Identification of Autoregressive Systems at a Very Low SNR Using Damped Cosine Model of Autocorrelation Function

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by

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Dedication

To my beloved parents.

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List of Principal Symbols

$x(n)$	Discrete time noise-free AR signal
$u(n)$	Input driving sequence (white noise) to AR system
$v(n)$	Additive white observation noise
σ_u^2	Variance of the sequence $u(n)$
σ_v^2	Variance of the sequence $v(n)$
$E[\cdot]$	Expectation operator
$y(n)$	Observed noisy signal
$R_{xx}(l)$	l-th lag of Autocorrelation function of noise-free signal $x(n)$
$R_{yy}(l)$	l-th lag of Autocorrelation function of noisy signal $y(n)$
$A(z)$	True order AR polynomial
a_i	i-th Autoregressive (AR) parameter
p	True AR model order
N	Number of data points
z_k	k-th pole of the AR system
r_k	Magnitude of the pole corresponds to z_k
ω_k	Angular position of the pole corresponds to z_k
C_k	Partial fraction coefficient corresponds to z_k
$h(n)$	Impulse response
$x^M(n)$	Proposed new AR signal representation
$R_{xx}^M(l)$	l-th lag of proposed autocorrelation function for noise-free signal
P_j, Q_j	Amplitudes of the proposed autocorrelation function

Abstract

This thesis work is concerned with the development of a new method for autoregressive (AR) system identification at a very low signal to noise ratio (SNR) from output observations corrupted by noise. The identification of AR systems at a very low SNR is still a challenging problem for researchers and no effective method has yet been reported. In all the existing methods it is invariably assumed that the AR signal and additive noise are uncorrelated. At a very low SNR this assumption is violated and the estimate of the autocorrelation function using the conventional method includes significant error. Thus all the autocorrelation based techniques fail to estimate the AR parameters below a certain positive value of SNR.

Intensive analysis of the underlying problem reveals that the basic assumption of uncorrelation between the AR signal and additive noise be relaxed. Unlike conventional approaches, in this thesis we present a novel method for computing the autocorrelation function using the convolution sum representation of the noise-free AR signal. Here, we express autocorrelation of the AR signal as a function of the system roots leading to a damped cosine model instead of conventional autoregressive model consisting of system parameters. Basically, the magnitudes and angular positions of the system poles are the unknowns of the new model. A least-square based technique is used to estimate these model parameters. The desired AR system parameters can then be obtained directly from these estimated model parameters. Accuracy of estimation of the AR parameters by the damped cosine model depends on the accuracy of estimation of the model parameters. Therefore, a total search technique is adopted to scan the entire domain of the unknown parameters. The proposed method guarantees the stability of the estimated AR system. The simulation results show that the method presented in this work can estimate the AR system parameters with high accuracy even at an SNR as low as -5 dB where no comparable results yet exist.

Chapter 1

Introduction

1.1 System Identification : Background

Mathematical modeling of a physical system from observed data is called system identification. It has acquired widespread applications in many areas. In control and system engineering, system identification methods are used to get appropriate models for characterizing the dynamic behaviour of a given system, and ultimately synthesizing a controller for the system [1]. In signal processing applications (such as in speech and image analysis, communications and mechanical engineering) models obtained by system identification are used for spectral analysis, fault detection, pattern recognition, adaptive filtering, linear prediction, image restoration, speaker recognition and other purposes. In economics, meteorology, astronomy and several other fields, with the help of system identification the hidden periodicity in the studied data may be determined, which are to be associated with cyclic behaviour or recurring process. In radar and sonar systems, the proper model of the system can provide information about the location of the sources (or targets) situated in the field of view by analyzing the received signals. In biomedical signal processing, system identification from various signals measured from a patient, such as electrocardiogram (ECG) signals, can provide useful material for diagnosis. In seismology, system identification helps in predicting some major events like volcano eruption or an earthquake. Seismic spectral estimation is also used to predict subsurface geologic structure in gas and oil exploration.

A dynamic system can be conceptually described as in Fig. 1.1. The system is driven by input signal $u(n)$ uncorrelated with the observation noise $v(n)$. The

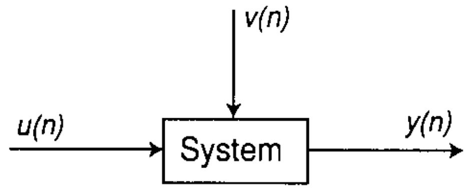


Fig. 1.1: General model of a dynamic system

user can control $u(n)$ but not $v(n)$. However two different cases may arise.

i. Both the input sequence $u(n)$ and the corresponding output sequence $y(n)$ are known.

ii. The input sequence $u(n)$ are totally unknown and only the output sequence $y(n)$ are known.

In the first case a definite mathematical model can be achieved by exploiting input and corresponding output sequences. Modeling of a motor or an electric heater are this type of problem. In the second case the system can be taken as a random process. No exact model can be achieved but only can be predicted by exploiting some statistical properties of observable output sequence $y(n)$. Speaker identification, pattern recognition, modeling the temperature of a city etc. are this type of problems. In both the cases the identification problem is to recover the parameters of the unknown systems.

In general, a finite-dimensional system is modeled as an autoregressive (AR) system, moving average (MA) system, or autoregressive moving average (ARMA) system. For a single input single output (SISO) system, the transfer function of the linear system is given by $B(z)/A(z)$, where $A(z)$ and $B(z)$ denote two finite-order coprime polynomials. When $B(z) = 1$ the system is called an AR system and for $A(z) = 1$ the system is called an MA system. Otherwise, the system is termed as an ARMA system. The first two systems, namely the AR and MA systems, may be viewed as the special cases of the ARMA system. The choice of an appropriate system depends upon the application. The problem of identifying such a system from noisy observations of its output response to an unknown input excitation, is of fundamental importance in many areas of engineering. For example, in spectral analysis of speech signal using linear prediction (LP) model,

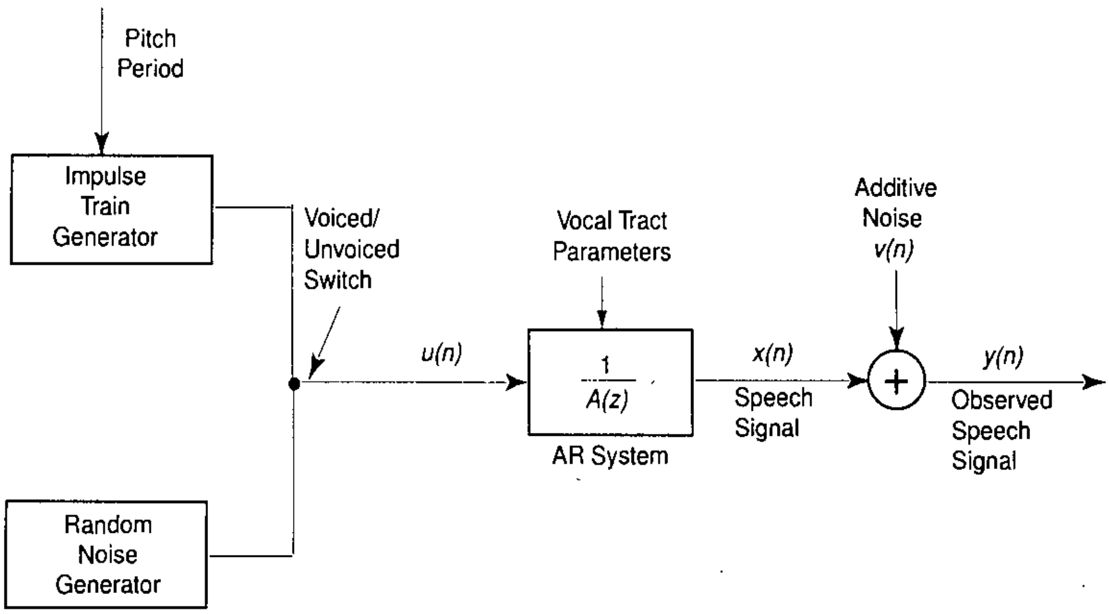


Fig. 1.2: Speech synthesis model based on AR system

the observed speech samples are assumed to be the output of an AR system driven by a quasiperiodic train of pulses for voiced sounds, or a random noise sequence for unvoiced sounds. Fig. 1.2 shows the model of speech signal corrupted by noise.

1.2 Literature Review

Numerous works have focused on this topic of system parameter estimation so far when the observations are noise-free [2]-[10]. In practical cases, however, observations contain additive noise and its effect cannot be neglected. Most of the recent works on system identification are for noise-corrupted observations [11]-[30]. In particular, the correlation-based methods are widely used in estimating the parameters of AR systems.

The problem of AR parameter estimation for the AR plus noise case was first examined by Walker [11] who evaluated the variance for the parameter estimate of a first-order AR process. Pagano [12], noted that the correct model for an AR(p) plus noise process is an autoregressive moving average (ARMA) process of order (p, p). In [12], an efficient AR parameter estimation method is described by using nonlinear regression.

It is shown in [13] that the displacement in estimated AR poles of an AR process is due to the introduction of spectral zeros caused by noise. To overcome the problem a large-order AR model is suggested. However, the order required for an accurate representation of the spectrum will depend upon the AR process and the SNR. AR process possessing a peaky power spectral density (PSD) will require a larger order model than an AR process with a smooth PSD. Also, the maximum model order is limited by the data record length. Too large a model order will result in spurious peaks. It is shown in [13] that these spurious peaks are a result of slight perturbations in the estimated noise poles, i.e. the poles which attempt to model the flat noise background.

The noise compensation based identification schemes require a *priori* knowledge of the additive noise variance [14], [15]. In [14], a noise compensation based approach is proposed which gives better results when the model order is larger than the actual AR order. This would seem to indicate that noise compensation is capable of only removing some of the noise effects and that to model the resulting autocorrelation function requires slightly larger order.

The noise compensated lattice filter (LF) algorithm proposed in [15] totally fails to estimate the AR parameters without a *priori* knowledge of the additive noise variance. Another shortcoming is that the stability of the noise compensated LF cannot be guaranteed [14].

In [16], it is shown that the successive autocorrelation will improve the signal-to-noise ratio. This improvement is signal dependent, with long correlation length signals (with respect to the sampling rate) showing the most pronounced improvement. The use of a successive autocorrelation operation as a preprocessor to a conventional all-pole parameter estimator can improve the parameter estimates significantly. As some poles are improved readily than others, and some are not improved at all, the estimates of the AR parameters do not demonstrate reduced bias. This method gives better results only in the cases where the poles are located near the unit circle in the z-plane.

The high-order Yule-Walker (HOYW) equations which do not require a *priori* knowledge of the additive noise variance can be used to estimate the AR parameters [17], [18]. However, this approach may suffer from singularity problem [31]. The possible singularity of the autocorrelation matrix leads to a substantial increase in the variance of the AR spectral estimate [14]. It has been shown in

[19] that the above method by Gingras [18] results in increased parameter estimation variance, which limits its application. In [20], [21] these problems were addressed by considering a least-squares solution to a combination of more than the minimal set of p HOYW equations.

Most of the previous methods either assume that the noise variance is known [14], [15] or subtract a suboptimal amount of noise power [22] to compensate for the noise effect. Determination of the optimal amount of noise power to remove the bias effect of noise is of utmost importance for AR parameter estimation from noisy observations. To combat this problem, Yahagi and Hasan [23] have proposed an iterative method using low-order Yule-Walker (LOYW) equations to compensate for the influence of noise in determining the AR parameters and to estimate the noise variance from a given set of noisy observations. But the case of strong nonlinearity of LOYW equations at low SNR was not addressed. Unfortunately, due to the inherent nonlinearity of LOYW equations when used for noisy observations with unknown noise variance and system parameters, the solutions found may not be unique depending upon the SNRs and the system characteristics. In these cases, the method fails to estimate the actual solution from the set of multiple solutions. Moreover, the method is computationally expensive as very small constant step size is maintained throughout the total range of search for better accuracy.

Davila [24] addressed a method of estimating the AR parameters and the noise variance by solving a matrix pencil called the noise-compensated Yule-Walker (NCYW) equations. For a p-th order AR system NCYW equations provide $(p+q)$ equations where the minimum value of q that will ensure a unique solution was not established. Among the $(p+q)$ equations, the first p equations are nonlinear and include the unknown AR parameters and the noise variance. The next q equations, however, are linear and include only the AR parameters. There are $(p+1)$ number of unknowns, with p number of AR parameters and a noise variance. One might expect that since there are a total of $(p+q)$ equations in $(p+1)$ unknowns, fewer than $q = p$ linear equations are needed. But Davila in his recent paper [25] has shown that this is not true and that $q \geq p$ is also a necessary condition for there to exist a unique solution.

Recently, a joint technique has been proposed in [26] which efficiently estimates the AR parameters and noise variance simultaneously. It is based on the high order and true order AR model fitting to the observed noisy process. In this method the first approach utilizes the uncompensated lattice filter algorithm to estimate the parameters of the over-fitted AR model. The second approach is iterative and it uses the noise compensated LOYW equations to estimate the true order AR model parameters. The desired AR parameters, equivalently the roots, are extracted from the over-fitted model roots using a root matching technique that utilizes the results obtained from the second approach. The major drawback of this method is its high computational complexity.

Among different Least-Squares (LS) based methods, the ILSNP (improved least-squares method with no prefiltering) method proposed in [27], provides accurate estimate of the AR parameters with less computational complexity but at high SNR. The basic idea of the ILSNP method is to increase the order of the underlying AR model by one, with the resulting augmented model containing a known parameter whose true value is just zero. The parameters of the augmented AR model are estimated by applying the standard least-square technique directly to the noisy observations. The least-square estimate of the augmented parameters is then shown to be related to the true augmented parameters and the observation noise variance in a very simple fashion, which forms a basis for estimating the observation noise variance. Using this estimated noise variance, the consistent estimate of AR parameters is obtained. The primary merits of this method are that it makes direct use of noisy observations without prefiltering and produces a direct estimate of the AR parameters and noise variances without parameter extraction.

In fact, all of the methods discussed can only identify an AR system up to a certain positive value of SNR. The phase matching technique, proposed in [28], claims that it can estimate the AR parameters at a low SNR. It minimizes the difference between the phase of the all-zero model and the phase of the maximum phase signal reconstructed from the power spectrum of the observed signal. The parameters of the AR model are obtained from the finite length sequence of the estimated all-zero model. This method involves nonlinear optimization. Due to strong non-linearity of the objective function, however, the convergence of this method is highly dependent on good initial conditions.

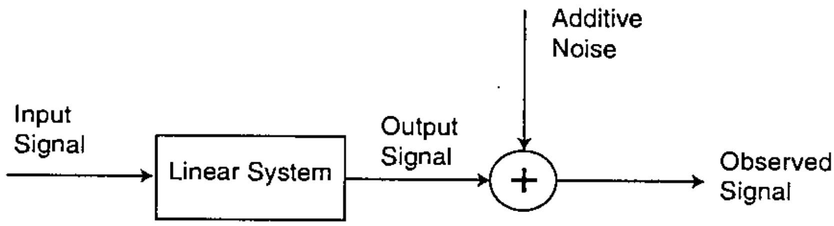


Fig. 1.3: A linear system with noise

In this research work, we consider identification of an AR system that can be modeled as in Fig. 1.3. The input signal is assumed to be white noise and is independent of the additive white observation noise. The problem of system identification is more critical in case of very low SNR (below 10 dB). In this work, we consider identification of the coefficients of the AR systems at a very low SNR using only the observed noisy signal.

1.3 Objective of This Research

The objective of this research is to propose a new technique for autoregressive (AR) system identification at a very low signal to noise ratio (SNR) using the *damped cosine model* for the autocorrelation function of the noise-free AR signal. Most of the system identification methods fail to estimate the system parameters at a low SNR, i.e., below 0 dB. Our aim was to overcome this situation. Intensive analysis of the underlying problems of system identification at a very low SNR, leads us to this new innovation.

In this research work, we have investigated the proposed method of AR parameter estimation for both low and high SNRs. The parameters of the *damped cosine model* are estimated using the given noisy observations. Then the AR system parameters can be directly obtained from this model parameters. With the estimated system parameters we can also estimate the AR spectrum quite accurately. A comparative study with the improved least-squares method with no prefiltering (ILSNP) [27] is provided to demonstrate the effectiveness of the proposed scheme. The proposed *damped cosine method* can estimate the system parameters with high accuracy even at an SNR as low as -5 dB.

1.4 Organization of the Thesis

In Chapter 2, a brief review of system identification techniques are presented. A brief description of the three linear models and the reason behind the selection of AR model are given. At first the deterministic models, where both the inputs and outputs are known, are described. Then the random process identification techniques are discussed where only the output sequences are known. The spectral density definition and the parametric methods of spectral estimation are also illustrated that are necessary to analyze the validation of the system identification methods. The effect of additive white noise on AR system identification is also elaborated in this chapter. A least-square based method [27] of AR system identification from noisy observations is explained in detail as we compare the results obtained by this method with the one proposed in this thesis.

In Chapter 3, a new method of AR system identification at a very low SNR is proposed. The reason behind the failure of other autocorrelation based methods at a very low SNR is explained. By the convolution sum implementation of the noise-free signal an alternative representation of the autoregressive signal is introduced. Using this signal representation the *damped cosine model* for the autocorrelation function of the noise-free AR signal is proposed. The method of determining the *damped cosine model* parameters which in turn gives the estimated AR parameters is presented.

In Chapter 4, different simulation results with figures and comments are included in detail. To examine the effectiveness of the *damped cosine model*, thirteen different AR systems are identified at various SNRs. Comparison with the results of other methods is presented in tabular form to show the superiority of the proposed method specially at very low SNRs.

The thesis concludes by presenting an overall discussion on the work and pointing out some unsolved problems for future work in Chapter 5.

In Appendix, the detail derivation of the convolution sum representation of the AR signal model and the *damped cosine model* are given.

Chapter 2

System Identification Methods

2.1 Classification of Systems

The transfer function of a system with input $u(n)$ and output $x(n)$ can be expressed as,

$$H(z) = \frac{X(z)}{U(z)} \tag{2.1}$$

where,

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$$

In general, a system whose output $x(n)$ at time n depends on any number of past output values $x(n-1)$, $x(n-2)$,... is called a *recursive system.* The input output relationship of a recursive system, described by a linear constant-coefficient difference equation, is linear and time-invariant. The system transfer function given in Eqn. (2.1) can be described by a linear constant coefficient *difference equation* of the form

$$x(n) = -\sum_{k=1}^{p} a_k x(n-k) + \sum_{k=0}^{q} b_k u(n-k) \tag{2.2}$$

Computing the z-transform of both sides, we obtain

$$X(z) = -\sum_{k=1}^{p} a_k X(z)z^{-k} + \sum_{k=0}^{q} b_k U(z)z^{-k} \tag{2.3}$$

The system transfer function can be expressed as

$$\begin{aligned} H(z) &= \frac{X(z)}{U(z)} \\ &= \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 + \sum_{k=1}^{p} a_k z^{-k}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_p z^{-p}} \\
&= \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 + \sum_{k=1}^{p} a_k z^{-k}} \\
&= \frac{B(z)}{A(z)} \\
&= b_0 z^{p-q} \frac{\prod_{k=1}^{q} (z - g_k)}{\prod_{k=1}^{p} (z - z_k)} \qquad (2.4)
\end{aligned}
$$

In Eqn. (2.4), if $b_k = 0$ for $k > 0$, the system is called Autoregressive or AR process. Then the system transfer function reduces to

$$
\begin{aligned}
H(z) &= \frac{b_0}{1 + \sum_{k=1}^{p} a_k z^{-k}} \\
&= z^p \frac{b_0}{\prod_{k=1}^{p} (z - z_k)} \qquad (2.5)
\end{aligned}
$$

The difference equation for input-output relationship under this condition can be obtained from Eqn. (2.2) as

$$
x(n) = -\sum_{k=1}^{p} a_k x(n-k) + u(n), \; b_0 = 1 \qquad (2.6)
$$

In this case, $H(z)$ consists of p poles, whose values are determined by the system parameters $\{a_k\}$ and a p-th order zero at the origin $z = 0$. We usually do not make reference to these trivial zeros, consequently, the system transfer function (Eqn. (2.5)) contains only nontrivial poles and the corresponding system is called an *all-pole* system [16]. Due to the presence of poles, the impulse response of such system is infinite in duration, and hence it is an IIR (Infinite Impulse Response) system.

In Eqn. (2.4) if $a_k = 0$ for $k \geq 1$, the system is called Moving Average (MA) system. The system transfer function in this case reduces to

$$
\begin{aligned}
H(z) &= \sum_{k=0}^{q} b_k z^{-k} \\
&= b_0 z^{-q} \prod_{k=1}^{q} (z - g_k) \qquad (2.7)
\end{aligned}
$$

The difference equation describing the input-output relationship under this condition is given by

$$x(n) = \sum_{k=0}^{q} b_k u(n-k) \tag{2.8}$$

In this case, $H(z)$ consists of q zeros, whose values are determined by the system parameters b_k and a q-th order pole at the origin $z = 0$. Since the system contains only trivial poles (at $z = 0$) and q nontrivial zeros, it is called *all-zero* system [29]. Due to the presence of zeros, the impulse response of such system is finite in duration, and hence it is an FIR (Finite Impulse Response) system.

The general form of the system transfer function given by Eqn. (2.4) contains both poles and zeros, and hence the corresponding system is called a *pole-zero* system or an ARMA (Autoregressive Moving Average) system, with p poles and q zeros. Poles and/or zeros at $z = 0$ and $z = \infty$ are implied but are not counted explicitly. Due to the presence of poles the pole-zero system is an also IIR system.

2.2 Why AR Model is Widely Used

Among the three linear models the AR model is by far the most widely used. The main reasons are given below.

1. The AR model is suitable for representing spectra with narrow peaks.

2. The AR model results in very simple linear equations for the AR parameters. It is possible to obtain reasonably good suboptimal estimates of the unknown AR parameters by solving a simultaneous set of linear equations [30].

3. Any minimum phase transfer function can be represented by a possibly infinite order, stable minimum phase AR-model [32]. If an AR model is picked erroneously, the unknown power spectral density can still be matched closely as long as a large enough AR model order is chosen.

On the other hand, the MA model, as a general rule, requires many more coefficients to represent a narrow spectrum. Consequently, it is rarely used by itself as a model for spectrum estimation. By combining poles and zeros, the ARMA model provides a more efficient representation, from the viewpoint of the number of model parameters, of the spectrum of a random process.

The decomposition theorem due to Wold (1938) asserts that any ARMA or MA process can be represented uniquely by an AR model of possibly infinite order, and any ARMA or AR process can be represented by an MA model of possibly infinite order. In view of this theorem, the issue of model selection reduces to selecting the model that requires the smallest number of parameters that are also easy to compute. Usually, the choice in practice is the AR model. The ARMA model usually gets the second preference due to the difficulties in finding the MA parameters especially when the output is corrupted by noise.

