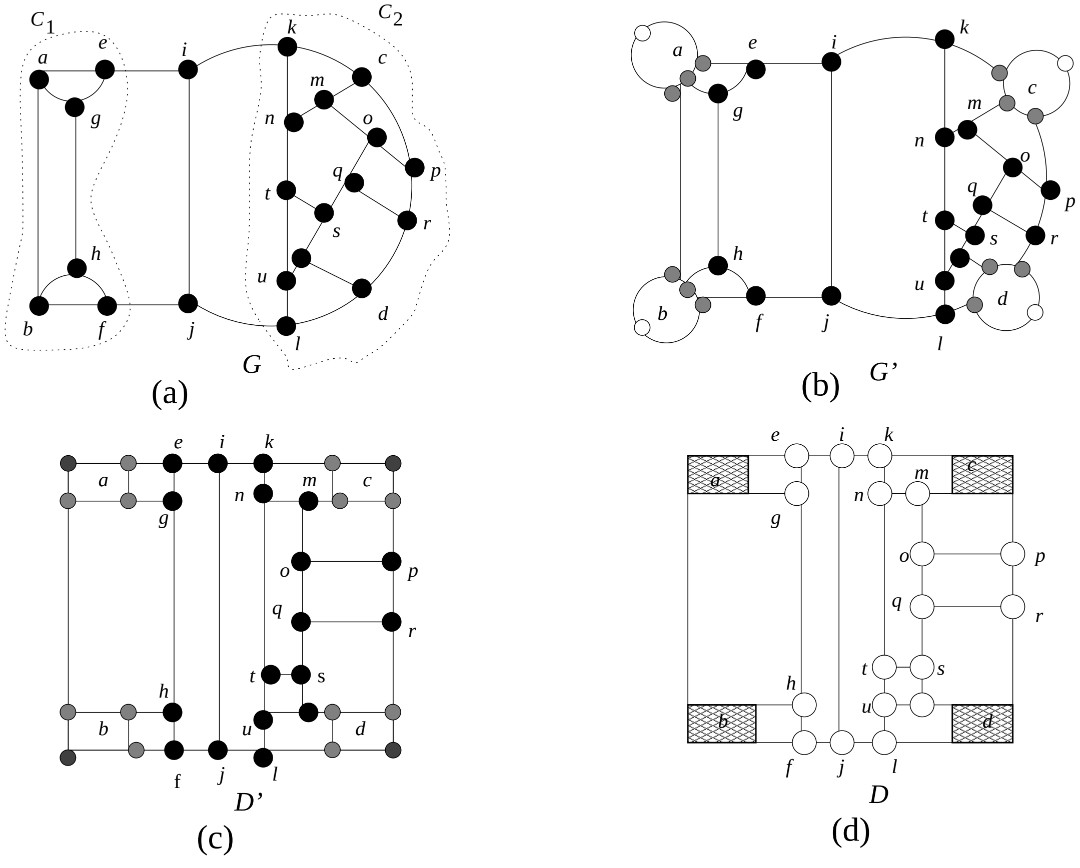


Figure 2.17: Illustration of Case 2.

We choose two vertices on each $C_i, 1 \leq i \leq 2$, as follows. For each of the k_i, minimal 3-legged cycles in $G(C_i)$ we arbitrarily choose exactly one vertex on $C_O(G)$. If $k < 2$, then we arbitrarily choose $2 - k_i$ vertices in $V(C_i) \bigcap V(C_O(G))$ which have not chosen so far. This can be done because C_i has at least two vertices on $C_O(G)$. Thus we have chosen four vertices on $C_O(G)$, and we ragard them as designated vertices for a box-rectangular drawing of G. In Fig. 2.17(a) G has a pair of independent 2-legged cycles C_1 and C_2, and four vertices a, b, c, and d on $C_O(G)$ are chosen as the designated vertices. Vertices a nad b are chosen from the vertices on C_1; each on a minimal 3-legged cycle in $G(C_1)$. Vertices c and d are chosen from the vertices on C_2; d is on a minimal 3-legged cycle in $G(C_2)$ and c is an arbitrarily vertex in $V(C_2) \bigcap V(C_O(G))$ other than d.

We now claim that any 2-legged cycle C in G has two designated vertices. If C is C_1 or C_2 then clearly C has exactly two designated vertices. Otherwise, $G(C)$ has either cycle C_1 or C_2, and hence C has exactly two designated vertices. We then claim that any 3-legged cycle C_3 in G has a designated vertex. By Condition (br2) $\{C_1, C_2$, and $C_3\}$ is not independent, and hence either $G(C_3)$ contains C_1 or C_2, or C_3 is contained in $G(C_1)$ or $G(C_2)$. If $G(C_3)$ contains C_1 or C_2, then C_3 contains a designated vertex. Otherwise, C_3 is contained in either $G(C_1)$ or $G(C_2)$. In this case C_3 contains a designated vertex, since we have chosen a designated vertex on each minimal 3-legged cycle inside $G(C_1)$ and $G(C_2)$.

We can find a box-rectangular drawing as follows. We replace each of the four designated vertices by a cycle and put a dummy white vertex of degree 2 on $C_o(G')$, and all other vertices of G' have degree 3. Since G has no 1-legged cycle, G' has no 1-legged cycle. Since any 2-legged cycle in G contains two designated vertices, G' satisfies Condition (r2) in Lemma 2.2.1. Since any 3-legged cycle in G contains a designated vertex, G' satisfies Condition (r3) in Lemma 2.2.1. Thus G' has a rectangular drawing D' by Lemma 2.2.1 as illustrated in Fig. 2.17(c). Regarding each face in D' corresponding to a replaced cycle as a box, we immediately obtain a box-rectangular drawing D of G from D' as illustrated in Fig. 2.17 (d).

Using the Lemma 2.5.3 one can find a box-rectangular drawing of G if G satisfies the Conditions in Lemma 2.2.2.

Lemma 2.5.3 implies the following corollary.

Corollary 2.5.4 *[RNN00] A plane connected graph G with $\Delta \leq 3$ has a box-rectangular drawing if and only if G satisfies the following four conditions.*

(c1) every 2- or 3- legged cycle in G has an edge on $C_O(G)$;

(c2) at most two 2-legged cycles of G are independent of each other.

(c3) at most four 3-legged cycles of G are independent of each other;

and

(c4) if G has a pair of independent 2-legged cycles C_1 and C_2, then $\{C_1, C_2, C_3\}$ is not independent for any 3-legged cycle C_3 in G, and neither $G(C_1)$ nor $G(C_2)$ has more than two independent 3-legged cycles of G.

2.5.3 Box-Rectangular Drawings of Plane Graphs with No Designated Corner Boxes for $\Delta \geq 4$

In this subsection we give a necessary and sufficient condition for a plane connected graph G with $\Delta \geq 4$ to have a box-rectangular drawing where no vertex is designated as a corner box. We also give a linear-time algorithm to find the drawing if it exists.

Let G be a plane graph with $\Delta \geq 4$. We construct a new plane graph Φ from G by replacing each vertex v of degree four or more in G by a cycle. Figures 2.18(a) and 2.18(b) illustrate G and Φ respectively. A replaced cycle corresponds to a real box in a box-rectangular drawing of G. We do not replace a vertex of degree 2 or 3 by a cycle since such a vertex may be drawn as a point. Thus $\Delta(\Phi) \leq 3$. The following lemma is the main result of this subsection.

Lemma 2.5.5 *[RNN00] Let G be a plane connected graph with $\Delta \geq 4$, and let Φ be the graph transformed from G as above. Then G has a box-rectangular drawing if and only if Φ has a box-rectangular drawing.*

If Φ has a box-rectangular drawing, then there exists a box-rectangular drawing for G. But there is no easy method which directly transforms a box-rectangular drawing D_Φ of the plane graph Φ to a box-rectangular drawing D_G of the plane graph G. We give some definitions before giving the approach to

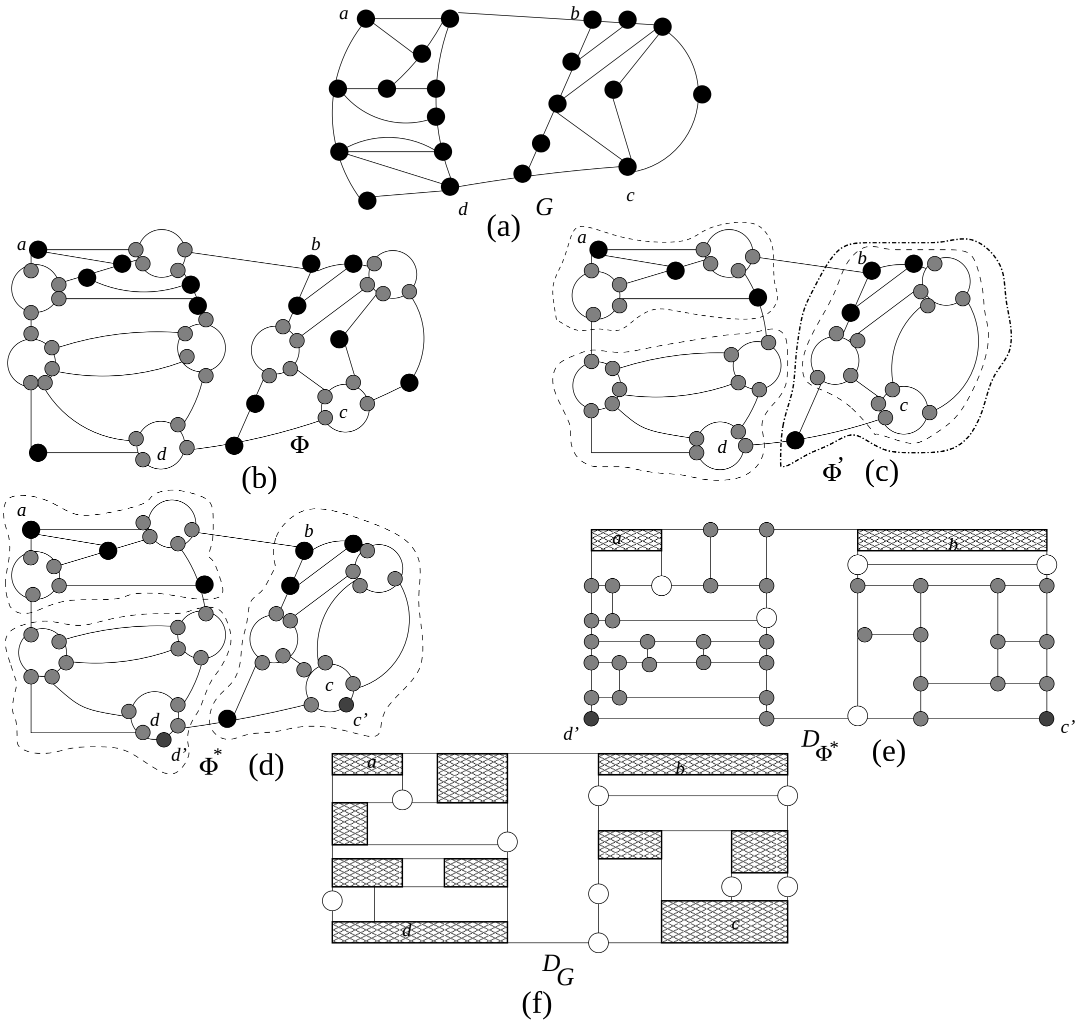


Figure 2.18: Box-rectangular drawings of plane graphs with no designated corner boxes for $\Delta \geq 4$

find D_G from D_Φ. We replace the vertices of degree 4 or more in G by cycles like Figure 2.5. We call a vertex of degree 3 on a replaced cycle a *replaced vertex.* The replaced cycle on $C_O(\Phi)$ corresponding to a vertex of degree 4 or more in G contains exactly one edge on $C_O(\Phi)$. We call such an edge in Φ a *green edge.* Each vertex of degree 2 or 3 in G has a corresponding vertex of the same degree in Φ, and we call such a vertex in Φ an *original vertex.* Now each vertex in Φ is either a replaced vertex or an original vertex. In the plane embedding Φ, for a green edge e and a cycle C in Φ, we call e a *green edge for* C if both ends of e are on C. Assume the plane graph Φ has a box-rectangular drawing D_Φ, then Φ satisfies (br1) and (br2) of Lemma 2.2.2. We can easily transform D_Φ to a box-rectangular drawing D_G of the plane graph G if only original vertices are drawn as corner boxes in D_Φ, because then each replaced vertex is a point in D_Φ, and each replaced cycle in Φ is a rectangular face in D_Φ, and hence D_Φ can be transformed to D_G by regarding each replaced cycle as a box. The problem is the case where a replaced vertex is drawn as a corner box in D_Φ. Because such a drawing D_Φ cannot always be transformed to a box-rectangular drawing D_G of G. However we show that a plane graph Φ^* obtained from Φ with slight modification has a particular box-rectangular drawing D_Φ^* which can be easily transformed to a box-rectangular drawing of G. We need the support of the following Lemma 2.5.6 given by [RNN00].

