A STUDY ON NIL AND NILPOTENT RINGS AND MODULES

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SUBMITTED BY

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ABSTRACT

A submodule X of a right R-module M is called a nilpotent submodule of M if I_X is a right nilpotent ideal of S and X is a nil submodule of M if I_X is a right nil ideal of S. By definition, a nilpotent submodule is a nil submodule. It is seen that X is a fully invariant nilpotent submodule of M if and only if I_X is a two-sided nilpotent ideal of S. Modifying the structure of nil and nilpotent right ideals over associative arbitrary rings, present study develops some properties of nil and nilpotent submodules over associative endomorphism rings. Some characterizations of nil and nilpotent submodules over associative endomorphism rings are also investigated in the present study.

CONTENTS

CHAPTER I

INTRODUCTION

The word algebra derives from the word al-jabr which appears in the little of a book written in the 9^{th} Century by the Persian mathematician Mohammed Al-Khowarizmi. This book, in a Latin translation, had great influence in Europe. Its concerns with problems equivalent to those of solving polynomial equations, especially those of degree 2, led to the word algebra eventually becoming synonymous with the science of equations. This state of affairs persisted into the 9^{th} Century, Serret, in 1894, observing that, "Algebra is, properly speaking, the analysis of equations".

Background and present state of the problem

Ring theory is an important part of algebra. It has been widely used in Electrical and Computer Engineering [1]. Historically, some of the major discoveries in ring theory have helped shape the course of development of modern abstract algebra. Modern ring theory begins when Wedderburn in 1907 proved his celebrated classification theorem for finite dimensional semi-simple algebras over fields. Twenty years later, E. Noether and E. Artin introduced the ascending chain condition and descending chain condition as substitutes for finite dimensionality.

We know that Module theory appeared as a generalization of theory of vector spaces over a field. Every field is a ring and every ring may be considered as a module. Köthe [2] first introduced and investigated the notion of nil ideals in commutative ring theory. Amitsur [3] investigated radicals of polynomial rings. There were some historical notes on nil ideals and nil radicals due to Amitsur [4]. Radicals of graded rings were introduced and investigated by Jespers et al. [5]. There is another notion of radicals in nil and Jacobson radicals in graded rings due to Smoktunowicz [6]. Puczylowski ([7],[8]) investigated some results concerning radicals of associative rings related to Köthe's nil ideal problems. Chebotar et al. [9] and Klein [10] investigated some results concerning nil ideals of associative rings which do not necessarily have identities. Sanh et al. [11] introduced the notion of fully invariant submodules and characterized their properties.

In the light of the above literature, it is pointed out that many authors have published their works on nil ideals and nil radicals over commutative and associative rings. Also it was found that Andrunakievich's Chain is equivalent to Köthe's problem.

In this thesis, Chapter I deals with the early brief history of nil and nilpotent rings and modules. All essential basic definitions, examples and their properties are given in Chapter II. In Chapter III, we described some properties of nil and nilpotent ideals in associative arbitrary rings. Some properties of nil and nilpotent ideals and modules are investigated in associative arbitrary rings in Chapter IV.

CHAPTER II

BASIC KNOWLEDGE

Overview

Throughout this thesis, all rings are associative with identity and all modules are unitary right R-modules. We denote by R an arbitrary ring and by mod-R, the category of all right R-modules. The notation M_R indicates the right R-module M which when $1 \in R$ is assumed to be unital (i.e. to have the property that $1.m = m$ for any $m \in M$). The set $Hom(M, N)$ denotes the set of right R-module homomorphism between two right R-modules M and N and if further emphasis is needed that the notation $Hom_R(M, N)$ is used. The kernel of any $f \in Hom_R(M, N)$ is denoted by $\ker(f)$ and the image of f by $\mathrm{Im}(f)$. In particular, $End_R(M)$ denotes the ring of endomorphism of a right R-module M.

A submodule X of M is indicated by writing $X \subsetneq M$. Also, $I \subsetneq M_R$ means that I is a right ideal of R and $I \subsetneq_R R$ that I is a left ideal. The notation $I \subsetneq R$ is reserved for two-sided ideal.

2.1 Preliminaries

Before dealing with deeper results on the structure of rings with the help of module theory, we provide first some essential elementary definitions, examples and properties.

2.1.1 Definition

Let R be a nonempty set and let $+$ and $\cdot$ denote two binary operations on R which we refer to as addition and multiplication, respectively. Then $(R, +, \cdot)$ is called a ring if the following conditions hold:

(i) $\langle R, + \rangle$ is an abelian group.
(ii) multiplication is associative, that is, $(a \cdot b) \cdot c = a \cdot (b \cdot c),\ \forall\, a, b, c \in R$.
(iii) multiplication is distributive over addition, that is,

$a \cdot (b + c) = a \cdot b + a \cdot c$ (left distributive law)

and $(a + b) \cdot c = a \cdot c + b \cdot c$ (right distributive law) , $\forall\, a, b, c \in R$.

Example

(i) The set of all integers $\mathbb{Z}$ is a ring under addition and multiplication. Similarly, the sets $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ of rational numbers, real numbers and complex numbers, respectively are rings under usual addition and multiplication.

(ii) The set R of all matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c, d are real numbers, with matrix addition and multiplication, is a ring.

(iii) If $a, b, c \in \mathbb{Z}(\text{or}, \mathbb{Q}, \mathbb{R}, \mathbb{C})$ then the system $(R, +, \cdot)$ is not a ring, where $R = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \right\}$, because $A, B \in R$ such that $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & 0 \end{bmatrix}$ imply $AB = \begin{bmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 \\ c_1 a_2 & c_1 b_2 \end{bmatrix}$.

Here AB is not a matrix of the form $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ and therefore, $AB \notin R$.

2.1.2 Definition

Let the set R contains only the zero element, that is, $R = \{0\}$, then $(R, +, \cdot)$ is called a zero ring.

Example

The system $(R, +, \cdot)$ is a zero ring, where $R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

2.1.3 Definition

A ring R is called a ring with unity if there exists an element $0 \neq 1 \in R$ such that $a.1 = 1.a = a, \forall a \in R$, 1 is called the multiplicative identity or unity.

Example

(i) The sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are all rings with unity.

(ii) Let $E = \{\cdots, -4, -2, 0, 2, 4, \cdots\}$ be a set of even integers. Then E is a ring without unity.

2.1.4 Definition

A ring R is called a ring with zero divisor if there exist two elements a, b in R such that $ab = 0$ where $a \neq 0$ and $b \neq 0$.

Example

(i) The ring $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ is a ring with zero divisors.

For, let, $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in R$, $B = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \in R$, where $a \neq 0$ and $b \neq 0$,

here $AB = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$, where $A \neq 0$ and $B \neq 0$.

(ii) In the ring of residue classes modulo six

$$R = \mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \ \bar{5}\}.$$

Here $\bar{2}.\bar{3} = \bar{0}$ but $\bar{2} \neq \bar{0}$ and $\bar{3} \neq \bar{0}$.

2.1.5 Definition

A ring R is called a ring without zero divisor if the product of two nonzero elements of R is not zero, that is, if $ab = 0$ then $a = 0$ or $b = 0$, for $a, b \in R$.

Example:

All the sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are rings without zero divisors.

2.1.6 Definition

Let R be a ring. Then a nonempty subset S of R is said to be a subring, if under the addition and multiplication operations in R, S itself forms a ring.

Example

(i) The set of even integers $E = \{\cdots, -4, -2, 0, 2, 4, \cdots\}$ is a subring of the ring of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots.\}$.

(ii) $\mathbb{Z}$ is a subring of $\mathbb{Q}$ and $\mathbb{Q}$ is subring of $\mathbb{R}$.

2.1.7 Definition

A nonempty subset S of a ring R is called a left ideal of R if

(i) S is a subring of R,

(ii) $rs \in S$ for all $s \in S$ and $r \in R$.

2.1.8 Definition

A nonempty subset S of a ring R is called a right ideal of R if

(i) S is a subring of R,

(ii) $sr \in S$ for all $s \in S$ and $r \in R$.

2.1.9 Definition

A nonempty subset S of a ring R is called an ideal (two-sided ideal) of R if

(i) S is a subring of R,

(ii) $rs \in S$ and $sr \in S$ for all $s \in S$ and $r \in R$.

Example

(i) The subring $E = \{\cdots,-4,-2,0,2,4,\cdots\}$ of even integers is an ideal of the ring of integers $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots.\}$.

(ii) Let $R = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z}\right\}$ is a ring. Then $S = \left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} : a,b \in \mathbb{Z}\right\}$ is a left ideal, but $T = \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a,b \in \mathbb{Z}\right\}$ is not a left ideal of R.

2.1.10 Definition

A ring R is called a simple ring if it has no proper ideal.

Example

Each of the rings $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ is a simple ring.

2.1.11 Definition

An ideal S of a ring R is called the principal ideal of R if the ideal S is generated by a single element 'a' of S and we write $S = (a)$ or $S = \langle a \rangle$.

Example

In the ring $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots.\}$, the ideal $S = (5) = \{\ldots,-10,-5,0,5,10,\ldots\}$ is a principal ideal.

2.1.12 Definition

A nonempty ideal I of a ring R, where $I \neq R$, is called a maximal ideal of R, if there exist no proper ideal of R containing I, that is, I will be maximal ideal if it is impossible to find another ideal which lies between I and the whole ring R.

Example

Consider the ring $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots.\}$ of integers.
Choose $I = (6) = \{\ldots, -12, -6, 0, 6, 12, \ldots\}$ and $J = (3) = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$.
Here I is not a maximal ideal as there exists an ideal J lying between I and $\mathbb{Z}$.

But if we choose $I = (5) = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$, then I is a maximal ideal, because the only ideal containing I is $\mathbb{Z}$ itself.

2.1.13 Definition

An ideal I of a ring R is called a prime ideal of R if $ab \in I$ implies $a \in I$ or $b \in I$.

Example

In the ring $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots.\}$, the ideal $I = (10) = \{\ldots, -20, -10, 0, 10, 20, \ldots\}$ is not a prime ideal since $30 = 6.5 \in \mathrm{I}$ but neither 6 nor 5 belongs to I.

But $I = (5) = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$ is a prime ideal, since $10 = 5.2 \in \mathrm{I} \Rightarrow 5 \in \mathrm{I}$ but $2 \notin \mathrm{I}$.

Again a prime ideal in a ring R is any proper ideal P of R such that, whenever I and J are ideals of R with $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$. An ideal I of a ring R is called strongly prime if for any $x, y \in R$ with $x\,y \in I$, then either $x \in I$ or $y \in I$. A prime ring is a ring in which 0 is a prime ideal or a ring R is called a prime ring if there are no nonzero two-sided ideals I and J of R such that $IJ = 0$.

2.1.14 Proposition [12]

For a proper ideal P in a ring R, the following conditions are equivalent:

(a) P is a prime ideal.
(b) If I and J are any ideals of R properly containing P, then $I\,J \not\subset P$.
(c) $R\,/\,P$ is a prime ring.
(d) If I and J are any right ideals of R such that $I\,J \subseteq P$, then either $I \subseteq P$ *or* $J \subseteq P$.
(e) If I and J are any left ideals of R such that $I\,J \subseteq P$, then either $I \subseteq P$ *or* $J \subseteq P$.
(f) If $x, y \in R$ with $x\,R\,y \subseteq P$, then either $x \in P$ or $y \in P$.
(g) For any $x \in R$ and any ideal I of R such that $x\,I \subset P$, then either $x\,R \subset P$ or $I \subset P$.

By induction, we know that if P is a prime ideal in a ring R and $J_1, \ldots, J_n$ are right ideals of R such that $J_1, \ldots, J_n \subset P$, then some $J_i \subset P$. A maximal ideal in a ring is meant a maximal proper ideal, i.e., an ideal which is maximal in the collection of proper ideals.

2.1.15 Theorem

Let R be a commutative ring. Then the following conditions are equivalent:
(a) A maximal ideal is prime.
(b) An ideal P is prime if and only if R / P is an integral domain.
(c) An ideal M is maximal if and only if R/M is a field.

2.1.16 Definition

A minimal prime ideal in a ring R is any prime ideal of R that does not properly contain any other prime ideals. For instance, if R is a prime ring, then 0 is the unique minimal prime ideal of R.

Examples

(i) In a commutative artinian ring, every maximal ideal is a minimal prime ideal.

(ii) In an integral domain, the only minimal prime ideal is the zero ideal.

2.1.17 Definition

A semiprime ideal in a ring R is any ideal of R which is an intersection of prime ideals. A semiprime ring is any ring in which 0 is a semiprime ideal. Note that an ideal P in a ring R is semiprime if and only if R/P is a semiprime ring. The intersection of any finite list $p_1\mathbb{Z}, \dots, p_k\mathbb{Z}$ of prime ideals, where $p_1, \dots, p_k$ are distinct prime integers, is the ideal $p_1\mathbb{Z}, \dots, p_k\mathbb{Z}$. Hence the nonzero semiprime ideals of $\mathbb{Z}$ consist of the ideals $n\mathbb{Z}$, where n is any square-free positive integer including $n = 1$.