2.3 Random Process Modeling

In this case, the input sequence and the system transfer function both are unknown, only the output sequence is known. The unknown input sequence $u(n)$ is random in nature. Hence the AR system output $x(n)$ will also be random. We can write

$$\begin{aligned} x(n) &= h(n) * u(n) \\ &= \sum_{k=-\infty}^{\infty} h(k)u(n-k) \end{aligned} \tag{2.9}$$

2.3.1 Yule-Walker equations

Correlation-based methods are widely used in estimating the parameters of AR systems. The objective in computing the correlation between the two signals is to measure the degree to which the two signals are similar and thus to extract some information that depends to a large extent on the application. For the two signal $x(n)$ and $u(n)$, both of them having the same length of data N, the crosscorrelation sequence is defined as

$$R_{xu}(l) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)u(n-l) \tag{2.10}$$

If $x(n)$ and $u(n)$ are random signals, Eqn. (2.10) can be written as

$$R_{xu}(l) = E\left[x(n)u(n-l)\right], \quad \text{for } l \geq 0, \tag{2.11}$$

where $E\left[\cdot\right]$ is the expectation operator. When $x(n) = u(n)$, the autocorrelation of $x(n)$ is defined as

$$R_{xx}(l) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)x(n-l) \tag{2.12}$$

or,

$$R_{xx}(l) = E\left[x(n)x(n-l)\right], \quad \text{for } l \geq 0, \tag{2.13}$$

Multiplying both sides of Eqn. (2.2) by $x(n-l)$ and taking expected value we get,

$$\begin{aligned} E\left\{x(n)x(n-l)\right\} &= -\sum_{k=1}^{p} a_k E\left\{x(n-k)x(n-l)\right\} \\ &\quad + \sum_{k=0}^{q} b_k E\left\{u(n-k)x(n-l)\right\} \end{aligned} \tag{2.14}$$

or, using the definition given in Eqn. (2.13) we get,

$$R_{xx}(l) = -\sum_{k=1}^{p} a_k R_{xx}(l-k) + \sum_{k=0}^{q} b_k R_{ux}(l-k) \tag{2.15}$$

Using Eqn. (2.9) and Eqn. (2.11) we can write

$$\begin{aligned} R_{ux}(l) &= E\left\{\left[\sum_{k=0}^{\infty} h(k)u(n-k)\right] u(n+l)\right\} \\ &= \sum_{k=0}^{\infty} h(k)E\left\{u(n-k)u(n+l)\right\} \end{aligned} \tag{2.16}$$

We consider $u(n)$ as a *white noise*. In that case we can use the following relation

$$E\left\{u(n)u(n+l)\right\} = \sigma_u^2 \delta(l) \tag{2.17}$$

Where $\delta(l)$ is the Kronecker delta function, i.e.,

$$\begin{aligned} \delta(l) &= 1, \ \text{for } l = 0 \\ &= 0, \ \text{for } l > 0 \end{aligned} \tag{2.18}$$

Using Eqn. (2.17), we can write Eqn. (2.16) as

$$R_{ux}(l) = \sum_{k=0}^{\infty} h(k)\sigma_u^2 \delta(k+l) \tag{2.19}$$

By the definition given in Eqn. (2.18), $\delta(k+l)$ is nonzero only at $k = -l$. Hence Eqn. (2.19) can be written as

$$\begin{aligned} R_{ux}(l) &= \sigma_u^2 h(-l), \qquad \text{for } \ l \leq 0 \\ &= 0, \qquad\qquad \text{for } \ l > 0 \end{aligned} \tag{2.20}$$

Now the autocorrelation sequence of Eqn. (2.15) can be written as

$$R_{xx}(l) = -\sum_{k=1}^{p} a_k R_{xx}(l-k) + \sigma_u^2 \sum_{k=0}^{q} b_k h(k-l) \tag{2.21}$$

If we change the variable $(k-l)$ to $\bar{k}$, the last term of the right hand side of Eqn. (2.21) becomes $\sigma_u^2 \sum_{\bar{k}=-l}^{q-l} b_{\bar{k}+l} h(\bar{k})$. Imposing causality condition and replacing $\bar{k}$ by k, we can write this term as

$$\begin{aligned} \sigma_u^2 \sum_{k=0}^{q} b_k h(k-l) &= \sigma_u^2 \sum_{k=0}^{q-l} b_{k+l} h(k), \text{ for } 0 \le l \le q \\ &= 0 \qquad\qquad\qquad\qquad \text{for } l > q \end{aligned} \tag{2.22}$$

Substituting Eqn. (2.22) into Eqn. (2.21) gives,

$$\begin{aligned} R_{xx}(l) &= -\sum_{k=1}^{p} a_k R_{xx}(l-k), \qquad\qquad \text{for } l > q \\ &= -\sum_{k=1}^{p} a_k R_{xx}(l-k) + \sigma_u^2 \sum_{k=0}^{q-l} b_{k+l} h(k), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 \le l \le q \\ &= R_{xx}(-l), \qquad\qquad\qquad\qquad \text{for } l < 0 \end{aligned} \tag{2.23}$$

The relationship in Eqn. (2.23) applies, in general, to the ARMA(p, q) process. For an AR(p) process, setting $q = 0$ in Eqn. (2.23), we obtain

$$\begin{aligned} R_{xx}(l) &= -\sum_{k=1}^{p} a_k R_{xx}(l-k), \qquad \text{for } l > 0 \\ &= -\sum_{k=1}^{p} a_k R_{xx}(l-k) + \sigma_u^2, \text{ for } l = 0 \\ &= R_{xx}(-l), \qquad\qquad\qquad \text{for } l < 0 \end{aligned} \tag{2.24}$$

Thus we have a linear relationship between $R_{xx}(l)$ and $\{a_k\}$ parameters. Eqn. (2.24) is the *Yule-Walker* equation. To determine the AR parameters $\{a_k\}$, any p equations from Eqn. (2.24) for $l > 0$ may be solved. Then it is also possible to estimate σ_u^2 from Eqn. (2.24) for $l = 0$. The set of equations, for $l = 1, 2, \cdots, p$ can be expressed in the matrix form as

$$\begin{bmatrix} R_{xx}(0) & R_{xx}(-1) & \cdots & R_{xx}(-(p-1)) \\ R_{xx}(1) & R_{xx}(0) & \cdots & R_{xx}(-(p-2)) \\ \vdots & \vdots & \ddots & \vdots \\ R_{xx}(p-1) & R_{xx}(p-2) & \cdots & R_{xx}(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = - \begin{bmatrix} R_{xx}(1) \\ R_{xx}(2) \\ \vdots \\ R_{xx}(p) \end{bmatrix} \tag{2.25}$$

Using vector matrix notation Eqn. (2.25) can be written as

$$\mathbf{Ra} = -\boldsymbol{\Upsilon} \tag{2.26}$$

where $\mathbf{R}$ is the correlation matrix of dimension $(p \times p)$Eqn. (2.25) is known as the low-order Yule-Walker (LOYW) equation. Taking $l = p+1, p+2, \cdots, 2p$ in Eqn. (2.24), the resulting equation will not involve $R_{xx}(0)$, i.e.,

$$\begin{bmatrix} R_{xx}(p) & R_{xx}(p-1) & \cdots & R_{xx}(1) \\ R_{xx}(p+1) & R_{xx}(p) & \cdots & R_{xx}(2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{xx}(2p-1) & R_{xx}(2p-2) & \cdots & R_{xx}(p) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = - \begin{bmatrix} R_{xx}(p+1) \\ R_{xx}(p+2) \\ \vdots \\ R_{xx}(2p) \end{bmatrix} \tag{2.27}$$

Eqn. (2.27) is known as the high-order Yule-Walker (HOYW) equation. The Levinson-Durbin algorithm [33]-[38] provides an efficient technique for solving matrix Eqn. (2.25).

2.3.2 Power spectral density of random signals

In applications most of the signals encountered are indeterministic, i.e., their variation in the future cannot be determined exactly. Only the probabilistic statements can be made about the variation. Signals of such category are described by random sequence which consists of an ensemble of possible realizations, each of which has some associated probability of occurrence. They can be categorized as wide sense stationary (WSS), stochastic process with zero mean. In stochastic system identification problems, only the output sequence is known. As we don't know about the input sequence and the system transfer function, the variation of the output sequences in future cannot be predicted exactly.

In addition to the system parameter extraction we are also interested in determining the spectral behaviour of the system. Spectral analysis considers the problem of determinig the spectral content (i.e., the distribution of power over frequency) of a time series from a finite set of measurements, by means of either nonparametric or parametric techniques. In this case the autocorrelation function

$$R_{xx}(l) = E\left[x(n+l)x^*(n)\right] \tag{2.28}$$

provides the basis for spectrum analysis, rather than the random process itself. The Wiener-Khinchin theorem gives us the power density spectrum (PSD) $P_x(f)$

as

$$P_x(f) = \int_{-\infty}^{\infty} R_{xx}(l) exp(-j2\pi f l) dl \tag{2.29}$$

For an ergodic process Eqn. (2.28) becomes

$$R_{xx}(l) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(n+l)x^*(n) dn \tag{2.30}$$

With the help of Eqn. (2.30) an alternate form of Eqn. (2.29) can be obtained as [40]-[42]

$$P_x(f) = \lim_{T\to\infty} E\left\{ \frac{1}{2T} \left| \int_{-T}^{T} x(n) exp(-j2\pi f n) dn \right|^2 \right\} \tag{2.31}$$

The indirect approach of PSD estimation using the autocorrelation function was introduced by Blackman and Tukey [43] and the spectral estimator is known as *correlogram.* The other PSD estimator, based on the direct approach using the FFT (Fast Fourier Transform) algorithm [44] is known as the periodogram. The high variance of the periodogram and correlogram methods motivates the development of modified methods that have lower variance.

In the parametric or model-based method of PSD estimation at first an estimate of the system parameters are determined. The signal's spectral characteristics of interest are then derived from the estimated system parameters. Model-based method is better in case of shorter data length.

The system function $H(z)$ for the ARMA process is described in Eqn. (2.4). In this case, the power spectrum at the output, $P_x(z)$, is related to the power spectrum of the input stochastic process, $P_u(z)$, as

$$P_x(z) = H(z)H^*(1/z^*)P_u(z) = \frac{B(z)B^*(1/z^*)}{A(z)A^*(1/z*)} \cdot P_u(z) \tag{2.32}$$

If $u(n)$ is assumed to be a white-noise sequence of zero mean and variance σ_u^2, then $P_u(z) = \sigma_u^2$. Considering $z = exp(j2\pi f)$, where f is the normalized frequency, the PSD of an ARMA process is given by

$$P_{\text{ARMA}}(f) = P_x(f) = \sigma_u^2 \left| \frac{B(f)}{A(f)} \right|^2 \tag{2.33}$$

where $A(f) = A(exp[j2\pi f])$ and $B(f) = B(exp[j2\pi f])$. The PSD of an MA process will be

$$P_{\text{MA}}(f) = \sigma_u^2 \left| B(f) \right|^2 \tag{2.34}$$

The PSD of an AR process will be

$$P_{\text{AR}}(f) = \frac{\sigma_u^2}{|A(f)|^2} \tag{2.35}$$

Though the primary objective of this work is to propose a method for AR parameter estimation, it is equally applicable for spectral estimation from noisy observations. Conventional periodogram and Blackman-Tukey analysis lead to spectral estimates that are characterized by many "hills and valleys", as it involves the Fourier transform of a zero mean random process though a window may be used for smoothing. A p-th order AR spectral estimate is supposed to have peaks less than or equal to p.

2.4 Effect of Noise on AR System Identification

A very important problem with the AR system identification is its sensitivity to the addition of noise to the observations [12]. As shown in Fig. 2.1, if a signal $x(n)$ is contaminated by zero mean white noise $v(n)$, the observed signal $y(n)$ is described as

$$y(n) = x(n) + v(n) \tag{2.36}$$

Assuming that $x(n)$ is the output signal of a p-th order AR model excited by

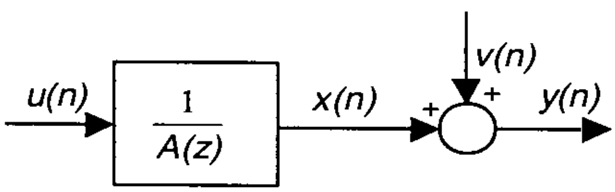


Fig. 2.1: Autoregressive process with noise

white noise $u(n)$, substitution of Eqn. (2.6) into Eqn. (2.1) results

$$y(n) = -\sum_{k=1}^{p} a_k x(n-k) + u(n) + v(n) \tag{2.37}$$

where, $u(n)$ is the zero mean white noise uncorrelated with $v(n)$ and $E\{u^2(n)\} = \sigma_u^2$ and $E\{v^2(n)\} = \sigma_v^2$. Here $E[\cdot]$ denotes the expectation operator.

Let us consider N samples of $y(n)$, $0 \leq n \leq N-1$. The autocorrelation sequence of $y(n)$ is given by,

$$R_{yy}(l) = \frac{1}{N} \sum_{n=0}^{N-1} y(n)y(n-l) \tag{2.38}$$

Substituting Eqn. (2.36) into Eqn. (2.38), we obtain

$$\begin{aligned} R_{yy}(l) &= \frac{1}{N}\sum_{n=0}^{N-1}[x(n)+v(n)][x(n-l)+v(n-l)] \\ &= \frac{1}{N}\sum_{n=0}^{N-1}x(n)x(n-l) \\ &\quad +\frac{1}{N}\sum_{n=0}^{N-1}[x(n)v(n-l)+v(n)x(n-l)] \\ &\quad +\frac{1}{N}\sum_{n=0}^{N-1}v(n)v(n-l) \\ &= R_{xx}(l)+R_{xv}(l)+R_{vx}(l)+R_{vv}(l) \end{aligned} \tag{2.39}$$

The first term $R_{xx}(l)$ on the right-hand side of the Eqn. (2.39) is the autocorrelation sequence of $x(n)$. It has significant values when shift l is small in comparison to N. However, as l approaches to N, the peaks are reduced in amplitude due to the fact that we have a finite data record of N samples so that many of the products $x(n)x(n-l)$ are zero. The crosscorrelations $R_{xv}(l)$ and $R_{vx}(l)$ between the signal $x(n)$ and the additive random noise $v(n)$ are expected to be relatively small as they are mutually uncorrelated. Hence these two terms can be neglected. Finally, the last term on the right-hand side of Eqn. (2.39) is the autocorrelation sequence of the random sequence $v(n)$. This correlation sequence will certainly contain a peak at $l = 0$, but because of its random characteristics, $R_{vv}(l)$ is expected to decay rapidly toward zero. Hence $R_{vv}(l)$ for $l > 0$ can be neglected. Now Eqn. (2.39) can be written as

$$\begin{aligned} R_{yy}(l) &= R_{xx}(l)+\sigma_v^2, \quad \text{for } l=0 \\ &= R_{xx}(l), \qquad\quad \text{for } l>0 \end{aligned} \tag{2.40}$$

Fig. 2.2 displays all the autocorrelation terms involve in Eqn. (2.39) at an SNR=-5 dB for a sixth order AR system with the parameters $a_1 = -0.8600, a_2 = 1.0494, a_3 = -0.6680, a_4 = 0.9592, a_5 = -0.7563$ and $a_6 = 0.5656$. The detail analysis of this system is given in chapter 4 where the system is termed as **system 12**. From Fig. 2.2, it is clear that at a very low SNR even for $l > 0$, $R_{xv}(l)$, $R_{vx}(l)$ and $R_{vv}(l)$ cannot be neglected. With the increase in noise level the effect of $R_{xv}(l)$, $R_{vx}(l)$ and $R_{vv}(l)$ are pronounced. At a very low SNR the autocorrelation function of the noisy signal includes significant error at all lags resulting from the

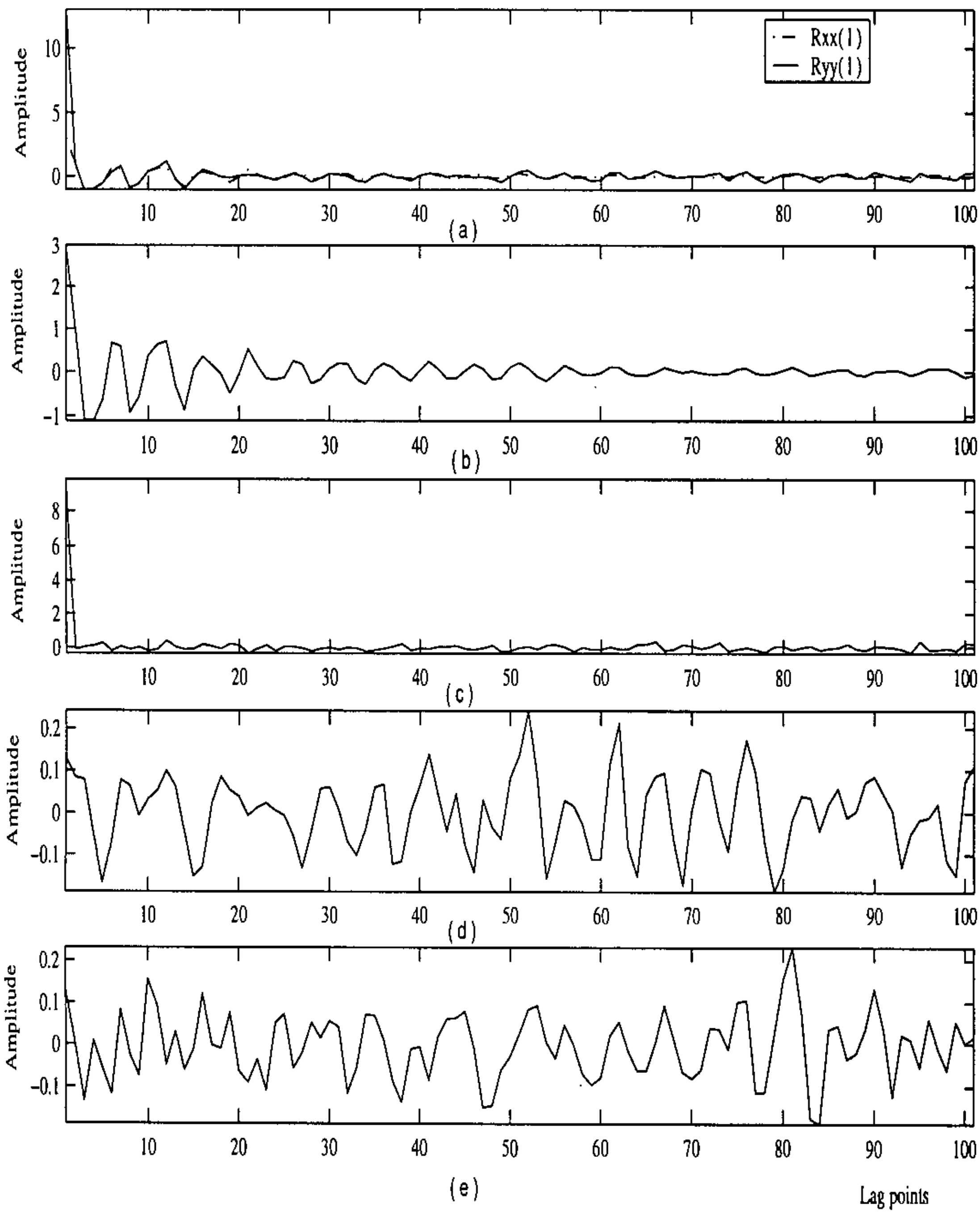


Fig. 2.2: Effect of noise on autocorrelation function. AR **System 12** is considered at an SNR=−5 dB : (a) Conventional autocorrelation function of the noise-free signal ($R_{xx}(l)$) obtained by Eqn. (2.12) and autocorrelation function of the noisy signal ($R_{yy}(l)$) obtained by Eqn. (2.39); (b) R_{xx}; (c) R_{vv}; (d) R_{xv} and (e) R_{vx}.

violation of assumption that the output AR signal $x(n)$ and additive noise $v(n)$ are uncorrelated.

In Fig. 2.3 the effect of increase in noise on nonzero lags of the autocorrelation function of the noisy signal is shown. For **system 12**, autocorrelation function of the noise-free signal and the autocorrelation function of the noisy signal at different SNRs are observed. The different SNRs considered are 10 dB, −5 dB and −10 dB. At SNR=−10 dB, the pattern of the $R_{yy}(l)$ totally differs from $R_{xx}(l)$. Even at SNR=−5 dB, $R_{yy}(l)$ includes significant error at nonzero lags. Hence it is very difficult to identify a system at low SNR using the noisy autocorrelation function.

Our research is carried out to estimate the AR system parameters from the noisy observations. The noise power σ_v^2 is also assumed to be unknown. Noise cancellation schemes that compensate the autocorrelation lags for the noise can be found in [14]. A serious deficiency is that, one does not know how much noise power to remove. Thus, if the subtracted noise power is inaccurate, there will be wrong estimation of system parameters.

2.5 A Least-Square Based Method for Identification of AR Systems in the Presence of Noise

In this section, we describe the improved least-square method with no prefiltering (ILSNP) [27] that deals with the problem of estimating the unknown parameters of an AR system corrupted by white noise. For the noisy AR system shown in Fig. 2.1 the covariances $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are

$$\begin{aligned} \mathbf{\Psi} &= \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{y}_n \mathbf{y}_n^T \\ \mathbf{\Phi} &= \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{y}_n y(n) \\ \mathbf{y}_n^T &= [y(n-1) \;\; y(n-2) \cdots y(n-p)] \end{aligned} \tag{2.41}$$

where $y(n)$ is described in Eqn. (2.37). The transfer function of this AR process is described in Eqn. (2.5). Our aim is to determine the AR parameters $\mathbf{a} = [a_1 \;\; a_2 \cdots a_p]$.

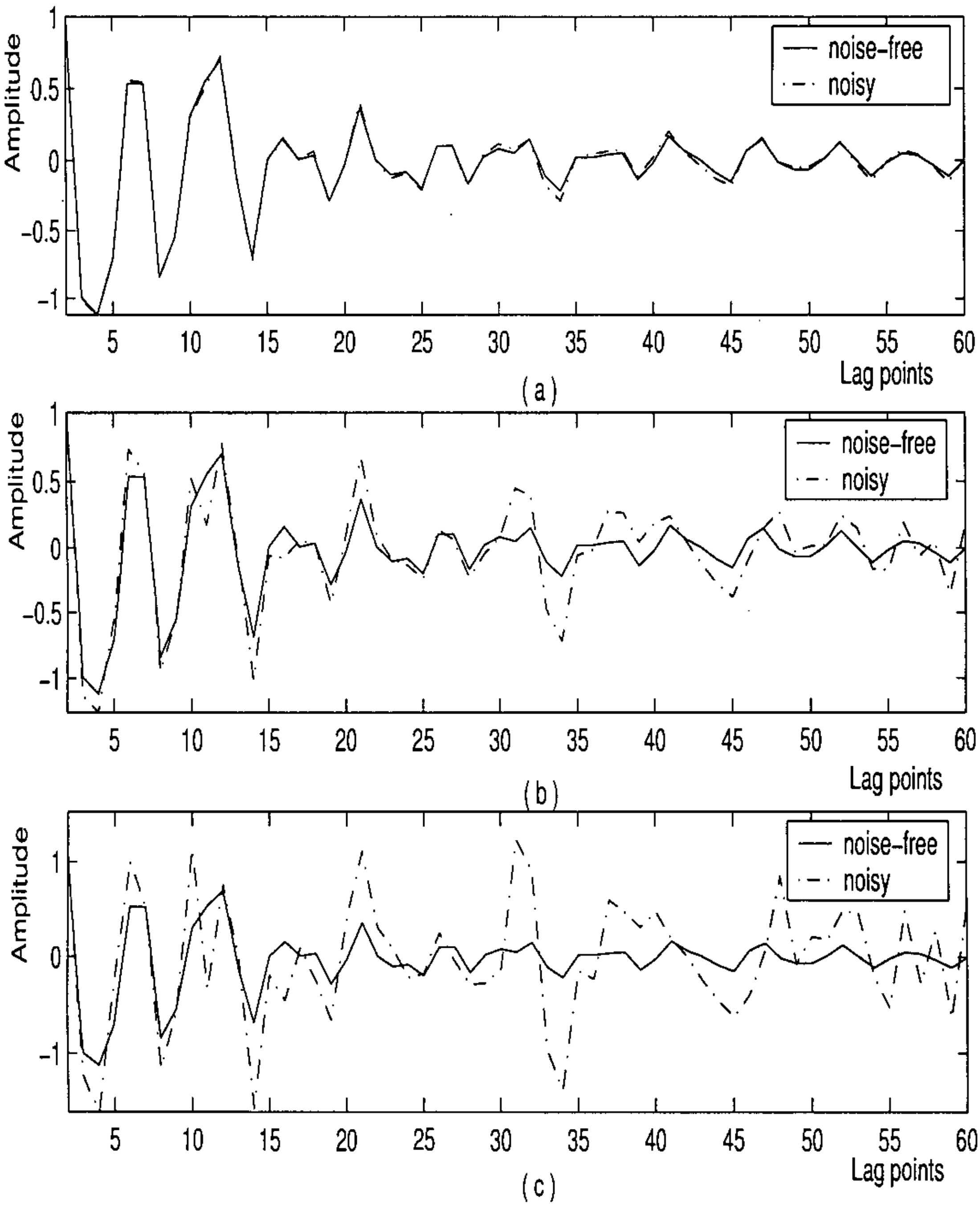


Fig. 2.3: Effect of increase in noise on the autocorrelation function of the noisy signal for nonzero positive lags. The autocorrelation functions of the noise-free and noisy signals for nonzero positive lags are shown for AR **System 12** : (a) SNR=10 dB; (b) SNR=−5 dB and (c) SNR=−10 dB.