Lemma 2.5.6 *[RNN00] Assume that the plane embedding Φ has a box-rectangular drawing. Then the following (a)-(c) hold:*

(a) Φ has four or more vertices of degree 3 on $C_O(\Phi)$.

(b) If a 3-legged cycle C in Φ contains at least two replaced vertices on $C_O(\Phi)$, then there exists a green edge for C.

(c) If a 2-legged cycle C in Φ contains at least one replaced vertex on $C_O(\Phi)$, then there exists a green edge for C.

We are now ready to give the approach to find D_G from D_Φ

Assume that Φ has a box-rectangular drawing. Then Φ satisfies Conditions (br1) and (br2) in Lemma 2.2.2. By Lemma 2.5.6(a) Φ has four or more vertices of degree 3 on $C_O(\Phi)$.

Let Φ' be the minimal graph homeomorphic to Φ as illustrated in Fig. 2.18(c); then Φ' is a cubic graph and satisfies Conditions (br1) and (br2). Using the algorithm mentioned in the description of Lemma 2.5.3, we choose four designated vertices for a box-rectangular drawing of Φ'; as illustrated in Fig. 2.18(c). Then each 2-legged cycle in Φ' contains exactly two designated vertices and each 3-legged cycle in Φ' contains at least one designated vertex.

If a designated vertex x in Φ', such as vertex a or b in Fig. 2.18(c), is an original vertex, then there is a vertex in the plane embedding G corresponding to x, and hence we consider the vertex as a corner box in a box-rectangular drawing D_G of G.

On the other hand, if a designated vertex x is a replaced vertex such as vertex c or d in Fig. 2.18(c), then instead of x we choose another vertex x' (probably a dummy vertex on a green edge) as a corner box of D_G. This can be done as in the following three cases, depending on how x was chosen. Note that the algorithm in the description of the Lemma 2.5.3 chooses a designated vertex x either on a minimal 3-legged cycle, or on a minimal 2-legged cycle, or arbitrarily.

Case 1 The replaced vertex x was chosen on a minimal 3-legged cycle C.

If C has a nondesignated original vertex on $C_O(\Phi')$, then we choose it as a designated vertex x' instead of x. Otherwise, C has either at least two replaced vertices on $C_O(\Phi')$ or a designated original vertex.

In the former case where C has at least two replaced vertices on $C_O(\Phi')$, by Lemma 2.5.6(b) there exists a green edge on $C_O(\Phi')$ for C and choose x' as designated vertex instead of x. (In Fig. 2.18(c) a designated vertex d is a replaced vertex on a minimal 3-legged cycle indicated by a dotted line, and we choose the dummy vertex d' on a green edge as a designated vertex instead of d, as illustrated in Fig. 2.18(d).)

In the latter case where C contains a designated original vertex, it is not necessary to choose a vertex on C as a corner of D_G. In this case we need to consider the following two subcases. We first consider the subcase where x is on a minimal 2-legged cycle C'. By Lemma 2.5.6(c) there is a green edge for C'. We put a dummy vertex x' on a green edge for C' in such a way that any green edge of Φ' contains at most two dummy vertices. We choose x' as a designated vertex instead of x. (In Fig. 2.18(c) a designated vertex c is a replaced vertex on a minimal 2-legged cycle indicated by a dotted line, and we choose the dummy vertex c' as a designated vertex instead of c as illustrated in Fig.2.18(d)). We now consider the other subcase where x is not on a minimal 2-legged cycle C'. Since x is replaced vertex on $C_O(\Phi')$, $C_O(\Phi')$ has a green edge. If $C_O(\Phi')$ has nondesignated original vertex, then we choose it as a designated vertex x' instead of x. Otherwise, we put a dummy vertex x' of degree 2 on a green edge in a way that any green edge contains at most two dummy vertices. This can be done since $C_O(\Phi')$ contains exactly four designated vertices. We choose x' as a designated vertex instead of x.

Case 2 x was chosen on a minimal 2-legged cycle C but not on a minimal

3-legged cycle.

In this case, the replaced vertex x is on C and on $C_O(\Phi)$. Then by Lemma 2.5.6 there is a green edge for C. We put a dummy vertex x' of degree 2 on the green edge in such a way that any green edge of Φ' contains at most two dummy vertices. This can be done since C contains exactly two designated vertices. We now choose x' as designated vertex instead of x.

Case 3 x was chosen arbitrarily on $C_O(\Phi')$ but not particularly chosen on a minimal 2- or 3-legged cycle.

Since x is replaced vertex on $C_O(\Phi')$, $C_O(\Phi')$ has a green edge. If $C_O(\Phi')$ has a nondesignated original vertex, then we choose it as a designated vertex x' instead of x. Otherwise, we put a dummy vertex x' of degree 2 on a green edge in such a way that any green edge contains at most two dummy vertices and choose x' as a designated vertex instead of x.

Thus, instead of each designated replaced vertex x, we have chosen another vertex x'. Let Φ^* be the resulting graph as illustrated in Fig. 2.18(d). Note that each of the four designated vertices in Φ^* is either an original vertex or a dummy vertex of degree 2 on a green edge of Φ'. Clearly, every 2-legged cycle in Φ^* contains at least two designated vertices and every 3-legged cycle in Φ^* contains at least one designated vertex. Hence, Φ^* has a box-rectangular drawing $D_{\Phi^*}$with the four designated vertices as corner boxes, as illustrated in Fig. 2.18(e). Inserting the removed vertices of degree 2 on some vertical and horizontal line segments in D_{Φ^*} and regarding the drawing of each replaced cycle as a box, we immediately obtain a box-rectangular drawing D_G of the plane embedding G from D_{Φ^*} as illustrated in Fig. 2.18(f).

2.6 Complexity of Algorithms

In this section we briefly introduce some terminologies related to *complexity* of algorithms. For interested readers, we refer the book of Garey and Johnson [GJ79].

The most widely accepted complexity measure for an algorithm is the *running time*, which is expressed by the number of operations it performs before producing the final answer. The number of operations required by an algorithm is not the same for all problem instances. Thus, we consider all inputs of a given size together, and we define the *complexity of the algorithm for that input size* to be the worst case behavior of the algorithm on any of these inputs. Then the running time is a function of size n of the input.

2.6.1 The Notation $O(n)$

In analyzing the complexity of an algorithm, we are often interested only in the "asymptotic behavior", that is, the behavior of the algorithm when applied to very large inputs. To deal with such a property of functions we shall use the following notations for asymptotic running time. Let $f(n)$ and $g(n)$ are the functions from the positive integers to the positive reals, then we write $f(n) = O(g(n))$ if there exists positive constants c_1 and c_2 such that $f(n) \leq c_1 g(n) + c_2$ for all n. Thus the running time of an algorithm may be bounded from above by phrasing like "takes time $O(n^2)$".

2.6.2 Polynomial Algorithms

An algorithm is said to be *polynomially bounded* (or simply *polynomial*) if its complexity is bounded by a polynomial of the size of a problem instance. Examples of such complexities are $O(n)$, $O(nlogn)$, $O(n^{100})$, etc. The remaining algorithms are usually referred as *exponential* or *non-polynomial*. Examples of such complexity are $O(2^n)$, $O(n!)$, etc. When the running time of an algorithm is bounded by $O(n)$, we call it a *linear-time* algorithm or simply a *linear* algorithm.

2.6.3 NP-complete Problems

There are a number of interesting computational problems for which it has not been proved whether there is a polynomial time algorithm or not. Most of them are "NP-complete", which we will briefly explain in this section.

The state of algorithms consists of the current values of all the variables and the location of the current instruction to be executed. A *deterministic algorithm* is one for which each state, upon execution of the instruction, uniquely determines at most one of the following state (next state). All computers, which exist now, run deterministically. A problem Q is in the *class P* if there exists a deterministic polynomial-time algorithm which solves Q. In contrast, a *non-deterministic algorithm* is one for which a state may determine many next states simultaneously. We may regard a non-deterministic algorithm as having the capability of branching off into many copies of itself, one for the each next state. Thus, while a deterministic algorithm must explore a set of alternatives one at a time, a non-deterministic algorithm examines all alternatives at the same time. A problem Q is in the *class NP* if there exists a non-deterministic polynomial-time algorithm which solves Q. Clearly, $P \subseteq NP$.

Among the problems in *NP* are those that are hardest in the sense that if one can be solved in polynomial-time then so can every problem in *NP*. These are called *NP-complete* problems. The class of *NP*-complete problems has the following interesting properties.

(a) No *NP*-complete problem can be solved by any known polynomial algorithm.

(b) If there is a polynomial algorithm for any *NP*-complete problem, then there are polynomial algorithms for all *NP*-complete problems.

Sometimes we may be able to show that, if problem Q is solvable in polynomial time, all problems in *NP* are so, but we are unable to argue that $Q \in NP$. So Q does not qualify to be called *NP*-complete. Yet, undoubtedly Q is as hard as any problem in *NP*. Such a problem Q is called *NP-hard*.

Chapter 3

Box-Rectangular Drawings of Planar Graphs with $\Delta \leq 3$

In this chapter we give a necessary and sufficient condition for a planar graph G with $\Delta \leq 3$ to have a box-rectangular drawing. We also give a linear-time algorithm to find the drawing if it exists. Section 3.1 describes the necessary and sufficient condition and Section 3.2 illustrates the algorithm.

3.1 Necessary and Sufficient Condition

In Subsection 3.1.1 we consider the case where G is a subdivision of a planar 3-connected cubic graph, and in Subsection 3.1.2 we consider the other case.

3.1.1 Case for a Subdivision of a Planar 3-Connected Cubic Graph

Let G be a subdivision of a planar 3-connected cubic graph. Then by Fact 2.2.3 G has an $O(n)$ number of embeddings, one for each chosen as outer face. Examining by the linear algorithm in Lemma 2.2.2 whether the two conditions (br1) and (br2) hold for each of the $O(n)$ embeddings, one can examine in time $O(n^2)$ whether the planar graph G has a box-rectangular drawing. However, we obtain the following necessary and sufficient condition for G to have a box-rectangular drawing, which leads to a linear-time algorithm.

Theorem 3.1.1 *Let G be a subdivision of a planar 3-connected cubic graph, and let Γ be an arbitrary plane embedding of G.*

(a) Suppose first that G is cyclically 4-edge-connected, that is, Γ has no regular 3-legged cycle. Then the planar graph G has a box-rectangular drawing.

(b) Suppose next that G is not cyclically 4-edge-connected, that is, Γ has a regular 3-legged cycle C. Let F_1, F_2, and F_3 be the three peripheral faces for C, and let Γ_1, Γ_2, and Γ_3 be the plane embeddings of G taking F_1, F_2, and F_3 respectively as the outer face. Then the planar graph G has a box-rectangular drawing if and only if at least one of the three embeddings Γ_1, Γ_2, and Γ_3 has a box-rectangular drawing.

Proof of Theorem 3.1.1(a). Before giving a proof of Theorem 3.1.1(a), we observe the following lemmas on subdivisions of planar 3-connected cubic graphs, which are cyclically 4-edge-connected.

Lemma 3.1.2 *Let G be a subdivision of planar 3-connected cubic graph, and Γ be an arbitrary plane embedding of G. If G is cyclically 4-edge-connected, then Γ does not have a set of independent 2- and 3-legged cycles. That is, $2c_2+c_3 \leq 2$ for any independent set ξ of cycles in Γ, where c_2 and c_3 are the numbers of 2- and 3-legged cycles in ξ, respectively.*

Proof. Let G be a subdivision of planar 3-connected cubic graph, and Γ be an arbitrary plane embedding of G. Let G be cyclically 4-edge-connected. Assume Γ has two independent 2-legged cycles, C_1 and C_2. Removal of the two legs of either C_1 or C_2 leaves a graph with two connected components, each of which has a cycle, contrary to the definition of cyclically 4-edge-connected graph. That is, Γ can not have two independent 2-legged cycles. Similarly we can prove that Γ can not have two independent 3-legged cycles. We can also prove that Γ can not have two cycles, one is 2-legged, and another is 3-legged, which are independent. If Γ has so, then removal of the legs either of the 2-legged cycle, or of the 3-legged cycle leaves a graph with two connected components, each of which has a cycle, contrary to the definition of cyclically 4-edge-connected graph. That is, $2c_2 + c_3 \leq 2$ for any independent set ξ of cycles in Γ, where c_2 and c_3 are the numbers of 2- and 3-legged cycles in ξ, respectively. $\mathcal{Q.E.D.}$

Lemma 3.1.3 *Let G be a subdivision of planar* 3*-connected cubic graph. If G is cyclically* 4*-edge-connected, then all the plane embeddings of the planar graph G, satisfy (br*1*) and (br*2*) of Lemma 2.2.2.*

Proof. Let G be a subdivision of planar 3-connected cubic graph. Let G be cyclically 4-edge-connected, and let Γ be an arbitrary plane embedding of G. Assume Γ has a 2-legged cycle C which has not an edge on $C_O(\Gamma)$, a contradiction to Lemma 2.2.2(a). Removal of the two legs of C leaves a graph with 2 connected components. One is the inner subgraph $\Gamma_1(C)$ for C, and another is $\Gamma - \Gamma_1(C)$ for C. Each of the component has a cycle, contrary to the definition of cyclically 4-edge-connected graph. That is, every 2-legged cycle has an edge on $C_O(\Gamma)$. Similarly we can prove for Γ, that every 3-legged cycle has an edge on $C_O(\Gamma)$.