2.1.18 Proposition [12]

If R is a commutative ring, then
(a) The intersection of all prime ideals of R is precisely the set of nilpotent element of R.
(b) For every ideal I of R, the intersection $r \in R$ such that $r^n \in I$ for some positive integer n.
(c) The ring R is semiprime if and only if it contains no nonzero nilpotent elements.

2.1.19 Corollary [12]

For an ideal I in a ring R, the following conditions are equivalent:

(a) I is a semiprime ideal.
(b) If J is any ideal of R such that $J^2 \leq I$, then $J \leq I$.
(c) If J is any right ideal of R such that $J^2 \leq I$, then $J \leq I$.
(d) If J is any left ideal of R such that $J^2 \leq I$, then $J \leq I$.

Note

R is a prime ring if and only if *0* is a prime ideal. *R* is a semiprime ring if and only if *0* is a semiprime ideal.

2.2 Properties of Ring

2.2.1. Definition

An element x of a ring R is called nilpotent if $x^n = 0$ for some positive integer n.

The element 0 (zero) of a ring is trivially nilpotent.

Example

The nilpotent elements of the ring $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$ of integers modulo 8 are $\bar{0}, \bar{2}, \bar{4}, \bar{6}$, since $\bar{2}^3 = \bar{0},\ \bar{4}^2 = \bar{0},\ \bar{6}^3 = \bar{0}$.

2.2.2 Definition

An ideal I in a ring R is said to be nilpotent if for some positive integer n, $I^n = (0)$.
Again, an ideal I is called left t-nilpotent if for any sequence $x_1, x_2, \ldots, x_n$ of elements of I, there exists $k \in \mathbb{N}$ such that $x_1 x_2 \ldots x_k = 0$.

Example

We have $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$, then the ideal $I = 2\mathbb{Z}_8 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ in $\mathbb{Z}_8$ is nilpotent, since $I^3 = (0)$.

2.2.3 Definition

An ideal I in a ring R is called a nil ideal if each element in I is nilpotent, that is, for each $a \in I$ there is some positive integer n such that $a^n = 0$.

Example

The ideal $I = 2\mathbb{Z}_8 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is a nil ideal, since every element in I is nilpotent:
$\bar{2}^3 = \bar{0},\ \bar{4}^2 = \bar{0},\ \bar{6}^3 = \bar{0}$.

Remarks

Every nilpotent ideal is a nil ideal, since if I is a nilpotent ideal, then there exists a positive integer n such that $I^n = (0)$. So for each $a \in I, a^n \in I^n = (0)$ implies that $a^n = 0$. Hence I is a nil ideal. But the converse is not true.

The notion of a nil ideal has a connection with that of a nilpotent ideal and in some classes of rings, the two notions coincide. If an ideal is nilpotent, it is of course nil. There are two main barriers for nil ideals to be nilpotent.

1. There need not be an upper bound on the exponent required to annihilate the elements. Arbitrarily high exponents may be required.
2. The product of n nilpotent elements may be nonzero for arbitrarily high n.

Both of these barriers must be avoided for a nil ideal to qualify as nilpotent.

2.2.4 Definition

A ring R is called an integral domain if it is a commutative ring with unity and without zero divisors.

Example

(i) The rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are integral domains.

(ii) The ring of even integers is not an integral domain since it does not contain the unit element.

2.2.5 Definition

A ring R is called a field if it is a commutative ring with unity and every nonzero element in R has a multiplicative inverse.

Example

(i)The sets $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields with respect to addition and multiplication.

(ii) The set $\mathbb{Z}$ of all integers is not a field, because all nonzero elements have not multiplicative inverses except 1and -1.

2.2.6 Definition

A nonempty set V is called a vector space over a field F, if for any $a, b \in F$ and $v, w \in V$, the following conditions are satisfied:

(1) V is an abelian additive group.
(2) $a \in F, v \in V \Rightarrow av \in V$, that is, V is closed under scalar multiplication.
(3) the following four laws of scalar multiplication are satisfied:

(i) $a(v+w) = av + aw$
(ii) $(a+b)v = av + bv$
(iii) $a(bv) = (ab)v$
(iv) $1v = v$, where 1 is the unity of F.

The elements of V are called vectors and the elements of F are called scalars. A vector space V over a field F is denoted by $V(F)$.

Example

$\mathbb{R}(\mathbb{R}), \mathbb{C}(\mathbb{R})$ and $\mathbb{C}(\mathbb{C})$, where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively, are vector spaces with respect to usual addition and multiplication.
But $\mathbb{R}(\mathbb{C})$ is not a vector space, because if $a \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ then $a\alpha \notin \mathbb{R}$.
For example, if $a = 3$ and $\alpha = 5 + 3i$, then $a\alpha = 3(5 + 3i) = 15 + 9i \notin \mathbb{R}$.

2.3 Module and different kind of submodules

2.3.1 Definition

Let R be a ring and M be an additive abelian group. Then M is called a right R-module if the mapping $M \times R \to M$ satisfies the following conditions:

(i) $(a + b)r = ar + br$
(ii) $a(r + s) = ar + as$
(iii) $a(rs) = (ar)s, \forall\, r, s \in R,$ and $a, b \in M$

Similarly, we can define left R-module by operating to the left side of M.

2.3.2 Definition

If M is a left R-module and also a right R-module then M is called an R-module or simply a module over R.

Example

(i) Every ring R is an R-module over itself.
(ii) Every additive group is a module over the ring of integers.
(iii) Let R be a ring and M a left ideal of R. Then M is an R-module.
(iv) The polynomial ring $R[x]$ over a ring R is an R-module.

2.3.3 Definition

Let M and N be two R-modules. If $N \subseteq M$, then N is called an R-submodule of M or simply, a submodule of M.

If N is a submodule of an R-module M, then $\forall r \in R$ and $\forall n \in N$, we have $rn \in N$.

Example

Let $A = \{\dots\dots, -12, -8, -4, 0, 4, 8, 12, \dots\dots.\}$ and $E = \{\dots\dots, -6, -4, -2, 0, 2, 4, 6, \dots\dots\}$, then A is a submodule of E over $\mathbb{Z}$.

2.3.4 Definition

Let R be a ring and S be a left ideal of R. Let $M = \{a + S : a \in R\}$ be the set of all cosets of S in R. Then M is an R-module with the compositions, defined by

$$(a + S) + (b + S) = (a + b) + S \text{ and}$$
$$r(a + S) = ra + S.$$

This module M is known as quotient module or factor module of R by S, which is usually written as R/S or $R - S$.

2.3.5 Definition

A submodule A of a right R-module M is called essential or large in M if for any nonzero submodule U of M, $A \cap U \neq 0$. If A is essential in M , we denote $A \subset_{>}^{*} M$.

A right ideal I of a ring R is called essential if it is essential in R.

For any right R-module M, we always have $M \subset_{>}^{*} M$. Any finite intersection of essential submodules of M is again essential in M, but it is not true in general.

Example

Consider the ring $\mathbb{Z}$ of integers. Every nonzero ideal of $\mathbb{Z}$ is essential in $\mathbb{Z}$, but the intersection of all ideals of $\mathbb{Z}$ is 0 which is not essential in $\mathbb{Z}$.

2.3.6 Proposition

In $\mathbb{Z}$, every nonzero ideal is essential.

Proof:

Let I be a nonzero ideal $\mathbb{Z}$. Then there exists $m \in \mathbb{Z}$ such that $I = m\mathbb{Z}$. For any nonzero ideal $J \subset \mathbb{Z}$, we can find an $n \in \mathbb{Z}$ such that $J = n\mathbb{Z}$ .Thus $I \cap J = m\mathbb{Z} \cap n\mathbb{Z} = mn\mathbb{Z}$, so $0 \neq mn \in I \cap J$ and so $I \cap J \neq 0$. Therefore, $I \subset_{>}^{*} \mathbb{Z}$.

2.3.7 Proposition

Let M be a right R-module. Then for any $A \subset_{>} M$, $A \subset_{>}^{*} M$ if and only if $0 \neq m \in M, \exists\, r \in R$ such that $0 \neq mr \in A$.

Proof:

Assume that $A \subset_{>}^{*} M$. Choose $0 \neq m \in M$. Then $mR \neq 0$ and so $A \cap mR \neq 0$. Then there exists $0 \neq x \in A \cap mR$. This means that $0 \neq x \in A$ and there exists $r \in R$ such that $x = mr$. Therefore, $0 \neq x = mr \in A$.

Conversely, let U be a nonzero submodule of M. Choose $0 \neq m \in U$. By hypothesis, there exists $r \in R$ with $mr \neq 0$ and $mr \in A$. But then since $mr \in U$, we have $mr \neq 0$ and $mr \in A \cap U$. Hence $A \subset_{>}^{*} M$.

2.3.8 Proposition

For any $M \in$ Mod-R, let $A \subset_{>} B \subset_{>} M$. If $A \subset_{>}^{*} M$, then (i) $A \subset_{>}^{*} B$, and (ii) $B \subset_{>}^{*} M$.

Proof:

(i) Let $U \subset_{>} B$ be such that $U \neq 0$. Then U is a submodule of M, since $A \subset_{>}^{*} M$, $U \cap A \neq 0$. Hence $A \subset_{>}^{*} B$.

(ii) Let $U \subset_{>} M$ be such that $U \neq 0$. Then $0 \neq A \cap U \subseteq B \cap U$, because $A \cap U \neq 0$ and so $B \subset_{>}^{*} M$.

2.3.9 Proposition

Let A and B be essential submodules in M_R. Then $A \oplus B \subset_>^* M$ and $A \cap B \subset_>^* M$.

Proof:

Let $U \subset_> M$ be such that $U \neq 0$. Then $U \cap (A \cap B) = (U \cap A) \cap B \neq 0$.

Hence $A \cap B \subset_>^* M$. We have $A \subset_> A \oplus B \subset_> M$ and $A \subset_>^* M$ implying that $A \oplus B \subset_>^* M$.

Note

Every nonzero submodule of M is essential in M, i.e., a nonzero submodule A of M is called essential in M if A has nonzero intersection with any nonzero submodule of M.

2.3.10 Definition

An R-module M is said to be a cyclic module generated by an element $a \in M$, if each element $m \in M$ is expressible as $m = ra$ for some $r \in R$. The element a in this case is called the generator of M and we write $M = (a)$.

Example

Let M be a unital R-module and for a fixed element $a \in M$, let $W = \{ra : r \in R\}$. Then W is a cyclic submodule of M generated by a.

Remark

For a ring R, the ring of integers, a cyclic R-module is nothing more than a cyclic group.

2.3.11 Theorem

Any unital, irreducible R-module is cyclic.

Proof:

Let M be a unital, irreducible R-module. Then the only submodules of M are M and $\{0\}$.

If $M = \{0\}$, it is clearly cyclic.

So, let $M \neq \{0\}$, choose $a \in M$ such that $a \neq 0$.

Let $W = \{ra: r \in R\}$, then W is submodule of M.

Also $a = 1.a \in W$, since $1 \in R$.

So $W \neq \{0\}$. Thus W is a nonzero submodule of M. Therefore, $W = M$, since the only nonzero submodule of M is M. It is also clear that W is a cyclic module generated by a. Hence M is cyclic. Thus the theorem is proved.

2.3.12 Definition

If $M_1$and M_2 are two submodules of an R-module M, then their linear sum, denoted by $M_1 + M_2$, is defined as $M_1 + M_2 = \{a + b: a \in M_1, b \in M_2\}$.

2.3.13 Theorem

If M_1 and M_2 are two submodules of an R-module M, then $M_1 + M_2$ is also a submodule of M.

Proof:

Let $c = a_1 + b_1$ and $d = a_2 + b_2$ be any two elements of $M_1 + M_2$. Then $a_1, a_2 \in M_1$ and $b_1, b_2 \in M_2$ and we have

$$c - d = (a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2).$$

Since $M_1$and M_2 are additive subgroups of M, $a_1, a_2 \in M_1$ implies $a_1 - a_2 \in M_1$ and $b_1, b_2 \in M_2$ implies $b_1 - b_2 \in M_2$.

Therefore, $(a_1 - a_2) + (b_1 - b_2) \in M_1 + M_2$.
So, $M_1 + M_2$ is an additive subgroup of M.

Now let $r \in R$ and $c = a_1 + b_1 \in M_1 + M_2$, then
$rc = r(a_1 + b_1) = ra_1 + rb_1$. Since M_1 is a submodule of M, for $r \in R$, $a_1 \in M_1$ we have $ra_1 \in M_1$.
Similarly, $rb_1 \in M_2$.