$R_{yy}(l)$ of Eqn (2.38) and $\mathbf{\Phi}$ of Eqn. (2.41) produce the similar results. On the other hand $\mathbf{R}$ introduced in Eqn. (2.26) and $\mathbf{\Psi}$ of Eqn. (2.41) are also similar. Hence like the low-order Yule-Walker equations described in Eqn. (2.26) we can write

$$\mathbf{a}_{LS} = -\mathbf{\Psi}^{-1}\mathbf{\Phi} \tag{2.42}$$

where $\mathbf{a}_{LS}$ are the estimated AR parameters. But there exists a basic difference between Eqn. (2.26) and Eqn. (2.42). Eqn. (2.42) uses the noisy observations $y(n)$ instead of actual output $x(n)$ that should be used in Eqn. (2.26). In the previous section we have discussed that if we use noisy autocorrelation $(R_{yy}(l))$ instead of the true autocorrelation $(R_{xx}(l))$, there will be inaccurate estimation of AR parameters at low SNR. Similarly the use of noisy covariances instead of the true covariances will introduce error in the estimated AR parameters, $\mathbf{a}_{LS}$. To overcome the problem, in [27] a least-square based technique is described. Let us introduce an additional parameter in the AR model polynomial $A(z)$ and thus resulting augmented model polynomial $\bar{A}(z)$ can now be defined as

$$\bar{A}(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_p z^{-p} + \bar{a}_{p+1} z^{-(p+1)} \tag{2.43}$$

where

$$\bar{\mathbf{a}}^T = [a_1 \;\; a_2 \cdots\cdots a_p \;\; \bar{a}_{p+1}] = \left[\mathbf{a}^T \;\; \bar{a}_{p+1}\right] \tag{2.44}$$

For a p-th order AR system the true value of the introduced parameter $\bar{a}_{p+1} = 0$. Using the least-square methods the augmented AR parameters can be obtained as

$$\bar{\mathbf{a}}_{LS} = \bar{\mathbf{\Psi}}^{-1}\bar{\mathbf{\Phi}} \tag{2.45}$$

where

$$\begin{aligned}
\bar{\mathbf{\Psi}} &= \frac{1}{N}\sum_{n=0}^{N-1} \bar{\mathbf{y}}_n \bar{\mathbf{y}}_n^T \\
\bar{\mathbf{\Phi}} &= \frac{1}{N}\sum_{n=0}^{N-1} \bar{\mathbf{y}}_n y(n) \\
\bar{\mathbf{y}}_n^T &= [y(n-1) \; y(n-2) \cdots y(n-p) \;\; y(n-p-1)]
\end{aligned} \tag{2.46}$$

According to the whiteness properties of $u(n)$ and $v(n)$ and the uncorrelativity between them, we can obtain from Eqn. (2.45) the following expression:

$$\bar{\mathbf{a}} = \bar{\mathbf{a}}_{LS} - \sigma_v^2 \bar{\mathbf{\Psi}}^{-1}\bar{\mathbf{a}} \tag{2.47}$$

where σ_v^2 is the observation noise variance. Let us consider a $((p+1)\times 1)$ vector μ. If $\mu^T = [0\ 0 \cdots 0\ 1]$, Eqn. (2.44) gives

$$\mu^T \bar{\mathbf{a}} = 0 \tag{2.48}$$

Premultiplying Eqn. (2.47) by μ^T and using Eqn. (2.48), an estimate of σ_v^2 can be obtained as

$$\sigma_v^2 = \frac{\mu^T \bar{\mathbf{a}}_{LS}}{\mu^T \bar{\Psi}^{-1} \bar{\mathbf{a}}} \tag{2.49}$$

Now the iterative method proposed in [27] for estimating the AR parameters can be summarized as follows.

1. From a finite number of noisy observations $\{y(0), y(1), \cdots, y(N-1)\}$ we can evaluate $\bar{\Psi}$ and $\bar{\Phi}$ using Eqn. (2.46). Then the initial value of AR parameters can be determined using Eqn. (2.45). This value will be the initial estimate of the AR parameters by the ILSNP method, i.e.

$$\bar{\mathbf{a}}_{ILSNP}^{(i)} = \bar{\mathbf{a}}_{LS},\ i = 0 \tag{2.50}$$

   where the superscript i denotes the iteration number.

2. Using Eqn. (2.49), σ_v^2 can be computed as

$$\sigma_v^{2(i)} = \frac{\mu^T \bar{\mathbf{a}}_{LS}}{\mu^T \bar{\Psi}^{-1} \bar{\mathbf{a}}_{ILSNP}^{(i-1)}} \tag{2.51}$$

3. Then the ILSNP estimate for $\bar{\mathbf{a}}$ can be obtained from Eqn. (2.47) as

$$\bar{\mathbf{a}}_{ILSNP}^{(i)} = \bar{\mathbf{a}}_{LS} - \sigma_v^{2(i)} \bar{\Psi}^{-1} \bar{\mathbf{a}}_{ILSNP}^{(i-1)} \tag{2.52}$$

4. Repeat steps (2) and (3) until the termination criterion is satisfied. The criterion may be chosen as

$$\frac{\|\bar{\mathbf{a}}_{ILSNP}^{(i)} - \bar{\mathbf{a}}_{ILSNP}^{(i-1)}\|}{\|\bar{\mathbf{a}}_{ILSNP}^{(i)}\|} \leq 10^{-3} \tag{2.53}$$

   or when the number of iterations exceeds 20. With the fulfillment of the chosen criterion, we can get the desired estimated AR parameters from $\bar{\mathbf{a}}_{ILSNP}^{(i)}$.

2.6 Conclusion

In this chapter different methods for estimating the parameters of AR systems have been discussed. Three linear models, ARMA, MA and AR, are explained and the logic behind the popularity of AR model is stated. The detail analysis of the random process modeling is given. Yule-Walker equation based on autocorrelation function is derived. In addition the effect of noise on the autocorrelation function is explained. The least-square based technique is also analyzed in detail as we intend to give a comparative study of the results obtained by our method with that of the improved least-squares method with no prefiltering (ILSNP) [27].

Chapter 3

AR System Identification by Damped Cosine Method

3.1 Introduction

The problem of identifying an AR system from noisy observations of its output response to an unknown white noise input excitation, is of fundamental importance in several areas such as speech processing, spectral estimation, economics, seismology and biomedical signal processing. Several methods have focused on this topic of AR parameter estimation from noise corrupted observations [14]-[30]. The noise compensated lattice filter (LF) algorithm proposed in [15] fail to estimate the AR parameters without *a priori* knowledge of the additive noise variance. Another shortcoming is that the stability of the noise compensated LF cannot be guaranteed [14]. The high-order Yule-Walker (HOYW) equations which do not require *a priori* knowledge of the additive noise variance can be used to estimate the AR parameters [23]. However, this approach may suffer from singularity problem [31]. The possible singularity of the autocorrelation matrix leads to a substantial increase in the variance of the AR spectral estimate [14]. Recently, an iterative method using the low order Yule-Walker (LOYW) equations has been proposed in [26] which estimates the AR parameters and noise variance simultaneously. The major drawback of this method is its high computational complexity. Among other methods, the improved least-squares method with no prefiltering (ILSNP) of noisy observations reported in [27] provides accurate estimate of the AR parameters at high SNR with less computational complexity. In fact, all of the methods discussed can only identify an AR system up to a certain positive value of SNR. The phase matching technique proposed in [28] claims that it can estimate the AR parameters at a low SNR. However, due to strong non-linearity of the objective function, the convergence of this method is strongly dependent on good initial conditions.

In this paper, we investigate a new method of AR parameter estimation from noisy observations of very low to high SNRs. A *damped cosine model* for the autocorrelation function of the noise-free signal is adopted for AR parameter estimation. The parameters of the *cosine model* are estimated using the given noisy observations. The desired AR system parameters are directly obtained from this model parameters.

3.2 Problem Formulation

If a signal $x(n)$ is contaminated by a white noise process $v(n)$ with distribution $\mathcal{N}(0, \sigma_v^2)$, the observed signal $y(n)$ is obtained as

$$y(n) = x(n) + v(n) \tag{3.1}$$

Assume that $x(n)$ is the output signal of a p-th order AR system excited by a sequence of white noise $u(n)$ with distribution $\mathcal{N}(0, \sigma_u^2)$ and is given by

$$x(n) = -\sum_{i=1}^{p} a_i x(n-i) + u(n) \tag{3.2}$$

The observation noise $v(n)$ is assumed to be independent of the input noise $u(n)$, i.e., $E[u(n)v(n-t)] = 0$ for all t, where $E[\cdot]$ denotes the expectation operator. The order p of the AR system is assumed to be known.

For the noise-free case, $\{a_k\}$ can be obtained from the Yule-Walker Eqn. (2.24) that is written here again

$$R_{xx}(l) = -\sum_{k=1}^{p} a_k R_{xx}(l-k), \qquad l \geq 1 \tag{3.3}$$

where $R_{xx}(l)$, the auto-correlation function of the signal $x(n)$, is generally computed as

$$R_{xx}(l) = \frac{1}{N} \sum_{n=0}^{N-1-|l|} x(n)x(n+|l|) \tag{3.4}$$

and N is the number of data points. Clearly, any p equations are sufficient to determine the AR parameters. Generally, $l = 1, 2, \cdots, p$ is chosen which results in a set of symmetric Toeplitz equations. When noise is present, $R_{xx}(l)$ is unknown and is to be estimated from the noisy data sequence $y(n)$. Exploiting the relation

$$R_{xx}(l) = \begin{cases} R_{yy}(0) - \sigma_v^2 & \text{for } l = 0 \\ R_{yy}(l), & \text{for } l \neq 0 \end{cases} \tag{3.5}$$

where, $R_{yy}(l)$, the autocorrelation function of the noisy signal $y(n)$, is calculated as

$$R_{yy}(l) = \frac{1}{N} \sum_{n=0}^{N-1-|l|} y(n)y(n+|l|) \tag{3.6}$$

We can calculate $R_{xx}(l)$ for all values of l except $l = 0$ as the noise-variance (σ_v^2) is unknown. Although the high-order Yule-Walker equations, where $R_{xx}(0)$ is absent, can be used to estimate the AR parameters [14], this approach suffers from singularity constraint [31]. The possible singularity of the auto-correlation matrix leads to a substantial increase in the variance of the AR spectral estimate [14].

It is a standard practice to assume that $R_{xx}(l)$ obtained from Eqn. (3.5) and Eqn. (3.6) for all l, with true value of σ_v^2 substituted at $l = 0$, result in noise-free autocorrelation sequence that satisfy Eqn. (3.3). In chapter 4 this standard practice is termed as **Method 1** and the result obtained by this method is presented. However, it is worth mentioning that at a very low SNR the autcorrelation function of the noisy signal includes significant error at all lags other than zero resulting from the non-ideal nature of the autocorrelation sequence of the additive noise. Under such a noisy condition, the conventional methods of computing autocorrelation sequence from a finite set of noisy data fail to estimate the AR parameters with acceptable level of accuracy.

The objective of this paper is to propose a novel method using *damped cosine model* of the autocorrelation function to estimate the AR parameters especially at a very low SNR. Using $R_{yy}(l)$, calculated from a finite set of noisy observations, we can estimate the *damped cosine model* parameters. Even at a very low SNR the model parameters can be estimated quite accurately. Thus the problem of AR parameter estimation at a very low SNR reduces to the problem of cosine model parameter estimation.

3.3 Estimation of Damped cosine Model Parameters

To develop a mathematical model for better estimation of the autocorrealtion function of $x(n)$ from a finite set of noisy signal $y(n)$ we introduce an alternative representation for $x(n)$. The transfer function of a p-th order AR system in the z-domain can be expressed as

$$H(z) = \frac{1}{A(z)} = \sum_{k=1}^{p} \frac{C_k}{1 - z_k z^{-1}} \tag{3.7}$$

where $A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_p z^{-p}$, z_k denotes the k-th pole of the AR system and C_k is the partial fraction coefficient corresponding to the k-th pole.

The unit impulse response $h(n)$ of the causal AR system described in Eqn. (3.7) can be expressed as

$$h(n) = \sum_{k=1}^{p} C_k (z_k)^n, n = 0, 1, 2, \cdots, N-1 \tag{3.8}$$

If this relaxed AR system is excited by a sequence of white noise $u(n)$ with distribution $\mathcal{N}(0, \sigma_u^2)$, the response $x^M(n)$ is given by

$$x^M(n) = u(n) * h(n) = \sum_{m=0}^{n} u(m) h(n-m) \tag{3.9}$$

Using Eqn. (3.8), Eqn. (3.9) can be written as

$$x^M(n) = \sum_{k=1}^{p} \sum_{m=0}^{n} C_k u(m) (z_k)^{n-m} \tag{3.10}$$

Eqn. (3.10) is the proposed autoregressive signal model. It is found that first $(p-1)$ terms of $x^M(n)$ are zero and after that $x(n)$ of Eqn. (3.2) and $x^M(n)$ of Eqn. (3.10) are identical. Hence their relationship can be expressed as

$$z^{-(p-1)} x^M(n) = x(n) \tag{3.11}$$

Thus Eqn. (3.2) is the difference equation implementation of $x(n)$ using the system parameters and Eqn. (3.10) is the convolution sum implementation of $x(n)$ using the system roots. The detail derivation of this convolution sum representation of AR signal is given in the Appendix A.

Using Eqn. (3.10), the autocorrelation of the noise-free signal $x^M(n)$ can be obtained as

$$R_{xx}^M(l) = R_{xx}(l) = \sum_{k=1}^{p} \beta_k (z_k)^l \tag{3.12}$$

$$\text{and} \quad \beta_k = \sigma_u^2 \left[\frac{C_k^2}{1 - z_k^2} + \sum_{q=1, q \neq k}^{p} \frac{C_k C_q}{1 - z_k z_q} \right] \tag{3.13}$$

The coefficient β_k may be real or complex depending on whether the pole is real or complex. Since $x(n)$ is real, in the latter case, a complex pole will always be accompanied by its complex conjugate pole. The detail derivation of this model is given in the Appendix B.

Considering the effect of complex and real poles, Eqn. (3.12) can be expressed as a summation of cosine terms that is given below

$$R_{xx}(l) = \sum_{j=1}^{g} G_j (r_j)^l \cos(\omega_j l + \phi_j) \tag{3.14}$$

where G_j and ϕ_j are constants that depend on β_j, r_j is the magnitude of the j-th pole, ω_j is the angular position of the j-th pole and g= Number of pair of complex conjugate poles + Number of real poles. For a stable system, r_j is always less than one. With the increase in lag points l, the factor $(r_j)^l$ always decreases. As a result the cosine function will be decaying in nature. Hence Eqn. (3.14) is termed as the **damped cosine model** of autocorrelation function for noise-free signal.

For the computational convenience we have decomposed the model in the following form.

$$R_{xx}(l) = \sum_{j=1}^{g} (r_j)^l \left[P_j \cos(\omega_j l) + Q_j \sin(\omega_j l) \right], \quad \text{for } l \geq 0 \tag{3.15}$$

where $P_j = G_j \cos\phi_j$ and $Q_j = -G_j \sin\phi_j$ are constants that depend on β_j. In general, r_j governs the decay rate of the AR system response and ω_j determines the angular position of the pole of the AR system in the z-plane.

We estimate each of the damped cosine function of the alternative representation of $R_{xx}(l)$ described in Eqn. (3.15) in an iterative fashion. At first from the given set of noisy data points $y(n)$, the autocorrelation function of the noisy signal $R_{yy}(l)$ is calculated using Eqn. (3.6). It is sufficient to consider only a few nonzero positive lags of $R_{yy}(l)$, where $l = 1, 2, \cdots, M$. The j-th component function $R^j_{xx}(l)$ of Eqn. (3.15) can be referred as

$$R^j_{xx}(l) = \left\{(r_j)^l \left[P_j \cos(\omega_j l) + Q_j \sin(\omega_j l)\right]\right\} \tag{3.16}$$

The component function $R^j_{xx}(l)$ of Eqn. (3.15) is then estimated by optimally fitting a finite sequence of this function with $R_{yy}(l)$ for $l > 0$. The fitted parameters at the first step will give an estimate of r_j and ω_j, $j = 1$. The corresponding fitted function is then subtracted from $R_{yy}(l)$ to obtain the first residue function $\Re_1(l)$. In the second step, another function of the proposed model is fitted to this residue function to get the second set of r_j and ω_j, $j = 2$. Then a second residue function $\Re_2(l)$ is calculated by subtracting the second fitted function from the first residue function. The k-th residue function is thus defined as

$$\Re_k(l) = \begin{cases} R_{yy}(l), & k = 0 \\ \Re_{k-1}(l) - (r_k)^l F_k, & k = 1, 2, \cdots, g-1 \end{cases} \tag{3.17}$$

where $F_k = P_k \cos(\omega_k l) + Q_k \sin(\omega_k l)$. For $0 < \omega_k < \pi$, we obtain $r_k e^{(\pm j\omega_k)}$ as one pair of complex conjugate poles of the AR system. However, $\omega_k = 0$ or π represent a real pole given by r_k or $-r_k$, respectively. Proceeding this way when all the p poles are identified no further steps are required. As for example, in case of a fourth order system with two real poles and a pair of complex-conjugate poles we need three steps. Once the poles are estimated, the AR system parameters can be obtained from their unique relationship [26].

In the proposed method, the parameters ω_k, r_k, P_k, and Q_k of the k-th component function are chosen such that the sum-squared error ($J^{(i)}_k$), between the $(k-1)$-th residue function and the k-th component function is minimized, where

$$\begin{aligned} J^{(i)}_k &= \sum_l \left|\Re_{k-1}(l) - (r^{(i)}_k)^l F^{(i)}_k\right|^2, \\ & \quad l = 1, 2, \cdots, M, \; k = 1, 2, \cdots, g-1 \end{aligned} \tag{3.18}$$

and

$$F^{(i)}_k = P^{(i)}_k \cos(\omega^{(i)}_k l) + Q^{(i)}_k \sin(\omega^{(i)}_k l) \tag{3.19}$$

Since the proposed method is iterative, the superscript '(i)' denotes the iteration index, i.e., $\omega^{(i)}_k$ denotes the angle of the k-th pole at iteration i. The optimum parameters are found as $P_k = P^{(i)}_k$, $Q_k = Q^{(i)}_k$, $r_k = r^{(i)}_k$, and $\omega_k = \omega^{(i)}_k$ for the value of i at which $J^{(i)}_k$ is minimum . For arbitrary values of $r^{(i)}_k$ and $\omega^{(i)}_k$, $P^{(i)}_k$ and

$Q_k^{(i)}$ can be obtained by minimizing $J_k^{(i)}$ in the least-squares sense. As in this case only $P_k^{(i)}$ and $Q_k^{(i)}$ will be the variable terms in $J_k^{(i)}$, to minimize $J_k^{(i)}$ the following two equations are sufficient.

$$\frac{\partial J_k^{(i)}}{\partial P_k^{(i)}} = 0 \tag{3.20}$$

$$\frac{\partial J_k^{(i)}}{\partial Q_k^{(i)}} = 0 \tag{3.21}$$

From Eqn. (3.20) we get,

$$\begin{aligned} &2\sum_{l=1}^{M}\left[\Re_{k-1}(l) - (r_k^{(i)})^l F_k^{(i)}\right](r_k^{(i)})^l \frac{\partial F_k^{(i)}}{\partial P_k^{(i)}} = 0 \\ \Longrightarrow\ & P_k^{(i)}\sum_{l=1}^{M}(r_k^{(i)})^{2l}\cos^2(\omega_k^{(i)}l) + Q_k^{(i)}\sum_{l=1}^{M}(r_k^{(i)})^{2l}\sin(\omega_k^{(i)}l)\cos(\omega_k^{(i)}l) \\ &\qquad = \sum_{l=1}^{M}\Re_{k-1}(l)(r_k^{(i)})^l\cos(\omega_k^{(i)}l) \end{aligned} \tag{3.22}$$

Similarly from Eqn. (3.21) we get,

$$\begin{aligned} &2\sum_{l=1}^{M}\left[\Re_{k-1}(l) - (r_k^{(i)})^l F_k^{(i)}\right](r_k^{(i)})^l \frac{\partial F_k^{(i)}}{\partial Q_k^{(i)}} = 0 \\ \Longrightarrow\ & P_k^{(i)}\sum_{l=1}^{M}(r_k^{(i)})^{2l}\cos(\omega_k^{(i)}l)\sin(\omega_k^{(i)}l) + Q_k^{(i)}\sum_{l=1}^{M}(r_k^{(i)})^{2l}\sin^2(\omega_k^{(i)}l) \\ &\qquad = \sum_{l=1}^{M}\Re_{k-1}(l)(r_k^{(i)})^l\sin(\omega_k^{(i)}l) \end{aligned} \tag{3.23}$$

Combining Eqn. (3.22) and Eqn. (3.23) we get

$$\begin{aligned} &\begin{bmatrix} \sum_{l=1}^{M}(r_k^{(i)})^{2l}\cos^2(\omega_k^{(i)}l) & \sum_{l=1}^{M}(r_k^{(i)})^{2l}\sin(\omega_k^{(i)}l)\cos(\omega_k^{(i)}l) \\ (r_k^{(i)})^{2l}\cos(\omega_k^{(i)}l)\sin(\omega_k^{(i)}l) & \sum_{l=1}^{M}(r_k^{(i)})^{2l}\sin^2(\omega_k^{(i)}l) \end{bmatrix} \\ &\qquad \times \begin{bmatrix} P_k^{(i)} \\ Q_k^{(i)} \end{bmatrix} = \begin{bmatrix} \sum_{l=1}^{M}\Re_{k-1}(l)(r_k^{(i)})^l\cos(\omega_k^{(i)}l) \\ \sum_{l=1}^{M}\Re_{k-1}(l)(r_k^{(i)})^l\sin(\omega_k^{(i)}l) \end{bmatrix} \end{aligned} \tag{3.24}$$

Eqn. (3.24) can be rewritten with simpler matrix notations as

$$\mathbf{DU} = \mathbf{V} \tag{3.25}$$

where the elements of the (2×2) matrix $\mathbf{D}$ are defined by $D_{11} = \sum_l (r_k^{(i)})^{2l} \cos^2(\omega_k^{(i)} l)$, $D_{22} = \sum_l (r_k^{(i)})^{2l} \sin^2(\omega_k^{(i)} l)$, $D_{12} = D_{21} = \sum_l (r_k^{(i)})^{2l} \cos(\omega_k^{(i)} l) \sin(\omega_k^{(i)} l)$, and $\mathbf{U}^T = \left[P_k^{(i)} \; Q_k^{(i)}\right]$, $\mathbf{V}^T = [V_1 \; V_2]$ with $V_1 = \sum_l \Re_{k-1}(l)(r_k^{(i)})^l \cos(\omega_k^{(i)} l)$, and $V_2 = \sum_m \Re_{k-1}(l) \; (r_k^{(i)})^l \sin(\omega_k^{(i)} l)$.

3.4 Conclusion

In this research work, a novel method for the identification of AR systems at a very low SNR using *damped cosine model* of autocorrelation function of the noise-free AR signal has been proposed. The conventional correlation based techniques fail to estimate the AR parameters below a certain positive value of SNR due to inaccurate estimation of the autocorrelation function from a finite set of noisy data. This research results have shown that the calculation of autocorrelation function using a *damped cosine model* can alleviate this problem and can identify AR systems even at an SNR as low as -5 dB.

Chapter 4

Simulation Results

4.1 Introduction

In this chapter, we evaluate the performance of the *damped cosine method* of AR system identification by presenting several numerical examples. First, we verify the convolution sum representation of the AR signal with a known system. Next we demonstrate the intermediate steps of the proposed method using numerical examples. Then we present the results obtained by the proposed method in the identification of various AR systems at different SNRs. Finally, a comparison between the proposed method and the improved least-square method with no prefiltering (ILSNP) [27] is given in tabular form.

4.2 Different AR Systems Used in Simulation

We have considered various AR systems on the basis of the following factors.

1. Order of the system.

2. Pole position in the z-plane.

3. Type of a pole, i.e., real or complex. In fact, the angular position of a pole determines whether the pole is real or complex.

4. Impulse response of the system.

In the proposed method the order of the AR system is assumed to be known. Considering the other three effects we have taken twelve various AR systems for simulation. For each AR system, the difference equation and AR polynomial ($A(z)$) are given below.