It is seen from Lemma 3.1.2, that an arbitrary plane embedding Γ of a subdivision of planar 3-connected cubic graph G, which is cyclically 4-edge-connected, satisfies $2c_2 + c_3 \leq 2$ for any independent set ξ of cycles in Γ, where c_2 and c_3 are the numbers of 2- and 3-legged cycles in ξ, respectively. This implies the (br2) of Lemma 2.2.2.

Thus we find that, Γ satisfies both (br1) and (br2) of Lemma 2.2.2. $\mathcal{Q.E.D.}$

As all the plane embeddings of a subdivision of planar 3-connected cubic graph G which is cyclically 4-edge-connected satisfy both (br1) and (br2) of Lemma 2.2.2, every plane embedding of the planar graph G has a box-rectangular drawing. $\mathcal{Q.E.D.}$

It is seen that all the plane embeddings of a subdivision of planar 3-connected cubic graph G which is cyclically 4-edge-connected, have box-rectangular drawings, where Rahman et al. [RNG04] showed that any arbitrary plane embedding Γ of a subdivision of planar 3-connected cubic graph G that is cyclically 4-edge connected must satisfy three necessary and sufficient conditions to have a rectangular drawing.

Proof of Theorem 3.1.1(b). Since the proof for the sufficiency is obvious, we give a proof for the necessity. Suppose that Γ has a regular 3-legged cycle C and that the planar graph G has a box-rectangular drawing. Then there is a plane embedding Γ' of G which has a box-rectangular drawing. Let F be the face of Γ corresponding to $F_O(\Gamma')$. It suffices to show that F is one of the three peripheral faces F_1, F_2, and F_3 for C in Γ.

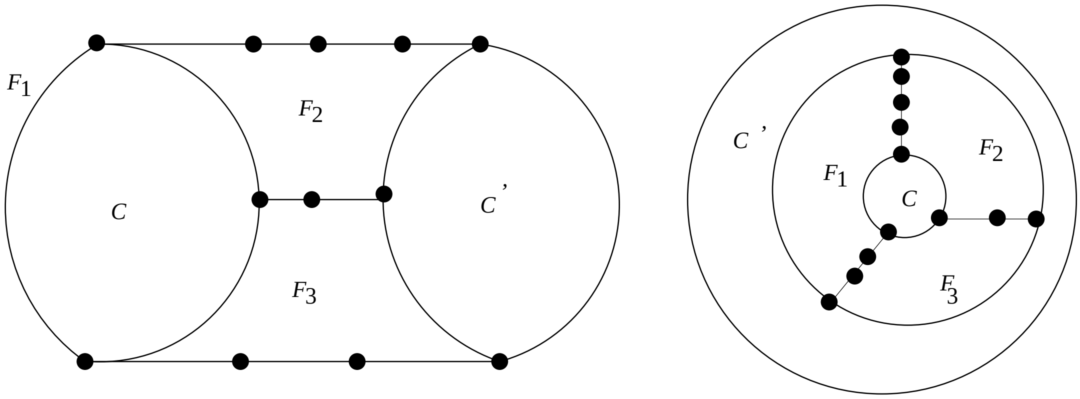


Figure 3.1: Cycles C and C' in Γ.

We first consider the case where C contains an edge on $F_O(\Gamma)$. Let C' be the cycle in $\Gamma - \Gamma_1(C)$ such that $\Gamma_1(C')$ has the maximum number of edges.(See Fig. 3.1(a).) One can observe that C' is a 3-legged cycle in Γ, and any face of Γ other than F_1, F_2, and F_3 is in $\Gamma_1(C)$ or in $\Gamma_1(C')$. Therefore it is sufficient to prove that F is neither in $\Gamma_1(C)$ nor in $\Gamma_1(C')$. If F is in $\Gamma_1(C)$, then C' is a 3-legged cycle in Γ' and contains no vertex on $F_O(\Gamma') = F$, a contradiction to (br1) of Lemma 2.2.2. Similarly, if F is in $\Gamma_1(C')$, then C is a 3-legged cycle in Γ' and no vertex on $F_O(\Gamma') = F$, a contradiction to (br1) of Lemma 2.2.2.

We next consider the case where C does not contain any edge on $F_O(\Gamma)$. Let C' be the cycle in $\Gamma - \Gamma_1(C)$ such that $\Gamma_1(C')$ includes $\Gamma_1(C)$ and has the minimum number of edges. (See Fig. 3.1(b)) Any face other than F_1, F_2 and F_3 is in $\Gamma_1(C)$ or in $\Gamma_O(C')$. Therefore it is sufficient to prove that F is neither in $\Gamma_1(C)$ nor in $\Gamma_O(C')$. If F is in $\Gamma_1(C)$, then C' is a 3-legged cycle in Γ' and contains no vertex on $F_O(\Gamma') = F$, a contradiction to (br1) of Lemma 2.2.2. If F is in $\Gamma_O(C')$, then C is a 3-legged cycle in Γ' contains no vertex on $F_O(\Gamma') = F$, a contradiction to (br1) of Lemma 2.2.2. $\mathcal{Q.E.D.}$

Theorem 3.1.1 immediately yields the following algorithm to examine whether a subdivision of a planar 3-connected cubic graph G has a box-rectangular drawing and to find a box-rectangular drawing of G if it exists.

Algorithm Subdivision-Draw-Δ-3-or-Less(G)

begin

Let Γ be any arbitrary plane embedding of G;

Examine whether Γ has a regular 3-legged cycle C;

if Γ has no 3-legged cycle **then** $\{G$ is cyclically 4-edge-connected.$\}$

  Find the box-rectangular drawing of Γ using the method stated in Subsection 2.5.2.

else $\{$ Γ has a regular 3-legged cycle, and G is not cyclically 4-edge-connected.$\}$

  begin

    Let C be any regular 3-legged cycle;

    Let F_1, F_2, and F_3 be the three peripheral faces of C;

    Γ_1, Γ_2, and Γ_3 be three plane embeddings of G taking F_1, F_2, and F_3 as the outer face respectively;

    Examine whether Γ_1, Γ_2, and Γ_3 satisfy both (br1) and (br2) of Lemma 2.2.2, that is, they have box-rectangular drawings.;

      if Γ_1, Γ_2, or Γ_3, say Γ_2 has a box-rectangular drawing **then**

        Find a box-rectangular drawing of Γ_2 by the method stated in Subsection 2.5.2;

      else

        G has no box-rectangular drawing;

  end

end.

Theorem 3.1.4 *Algorithm Subdivision-Draw-Δ-3-or-Less examines in linear time whether a subdivision of a planar 3-connected cubic graph G has a box-rectangular drawing, and finds a box-rectangular drawing of G in linear time if it exists.*

Proof. Using a method similar to that in [RNN99, RNN00], one can examine in linear time whether Γ has a regular 3-legged cycle. The method stated in Subsection 2.5.2 takes linear time. Therefore, the overall time complexity of the **Algorithm Subdivision-Draw-Δ-3-or-Less** is linear. Clearly, **Algorithm Subdivision-Draw-Δ-3-or-Less** correctly examines whether G has a box-rectangular drawing or not, and finds a box-rectangular drawing of G if it exists. $\mathcal{Q.E.D.}$

3.1.2 The Other Case for a Planar Graph G with $\Delta \leq 3$

In this subsection we assume that G is a planar biconnected with $\Delta \leq 3$ but is not a subdivision of a 3-connected cubic graph. We give a linear-time algorithm to examine whether G has a box-rectangular drawing and to find a box-rectangular drawing of G if it exists.

If the planar graph G has all the vertices of degree 2 then we have the following theorem.

Theorem 3.1.5 *Let G be a planar graph. If G has all the vertices of degree two, then all the plane embeddings of G are unique, that is, there is only one plane embedding of G. G has a box-rectangular drawing for that plane embedding.*

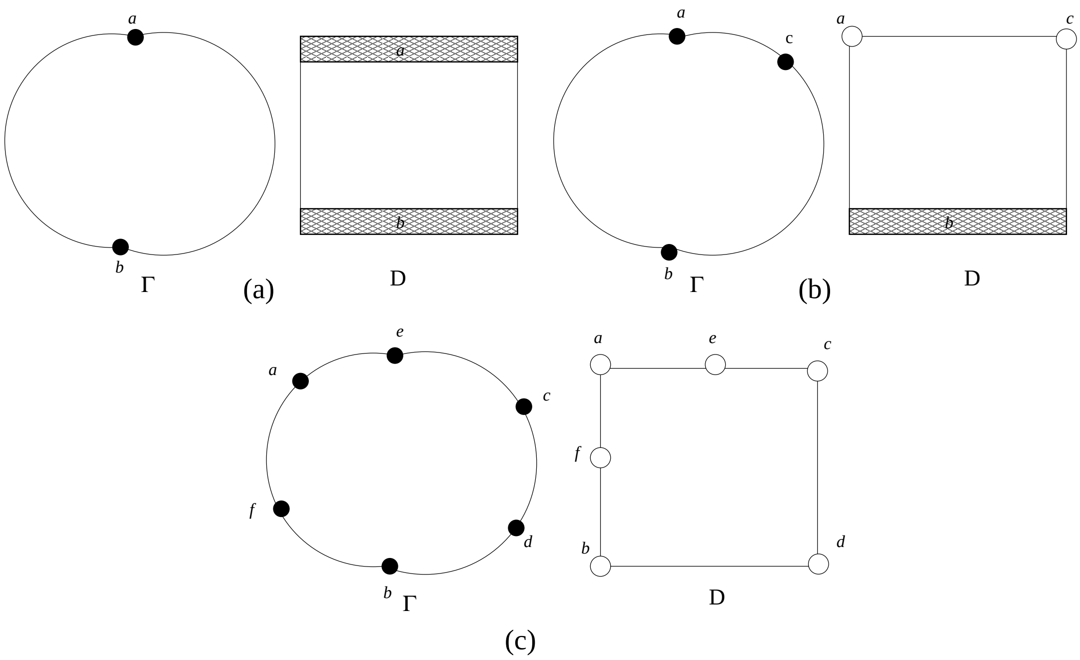


Figure 3.2: Planar graphs with all vertices of degree 2, and their corresponding box-rectangular drawings.

Proof. Let G be a planar graph. G has all the vertices of degree 2. In each of the Fig. 3.2(a, b, or c) one can easily observe that Γ is the only one plane embedding of G and also that Γ has a box-rectangular drawing. $\mathcal{Q.E.D.}$

.

Similarly if the planar graph G has at most two vertices of degree 3 then we have the following theorem.

Theorem 3.1.6 *Let G be a planar 2-connected graph with $\Delta \leq 3$ but not a subdivision of a planar 3-connected cubic graph. If G has at most two vertices of degree three, then G has a box-rectangular drawing.*

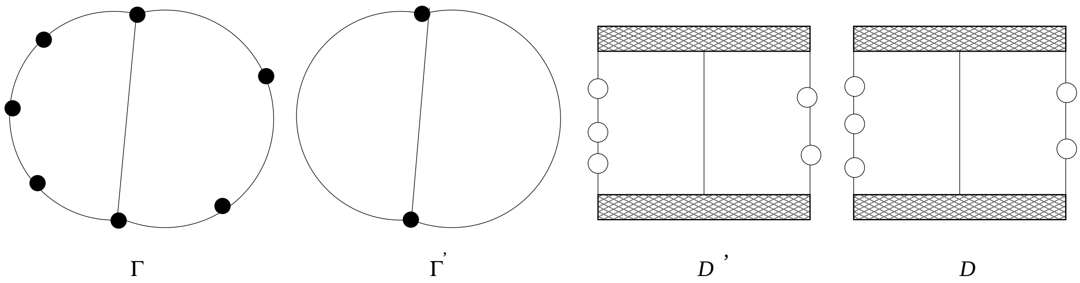


Figure 3.3: A Planar graph with exactly two vertices of degree 3, its transformations, and its corresponding box-rectangular drawing.