Thus $ra_1 + rb_1 \in M_1 + M_2$.

So, $r \in R, c \in M_1 + M_2$ implies $rc \in M_1 + M_2$.

Hence $M_1 + M_2$ is a submodule of M.

2.3.14 Definition

Let M be an R-module. Then M is said to satisfy the descending chain condition (DCC) on submodules if whenever $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ for submodules M_i of M, then there exists an integer N such that $M_k = M_N$ for all $k \geq N$.

2.3.15 Definition

Let M be an R-module. Then M is said to satisfy the ascending chain condition (ACC) on submodules if whenever $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$ for submodules M_i of M, then there exists an integer N such that $M_k = M_N$ for all $k \geq N$.

2.3.16 Definition

The ring *R* is semisimple if *R* is semisimple as a right *R*-module. A right ideal of *R* which is simple as an *R*-module is called a minimal right ideal. A semisimple ring is thus a direct sum of minimal right ideals, and every simple module is isomorphic to a minimal right ideal of *R*.

Note

The prime radical of M is the intersection of all prime submodules of M and is denoted by $P(M)$. The prime radical of a ring R is the intersection of all prime ideals of R and is denoted by $P(R)$.

2.3.17 Lemma [13]

For a ring *R* with identity, the following conditions are equivalent:

(b) *0* is the only nilpotent ideal in *R;*
(a) *R* is a semiprime ring (*i.e., P(R) =0*);
(c) For ideals *I, J* in *R* with *I J = 0* implies $I \cap J = 0$.

Note

In noetherian rings, all nil one-sided ideals are nilpotent. If R is a nonzero ring, it has no prime ideal and so, *P(R) = R*. If *R* is nonzero, then it has at least one maximal ideal. A ring is semiprime if and only if *P(R) = 0*. In any case, $P(R)$ is the smallest semiprime ideal of R, and because *P*(*R*) is semiprime, it contains all nilpotent one-sided ideals of *R*.

2.3.18 Corollary [14]

(a) Every one-sided or two-sided nilpotent ideal is a nil ideal.
(b) The sum of two nilpotent right, left or two-sided ideals is again nilpotent.
(c) If R_R is noetherian, then every two-sided nil ideal is nilpotent.

2.3.19 Proposition [15]

The following properties of a module M are equivalent:

(a) M is semisimple.
(b) M is a direct sum of simple modules.
(c) Every submodule of M is a direct summand.

2.3.20 Proposition [15]

For a ring R, the following are equivalent:

(a) R is a semisimple ring and has no two-sided ideal except 0 and R.
(b) R is a semisimple ring, there is only one isomorphism class of simple modules.

2.3.21 Definition

Let X be a subset of a right R-module M. The right annihilator of X is the set

$$r_R(X) = \{r \in R: x\, r = 0 \text{ for all } x \in X\}, \text{ which is a right ideal of } R.$$

If X is a submodule of M, then $r_R(X)$ is a two-sided ideal of R. If $M = R$, then the right annihilator of X is $r_R(X) = \{ r \in R : x\, r = 0$ for all $x \in X \}$ as well as a left annihilator of X is

$$l_R(X) = \{ r \in R : \ r\, x = 0 \text{ for all } x \in X \}.$$

A right annihilator is a right ideal of R which is of the form $r_R(X)$ (or simply $r(X)$) for some X and a left annihilator is a left ideal of the form $l_R(X)$. An element c of a ring R is called right regular if $r_R(c) = 0$, left regular if $l_R(c) = 0$, and regular if $r_R(c) = l_R(c) = 0$.

Example

Every nonzero element of an integral domain is regular.

If M is a right R-module and $m \in M$, then it is an annihilator if $r_R(m) = \{r \in R : mr = 0\} \subset R_R$ and $l_R(m) = \{r \in R : rm = 0\} \subset {}_RR$.

2.3.22 Definition

A ring R is a prime ring if for any two elements $a, b \in R$, $a\, r\, b = 0$ for all r in R implies $a = 0$ or $b = 0$.

Examples

(i) Any integral domain.

(ii) Any primitive ring.
(iii) A matrix ring over an integral domain. In particular, the ring of 2 x 2 integer matrices is a prime ring.

2.3.23 Definition

A submodule X of a right R-module M is called a simple submodule (or minimal submodule) if X is a simple module.

2.3.24 Definition

A submodule X is called a maximal submodule of M if $X \neq M$ and for any submodule Y of M, if $X \subset_{>} Y$ then $Y = X$ or $Y = M$.

2.3.25 Theorem

The following statements hold:
(a) Every finitely generated right R-module contains at least one maximal submodule. Therefore, every ring with identity contains at least one maximal right ideal.
(b) For any submodule X of M, X is maximal if and only if M / X is simple.
(c) M is simple if and only if for any $0 \neq m \in M$, $M = mR$.

2.4 Noetherian and artinian modules

2.4.1 Definition

A ring which satisfies the descending chain condition (DCC) for left (resp. right) ideals is called a left (resp. right) artinian ring.

A ring which is both left artinian and right artinian is called an artinian ring.

Example

(i) Every finite ring is artinian.
(ii) Every divisor ring D is right artinian as its only right ideals are (0) and D itself. Because of similar reason, D is also left artinian.

2.4.2 Definition

A ring which satisfies the ascending chain condition (ACC) for left (resp. right) ideals is called a left (resp. right) noetherian ring.

A ring which is both left noetherian and right noetherian is called a noetherian ring.

Example

(i) Every finite ring is both left and right noetherian.

(ii) Every principal ideal ring is a noetherian ring.
(iii) The set $\mathbb{Z}$ of all integers is a noetherian ring because it is a principal ideal domain (PID) and every PID is a noetherian ring.
(iv) For a divisor ring D, the only right ideals of D are (0) and D itself. So, D is right noetherian. For similar reasons, D is also left noetherian.

2.4.3 Definition

A module M is called artinian if DCC (or minimum condition) holds for M.

Example

(i) A module which has only finitely many submodules is artinian. In particular, finite abelian groups are artinian as modules over $\mathbb{Z}$.

(ii) Infinite cyclic groups are not artinian. For instance, $\mathbb{Z}$ has a non stationary descending chain of subgroups, namely, $\mathbb{Z} = (1) \supset (2) \supset (4) \supset \cdots \supset (2^n) \supset \cdots$.

2.4.4 Noetherian module

A module M is called noetherian if ACC (or maximum condition) holds for M.

Example

(i) A module which has only finitely many submodules is artinian. In particular, finite abelian groups are artinian as modules over $\mathbb{Z}$.
(ii) Unlike the artinian case, infinite cyclic groups are noetherian, because every subgroup of a cyclic group is cyclic.

2.4.5 Definition

A module M of which the 0 submodule is a prime submodule is called a prime module. It is proved that M is a prime module if and only if $Ann\ (N) = Ann(M)$, for all nonzero submodules N of M. It is proved that an artinian faithful multiplication R-module is a prime module if and only if R is a Dedekind domain.

2.4.6 Proposition

An artinian R-module M is a prime module if and only if $R/Ann\ (M)$ is a field.

Proof:

Let $T = \{N : N$ is a non trivial submodule of $M\}$. Suppose that N_0 is a minimal element of T. Obviously N_0 is a nonzero simple module. Hence there exists an element $0 \neq a \in M$ such that

$N_0 = Ra \cong R/Ann(a)$ and $Ann(a)$ is a maximal ideal of R. Since M is a prime module, $Ann(a) = Ann(M)$. Consequently, $Ann(M)$ is a maximal ideal of R. Conversely, note that in a vector space every proper submodule (subspace) is a prime submodule. Now since 0 is a prime submodule of M as an $R/Ann(M)$-module, obviously it is a prime submodule of M as an R-module.

2.4.7 Proposition [12]

For a ring R, the following conditions are equivalent:
(a) R is right artinian and $J(R) = 0$.
(b) R is left artinian and $J(R) = 0$.
(c) R is semisimple.

2.4.8 Definition

A right R-module M is called artinian if every nonempty family of submodules has a minimal element by inclusion.

A right R- module is called right artinian if R_R is artinian as a right R-module.

2.4.9 Corollary [12]

For a right artinian ring R, the Jacobson radical equals the prime radical.

2.4.10 Corollary [12]

For a ring R, the following conditions are equivalent:
(a) R is right artinian and semiprime.
(b) R is left artinian and semiprime.
(c) R is semisimple.

2.4.11 Corollary [12]

For a ring R, the following conditions are equivalent:
(a) R is prime and right artinian.
(b) R is prime and left artinian.
(c) R is simple and right artinian.
(d) R is simple and left artinian.
(e) R is simple and semiprime.
(f) $R \cong M_n(D)$ for some positive integer n and some division ring D.

2.4.12 Proposition [12]

If R is a nonzero right or left artinian ring, then all prime ideals in R are maximal.

2.4.13 Theorem [16]

Let M be a right R-module and let X be its submodule. Then the following statements are equivalent:

(a) M is noetherian;
(b) X and M/X are noetherian;
(c) Any ascending chain $M_1 \subset M_2 \subset ... \subset M_n \subset ...$ of submodules of M is stationary;
(d) Every submodule of M is finitely generated;
(e) For every nonempty family $\{M_i : i \in I\}$ of submodules M_i of M, there exists a finite subfamily $\{M_i : i \in I_0\}$ where $I_0 \subset I$ with I_0 finite such that $\sum_{i \in I} M_i = \sum_{i \in I_0} M_i$.

Theorem 2.4.14 [17]

Let M be a right R-module and let X be its submodule. Then the following statements are equivalent:

(a) M is artinian;
(b) X and M/X are artinian;
(c) Any descending chain $M_1 \supset M_2 \supset ... \supset M_n \supset ...$ of submodules of M is stationary;
(d) Every factor module of M is finitely cogenerated;
(e) For every nonempty family $\{M_i : i \in I\}$ of submodules M_i of M, there exists a finite subfamily $\{M_i : i \in I_0\}$ where $I_0 \subset I$ with I_0 finite such that $\bigcap_{i \in I} M_i = \bigcap_{i \in I_0} M_i$.

2.5 Prime submodules and semiprime submodules

Sanh et al. [11] introduced the notion of prime and semiprime submodules of a given right R-module over S.

Let M be a right R-module and $S = End_R(M)$, its endomorphism ring. Ahmed et al. [19] investigated some results on prime submodules and semiprime submodules.

2.5.1 Proposition [19]

Let M be a right R-module which is a self-generator. Then we have the followings:
(1) If X is a minimal prime submodule of M, then I_X is a minimal prime ideal of S.
(2) If P is a minimal prime ideal of S, then $X = P(M)$ is a minimal prime submodule of M and $I_X = P$.

2.5.2 Theorem [19]

Let M be a right R-module which is a self-generator. Let X be a fully invariant submodule of M. Then the following conditions are equivalent:

(1) X is a semiprime submodule of M;
(2)If J is any ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;
(3) If J is any ideal of S such that $J(M) \supsetneqq X$, then $J^2(M) \not\subset X$;
(4) If J is any right ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;
(5) If J is any left ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$.

Now we have more properties about prime and semiprime submodules.

2.5.3 Proposition [17]

Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a noetherian module, then there exists only finitely many minimal prime submodules.

Proof:

If M is a noetherian module, then S is a right noetherian ring. Indeed, suppose that we have an ascending chain of right ideal of S, say $I_1 \subset I_2 \subset \dots$. Then we have $I_1(M) \subset I_2(M) \subset \dots$ is an ascending chain of submodules of M. Since M is a noetherian module, there is an integer n such that $I_n(M) = I_k(M)$, for all $k > n$.

Then we have $I_n = \text{Hom}\,(M, I_n(M) = \text{Hom}\,(M, I_k(M)) = I_k$.

Thus the chain $I_1 \subset I_2 \subset \dots$is stationary and so S is a right noetherian ring. Thus $P_1(M), \dots, P_t(M)$ are the only minimal prime submodules of M.

2.5.4 Lemma [17]

Let M be a quasi-projective, finitely generated right R-module which is a self-generator and X, a minimal submodule of M. Then I_X is a minimal right ideal of S.

Proof:

Let J be a right ideal of S such that $0 \neq J \subset I_X$. Then $J(M)$ is a nonzero submodule of M and $J(M) \subset X$. Thus $J(M) = X$ and it follows that $J = I_X$.

2.5.5 Proposition [17]

Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Let X be a minimal submodule of M. Then either $I_X^2 = 0$ or $X = f(M)$ for some idempotent $f \in I_X$.