1. **System 1:** Second order AR system

$$\begin{aligned} &x(n) - 1.8x(n-1) + 0.97x(n-2) = u(n) \\ &A(z) = 1 - 1.8z^{-1} + 0.97z^{-2} \\ &Poles = 0.9849e^{\pm j0.4182} \end{aligned} \tag{4.1}$$

2. **System 2:** Second order AR system

$$\begin{aligned} &x(n) - 0.15x(n-1) - 0.76x(n-2) = u(n) \\ &A(z) = 1 - 0.15z^{-1} - 0.76z^{-2} \\ &Poles = 0.95, -0.85 \end{aligned} \tag{4.2}$$

3. **System 3:** Second order AR system

$$\begin{aligned} &x(n) + x(n-1) + 0.5x(n-2) = u(n) \\ &A(z) = 1 + z^{-1} + 0.5z^{-2} \\ &Poles = 0.7071e^{\pm j2.3562} \end{aligned} \tag{4.3}$$

4. **System 4:** Third order AR system

$$\begin{aligned} &x(n) - 0.65x(n-1) - 0.72x(n-2) + 0.76x(n-3) = u(n) \\ &A(z) = 1 - 0.65z^{-1} - 0.72z^{-2} + 0.76z^{-3} \\ &Poles = -0.95, 0.8944e^{\pm j0.4636} \end{aligned} \tag{4.4}$$

5. **System 5:** Third order AR system

$$\begin{aligned} &x(n) - 0.5x(n-1) - 0.61x(n-2) + 0.585x(n-3) = u(n) \\ &A(z) = 1 - 0.5z^{-1} - 0.61z^{-2} + 0.585z^{-3} \\ &Poles = -0.9, 0.8062e^{\pm j0.5191} \end{aligned} \tag{4.5}$$

6. **System 6:** Third order AR system

$$\begin{aligned} &x(n) - 0.45x(n-1) - 0.68x(n-2) + 0.6175x(n-3) = u(n) \\ &A(z) = 1 - 0.45z^{-1} - 0.68z^{-2} + 0.6175z^{-3} \\ &Poles = -0.95, 0.8062e^{\pm j0.5191} \end{aligned} \tag{4.6}$$

7. **System 7:** Fourth order AR system

$$x(n) - 2.7607x(n-1) + 3.8106x(n-2) - 2.6535x(n-3)$$
$$+0.9238x(n-4) = u(n)$$
$$A(z) = 1 - 2.7607z^{-1} + 3.8106z^{-2} - 2.6535z^{-3} + 0.9238z^{-4}$$
$$Poles = 0.9805e^{\pm j0.8798}, 0.9803e^{\pm j0.6909} \tag{4.7}$$

8. **System 8:** Fourth order AR system

$$x(n) - 2.595x(n-1) + 3.339x(n-2) - 2.2x(n-3)$$
$$+0.7310x(n-4) = u(n)$$
$$A(z) = 1 - 2.595z^{-1} + 3.339z^{-2} - 2.2z^{-3} + 0.7310z^{-4}$$
$$Poles = 0.9498e^{\pm j0.6289}, 0.9002e^{\pm j0.9420} \tag{4.8}$$

9. **System 9:** Fourth order AR system

$$x(n) - 0.55x(n-1) - 0.155x(n-2) + 0.5495x(n-3)$$
$$-0.6241x(n-4) = u(n)$$
$$A(z) = 1 - 0.55z^{-1} - 0.155z^{-2} + 0.5495z^{-3} - 0.6241z^{-4}$$
$$Poles = -0.95, 0.9, 0.8544e^{\pm j1.212} \tag{4.9}$$

10. **System 10:** Fourth order AR system

$$x(n) + 0.55x(n-1) - 0.155x(n-2) - 0.5495x(n-3)$$
$$-0.6241x(n-4) = u(n)$$
$$A(z) = 1 + 0.55z^{-1} - 0.155z^{-2} - 0.5495z^{-3} - 0.6241z^{-4}$$
$$Poles = 0.95, -0.9, 0.8544e^{\pm j1.9296} \tag{4.10}$$

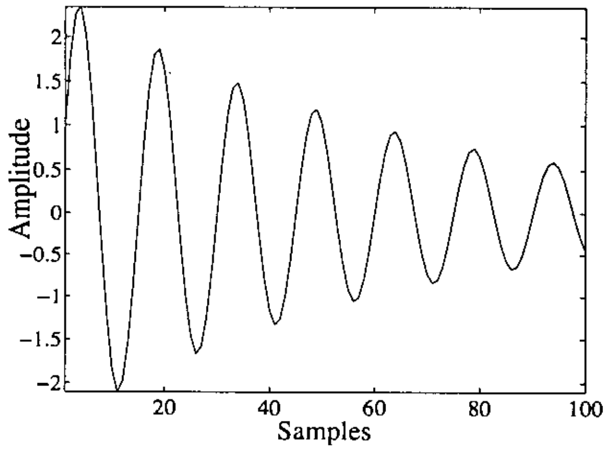
11. **System 11:** Fifth order AR system

$$x(n) + 0.6x(n-1) - 0.2975x(n-2) - 0.1927x(n-3)$$
$$+0.6329x(n-4) + 0.7057x(n-5) = u(n)$$
$$A(z) = 1 + 0.6z^{-1} - 0.2975z^{-2} - 0.1927z^{-3} + 0.6329z^{-4}$$
$$+0.7057z^{-5}$$
$$Poles = -0.9, 0.9605e^{\pm j0.6747}, 0.9219e^{\pm j2.2794} \tag{4.11}$$

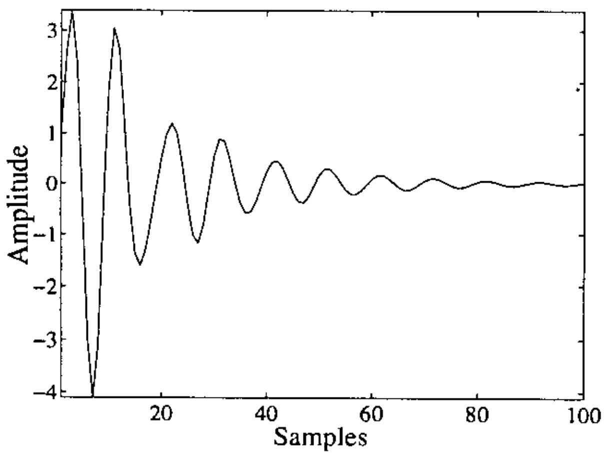
12. **System 12:** Sixth order AR system

$$
\begin{aligned}
& x(n) - 0.86x(n-1) + 1.0494x(n-2) - 0.6680x(n-3) \\
& +0.9592x(n-4) - 0.7563x(n-5) + 0.5656x(n-6) = u(n) \\
& A(z) = 1 - 0.86z^{-1} + 1.0494z^{-2} - 0.6680z^{-3} + 0.9592z^{-4} \\
& \qquad\qquad\qquad\qquad\qquad -0.7563z^{-5} + 0.5656z^{-6} \\
& Poles = 0.95e^{\pm j2.2143}, 0.9203e^{\pm j1.2387}, 0.8602e^{\pm j0.6203}
\end{aligned}
\tag{4.12}
$$

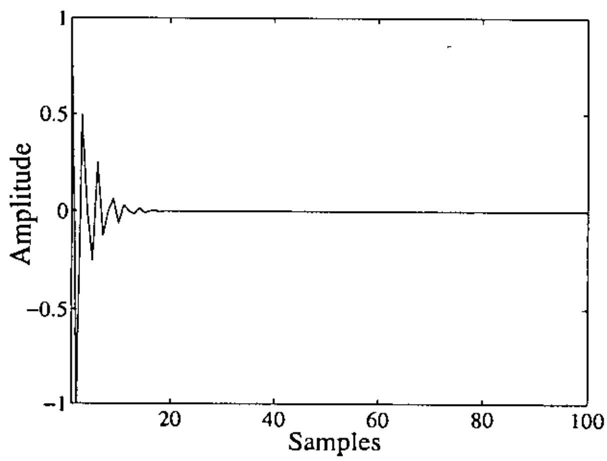
Impulse response (IR) of an AR system strongly depends on the type and magnitude of its poles. For an AR system, with a pole (or a complex conjugate pair of poles) near the origin, the impulse response decays more rapidly than one associated with a pole near (but inside) the unit circle. It is expected that the pattern of the autocorrelation function of the AR signal will follow the same nature. Hence before selecting the AR systems for simulation, we have analyzed their impulse responses. Fig. 4.1 shows the impulse response of three different AR systems. Fig. 4.1(a) shows the impulse response of **System 1** that consists of a pair of complex conjugate poles with magnitude 0.9849. The impulse response of this system decays very slowly and it retains significant percentage of its initial value even after 100 instances. Impulse response of this system can be categorized as 'long IR'. Fig. 4.1(b) shows the impulse response of **System 8** that consists of two pair of complex conjugate poles with magnitude 0.9498 and 0.9002. The impulse response of this system decays moderately and after 60 instances it reaches almost to zero. Impulse response of this system can be categorized as 'medium IR'. Fig. 4.1(c) shows the impulse response of **System 3** that consists of a pair of complex conjugate poles with magnitude 0.7071. The impulse response of this system decays very rapidly and it reaches almost to zero just after the 15 instances. Impulse response of this system can be categorized as 'short IR'. For simulation, we have chosen various AR systems covering all the three categories of impulse responses.



(a)



(b)



(c)

Fig. 4.1: Impulse response of AR Systems : (a) **System 1**; (b) **System 8** and (b) **System 3**

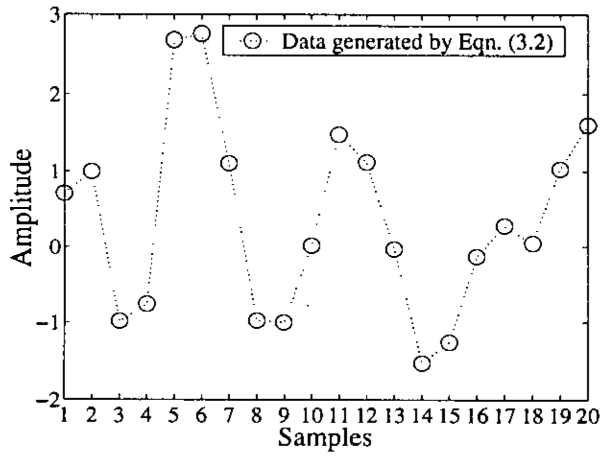
4.3 Results on the Convolution Sum Representation of AR Signal

In chapter 3, we proposed the convolution sum representation of AR signal. The relationship between the difference equation representation and the convolution sum representation of $x(n)$ is established in Eqn. (3.11). We have implemented Eqn. (3.2) and Eqn. (3.10) for the **System 12** which is an AR(6) system. Fig. 4.2 shows the simulation results obtained using these equations. For both the representations we have used the same random white noise input $u(n)$ with distribution $\mathcal{N}(0, \sigma_u^2)$. Fig. 4.2(a) shows the response of the AR system using Eqn. (3.2). Fig. 4.2(b) shows the response of the AR system using Eqn. (3.10). It can be seen that in this case, first 5 responses are zero and from sixth instant it follows the pattern as in Fig. 4.2(a). Fig. 4.2(c) shows that the results obtained by these two equations are essentially identical.

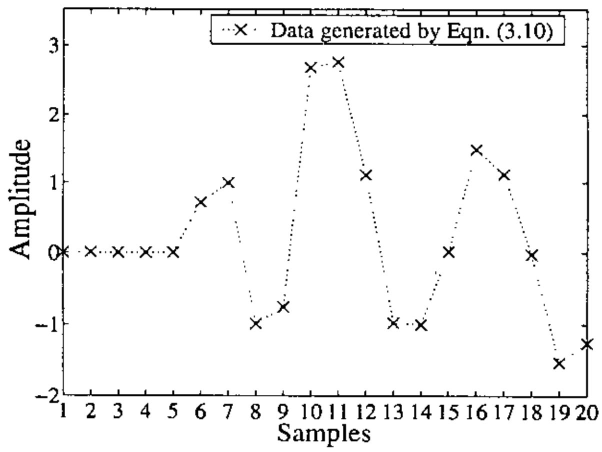
4.4 Effect of Data Length on the Damped Cosine Model

To test the validity of the *damped cosine model* we have implemented Eqn. (3.12) with a known system. We have considered that the system parameters are known i.e., the transfer function coefficients C_i and the poles z_i can be calculated. In the derivation of Eqn. (3.12), the data length is considered infinity. In the practical case we consider the length of data as long as possible. To observe the effect of data length, Eqn. (3.12) is simulated for different data lengths. In Fig. 4.3 we have shown the effect of increase in data lengths on the proposed autocorrelation model.

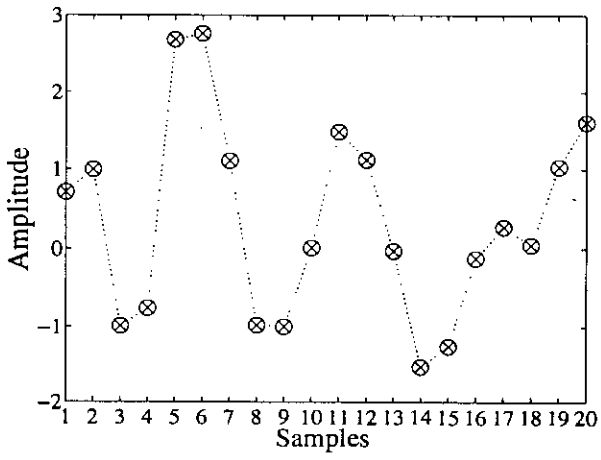
Autocorrelation function determined by Eqn. (3.4) is termed as the conventional estimation of the autocorrelation function and the simulation result of Eqn. (3.12) is termed as the proposed model based estimate. Number of data points used are 128, 512, 4096 and 2,62,144. From Fig. 4.3 it is clear that, with the increase in data lengths the proposed model based estimate matches the conventional estimate more accurately.



(a)



(b)



(c)

Fig. 4.2: System response generated by difference Eqn. (3.2) and convolution sum representation given in Eqn. (3.10). For **System 12** : (a) response generated by Eqn. (3.2); (b) response generated by Eqn. (3.10) and (c) response generated by Eqn. (3.10) is plotted from 6-th instant over the response generated by Eqn. (3.2).

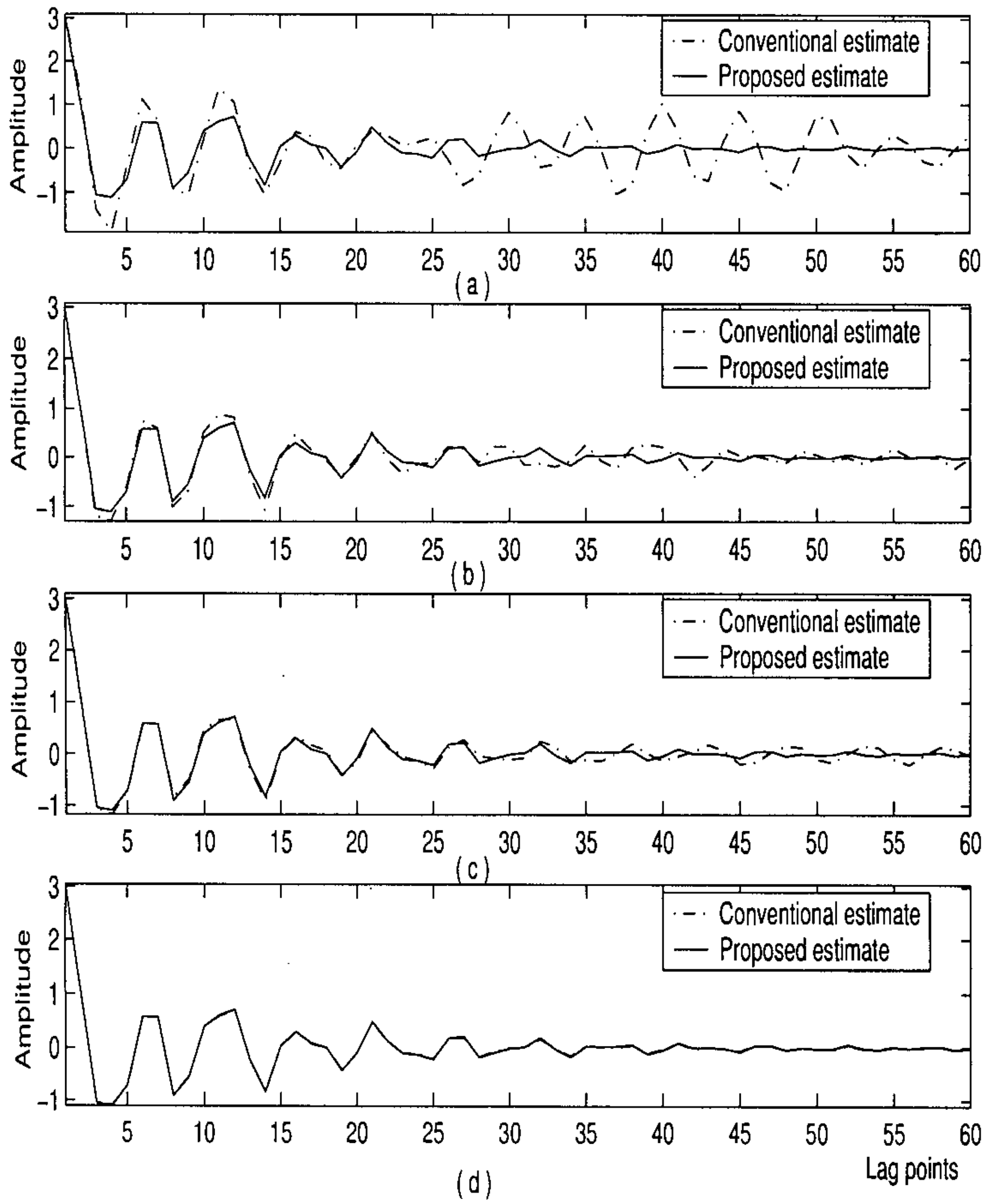


Fig. 4.3: Effect of increasing data points on the proposed autocorrelation model. Conventional estimate of autocorrelation function obtained by Eqn. (3.4) and autocorrelation function obtained by Eqn. (3.12) are plotted for **System 12** : (a) Number of data points N=128, (b) N=512, (c) N=4096 and (d) N=2,62,144.

4.5 General Considerations for All Systems

Data was generated according to Eqns. (3.1) and (3.2) given by

$$\begin{aligned} x(n) &= -\sum_{i=1}^{p} a_i x(n-i) + u(n) \\ y(n) &= x(n) + v(n) \end{aligned} \qquad (4.13)$$

where $x(n)$ is the output signal of a p-th order AR system excited by a sequence of white noise $u(n)$ with distribution $\mathcal{N}(0, \sigma_u^2)$, $v(n)$ is the additive white noise with distribution $\mathcal{N}(0, \sigma_v^2)$ and $y(n)$ is the observed noisy signal. The variance of the input signal is fixed at $\sigma_u^2 = 1$ and the variance σ_v^2 of the observation noise $v(n)$ is selected to give different SNRs defined as

$$\text{SNR} = 10 \log \frac{\sigma_x^2}{\sigma_v^2} \qquad \text{(dB)} \qquad (4.14)$$

where σ_x^2 is the variance of $x(n)$. In all the simulations, N = 4000 samples of the noisy data were used. As explained in chapter 3, we calculate the desired AR system parameters from the *damped cosine model* parameters. With this view, for determining the *damped cosine model* parameters, we use $R_{yy}(l)$ for $l = 1, 2, \cdots, M$ with $M = 10p$ in all the simulations. In order to estimate ω_k and r_k, a domain of ω_k from 0 to π was scanned at a resolution of 0.001 for different values of r_k. Scanning interval of r_k was taken to be $0 < r_k \leq 1$ and scanning resolutions were chosen to be 0.001. For all the systems, the parameters were estimated for 25 independent runs. The standard deviations from true (SDT) values and the standard deviations from the mean (SDM) are used as two indices for comparing the consistency of the proposed method with other methods. SDM and SDT can be defined as

$$\text{SDM} = \sqrt{\frac{\sum_{j=1}^{\varepsilon} \sum_{i=1}^{p} (\hat{a}_i^j - \bar{a}_i)^2}{\varepsilon - 1}} \qquad (4.15)$$

$$\text{SDT} = \sqrt{\frac{\sum_{j=1}^{\varepsilon} \sum_{i=1}^{p} (\hat{a}_i^j - a_i)^2}{\varepsilon - 1}} \qquad (4.16)$$

where a_i is the true AR parameter, $\hat{a}_i^j$ is the estimated AR parameter at the j-th run, $\bar{a}_i$ is the mean value of the i-th estimated parameter, ε is the total number of runs and p is the system order.

4.6 Simulation Results for Yule-Walker Equations

In chapter 2, we have described modified Yule-Walker equations for identifying AR systems. Here, we present simulation results for three different methods based on the Yule-Walker equations.

1. **Method 1** : It is based on the noise compensated lower-order Yule-Walker (LOYW) equations. True value of the noise variance, σ_v^2 is used in Eqn. (2.40). Then substituting $R_{xx}(l)$, in Eqn. (2.25) by $R_{yy}(l)$ according to Eqn. (2.40), AR parameters may be estimated.

2. **Method 2** : It is based on the noise compensated higher-order Yule-Walker (HOYW) equations. Substituting $R_{xx}(l)$, in Eqn. (2.27) by $R_{yy}(l)$, AR parameters may be estimated.

3. **Method 3** : In this case modified HOYW equation is used. It differs from the basic HOYW equations in that, more than p equations are used. Here we take $10p$ equations resulting the correlation matrix of dimension $(10p \times p)$. In this case also $R_{xx}(l)$ is substituted by $R_{yy}(l)$ in the modified HOYW equations to identify the AR parameters.

Table 4.1: AR Parameter Estimation by **Method 1** for **System 7** (Under each estimated parameter first SDT and then SDM are shown)

True AR parameters	Estimated parameters		
	20 dB	10 dB	0 dB
a_1=−2.7607	−2.7223	−0.9248	−2.8736
	±0.7974	±2.7810	±8.7047
	±0.7964	±2.0551	±8.7040
a_2=3.8106	3.7349	−0.6378	5.6645
	±1.9011	±6.7256	±23.2631
	±1.8995	±4.9619	±23.1860
a_3=−2.6535	−2.5863	1.8092	−5.2122
	±1.8950	±6.7429	±24.6393
	±1.8938	±4.9720	±24.5004
a_4=0.9238	0.9043	−0.3435	2.5612
	±0.7834	±2.8197	±11.6071
	±0.7831	±2.0780	±11.4862

Table 4.2: AR Parameter Estimation by **Method 2** for **System 7** (Under each estimated parameter first SDT and then SDM are shown)

True AR parameters	Estimated parameters 20 dB	10 dB	0 dB
a_1=−2.7607	−2.7583	−1.4414	−2.8736
	±0.5146	±5.3781	±8.7047
	±0.5146	±5.2068	±8.7040
a_2=3.8106	3.8026	0.4115	5.6645
	±1.2025	±13.0302	±23.2631
	±1.2024	±12.5599	±23.1860
a_3=−2.6535	−2.6436	0.7678	−5.2122
	±1.1622	±12.8007	±24.6393
	±1.1622	±12.3153	±24.5004
a_4=0.9238	0.9178	−0.5774	2.5612
	±0.4693	±5.3637	±11.6071
	±0.4692	±5.1402	±11.4862

Tables 4.1, 4.2 and 4.3 show the results obtained by these methods for **System 7.** The first two methods fail totally below 20 dB. Even at 20 dB the value of SDT

Table 4.3: AR Parameter Estimation by **Method 3** for **System 7** (Under each estimated parameter first SDT and then SDM are shown)

True AR parameters	Estimated parameters 20 dB	10 dB	0 dB
a_1=−2.7607	−2.7598	−2.7070	−1.5324
	±0.0076	±0.0620	±1.2623
	±0.0076	±0.0290	±0.1477
a_2=3.8106	3.8080	3.6848	1.0992
	±0.0161	±0.1456	±2.7841
	±0.0159	±0.0685	±0.3053
a_3=−2.6535	−2.6510	−2.5323	−0.1210
	±0.0154	±0.1405	±2.5994
	±0.0152	±0.0666	±0.2754
a_4=0.9238	0.9226	0.8760	−0.0162
	±0.0066	±0.0560	±0.9644
	±0.0065	±0.0274	±0.0986

and SDM are very high that reflect the inconsistency of estimation at different runs. **Method 3** performs better than these two methods. Its performance at 10 dB is satisfactory but below 10 dB it fails to identify the system. For other systems also similar performances of these methods are expected.

4.7 Step by Step Estimation of Damped Cosine Model

In chapter 3, we have explained each step involved in the *damped cosine method* in detail. Here we explain the step by step estimation procedure for **System 9**. For this system there exists two real poles and a pair of complex conjugate poles. *Damped cosine method* estimates one angle (ω_k) and its corresponding magnitude of pole (r_k) at each step. If $0 < \omega_k < \pi$, we construct a pair of complex conjugate poles $r_k exp(\pm j\omega_k)$. If $\omega_k = 0$ or $\omega_k = \pi$, we construct a single pole r_k or $-r_k$ respectively. Hence to identify all the poles of **System 9**, we need three steps. Fig. 4.4 shows the different steps of *damped cosine method* for this system at SNR=−5 dB. In Table 4.4 the estimated model parameters at each step are summarized. The three steps involved in this case are explained below.

Table 4.4: Estimated Damped Cosine Model Parameters at Each Step for **System 9** at SNR=−5 dB

Model parameters	Estimated values		
	1st step	2nd step	3rd step
ω_k	0	3.1416	1.2706
r_k	0.9512	0.9512	0.8840
P_k	0.4589	0.5075	0.4931
Q_k	0	0	0.2123

1. **Step 1** : Performing the fitting operation on the noisy autocorrelation function $R_{yy}(l)$, the first component function of the *damped cosine model* is estimated. This gives the model parameters of the first step. In Fig. 4.4(a) the 1st component function is plotted with $R_{yy}(l)$. The true values of ω_k and r_k are stated in Eqn. (4.9). From Table 4.4 we see that in this case the angle chosen is zero which is absolutely correct but the estimated $r_k = 0.9512$ differs from the original $r_k = 0.9$. Number of pole obtained is one and it is 0.9512. The **1st residue** function ($\Re_1$) is obtained by subtracting the 1st component function from $R_{yy}(l)$.