Proof. Let G be a planar 2-connected graph with $\Delta \leq 3$ but not a subdivision of a planar 3-connected cubic graph. G has at most two vertices of degree 3. Let Γ be an arbitrary plane embedding of G. Then one can easily observe that Γ has a box-rectangular drawing and find the box-rectangular drawing as illustrated in Fig. 3.3. $\mathcal{Q.E.D.}$

We may thus assume that any arbitrary plane embedding Γ of a planar 2-connected graph G has three vertices of degree 3. Then Γ has a regular 2-legged cycle; otherwise, G would be a subdivision of a 3-connected cubic graph.

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)$ be all pairs of vertices such that x_i and y_i, $1 \leq i \leq l$, are the leg vertices of a regular 2-legged cycle or the hand-vertices of a regular 2-handed cycle. If there is a plane embedding Γ' of G having a box-rectangular drawing, then the outer face $F_O(\Gamma')$ must contain all vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)$; otherwise, Γ' would have a 2-legged cycle containing no vertex on $F_O(\Gamma')$ and hence by (br1) of Lemma 2.2.2 Γ' would not have a box-rectangular drawing. Construct a graph G^+ from G by adding a dummy vertex z and dummy edges (x_i, z) and (y_i, z) for all indices i, $1 \leq i \leq l$. Then G has a plane embedding whose outer face contains all vertices $x_1, y_1, x_2, y_2, \ldots, x_l, y_l$ if and only if G^+ is planar. (Figure 3.4(b) illustrates G^+ for G in Fig. 3.4(a).)

We may thus assume that G^+ is planar. Let Γ^+ be an arbitrary plane embedding of G^+ such that z is embedded on the outer face, as illustrated in Fig. 3.4(c). We delete from Γ^+ the dummy vertex z and let Γ^* be the resulting plane

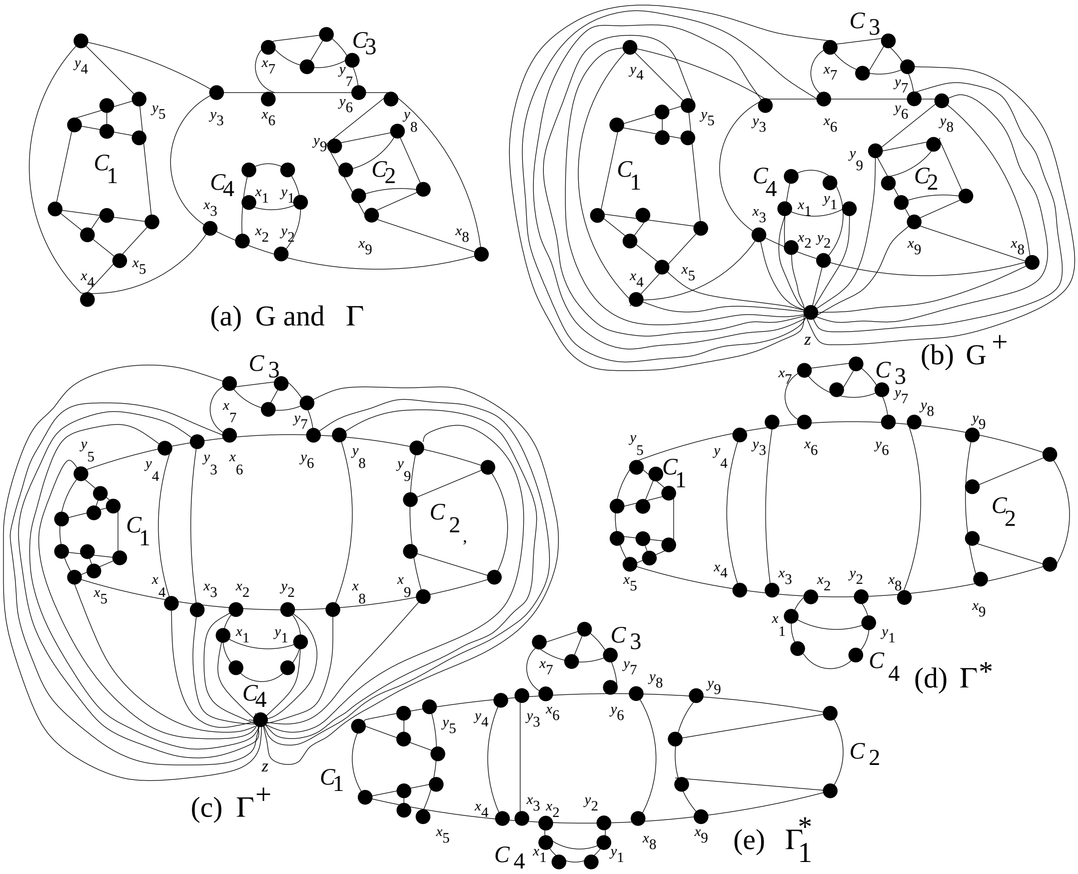


Figure 3.4: $G, \Gamma, G^+, \Gamma^+, \Gamma^*$, and $\Gamma_1{}^*$.

embedding of G, in which $F_O(\Gamma^*)$ contains all vertices $x_1, y_1, x_2, y_2, \ldots, x_l, y_l$, as illustrated in Fig. 3.4(d). One can observe that every 2-legged cycle in Γ^* has the leg-vertices on $F_O(\Gamma^*)$.

Let p be the largest integer such that a number p of 2-legged cycle in Γ^* are independent with each other. Then $p \geq 2$ since Γ and hence Γ^* has a regular 2-legged cycle. Γ^* has a number p of minimal 2-legged cycles. If Γ_1^* is a plane embedding obtained from Γ^* by flipping $\Gamma_1{}^*(C)$ for a minimal 2-legged cycle C, then the leg vertices of all 2-legged cycles in $\Gamma_1{}^*$ are on $F_O(\Gamma_1{}^*)$ (The embedding $\Gamma_1{}^*$ in Fig. 3.4(e) is obtained from Γ^* in Fig. 3.4(d) by flipping $\Gamma_1{}^*(C_1)$). One can observe that only the plane embeddings of G that can be obtained from Γ^* by flipping $\Gamma_1{}^*(C)$ for some minimal 2-legged cycles C have $x_1, y_1, x_2, y_2, \ldots, x_l, y_l$ on the outer face. We now have $p = 2$; otherwise, any plane embedding of G whose outer face contains all vertices $x_1, y_1, x_2, y_2, \ldots, x_l, y_l$ has three or more independent 2-legged cycles, and hence

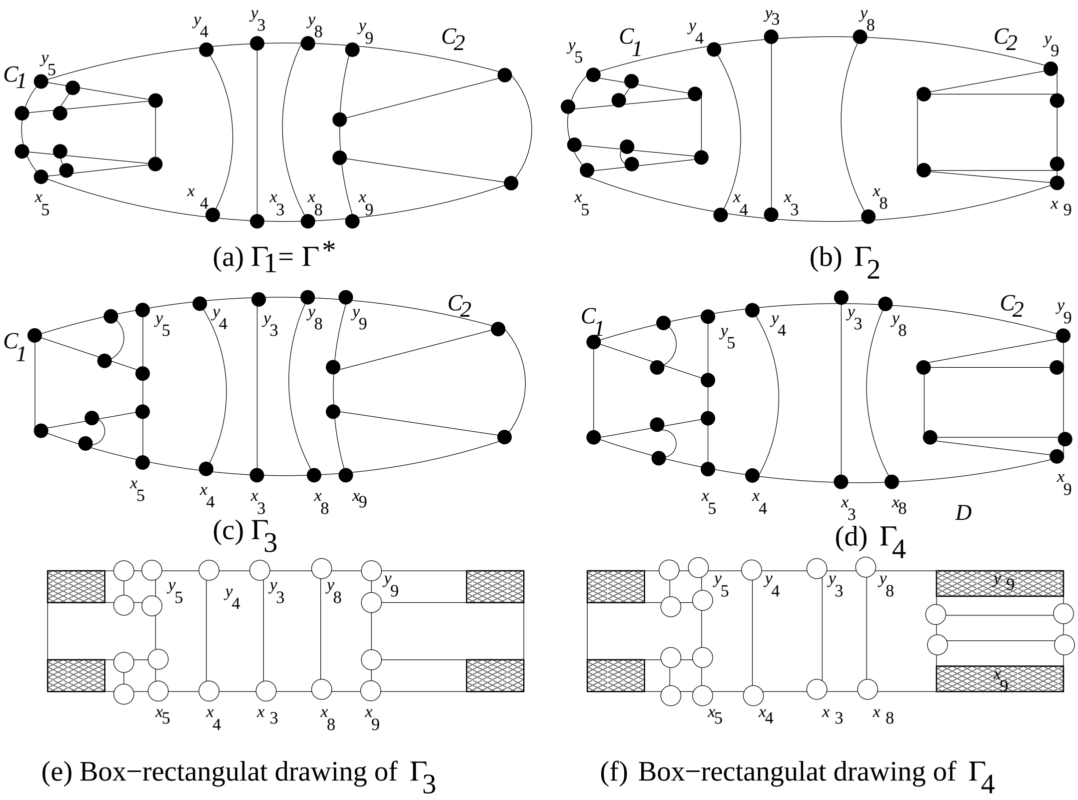


Figure 3.5: Four different embeddings $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4, and box-rectangular drawings of Γ_3 and Γ_4.

by (br2) of Lemma 2.2.2 the embedding has no box-rectangular drawing.

Since $p = 2$, Γ^* has exactly two independent 2-legged cycles C_1 and C_2. We may assume without loss of generality that C_1 and C_2 are minimal 2-legged cycles, as illustrated in Fig. 3.5. By flipping ${\Gamma_1}^*(C_1)$ or ${\Gamma_1}^*(C_2)$ around the leg vertices of C_1 or C_2, we have at most four different embeddings $\Gamma_1(= \Gamma^*)$, Γ_2, Γ_3, and Γ_4 such that each $F_O(\Gamma_j)$, $1 \leq j \leq 4$, contains all vertices $x_1, y_1, x_2, y_2, \ldots, x_l, y_l$ as illustrated in Fig. 3.5. Since only the four embeddings $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 of G have all vertices $x_1, y_1, x_2, y_2 \ldots, x_l, y_l$ on the outre face, G has a box-rectangular drawing if and only if at least one of $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 has a box-rectangular drawing. (None of the embeddings Γ_1 and Γ_2 in Figs. 3.5(a and b) has a box-rectangular drawing since there are no four vertices on the outer face to choose as corner boxes according to (br1) of Lemma 2.2.2, two at each 2-legged cycle, and at least one at each 1-legged cycle, while each of the embeddings Γ_3 and Γ_4 in Figs. 3.5(c and d) has a box-rectangular drawing as illustrated in Fig. 3.5(e) and in Fig. 3.5(f) respectively). We thus have the following theorem.

Theorem 3.1.7 *Let G be a planar biconncted graph with $\Delta \leq 3$ which is not a subdivision of a planar 3-connected cubic graph. Let Γ be a plane embedding of G such that every 2-legged cycle in Γ has leg-vertices on $F_O(\Gamma)$, let Γ have exactly two independent 2-legged cycles, and let C_1 and C_2 be the two minimal 2-legged cycles in Γ. Let $\Gamma_1(=\Gamma)$, Γ_2, Γ_3, and Γ_4 be the four embeddings of G obtained from Γ by flipping $\Gamma_1(C_1)$ or $\Gamma_1(C_2)$ around the the leg vertices of C_1 and C_2. Then G has a box-rectangular drawing if and only if at least one of the four embeddings Γ_1, Γ_2, Γ_3, and Γ_4 has a box-rectangular drawing.*

3.2 Algorithm

In this section we formally describe our **Algorithm Planar-Box-Rectangular-Draw-Δ-3-or-Less** to examine whether a planar graph G with $\Delta \leq 3$ has a box-rectangular drawing and to find a box-rectangular drawing of G if it exists.