Proof:

Since X is a minimal submodule of M, I_X is a minimal right ideal of S. Suppose that $I_X^2 \neq 0$. Then there is $g \in I_X$ such that $g I_X \neq 0$. Since $g I_X$ is a right ideal of S and g $I_X \subset I_X$, we have $g I_X = I_X$. Then there exists $f \in I_X$ such that $gf = g$. Then set $I = \{h \in I_X : gh = 0\}$ is a right ideal of S and I is properly contained in I_X since $f \notin I$. By the minimality of I_X, we must have $I = 0$. We have $f^2 - f \in I_X$ and $g(f^2 - f) = 0$, so $f^2 = f$. Since $f(M) \subset X$ and $f(M) \neq 0$, we have $f(M) = X$.

2.5.6 Corollary [17]

Let M be a quasi-projective, finitely generated right R-module which is a self-generator and let X be a minimal submodule of M. If M is a semiprime module, then $X = f(M)$ for some idempotent $f \in I_X$.

2.5.7 Proposition [17]

Let M be a quasi-projective, finitely generated right R-module which is a self-generator. The$Z(S)\ (M) \subset Z(M)$ where $Z(S)$ is a singular ideal of S and $Z(M)$ is a singular submodule of M.

Proof:

Let $f \in Z(S)$ and $x \in M$. We show that $f(x) \in Z(M)$. Since $f \in Z(S)$, there exists an essential right ideal K of S such that $fK = 0$. Then $f\ K(M) = 0$. Since K is an essential right ideal of S, we have $K(M)$ is an essential submodule of M and so $x^{-1} K(M)$ is an essential right ideal of R. We have $f(x)\ (x^{-1} K(M)) = f(x(x^{-1} K(M))) \subset f\ K(M) = 0$, proving that $f(x) \in Z(M)$.

2.5.8 Corollary [17]

Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a nonsingular module, then S is a right nonsingular ring.

2.5.9 Proposition [17]

Let M be a right R-module which is a self-generator. If M is a semiprime module with the ACC for M-annihilators, then M has only a finite number of minimal prime submodules. If $P_1, \dots, P_n$ are minimal prime submodules of M, then $P_1 \cap \dots \cap P_n = 0$. Also, a prime submodule P of M is minimal if and only if I_P is an annihilator ideal of S.

Proof:

Since M is a semiprime module, S is a semiprime ring. If satisfies the ACC for M-annihilators, then S satisfies the ACC for right annihilators. Then, S has only a finite number of minimal prime ideals. Therefore M has only a finite number of minimal prime submodules. If $P_1, \dots, P_n$ are minimal prime submodules of M, then $I_{P_1}, \dots, I_{P_n}$ are minimal prime ideals of S. Thus $I_{P_1} \cap \dots \cap I_{P_n} = 0$, but $I_{P_1} \cap \dots \cap I_{P_n} = I_{P_1 \cap \dots \cap P_n}$, we have $P_1 \cap \dots \cap P_n = 0$. Finally, a prime submodule P of M is minimal if and only if I_P is a minimal prime ideal of S.

2.5.10 Proposition [17]

Let M be a quasi-projective right R-module and X, a fully invariant submodule of M. Then the following are equivalent:

(1) X is a semiprime submodule of M.
(2) M / X is a semiprime module.

2.6 Homomorphism and endomorphism

2.6.1 Definition

Let M and N be two R-modules. Then a mapping $f: M \to N$ is called a homomorphism (R-homomorphism or module homomorphism) if

(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$, for any $m_1, m_2 \in M$
(ii) $f(rm) = rf(m), \forall r \in R,\ m \in M$

If f is one-one and onto, it is called an isomorphism of M into N.

Remark

If $f: M \to N$ is a homomorphism, then

(i) $f(0) = 0$.
(ii) $f(-m) = -f(m), \forall m \in M$.
(iii) $f(m_1 - m_2) = f(m_1) - f(m_2)\ \forall m_1, m_2 \in M$.
(iv) If R is a divisor ring, then an R-module homomorphism is called a linear transformation.

Example

Let M be an R-module. Then the mapping $f: x \mapsto x$ of M onto M is clearly an R-homomorphism of M onto M.

2.6.2 Definition

A homomorphism $f: M \rightarrow N$ of R-modules M and N is called

(i) a monomorphism if f is injective.
(ii) an epimorphism if f is surjective.
(iii) an isomorphism if f is bijective.
(iv) an endomorphism if $M = N$.
(v) an automorphism if $M = N$ and f is an isomorphism.

2.6.3 Definition

Let $f: M \rightarrow N$, then the kernel of f is denoted by $kerf$ and is defined by

$kerf = \{m \in M: f(m) = 0,$ where 0 is the additive identity of $N\}$.

2.6.4 Theorem [14]

The kernel of a module homomorphism is a submodule.

Proof:

If f is a homomorphism of an R-module M into an R-module N, then the kernel of f is $kerf = \{m \in M: f(m) = 0,$ where 0 is the additive identity of $N\}$.
Since $f(0) = 0$, it follows that $0 \in kerf$. So $kerf \neq \emptyset$.
Let $m_1, m_2 \in kerf$. Then $f(m_1) = 0, f(m_2) = 0$ and

$$f(m_1 - m_2) = f(m_1) - f(m_2) = 0.$$

Thus $m_1, m_2 \in kerf$ implies $(m_1 - m_2) \in kerf$.

Also, if $r \in R$ and $m \in kerf$, then $f(rm) = rf(m) = r.0 = 0$.
So, $r \in R$ and $m \in kerf$ implies $rm \in kerf$.

Hence, $kerf$ is a submodule of M.

2.6.5 Definition

Let f be a homomorphism of an R-module M into an R-module N, then the image of f is denoted by $Im(f)$ and is defined by $Im(f) = \{f(m): m \in M\}$.

2.6.6 Theorem [14]

The image of a homomorphism is a submodule.

Proof:

If f is a homomorphism of an R-module M into an R-module N, then the image of f is $Im(f) = \{f(m): m \in M\}$.

We have to prove that $Im(f)$ is a submodule of N.

Let $f(m_1), f(m_2)$ be any two elements of $Im(f)$, where $m_1, m_2 \in M$.

Now $f(m_1) - f(m_2) = f(m_1 - m_2) \in Im(f)$, since $m_1 - m_2 \in M$.

Therefore, $Im(f)$ is an additive subgroup of N.

Again, let $r \in R$ and $f(m) \in Im(f)$, then $f(rm) = rf(m) \in Im(f)$, since $rm \in M$.

Hence $Im(f)$ is a submodule of N.

2.6.7 Theorem [14]

Let f be a module homomorphism. Then f is an injective (or, monomorphism) if and only if $kerf = \{0\}$.

Proof:

Let f be a homomorphism of an R-module M into and R-module N and let $kerf = \{0\}$. We have to show that f is injective.

Now, if $m_1, m_2 \in M$, then $f(m_1) = f(m_2)$ implies $f(m_1) - f(m_2) = 0$

$$=> f(m_1 - m_2) = 0$$
$$=> (m_1 - m_2) \in kerf$$
$$=> m_1 - m_2 = 0$$
$$=> m_1 = m_2$$

This shows that f is injective.

Conversely, let f be injective. Then $m \in kerf => f(m) = 0 = f(0)$.
Since f is one-one, we have $f(m) = f(0) => m = 0$.
Hence, $kerf = \{0\}$. Thus the theorem is proved.

2.6.8 Proposition [17]

Let M and N be left R-modules and let $f : M \to N$ be an R-homomorphism. Then the following statements are equivalent:

(a) f is an epimorphism onto N;
(b) $Im(f) = N$;

2.6.9 Proposition [17]

Let M and N be left R-modules and let $f : M \to N$ be an R-homomorphism. Then the following statements are equivalent:

(a) f is a monomorphism ;
(b) $Ker\, f = 0$;

2.6.10 Definition

Let M be a right R-module. A homomorphism $f : M \to M$ is called an endomorphism. The abelian group $\mathrm{Hom}_R(M, M)$ becomes a ring if we use the composition of maps as multiplication. This ring is called the endomorphism ring of M, and we denoted by $End_R(M)$

2.6.11 Proposition [17]

Let R and S be a rings and M an abelian group. If M is a left R-module via $f : R \to End^l(M)$ and a right S-module via $g : S \to End^\gamma(M)$ then the following are equivalent :
(a) ${}_R M_S$;
(b) f: $R \to End(M_S)$ is a ring homomorphism;
(c) g: $S \to End({}_R M)$ is a ring homomorphism.

2.6.12 Definition

Let M be a right R-module and $S = End(M_R)$. Suppose that X is a fully invariant submodule of M. Then the set $I_X = \{f \in S : f(M) \subset X\}$ is a two-sided ideals of S. By the definition, the class of all fully invariant submodules of M is nonempty and closed under intersections and sums. Indeed, if X and Y are fully invariant submodules of M, then for every $f \in S$, we have $f(X + Y) = f(X) + f(Y) \subset X + Y$ and $f(X \cap Y) \subset f(X) \cap f(Y) \subset X \cap Y$. In general, if $\{X_i : I \in I\}$ where I is an index set, is a family of fully invariant submodules of M, then $\sum_{i \in I} X_i$ and $\bigcap_{i \in I} X_i$ are fully invariant submodules of M.

2.6.13 Proposition [17]

If R is a ring and λ and ρ denote respectively the left and right multiplication, then
λ : R $\to$ End(M_R) and ρ : R $\to$ End (${}_R M$) are ring isomorphisms.

2.6.14 Definition

Let M be a right R-module and X, a proper fully invariant submodule of M. Then X is called a prime submodule of M if for any ideal I of S and any fully invariant submodule U of M, if $I(U) \subset X$ then either $I(M) \subset X$ or $U \subset X$. A fully invariant submodule X of M is called strongly prime if for any $f \in S$ and any $m \in M$, $f(m) \in X$ implies $f(M) \subset X$ or $m \in X$.

2.6.15 Theorem ([11], [18])

Let M be a right R-module and P be a proper fully invariant submodule of M. Then the following conditions are equivalent:

(a) P is a prime submodule of M;
(b) For any right ideal I of S and any submodule U of M, if $I(U) \subset P$, then either $I(M) \subset P$ or $U \subset P$;
(c) For any $\varphi \in S$ and any fully invariant submodule U of M, if $\varphi(U) \subset P$, then either $\varphi(M) \subset P$ or $U \subset P$;
(d) For any left ideal I of S and any subset A of M, if $IS(A) \subset P$, then either $I(M) \subset P$ or $A \subset P$;
(e) For any $\varphi \in S$ and any $m \in M$, if $\varphi(S(m)) \subset P$, then either $\varphi(M) \subset P$ or $m \in P$.
Moreover, if M is quasi-projective, then the above conditions are equivalent to:
(f) M / P is a prime module.

In addition, if M is quasi-projective and a self-generator, then the above conditions are equivalent to:

(g) If I is an ideal of S and U, a fully invariant submodule of M such that $I(M)$ and U properly contain P, then $I(U) \not\subset P$.

2.6.16 Theorem [14]

A homomorphic image of a right noetherian ring is a noetherian ring.

Proof:
Let S be a homomorphic image of a right noetherian ring R. Then $S \cong R/I$ for some ideal I of R. So it is sufficient to prove that R/I is right noetherian.
Let $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$ be an ascending chain of right ideals of R/I. Now each J_i is of the form K_i/I, where K_i is a right ideal of R containing I. Also $J_i \subseteq J_{i+1} \Rightarrow K_i \subseteq K_{i+1}$. So the above ascending chain gives the rise to the ascending chain $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ of right ideals of R. But as R is right noetherian, there exists a positive integer n such that $K_m = K_n \ \forall m \geq n$. This implies that $J_m = J_n \ \forall m \geq n$.
Hence R/I is right noetherian.

2.6.17 Theorem [14]

If R is a left noetherian ring, then any homomorphic image of R is a left noetherian ring.

Proof:

Let R is a left noetherian ring and $f: R \to S$ be an epimorphism of rings.
Let $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$ be an ascending chain of S. Let $I_k = f^{-1}(J_k)$ for all $k \geq 1$. Then I_k is a left ideal of R for all k and $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$.
Since R is left noetherian, there exists positive integer n such that $I_n = I_{n+i}$ $\forall i \geq 1$.
Let $y \in J_{i+1}, i \geq 1$. Since f is onto, there exists $x \in R$ such that $f(x) = y$. Then $x \in I_n = I_{n+i}$ and so $y \in J_n$. Therefore, $J_n = J_{n+i}$ $\forall i \geq 1$, proving that S is left noetherian.

2.7 Injective and Projective modules

2.7.1 Definition

A sequence is a function whose domain is the set of positive integers, that is, a sequence in a set is a function $f: \mathbb{N} \to S$ where $\mathbb{N}$ is the set of natural numbers and is written as $(f_i), i = 1,2, \ldots$ or $(f_1, f_2, \ldots)$, where $f_i = f(i)$.

2.7.2 Definition

Let R be a ring. A sequence (finite or infinite) $\ldots . \to A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \to \cdots$. of R-modules and R-module homomorphisms is called an exact sequence if $Im(f_{n-1}) = ker f_n$ for all n.