2. **Step 2** : Performing the fitting operation on the 1st residue function ($\Re_1$), the second component function of the *damped cosine model* is estimated.

This gives the model parameters of the second step. In Fig. 4.4(b) the 2nd component function is plotted with 1st residue function. The true values of ω_k and r_k are stated in Eqn. (4.9). From Table 4.4 we see that in this case the angle chosen is π which is absolutely correct and also the estimated $r_k = 0.9512$ almost exactly matches with the original $r_k = 0.95$. Number of pole obtained is one and it is -0.9512. The **2nd residue** function ($\Re_2$) is obtained by subtracting the 2nd component function from 1st residue.

3. **Step 3** : Performing the fitting operation on the 2nd residue function ($\Re_2$), the third component function of the *damped cosine model* is estimated. This gives the model parameters of the third step. In Fig. 4.4(c) the 3rd component function is plotted with 2nd residue function. The true values of ω_k and r_k are stated in Eqn. (4.9). From Table 4.4 we see that in this case $\omega_k = 1.2706$ which is very closer to the actual value of 1.212. The estimated $r_k = 0.8840$ almost matches with the original $r_k = 0.8544$. A pair of complex conjugate poles are obtained which is $0.884e^{\pm j1.2706}$. As now the total number of poles obtained is 4, no further step is required.

Summing the three component functions obtained at three steps, an estimate of the autocorrelation function using the proposed model is obtained. In Fig. 4.4(d) autocorrelation function of the noise-free signal calculated using Eqn. (3.4) and estimated autocorrelation function using *damped cosine model* are plotted. In this case the true and estimated AR parameters are given below.

$$\begin{aligned} \text{True parameters} &= [1 \;\; -0.5500 \;\; -0.1550 \;\; 0.5495 \;\; -0.6241] \\ \text{Estimated parameters} &= [1 \;\; -0.5227 \;\; -0.1234 \;\; 0.4730 \;\; -0.7070] \end{aligned}$$

At this low SNR condition, the estimated autocorrelation function matches with the noise-free autocorrelation function quite accurately. Unlike conventional methods, estimated autocorrelation function is not directly used for estimating the AR parameters. Rather, the values of ω_k and r_k, obtained at each step, are used to estimate the AR parameters in the proposed method. The estimated AR parameters also show the efficiency of the proposed method at this low SNR of -5 dB. The results shown in Table 4.4 is just the result of a single run. In our simulation, we always use 25 independent runs and then take their average.

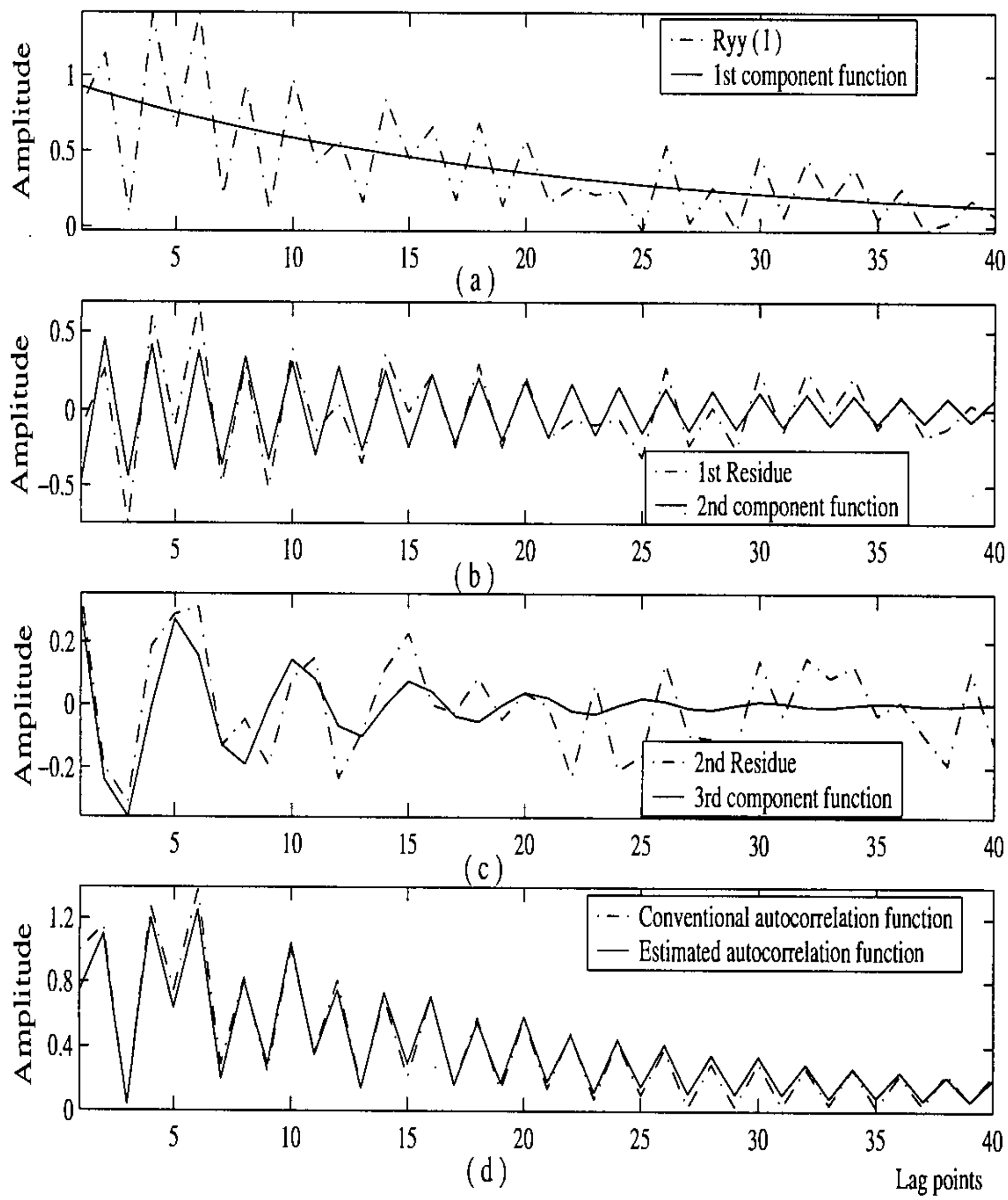


Fig. 4.4: Estimation of autocorrelation function at different steps of damped cosine model for AR **System 9** at SNR=−5 dB : (a) $R_{yy}(l)$ and 1st component function of the proposed damped cosine model; (b) 1st residue and 2nd component function of the proposed damped cosine model; (c) 2nd residue and 3rd component function of the proposed damped cosine model; (d) Conventional autocorrelation function of the noise-free signal obtained by using Eqn. (3.4) and estimated autocorrelation function using damped cosine model.

4.8 Results on AR Parameter Estimation Using Damped Cosine Method

Different systems are analyzed by the *damped cosine method* in noisy observations. Our main interest is in extreme noisy situations where most of the system identification methods fail. From Table 4.5 to Table 4.10 we have listed the simulation results for **System 1** through **System 12**. We have estimated the AR parameters by using both the *damped cosine method* and ILSNP method [27] for all of these systems for SNR $= 10$ dB and -5 dB. In each Table, the average value of the estimated AR parameters and their standard deviations from the mean (SDM) and standard deviations from the true values (SDT) using the proposed and the ILSNP methods are summarized. In Fig. 4.5 to Fig. 4.16, we have shown the true and estimated poles on the z-plane and the power spectra obtained by the proposed method for all the twelve systems. For each AR system, at first we have depicted the true spectrum with the spectrum obtained by the average value of the estimated AR parameters. As stated earlier, that we have taken the average of 25 independent runs. In each figure we have also shown the plot of 25 estimated AR spectra obtained in each step to show the variance of spectral estimates.

4.9 Conclusion

The results show that the proposed method can estimate complex poles as well as real poles on either side of the z-plane quite accurately both at low and high SNRs. It can also be seen that the estimated AR parameters, SDT and SDM of both the methods are comparable at 10 dB, but at -5 dB the ILSNP method fails to estimate the AR parameters. Also notice that SDM and SDT of the estimated AR parameters using the proposed method are consistently satisfactory. The spectrum estimation and the z-plane plot vividly show the accuracy of estimation of the proposed method.

Table 4.5: Performance Comparison of the Proposed and ILSNP Estimators for **System 1** to **System 3** (Under each estimated parameter first SDT and then SDM are shown)

System	True values	Proposed Method 10dB	Proposed Method −5dB	ILSNP Method 10dB	ILSNP Method −5dB
1	$a_1=$	−1.7983	−1.8003	−1.8006	−0.3093
	−1.8000	±0.0049	±0.0105	±0.2091	±0.0947
		±0.0046	±0.0105	±0.2091	±2.2310
	$a_2=$	0.9683	0.9700	0.9671	−0.1421
	0.9700	±0.0049	±0.0115	±0.1410	±0.0354
		±0.0046	±0.0115	±0.1410	±1.2380
2	$a_1=$	−0.1462	−0.1031	−0.1527	−0.1907
	−0.1500	±0.0265	±0.1572	±0.0274	±0.4030
		±0.0262	±0.1497	±0.0007	±0.1576
	$a_2=$	−0.7646	−0.7741	−0.7559	−0.7505
	−0.7600	±0.0250	±0.0652	±0.0624	±0.5345
		±0.0245	±0.0636	±0.0038	±0.2744
3	$a_1=$	1.0015	0.9969	0.9901	1.0125
	1.0000	±0.0658	±0.1652	±0.0347	±0.0365
		±0.0658	±0.1652	±0.0013	±0.0014
	$a_2=$	0.5033	0.5271	0.4945	−0.5093
	0.5000	±0.0257	±0.0743	±0.0221	±0.0313
		±0.0255	±0.0690	±0.0005	±0.0010

Table 4.6: Performance Comparison of the Proposed and ILSNP Estimators for **System 4** to **System 6** (Under each estimated parameter first SDT and then SDM are shown)

System	True values	Proposed Method		ILSNP Method	
		10dB	−5dB	10dB	−5dB
4	$a_1=$ −0.6500	−0.6556 ±0.0227 ±0.0220	−0.6570 ±0.0403 ±0.0397	−0.6467 ±0.0582 ±0.0033	−0.2778 ±0.2908 ±0.2197
	$a_2=$ −0.7200	−0.7277 ±0.0184 ±0.0166	−0.7362 ±0.0372 ±0.0333	−0.7168 ±0.0310 ±0.0009	−0.3881 ±0.4962 ±0.3465
	$a_3=$ 0.7600	0.7885 ±0.0350 ±0.0195	0.7961 ±0.0504 ±0.0344	0.7521 ±0.0537 ±0.0028	−0.2966 ±0.6446 ±0.6136
5	$a_1=$ −0.5000	−0.5023 ±0.0542 ±0.0541	−0.4822 ±0.1938 ±0.1929	−0.5054 ±0.0665 ±0.0667	−0.2766 ±0.3360 ±0.1616
	$a_2=$ −0.6100	−0.5979 ±0.0297 ±0.0270	−0.5790 ±0.1380 ±0.1343	−0.6103 ±0.0499 ±0.0499	−0.2911 ±0.4471 ±0.2996
	$a_3=$ 0.5850	0.6467 ±0.0675 ±0.0243	0.6412 ±0.0985 ±0.0800	0.5887 ±0.0855 ±0.0856	−0.2000 ±0.5524 ±1.4503
6	$a_1=$ −0.4500	−0.4605 ±0.0671 ±0.0662	−0.4549 ±0.1456 ±0.1455	−0.3385 ±0.0871 ±0.0197	−0.2757 ±0.2091 ±0.0723
	$a_2=$ −0.6800	−0.6595 ±0.0876 ±0.0850	−0.6408 ±0.1309 ±0.1246	−0.6156 ±0.0655 ±0.0083	−0.6069 ±0.1995 ±0.0435
	$a_3=$ 0.6175	0.6502 ±0.0472 ±0.0333	0.6559 ±0.0774 ±0.0668	0.4909 ±0.1108 ±0.0278	−0.4427 ±0.2748 ±0.1030

Table 4.7: Performance Comparison of the Proposed and ILSNP Estimators for **System 7** and **System 8** (Under each estimated parameter first SDT and then SDM are shown)

System	True values	Proposed Method		ILSNP Method	
		10dB	−5dB	10dB	−5dB
7	$a_1=$	−2.7285	−2.7433	−0.5489	−1.2390
	−2.7607	±0.0364	±0.0802	±0.7014	±3.2250
		±0.0157	±0.0782	±5.3644	±12.300
	$a_2=$	3.7722	3.7867	0.0739	0.3478
	3.8106	±0.0504	±0.1341	±0.4425	±1.0434
		±0.0316	±0.1341	±14.151	±13.036
	$a_3=$	−2.6408	−2.6457	0.3768	0.7234
	−2.6535	±0.0340	±0.1005	±0.1209	±2.0638
		±0.0314	±0.1005	±9.1965	±15.492
	$a_4=$	0.9298	0.9258	0.2866	−0.4142
	0.9238	±0.0170	±0.0290	±0.5125	±3.6782
		±0.0159	±0.0289	±0.6581	±14.778
8	$a_1=$	−2.6194	−2.6764	−0.5828	−0.4987
	−2.5950	±0.0992	±0.1764	±0.6756	±0.5010
		±0.0961	±0.1556	±4.4873	±4.6354
	$a_2=$	3.4545	3.5805	−0.0498	−0.0368
	3.3390	±0.2092	±0.3593	±0.4903	±0.4304
		±0.1728	±0.2614	±11.715	±11.574
	$a_3=$	−2.3348	−2.4472	0.3121	0.2279
	−2.2000	±0.1894	±0.3108	±0.1279	±0.2533
		±0.1302	±0.1814	±6.3265	±5.9562
	$a_4=$	0.7906	0.8272	0.2812	−0.3656
	0.7310	±0.0679	±0.1058	±0.4313	±0.3251
		±0.0302	±0.0396	±0.3809	±0.2349

Table 4.8: Performance Comparison of the Proposed and ILSNP Estimators for **System 9** and **System 10** (Under each estimated parameter first SDT and then SDM are shown)

System	True values	Proposed Method 10dB	Proposed Method −5dB	ILSNP Method 10dB	ILSNP Method −5dB
9	$a_1=$ −0.5500	−0.5425 ±0.0375 ±0.0367	−0.5839 ±0.01820 ±0.01787	−0.5532 ±0.0451 ±0.0020	−0.4322 ±0.3394 ±0.1242
	$a_2=$ −0.1550	−0.1347 ±0.0364 ±0.0360	−0.1059 ±0.0941 ±0.0797	−0.1518 ±0.0258 ±0.0006	−0.0205 ±0.3402 ±0.1292
	$a_3=$ 0.5495	0.5447 ±0.0312 ±0.0308	0.5714 ±0.1536 ±0.1520	0.5569 ±0.0434 ±0.0019	0.2825 ±0.7226 ±0.5726
	$a_4=$ −0.6241	−0.6711 ±0.0546 ±0.0262	−0.6990 ±0.1045 ±0.0713	−0.6316 ±0.0406 ±0.0016	−0.4619 ±0.4290 ±0.2030
10	$a_1=$ 0.5500	0.5071 ±0.0555 ±0.0342	0.5356 ±0.1136 ±0.1126	0.5527 ±0.0423 ±0.0424	0.4113 ±0.5449 ±0.3132
	$a_2=$ −0.1550	−0.1428 ±0.0331 ±0.0301	−0.1090 ±0.0931 ±0.0804	−0.1547 ±0.0228 ±0.0228	−0.1912 ±0.4124 ±0.1697
	$a_3=$ −0.5495	−0.5363 ±0.0348 ±0.0321	−0.5634 ±0.1054 ±0.1044	−0.5512 ±0.0324 ±0.0325	−0.4054 ±1.0839 ±1.1840
	$a_4=$ −0.6241	−0.6625 ±0.0508 ±0.0324	−0.6913 ±0.0996 ±0.0723	−0.6255 ±0.0319 ±0.0319	−0.5226 ±0.6757 ±0.4623

Table 4.9: Performance Comparison of the Proposed and ILSNP Estimators for **System 11** (Under each estimated parameter first SDT and then SDM are shown)

System	True values	Proposed Method		ILSNP Method	
		10dB	−5dB	10dB	−5dB
11	a_1=	0.5573	0.5839	0.6003	0.4989
	0.6000	±0.0619	±0.0641	±0.0424	±0.3333
		±0.0441	±0.0620	±0.0017	±0.1168
	a_2=	−0.2957	−0.2947	−0.2997	−0.2214
	−0.2975	±0.0211	±0.0497	±0.0177	±0.5825
		±0.0211	±0.0497	±0.0003	±0.3316
	a_3=	−0.1778	−0.1895	−0.1980	−0.1368
	−0.1927	±0.0347	±0.0665	±0.0310	±0.4632
		±0.0312	±0.0664	±0.0010	±0.2091
	a_3=	0.6015	0.6125	0.6304	0.5004
	0.6329	±0.0481	±0.0498	±0.0336	±0.6452
		±0.0358	±0.0453	±0.0011	±0.4172
	a_4=	0.7671	0.7529	0.7081	0.5748
	0.7057	±0.0652	±0.0577	±0.0466	±0.7653
		±0.0181	±0.0318	±0.0021	±0.5794

Table 4.10: Performance Comparison of the Proposed and ILSNP Estimators for **System 12** (Under each estimated parameter first SDT and then SDM are shown)

System	True values	Proposed Method		ILSNP Method	
		10dB	−5dB	10dB	−5dB
12	a_1=	−0.9027	−0.9738	−0.8577	−0.6984
	−0.8600	±0.0679	±0.1576	±0.0369	±0.5453
		±0.0520	±0.1066	±0.0013	±0.3115
	a_2=	1.0033	1.0567	1.0451	0.4973
	1.0494	±0.0617	±0.1207	±0.0407	±1.0767
		±0.0398	±0.1205	±0.0016	±1.4177
	a_3=	−0.6880	−0.7145	−0.6596	−0.1145
	−0.6680	±0.0454	±0.1515	±0.0451	±1.3772
		±0.0405	±0.1438	±0.0020	±2.1271
	a_4=	0.9525	1.0180	0.9596	0.4983
	0.9592	±0.0368	±0.1501	±0.0400	±0.9481
		±0.0361	±0.1376	±0.0015	±1.0754
	a_5=	−0.7864	−0.8075	−0.7518	−0.7274
	−0.7563	±0.0479	±0.1070	±0.0371	±0.3716
		±0.0368	±0.0934	±0.0013	±0.1334
	a_6=	0.6202	0.6880	0.5709	0.4616
	0.5656	±0.0626	±0.1324	±0.0225	±0.5080
		±0.0285	±0.0624	±0.0005	±0.258 6

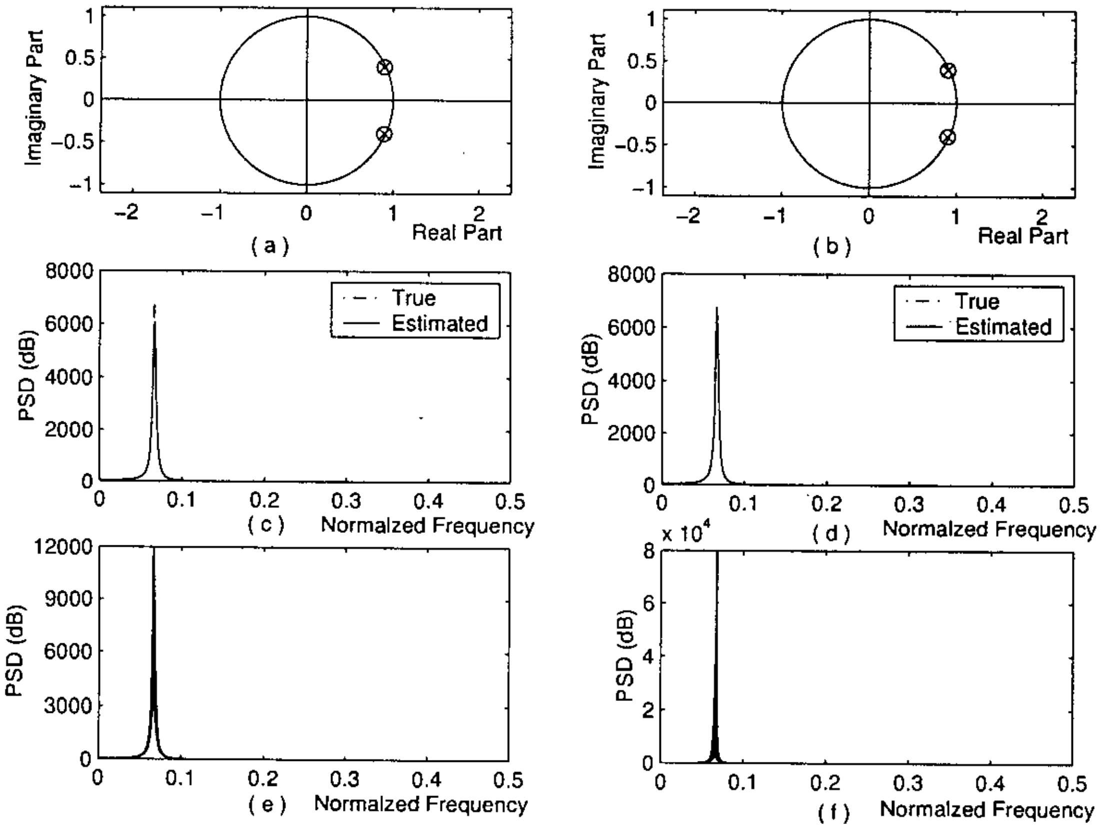


Fig. 4.5: Estimated poles and power spectra obtained by proposed method for AR **System 1** : (a) estimated poles at SNR=10 dB (o: true, x: estimate), (b) estimated poles at SNR=−5 dB (o: true, x: estimate), (c) true and estimated average power spectrum at SNR=10 dB, (d) true and estimated average power spectrum at SNR=−5 dB, (e) estimated power spectrum of 25 runs at SNR=10 dB, (f) estimated power spectrum of 25 runs at SNR=−5 dB.

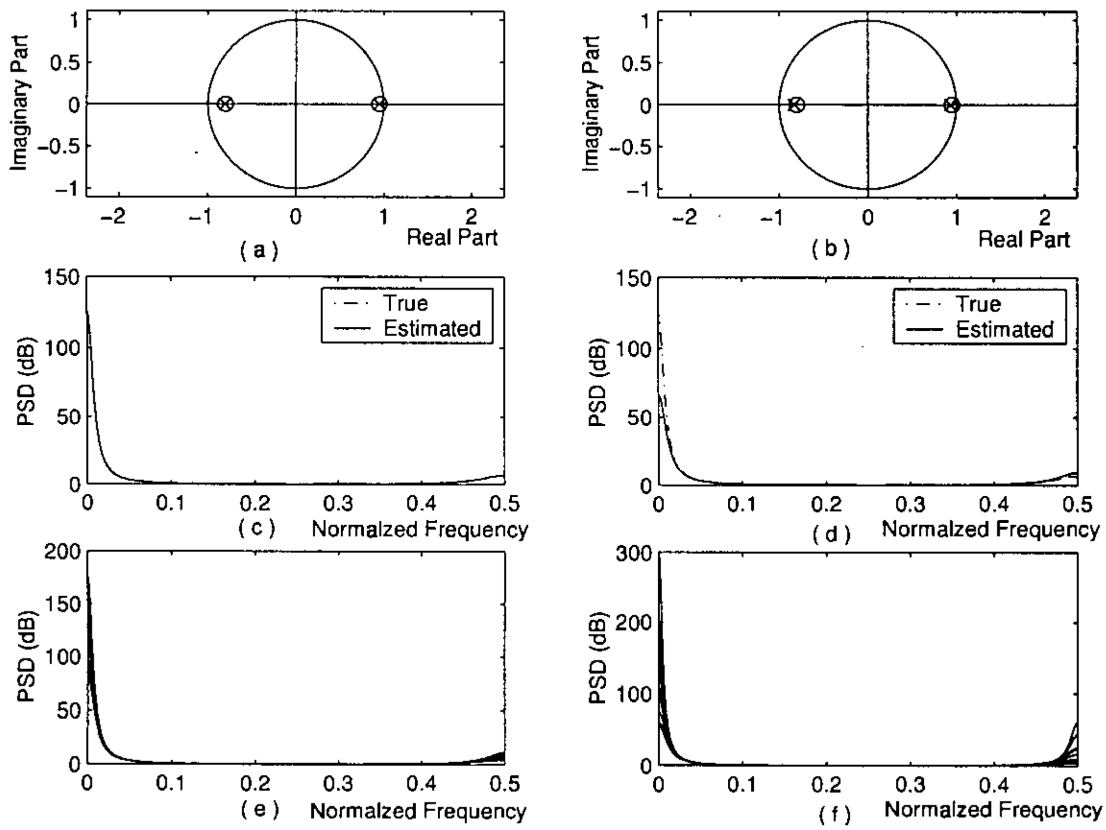


Fig. 4.6: Estimated poles and power spectra obtained by proposed method for AR **System 2** : (a) estimated poles at SNR=10 dB (o: true, x: estimate), (b) estimated poles at SNR=−5 dB (o: true, x: estimate), (c) true and estimated average power spectrum at SNR=10 dB, (d) true and estimated average power spectrum at SNR=−5 dB, (e) estimated power spectrum of 25 runs at SNR=10 dB, (f) estimated power spectrum of 25 runs at SNR=−5 dB.

