EFFECTS OF TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY ON MAGNETOHYDRODYNAMIC FREE CONVECTION FLOW ALONG A VERTICAL FLAT PLATE

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DEDICATION

This work is dedicated

To

My Respected Parents

Abstract

The effects of temperature dependent thermal conductivity on Magnetohydrodynamic (MHD) free convection flow along a vertical flat plate have been investigated. The governing equations with associated boundary conditions for this phenomenon are converted to dimensionless form using suitable transformations. The transformed non-linear partial differential equations are then solved using the implicit finite difference method with Keller–box scheme. FORTRAN 90 is used to perform computational job and the post processing software TECPLOT has been used to display the numerical results graphically. Numerical results of the velocity profiles, temperature profiles, skin friction coefficient and surface temperature distribution for different values of the magnetic parameter M, thermal conductivity variation parameter γ, Prandtl number Pr, heat generation parameter Q, joule heating parameter J and viscous dissipation parameter N are presented graphically. The comparison of the present numerical results in terms of skin friction coefficient and surface temperature with some published results is shown in tabular forms.

Author's Declaration

I hereby declare that this thesis work submitted to the Department of Mathematics, Bangladesh University of Engineering and Technology (BUET) in partial fulfillment of the requirements for the degree of Master of Philosophy in Mathematics has not been submitted elsewhere (Universities or Institutions) for the any other degree.

A.K.M. Safiqul Islam
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Contents

Nomenclature

b	Plate thickness
$\overline{B}$	Magnetic induction vector
C_{fx}	Local skin friction coefficient
C_p	Specific heat at constant pressure
$\overline{F}$	Body force per unit volume
f	Dimensionless stream function
g	Acceleration due to gravity
Gr	Grashof number
h	Dimensionless temperature
H_0	Applied Magnetic field strength
$\overline{J}$	Current density vector
l	Length of the plate
M	Magnetic parameter
N	Viscous dissipation parameter
Q	Heat generation parameter
J	Joule heating parameter
P	Conjugate conduction parameter
Pr	Prandtl number
T	Temperature of the interface
T_b	Temperature at outer surface of the plate
T_f	Temperature of the fluid
T_∞	Temperature of the ambient fluid
$\overline{u}$	Velocity component in x- direction
$\overline{v}$	Velocity component in y- direction
$\overline{V} \times \overline{B}$	Electrical fluid vector
u	Dimensionless velocity component in x- direction
v	Dimensionless velocity component in y- direction
$\overline{x}$	Cartesian coordinates
$\overline{y}$	Cartesian coordinates
x	Dimensionless Cartesian coordinate in x direction
y	Dimensionless Cartesian coordinate in y direction

Greek Symbols

β	Coefficient of thermal expansion
γ	Thermal conductivity variation parameter
∇	Vector differential operator
η	Similarity variable
$\theta(x,0)$	Surface temperature distribution
κ_∞	Thermal conductivity of the ambient fluid
κ_s	Thermal conductivity of the solid
κ_f	Thermal conductivity of the fluid
μ	Viscosity of the fluid
μ_e	Magnetic permeability of the fluid
υ	Kinematic viscosity
ρ	Density of the fluid inside the boundary layer
σ	Electrical conductivity of the fluid
τ_{w}	Shearing stress
ψ	Stream function

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Chapter 1

1 Introduction

1.1 General

Fluid dynamics is one of the oldest branches of applied mathematics which is concerned with the study of the motion of fluids or that of bodies in contact with fluids. It is also the branch in which some of the most significant advances have been made during the last fifty years. These advances have been motivated by exciting development in science and technology and facilitated by growth of computer capabilities and development of sophisticated mathematical techniques.

Heat is the form of energy that can be transferred from one system to another as a result of temperature difference. A thermodynamic analysis is concerned with the amount of heat transfer as a system undergoes a process from one equilibrium state to another. The science that deals with the determination of the rates of such energy transfers is the heat transfer. The transfer of energy as heat is always from the higher temperature medium to the lower temperature one, and heat transfer stops when the two mediums reach the same temperature. Heat transfer is that science which seeks to predict the energy transfer which may take place between material bodies as a result of a temperature difference. Thermodynamics teaches that this energy is defined as heat. The science of heat transfer seeks not merely to explain how heat energy may be transferred, but also to predict the rate at which the exchange will take place under certain specified conditions.

The phenomenon of heat transfer was known to human being even in the primitive age when they used to use solar energy as a source of heat. Heat transfer in its initial stage was conceived with the invention of fire in the early age of human civilization. Since then its knowledge and use has been progressively increasing each day as it is directly related to the growth of human civilization. With the invention of stream engine by James Watt in 1765 A. D., the phenomenon of heat transfer got its first industrial recognition and after that its use extended to a great extent and spread out in different spheres of engineering fields. In the past three decades, digital computers, numerical techniques and development of numerical models of heat transfer have made it possible to calculate heat transfer of considerable complexity and thereby create a new approach to the design of heat transfer equipment. Heat transfer processes have always been an integral part of our environment.

The study of temperature and heat transfer is of great importance to the engineers because of its almost universal occurrence in many branches of science and engineering. Although heat transfer analysis is most important for the proper sizing of fuel elements in the nuclear reactors cores to prevent burnout, the performance of aircraft also depends upon the case with which the structure and engines can be cooled. The design of chemical plants is usually done on the basis of heat transfer analysis and the analogous mass transfer processes. The transfer and conversion of energy from one form to another is the basis to all heat transfer process and hence, they are governed by the first as well as the second law of thermodynamics. Heat transfer is commonly associated with fluid dynamics. The knowledge of temperature distribution is essential in heat transfer studies because of the fact that the heat flow takes place only wherever there is a temperature gradient in a system. The heat flux which is defined as the amount of heat transfer per unit area in per unit time can be calculated from the physical laws relating to the temperature gradient and the heat flux.

Heat can be transferred in three different mechanisms or modes: conduction, convection and radiation. All modes of heat transfer require the existence of a temperature difference, and all modes are from the high temperature medium to a lower temperature one. In reality, the combined effect of these three modes of heat transfer control temperature distribution in a medium. Heat conduction and thermal conduction are the spontaneous transfer of thermal energy through matter, from a region of high temperature to a region of lower temperature. The same force that act to support the structure of matter can be said to move by physical contact between the particles transfer and the thermal energy, in the form of continuous random motion of particles of the matter. Conduction is most effective in solids but it can happen in fluids. Radiation is electromagnetic waves which directly transports energy through space. Sunlight is a form of radiation that is radiated through space to our planet without the aid of fluids or solids. The Sun transfers heat through 93 million miles of space. Because there are no solids touching the Sun and our planet, conduction is not responsible for transferring the heat. Thus radiation brings heat to our planet.

1.2 Some Definitions

Some basic definitions that are related to this study are presented below:

Boundary Layer

Since fluid motion is the distinguishing feature of heat convection, it is necessary to understand some of the principles of fluid dynamics in order to describe adequately the processes of convection. When a fluid flows over a body, the velocity and the temperature distribution at the immediate vicinity of the surface are strongly influenced by the convective heat transfer. In order to simplify the analysis of convective heat transfer the boundary layer concept frequently is introduced to model the velocity and the temperature fields near the solid surface. So we are concerned with two different kinds of boundary layers, the velocity boundary layer and the thermal boundary layer.

The velocity boundary layer is defined as the narrow region, near the solid surface, over which velocity gradients and shear stresses are large, but in the region outside the boundary layer, called the potential flow region, the velocity gradients and shear stresses are negligible. The exact limit of the boundary layer cannot be precisely defined because of the asymptotic nature of the velocity variation. The limit of the boundary layer is usually taken to be at the distance from the surface, at which the fluid velocity is equal to a predetermined percentage of the free stream value, U_∞ .This percentage depends on the accuracy desired, 99% or 95% being customary. Although, outside the boundary layer region the flow is assumed to be inviscid, but inside the boundary layer the viscous flow may be either laminar or turbulent. In the case of laminar boundary layer, fluid motion is highly ordered and it is possible to identify streamlines along which particles move. Fluid motion along a streamline is characterized by velocity components in both the x and y directions. Since the velocity component v is in the direction normal to the surface, it can contribute significantly the transfer of momentum and energy through the boundary layer. Fluid motion normal to the surface is necessitated by boundary layer growth in the x direction. In contrast, fluid motion in the turbulent boundary layer is highly irregular and is characterized by velocity fluctuations. These fluctuations enhance the transfer of momentum and energy and hence increase surface friction, as well as convection transfer rates. Due to fluid mixing resulting from the fluctuations, turbulent boundary layer thicknesses are larger and boundary layer profiles are flatter than in laminar flow. The thermal boundary layer may be defined (in the same sense that the velocity boundary layer was defined above) as the narrow region between the surface and the point at which the fluid temperature has reached a certain percentage of ambient temperature T_∞. Outside the thermal boundary layer the fluid is assumed to be a heat sink at a uniform temperature T_∞. The thermal boundary layer is generally not coincident with the velocity boundary layer, although it is certainly dependent on it. If the fluid has high thermal conductivity, it will be thicker than the velocity boundary layer, and if conductivity is low, it will be thinner than the velocity boundary layer.