In particular, a pair of module homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$, is said to be exact at B provided $Im(f) = ker g$.

An exact sequence of special form $\{0\} \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to \{0\}$ is called a short exact sequence.
Here the exactness means that α is injective, β is surjective and $Im(\alpha) = ker\beta$.

Example

(i) If C is a submodule of D, then the sequence $0 \to C \xrightarrow{i} D \xrightarrow{\nu} D/C \to 0$ is exact, where i is the inclusion map and ν is the canonical epimorphism.

(ii) Let A, C be R-modules, then $\{0\} \to A \to A \oplus C \to C \to \{0\}$ is a short exact sequence.
Now let A, B, C, D be modules with homomorphisms $f: A \to B$, $g': A \to C$, $f': C \to D$ and $g: B \to D$. This is given in the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\
C \arrow[r, "f'"'] & D
\end{tikzcd}$$

We say this diagram is commutative if $f \circ g = g' \circ f'$.

2.7.3 Lemma (The short five lemma) [17]

Let R be a ring and

$$\begin{tikzcd} O \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & O \\ O \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & O \end{tikzcd}$$

a commutative diagram of R-module homomorphisms such that each row is a short exact sequence. Then

(i) If α, γ monomorphisms then β is a monomorphism.
(ii) If α, γ epimorphisms then β is an epimorphism.
(iii) If α, γ isomorphisms then β is an isomorphism.

2.7.4 Definition

Two short exact sequences are said to be isomorphic if there is a commutative diagram of module homomorphisms

$$\begin{tikzcd} O \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & O \\ O \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & O \end{tikzcd}$$

such that f, g and h are isomorphisms. It is easy to verify the the diagram

$$\begin{tikzcd} O \arrow[r] & A \arrow[r] \arrow[d, "f^{-1}"] & B \arrow[r] \arrow[d, "g^{-1}"] & C \arrow[r] \arrow[d, "h^{-1}"] & O \\ O \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & O \end{tikzcd}$$

is also commutative. In fact, isomorphism of short exact sequences is an equivalence relation.

2.7.5 Definition

A module A over a ring R is said to be projective if given any diagram of R-module homomorphisms

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & O \end{tikzcd}$$

with bottom row exact (that is an epimorphism), there exists an R-module homomorphism $h: P \to A$ such that the diagram

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, "h"'] & \\ A \arrow[r, "g"'] & B \arrow[r] & O \end{tikzcd}$$

is commutative (that is, $gh = f$).

Example

If the ring R has an identity and P is unitary, then P is projective if and only if for every pair of unitary modules A, B and diagram of R-module homomorphisms

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & O \end{tikzcd}$$

with g an epimorphism, there exists a homomorphism $h: P \to A$ with $gh = f$.

2.7.6 Theorem [17]
Every free module F over a ring R with identity is projective.

Proof:
Consider a diagram of homomorphisms of unitary R-modules:

$$\begin{tikzcd} & F \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & O \end{tikzcd}$$

with g an epimorphism and F a free R-module on the set X $(\lambda: X \to F)$. For each $x \epsilon X, f(\lambda(x)) \epsilon B$. Since g is an epimorphism, there exists $a_x \epsilon A$ with $g(a_x) = f(\lambda(x))$. Since F is free, the map $X \to A$ given by $x \mapsto a_x$ induces an R-module homomorphism $h: F \to A$ such that $h(\lambda(x)) = a_x$ for all $x \epsilon X$. Consequently, $gh(\lambda(x)) = g(a_x) = f\lambda(x)$ for all $x \epsilon X$ so that $gh\lambda = f\lambda: X \to B$ which implies $gh = f$.

Therefore, F is projective. Thus the theorem is proved.

2.7.7 Theorem [17]

Let R be a ring. The following conditions on an R-module P are equivalent.

(i) P is projective

(ii) Every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ is split exact (hence $B \cong A \oplus P$);

(iii)Here is a free module F and an R-module K such that $F \cong K \oplus P$

2.7.8 Definition

A module J over a ring R is said to be injective if given any diagram of R-module homomorphisms

$$\begin{tikzcd} O \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"'] & B \\ & J & \end{tikzcd}$$

with top row exact (that is, g a monomorphism), then there exists an R-module homomorphism $h: B \to J$ that the diagram

$$\begin{tikzcd} O \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[dl, "h"] \\ & J & \end{tikzcd}$$

is commutative (that is, $hg = f$).

2.7.9 Theorem [17]

Let R be a ring with identity, then the following conditions are equivalent:

(i) R is semisimple.

(ii) Every R-module is projective.

(iii)Every R-module is injective.

2.7.10 Proposition [17]

The following properties of a module M are equivalent:

(i) M is projective.

(ii) M is a direct summand of a free module.

(iii) Every exact sequence $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} P \to 0$ splits.

2.7.11 Definition

An element $c \in R$ is called right regular (resp. left regular) if for any $r \in R$, $cr = 0 \Rightarrow r = 0$ (resp. $rc = 0 \Rightarrow r = 0$). If $cr = 0 = rc$, then c is called a regular element. For example, every non-zero element of an integral domain is regular and if F is a field, then any element of the set $M_n(F)$ is regular if and only if its determinant values is zero. Elements which are regular on one side need not be regular.

2.7.12 Proposition [17]

The following properties of a ring R are equivalent:

(i) R is regular.

(ii) Every principal right ideal of R is generated by an idempotent element.

(iii) Every finitely generated right ideal of R is generated by an idempotent element.

(iv) Every left R-module is flat.

CHAPTER III

NIL AND NILPOTENT RINGS

Overview

In this chapter, a ring will be defined as an algebraic structure with a commutative addition, and a multiplication which may or may not be commutative. This distinction yields two quite different theories: the theory of commutative and noncommutative rings. This chapter is mainly concerned with commutative rings.

It is Dedekind who extracted the important properties of "ideal numbers", defined an "ideal" by its modern properties: namely that of being a subgroup which is closed under multiplication by any ring element. He further introduced prime ideals as a generalization of prime numbers. Note that today we still use the terminology "Dedekind rings" to describe rings which have in particular a good behavior with respect to factorization of prime ideals.

3.1 Definition

Let R be a ring with DCC on right ideals. Let $\{I_\alpha\}$ be the collection of all nilpotent right ideals of R. Then $N = \sum I_\alpha$ is called the radical of R.

3.2 Theorem [18]

Let R be a ring with DCC on right ideals and let N be the radical of R. Then N is nilpotent.

Proof:

Clearly, N is a right ideal, so $N \supseteq N^2 \supseteq N^3 \supseteq \cdots$ is a descending sequence of right ideals. By the DCC, there exists an integer k such that $N^k = N^{k+1} = \cdots = N^{2k}$. Thus $N^k = N^k . N^k$.

If $N^k = 0$, the proof is finished.

Otherwise, there exists right ideals I such that $IN^k \neq 0$. BY the minimum condition (equivalent to the DCC), there exists a right ideal I_0 minimal with respect to the property $I_0 N^k \neq 0$.

Since $I_0 N^k \neq 0$, there exists an element $x \in I_0$, $x \neq 0$ such that $xN^k \neq 0$. But then $(xN^k)N^k = xN^{2k} = xN^k \neq 0$, so by minimality of I_0, $xN^k = I_0$.

Thus there exists an element $y \in N^k$ such that $xy = x$. Now $y \in N$, so y also is contained in the sum of finitely many nilpotent right ideals. Therefore, y is nilpotent, that is, $y^m = 0$ for some integer m. But then $xy = x$ implies $xy^n = x$, for all n, where as $0 = xy^m = x \neq 0$, a contradiction.
This contradiction shows that $N^k = 0$ and N is nilpotent.

3.3 Corollary [14]

(1) Every one-sided or two-sided nilpotent ideal is a nil ideal.

(2) The sum of two nilpotent right, left or two-sided ideals is again nilpotent.

(3) If R_R is noetherian, then every two-sided nil ideal is nilpotent.

Proof:

(1) By definition, it follows the result.

(2) Let $A \subset_> R_R$, $B \subset_> R_R$ and $A^m = 0$, $B^n = 0$.

We assert that $(A+B)^{m+n} = 0$. Let $a_i \in A$, $b_i \in B$, $i = 1,2,3,\ldots, m+n$ then by binomial theorem $\prod_{i=1}^{m+n}(a_i + b_i)$ is a sum of products of $m + n$ factors of which either at least m factors are from A or at least n factors are from B. Since A and B are right ideals, the assertion follows.

(3) Let N be a two-sided nil ideal of R. Since R_R is Noetherian, among the nilpotent right ideals contained in N, there is a maximal one. Let A be one such and suppose we have $A^n = 0$. By (2), A is indeed the largest nilpotent right ideal contained in N. Since for $x \in R$, xA is also a nilpotent right ideal contained in N, A is in fact a two-sided ideal. If for an element $b \in N$ we have $(bR)^K \subset_> A$, then it follows that $(bR)^{Kn} = 0$, thus $bR \subset_> A$.

The following corollary is an extension of the above corollary over associative arbitrary rings.

3.4 Corollary

Let R be a right noetherian ring. Then each nil one-sided ideal of R is nilpotent.

Proof:

Let S be the sum of all the nilpotent right ideals of R. Then S is an ideal. Since R is right noetherian, S is the sum of a finite number of nilpotent right ideals and hence S is nilpotent. It follows that the quotient ring R/S has no nonzero nilpotent right ideals. Let I be a nil one-sided ideal of R. Then the image of I in R/S is zero. Hence $I \subseteq S$.

3.5 Proposition [14]

Let R be a ring with DCC on right ideals. Let N be the radical of R. Then

(1). N is a nilpotent ideal.
(2). N is the sum of all nilpotent left ideals
(3). N is the unique ideal of R maximal with respect to being nilpotent.

3.6 Theorem [18]

Let R be a ring with DCC on right ideals. Then the radical of R/N is zero.

Proof:

Let I' be a nilpotent right ideal of R/N and $I = \{r \in R : r + N \in I'\}$. Then I is a right ideal in R.

Since I' and N are nilpotent, there exist integers m, n such that
(i) $(I')^m = 0$ *in* R/N , that is, $(r_1 + N) \dots (r_m + N) = N$, where $r_1, \dots, r_m \in I$.
(ii) $N^n = 0$

Now let $r_1, \dots, r_{mn} \in I$. Then $a_1 = r_1 \dots r_m \in N$, $a_2 = r_{m+1} \dots r_{2m} \in N, \dots$, where the product $a_1 a_2 \dots a_n$ of these n elements of N is zero, that is, $r_1 \dots . r_{mn} = 0$ and so $I^{mn} = 0$.

Since I is nilpotent $I \subseteq N$ and so I' equals zero in R/N.

Thus R/N has radical zero.

Hence the theorem is proved.

3.7 Proposition [14]

Let R be a semisimple ring. Then
(1). R has minimal right ideals
(2). No minimal right ideal is nilpotent
(3). If I is a minimal right ideal of R, there exists an idempotent element $e \in I$ $(e^2 = e \neq 0)$ such that $I = eR$.

3.8 Proposition [15]

If R is a semisimple ring then it has no two-sided ideals except zero and R.

The following proposition is an extension of the above proposition over associative arbitrary rings.

3.9 Proposition

Let R be a semiprime ring with the *ACC* for right annihilators. Then R has no nonzero nil one-sided ideals.

Proof:

Let I be a nonzero one-sided ideal of R and let $0 \neq a \in I$ with $r_R(a)$ as large as possible. Since R is semiprime, there is an element $x \in R$ such that $axa \neq 0$. Thus axa is a nonzero element of I such that $r_R(a) \subseteq r_R(axa)$. So $r_R(a) = r_R(axa)$. We have $ax \neq 0$, i.e., $x \notin r_R(a)$. Thus

$x \notin r_R(axa)$. So, $(ax)^2 \neq 0$. Hence $xax \notin r_R(a)$ implying that $(ax)^3 \neq 0$. Therefore, ax and hence, also xa is not nilpotent and $ax \in I$ or $xa \in I$.

3.10 Definition

The nil radical of a ring R is defined to be the radical ideal with respect to the property that "a two-sided ideal is nil" and is denoted by $N(R)$. That is, $N(R)$ is the largest two-sided ideal of R such that every element of $N(R)$ is nilpotent.

Note

Recall that the prime radical of M is the intersection of all prime submodules of M and is denoted by $P(M)$. The prime radical of a ring R is the intersection of all prime ideals of R and is denoted by $P(R)$.

3.11 Theorem [18]

Let R be a simple ring with DCC on right ideals. Then N, the radical of R, is zero.

Proof:

Since N is an ideal of R, then $N = R$ or $N = 0$. Since N is nilpotent, $N = R$ implies $R^n = 0$ for some, contradicting that $R^2 = R$ for $R^2 \neq 0$ implies $R^2 = R$, since R^2 is an ideal.
Thus $N = 0$.