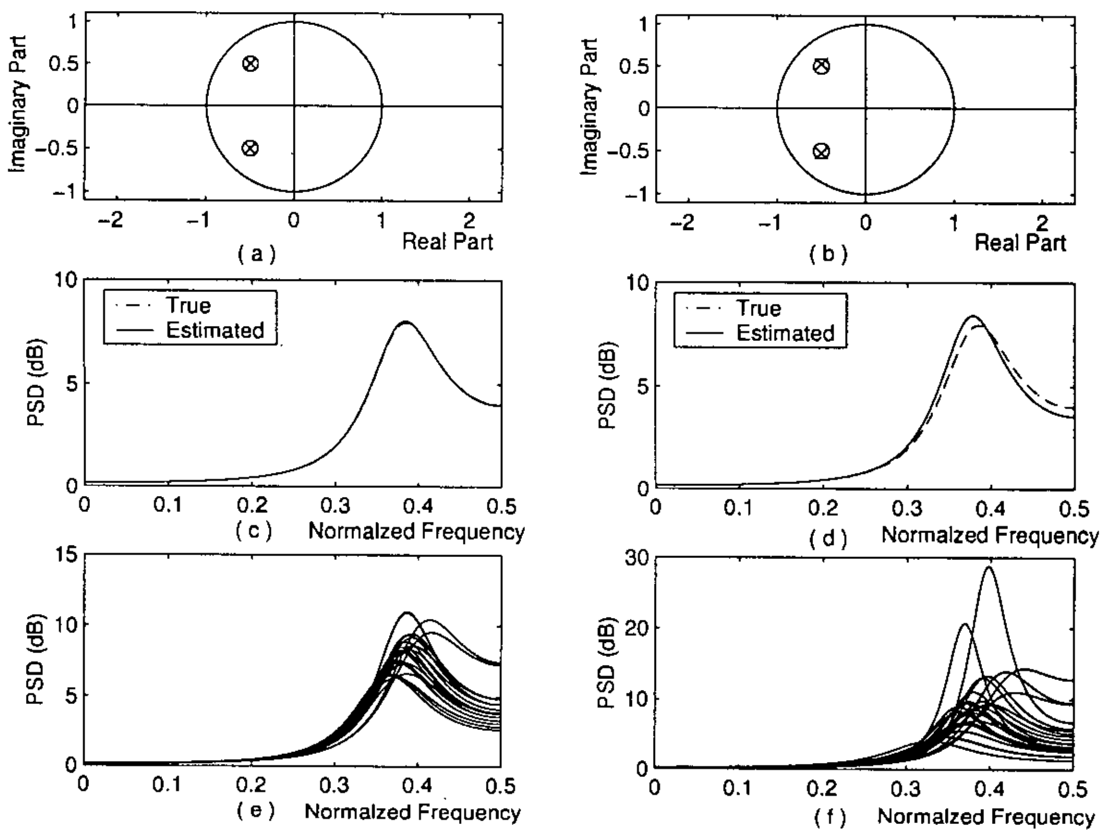


Fig. 4.7: Estimated poles and power spectra obtained by proposed method for AR **System 3** : (a) estimated poles at SNR=10 dB (o: true, x: estimate), (b) estimated poles at SNR=−5 dB (o: true, x: estimate), (c) true and estimated average power spectrum at SNR=10 dB, (d) true and estimated average power spectrum at SNR=−5 dB, (e) estimated power spectrum of 25 runs at SNR=10 dB, (f) estimated power spectrum of 25 runs at SNR=−5 dB.

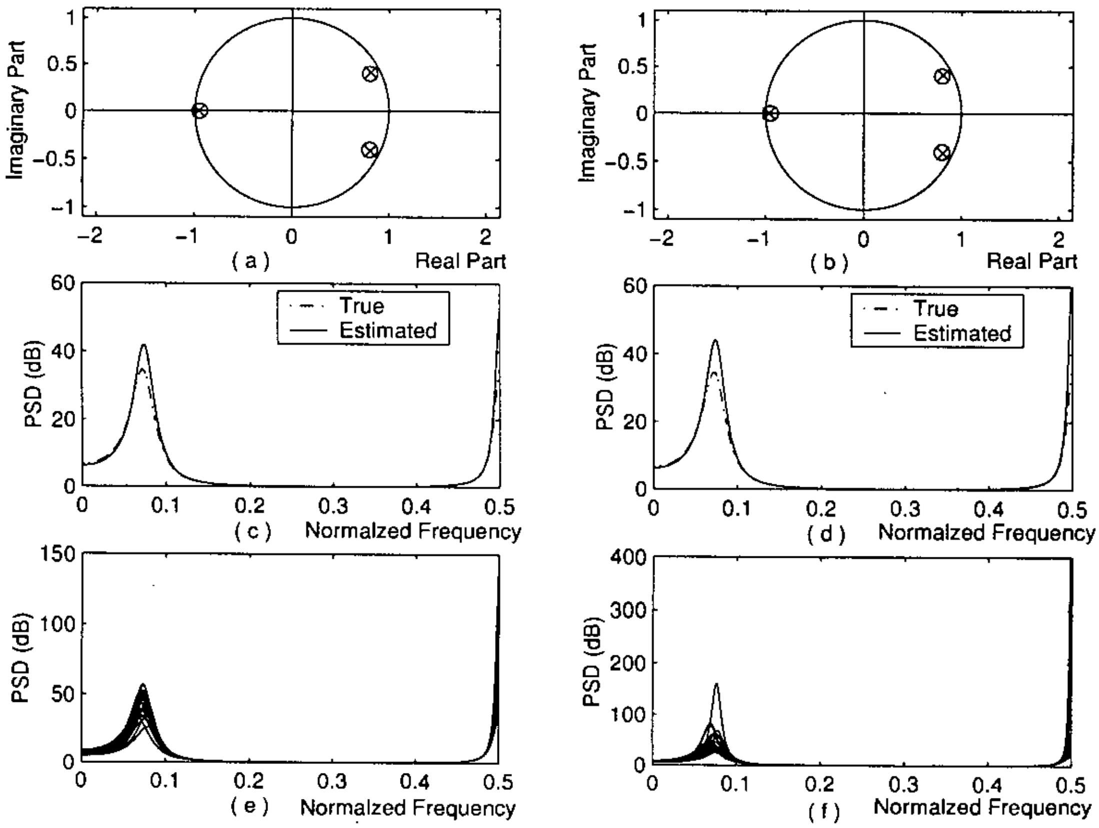


Fig. 4.8: Estimated poles and power spectra obtained by proposed method for AR **System 4** : (a) estimated poles at SNR=10 dB (o: true, x: estimate), (b) estimated poles at SNR=−5 dB (o: true, x: estimate), (c) true and estimated average power spectrum at SNR=10 dB, (d) true and estimated average power spectrum at SNR=−5 dB, (e) estimated power spectrum of 25 runs at SNR=10 dB, (f) estimated power spectrum of 25 runs at SNR=−5 dB.

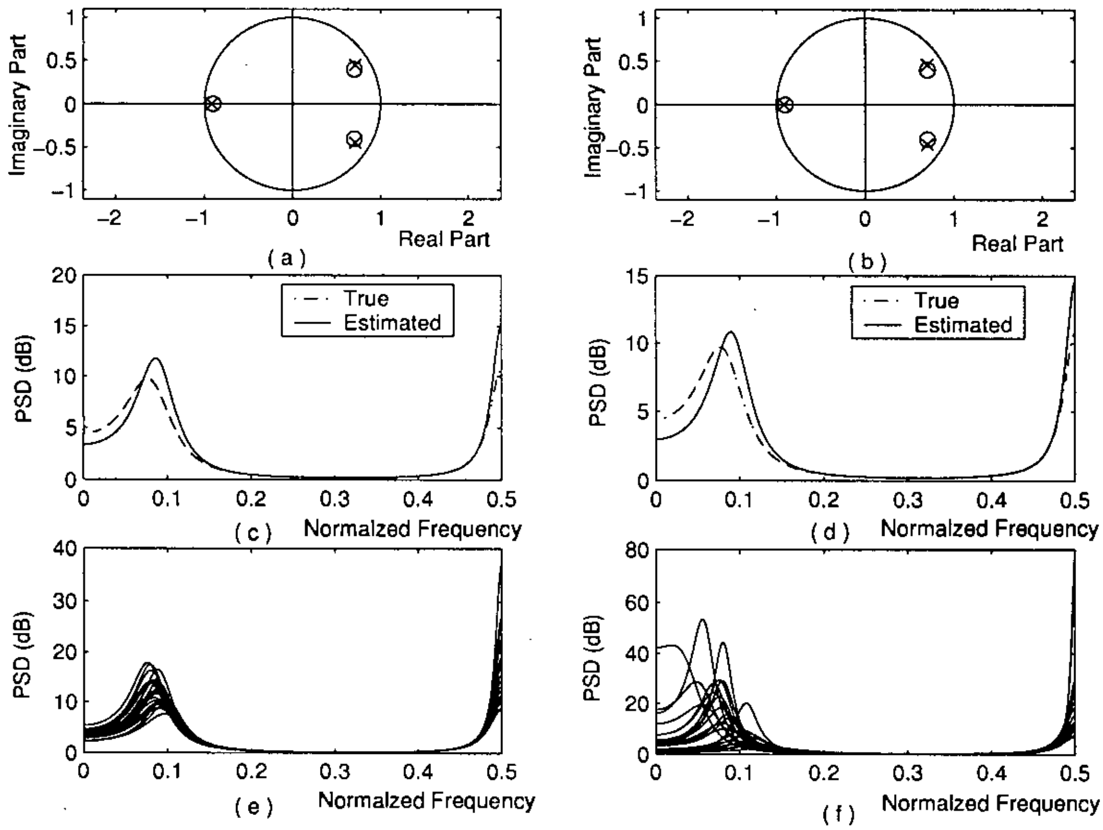


Fig. 4.9: Estimated poles and power spectra obtained by proposed method for AR **System 5** : (a) estimated poles at SNR=10 dB (o: true, x: estimate), (b) estimated poles at SNR=−5dB (o: true, x: estimate), (c) true and estimated average power spectrum at SNR=10 dB, (d) true and estimated average power spectrum at SNR=−5 dB, (e) estimated power spectrum of 25 runs at SNR=10 dB, (f) estimated power spectrum of 25 runs at SNR=−5 dB.

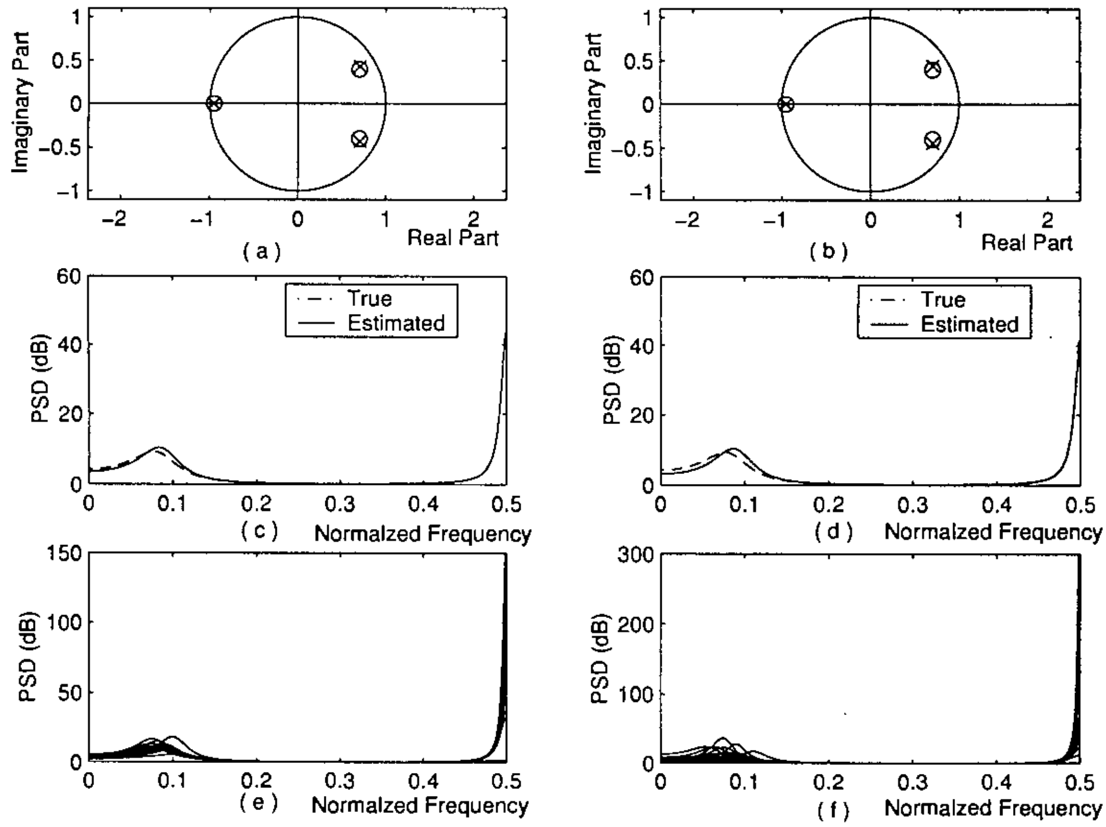


Fig. 4.10: Estimated poles and power spectra obtained by proposed method for AR **System 6** : (a) estimated poles at SNR=10 dB (o: true, x: estimate), (b) estimated poles at SNR=−5 dB (o: true, x: estimate), (c) true and estimated average power spectrum at SNR=10 dB, (d) true and estimated average power spectrum at SNR=−5 dB, (e) estimated power spectrum of 25 runs at SNR=10 dB, (f) estimated power spectrum of 25 runs at SNR=−5 dB.

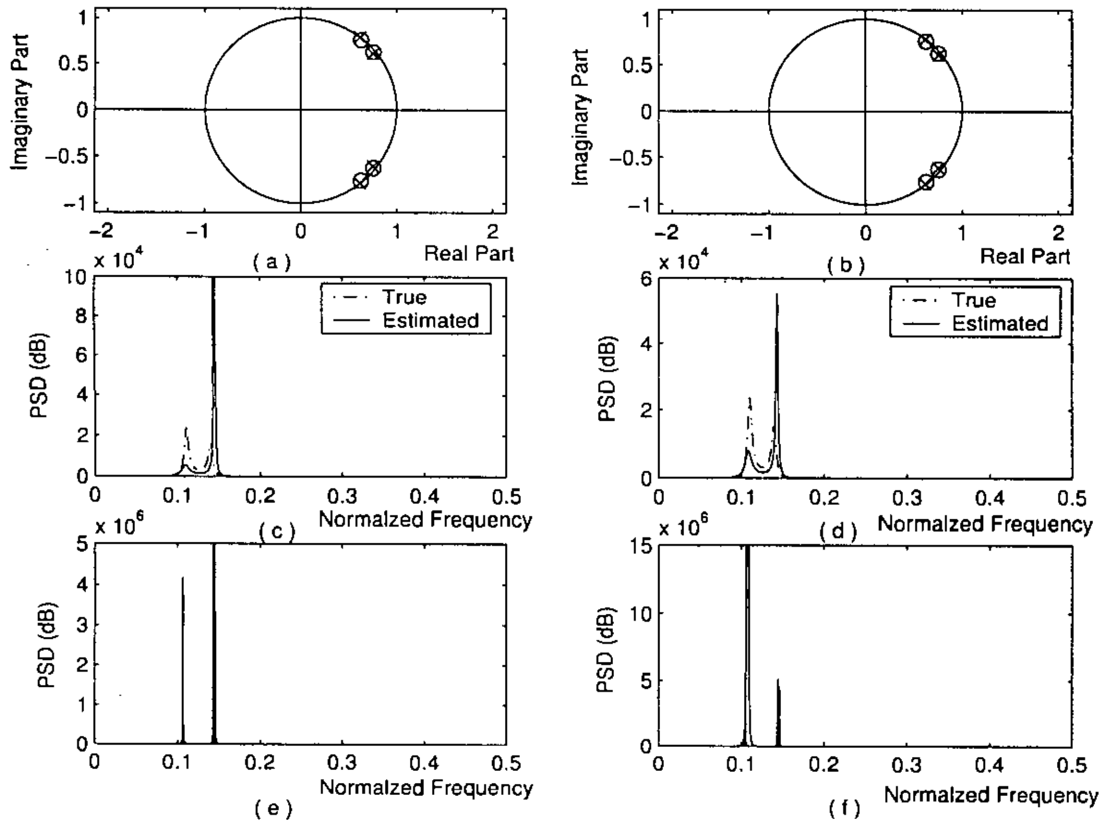


Fig. 4.11: Estimated poles and power spectra obtained by proposed method for AR **System 7** : (a) estimated poles at SNR=10 dB (o: true, x: estimate), (b) estimated poles at SNR=−5 dB (o: true, x: estimate), (c) true and estimated average power spectrum at SNR=10 dB, (d) true and estimated average power spectrum at SNR=−5 dB, (e) estimated power spectrum of 25 runs at SNR=10 dB, (f) estimated power spectrum of 25 runs at SNR=−5 dB.

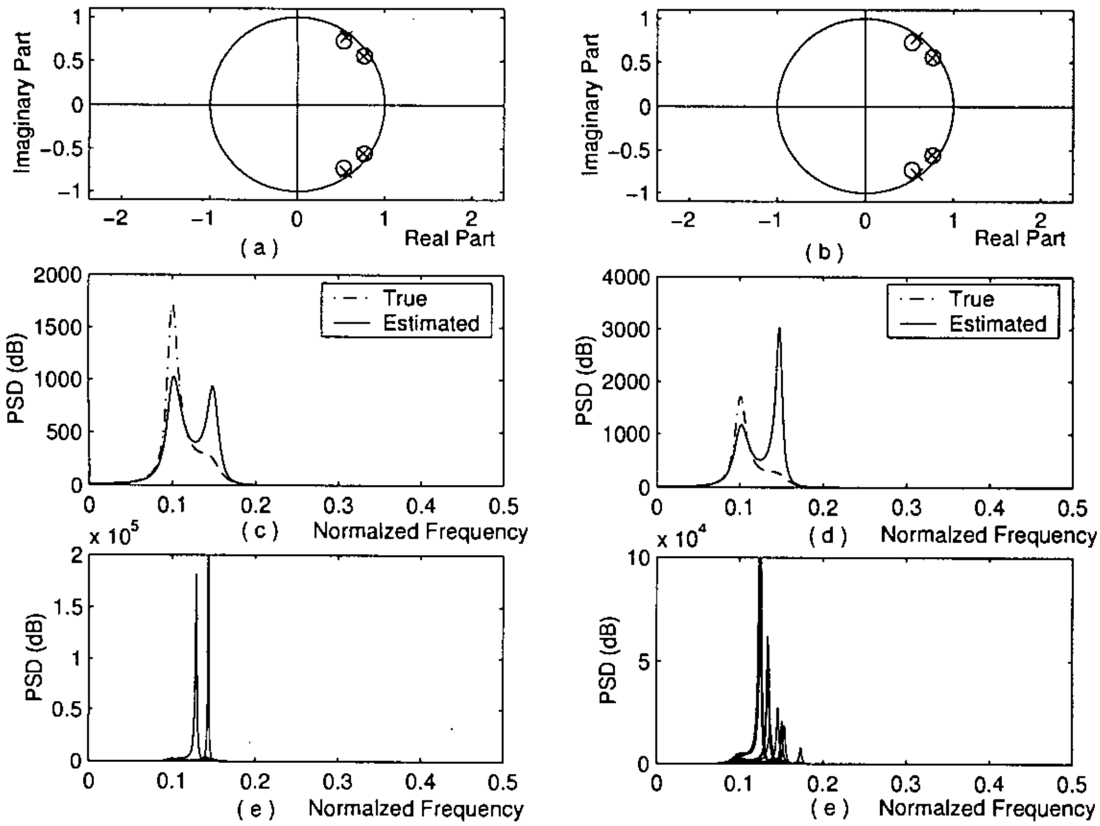


Fig. 4.12: Estimated poles and power spectra obtained by proposed method for AR **System 8** : (a) estimated poles at SNR=10 dB (o: true, x: estimate), (b) estimated poles at SNR=−5 dB (o: true, x: estimate), (c) true and estimated average power spectrum at SNR=10 dB, (d) true and estimated average power spectrum at SNR=−5 dB, (e) estimated power spectrum of 25 runs at SNR=10 dB, (f) estimated power spectrum of 25 runs at SNR=−5 dB.

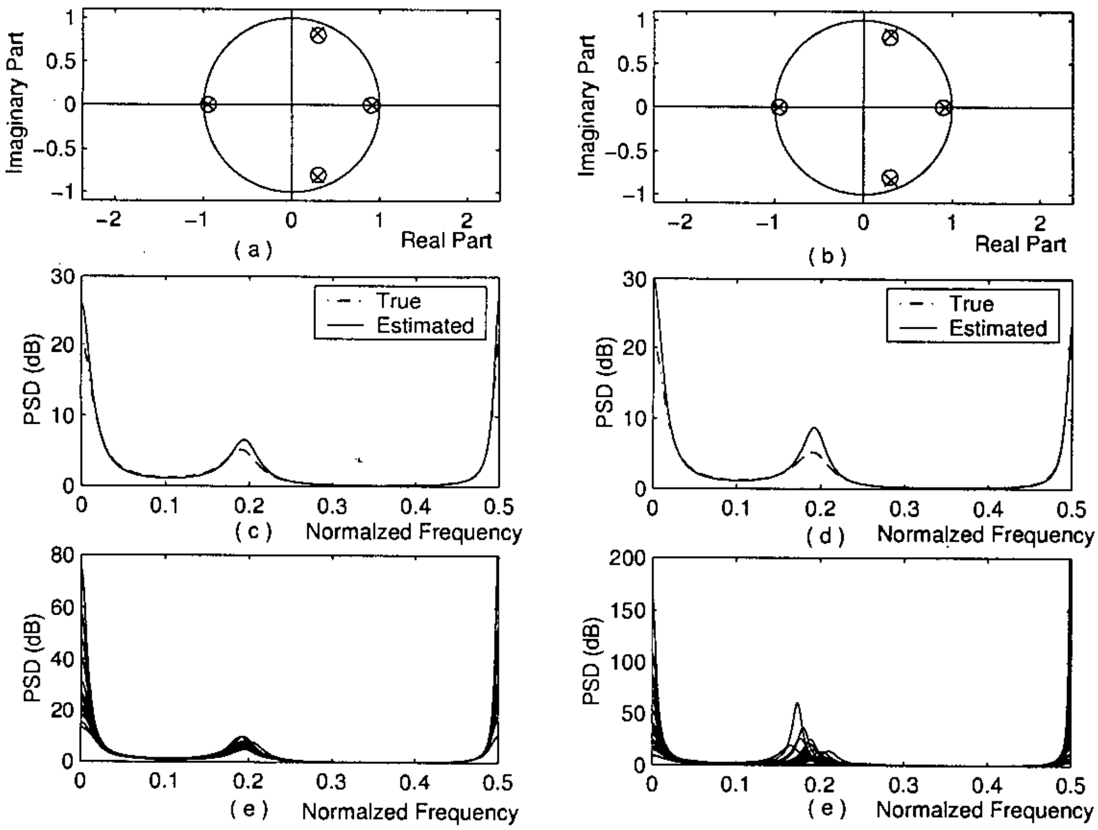


Fig. 4.13: Estimated poles and power spectra obtained by proposed method for AR **System 9** : (a) estimated poles at SNR=10 dB (o: true, x: estimate), (b) estimated poles at SNR=−5 dB (o: true, x: estimate), (c) true and estimated average power spectrum at SNR=10 dB, (d) true and estimated average power spectrum at SNR=−5 dB, (e) estimated power spectrum of 25 runs at SNR=10 dB, (f) estimated power spectrum of 25 runs at SNR=−5 dB.