Convection

Convection is the transfer of heat by the actual movement of the warmed matter. Heat leaves the coffee cup as the currents of steam and air rise. Convective heat transfer or, simply, convection is the study of heat transport processes affected by the flow of fluids. Convection is that mode of heat transfer where energy exchange occurs between the particles by convection current. It may be explained as; when fluid flows over a solid body or inside a channel while temperature of the fluid and the solid surface are different, heat transfer between the fluid and the solid surface takes place as a consequences of the motion of the fluid relative to the surfaces; this mechanism of heat transfer called convection. If the motion of the fluid arises due to an external agent, such as the externally imposed flow of fluid stream over a heated object, the process is termed as forced convection. The fluid flow may be the result of, for instance, a fan, a blower, the wind or the motion of the heated object itself. Such problems are very frequently encountered in technology where heat transfer to, or from, a body is often due to an imposed flow of a fluid at a temperature difference from that of the body. It has wide applications in compact heat exchanger, central air conditioning system, cooling tower, gas turbine blade, internal cooling passage, chemical engineering process industries, nuclear reactors and many other cases. If, on the other hand, no such externally induced flow is provided and the flow arises naturally, simply due to the effect of a density difference, resulting from a temperature difference, in a body force field, such as gravitational field, the process is termed as natural or free convection. The density difference gives rise to buoyancy effects due to which the flow is generated. A heated body cooling in ambient air generates such a flow in the region surrounding it. Similarly, the buoyant flow, which arises due to heat rejection into the atmosphere and the other ambient media. Heat transfer by free convection occurs in many engineering application, such as heat transfer from hot radiators, refrigerator coils, transmission lines, electric transformers, electric heating elements and electronic equipment etc.

The effects of these buoyancy forces are however; usually negligible when there is a forced flow. In some cases, however, these buoyancy forces do have a significant influence on the flow and consequently on the heat transfer rate. In such cases, the flow about the body is a combination of forced and free convection; such flows are referred to as mixed convection. For example, heat transfer from one fluid to another fluid through the walls of pipe occurs in many practical devices. In this case, heat is transferred by convection from the hotter fluid to the one surface of the pipe. Heat is then transferred by conduction through the walls of the pipe. Finally, heat is transferred by convection from the other surface to the colder fluid.

Magneto hydrodynamics

Magneto hydrodynamics involves magnetic fields (Magneto) and fluids (hydro) that conduct electricity and interact (dynamics). MHD technology is based on a fundamental law of electromagnetism. When a magnetic field and an electric current intersect in a liquid, their repulsive intersection propels the liquid in a direction perpendicular to both the field and the current.

Magneto hydrodynamics is that branch of science, which deals with the motion of highly conducting ionized (electric conductor) fluid in presence of magnetic field. The motion of the conducting fluid across the magnetic field generates electric currents which change the magnetic field and the action of the magnetic field on these currents give rise to mechanical forces, which modify the fluid. It is possible to attain equilibrium in a conducting fluid if the current is parallel to the magnetic field. Then the magnetic forces vanish and the equilibrium of the gas is the same as in the absence of magnetic fields. But most liquids and gases are poor conductors of electricity. In the case when the conductor is either a liquid or a gas, electromagnetic forces will be generated which may be of the same order of magnitude as the hydro dynamical and inertial forces. Thus the equation of motion as well as the other forces will have to take these electromagnetic forces into account. The MHD was originally applied to astrophysical and geophysical problems, where it is still very important but more recently applied to the problem of fusion power where the application is the creation and containment of hot plasmas by electromagnetic forces, since material walls would be destroyed. Astrophysical problems include solar structure, especially in the outer layers, the solar wind bathing the earth and other planets and interstellar magnetic fields. The primary geophysical problem is planetary magnetism, produced by currents deep in the planet, a problem that has not been solved to any degree of satisfaction.

Thermal Conductivity

Thermal conductivity of a material can be defined as the rate of heat transfer through a unit thickness of the material per unit area per unit temperature difference. Therefore the thermal conductivity is the intensive property of a material which is a measure of the ability of the material to conduct heat. Thermal conductivity approximately tracks electrical conductivity, as freely moving valence elections transfer not only electric current but also heat energy. A high value for thermal conductivity indicates that the material is a good heat conductor, and a low value for thermal conductivity indicates that the material is a poor heat conductor or insulator. For example the materials such as copper and silver that are good electric conductors are also good thermal conductors, and have high values of thermal conductivity. Materials such as rubber, wood are poor conductors of heat and have low conductivity values. The rate of heat conduction through a medium depends on the geometry of the medium, its thickness, and the material of the medium, as well as the temperature difference across the medium. According to Fourier's law (1955), the rate of heat conduction is proportional to the area measured normal to the direction of heat flow and to the temperature gradient in that direction

$$Q = -kA\frac{\partial T}{\partial n} \text{ or } q = -k\frac{\partial T}{\partial n} \tag{1}$$

The constant of proportionality k is called the coefficient of thermal conductivity, which is a physical property of the substance and is defined as the ability of a substance to conduct heat. Since conduction is a molecular phenomenon, Fourier's law (1) is similar to Newton's law of viscosity for laminar flow

$$\tau = \mu\frac{\partial u}{\partial y} \tag{2}$$

Comparing equations (1) and (2) we note that the viscosity in fluid motion is analogous to the thermal conductivity in heat transfer. Thermal conductivity like viscosity is primarily a function of temperature and or position and nature of the substance. It varies significantly with pressure only in the case of gases subjected to high pressure. However, for many engineering problems, materials are often considered to possess a constant thermal conductivity (isotropic).Pure metals have the high values of thermal conductivity while gases and vapors have the lowest. For most pure metals thermal conductivity decreases with increasing temperature where as for gases and insulating material it increases with increasing temperature.

Heat Generation

A medium through which heat is conducted may involve the conversation of electrical, nuclear or chemical energy into heat (or thermal) energy. In heat conduction analysis, such conversion processes are characterized as heat generation. For example, the temperature of a resistance wire rises rapidly when electric current passes through it, as a result of the electrical energy are converted to heat at a rate of I^2R, where I is the current and R is the electrical resistance of the wire. The safe and effective removal of this heat away from the sites of heat generation (the electronic circuits) is the subject of electronics cooling which is one of the modern application areas of heat transfer. Likewise, a large amount of heat is generated in the fuel elements of nuclear reactors as a result of nuclear fission that serves as the heat source for the nuclear power plants. The heat generated in the Sun as a result of the fusion of hydrogen into helium makes the Sun a large nuclear reactor that supplies heat to the earth. Another source of heat generation in a medium is exothermic chemical reactions that may occur throughout the medium. The chemical reaction in this case serves as a heat source for the medium. In the case of endothermic reactions, however, heat is absorbed instead of being released during reaction and thus the chemical reaction serves as a heat sink. The heat generation term becomes a negative quantity in this case. Heat generation is a volumetric phenomenon. That is, it occurs throughout the body of a medium. Therefore, the rate of heat generation in a medium is usually specified per unit volume. Heat generation is the ability to emit greater-than-normal heat from the body. Joule heating is the predominant heat mechanism for heat generation in integrated circuits and is an undesired effect.

Joule Heating

In electronics in particular and in physics broadly, Joule heating is the heating effect of conductors carrying currents. It refers to the increase in temperature of a conductor as a result of resistance to an electrical current flowing through it. At an atomic level, Joule heating is the result of moving electrons colliding with atoms in a conductor, whereupon momentum is transferred to the atom, increasing its kinetic or vibrational energy. When similar collisions cause a permanent structural change, rather than an elastic response, the result is known as electro migration. Joule heating is caused by interactions between the moving particles that form current (usually, but not always, electrons) and the atomic ions that make up the body of the conductor. Charged particles in an electric circuit are accelerated by an electric field but give up some of their kinetic energy each time they collide with an ion. The increase in the kinetic energy of the ions manifests itself as heat and a rise in the temperature of the conductor. Hence energy is transferred from electrical power supply to the conductor and any materials with which it is in thermal contact.

James Prescott Joule studied first Joule heating in 1841. It is the process by which the passage of an electric current through a conductor releases heat. Joule's first law is also known as Joule effect. It states that heat generation by a constant current through a resistive conductor for a time whose unit is joule. It is also related to Ohm's first law. Joule heating is also referred to as Ohmic heating or Resistive heating because of its relationship to Ohm's law. The SI unit of energy was subsequently named the joule and given the symbol *J*. The commonly known unit of power, the watt, is equivalent to one joule per second.

Viscous Dissipation

The force per unit volume or per unit mass, which arising from the action of tangential stresses in a moving viscous fluid is viscous dissipation. This force may be introduced as a term in the equations of motion. Viscous dissipation is also important in the flow of fluids having high viscosities. Temperature of the fluid increases because of it.