3.12 Theorem [18]

For any ring R, the nil radical $N(R)$ exists and it is characterized by
$N(R) = \{a \in R$: the principal two-sided ideal (a) is a nil ideal $\}$.

Proof:

First we have to prove that $N = N(R)$ as above is a two-sided ideal and second that it is the largest for that property.

(1) Since $0 \in N, N \neq \phi$. If $a \in N$ and $\in R$, then $(xa) \subseteq (a)$ and $(ax) \subseteq (a)$, and so, both (xa) and (ax) are nil ideal; hence $xa, ax \in N$. Thus we have only to prove the following:
(2) N is an additive subgroup of R.

To see this, for $a, b \in N$, we have to show that $(a - b)$ is a nil ideal. Since $(a - b) \subseteq (a) + (b)$, every element $x \in (a - b)$ can be written as $x = y + z$ for some $y \in (a)$ and $z \in (b)$. Since (a) and (b) are nil ideals, both y and z are nilpotent, say $y^n = 0$ and $z^n = 0$ for some $n \gg 0$. Now look at $x^n = (y + z)^n = y^n + z' = 0 + z'$, where z' is a sum of monomials in y and z in each of which z is a factor, that is, $z' \in (z) \subseteq (b)$, and so, z' is nilpotent and hence x is nilpotent, that is, $(a - b)$ is a nil ideal, as required.

Finally, let I be any two-sided nil ideal of R . Then trivially, $(a) \subseteq I, \forall a \in I$, and hence, (a) is a nil ideal, that is, $I \subseteq N$, as required.

3.13 Definition

The Jacobson radical of a ring R with identity is defined as the radical ideal of R with respect to the property that "a two-sided ideal I is such that $1 - a$ is a unit in R for all $a \in I$" and it is denoted by $J(R)$. In other words, $J(R)$ is the largest two-sided ideal of R such that $1 - a$ is a unit all $a \in J(R)$.

3.14 Proposition [14]

For any ring R, $N(R) \subseteq J(R)$ and equality need not hold.

Proof:
Let $a \in N(R)$. Since $N(R)$ is a nil ideal, a is nilpotent, say $a^n = 0$ for some $n \in \mathbb{N}$. Now we have $1 = 1 - a^n = (1 - a)(1 + a + a^2 + \cdots + a^{n-1})$ implies that $1 - a$ is a unit in R and so $a \in J(R)$, as required.

3.15 Proposition [18]

The Jacobson radical of an artinian ring is the intersection of some finitely maximal left (resp. right) ideals.

Proof:

Let R be an artinian ring. Let $\mathcal{M}$ be the set of all maximal left ideals of R. Let $\mathcal{F}$ be the family of all left ideals of R each of which is an intersection of finitely many maximal left ideals of R. Obviously, this family is nonempty, since $\mathcal{M} \in \mathcal{F}$. Since R is artinian, $\mathcal{F}$ has a minimal member, say $J_0 = \cap_{i=1}^{n} M_i$, $M_i \in \mathcal{M}$. We have $J \subseteq J_0$ where $J = J(R)$. On the other hand, if $M \in \mathcal{M}$, then $J_0 \cap M$ being a member of $\mathcal{F}$ must be equal to J_0 by the minimality of J_0 which means that $J_0 \subseteq M$, $\forall M \in \mathcal{M}$. Thus we get that $J \subseteq J_0 \subseteq \cap_{M \in \mathcal{M}} M = J$ and hence $J = J_0$, as required.

3.16 Theorem [18]

The Jacobson radical of an artinian ring R is nilpotent. In fact, $J(R)$ is the largest nilpotent (left or right or two-sided) ideal of R and consequently $N(R) = J(R)$.

Proof:

Since R is artinian, the descending chain of ideals
$J \supseteq J^2 \supseteq \cdots \supseteq J^n \supseteq \cdots$
is stationary where $J = J(R)$. Say, $J^m = J^{m+1} = \cdots$ for some $m \gg 0$. Write $I = J^m$. Now we have $I = I^2$ and $JI = I$.

Assume, if possible, that $I \neq (0)$. Consider the family $\mathcal{F}$ of all left ideals K of R such that $IK \neq (0)$. Since $I^2 = I = (0)$, $I \in \mathcal{F}$, and so, $\mathcal{F} \neq 0$. Note that $(0) \notin \mathcal{F}$. Since R is artinian, $\mathcal{F}$ has a minimal member, say K, that is, K is a left ideal of R such that $IK \neq (0)$ and K is minimal for this property. On the other hand, since $IK \neq (0)$, we find $a \in I$ and $b \in K$ such that $ab \neq 0$ which implies that $I(Rb) \neq (0)$, that is, $Rb \in \mathcal{F}$. But $Rb \subseteq K$, and so, $Rb = K$ by minimality of K. Thus K is a principal left ideal of R.

Finally, we have $(IJ).Rb = I.Rb = Ib \neq (0)$ and $J.Rb = J.b \subseteq Rb$ and $J.Rb \neq (0)$ which gives that $J.Rb = Rb$. Now we know that $K = Rb = (0)$, a contradiction to the assumption that $I \neq (0)$. Hence $I = J^n = (0)$.

3.17 Theorem [18]

In a right artinian ring every nil right ideal is nilpotent.

Proof:

Let I be a nil right ideal of a right artinian ring R. For the descending chain of right ideal $I \supseteq I^2 \supseteq I^3 \supseteq \cdots$ there exists a positive integer n such that $I^m = I^n, \forall m \geq n$. In particular, $I^{2n} = I^n$. We claim that $I^n = 0$, if not then $I^n \neq (0)$.

Let $\mathcal{F} = \{A: A$ is a right ideal of R such that $AI^n \neq (0)\}$

Here $\mathcal{F}$ is nonempty as $I^n \in \mathcal{F}$. Since R is right artinian, $\mathcal{F}$ has a minimal element. Let K be the minimal element of $\mathcal{F}$ .Then $KI^n \neq (0)$=> there exists $k \in K$ such that $kI^n \neq 0$. But $kI^n \subseteq K$, as K is a right ideal of R. Let $kI^n < K$.

Again, $(kI^n)I^n = kI^{2n} = kI^n \neq (0) => kI^n \in \mathcal{F}$ as kI^n is a right ideal of R. This violates the minimality of K . Hence $kI^n = K$.

Since $k \in K$, there exists $a \in I^n$ such that $ka = k$. But as I is a nil right ideal and $I^n \in I$, a is nilpotent. So there exists a positive integer t such that $a^t = 0$.

Then $k = ka = ka^2 = ka^3 = \cdots = ka^t = 0 => kI^n = (0)$, which is a contradiction to the choice of k.

Hence $I^n = (0)$. Consequently, I is a nilpotent right ideal. Thus the theorem is proved.

3.18 Lemma [12]

Let *R* be a commutative ring. Then the right singular ideal *Z(R)* of *R* is zero if and only if *R* is semiprime.

Proof:

Suppose that R is a semiprime ring. Let $z \in Z(R)$. We show that $z = 0$. Set $I = zR \cap r_R(z)$. We have $zR \cdot r_R(z) = 0$. In fact, for any $t \in R$ and any $t_1 \in r_R(z)$, we have $t_1 z = 0$. So, $z\, tt_1 = tt_1 z = t.0 = 0$, showing that for any $t \in R$, $zR.r_R(z) = 0$. We have $I^2 \subseteq I = zR \cap r_R(z) = 0$. So $I^2 = 0$. Since R is a semiprime ring, *0* is a semiprime ideal. It follows that $I = 0$. But $r_R(z)$ is an essential right ideal of R. This implies that $zR = 0$. Thus $z = 0$.

Conversely, suppose that $Z(R) = 0$. Let a be an element of R such that $a^2 = 0$. We show that $a = 0$ from which it follows that R has no nonzero nilpotent element. Let $0 \neq x \in R$. Then we need to consider two cases: (i) $ax = 0 \Rightarrow x \in r_R(a)$; (ii) $ax \neq 0 \Rightarrow a(ax) = a^2 x = 0 \Rightarrow ax \in r_R(a)$. Hence $xR \cap r_R(a) \neq 0$. Therefore, $r_R(a)$ is an essential right ideal of R. This implies that $a \in Z(R)$. Thus $a = 0$. This completes the proof.

3.19 Theorem [12]

Let R be a ring with the *ACC* for right annihilators. Then the right singular ideal $Z(R)$ of R is nilpotent.

Proof:

We write Z rather than $Z(R)$ for the right singular ideal of R. Since $Z \supseteq Z^2 \supseteq Z^3 \supseteq \cdots$, we have $r_R(Z) \subseteq r_R(Z^2) \subseteq r_R(Z^3) \subseteq \cdots$. So, there exists a positive integer n such that $r_R(Z^n) = r_R(Z^{n+1})$. Suppose that $Z^{n+1} \neq 0$. We obtain a contradiction. There is an element $a \in Z$ such that $Z^n a \neq 0$. Choose such an element a with $r_R(a)$ large enough. Take any $b \in Z$, then $r_R(b)$ is an essential right ideal of R whence $r_R(b) \cap aR \neq 0$. Thus there exists an element $r \in R$ such that $ar \neq 0$ and $ar \in r_R(b)$. We have $ba \in Z$ and $r_R(a) \subseteq r_R(ba)$. But $ar \neq 0$ and $bar = 0$. Therefore, $r_R(a)$ is strictly contained in $r_R(ba)$. It follows from the choice of a that $Z^n ba = 0$. But b is an arbitrary element of Z. Hence $Z^{n+1} a = 0$, and so, $Z^n a = 0$. This completes the proof of the theorem.

3.20 Theorem [18]

Let I_1 and I_2 be two ideals of a ring R and let $I_1 + I_2 = \{a_1 + a_2 : a_1 \in I_1, a_2 \in I_2\}$. Then $I_1 + I_2$ is an ideal of R.

The following theorem is an extension of the above theorem over associative arbitrary rings.

3.21 Theorem

If R is a ring and I, J are two nil right ideals of R, then the sum $(I + J)$ is a nil right ideal.

Proof:

Let $I = \{a_1, a_2, a_3, \ldots, a_s\}$ and $J = \{b_1, b_2, b_3, \ldots, b_t\}$ be such that

$$a_1^{n1} = 0, a_2^{n2} = 0, a_1^{n3} = 0, \ldots, a_s^{n_s} = 0 \text{ where } n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_s$$

and $b_1^{m} = 0, b_2^{m2} = 0, b_1^{m3} = 0, \dots, b_t^{m_t} = 0$ where $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_t$.

Let n and m be positive numbers such that

$n = n_1 \geq n_i$, $\forall i$ and $m = m_1 \geq m_j$, $\forall j$, hence $a_i^n = 0$, $\forall i$ and $b_j^m = 0$, $\forall j$.

Since $\forall a_i \in I$ we have $a_i r \in I$ implies $a_i r = a_k$ where $a_k^k = a_k^n = 0$; $n \geq k$.
Also, $\forall b_j \in J$ we have $b_j r \in J$ implies $b_j r = b_t$ where $a_t^t = a_t^m = 0$; $m \geq t$.

Take for example $n = 3, m = 2$.

Also let $a \in I$ such that $a^3 = 0$ and $b \in J$ such that $b^2 = 0$.
So, as $n = 3$, we get $(ab)^3 = (a^2 b)^3 = (aba)^3 = \cdots = 0$, where $ab, a^2 b, aba \in I$.

Similarly, as $m = 2$, we get $(ba)^2 = (ba^2)^2 = (b^2 a)^2 = \cdots = 0$, where $ba, ba^2, b^2 a \in J$.

Now $I + J = \{a + b : a \in I, b \in J\}$. Then

$$
\begin{aligned}
(a+b)^2 &= a^2 + ab + ba + b^2 = a^2 + ab + ba \\
(a+b)^3 &= a^3 + a^2 b + aba + ab^2 + ba^2 + bab + b^2 a \\
&= a^2 b + aba + ab^2 + ba^2 + bab \\
(a+b)^4 &= a^2 ba + aba^2 + abab + ba^2 b \\
(a+b)^5 &= a^2 ba^2 + a^2 bab + aba^2 b + ba^2 ba \\
(a+b)^6 &= a^2 ba^2 b + aba^2 ba + aba^2 b + ba^2 ba \\
(a+b)^7 &= a^2 ba^2 ba \\
(a+b)^8 &= ba^2 ba^2 ba = (ba^2) ba = 0
\end{aligned}
$$

If we take $n = 3$ and $m = 3$, then we get
$(a + b)^{19} = 0$.

So if $a_i^n = 0$; $i = 1,2,3, \dots, s$ and $b_j^m = 0$; $j = 1,2,3, \dots, t$

Then there exists $n \geq n_i$ $\forall i$ and $m \geq m_j$ $\forall j$ such that

$a_i^n = (a_i r)^n = 0, \forall i;\ a_i r \in I,\ \ b_j^m = (b_j r)^m = 0, \forall j;\ b_j r \in J$

Then for any $a \in I, b \in J$ there exists k such that $(a + b)^k = 0$.