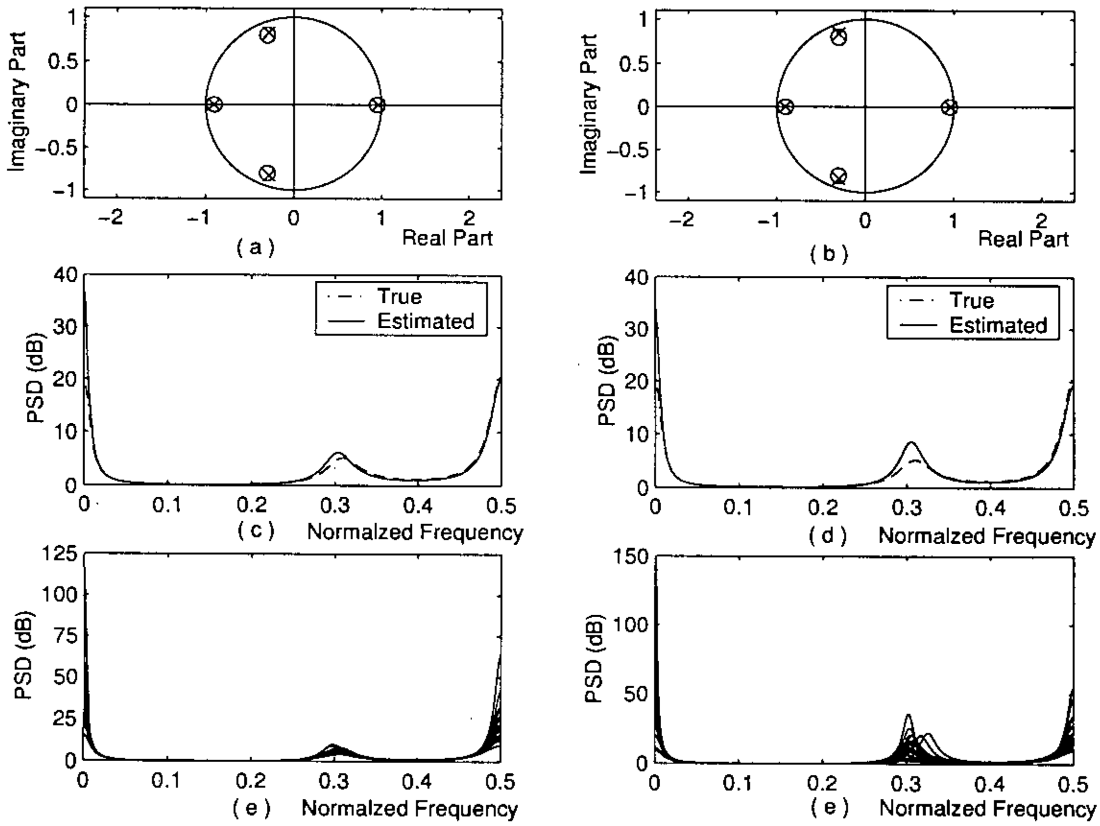


Fig. 4.14: Estimated poles and power spectra obtained by proposed method for AR **System 10** : (a) estimated poles at SNR=10 dB (o: true, x: estimate), (b) estimated poles at SNR=−5 dB (o: true, x: estimate), (c) true and estimated average power spectrum at SNR=10 dB, (d) true and estimated average power spectrum at SNR=−5 dB, (e) estimated power spectrum of 25 runs at SNR=10 dB, (f) estimated power spectrum of 25 runs at SNR=−5 dB.

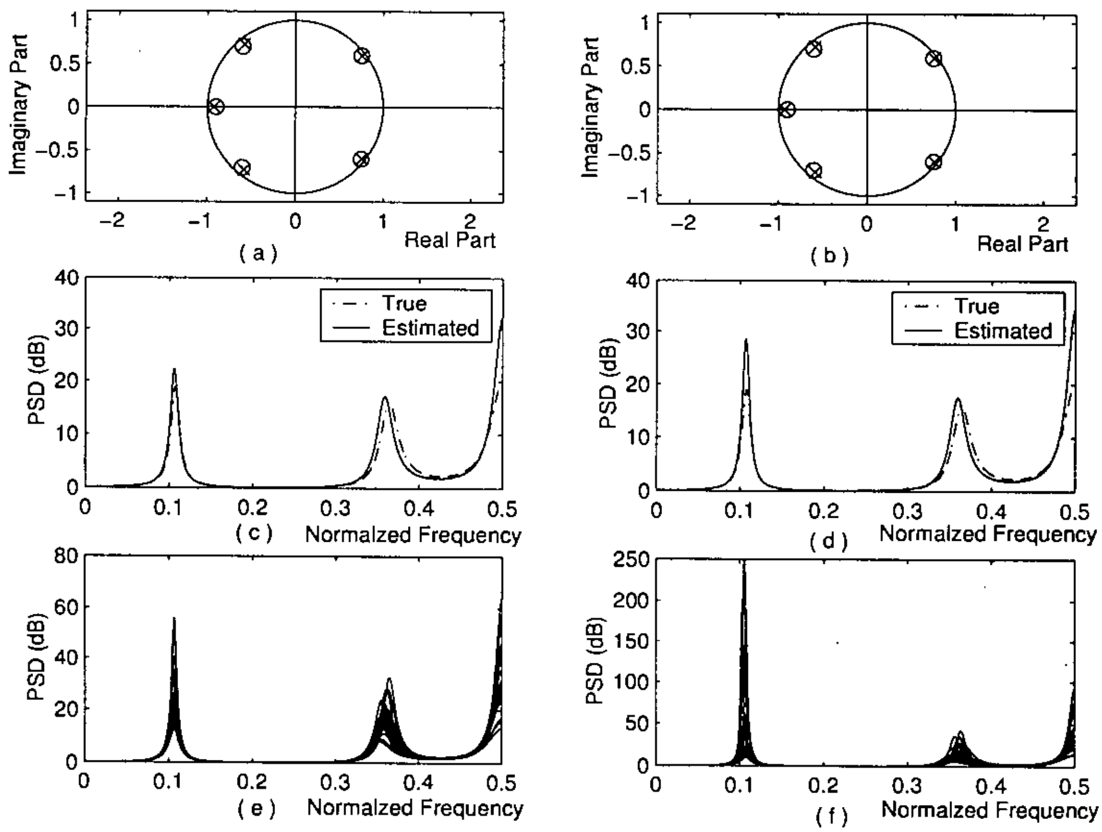


Fig. 4.15: Estimated poles and power spectra obtained by proposed method for AR **System 11** : (a) estimated poles at SNR=10 dB (o: true, x: estimate), (b) estimated poles at SNR=−5 dB (o: true, x: estimate), (c) true and estimated average power spectrum at SNR=10 dB, (d) true and estimated average power spectrum at SNR=−5 dB, (e) estimated power spectrum of 25 runs at SNR=10 dB, (f) estimated power spectrum of 25 runs at SNR=−5 dB.

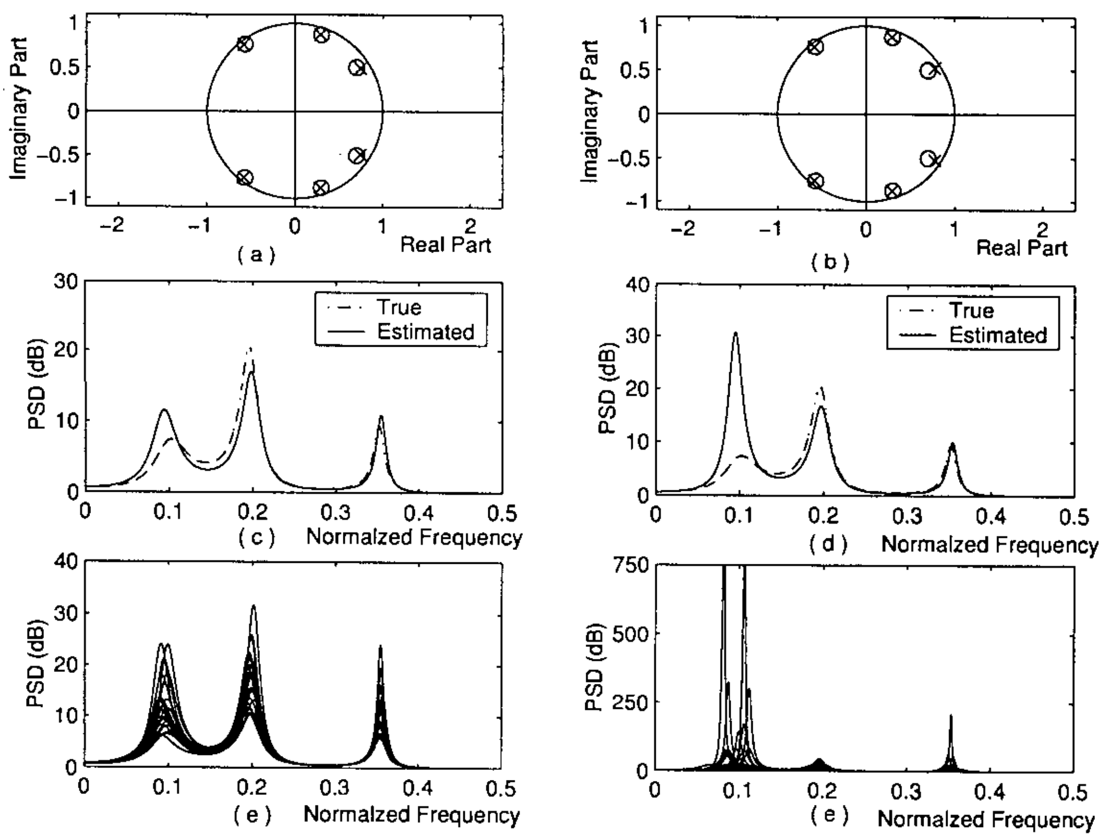


Fig. 4.16: Estimated poles and power spectra obtained by proposed method for AR **System 12** : (a) estimated poles at SNR=10 dB (o: true, x: estimate), (b) estimated poles at SNR=−5 dB (o: true, x: estimate), (c) true and estimated average power spectrum at SNR=10 dB, (d) true and estimated average power spectrum at SNR=−5 dB, (e) estimated power spectrum of 25 runs at SNR=10 dB, (f) estimated power spectrum of 25 runs at SNR=−5 dB.

Chapter 5

Conclusion

A novel method for the identification of autoregressive (AR) system from noisy measurements of its output response has been proposed. The major focus of the research was to develop an accurate estimation technique for AR parameters specially at low SNR. To achieve this goal, unlike conventional approaches, a *damped cosine model* of autocorrelation function of the noise-free AR signal is employed. A least-square based technique is used to estimate the model parameters from the autocorrelation function of the noisy signal. Even at a low SNR the model parameters can be estimated quite accurately. Once the model parameters are estimated, the AR system parameters can be obtained directly from these parameters. The conventional correlation based techniques fail to estimate the AR parameters below a certain positive value of SNR due to inaccurate estimation of the autocorrelation function from a finite set of noisy data. This research results have shown that the calculation of autocorrelation function using a *damped cosine model* can alleviate this problem and can identify AR systems even at an SNR as low as -5 dB.

Accuracy of the estimation of AR parameters by the *Damped cosine model* depends on the accuracy of the estimation of the model parameters, e.g. , the magnitude (r) and the angular position (ω) of the poles. With a view to achieve the desired accuracy, the entire domain of these two unknown parameters for each pole is scanned at a high resolution. Thus the proposed searching technique may be considered as computationally expensive.

The proposed method strongly depends on the pole position of the AR system. Poles located near the unit circle of the z-plane can be identified with high accuracy even at a very low SNR, e.g. -5 dB. It has been observed that the performance of the proposed method alike the conventional methods deteriorates for the systems with poles away from the unit circle. As with the other methods, the proposed one may fail to identify AR systems with very short duration impulse response, or in other words for systems having poles closer to the origin. The main reason behind this failure is that at a very low SNR the observed signal autocorrelation function includes pronounced error even after the zero lag due to the existence of correlation between the AR signal and the noise and also because of the fact that, except for first few lags, the values of the autocorrelation sequence are comparable to this error. Further investigation is required to solve this problem.

We have considered that the observation noise is a white noise. In some practical cases the additive noise may not be white. Recently, Zheng [45] has extended the ILSNP method [27] to AR system identification from white noise corrupted observations for AR signals corrupted by colored noise. Extension of the damped cosine method for estimation of AR parameters in presence of colored noise is to be investigated.

Modeling human vocal-tract as all-pole system and the corresponding speech signal as AR process is one of the most important applications of AR modeling [46]. Using the damped cosine model proposed in this work the two important phenomena, speaker identification and pitch detection, may be analyzed in future.

Another important topic is the estimation of the order of AR systems. In this work, order of the AR system has been assumed to be known. A great deal of work has been done on the AR system order estimation [47]-[51] when the observations are noise-free. However, the accurate estimation of the system order from noisy signal is yet a challenging problem.

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Appendix A

Derivation of The New AR Signal Model

The transfer function of an AR system in z-domain can be expressed as

$$H(z) = \frac{1}{A(z)} = \sum_{k=1}^{p} \frac{C_k}{1 - z_k z^{-1}} \tag{A.1}$$

where,

$$\begin{aligned} p &= \text{Order of the AR system, assumed to be known} \\ z_k &= \text{k-th pole of the AR system} \\ &= r_k \; e^{(j\omega_k)} \\ r_k &= \text{magnitude of pole } z_k \\ \omega_k &= \text{angular position of pole } z_k \\ C_k &= \text{Partial fraction coefficient corresponds to the} \\ & \quad\; k\text{-th pole of the AR system} \end{aligned} \tag{A.2}$$

We shall consider a causal, stable and minimum phase linear time-invariant (LTI) system whose unit impulse response $h(n)$ satisfies the condition

$$h(n) = 0, \quad \text{for } n < 0 \tag{A.3}$$

The unit impulse response $h(n)$ of the system described in Eqn. (A.1) can be expressed as

$$h(n) = \sum_{k=1}^{p} C_k (z_k)^n \tag{A.4}$$

This p-th order relaxed AR system is excited by a sequence of white noise $u(n)$ with distribution $\mathcal{N}(0, \sigma_u^2)$. The response to the white noise input can be expressed as the convolution of $u(n)$ and $h(n)$. Using convolution summation, the response $x^M(n)$ can be written as

$$\begin{aligned} x^M(n) &= u(n) * h(n) \\ &= \sum_{m=-\infty}^{\infty} u(m)h(n-m) \end{aligned} \tag{A.5}$$

Using causality condition as in Eqn. (A.3) and replacing $h(n-m)$ according to Eqn. (A.4), Eqn. (A.5) can be written as

$$x^M(n) = \sum_{m=0}^{n} u(m) \left[\sum_{k=1}^{p} C_k (z_k)^{n-m} \right] \tag{A.6}$$

Thus the proposed signal model for the p-th order AR system is

$$x^M(n) = \sum_{k=1}^{p} \sum_{m=0}^{n} C_k u(m) (z_k)^{n-m} \tag{A.7}$$

For clear visualization of this model, the matrix representation of Eqn. (A.7) is given below

$$\begin{aligned} x^M(n) &= \sum_{k=1}^{p} C_k \left[u(0)(z_k)^n \;\; u(1)(z_k)^{n-1} \cdots u(n-1) \;\; u(n-1)z_k \;\; u(n) \right] \\ &= \sum_{k=1}^{p} C_k \left[u(0) \;\; u(1) \cdots u(n) \right] \begin{bmatrix} (z_k)^n \\ (z_k)^{n-1} \\ \vdots \\ z_k \\ 1 \end{bmatrix} \end{aligned} \tag{A.8}$$

Let us now investigate the responses at different instances generated by this alternate representation of the autoregressive signal. For a first order system, the output at the first instant, i.e. $x_1^M(0)$ can be found from Eqn. (A.8)

$$x_1^M(0) = \sum_{k=1}^{1} C_k u(0) = C_1 u(0) \tag{A.9}$$

The subscript 1 is added with x^M to indicate the order of the system. It is obvious that the partial fraction coefficient C_1 for a first order AR system, will be always 1. Hence

$$x_1^M(0) = u(0) \tag{A.10}$$

As $x^M(n)$ is real, the AR system parameters $\{a_i\}$ are real. Hence the complex poles of the AR system must come in conjugate and the corresponding coefficients in the partial fraction expansion of the AR system transfer function are also complex conjugates. However, in case of a second order system the two poles are either two real poles or a pair of complex conjugate poles. For both the case sum of partial fraction coefficients is zero. For a second order system, the output at the first instant, i.e. $x_2^M(0)$ as found from Eqn. (A.8) is

$$\begin{aligned} x_2^M(0) &= \sum_{k=1}^{2} C_k u(0) \\ &= [C_1 + C_2]\, u(0) \\ &= 0 \times u(0) \\ &= 0 \end{aligned} \tag{A.11}$$

The output at the second instant, i.e. $x_2^M(1)$, obtained from Eqn. (A.8) is

$$\begin{aligned} x_2^M(1) &= \sum_{k=1}^{2} C_k \left[u(0) z_k + u(1)\right] \\ &= [C_1 z_1 + C_2 z_2]\, u(0) + [C_1 + C_2]\, u(1) \\ &= [C_1 z_1 + C_2 z_2]\, u(0) + 0 \end{aligned} \tag{A.12}$$

Let us consider a second order system containing a pair of complex conjugate poles z_1 and z_2, where $z_1 = \alpha_1 + j\gamma_1$ and $z_2 = \alpha_1 - j\gamma_1$. The corresponding partial fraction coefficients C_1 and C_2 must be a complex conjugate pair. Evaluating the partial fraction coefficients we get,

$$\begin{aligned} C_1 &= \frac{1}{z_1 - z_2} = -\frac{j}{2\gamma_1} \\ C_2 &= \frac{1}{z_2 - z_1} = \frac{j}{2\gamma_1} = C_1^* \end{aligned} \tag{A.13}$$

With this C_1 and C_2 we get,

$$\begin{aligned} C_1 z_1 + C_2 z_2 &= \left(-\frac{j}{2\gamma_1}\right)(\alpha_1 + j\gamma_1) + \left(\frac{j}{2\gamma_1}\right)(\alpha_1 - j\gamma_1) \\ &= \left(\frac{1}{2} - j\frac{\alpha_1}{2\gamma_1}\right) + \left(\frac{1}{2} + j\frac{\alpha_1}{2\gamma_1}\right) \\ &= 1 \end{aligned} \tag{A.14}$$

Using Eqn. (A.14), we can find $x_2^M(1)$ in Eqn. (A.12) as

$$x_2^M(1) = u(0) \tag{A.15}$$

For two real roots ρ_1 and ρ_2, $C_1 = -C_2 = 1/(z_1 - z_2) = 1/(\rho_1 - \rho_2)$. Again the right hand side of the Eqn. (A.14) will be 1 and as a result, we get $x_2^M(1) = u(0)$.

Let us now analyze a third order system, i.e. $p = 3$. The system contains a pair of complex conjugate poles z_1, z_2 and a real pole ρ_3. The corresponding partial fraction coefficients are C_1, C_2 and C_3. We take $z_1 = \alpha_1 + j\gamma_1$ and $z_2 = \alpha_1 - j\gamma_1$. Evaluating the partial fraction coefficients we get,

$$
\begin{aligned}
C_3 &= \frac{1}{(z_3 - z_1)(z_3 - z_2)} \\
&= \frac{1}{\{(\rho_3 - \alpha_1) - j\gamma_1\}\{(\rho_3 - \alpha_1) + j\gamma_1\}} \\
&= \frac{1}{(\rho_3 - \alpha_1)^2 + \gamma_1^2} \\
C_1 &= \frac{1}{(z_1 - z_2)(z_1 - z_3)} \\
&= \frac{1}{\{j2\gamma_1\}\{(\alpha_1 - \rho_3) + j\gamma_1\}} \\
&= \frac{-\gamma_1 - j(\alpha_1 - \rho_3)}{2\gamma_1\{\gamma_1^2 + (\alpha_1 - \rho_3)^2\}} \\
&= \left(\frac{-C_3}{2\gamma_1}\right)[\gamma_1 + j(\alpha_1 - \rho_3)] \\
C_2 &= C_1^* \\
&= \left(\frac{-C_3}{2\gamma_1}\right)[\gamma_1 - j(\alpha_1 - \rho_3)]
\end{aligned}
\tag{A.16}
$$

As in the case of a second order system, here we also see that the sum of the partial fraction coefficients is zero. We have also analyzed higher order systems with different combination of poles and evrywhere we get the same result. Therefore, we can conclude in general that for an AR system with real $x^M(n)$, the sum of the partial fraction coefficients of the system transfer function is zero, i.e.

$$\sum_{i=1}^{p} C_i = 0 \tag{A.17}$$

For a third order system, the output at the first instant, i.e. $x_3^M(0)$ as found from Eqn. (A.8) is

$$x_3^M(0) = \sum_{k=1}^{3} C_k u(0) = 0 \tag{A.18}$$

The output at the second instant, i.e. $x_3^M(1)$, obtained from Eqn. (A.8) is

$$x_3^M(1) = \sum_{k=1}^{3} C_k [u(0) z_k + u(1)]$$

$$= \sum_{k=1}^{3} C_k z_k u(0) \tag{A.19}$$

The output at the third instant, i.e. $x_3^M(2)$, obtained from Eqn. (A.8) is

$$\begin{aligned} x_3^M(2) &= \sum_{k=1}^{3} C_k \left[u(0)z_k^2 + u(1)z_k + u(2)\right] \\ &= \sum_{k=1}^{3} C_k z_k^2 u(0) + \sum_{k=1}^{3} C_k z_k u(1) \end{aligned} \tag{A.20}$$

Now we evaluate $\sum_{i=1}^{3} C_i z_i$ and $\sum_{i=1}^{3} C_i z_i^2$ as follows

$$\begin{aligned} \sum_{i=1}^{3} C_i z_i &= \left(\frac{-C_3}{2\gamma_1}\right)[\gamma_1 + j(\alpha_1 - \rho_3)][\alpha_1 + j\gamma_1] + \\ &\quad \left(\frac{-C_3}{2\gamma_1}\right)[\gamma_1 - j(\alpha_1 - \rho_3)][\alpha_1 - j\gamma_1] + C_3\rho_3 \\ &= \left(\frac{-C_3}{2\gamma_1}\right)[2\{\gamma_1\alpha_1 - (\alpha_1 - \rho_3)\gamma_1\}] + C_3\rho_3 \\ &= \left(\frac{-C_3}{2\gamma_1}\right)[2\gamma_1\rho_3] + C_3\rho_3 \\ &= 0 \end{aligned} \tag{A.21}$$

$$\begin{aligned} \sum_{i=1}^{3} C_i z_i^2 &= \left(\frac{-C_3}{2\gamma_1}\right)[\gamma_1 + j(\alpha_1 - \rho_3)][\alpha_1 + j\gamma_1]^2 + \\ &\quad \left(\frac{-C_3}{2\gamma_1}\right)[\gamma_1 - j(\alpha_1 - \rho_3)][\alpha_1 - j\gamma_1]^2 + C_3\rho_3^2 \\ &= \left(\frac{-C_3}{2\gamma_1}\right)\left[2\left\{\alpha_1^2\gamma_1 - \gamma_1^3 - 2\alpha_1^2\gamma_1 + 2\alpha_1\rho_3\gamma_1\right\}\right] + C_3\rho_3^2 \\ &= C_3(\alpha_1^2 + \gamma_1^2 - 2\alpha_1\rho_3) + C_3\rho_3^2 \\ &= C_3(\rho_3^2 - 2\alpha_1\rho_3 + \alpha_1^2 + \gamma_1^2) \\ &= \left[\frac{1}{(\rho_3 - \alpha_1)^2 + \gamma_1^2}\right]\left[(\rho_3 - \alpha_1)^2 + \gamma_1^2\right] \\ &= 1 \end{aligned} \tag{A.22}$$

Using Eqn. (A.21) and Eqn. (A.19), $x_3^M(1)$ can be found as

$$x_3^M(1) = 0 \tag{A.23}$$

Similarly using Eqn. (A.21), Eqn. (A.22) and Eqn. (A.20), $x_3^M(2)$ can be found as

$$x_3^M(2) = u(0) \tag{A.24}$$

We now compare this signal model with the difference equation of the AR system given by

$$x(n) = -\sum_{k=1}^{p} a_k x(n-k) + u(n) \tag{A.25}$$

From the difference Eqn. (A.25), we see that $x(0)$ is always $u(0)$. To comment about the signal model described in Eqn. (A.7), we rewrite the responses that we have found by using Eqn. (A.7) for the first, second and third order AR systems. For a first order system,

$$x_1^M(0) = u(0) \tag{A.26}$$

For a second order system,

$$\begin{aligned} x_2^M(0) &= 0 \\ x_2^M(1) &= u(0) \end{aligned}$$

For a third order system,

$$\begin{aligned} x_3^M(0) &= 0 \\ x_3^M(1) &= 0 \\ x_3^M(2) &= u(0) \end{aligned}$$