1.3 Dimensionless Parameters

The dimensionless parameters can be thought of as measures of the relative importance of certain aspects of the flow. Some dimensionless parameters related to our study are discussed below:

Grashof Number, Gr

The flow regime in free convection is governed by the dimensionless parameter Grashof number, in momentum equation, which represent the ratio of the buoyancy force to the viscous force acting on the fluid, and is defined as $Gr = \dfrac{g\beta L^3 (T_s - T_\infty)}{\upsilon^2}$

Where, g is the acceleration due to gravity, β is the volumetric thermal expansion coefficient, T_s is the surface temperature, T_∞ is the ambient temperature, L is the characteristic length and υ is the kinematic viscosity. The Grashof number *Gr* plays same role in free convection as the Reynolds number *Re* plays in forced convection. As such, the Grashof number provides the main criterion in determining whether the fluid flow is laminar or turbulent in free convection. For vertical plates, the critical value of the Grashof number is observed to be about 10^9. Therefore, the flow regime on a vertical plate becomes turbulent at Grashof number greater than 10^9.

Prandtl Number, Pr

The Prandtl number is a measure of the relative importance of heat conduction and viscosity of the fluid and is defined as $\Pr = \frac{\mu C_p}{\kappa}$

Where, C_p is the specific heat of the fluid at constant pressure, and κ is the coefficient of thermal conductivity of the fluid. The Prandtl number may also be written as follows:

$$\Pr = \frac{\frac{\mu}{\rho}}{\frac{\kappa}{\rho C_p}} = \frac{\nu}{\frac{\kappa}{\rho C_p}}$$

= kinematic viscosity/ thermal diffusivity.

The value of υ shows the effect of viscosity of a fluid. If other things are the same, the smaller the value of υ, the narrower will be the region affected by viscosity. This region is known as the boundary layer region when υ is very small. While υ shows the momentum diffusivity due to viscosity effect, $\kappa / \rho C_p$ shows the thermal diffusivity due to heat conduction. The smaller value of $\kappa / \rho C_p$, the narrower will be the region affected by heat conduction. This region is known as thermal boundary layer when $\kappa / \rho C_p$ is very small. Thus the Prandtl number shows the relative importance of heat conduction and viscosity of the fluid. Since for a gas the Prandtl number is of the order of unity, whenever the effect of viscosity is considered we must simultaneously take into account of the influence of the thermal conductivity of the gas. The Prandtl numbers of the fluids range from less than 0.01 for liquid metals to more than 100,000 for heavy oils. Note that the Prandtl number is in the order of 7 for water. The Prandtl numbers of gases are about 1, which indicates that both momentum and heat dissipate through the fluid at about the same rate. Consequently the thermal boundary layer is much thicker for liquid metals and much thinner for oils relative to the velocity boundary layer.

The relative thickness of the velocity and the thermal boundary layers is best described by the dimensionless parameter, Prandtl number.

1.4 Main Objectives of the Work

This research is to investigate the effects of temperature dependent thermal conductivity on MHD free convection flow along a vertical flat plate.

Solutions are obtained and analyzed in terms of skin friction coefficient, surface temperature distribution along the flat plate and the velocity profiles and temperature profiles over the whole boundary layer for a selection of set of parameters consisting of the magnetic parameter M, thermal conductivity variation parameter γ, Prandtl number Pr, heat generation parameter Q, joule heating parameter J and viscous dissipation parameter N.

The major objectives of this study are:

1. To study the effects of the magnetic parameter M on velocity and temperature, skin friction coefficient and surface temperature.
2. To analyze the effects of the thermal conductivity variation parameter γ on velocity and temperature, skin friction coefficient and surface temperature.
3. To investigate the effects of the Prandtl number Pr on velocity and temperature, skin friction coefficient and surface temperature.
4. To investigate the effects of the heat generation parameter Q on velocity and temperature, skin friction coefficient and surface temperature.
5. To study the effects of the joule heating parameter J on velocity and temperature, skin friction coefficient and surface temperature.
6. To assess the effects of the viscous dissipation parameter N on velocity and temperature, skin friction coefficient and surface temperature.
7. To compare the present results with other published works.

1.5 Literature Review

Model studies of the free and mixed convection flows have earned reputations because of their applications in geophysical, geothermal and nuclear engineering problems. Electrically conducting fluid flow in presence of magnetic field and the effect of temperature dependent thermal conductivity on MHD free convection flow and heat generation, joule heating problems are important from the technical point of view and such types of problems have been received much attention by many researchers. Experimental and theoretical works on MHD free and forced convection flows have been done extensively but a few works have been done on the conjugate effects of convection and conduction problems.

Sparrow et al. (1959), were the first investigator, who dealt with the combined forced and free convective boundary layer flow. The combined free and forced convection flow about inclined surfaces in porous media were studied by Cheng (1977). The combined forced and free convection in boundary layer flow of a micro polar fluid over a horizontal plate was investigated by Hassanien (1977); similarity solutions were acquired in his work for the case of wall temperature, which is inversely proportional to the square root of the distance from the leading edge.

Free convective steady hydromagnetic flow about a heated vertical flat plate was considered by Gupta (1961), Poots (1961), Osterle and Yound (1961). A transformation of the boundary layer equations for free convection past a vertical plate with arbitrary blowing and wall temperature variations was studied by Vedhanayagam et al. (1980). The case of a heated isothermal horizontal surface with transpiration was discussed in some detail first by Clarke and Riley (1975, 1976) and then by Lin and Yu (1988). The problem of the free convection boundary layer on a vertical plate with prescribed surface heat flux was studied by Merkin and Mahmood (1990).

The effect of magnetic field on free convection heat transfer was studied by Sparrow and Cess (1961). MHD mixed convection flow investigated by Yu (1965), Gardner and Lo (1975) and Al-Khawaja et al. (1999). Kuiken (1970) studied the problem of magnetohydrodynamic free convection in a strong cross-field. Takhar and Soundalgekar (1980) studied the dissipation effects on MHD free convection flow past a semi-infinite vertical plate. Moreover, Raptis and Kafoussias (1982) investigated the problem of magnetohydrodynamic free convection flow and mass transfer through a porous medium bounded by an infinite vertical porous plate with constant heat flux. Hossain (1992) analyzed the viscous and Joule heating effects on MHD free convection flow with variable plate temperature. Hossain et al. (1999) analyzed the effect of radiation on free convection from a porous vertical plate. Hossain et al. (1997) studied the free convection boundary layer flow along a vertical porous plate in presence of magnetic field. Also Hossain et al. (1998) investigated the heat transfer response of MHD free convection flow along a vertical plate to surface temperature oscillation. MHD free convection flow of visco-elastic fluid past an infinite porous plate was investigated by Chowdhury and Islam (2000). Elbashbeshy (2000) also discussed the effect of free convection flow with variable viscosity and thermal diffusivity along a vertical plate in presence of magnetic field. Ahmad and Zaidi (2004) investigated the magnetic effect on over back convection through vertical stratum.On the conjugate problems of forced convection heat transfer, lots of researches have been done both experimentally and theoretically but a small number of works have been dedicated to the conjugate problem of natural convection. Gebhart (1962) investigated the effect of dissipation on natural convection. On the other hand, the effect of axial heat conduction in a vertical flat plate on free convection heat transfer was studied by Miamoto et al. (1980). Pozzi and Lupo (1988) investigated the coupling of conduction with laminar convection along a flat plate. Pop et al. (1995) then extended the analysis to conjugate mixed convection on a vertical surface in porous medium. Moreover, the thermal interaction between laminar film condensation and forced convection along a conducting wall was investigated by Chen Chang (1996). Merkin and Pop (1996) analyzed the Conjugate free convection on a vertical surface. Shu and Pop (1999) analyzed the thermal interaction between free convection and forced convection along a vertical conducting wall. Khan (2002) investigated the conjugate effect of conduction and convection with natural convection flow from a vertical flat plate. Chen (2006) analyzed a numerical simulation of micro polar fluid flows along a flat plate with wall conduction and buoyancy effects. Alam et al. (2007) studied the viscous dissipation effects on MHD natural convection flow over a sphere in the presence of heat generation. Alim et al. (2007) investigated the Joule heating effect on the coupling of conduction with Magnetohydrodynamic free convection flow from a vertical flat plate. Alim et al. (2008) investigated the combined effect of viscous dissipation & joule heating on the coupling of conduction & free convection along a vertical flat plate. Rahman et al. (2008) investigated the effects of temperature dependent thermal conductivity on MHD free convection flow along a vertical flat plate with heat conduction. Rahman and Alim (2009) analyzed numerical study of MHD free convective heat transfer flow along a vertical flat plate with temperature dependent thermal conductivity. Nasrin and

Alim (2009) studied the combined effects of viscous dissipation and temperature dependent thermal conductivity on MHD free convection flow with conduction and joule heating along a vertical flat plate. In all the aforementioned analyses the effect of temperature dependent thermal conductivity and viscous dissipation on free convective flow along a vertical plate has not been considered. Also in all the aforementioned analyses the effects of heat generation and joule heating have not been considered combinedly. The present study is to incorporate the idea of the effects of temperature dependent thermal conductivity on MHD free convection flow along a vertical flat plat with heat generation and joule heating.

In the present work, the effects of temperature dependent thermal conductivity on MHD free convection flow along a vertical flat plate have been investigated. The results have been obtained for different values of relevant physical parameters. The governing partial differential equations are reduced to locally non similarity partial differential forms by adopting appropriate transformations. The transformed boundary layer equations are solved numerically using implicit finite difference method together with Keller (1978) box technique and later used by Cebeci and Bradshaw (1984). Numerical results of the velocity profiles, temperature profiles, skin friction coefficient and surface temperature distribution for different values of the magnetic parameter M, thermal conductivity variation parameter γ, Prandtl number Pr, heat generation parameter Q, joule heating parameter J and viscous dissipation parameter N are presented graphically.