Thus the theorem is proved.

CHAPTER IV

NIL AND NILPOTENT MODULES

Overview

In this chapter, we study the mathematical objects called modules. The use of modules was pioneered by one of the most prominent mathematicians of the first part of this century, Emmy Noether, who led the way in demonstrating the power and elegance of this structure. We shall see that the vector spaces are just special types of modules which arise when the underlying ring is a field. If R is a ring, the definition of an R-module M is closely analogous to the definition of a group action where R plays the role of the group and M the role of the set. The additional axioms for a module require that M itself have more structure (namely that M is an abelian group). Modules are the "representation objects" for rings, that is, they are, by definition, algebraic objects on which rings act. As the theory develops, it will become apparent how the structure of the ring R is reflected by the structure of its modules and vice versa.

4.1 Theorem [18]

The followings are equivalent for an R-module M.

(1) Descending chain condition (DCC) holds for submodules of M , that is, any descending chain $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \cdots \supseteq \cdots$ of submodules of M is stationary in the sense that $M_r = M_{r+1}$ for some r (We write this as $M_r = M_{r+1}$, $\forall\, r \gg 0$).

(2) Minimum condition for submodules holds for M , in the sense that any nonempty family of submodules of M has a minimal element.

Proof:

(1)=>(2): Let $\mathcal{F} = \{M_i : i \in I\}$ be a nonempty family of submodule of M . Pick any index $i_1 \in I$ and look at M_{i_1}. If M_{i_1} is minimal in $\mathcal{F}$ we are through. Otherwise, there is an $i_2 \in I$ such that $M_{i_1} \supset M_{i_2}$, $M_{i_1} \neq M_{i_2}$. If this M_{i_2} is minimal in $\mathcal{F}$, we are through again. Proceeding in this way, if we do not find a minimal element at any finite stage, we would end up with a non-stationary descending chain of submodules of M, namely $M_{i_1} \supset M_{i_2} \supset \cdots \supset M_{i_n} \supset \cdots$, contradicting (1).

(2)=>(1): Let $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \cdots \supseteq \cdots$ be a descending chain of submodules of M. Consider the nonempty family $\mathcal{F} = \{M_i : i \in \mathbb{N}\}$ of submodules of M. This must have a minimal element, say M_r, for some r. Now we have $M_s \subseteq M_r, \forall\, s \geq r$ which implies by minimality of M_r that $M_s = M_r, \forall\, s \geq r$.

4.2 Theorem [18]

Submodules and quotient modules of artinian modules are artinian.

Proof:

Let M be artinian and N a submodule of M. Any family of submodules of N is also one in M and hence the result follows. On the other hand, any descending chain of submodules of M/N corresponds to one in M (wherein each member contains N) and hence the result.

4.3 Theorem [18]

If a module M is such that it has a submodule N with both N and M/N are artinians, then M is artinian.

Proof:

Let $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \cdots \supseteq \cdots$ be a descending chain of M. Intersecting with N gives the descending chain in M , namely, $N \cap M_1 \supseteq N \cap M_2 \supseteq \cdots \supseteq N \cap M_n \cdots \supseteq \cdots$ which must be stationary, say $N \cap M_r = N \cap M_{r+1} = \cdots$ for some r. On the other hand, we have the descending chain in M/N, namely, $(N + M_1)/N \supseteq (N + M_2)/N \supseteq \cdots \supseteq (N + M_n)/N \supseteq \cdots$ which must be also stationary, say $(N + M_s)/N = (N + M_{s+1})/N = \cdots$ for some s. Now we prove the following:

Claim: $M_n = M_{n+1}, \forall\, n \geq (r + s)$.

This is an immediate consequence of the four facts, namely,

(1) $M_n \supseteq M_{n+1}, \forall\, n \in \mathbb{N}$;
(2) $N \cap M_n = N \cap M_{n+1}, \forall\, n \geq r$;
(3) $(N + M_n)/N = (N + M_{n+1})/N, \forall\, n \geq s$ and
(4) $(N + M_n)/N \simeq M_n/(N \cap M_n), \forall\, n \in \mathbb{N}$.
Putting together we get that
$M_n/(N \cap M_n) = (N + M_n)/N = (N + M_{n+1})/N = M_{n+1}/(N \cap M_{n+1})$,
which implies the claim and hence the result.

4.4 Theorem [18]

Let R be an artinian ring with unity. Then we have the followings:

(1) Every nonzero divisor in R is a unit. In particular, an artinian integral domain is a divisor ring.
(2) If R is commutative, every prime ideal is maximal. (In particular, a commutative artinian integral domain is a field).

Proof:

(1) Let $x \in R$ be not a zero divisor. Note then that x^r is not a zero divisor for any $x \in \mathbb{N}$. Since R is artinian, the descending chain of principal left ideals, namely, $(x)_l \supset (x^2)_l \supset \cdots \supset (x^n)_l \supset \cdots$ must be stationary, say $(x^n)_l = (x^{n+1})_l = \cdots$ for some $r \in \mathbb{N}$. Since $x^r \in (x^{n+1})_l$,
we can write $x^r = yx^{r+1}$ for some $y \in R$. This gives $(1 - yx)x^r = 0$ and hence $1 = yx$ (on cancelling x^r which is not a zero divisor). Now we have $x = x(yx) = (xy)x$ and hence $(1 - yx)x = 0$ implying $1 = yx$ (on cancelling x). Thus we get that $yx = 1 = xy$.

(2) If R is commutative artinian and P is a prime ideal in R, then R/P is an artinian integral domain and hence every nonzero element (being not a zero divisor) is a unit, that is, R/P is a field, that is, P is a maximal ideal, as required.

4.5 Theorem [18]

The followings are equivalent for an R-module M.

(1) Ascending chain condition (ACC) holds for submodules of M , that is, any ascending chain $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \cdots \subseteq \cdots$ of submodules of M is stationary in the sense that $M_r = M_{r+1}$ for some r (We write this as $M_r = M_{r+1}$,$\forall\, r \gg 0$).
(2) Maximum condition holds for M in the sense that any nonempty family of submodules of M has a maximal element.
(3) Finiteness condition holds for M in the sense that every submodule of M is finitely generated.

Proof:

(1)=>(2): Let $\mathcal{F} = \{M_i : i \in I\}$ be a nonempty family of submodule of M . Pick any index $i_1 \in I$ and look at M_{i_1}. If $M_{i_1}$is maximal in $\mathcal{F}$, we are through. Otherwise, there is an $i_2 \in I$ such that $M_{i_1} \subset M_{i_2}$, $M_{i_1} \neq M_{i_2}$. If this M_{i_2} is maximal in $\mathcal{F}$, we are through again. Proceeding in this way, if we do not find a maximal element at any finite stage, we would end up with a non-stationary ascending chain of submodules of M, namely $M_{i_1} \subset M_{i_2} \subset \cdots \subset M_{i_n} \subset \cdots$, contradicting (1).

(2)=>(3): Let N be a submodule of M. Consider the family $\mathcal{F}$ of all finitely generated submodules of N. This family is nonempty since the submodule (0) is a member. This family has a maximal member, say $N_0 = (x_1, \dots, x_r)$. If $N_0 \neq N$, pick an $x \in N, x \in N_0$.
Now $N_1 = N_0 + (x) = (x, x_1, \dots, x_r)$ is a finitely generated submodule of N and hence $N_1 \in \mathcal{F}$. But then this contradicts the maximality of N_0 in $\mathcal{F}$, since $N_0 \subset N_1, N_0 \neq N_1$ and so $N_0 = N$ is finitely generated.

(3)=>(1): Let $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \cdots \subseteq \cdots$ be a ascending chain of submodules of M. Consider the submodule $N = \bigcup_{i=1}^{\infty} M_i$ of M which must be finitely generated, say $N = (x_1, \dots, x_n)$. It follows that $x_i \in M_r$,$\forall\, i$,$1 \leq i \leq n$ for some $r(\gg 0)$. Now we have $N \subseteq M_s \subseteq N$,$\forall s \geq r$ and so $M_r = M_{r+1} = \cdots$.

4.6 Theorem [14]

Submodules and quotient modules of noetherian module are noetherian.

Proof:

Let M be a noetherian and N be a submodule of M. Any family of submodules of N is also one in M and hence the result follows. On the other hand, any ascending chain of submodules of M/N corresponds to one in M (wherein each member contains N) and hence the result.

4.7 Theorem [14]

If a module M is such that it has a submodule N with both N and M/N are noetherians, then M is noetherian.

Proof:
Let $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \cdots \subseteq \cdots$ be a ascending chain of M. Intersection with N gives the ascending chain in M, namely, $N \cap M_1 \subseteq N \cap M_2 \subseteq \cdots \subseteq N \cap M_n \cdots \subseteq \cdots$ which must be stationary, say $N \cap M_r = N \cap M_{r+1} = \cdots$ for some r. On the other hand, we have the ascending chain in M/N , namely, $(N + M_1)/N \subseteq (N + M_2)/N \subseteq \cdots \subseteq (N + M_n)/N \subseteq \cdots$ which must be also stationary, say $(N + M_s)/N = (N + M_{s+1})/N = \cdots$ for some s. Now we prove the following:

Claim: $M_n = M_{n+1}$, $\forall\, n \geq (r + s)$.

This is an immediate consequence of the four facts, namely,

(1) $M_n \subseteq M_{n+1}$, $\forall\, n \in \mathbb{N}$;
(2) $N \cap M_n = N \cap M_{n+1}$, $\forall\, n \geq r$;
(3) $(N + M_n)/N = (N + M_{n+1})/N, \forall\, n \geq s$ and
(4) $(N + M_n)/N \simeq M_n/(N \cap M_n), \forall\, n \in \mathbb{N}$.
Putting together we get that
$M_n/(N \cap M_n) = (N + M_n)/N = (N + M_{n+1})/N = M_{n+1}/(N \cap M_{n+1})$,
which implies the claim and hence the result.

4.8 Theorem [18]

Let R be a ring, $R \neq 0$. Then a minimum right ideal I is either nilpotent or an irreducible R-module.

Proof:

Here $I^2 \subseteq I$, so either $I^2 = 0$ and I is nilpotent or $I^2 = I$. Since $I^2 = I$ implies $IR \neq 0$, I is an irreducible R-module.

4.9 Theorem [18]

Let M be an R-module, where R is a semisimple. Then M is the sum of irreducible submodules.

Proof:

We know that $1 + e_1 + \cdots + e_n$, where each e_i is idempotent and $e_i R$ is a minimal right ideal. Let $v \in M$, then $v = v.1 = ve_1 + \cdots + ve_n$.

Suppose $ve_i \neq 0$. Then $ve_i R \neq 0$ and $ve_i R \cong e_i R$ as R-module. Since $e_i R$ is irreducible, so is $ve_i R$. Thus $v \in R,\ ve_i R \subseteq M$ and M is clearly a sum of irreducible submodules.

Hence the theorem is proved.

The following theorem is an extension of the above theorem for modules over associative endomorphism rings.

4.10 Theorem

Let R be a ring with identity and with DCC on right ideals. Let N be the radical of R and let M be an R-module. Then $MN = 0$ if and only if M is the sum of irreducible submodules.

Proof:

Let M is the sum of irreducible submodules, then any $m \in M$ is in $\sum_{k=1}^{n} M_k$, where M_k are irreducible.

Now $M_k N = M_k$ or, $M_k N = 0$. If $N^j = 0$, then $M_k N = M_k$ implies $M_k = 0$, a contradiction. Thus $M_k N = 0$, where $mN = 0$ and so $MN = 0$.
Conversely, suppose that $MN = 0$. Then we can consider M as R/N-module, by putting $m(r + N) = mr$ for all $r \in R$. Now R/N is semisimple and so, M is the sum of irreducible R/N-modules. Now let $\bar{M}$ be an irreducible R/Nmodule, then since $\bar{M}N = 0,\ \bar{M}$ is an R-module, where $mr = m(r + N)$. Moreover, $\bar{M}$ has no non zero proper R-submodules, since this would induce proper non zero R/N-submodules. Thus $\bar{M}$ is an irreducible R-module .

4.11 Theorem [14]

Let R be a commutative local ring whose maximal ideal is nilpotent. Then R is artinian if and only if it is noetherian.

Proof:

Let M be the maximal ideal of R with $M^r = (0)$. Let $K = R/M$ be the residue field of R. It is obvious that R is artinian (resp. noetherian) if and only if M is so. Now M is artinian (resp. noetherian) if and only if both M/M^2 and M^2 are so, etc. Secondly, since M annihilates M^i/M^{i+1}, it is a vector space over the field K and the R-module structure is the same as the vector space structure. But then we know M^i/M^{i+1} is artinian (resp. noetherian) if and only if M^i/M^{i+1} is finite dimensional over$K$.