Algorithm Planar-Box-Rectangular-Draw-Δ-3-or-Less (G)
{ Assume that G has three or more vertices of degree 3. Otherwise, one can easily examine whether G has a box-rectangular drawing or not, as illustrated in Fig. 3.2 and in Fig. 3.3. }
begin
Let Γ be any plane embedding of G;
Examine whether Γ has a regular 2-legged cycle;
if Γ has no regular 2-legged cycle **then**
{ G is a subdivision of a planar 3-connected cubic graph.}
Examine by Algorithm **Subdivision-Draw-Δ-3-or-Less** whether G has a box-rectangular drawing and find a box-rectangular drawing of G if it exists;
else { G is not a subdivision of a planar 3-connected cubic graph.}
begin
Let $(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)$ be all pair of vertices such that x_i and y_i, $1 \leq i \leq l$, are the leg vertices of a regular 2-legged cycle or the hand vertices of a regular 2-handed cycle in Γ;
Construct a graph G^+ from G by adding a dummy vertex z and

dummy edges (x_i, z) and (y_i, z) for all indices i, $1 \leq i \leq l$;
Examine whether G^+ is planar;
if G^+ is not planar **then**
G has no box-rectangular drawing;
else
begin
Find a planar embedding Γ^+ of G^+ such that z is embedded on the outer face;
Delete from Γ^+ the dummy vertex z and let Γ^* be the resulting plane subgraph;
if Γ^* has three or more independent 2-legged cycles **then**
G has no box-rectangular drawing;
else
begin
Let C_1 and C_2 be the two minimal regular 2-legged cycles in Γ^*;
Let Γ_1, Γ_2, Γ_3 and Γ_4 be the four plane embeddings of G obtained from Γ^* by flipping ${\Gamma_1}^*(C_1)$ or ${\Gamma_1}^*(C_2)$ around the leg-vertices of C_1 or C_2;
Examine whether Γ_1, Γ_2, Γ_3, and Γ_4 have box-rectangular drawings by Lemma 2.2.2;
if Γ_1, Γ_2, Γ_3, or Γ_4, say Γ_2, has a box-rectangular drawing **then**
Find a box-rectangular drawing of Γ_2 by the method stated in Subsection 2.5.2;
else
G has no box-rectangular drawing;
end
end
end

We now have the following theorem on **Algorithm Planar-Box-Rectangular-Draw-Δ-3-or-Less**.

Theorem 3.2.1 *Let G be a planar biconnected graph with $\Delta \leq 3$. Then Algorithm Planar-Box-Rectangular-Draw-Δ-3-or-Less examines in linear time whether G has a box-rectangular drawing and finds a box-rectangular drawing of G if it exists.*

Proof. One can find all pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)$ of vertices in linear time using a method similar to the algorithm in [RNN00] to find 2-legged cycles. By theorem 3.1.4 **Algorithm Subdivision-Draw-Δ-3-or-Less** takes linear time. One can examine the planarity of G^+ in linear time and find Γ^+ in linear time [NC88]. The method stated in Subsection 2.5.2 takes linear time. Thus **Algorithm Planar-Box-Rectangular-Draw-Δ-3-or-Less** takes linear time in total.

The correctness of **Algorithm Planar-Box-Rectangular-Draw-Δ-3-or-Less** is immediate from Theorems 3.1.1, and 3.1.7. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

Chapter 4

Box-Rectangular Drawings of Planar Graphs with $\Delta \geq 4$

In this chapter we give a necessary and sufficient condition for a 2-connected planar graph G with $\Delta \geq 4$ to have a box-rectangular drawing. We also give a linear-time algorithm to find the drawing if it exists. Section 4.1 describes the necessary and sufficient condition, and Section 4.2 illustrates the algorithm.

4.1 Necessary and Sufficient Condition

In Subsection 4.1.1 we consider the case where G is a subdivision of a planar 3-connected graph with $\Delta \geq 4$, and in Subsection 4.1.2 we consider the cases where G with $\Delta \geq 4$ has at most two vertices, and where G is a planar 2-connected graph with $\Delta \geq 4$ but not a subdivision of a 3-connected graph.

Before entering into different cases, we observe the following lemmas on a planar graph with $\Delta \geq 4$.

Lemma 4.1.1 *Let G be a planar graph with $\Delta \geq 4$, and Γ be an arbitrary plane embedding of G. Let H be the transformed graph of Γ by replacing each vertex v of degree four or more in Γ by a cycle, and let Φ be an arbitrary plane embedding of H. Denote the total number of 2-legged and 3-legged cycles in Γ by l_Γ, and the total number of 2-handed and 3-handed cycles in Γ by h_Γ. Also denote the total number of 2-legged and 3-legged cycles in Φ by l_ϕ, and the total number of 2-handed and 3-handed cycles in Φ by h_Φ. If $p_\Gamma = l_\Gamma + h_\Gamma$ and $p_\Phi = l_\Phi + h_\Phi$, then $p_\Gamma = p_\Phi$.*

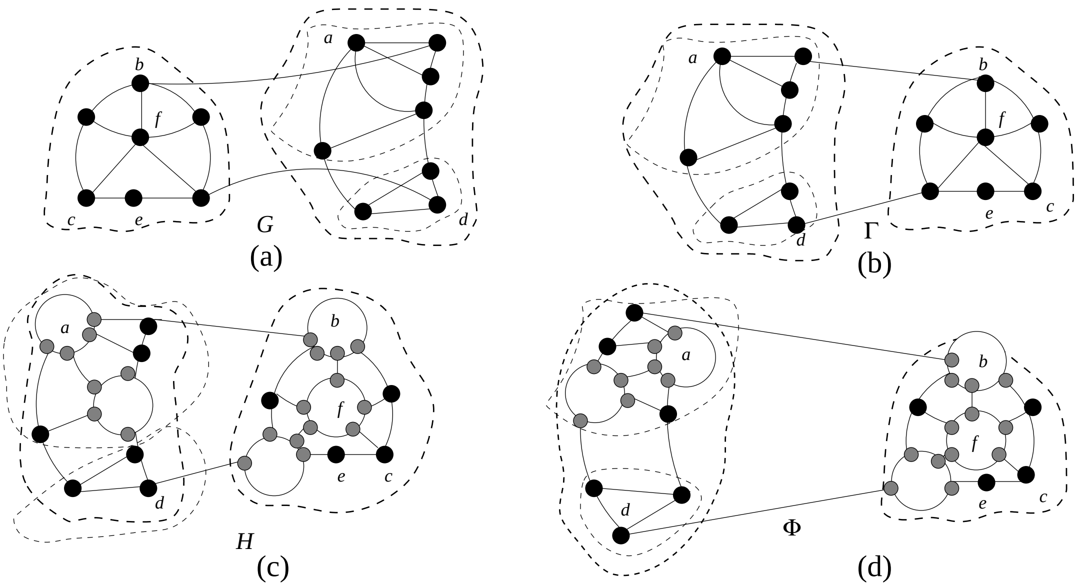


Figure 4.1: $p_\Gamma = p_\Phi$

Proof. Let G be a planar graph with $\Delta \geq 4$ as in Fig. 4.1(a), and let Γ be an arbitrary plane embedding of G as in Fig. 4.1(b). In Fig. 4.1(c), H is the transformed graph of Γ as above, and let Φ be an arbitrary plane embedding of H as in Fig. 4.1(d). 2-legged cycles and 2-handed cycles in Γ are shown by thick dotted lines in Fig. 4.1(b), and 3-legged cycles and 3-handed cycles in Γ are shown by dotted lines in Fig. 4.1(b). Similarly 2-legged cycles and 2-handed cycles in Φ are shown by thick dotted lines in Fig. 4.1(d), and 3-legged cycles and 3-handed cycles in Φ are shown by dotted lines as in Fig. 4.1(d). Since a 2-legged cycle in any plane embedding of G is a 2-legged or a 2-handed cycle in another plane embedding of G, and a 2-handed cycle in any plane embedding of G is a 2-handed or a 2-legged cycle in another plane embedding of G, different plane embeddings of a same planar graph do not change in total number of 2-legged cycles and 2-handed cycles. Similarly different plane embeddings of a same planar graph do not change in total number of 3-legged cycles and 3-handed cycles. One can easily observe that, the total number of 2-legged and 2-handed cycles in Φ remains same with the total number of 2-legged and 2-handed cycles in Γ after transformation. Similarly the total number of 3-legged and 3-handed cycles in Φ remains same with the total number 3-legged and 3-handed cycles in Γ, after transformation. Because in Φ every replaced cycle is either a 4- or more handed, or a 4- or more legged cycle. Let the total number of 2-legged and 3-legged cycles in Γ be l_Γ, the total number of 2-handed and 3-handed cycles in Γ be h_Γ, the total number of 2-legged and 3-legged cycles in Φ be l_Φ, the total number of 2-handed and 3-handed cycles in Φ be h_Φ, and if $p_\Gamma = l_\Gamma + h_\Gamma$ and $p_\Phi = l_\Phi + h_\Phi$ then $p_\Gamma = p_\Phi$. $\mathcal{Q.E.D.}$

Lemma 4.1.2 *Let G be a planar graph with $\Delta \geq 4$ and Γ be an arbitrary plane embedding of G. Let H be the transformed graph of Γ by replacing each vertex v of degree four or more in Γ by a cycle. Let Φ_R be any arbitrary plane embedding of H, where $C_O(\Phi_R)$ is the face of any replaced cycle in H. Then G is cyclically 4-edge-connected if and only if Φ_R has a box-rectangular drawing.*

Necessity of Lemma 4.1.2. Let G be a planar graph with $\Delta \geq 4$ and let Γ be an arbitrary plane embedding of G. Let H be the transformed graph of Γ as above. Let Φ_R be any arbitrary plane embedding of H, where $C_O(\Phi_R)$ is the face of any replaced cycle in H. Assume G is cyclically 4-edge-connected. By Lemma 3.1.3, a plane embedding Γ of the planar graph G is also cyclically 4-edge-connected. As all the replaced cycles in H are 4- or more legged cycles, H is also cyclically 4-edge-connected. By Lemma 3.1.3, a plane embedding Φ_R of the planar graph H is also cyclically 4-edge-connected. Then by Theorem 3.1.1(a), Φ_R has a box-rectangular drawing. $\mathcal{Q.E.D.}$

Sufficiency of Lemma 4.1.2. Let G be a planar graph with $\Delta \geq 4$ and let Γ be an arbitrary plane embedding of G. Let H be the transformed graph of Γ as above. Let Φ_R be any arbitrary plane embedding of H, where $C_O(\Phi_R)$ is the face of any replaced cycle in H. Assume Φ_R has a box-rectangular drawing. That is, Φ_R satisfies both (br1) and (br2) of Lemma 2.2.2. There is no 2- or 3-legged cycle in Φ_R, which does not contain an edge on $C_O(\Phi_R)$. $C_O(\Phi_R)$ is also a 4- or more handed cycle. That is, there is no independent set of 2- and 3-legged cycles in Φ_R. Removal of any 2 or 3 edges leaves a graph in Φ_R such that exactly one component has a cycle. So Φ_R is cyclically 4-edge-connected. Another plane embedding Φ of H, which does not have a replaced cycle as outer face is also cyclically 4-edge-connected, as Φ and Φ_R are the two different plane embeddings of the same planar graph H. By Lemma 4.1.1, Γ and Φ do not change in total number of 2-legged, 2-handed, 3-legged, and 3-handed cycles. So, Γ is cyclically 4-edge-connected. That is, G is a cyclically 4-edge-connected graph. $\mathcal{Q.E.D.}$