In chapter 2, the effects of temperature dependent thermal conductivity on MHD free convection flow along a vertical flat plate with heat generation and joule heating have been described. The non dimensional boundary layer equations have been solved by using implicit finite difference method by Keller (1978) and later by Cebeci and Bradshaw (1984). Numerical results of the velocity profiles, temperature profiles, skin friction coefficient and surface temperature distribution for different values of magnetic parameter M, thermal conductivity variation parameter γ, Prandtl number Pr, heat generation parameter Q and joule heating parameter J have been presented graphically. The comparisons of the present numerical results of the skin friction coefficient and the surface temperature distribution are also given with those obtained by Pozzi and Lupo (1988) and Merkin and Pop (1996) are presented.

In chapter 3, natural convection flows of an electrically conducting fluid along a vertical flat plate with temperature dependent thermal conductivity and viscous dissipation effects have been analyzed. The non dimensional boundary layer equations have been solved by using implicit finite difference method by Keller (1978) and later by Cebeci and Bradshaw (1984) .Velocity profiles, temperature profiles, skin friction coefficient and surface temperature distribution have been presented graphically for various values of the thermal conductivity variation parameter γ, Prandtl number Pr and viscous dissipation parameter N. The comparisons of the present numerical results of the skin friction coefficient and the surface temperature distribution are also given with those obtained by Pozzi and Lupo (1988) and Merkin and Pop (1996) are presented.

Chapter 2

Effects of Temperature Dependent Thermal Conductivity on Magnetohydrodynamic Free Convection Flow along a Vertical Flat Plate with Heat Generation and Joule Heating

2.1 Introduction

The effects of the temperature dependent thermal conductivity on MHD free convection flow along a vertical flat plate with heat generation and joule heating have been described in this chapter. The governing boundary layer equations are transformed into a non-dimensional form and the resulting non linear system of partial differential equations are reduced to local non similarity equations which are solved numerically by implicit finite difference method together with Keller-box technique. Numerical results are presented by graphically for velocity and temperature profiles, skin friction coefficient and surface temperature profiles for magnetic parameter M, thermal conductivity variation parameter γ, Prandtl number Pr, heat generation parameter Q and joule heating parameter J. In the following section detailed derivations of the governing equations for the flow and heat transfer and the method of solutions along with the results and discussions are presented.

2.2 Governing equations of the flow

The mathematical statement of the basic conservation laws of mass, momentum and energy for the steady two-dimensional viscous incompressible and electrically conducting flow are respectively

$$\nabla.\overline{V} = 0 \tag{2.1}$$

$$\rho(\overline{V}.\nabla)\overline{V} = -\nabla p + \mu\nabla^2\overline{V} + \overline{F} + (\bar{j}\times\overline{B})_x \tag{2.2}$$

$$\rho c_p(\overline{V}.\nabla)T_f = \nabla(\kappa_f \nabla T_f) \tag{2.3}$$

where $\overline{V} = (\bar{u},\bar{v})$,$\bar{u}$ and $\bar{v}$ are the velocity components along the $\bar{x}$ and $\bar{y}$ axes respectively, $\overline{F}$ is the body force per unit volume which is defined as $-\rho g$ the terms $\overline{J}$ and $\overline{B}$ are respectively the current density and magnetic induction vector and the term $\overline{J}\times\overline{B}$ is the force on the fluid per unit volume produced by the interaction of current and magnetic field in the absence of excess charges. T_f is the temperature of the fluid in the boundary layer; g is the acceleration due to gravity, κ_f is the thermal conductivity of the fluid and C_p is the specific heat at constant pressure and μ is the viscosity of the fluid. Here $\overline{B} = \mu_e H_0$, μ_e being the magnetic permeability of the fluid, H_0 is the applied magnetic field strength and ∇ is the vector differential operator and is defined by

$$\nabla = \hat{i}_x \frac{\partial}{\partial \overline{x}} + \hat{j}_y \frac{\partial}{\partial \overline{y}} \tag{2.4}$$

where $\hat{i}_x$ and $\hat{j}_y$ are the unit vector along $\overline{x}$ and $\overline{y}$ axes respectively. When the external electric conductivity of the fluid is zero and the induced electric field is negligible, the current density is related to the velocity by Ohm's law as follows

$$\overline{J} = \sigma(\overline{V} \times \overline{B}) \tag{2.5}$$

where $(\overline{V} \times \overline{B})$ is electrical fluid vector and σ denotes the electrical conductivity of the fluid next under the conduction that the magnetic Reynold's number is small, induced magnetic field is negligible compared with applied fields. This conduction is usually well satisfied in terrestrial application especially so in (low velocity) free convection flows, so it can write

$$\overline{B} = \bar{j}_y H_0 \tag{2.6}$$

Bringing together equations (2.4) and (2.5) the force per unit volume $\overline{J} \times \overline{B}$ acting along the $\overline{x}$-axis takes the form

$$(\overline{J} \times \overline{B})_x = -\,\sigma H_0^2\, \overline{u} \tag{2.7}$$

Consider a steady two-dimensional laminar natural convection flow of an electrically conducting, viscous and incompressible fluid along a vertical flat plate of length l and thickness b (Figure-2.1). It is assumed that the temperature at the outer surface of the plate is maintained at a constant temperature T_b, where $T_b > T_\infty$ the ambient temperature of the fluid outside the boundary layer, g is the acceleration due to gravity. A uniform magnetic field of strength H_0 is imposed along the $\overline{y}$-axis i.e. normal direction to the surface and $\overline{x}$-is taken along the flat plate. The coordinate system and the configuration are shown in Figure-2.1.

The governing equations by using the equations (2.4) to (2.6) with respect to above considerations into the basic equations (2.1) to (2.3), of such two dimensional laminar free convection boundary layer flow of a viscous incompressible fluid with heat generation, joule heating and also thermal conductivity variation along a vertical flat plate under the usual boundary layer and the Boussinesq approximations for the present problem of continuity, momentum and energy equations can be written as

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \qquad (2.8)$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \upsilon\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + g\beta(T_f - T_\infty) - \frac{\sigma H_0^2 \bar{u}}{\rho} \qquad (2.9)$$

$$\bar{u}\frac{\partial T_f}{\partial \bar{x}} + \bar{v}\frac{\partial T_f}{\partial \bar{y}} = \frac{1}{\rho C_p}\frac{\partial}{\partial \bar{y}}(\kappa_f \frac{\partial T_f}{\partial \bar{y}}) + \frac{Q_0}{\rho C_p}(T_f - T_\infty) + \sigma\frac{H_0^2 \bar{u}^2}{\rho C_p} \qquad (2.10)$$

Here β is coefficient of volume expansion. The temperature dependent thermal conductivity, which is proposed by Charraudeau (1975) and used by Rahaman (2008) as follows

$$\kappa_f = \kappa_\infty[1 + \delta(T_f - T_\infty)] \qquad (2.11)$$

where κ_∞ is the thermal conductivity of the ambient fluid and δ is a constant, defined as

$$\delta = \frac{1}{\kappa_\infty}\left(\frac{\partial \kappa}{\partial T}\right)_f$$

The appropriate boundary conditions to be satisfied by the above equations are (Luikov 1974, Chang 2006, Pop & Ingham 2001, Merkin & Pop 1996)

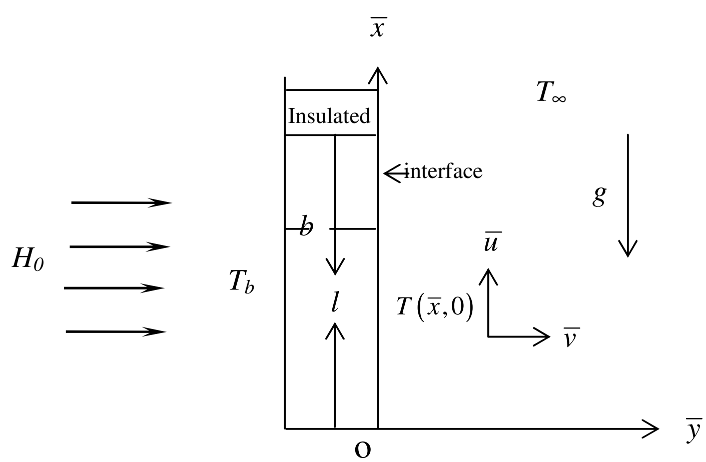

Figure.2.1: Physical model and co-ordinate system

$$\left.\begin{aligned} &\bar{u}=0, \qquad \bar{v}=0 \\ &T_f=T(\bar{x},0), \quad \frac{\partial T_f}{\partial \bar{y}}=\frac{\kappa_s}{b\kappa_f}(T_f-T_b) \end{aligned}\right\} \quad on \quad \bar{y}=0,\ \bar{x}>0 \tag{2.12}$$
$$\bar{u}\to 0,\ T_f\to T_\infty \ \ as \quad \bar{y}\to\infty, \quad \bar{x}>0$$

Observe that the equations (2.9) and (2.10) together with the boundary conditions (2.12) are non-linear partial differential equations. In the following sections the solution methods of these equations are discussed in detail.