Suppose R is artinian (resp. noetherian). Then M^i/M^{i+1} is artinian (resp. noetherian) and hence finite dimensional over K, for all $i = 0, 1, \dots, r-1$. Consequently, each is noetherian (resp. artinian). Now $M^{r-1} = M^{r-1}/M^r$ and M^{r-2}/M^{r-1} are both noetherian (resp. artinian) implies that M^{r-2} is noetherian (resp. artinian), etc. Proceeding thus we get that M is noetherian (resp. artinian).

4.12 Definition

By a composition series of a nonzero module M, we mean a finite descending chain of submodules of M starting with M and ending with (0), say
$M = M_0 \supset M_1 \supset \cdots \supset M_m = (0)$
such that the successive quotients $M_i = M_{i+1}$ are simple $\forall\, i$. The integer m is called the length of the series.

4.13 Corollaries [14]

(i) R_R is artinian => Rad (R) is the largest nilpotent right, left or two-sided ideal of R.
(ii) R is commutative and artinian => Rad (R) is the set of all nilpotent elements of R.
(iii) R_R is artinian => for every right R-module M (resp. for every left R-module M) we have
$Rad\ (R) = MRad(R) \hookrightarrow M$ (resp. $Rad\ (R) = Rad(R)M \hookrightarrow M$).

Proof:

(i) $Rad\ (R)$is nilpotent and every nilpotent ideal is contained in it.
(ii) Since $Rad\ (R)$ is nilpotent, every one of its elements is nilpotent. Let $a \in R, a^n = 0$. Then it follows that since R is commutative, $(aR)^n = a^n R^n = a^n R = 0R = 0$.
Thus aR is nilpotent, and consequently, $a \in aR \hookrightarrow Rad(R)$.
(iii) We have $Rad\ (M) = MRad(R)$[resp. $Rad\ (M) = Rad(R)M$].
Since $Rad(R)$ is nilpotent, there is an $n \in \mathbb{N}$ with $(Rad(R))^2 = 0$.
Now let for $U \hookrightarrow M$,

$$M = U + MRad(R).$$

Then by substituting the equality for $M(n-1)$ times into $MRad(R)$ it follows that on the right side of the equality we have $M = U + M(Rad(R))^n = U$.
Thus $MRad(R) \hookrightarrow M$ holds. This equally holds for left R-module.

4.14 Theorem [14]

Let $R/Rad(R)$ be semisimple and let $Rad(R)$ be nilpotent. Then the following are equivalent for a right R-module M.
(i) M is artinian.
(ii) M is noetherian.
(iii) M has finite length.

4.15 Theorem [14]

A module is of finite length if and only if it is both artinian and noetherian.

Proof:

Let M be a module of finite length. If $M = (0)$, then the result is obvious. Suppose $M \neq (0)$ and has a composition series, say $M = M_0 \supset M_1 \supset \cdots \supset M_m = (0)$.
Now proceed by induction on m. If $m = 1$, then M is simple and hence trivially M is both artinian and noetherian. Assume that $m \geq 2$ and the induction hypothesis that any module having some composition series of length at most $(m-1)$ is both artinian and noetherian. Now look at M_1 which has the composition series, namely, $M_1 \supset \cdots \supset M_m = (0)$, of length $(m-1)$. Hence M_1 is both artinian and noetherian. On the other hand, the quotient module M/M_1, being simple, is also both artinian and noetherian and it follows that M is both artinian and noetherian, as required.

Conversely, suppose M is both artinian and noetherian, we may assume that $M \neq (0)$. Since M is noetherian, it has a maximal submodule, say M_1. If $M_1 = (0)$, then M is simple and hence it is a module of finite length. Otherwise M_1, also being noetherian, has a maximal submodule, say M_2. If $M_2 = (0)$, we have a composition series for M, namely, $M = M_0 \supset M_1 \supset M_2 = (0)$. Proceeding thus, at any finite stage n, if $M_n \neq (0)$, we get a maximal submodule M_{n+1} of M_n and so on, yielding an infinitely descending chain of submodules of M, namely, $M = M_0 \supset M_1 \supset \cdots \supset M_n \supset \cdots \supset \cdots$, contradicting that M is artinian. Hence $M_m = (0)$ for some m.

4.16 Theorem [14]

Let R_1 and R_2 be two rings with DCC (ACC) on right ideals. Then $R_1 \oplus R_2$ also has DCC (ACC) on right ideals.

Proof:

Let us prove the theorem for DCC only.
Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be a descending chain of right ideals in $R_1 \oplus R_2$.
Let $A_i = \{a_i : (a_i, b_i) \in I_i\}$.
Then each A_i is a right ideal in R_1 and also $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$
Therefore, by the DCC in R_1, there exists an integer N such that $A_N = A_{N+1} = \cdots$.
Now for $i \geq N$, let $B_i = \{b_i : (0, b_i) \in I_i\}$.
Then B_i is a right ideal in R_2 and also $B_N \supseteq B_{N+1} \supseteq \cdots$
By the DCC in R_2, there exists an integer M such that $B_M = B_{M+1} = \cdots$.
We claim that $I_M = I_{M+1} = \cdots$. To show this, let $(a_M, b_M) \in I_M$.
Since $A_N = A_{N+1}$, there exists $(a_M, c_M) \in I_{M+1} \in I_M$.
Therefore $(a_M, b_M) - (a_M, c_M) \in I_M$, where $(0, b_M - c_M) \in I_M$.
Thus $(b_M - c_M) \in B_M = B_{M+1}$. So, $(0, b_M - c_M) \in I_{M+1}$.
Since $(a_M, c_M) \in I_{M+1}$,
$(a_M, c_M) + (0, b_M - c_M) = (a_M, b_M) \in I_{M+1}$.

This shows that $I_M \subseteq I_{M+1}$.
By hypothesis, $I_{M+1} \subseteq I_M$.
Therefore $I_M = I_{M+1}$ and so $R_1 \oplus R_2$ satisfies the DCC on right ideals.
Hence the theorem is proved for DCC.
Similarly, we can prove the theorem for ACC.

4.17 Theorem [18]

Let M be an R-module. Then M has a composition series if and only if M satisfies both the ACC and DCC on submodules.

Proof:

Suppose that M has a composition series of length n. If either the ACC or DCC fails, we can get a normal series of length$(n+1)$. Any refinement of this normal series clearly has length $\geq n+1$, so in particular, refining this normal series to a composition series leads to a composition series of length $\geq n+1$, contracting the existence of a composition series of length$n$. Thus both the ACC and DCC must hold.

Conversely, suppose the ACC and DCC, and hence the maximum and minimum conditions hold. If S is the collection of all proper submodules of $M = M_0$, S has a maximal element, say, M_1. Similarly, if $M_1 \neq 0$, there exists a proper submodule of M_1, say, M_2, maximal with respect to being a proper submodule. Continuing in this way, we get $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$, and clearly $\{M_0, M_1, \ldots, M_n\}$ is a composition series for M.
Thus the theorem is proved.

4.18 Proposition [12]

Let R be a noetherian ring and N be the prime radical of R. Then N is a nilpotent ideal of R containing all nilpotent right or left ideals of R.

4.19 Definition

Let M be a right R-module and $S = End_R(M)$, an endomorphism ring. Then X is a nil submodule of M if I_X is a right nil ideal of S.

From the definition, we see that X is a fully invariant nil submodule of M if and only if I_X is a two-sided nil ideal of S.

Note

If X is a nil submodule of M, then for any $f \in I_X$, $1-f$ is invertible in S.

4.20 Definition

Let M be a right R-module and X a submodule of M. We say that X is a nilpotent submodule of M if I_X is a right nilpotent ideal of S. By definition, a nilpotent submodule is a nil submodule.

From the definition, we see that X is a fully invariant nilpotent submodule of M if and only if I_X is a two-sided nilpotent ideal of S.

The following proposition is an extension of the above proposition for modules over associative endomorphism rings.

4.21 Proposition

Let M be a right R-module which is a self-generator. Let N be a semiprime submodule of M. Then N contains all nilpotent submodules of M.

Proof:

Let X be a nilpotent submodules of M. Then I_X is a right nilpotent ideal of S, so $I_X^n = 0$ for some positive integer n, and so $I_X^n(M) = 0 \subset N$. Since N is a semiprime submodule of M, then $I_X(M) \subset N$, so $X \subset N$.

4.22 Proposition [14]

Let I be a minimal left ideal in a ring R. Then either $I^2 = 0$ or $I = Re$, for some idempotent $e \in I$, that is, I is either nilpotent or generated by a idempotent.

The following proposition is an extension of the above proposition for modules over associative endomorphism rings.

4.23 Proposition

Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Let X be a simple submodule of M. Then either $I_X^2 = 0$ or $X = f(M)$ for some idempotent $f \in I_X$.

Proof:

Since X is a simple submodule of M, I_X is a minimal right ideal of S. Suppose that $I_X^2 \neq 0$. Then there is a $g \in I_X$ such that $gI_X \neq 0$. Since gI_X is a right ideal of S and $gI_X \subset I_X$, we have $gI_X = I_X$ by the minimality of I_X. Hence there exists $f \in I_X$ such that $gf = g$. The set $I = \{h \in I_X : gh = 0\}$ is a right ideal of S and I is properly contained in I_X since $f \notin I$. By the minimality of I_X, we must have $I = 0$. It follows that $f^2 - f \in I_X$ and $g(f^2 - f) = 0$, and hence $f^2 = f$. Note that $f(M) \subset X$ and $f(M) \neq 0$, and from this we have $f(M) = X$.

4.24 Proposition [15]

If a ring R satisfies ACC on two-sided ideals, then the prime radical $N(R)$ is a nilpotent ideal.

The following proposition is an extension of the above proposition for modules over associative endomorphism rings.

4.25 Proposition

Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M satisfies the ACC on fully invariant submodules, then $P(M)$ is nilpotent.

Proof:

If M satisfies the ACC on fully invariant submodules, then S satisfies the ACC on two-sided ideals. Indeed, $I_1 \subset I_2 \subset \cdots$ is an ascending chain of two-sided ideals of S, then $I_1(M) \subset I_2(M) \subset \cdots$ is an ascending chain of fully invariant submodules of M. Since M has the ACC on fully invariant submodules, there exists a positive integer n such that $I_n(M) = I_k(M)$ for all $k > n$. Thus $I_n = I_k$ for all $k > n$, showing that S satisfies the ACC on two-sided ideals. Therefore $P(S)$ is nilpotent. Since $(S) = I_{P(M)}$, we have $P(M)$ is nilpotent.

4.26 Theorem [12]

Let R be a noetherian ring and N be the prime radical of R. Then N is a maximal nilpotent ideal of R.

The following theorems are the extension of the above theorem for modules over associative endomorphism rings.

4.27 Theorem

Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then M is a semiprime module if and only if M contains no nonzero nilpotent submodules.

Proof:

By hypothesis, 0 is a semiprime submodule of M. If X is a nilpotent submodule of M, then $I_X^n = 0$ for some positive integer n, and hence $I_X^n(M) = 0$.

Note that $I_X(M) = 0$, we can see that $X = 0$.

Conversely, suppose that M contains no nonzero nilpotent submodules. Let I be an ideal of S such that $I^2(M) = 0$. Then we can write $I = I_{I(M)}$ and hence $I_{I(M)}^2 = 0$. It follows that $I(M)$ is a nilpotent submodule of M and we get $I(M) = 0$. Thus 0 is a semiprime submodule of M and thus M is a semiprime module.

4.28 Theorem

Let M be a quasi-projective, finitely generated right R-module which is a self-generator and $P(M)$ be the prime radical of M. If M is a noetherian module, then $P(M)$ is the largest nilpotent submodule of M.

Proof:

Let $\mathcal{F}$ be the family of all minimal submodules of M. Then we can we write $P(M) = \bigcap_{X \in \mathcal{F}} X$. But $P(M)$ contains all nilpotent submodules of M. Again $I_{P(M)} = \bigcap_{X \in \mathcal{F}} I_X = P(S)$. Note that from our assumption we can see that S is a right noetherian ring. Then there exist only finitely many minimal prime ideals of S and there is a finite product of them which is 0, says $P_1 \cdots P_n = 0$. Since $I_{P(M)}$ is contained in each $P_i, i = 1, \dots, n$, we have $I_{P(M)}^n = 0$. Thus $P(M)$ is nilpotent.

CONCLUSION

In this work, we developed the structure of nil and nilpotent submodules over associative endomorphism rings by modifying the structure of nil and nilpotent ideals over associative arbitrary rings. As generalizations of nil and nilpotent rings and modules over associative arbitrary rings, some characterizations of nil and nilpotent submodules over associative endomorphism rings are investigated in the present study.

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