4.1.1 Case for a Subdivision of a Planar 3-Connected Graph with $\Delta \geq 4$

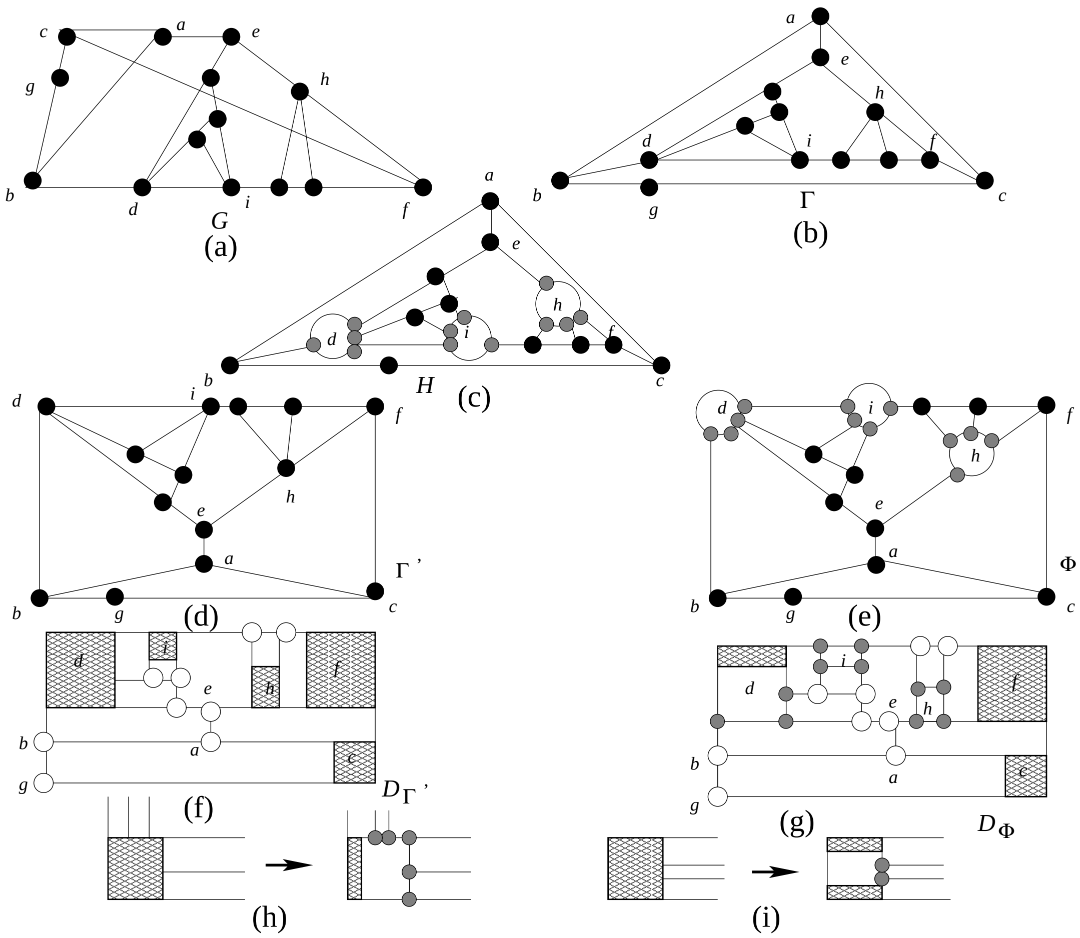


Figure 4.2: Illustration of G, Γ, H, Γ', Φ, $D_{\Gamma'}$, D_{Φ}, and the two transformations.

Let G be a subdivision of a planar 3-connected graph with $\Delta \geq 4$, and Γ be an arbitrary plane embedding of G. We construct a new planar graph H from Γ by replacing each vertex v of degree four or more in G by a cycle. Figures 4.2(a), 4.2(b), and 4.2(c) illustrate G, Γ, and H respectively. A replaced cycle corresponds to a real box in a box-rectangular drawing of G. We do not replace a vertex of degree 2 or 3 by a cycle since such a vertex may be drawn as a point. Thus $\Delta(H) \leq 3$. The following theorem is the main result of this subsection.

Theorem 4.1.3 *Let G be a subdivision of a planar 3-connected graph with $\Delta \geq 4$ and let Γ be an arbitrary plane embedding of G. Let H be the graph transformed* *from* Γ *as above. Then* G *has a box-rectangular drawing if and only if the planar graph* H *has a box-rectangular drawing.*

It is rather easy to prove the necessity of Theorem 4.1.3.

Necessity of Theorem 4.1.3. Let Γ in Fig. 4.2(b) be an arbitrary plane embedding of the planar graph G in Fig. 4.2(a). Assume that G has a box-rectangular drawing for any plane embedding Γ' in fig. 4.2(d) of G. Then by Lemma 2.4.5, Γ' has a box-rectangular drawing $D_{\Gamma'}$ in which every vertex of degree 4 or more is drawn as a real box, as illustrated in Fig. 4.2(f). Then, as illustrated in Fig. 4.2(g), one can obtain a box-rectangular drawing D_Φ for the plane graph Φ in Fig. 4.2(e) of the planar graph H from $D_{\Gamma'}$ by the following transformation:

(i) regard each noncorner real box in $D_{\Gamma'}$ as a face in D_Φ;

(ii) if a corner box in $D_{\Gamma'}$ corresponds to a vertex of degree 3 in Γ then regard it as a corner box in D_Φ; and

(iii) if a corner box in $D_{\Gamma'}$ corresponds to a vertex of degree four or more in Γ, then transform it to a drawing of a replaced cycle with one or more real boxes as illustrated in Figs. 4.2(h) and 4.2(i) where the box in $D_{\Gamma'}$ contains one corner in Fig. 4.2(h) and contains two corners in Fig. 4.2(i). $\mathcal{Q.E.D.}$

Figures 4.2(f) and 4.2(g) illustrate $D_{\Gamma'}$ and D_Φ, respectively. Box h in $D_{\Gamma'}$ is a noncorner real box, and it is regarded as a face in D_Φ. Corner boxes c and f in $D_{\Gamma'}$ correspond to a vertices of degree 3, and they remain as boxes in D_Φ. Corner box g in $D_{\Gamma'}$ corresponds to a vertex of degree 2, it remains as a degenerated box in D_Φ. Corner box d in $D_{\Gamma'}$ correspond to vertex of degree four or more is transformed to a drawing of a replaced cycle with one real box in D_Φ as illustrated in Fig. 4.2(h).

It is rather difficult to prove the sufficiency of Theorem 4.1.3, since there is no easy method which directly transforms a box-rectangular drawing D_Φ of any plane embedding Φ of H to a box-rectangular drawing $D_{\Gamma'}$ of any plane embedding Γ' of the planar graph Γ as well as of the planar graph G. We give some definitions before proving the sufficiency. Figures 4.3(a) and 4.3(b) illustrate G and Γ respectively. We replace the vertices of degree 4 or more in

Γ by cycles like Figure 2.5. We call a vertex of degree 3 on a replaced cycle a *replaced vertex*. The edges on the replaced cycle are called *born edges*. In any arbitrary plane embedding Φ of H, that does not contain a replaced cycle as outer face, the replaced cycle on $C_O(\Phi)$ corresponding to a vertex of degree 4 or more in G contains exactly one edge on $C_O(\Phi)$. We call such an edge in Φ a *green edge*. Each vertex of degree 2 or 3 in G has a corresponding vertex of the same degree in H, and we call such a vertex in H an *original vertex*. Now each vertex in H is either a replaced vertex or an original vertex. In any arbitrary plane embedding Φ of H,that does not contain a replaced cycle as outer face, for a green edge e and a cycle C in Φ, we call e a *green edge for* C if both ends of e are on C.

Assume an arbitrary plane embedding Φ of the planar graph H has a box-rectangular drawing D_Φ, then Φ satisfies (br1) and (br2) of Lemma 2.2.2. We can easily transform D_Φ to a box-rectangular drawing $D_{\Gamma'}$ of any plane embedding Γ' of the planar graph Γ as well as of the planar graph G if only original vertices are drawn as corner boxes in D_Φ, because then each replaced vertex is a point in D_Φ, and each replaced cycle in Φ is a rectangular face in D_Φ, and hence D_Φ can be transformed to $D_{\Gamma'}$ by regarding each replaced cycle as a box. The problem is the case where a replaced vertex is drawn as a corner box in D_Φ. Because such a drawing D_Φ cannot always be transformed to a box-rectangular drawing $D_{\Gamma'}$ of Γ'. However we show that a plane graph Φ^* in Fig. 4.3(f) obtained from Φ in Fig. 4.3(d) through an intermediate graph Φ' in Fig. 4.3(e) with slight modification has a particular box-rectangular drawing D_{Φ^*} which can be easily transformed to a box-rectangular drawing of Γ' as illutrated in Fig. 4.3(h). Transformation is also not possible when the outer face of Φ is a replaced cycle. It is found from Lemma 4.1.2(a) that, if a plane graph Φ_R of the planar graph H, containing a replaced cycle as outer face has a box-rectangular drawing, then the planar graph H is cyclically 4-edge connected. Then by Theorem 3.1.1(a), any arbitrary plane embedding Φ of the planar graph H, that does not contain a replaced cycle as outer face has a box-rectangular drawing.

We are now ready to prove the sufficiency of Theorem 4.1.3.

Sufficiency of Theorem 4.1.3. Let an arbitrary plane embedding of H as in Fig. 4.3(c) be Φ as in Fig. 4.3(d) that does not have a replaced cycle as outer face. Assume that Φ has a box-rectangular drawing. Let Φ' be the minimal graph homeomorphic to Φ as illustrated in Fig. 4.3(e); then Φ' is a cubic graph and satisfies Conditions (br1) and (br2) in Lemma 2.2.2. Using the similar approach used in Subsection 2.5.3, we can designate four vertices as corners vertices after slight modification in Φ'. Let Φ^* be the resulting graph as illustrated in Fig. 4.3(f). Note that each of the four designated vertices in Φ^* is either an original vertex or a dummy vertex of degree 2 on a green edge of Φ'. Clearly, every 3-legged cycle in Φ^* contains at least one designated vertex. Hence, Φ^* has a box-rectangular drawing with the four designated vertices as corner boxes, as illustrated in Fig. 4.3(g). Inserting the removed vertices of degree 2 on some vertical and horizontal line segments in $D_{\Phi}{}^*$ and regarding the drawing of each replaced cycle as a box, we immediately obtain a box-rectangular drawing $D_{\Gamma'}$ of the plane embedding Γ' of the planar graph Γ as well as of the planar graph G from $D_{\Phi}{}^*$, as illustrated in Fig. 4.3(h). $\mathcal{Q.E.D.}$

Theorem 4.1.3 immediately yields the following algorithm to examine whether a subdivision of a planar 3-connected graph G with $\Delta \geq 4$ has a box-rectangular drawing and to find a box-rectangular drawing of G if it exists.

```
Algorithm Subdivision-Draw-3-Connected-Δ-4-or-More(G)
begin
  Let Γ be any arbitrary plane embedding of G;
  Transform the graph from Γ to H by using the method as in Theorem
    4.1.3;
  Examine whether the planar graph H has a box-rectangular drawing or
    not by using the Algorithm Planar-Box-Rectangular-Draw-Δ-3-
    or-Less;
  if H has a box-rectangular drawing then
    Find the box-rectangular drawing of G by using the method stated in
    the constructive proof of sufficiency of Theorem 4.1.3;
  else
    G has no box-rectangular drawing;
end
```

Theorem 4.1.4 *Algorithm Subdivision-Draw-*3*-Connected-*Δ*-*4*-or-More examines in linear time whether a subdivision of a planar* 3*-connected graph* G *with*

$\Delta \geq 4$ has a box-rectangular drawing, and finds a box-rectangular drawing of G in linear time if it exists.

Proof. One can construct H from Γ in time $O(m)$, where m is the number of edges in G. By theorem 3.2.1, **Algorithm Planar-Box-Rectangular-Draw-Δ-3-or-Less** examines in linear time whether the planar graph H has a box-rectangular drawing and finds a box-rectangular drawing of H as well as of G if it exists. Thus **Algorithm Subdivision-Draw-3-Connected-Δ-4-or-More** takes linear time in total.

The correctness of **Algorithm Subdivision-Draw-3-Connected-Δ-4-or-More** is immediate from the Theorem 4.1.3. $\mathcal{Q.E.D.}$

4.1.2 Case for G with $\Delta \geq 4$, where G has at most two vertices, or where G is a planar 2-connected graph with $\Delta \geq 4$ but not a subdivision of a planar 3-connected graph

In this Subsection we assume that G with $\Delta \geq 4$ has at most two vertices, or G is a planar 2-connected graph with $\Delta \geq 4$ but not a subdivision of a planar 3-connected graph.

If the planar graph G with $\Delta \geq 4$ has at most two vertices then we have the following theorem.

Theorem 4.1.5 *Let G be a planar graph. If G with $\Delta \geq 4$ has at most two vertices, then all the plane embeddings of G are unique, that is, there is only one plane embedding of G. G has a box-rectangular drawing for that plane embedding.*

Proof. Let G be a planar graph with $\Delta \geq 4$. G has at most two vertices. In the Fig. 4.4 one can easily observe that Γ is the only one plane embedding of G and also that Γ has a box-rectangular drawing. $\mathcal{Q.E.D.}$

.

Then we assume that G is a planar 2-connected graph with $\Delta \geq 4$ but not a subdivision of a planar 3-connected graph.