2.3 Transformation of the governing equations

The non-dimensional governing equations and boundary conditions can be obtained from equation (2.8) - (2.10) using the following non-dimensional quantities:

$$x=\frac{\bar{x}}{l},\ y=\frac{\bar{y}}{l}Gr^{\frac{1}{4}},\ u=\frac{\bar{u}l}{\upsilon}Gr^{-\frac{1}{2}},\ v=\frac{\bar{v}l}{\upsilon}Gr^{-\frac{1}{4}},\theta=\frac{T_f-T_\infty}{T_b-T_\infty},\ Gr=\frac{g\beta l^3(T_b-T_\infty)}{\upsilon^2} \tag{2.13}$$

where l is the length of the plate , Gr is the Grashof number, θ is the non-dimensional temperature.

Now substituting the equation (2.13) into the equations (2.8) - (2.10) we obtain the following dimensionless equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.14}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+Mu=\frac{\partial^2 u}{\partial y^2}+\theta \tag{2.15}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\Pr}(1+\gamma\theta)\frac{\partial^2\theta}{\partial y^2}+\frac{\gamma}{\Pr}\left(\frac{\partial\theta}{\partial y}\right)^2+Q\theta+Ju^2 \tag{2.16}$$

where $\Pr=\dfrac{\mu C_p}{\kappa_\infty}$ is the Prandtl number, $M=\dfrac{\sigma H_0^2 l^2}{\mu Gr^{1/2}}$ is the dimensionless magnetic parameter, $\gamma=\delta(T_b-T_\infty)$ is the dimensionless thermal conductivity variation parameter, $Q=\dfrac{Q_\circ l^2}{\mu C_p Gr^{1/2}}$ is the non-dimensional heat generation parameter and $J=\dfrac{\sigma H_0^2\upsilon Gr^{1/2}}{\rho C_p(T_b-T_\infty)}$ is the dimensionless joule heating parameter. The corresponding boundary conditions (2.12) then take the following form

$$u = 0,\, v = 0,\;\; \theta - 1 = (1 + \gamma\theta)\; p\frac{\partial \theta}{\partial y} \quad on \;\; y = 0,\, x > 0$$
$$u \to 0,\, \theta \to 0 \quad as \;\; y \to \infty,\, x > 0 \qquad (2.17)$$

where $p = \left(\dfrac{\kappa_\infty\, b}{\kappa_s l}\right) Gr^{\frac{1}{4}}$ is the conjugate conduction parameter. The described problem is governed by the coupling parameter p. In actual fact, magnitude of $O(p)$ depends on b/l and $Gr^{1/4}$ being the order of unity. Since l is small, the term b/l becomes greater than one. For air, $\dfrac{\kappa_\infty}{\kappa_s}$ attains very small values if the plate is highly conductive and reaches the order 0.1 for materials such as glass. Therefore in different cases p is different but not always a small number. In the present investigation we have considered $p = 1$.

To solve the equations (2.15) and (2.16) subject to the boundary conditions (2.17) the following transformations are introduced

$$\psi = x^{\frac{4}{5}}(1+x)^{-\frac{1}{20}} f(x,\eta)$$
$$\eta = y\, x^{-\frac{1}{5}}(1+x)^{-\frac{1}{20}} \qquad (2.18)$$
$$\theta = x^{\frac{1}{5}}(1+x)^{-\frac{1}{5}} h(x,\eta)$$

here η is the similarity variable and ψ is the non-dimensional stream function which satisfies the continuity equation and is related to the velocity components in the usual way as $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$. Moreover, $h(x,\eta)$ represents the non-dimensional temperature. The momentum and energy equations (Equation (2.15) and (2.16), respectively) are transformed for the new coordinate system. At first, the velocity components are expressed in terms of the new variables for this transformation. Thus the following equations

$$f''' + \frac{16+15x}{20(1+x)} f\, f'' - \frac{6+5x}{10(1+x)} f'^2 - M\, x^{\frac{2}{5}}(1+x)^{\frac{1}{10}} f' + h = x\left(f' \frac{\partial f'}{\partial x} - f'' \frac{\partial f}{\partial x}\right) \qquad (2.19)$$

$$\frac{1}{\Pr} h'' + \frac{\gamma}{\Pr}\left(\frac{x}{1+x}\right)^{\frac{1}{5}} h\, h'' + \frac{\gamma}{\Pr}\left(\frac{x}{1+x}\right)^{\frac{1}{5}} h'^2 + \frac{16+15x}{20(1+x)} f\, h' +$$
$$J x^{7/5}(1+x)^{1/10} f'^2 + Q x^{2/3}(1+x)^{1/10} h - \frac{1}{5(1+x)} f' h = \left(f' \frac{\partial h}{\partial x} - h' \frac{\partial f}{\partial x}\right) \qquad (2.20)$$

where prime denotes partial differentiation with respect to η. The boundary conditions as mentioned in equation (2.17) then take the following form:

$$f(x,0) = f'(x,0) = 0$$
$$h'(x,0) = \frac{x^{\frac{1}{5}}(1+x)^{-\frac{1}{5}}h(x,0) - 1}{(1+x)^{-\frac{1}{4}} + \gamma x^{\frac{1}{5}}(1+x)^{-\frac{9}{20}}h(x,0)} \tag{2.21}$$
$$f'(x,\infty) \to 0,\ h(x,\infty) \to 0$$

The set of equations (2.19) and (2.20) together with the boundary conditions (2.21) are solved by applying implicit finite difference method with Keller box (1978) scheme. A good description of this method and its application to the boundary layer flow problems are given in the book by Cebeci and Bradshaw (1984).

From the process of numerical computation, in practical point of view, it is important to calculate the values of the surface shear stress in terms of the skin friction coefficient. This can be written in the non-dimensional form as Molla et.al (2005)

$$C_f = \frac{Gr^{-\frac{3}{4}} l^2}{\mu \upsilon} \tau_w \tag{2.22}$$

Where $\tau_w [= \mu(\partial \bar{u}/\partial \bar{y})_{\bar{y}=0}]$ is the shearing stress. Using the new variables described in (2.13), the local skin friction coefficient C_{fx} can be written as

$$C_{fx} = x^{\frac{2}{5}}(1+x)^{-\frac{3}{20}} f''(x,0) \tag{2.23}$$

The numerical values of the surface temperature distribution $\theta(x,0)$ are obtained from the relation

$$\theta(x,0) = x^{\frac{1}{5}}(1+x)^{-\frac{1}{5}} h(x,0) \tag{2.24}$$

We have also discussed the velocity profiles and the temperature distributions for different values of the magnetic parameter, thermal conductivity variation parameter, Prandtl number, heat generation parameter and joule heating parameter.

2.4 Results and Discussion

The objective of the present work is to analyze the effect of temperature dependent thermal conductivity on electrically conducting fluid on free convection flow along a vertical flat plate with heat generation and joule heating for strong magnetic field. In the simulation the values of the Prandtl number Pr are considered to be 0.73, 1.00, 1.50, 2.00 and 2.50 that corresponds to hydrogen, steam, sulfur dioxide, ammonia and methyl chloride respectively. Detailed numerical results of the velocity, temperature, skin friction coefficient and surface temperature profiles obtained for different values of the magnetic parameter, thermal conductivity variation parameter, Prandtl number, heat generation parameter and joule heating parameter are presented graphically. Numerical computation are carried out for a range of magnetic parameter M = 0.01, 0.20, 0.40, 0.60, 0.80, thermal conductivity variation parameter γ= 0.01, 0.15, 0.30, 0.45, 0.60, heat generation parameter Q = 0.01, 0.10, 0.20, 0.30, 0.40 and joule heating parameter J = 0.01, 0.10, 0.20, 0.30, 0.40.

The velocity and the temperature fields obtained from the solutions of the equations (2.19) to (2.20) and the local skin friction coefficient and surface temperature distribution fields obtained from the solutions of the equations (2.23) to (2.24) are depicted in figures 2 to 11.

The interaction of the magnetic field and moving electric charge carried by the flowing fluid induces a force, which tends to oppose the fluid motion.

In figure 2(a), it is shown that the magnetic field action along the horizontal direction retards the fluid velocity with γ= 0.01, Pr = 0.73, Q = 0.01 and J = 0.01. Here position of peak velocity moves toward the interface with the increasing values of M. From figure 2(b), it can be observed that the temperature within the boundary layer increases for the increasing values of M from 0.01 to 0.80. The magnetic field decreases the temperature gradient at the wall and increases the temperature in the flow region.

The effect of thermal conductivity variation parameter γ on the velocity and temperature profiles against η within the boundary layer with M = 0.01, Pr = 0.73, Q = 0.01, and J = 0.01 are shown in figures 3(a) and 3(b), respectively. It is seen from figures 3(a) and 3(b) that the velocity and temperature increase within the boundary layer with the increasing values of γ. It means that the velocity boundary layer and the thermal boundary layer thickness expand for large values of γ.