Proceeding in this way, we can write initial responses for a p-th order system

$$\begin{aligned} x_p^M(0) &= 0 \\ x_p^M(1) &= 0 \\ x_p^M(2) &= 0 \\ &\vdots \\ x_p^M(p-1) &= u(0) \end{aligned}$$

Hence the response obtained from the proposed signal model at the p-th instant $x_p^M(p-1)$ is similar to the response obtained from the first instant of the difference equation. Therefore, the two models generate the same sequences. It is to be mentioned that for a p-th order AR system the proposed signal model will generate $(p-1)$ number of zeros at the beginning and then it generates the same sequence as that would be generated by the difference equation. Therefore, we can relate the responses of the two representations as

$$z^{-(p-1)} x^M(n) = x(n) \tag{A.27}$$

Appendix B

Derivation of the Damped Cosine Model

The signal model proposed in Eqn. (3.10) is

$$x^M(n) = \sum_{k=1}^{p} \sum_{m=0}^{n} C_k u(m)(z_k)^{n-m} \tag{B.1}$$

To derive the general formula for the damped cosine model of autocorrelation function, let us consider a p-th order relaxed AR system excited by a sequence of white noise $u(n)$ with distribution $\mathcal{N}(0, \sigma_u^2)$. The response to the white noise at different instances are as follows:

$$\begin{aligned}
x_p^M(0) &= C_1 u(0) + C_2 u(0) + \cdots + C_p u(0) \\
x_p^M(1) &= C_1 \{u(0) z_1 + u(1)\} + C_2 \{u(0) z_2 + u(1)\} + \cdots\cdots \\
&\qquad + C_p \{u(0) z_p + u(1)\} \\
x_p^M(2) &= C_1 \left\{u(0) z_1^2 + u(1) z_1 + u(2)\right\} + C_2 \left\{u(0) z_2^2 + u(1) z_2 + u(2)\right\} + \\
&\qquad \cdots + C_p \left\{u(0) z_p^2 + u(1) z_p + u(2)\right\} \\
&\vdots \\
&\vdots \\
x_p^M(l) &= C_1 \left\{u(0) z_1^l + u(1) z_1^{l-1} + \cdots + u(l-1) z_1 + u(l)\right\} + \\
&\quad C_2 \left\{u(0) z_2^l + u(1) z_2^{l-1} + \cdots + u(l-1) z_2 + u(l)\right\} + \\
&\quad \cdots \qquad \cdots \qquad \cdots \\
&\quad \cdots \qquad \cdots \qquad \cdots \\
&\quad \cdots \qquad \cdots \qquad \cdots \\
&\quad + C_p \left\{u(0) z_p^l + u(1) z_p^{l-1} + \cdots + u(l-1) z_p + u(l)\right\}
\end{aligned}$$

$$
\begin{aligned}
x_p^M(l+1) &= C_1\left\{u(0)z_1^{l+1}+u(1)z_1^{l}+\cdots+u(l)z_1+u(l+1)\right\}+\\
&\quad C_2\left\{u(0)z_2^{l+1}+u(1)z_2^{l}+\cdots+u(l)z_2+u(l+1)\right\}+\\
&\quad \cdots \qquad \cdots \qquad \cdots\\
&\quad \cdots \qquad \cdots \qquad \cdots\\
&\quad +C_p\left\{u(0)z_p^{l+1}+u(1)z_p^{l}+\cdots+u(l)z_p+u(l+1)\right\}\\
x_p^M(l+2) &= C_1\left\{u(0)z_1^{l+2}+u(1)z_1^{l+1}+\cdots+u(l+1)z_1+u(l+2)\right\}+\\
&\quad C_2\left\{u(0)z_2^{l+2}+u(1)z_2^{l+1}+\cdots+u(l+1)z_2+u(l+2)\right\}+\\
&\quad \cdots \qquad \cdots \qquad \cdots\\
&\quad \cdots \qquad \cdots \qquad \cdots\\
&\quad +C_p\left\{u(0)z_p^{l+2}+u(1)z_p^{l+1}+\cdots+u(l+1)z_p+u(l+2)\right\}\\
&\vdots\\
x_p^M(n-1) &= C_1\left\{u(0)z_1^{n-1}+u(1)z_1^{n-2}+\cdots+u(n-2)z_1+u(n-1)\right\}+\\
&\quad C_2\left\{u(0)z_2^{n-1}+u(1)z_2^{n-2}+\cdots+u(n-2)z_2+u(n-1)\right\}+\\
&\quad \cdots \qquad \cdots \qquad \cdots\\
&\quad \cdots \qquad \cdots \qquad \cdots\\
&\quad +C_p\left\{u(0)z_p^{n-1}+u(1)z_p^{n-2}+\cdots+u(n-2)z_p+u(n-1)\right\}\\
x_p^M(n) &= C_1\left\{u(0)z_1^{n}+u(1)z_1^{n-1}+\cdots+u(n-1)z_1+u(n)\right\}+\\
&\quad C_2\left\{u(0)z_2^{n}+u(1)z_2^{n-1}+\cdots+u(n-2)z_2+u(n)\right\}+\\
&\quad \cdots \qquad \cdots \qquad \cdots\\
&\quad \cdots \qquad \cdots \qquad \cdots\\
&\quad +C_p\left\{u(0)z_p^{n}+u(1)z_p^{n-1}+\cdots+u(n-2)z_p+u(n)\right\}
\end{aligned}
\tag{B.2}
$$

Calculating the autocorrelation function of $x_p^M(n)$ as in Eqn. (3.4), we obtain

$$
R_{xx}^M(l)=\frac{1}{N}\left[x_p^M(0)x_p^M(l)+x_p^M(1)x_p^M(l+1)+\cdots+x_p^M(N-1-l)x_p^M(N-1)\right]
\tag{B.3}
$$

Next, we evaluate each terms in the autocorrelation function as given in the following

$$
\begin{aligned}
&x_p^M(0)x_p^M(l)=\\
&\sum_{i=1}^{p}C_i^2\left\{u(0)^2z_i^l+u(0)u(1)z_i^{l-1}+\cdots+u(0)u(l)\right\}+
\end{aligned}
$$

$$\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_j\left\{u(0)^2z_j^l+u(0)u(1)z_j^{l-1}+\cdots+u(0)u(l)\right\}$$

$$x_p^M(1)x_p^M(l+1)=$$
$$\sum_{i=1}^{p}C_i^2\left\{\begin{array}{l}u(0)^2z_i^{l+2}+u(0)u(1)z_i^{l+1}+\cdots+u(0)u(l+1)z_i\\+u(1)u(0)z_i^{l+1}+u(1)^2z_i^l+\cdots+u(1)u(l+1)\end{array}\right\}+$$
$$\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p}C_iC_j\left\{\begin{array}{l}u(0)^2z_iz_j^{l+1}+u(0)u(1)z_iz_j^l+\cdots+u(0)u(l+1)z_i\\+u(1)u(0)z_j^{l+1}+u(1)^2z_j^l+\cdots+u(1)u(l+1)\end{array}\right\}$$

$$x_p^M(2)x_p^M(l+2)=$$
$$\sum_{i=1}^{p}C_i^2\left\{\begin{array}{l}u(0)^2z_i^{l+4}+u(0)u(1)z_i^{l+3}+\cdots+u(0)u(l+2)z_i^2\\+u(1)u(0)z_i^{l+3}+u(1)^2z_i^{l+2}+\cdots+u(1)u(l+2)z_i\\+u(2)u(0)z_i^{l+2}+u(2)u(1)z_i^{l+1}+\cdots+u(2)u(l+2)\end{array}\right\}+$$
$$\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p}C_iC_j\left\{\begin{array}{l}u(0)^2z_i^2z_j^{l+2}+u(0)u(1)z_i^2z_j^{l+1}+\cdots+u(0)u(l+2)z_i^2\\+u(1)u(0)z_iz_j^{l+2}+u(1)^2z_iz_j^{l+1}+\cdots+u(1)u(l+2)z_i\\+u(2)u(0)z_j^{l+2}+u(2)u(1)z_j^{l+1}+\cdots+u(2)u(l+2)\end{array}\right\}$$

$$\vdots$$

$$\vdots$$

$$x_p^M(\mu)x_p^M(l+\mu)=$$
$$\sum_{i=1}^{p}C_i^2\left\{\begin{array}{lll}u(0)^2z_i^{l+2\mu}+u(0)u(1)z_i^{l+2\mu-1}+\cdots+u(0)u(l+\mu)z_i^{\mu}\\+u(1)u(0)z_i^{l+2\mu-1}+u(1)^2z_i^{l+2\mu-2}+\cdots+u(1)u(l+\mu)z_i^{\mu-1}+\\\cdots\quad\cdots\quad\cdots\\\cdots\quad\cdots\quad\cdots\\+u(\mu)u(0)z_i^{l+\mu}+\cdots+u(\mu)^2z_i^l+\cdots+u(\mu)u(l+\mu)\end{array}\right\}+$$
$$\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p}C_iC_j\left\{\begin{array}{l}u(0)^2z_i^{\mu}z_j^{l+\mu}+u(0)u(1)z_i^{\mu}z_j^{l+\mu-1}+\cdots+u(0)u(l+\mu)z_i^{\mu}\\+u(1)u(0)z_i^{\mu-1}z_j^{l+\mu}+\cdots\cdots+u(1)u(l+\mu)z_i^{\mu-1}+\\\cdots\quad\cdots\quad\cdots\\\cdots\quad\cdots\quad\cdots\\u(\mu)u(0)z_j^{l+\mu}+\cdots+u(\mu)^2z_j^l+\cdots+u(\mu)u(l+\mu)\end{array}\right\}$$
(B.4)

Here $\mu=N-1-l$ and theoretically N tends to infinity, i.e. we have to consider a large number of data points. We know that autocorrelation function is a decaying function. The rate of decay depends on the magnitude of pole r, i.e., on the system parameters. After few lags $l=l_z, l_z \ll N$, autocorrelation function decays to negligible values. Thus for significant values of autocorrelation function, it is sufficient to take $l \ll N$. Then $\mu \simeq N$, i.e., the value of μ is very large. As the magnitude of a pole of a stable AR system must be less than one, the value of z_i^{μ} corresponding to a pole z_i will be very small. In the derivation of the formula for autocorrelation function we need to consider all the factors discussed above. Summing all the terms in Eqn. (B.4), we get the following result.

$$
\begin{aligned}
&\sum_{i=1}^{p} C_i^2 z_i^l (u(0)^2 + u(1)^2 + \cdots + u(\mu)^2) + \\
&\sum_{i=1}^{p} C_i^2 z_i^{l+2} (u(0)^2 + u(1)^2 + \cdots + u(\mu-1)^2) + \\
&\ldots \qquad \ldots \qquad \ldots \\
&\ldots \qquad \ldots \qquad \ldots \\
&+ \sum_{i=1}^{p} C_i^2 z_i^{l-1} (u(0)u(1) + u(1)u(2) + \cdots + u(\mu)u(\mu+1)) + \\
&+ \sum_{i=1}^{p} C_i^2 z_i^{l+1} (u(0)u(1) + u(1)u(2) + \cdots + u(\mu-1)u(\mu)) + \\
&\ldots \qquad \ldots \qquad \ldots \\
&\ldots \qquad \ldots \qquad \ldots \\
&+ \sum_{i=1}^{p} C_i^2 z_i^{l-2} (u(0)u(2) + u(1)u(3) + \cdots + u(\mu)u(\mu+2)) + \\
&+ \sum_{i=1}^{p} C_i^2 z_i^{l} (u(0)u(2) + u(1)u(3) + \cdots + u(\mu-1)u(\mu+1)) + \\
&\ldots \qquad \ldots \qquad \ldots \\
&\ldots \qquad \ldots \qquad \ldots \\
&+ \sum_{i=1}^{p} C_i^2 z_i^{l+1} (u(1)u(0) + u(2)u(1) + \cdots + u(\mu)u(\mu-1)) + \\
&+ \sum_{i=1}^{p} C_i^2 z_i^{l+3} (u(1)u(0) + u(2)u(1) + \cdots + u(\mu-1)u(\mu-2)) + \\
&\ldots \qquad \ldots \qquad \ldots \\
&\ldots \qquad \ldots \qquad \ldots \\
&+ \sum_{i=1}^{p} C_i^2 z_i^{l+2} (u(2)u(0) + u(3)u(1) + \cdots + u(\mu)u(\mu-2)) + \\
&+ \sum_{i=1}^{p} C_i^2 z_i^{l+4} (u(2)u(0) + u(3)u(1) + \cdots + u(\mu-1)u(\mu-3)) + \\
&\ldots \qquad \ldots \qquad \ldots \\
&\ldots \qquad \ldots \qquad \ldots
\end{aligned}
$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_j^l(u(0)^2+u(1)^2+\cdots+u(\mu)^2)+$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_iz_j^{l+1}(u(0)^2+u(1)^2+\cdots+u(\mu-1)^2)+$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i} C_iC_jz_i^2z_j^{l+2}(u(0)^2+u(1)^2+\cdots+u(\mu-2)^2)+$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_j^{l-1}(u(0)u(1)+u(1)u(2)+\cdots+u(\mu)u(\mu+1))+$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_iz_j^{l}(u(0)u(1)+u(1)u(2)+\cdots+u(\mu-1)u(\mu))+$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_i^2z_j^{l+1}(u(0)u(1)+u(1)u(2)+\cdots+u(\mu-2)u(\mu-1))+$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_j^{l-2}(u(0)u(2)+u(1)u(3)+\cdots+u(\mu)u(\mu+2))+$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_iz_j^{l-1}(u(0)u(2)+u(1)u(3)+\cdots+u(\mu-1)u(\mu+1))+$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_i^2z_j^{l}(u(0)u(2)+u(1)u(3)+\cdots+u(\mu-2)u(\mu))+$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_j^{l+1}(u(1)u(0)+u(2)u(1)+\cdots+u(\mu)u(\mu-1))+$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_iz_j^{l+2}(u(1)u(0)+u(2)u(1)+\cdots+u(\mu-1)u(\mu-2))+$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_i^2z_j^{l+3}(u(1)u(0)+u(2)u(1)+\cdots+u(\mu-2)u(\mu-3))+$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_j^{l+2}(u(2)u(0)+u(3)u(1)+\cdots+u(\mu)u(\mu-2))+$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_iz_j^{l+3}(u(2)u(0)+u(3)u(1)+\cdots+u(\mu-1)u(\mu-3))+$$

$$+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_i^2z_j^{l+4}(u(2)u(0)+u(3)u(1)+\cdots+u(\mu-2)u(\mu-4))+$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$\cdots \qquad \cdots \qquad \cdots \tag{B.5}$$

According to the definition given in Eqn. (3.4), autocorrelation function of white noise input $u(n)$ for different lags can be determined.

$$\begin{aligned}
&\frac{1}{N}\left(u(0)^2+u(1)^2+\cdots+u(\mu)^2\right)=R_{uu}(0)=\sigma_u^2\\
&\frac{1}{N}\left(u(0)u(1)+u(1)u(2)+\cdots+u(\mu)u(\mu+1)\right)=R_{uu}(1)\\
&\frac{1}{N}\left(u(0)u(2)+u(1)u(3)+\cdots+u(\mu)u(\mu+2)\right)=R_{uu}(2)\\
&\cdots \qquad \cdots \qquad \cdots\\
&\cdots \qquad \cdots \qquad \cdots
\end{aligned} \tag{B.6}$$

Dividing the sum obtained in Eqn. (B.5) by N and using Eqn. (B.6), we get

$$\begin{aligned}
R_{xx}(l)= \quad & \sum_{i=1}^{p} C_i^2R_{uu}(0)z_i^l\left(1+z_i^2+z_i^4+\cdots\right)\\
&+\sum_{i=1}^{p} C_i^2R_{uu}(1)z_i^{l-1}\left(1+z_i^2+z_i^4+\cdots\right)\\
&+\sum_{i=1}^{p} C_i^2R_{uu}(2)z_i^{l-2}\left(1+z_i^2+z_i^4+\cdots\right)+\\
&\cdots \qquad \cdots \qquad \cdots\\
&\cdots \qquad \cdots \qquad \cdots\\
&+\sum_{i=1}^{p} C_i^2R_{uu}(1)z_i^{l+1}\left(1+z_i^2+z_i^4+\cdots\right)\\
&+\sum_{i=1}^{p} C_i^2R_{uu}(2)z_i^{l+2}\left(1+z_i^2+z_i^4+\cdots\right)+\\
&\cdots \qquad \cdots \qquad \cdots\\
&\cdots \qquad \cdots \qquad \cdots\\
&+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jR_{uu}(0)z_j^l\left(1+z_iz_j+z_i^2z_j^2+\cdots\right)\\
&+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jR_{uu}(1)z_j^{l-1}\left(1+z_iz_j+z_i^2z_j^2+\cdots\right)\\
&+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jR_{uu}(2)z_j^{l-2}\left(1+z_iz_j+z_i^2z_j^2+\cdots\right)+
\end{aligned}$$

$$
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots
\end{array}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jR_{uu}(1)z_j^{l+1}\left(1+z_iz_j+z_i^2z_j^2+\cdots\right)\\
&+\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jR_{uu}(2)z_j^{l+2}\left(1+z_iz_j+z_i^2z_j^2+\cdots\right)+\\
&\cdots \qquad \cdots \qquad \cdots\\
&\cdots \qquad \cdots \qquad \cdots
\end{aligned}
\tag{B.7}
$$

The autocorrelation sequence of $u(n)$ will certainly contain a peak at $l = 0$, but because of its random characteristics, $R_{uu}(l)$ is expected to decay rapidly toward zero. Considering $R_{uu}(l) \approx 0$, for $l > 0$, Eqn. (B.7) can be written as

$$
\begin{aligned}
R_{xx}(l) &= R_{uu}(0)\sum_{i=1}^{p} C_i^2 z_i^l\left(1+z_i^2+z_i^4+\cdots\right)+\\
&\qquad R_{uu}(0)\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_jz_j^l\left(1+z_iz_j+z_i^2z_j^2+\cdots\right)\\
&= R_{uu}(0)\sum_{i=1}^{p} C_i^2\frac{z_i^l}{1-z_i^2}+R_{uu}(0)\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_j\frac{z_j^l}{1-z_iz_j}
\end{aligned}
\tag{B.8}
$$

It is obvious that

$$
\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_j\frac{z_j^l}{1-z_iz_j}=\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_j\frac{z_i^l}{1-z_iz_j}
\tag{B.9}
$$

Using Eqn. (B.9) in Eqn. (B.8) we get

$$
\begin{aligned}
R_{xx}(l) &= R_{uu}(0)\sum_{i=1}^{p} C_i^2\frac{z_i^l}{1-z_i^2}+R_{uu}(0)\sum_{i=1}^{p}\sum_{j=1,j\neq i}^{p} C_iC_j\frac{z_i^l}{1-z_iz_j}\\
&= \sum_{i=1}^{p}(z_i)^l\sigma_u^2\left[\sum_{j=1,j\neq i}^{p}\left\{\frac{C_i^2}{1-(z_i)^2}+\frac{C_iC_j}{1-z_iz_j}\right\}\right]
\end{aligned}
\tag{B.10}
$$

Changing the notations i and j with k and q, respectively, we can rewrite Eqn. (B.10) as

$$
R_{xx}(l)=\sum_{k=1}^{p}(z_k)^l\sigma_u^2\left[\sum_{q=1,q\neq k}^{p}\left\{\frac{C_k^2}{1-(z_k)^2}+\frac{C_kC_q}{1-z_kz_q}\right\}\right]
\tag{B.11}
$$

Therefore, the autocorrelation function of the noise-free AR signal can be written as

$$R_{xx}(l) = \sum_{k=1}^{p} \beta_k (z_k)^l \tag{B.12}$$

where,

$$\beta_k = \sigma_u^2 \left[\frac{C_k^2}{1-(z_k)^2} + \sum_{q=1,q\neq k}^{p} \frac{C_k C_q}{1 - z_k z_q} \right] \tag{B.13}$$

The coefficient β_k may be real or complex depending on whether the pole is real or complex. Let us consider there are g_r number of real poles and remaining $(p - g_r)$ or g_c number of complex poles. Since $x(n)$ is real, complex pole will always be accompanied by its complex conjugate pole. Hence the number of pair of complex conjugate poles is $g_{cc} = g_c/2$. As g_{cc} cannot be a fraction, for a p-th order AR system with p being odd, g_r must be odd. Separating the terms with real poles from the terms with complex poles, we can write Eqn. (B.12) as

$$R_{xx}(l) = \sum_{k_c=1}^{g_c} \beta_{k_c} (z_{k_c})^l + \sum_{k_r=g_c+1}^{g_c+g_r} \beta_{k_r} (z_{k_r})^l \tag{B.14}$$

For a pair of complex conjugate poles corresponding β will also be complex conjugate pair. Let us consider the effect of a pair of complex conjugate poles z_1 and z_2, where $z_1 = r_1 e^{j\omega_1}$, $z_2 = r_1 e^{-j\omega_1}$ and corresponding $\beta_1 = \zeta_1 e^{j\phi_1}$ and $\beta_2 = \zeta_1 e^{-j\phi_1}$.

$$\begin{aligned} \beta_1(z_1)^l + \beta_2(z_2)^l &= \zeta_1 (r_1)^l e^{j(\omega_1 l + \phi_1)} + \zeta_1 (r_1)^l e^{-j(\omega_1 l+\phi_1)} \\ &= 2\zeta_1 (r_1)^l \cos(\omega_1 l + \phi_1) \\ &= G_1^c (r_1)^l \cos(\omega_1 l + \phi_1) \end{aligned} \tag{B.15}$$

where $G_1^c = 2\zeta_1$ is a constant that depends on β_1. Hence the sum of terms with complex poles in Eqn. (B.14) can be expressed as

$$\sum_{k_c=1}^{g_c} \beta_{k_c} (z_{k_c})^l = \sum_{j_c=1}^{g_{cc}} G_{j_c}^c (r_{j_c})^l \cos(\omega_{j_c} l + \phi_{j_c}) \tag{B.16}$$

where $G_{j_c}^c = 2\zeta_{j_c}$ is a constant that depends on β_{j_c}. As in the case of a complex pole, a real pole can be expressed as $z_k = r_k e^{j\omega_k}$ where ω_k is 0 or π and the corresponding $\beta_k = \zeta_k e^{j\phi_k}$ where ϕ_k is also 0 or π. Hence the sum of terms with real poles in Eqn. (B.14) can be expressed as

$$\begin{aligned}
\sum_{k_r=g_c+1}^{g_c+g_r} \beta_{k_r}(z_{k_r})^l &= \sum_{j_r=1}^{g_r} \beta_{j_r}(z_{j_r})^l \\
&= \sum_{j_r=1}^{g_r} \zeta_{j_r}(r_{j_r})^l e^{j(\omega_{j_r} l+\phi_{j_r})} \\
&= \sum_{j_r=1}^{g_r} \zeta_{j_r}(r_{j_r})^l \cos(\omega_{j_r} l+\phi_{j_r}) \quad [\text{As, } \omega_{j_r}=0 \text{ or } \pi,\ \phi_{j_r}=0 \text{ or } \pi] \\
&= \sum_{j_r=1}^{g_r} G^r_{j_r}(r_{j_r})^l \cos(\omega_{j_r} l+\phi_{j_r}) \qquad \text{(B.17)}
\end{aligned}$$

where $G^r_{j_r} = \zeta_{j_r}$ is a constants that depends on β_{j_r}. Combining Eqn. (B.16) and Eqn. (B.17), we can write Eqn. (B.14) as

$$\begin{aligned}
R_{xx}(l) &= \sum_{j_c=1}^{g_{cc}} G^c_{j_c}(r_{j_c})^l \cos(\omega_{j_c} l+\phi_{j_c}) + \sum_{j_r=1}^{g_r} G^r_{j_r}(r_{j_r})^l \cos(\omega_{j_r} l+\phi_{j_r}) \\
&= \sum_{j=1}^{g} G_j(r_j)^l \cos(\omega_j l+\phi_j) \qquad \text{(B.18)}
\end{aligned}$$

where G_j is a constant and $g = g_{cc} + g_r$ = Number of pair of complex conjugate poles + Number of real poles. Eqn. (B.18) can be further expanded as

$$\begin{aligned}
R_{xx}(l) &= \sum_{j=1}^{g}(r_j)^l \left[G_j \cos\phi_j \cos(\omega_j l) - G_j \sin\phi_j \sin(\omega_j l)\right] \\
&= \sum_{j=1}^{g}(r_j)^l \left[P_j \cos(\omega_j l) - Q_j \sin(\omega_j l)\right] \qquad \text{(B.19)}
\end{aligned}$$

where $P_j = G_j \cos\phi_j$ and $Q_j = -G_j \sin\phi_j$ are constants that depend on β_j.