Let G be a planar 2-connected graph with $\Delta \geq 4$ but not and a subdivision of a planar 3-connected graph, and Γ be an arbitrary plane embedding of

G. Let $(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)$ be all pairs of vertices such that x_i and y_i, $1 \leq i \leq l$, are the leg vertices of a regular 2-legged cycle or the hand-vertices of a regular 2-handed cycle. If there is a plane embedding Γ' of G having a box-rectangular drawing, then the outer face $F_O(\Gamma')$ must contain all vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)$; otherwise, Γ' would have a 2-legged cycle containing no vertex on $F_O(\Gamma')$. Because after replacing the vertices of degree 4 or more by cycles like Fig. 2.5 in Γ', according to Lemma 4.1.1, 2-legged cycles will remain same and the total number of 2-legged cycles will also remain same. The graph is named as Φ after transformation from Γ'. If Γ' has a 2-legged cycle containing no vertex on $F_O(\Gamma')$, then Φ also has a 2-legged cycle containing no vertex on $F_O(\Phi)$, and hence by Lemma 2.5.5 and by (br1) of Lemma 2.2.2, Γ' does not have a box-rectangular drawing. Similarly if there is a plane embedding Γ' of G having a box-rectangular drawing, then the outer face $F_O(\Gamma')$ must contain two leg vertices of every 3-legged cycle.

Let p be the largest integer such that a number p of minimal 2-legged and maximal 2-handed cycles in Γ are independent with each other, and q be the largest integer such that a number q of minimal 3-legged and maximal 3-handed cycles in Γ are independent with each other. By (br2) of Lemma 2.2.2, $p \leq 2$ and $q \leq 4$; otherwise, Γ' does not have a box-rectangular drawing. Assume the worst case, that is, $p = 2$ and $q = 4$ in Γ. Independent minimal 3-legged or maximal 3-handed cycles in Γ are denoted by C_1, C_2, C_3, and C_4. Let $\{a_k, b_k, c_k\}$ be the set of leg vertices or hand vertices in C_k, for $k = 1, 2, 3$, and 4. We can choose two vertices from each C_1, C_2, C_3, or C_4 in 3 ways. The combinations are $\{(a_k, b_k), (b_k, c_k)$, and $(c_k, a_k)\}$, for $k = 1, 2, 3$ or 4. If we want to choose eight vertices from 4 cycles, C_1, C_2, C_3, and C_4, two vertices from each C_k, for $k = 1, 2, 3$ or 4, we can choose in 3 x 3 x 3 x 3 = 81 number of ways. The combinations are $S_1 = \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4)\}$, $S_2 = \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (b_4, c_4)\}$, $S_3 = \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (c_4, a_4)\}$, $\ldots$, and $S_{81} = \{(c_1, a_1), (c_2, a_2), (c_3, a_3), (c_4, a_4)\}$.

Let G be a planar 2-connected graph with $\Delta \geq 4$ but not and a subdivision of a planar 3-connected graph, and Γ be an arbitrary plane embedding of G as in Fig. 4.5(a). Let $(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)$ be all pairs of vertices such that x_i and y_i, $1 \leq i \leq l$, are the leg vertices of a regular 2-legged cycle or the hand-vertices of a regular 2-handed cycle, and $\{a_k, b_k, c_k\}$ be the set of leg vertices or hand vertices in C_k, for $k = 1, 2, 3$ and 4. A dummy vertex z is added in the outer face of Γ. Construct a graph ${\Gamma_j}^+$, for any $j = 1, 2, 3, \ldots,$ or 81, by adding dummy edges (x_i, z) and (y_i, z) for all indices $i, 1 \leq i \leq l$, and by adding eight dummy edges from z to all vertices in the set S_j. In this way we can get 81 number of graphs ${\Gamma_j}^+$, for every $j = 1, 2, 3, \ldots,$ and 81. ${\Gamma_1}^+$ and ${\Gamma_2}^+$ are two such kinds of graphs as illustrated in Fig. 4.5(b) and in Fig. 4.5(c) respectively. G may have a box-rectangular drawing, only if, any one of the graphs ${\Gamma_j}^+$, for $j = 1, 2, 3, \ldots,$ and 81, has a planar embedding such that z is embedded in the outer face. ${\Gamma_{2P}}^+$ in Fig. 4.5(c) is such a planar embedding of the graph ${\Gamma_2}^+$, but ${\Gamma_1}^+$ in Fig. 4.5(b) has no such a planar embedding. That is why, the planar graph G in Fig. 4.5(a) may have a box-rectangular drawing. Delete the dummy vertex z from ${\Gamma_{2P}}^+$. The graph is then called ${\Gamma_{2P}}^*$ as in Fig. 4.5(d). Lastly by Lemma 2.5.5 and by the approach used in Subsecion 2.5.3, we can test whether the plane graph ${\Gamma_{2P}}^*$ has a box-rectangular drawing and find the drawing if it exists. $D_{{\Gamma_{2P}}^*}$ is a box rectangular of the plane graph ${\Gamma_{2P}}^*$ as well as of the planar graph G, as illustrated in Fig. 4.5(e).

We thus have the following theorem on a planar 2-connected graph G with $\Delta \geq 4$ but not a subdivision of a planar 3-connected graph.

Theorem 4.1.6 *Let G be a planar 2-connected graph with $\Delta \geq 4$ which is not a subdivision of a planar 3-connected graph, and Γ be an arbitrary plane embedding of G. Assume G has at most two independent minimal 2-legged and maximal 2-handed cycles, and at most four independent minimal 3-legged and maximal 3-handed cycles; otherwise, G has no box-rectangular drawing. Construct different graphs ${\Gamma_j}^+$, for every $j = 1, 2, 3, \ldots,$ and 81, as above. Test whether any of the graphs ${\Gamma_j}^+$, for $j = 1, 2, 3, \ldots,$ and 81, has a planar embedding, such that z is embedded in the outer face; otherwise, G has no box-rectangular drawing. Say, ${\Gamma_{1P}}^+$ is such a planar embedding of the graph ${\Gamma_1}^+$. Construct ${\Gamma_{1P}}^*$ from ${\Gamma_{1P}}^+$ as above. The planar graph G has a box-rectangular drawing if and only if the plane graph ${\Gamma_{1P}}^*$ has a box-rectangular drawing.*

Theorem 4.1.6 immediately yields the following algorithm for a planar 2-connected graph G with $\Delta \geq 4$ but not a subdivision of a planar 3-connected graph. The algorithm can examine whether the planar graph G has a box-rectangular drawing or not, and can find the box-rectangular drawing of G if it exists.

Algorithm Planar-Box-Rect-Draw-2-Conn-but-Not-Subdiv-3-Conn-Δ-4-or-More (G)

{ Assume that G with $\Delta \geq 4$ has three or more vertices. Otherwise, one can easily examine whether G has a box-rectangular drawing or not, as illustrated in Fig. 4.4. }

begin

1 Let Γ be any plane embedding of G and let $(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)$ be all pair of vertices such that x_i and y_i, $1 \leq i \leq l$, are the leg vertices of a regular 2-legged cycle or the hand vertices of a regular 2-handed cycle in Γ. Let p be the largest integer such that a number p of minimal 2-legged and maximal 2-handed cycles in G are independent with each other, and q be the largest integer such that a number q of minimal 3-legged and maximal 3-handed cycles in G are independent with each other;

2 **if** $p > 2$ or $q > 4$ **then**

G has no box-rectangular drawing;

3 **else** { Assume the worst case, that is, $p = 2$ and $q = 4$ }

begin

4 Independent minimal 3-legged or maximal 3-handed cycles in Γ are denoted by C_1, C_2, C_3, and C_4. Let $\{a_k, b_k, c_k\}$ be the set of leg vertices or hand vertices in C_k, for $k = 1, 2, 3$, and 4. One can choose two vertices from each C_1, C_2, C_3, or C_4 in 3 ways. The combinations are $\{(a_k, b_k), (b_k, c_k)$, and $(c_k, a_k)\}$, for $k = 1, 2, 3$ or 4. If one wants to choose eight vertices from 4 cycles, C_1, C_2, C_3, and C_4, two vertices from each C_k, for $k = 1, 2, 3$ or 4, he can choose in 3 x 3 x 3 x 3 = 81 number of ways. The combinations are

$S_1 = \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4)\}$,

$S_2 = \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (b_4, c_4)\}$,

$S_3 = \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (c_4, a_4)\}$, $\ldots$, and

$S_{81} = \{(c_1, a_1), (c_2, a_2), (c_3, a_3), (c_4, a_4)\}$;

5 Construct a graph ${\Gamma_j}^+$, for any $j = 1, 2, 3, \ldots$, or 81, by setting a dummy vertex z at the outer face of Γ, by adding dummy edges (x_i, z) and (y_i, z) for all indices $i, 1 \leq i \leq l$, and by adding eight dummy edges from z to all vertices in the set S_j. In this way one can get

81 number of graphs ${\Gamma_j}^+$, for every $j = 1, 2, 3, \ldots$, and 81;
if no one of the graphs ${\Gamma_j}^+$, for $j = 1, 2, 3, \ldots$, and 81, has a planar embedding such that z is embedded in the outer face **then**
  G has no box-rectangular drawing;
else
  begin
    Say, ${\Gamma_{1P}}^+$ is such a planar embedding of the graph ${\Gamma_1}^+$;
    Construct the graph ${\Gamma_{1P}}^*$ by deleting the vertex z from ${\Gamma_{1P}}^+$;
    By Lemma 2.5.5 and by the approach used in Subsecion 2.5.3, one can test whether the plane graph ${\Gamma_{1P}}^*$ has a box-rectangular drawing and find the drawing $D_{{\Gamma_{1P}}^*}$ if it exists; $\{D_{{\Gamma_{1P}}^*}$ is the box-rectangular drawing of the plane graph ${\Gamma_{1P}}^*$ as well as of the planar graph G. $\}$
  end
end
end

We now have the following theorem on **Algorithm Planar-Box-Rect-Draw-2-Conn-but-Not-Subdiv-3-Conn-Δ-4-or-More**.

Theorem 4.1.7 *Let G be a planar 2-connected graph G with $\Delta \geq 4$ which is not a subdivision of a planar 3-connected graph. Then Algorithm Planar-Box-Rect-Draw-2-Conn-but-Not-Subdiv-3-Conn-Δ-4-or-More examines in linear time whether G has a box-rectangular drawing and finds a box-rectangular drawing of G if it exists.*

Proof. One can find all pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)$, and all sets $\{(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3)$, and $(a_4, b_4, c_4)\}$ of vertices at Γ in linear time using a method similar to the algorithm in [RNN00] to find 2-legged cycles, and minimal 3-legged or maximal 3-handed cycles. Combinations $S_1, S_2, S_3, \ldots, S_{81}$ can be got in constant time. One can examine the planarity of ${\Gamma_j}^+$, for any j = 1, 2, 3, ..., or 81, and find the planar embedding ${\Gamma_{jP}}^+$ of ${\Gamma_j}^+$, for that j, in linear time [NC88], if it exists,. The method stated in Subsection 2.5.3 takes linear time to test whether ${\Gamma_{jP}}^*$ has a box-rectangular drawing and to find the drawing if it exists [RNN00]. Thus **Algorithm Planar-Box-Rect-Draw-2-Conn-but-Not-Subdiv-3-Conn-Δ-4-or-More** takes linear time in total. *Q.E.D.*

A planar graph G with maximum degree 4 may have a rectangular drawing. But no algorithm has yet been developed to test whether the planar graph G with maximum degree 4 has a rectangular drawing and for finding out the drawing if it exists. On the other hand Rahman et al. [RNN00] gave a necessary and sufficient condition for a plane graph with maximum degree 4 or more to have box-rectangular drawing and they also developed a linear-time algorithm for finding out the drawing if it exists. In this thesis we derived a necessary and sufficient condition for a planar graph G with maximum degree 4 or more to have a box-rectangular drawing, and we also developed a linear-time algorithm for finding out the drawing if it exists.

4.2 Algorithm

In this section we formally describe **Algorithm Planar-Box-Rectangular-Draw-2-Connected-Δ-4-or-More**, that is, for the general case to examine whether a 2-connected planar graph G with $\Delta \geq 4$ has a box-rectangular drawing or not and to find out the drawing if it exists. The algorithm is as follows.

Algorithm Planar-Box-Rectangular-Draw-2-Connected-Δ-4-or-More (G)

{ Assume that G with $\Delta \geq 4$ has three or more vertices. Otherwise, one can easily examine whether G has a box-rectangular drawing or not, as illustrated in Fig. 4.4. }

begin

Let Γ be an arbitrary plane embedding of G;

Remove the vertices of degree 2 in Γ, and call the graph Ξ;

Examine whether Ξ is 3-connected;

if Ξ is 3-connected **then**

{ G is a subdivision of a planar 3-connected graph.}

Examine by Algorithm **Subdivision-Draw-3-Connected-Δ-4-or-More** whether G has a box-rectangular drawing and find a box-rectangular drawing of G if it exists;

else { G is a planar 2-connected graph.}

Examine by Algorithm **Planar-Box-Rect-Draw-2-Conn-but-Not -Subdiv-3-Conn-Δ-4-or-More** whether G has a box-rectangular drawing and find a box-rectangular drawing of G if it exists;

end

We now have the following theorem on **Algorithm Planar-Box-Rectangular-Draw-2-Connected-Δ-4-or-More**.