Figures 4(a) and 4(b) illustrate the velocity and temperature profiles against η for different values of Prandtl number Pr with M = 0.01, γ= 0.01, Q = 0.01 and J = 0.01. From figure 4(a), it can be observed that the velocity decreases as well as its position moves toward the interface with the increasing Pr. From figure 4(b), it is seen that the temperature profiles shift downward with the increasing Pr.

Figures 5(a) and 5(b) describe the velocity and temperature profiles against η for different values of heat generation parameter Q with M = 0.01, γ= 0.01, Pr =0.73 and J = 0.01. From figure 5(a), it can be observed that the velocity increases as well as its position moves upward the interface with the increasing Q. From figure 5(b), it is seen that the temperature profiles also the same as increasing within the boundary layer. It means that the velocity boundary layer and the thermal boundary layer thickness expand for large values of Q.

The effect of joule heating parameter J on the velocity and temperature profiles against η within the boundary layer with M = 0.01, γ= 0.01, Pr = 0.73 and Q = 0.01 are shown in figures 6(a) and 6(b) respectively. The velocity and temperature profiles increase within the boundary layer with the increasing values of J. But the effect is small.

The variation of the local skin friction coefficient C_{fx} and surface temperature distribution $\theta(x,0)$ against x for different values of M with γ= 0.01, Pr = 0.73, Q = 0.01 and J = 0.01 at different positions are illustrated in figures 7(a) and 7(b), respectively. It is observed from figure 7(a) that the increased value of the magnetic parameter M leads to decrease the skin friction factor. Again figure 7(b) shows that the surface temperature increases due to the increased values of the magnetic parameter M. It can also be seen that the surface temperature increases along the upward direction of the plate for a particular M. The magnetic field acts against the flow and reduces the skin friction and produces the temperature at the interface.

Figures 8(a) and 8(b) illustrate the effect of the thermal conductivity variation parameter γ on the skin friction coefficient C_{fx} and surface temperature $\theta(x,0)$ against x with M = 0.01, Pr =0.73, Q = 0.01 and J = 0.01. It is seen that the skin friction increases monotonically along the upward direction of the plate for a particular values of γ. It is also seen that that the skin friction increases for the increasing values of γ. The same result is observed for the surface temperature from figure 8(b). This is to be expected because the higher value for the thermal conductivity variation parameter accelerates the fluid flow and increases the temperature as mentioned in figure 3(a) and 3(b), respectively.

Figures 9(a) and 9(b) deal with the effect of Prandtl number Pr on the local skin friction C_{fx} and surface temperature distribution $\theta(x,0)$ against x with M = 0.01, γ= 0.01, Q = 0.01 and J = 0.01.It can be observed from figure 9(a) that the skin friction increases monotonically for a particular value of Pr. It can also be noted that the skin friction coefficient decreases for the increasing values of Pr. From figure 9(b), it can be seen that the surface temperature decreases when Pr increases along the positive x direction for a particular Pr.

Figures 10(a) and 10(b) deal with the effect of Q on the skin friction coefficient C_{fx} and surface temperature $\theta(x,0)$ against x with M = 0.01, γ= 0.01, Pr = 0.73 and J = 0.01. It can be noted that the skin friction and surface temperature profile increase for the increasing values of Q.

Figures 11(a) and 11(b) deal with the effect of J on the local skin friction coefficient C_{fx} and surface temperature distribution $\theta(x,0)$ against x with M = 0.01, γ= 0.01, Pr = 0.73 and Q = 0.01. It can be noted that the skin friction and surface temperature profiles increase for the increasing values of J.

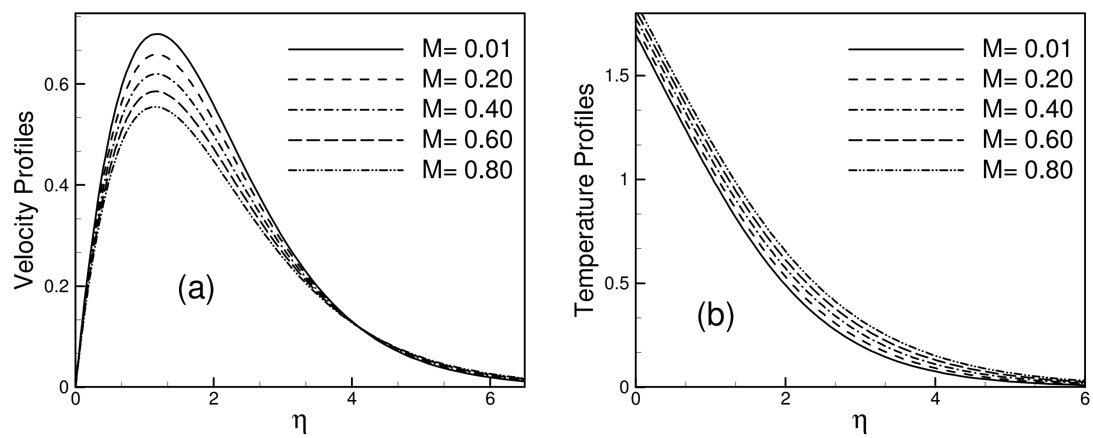

Figure 2(a) Velocity and (b) Temperature profiles against η for different values of M with γ = 0.01, Pr = 0.73, Q = 0.01 and J = 0.01.

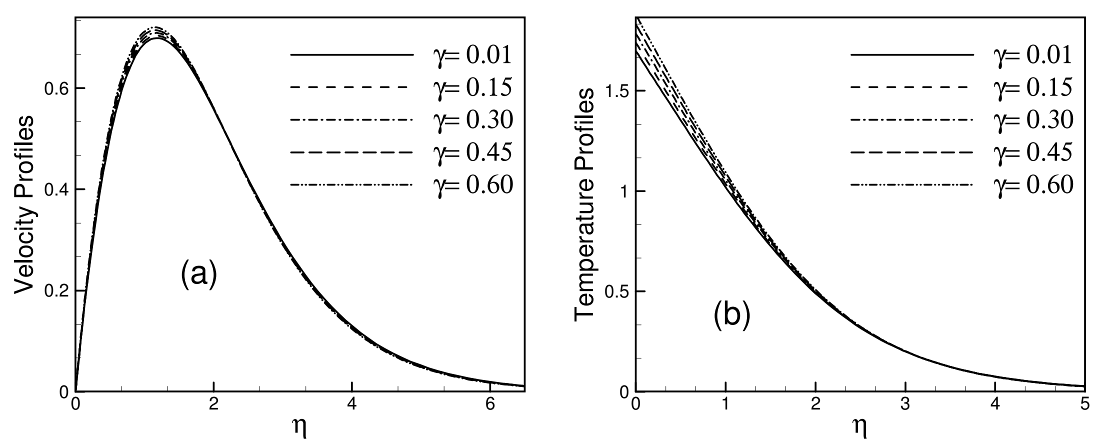

Figure 3(a) Velocity and (b) Temperature profiles against η for different values of γ with M = 0.01, Pr = 0.73, Q = 0.01 and J = 0.01.

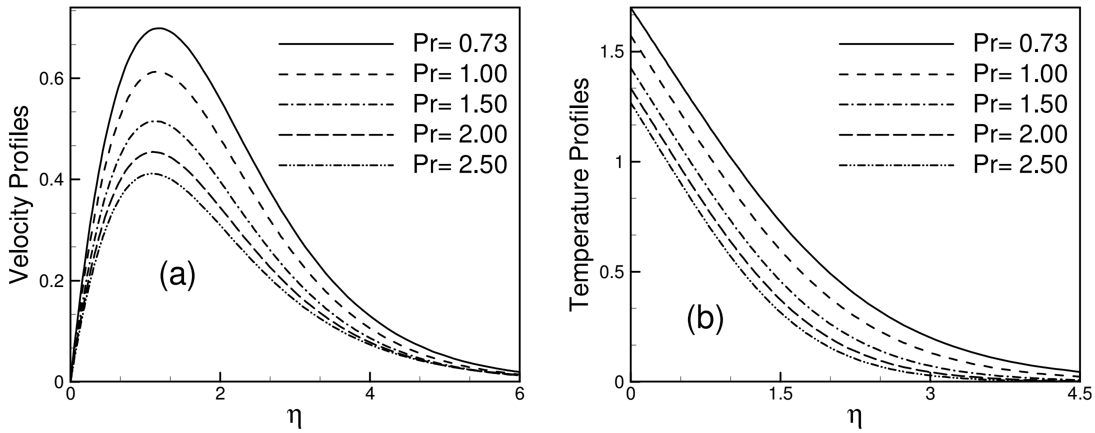

Figure 4(a) Velocity and (b) Temperature profiles against η for different values of Pr with $M = 0.01$, $\gamma = 0.01$, $Q = 0.01$ and $J = 0.01$.

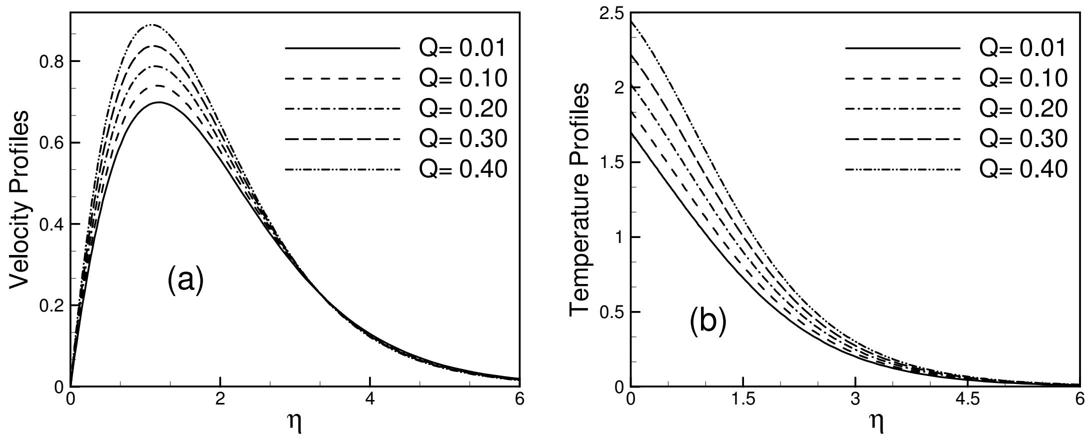

Figure 5(a) Velocity and (b) Temperature profiles against η for different values of Q with $M = 0.01$, $\gamma = 0.01$, $Pr = 0.73$ and $J = 0.01$.

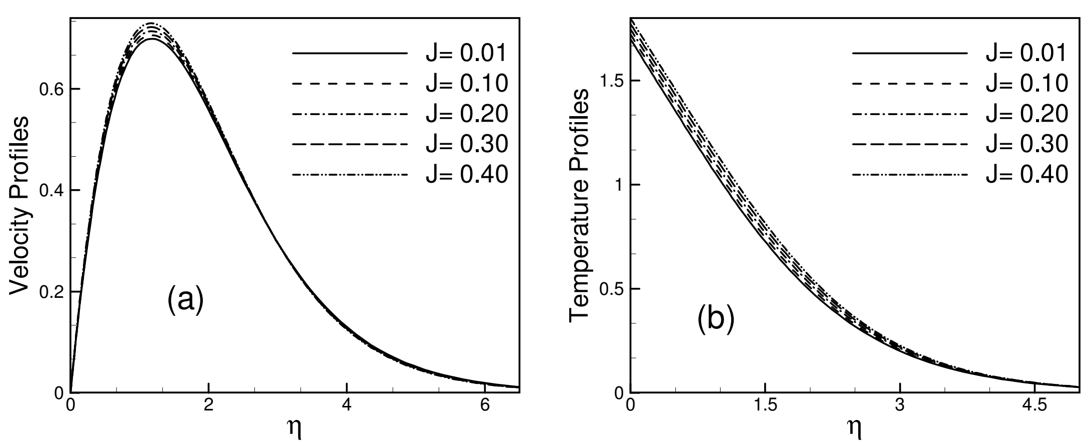

Figure 6(a) Velocity and (b) Temperature profiles against η for different values of J with $M = 0.01$, $\gamma = 0.01$, $Pr = 0.73$ and $Q = 0.01$.

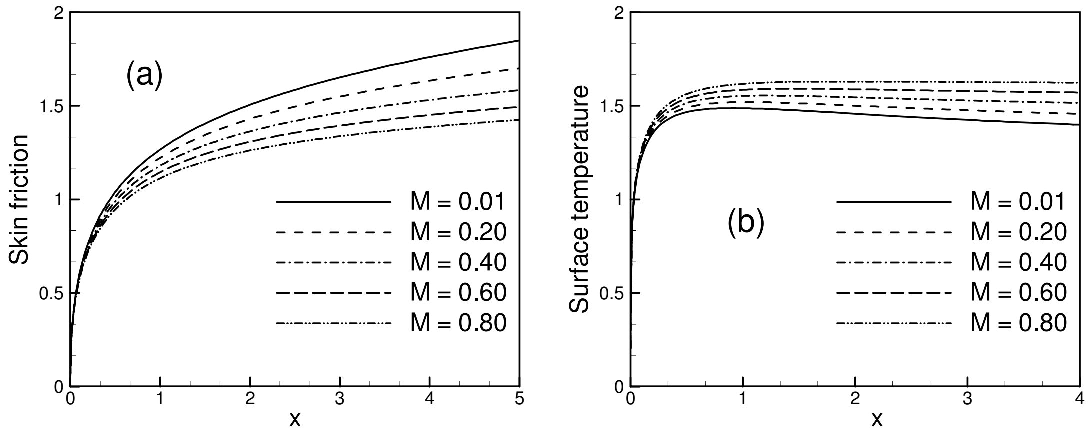

Figure 7(a) Local skin friction coefficient and (b) Surface temperature distribution against x for different values of M with $\gamma = 0.01$, $Pr = 0.73$, $Q = 0.01$ and $J = 0.01$.

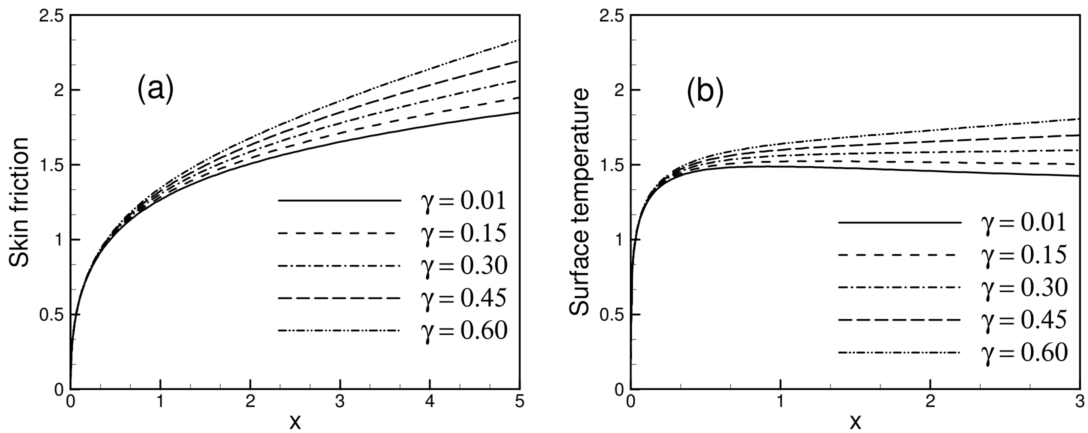

Figure 8(a) Local skin friction coefficient and (b) Surface temperature distribution against x for different values of γ with $M = 0.01$, $Pr = 0.73$, $Q = 0.01$ and $J = 0.01$.

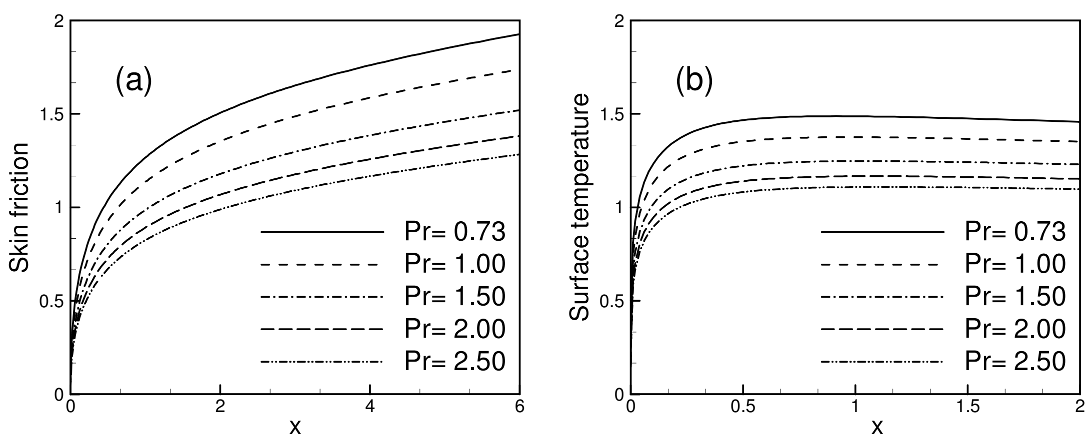

Figure 9(a) Local skin friction coefficient and (b) Surface temperature distribution against x for different values of Pr with $M = 0.01$, $\gamma = 0.01$, $Q = 0.01$ and $J = 0.01$.

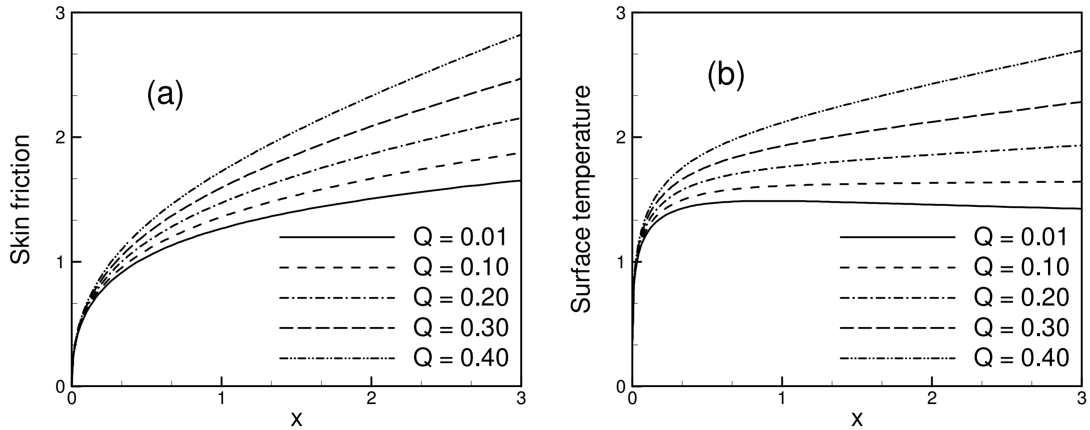

Figure 10(a) Local skin friction coefficient and (b) Surface temperature distribution against x for different values of Q with $M = 0.01$, $\gamma = 0.01$, $Pr = 0.73$ and $J = 0.01$.