Theorem 4.2.1 *Let G be a planar biconnected graph with $\Delta \geq 4$. Then Algorithm Planar-Box-Rectangular-Draw-2-Connected-Δ-4-or-More examines in linear time whether G has a box-rectangular drawing and finds a box-rectangular drawing of G if it exists.*

Proof. One can remove the vertices of vertices of degree 2 from Γ in linear time and call the graph Ξ. Using the algorithm by Hopcroft and Tarjan, one can decompose a graph Ξ into 3-connected components [HT73]. If Ξ has exactly only one component, then G is a subdivision of a 3-connected graph, and by Theorem 4.1.4, **Algorithm Subdivision-Draw-3-Connected-Δ-4-or-More** takes linear time. Otherwise G is a 2-connected graph, and by Theorem 4.1.7, **Algorithm Planar-Box-Rect-Draw-2-Conn-but-Not-Subdiv-3-Conn-Δ-4-or-More** takes also linear time. Thus **Algorithm Planar-Box-Rectangular-Draw-2-Connected-Δ-4-or-More** takes linear time in total.

Theorems 4.1.3 and 4.1.6 immediate the correctness of **Algorithm Planar-Box-Rectangular-Draw-2-Connected-Δ-4-or-More**. $\mathcal{Q.E.D.}$

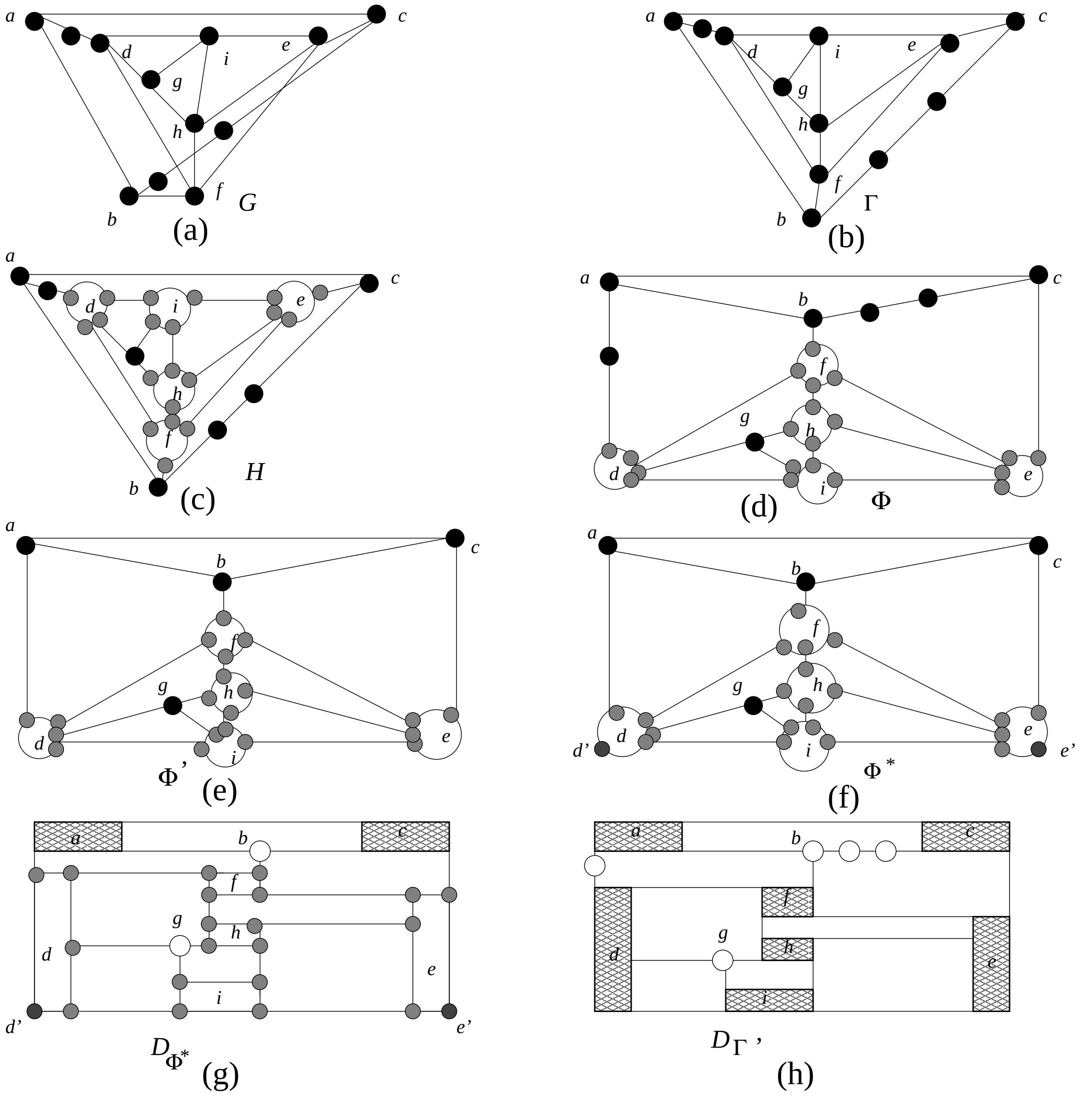


Figure 4.3: Illustration for a box-rectangular drawing of a subdivision of a planar 3- connected graph G with $\Delta \geq 4$.

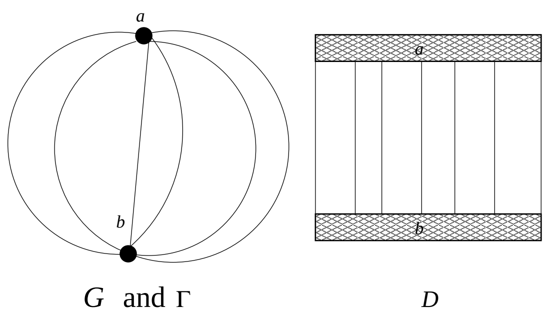


Figure 4.4: A planar graph with $\Delta \geq 4$ has at most two vertices, and its corresponding box-rectangular drawing.

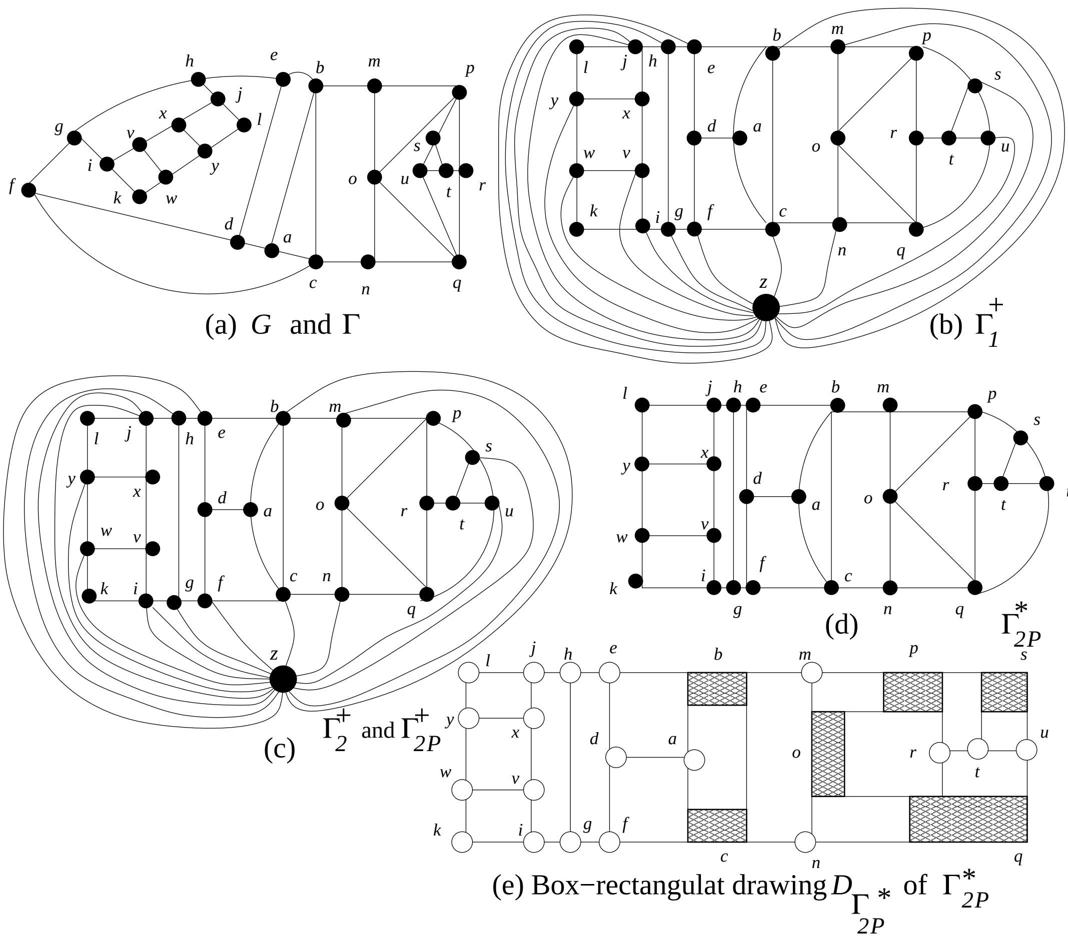


Figure 4.5: Illustration for a box-rectangular drawing of a biconnected graph G with $\Delta \geq 4$ but not a subdivision of a 3-connected graph .

Chapter 5

Conclusion

In this thesis we addressed the problem of finding box-rectangular drawings of planar graphs. We gave necessary and sufficient conditions for the different cases of planar graphs to have box-rectangular drawings, and then we developed linear-time algorithms for finding out the drawings if drawings exist. In this regard we first considered the case for planar graphs of maximum degree 3. Then we considered the general case where graphs have vertices of maximum degree 4 or more. Our first linear-time algorithm (described in Chapter 3) was for examining whether a planar graph G with $\Delta \leq 3$ has a box-rectangular drawing, and to find the drawing if it exists. Then we showed that, one can determine whether a subdivision of a planar 3-connected graph G with $\Delta \geq 4$ has a box-rectangular drawing or not by investigating whether the planar graph H (described in Chapter 4) has a box-rectangular drawing or not, which leads to a linear-time algorithm. We also derived a necessary and sufficient condition that runs in linear time for a planar 2-connected graph with $\Delta \geq 4$ but not a subdivision of a 3-connected graph to have a box-rectangular drawing. We developed a linear-time algorithm for finding out the box-rectangular drawing of a planar 2-connected graph with $\Delta \geq 4$ but not a subdivision of a planar 3-connected graph, if it exists. We gave a technique to determine whether a planar graph G with $\Delta \geq 4$ is cyclically 4-edge connected or not.

The problem of finding box-rectangular drawings of planar graphs is motivated by both theoretical interest and practical applications. We showed that, all the plane embeddings of a subdivision of planar 3-connected cubic graph G, that are cyclically 4-edge-connected have box-rectangular drawings, leads to a linear-time algorithm, where Rahman et al. [RNG04] showed that, any arbitrary plane embedding Γ of a subdivision of planar 3-connected cubic graph G must satisfy three necessary and sufficient conditions to have a rectangular drawing. In all other cases for the planar graph G of maximum degree 3 it was shown that, one needs to examine only a fixed number of embeddings of G to determine whether G has a box-rectangular drawing or not. A planar graph G with maximum degree 4 may have a rectangular drawing. But no algorithm has yet been developed to test whether the planar graph G with maximum degree 4 has a rectangular drawing, and for finding out the drawing if it exists. On the other hand Rahman et al. [RNN00] gave a necessary and sufficient condition for a plane graph with maximum degree 4 or more to have a box-rectangular drawing, and they also developed a liner-time algorithm for finding out the drawing if it exists. In this thesis we derived a necessary and sufficient condition for a planar graph G with maximum degree 4 or more to have a box-rectangular drawing, and we also developed a linear-time algorithm for finding out the drawing if it exists.

The following is a brief list of future works related to our results presented in this thesis.

- Applications of box-rectangular drawings in civil structural designs and in new fields may be a good research area.
- Concept of a box-rectangular drawing may be implemented in convex drawing.
- In this thesis we have shown that, at least 81 number of embeddings of a planar 2-connected graph with $\Delta \geq 4$ are required to be checked to take a decision whether the planar graph has a box-rectangular drawing or not. This is for the worst case. The planar graph has a box-rectangular drawing if and only if any of the embeddings has a box-rectangular drawing. In future one can try to minimize the number of embeddings required to be checked to take the decision whether the planar graph has a box-rectangular drawing or not.

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