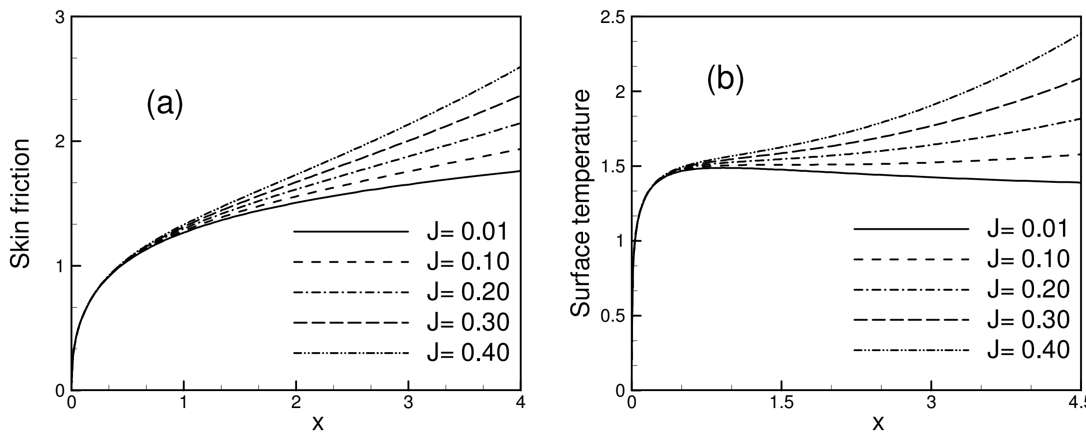

Figure 11(a) Local skin friction coefficient and (b) Surface temperature distribution against x for different values of J with $M = 0.01$, $\gamma = 0.01$, $Pr = 0.73$ and $Q = 0.01$.

2.5 Comparison of the Results

Table 2.1 and Table 2.2 depict the comparisons of the present numerical results of the skin friction coefficient C_{fx} and the surface temperature $\theta(x,0)$ with those obtained by Pozzi and Lopo(1988) and Merkin and Pop(1996) respectively. Here, the magnetic parameter M, thermal conductivity variation parameter γ, heat generation parameter Q and joule heating parameter J are ignored (i.e. $M = 0$, $\gamma = 0$, $Q = 0$ and $J = 0$) and the Prandtl number Pr = 0.733 with $x^{\frac{1}{5}} = \xi$ is chosen. It is clearly seen that there is an excellent agreement among the present results with the solutions of Pozzi and Lopo(1988) and Merkin and Pop(1996).

Table 2.1: Comparison of the present numerical results of skin friction coefficient C_{fx} with Prandtl number $Pr = 0.733$, $M = 0$, $\gamma = 0$, $Q = 0$ and $J = 0$ against x.

C_{fx}			
$x^{\frac{1}{5}} = \xi$	Pozzi and Lupo (1988)	Merkin and Pop (1996)	Present work
0.7	0.430	0.430	0.423
0.8	0.530	0.530	0.528
0.9	0.635	0.635	0.633
1.0	0.741	0.745	0.748
1.1	0.829	0.859	0.857
1.2	0.817	0.972	0.972

Table 2.2: Comparison of the present numerical results of surface temperature $\theta(x,0)$ with Prandtl number $Pr = 0.733$, $M = 0$, $\gamma = 0$, $Q = 0$ and $J = 0$ against x.

$\theta(x,0)$			
$x^{\frac{1}{5}} = \xi$	Pozzi and Lupo (1988)	Merkin and Pop (1996)	Present work
0.7	0.651	0.651	0.651
0.8	0.684	0.686	0.687
0.9	0.708	0.715	0.716
1.0	0.717	0.741	0.742
1.1	0.699	0.762	0.763
1.2	0.640	0.781	0.781

2.6 Summary and Conclusion of this Chapter

The effects of temperature dependent thermal conductivity on MHD free convection flow along a vertical flat plate with heat generation and joule heating have been studied numerically and presented graphically. From this investigation the following conclusions may be drawn

- The velocity within the boundary layer increases for decreasing values of M, Pr and for increasing values of γ, Q and J.
- The temperature within the boundary layer increases for increasing values of M, γ, Q and J and for decreasing values of Pr.
- The local skin friction coefficient decreases for the increasing values of M, Pr and increases for increasing values of γ, Q and J.
- An increase in the values of M, γ, Q and J leads to an increase in surface temperature. On the other hand, this decreases for increasing values of Pr.

Chapter 3

Effects of Temperature Dependent Thermal Conductivity and Viscous Dissipation on Free Convective Flow along a Vertical Flat Plate

3.1 Introduction

The effects of the temperature dependent thermal conductivity and viscous dissipation on free convection flow along a vertical flat plate have been described in this chapter. The governing boundary layer equations are transformed into a non-dimensional form and the resulting non linear system of partial differential equations are reduce to local non similarity equations which are solved numerically by very efficient implicit finite difference method together with Keller-box technique. Numerical results are presented graphically in the form of velocity and temperature profiles, skin friction coefficient and surface temperature profiles for thermal conductivity variation parameter γ, Prandtl number *Pr* and viscous dissipation parameter *N*. In the following section detailed derivations of the governing equations for the flow and heat transfer and the method of solutions along with the results and discussions are presented.

3.2 Governing equations of the flow

Consider a steady natural convection flow of an electrically conducting, viscous and incompressible fluid along a vertical flat plate of length l and thickness *b* (Figure.3-1). Over the work it is assumed that the temperature at the outside surface is maintained at a constant temperature T_b, where $T_b > T_\infty$, the ambient temperature of the fluid and *g* is the acceleration due to gravity.

To the above considerations into the steady two dimensional laminar free convection boundary layer flow of a viscous incompressible fluid with viscosity and also thermal conductivity variation along a vertical flat plate take the following electrically conducting form and under the usual Boussinesq approximations $\rho = \rho_\infty[1-\beta(T_b - T_\infty)]$, where ρ_∞ and T_∞ are the density and temperature respectively outside the boundary layer , β is the coefficient of thermal expansion. So for the present problem of continuity, momentum and energy equations can be written as

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{3.1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \upsilon\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + g\beta(T_f - T_\infty) \tag{3.2}$$

$$\bar{u}\frac{\partial T_f}{\partial \bar{x}} + \bar{v}\frac{\partial T_f}{\partial \bar{y}} = \frac{1}{\rho C_p}\frac{\partial}{\partial \bar{y}}(\kappa_f \frac{\partial T_f}{\partial \bar{y}}) + \frac{\upsilon}{C_p}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 \tag{3.3}$$

Here we consider the temperature dependent thermal conductivity, which is proposed by Charraudeau (1975), as follows

$$\kappa_f = \kappa_\infty[1 + \delta(T_f - T_\infty)] \tag{3.4}$$

where κ_∞ is the thermal conductivity of the ambient fluid and δ is a constant, defined as $\delta = \frac{1}{\kappa_f}\left(\frac{\partial \kappa}{\partial T}\right)_f$.

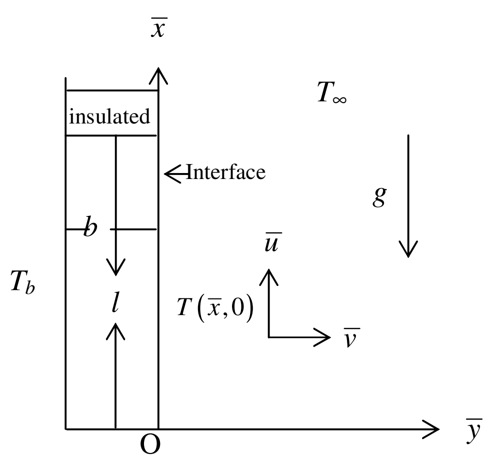

Figure.3.1: Physical model and co-ordinate system

The appropriate boundary conditions to be satisfied by the above equations are (Luikov 1974, Chang 2006, Pop & Ingham 2001, Merkin & Pop 1996)

$$\left.\begin{aligned} &\bar{u} = 0, && \bar{v} = 0 \\ &T_f = T(\bar{x},0), && \frac{\partial T_f}{\partial \bar{y}} = \frac{\kappa_s}{b\kappa_f}(T_f - T_b) \end{aligned}\right\} \quad on \quad \bar{y} = 0,\ \bar{x} > 0 \tag{3.5}$$
$$\bar{u} \to 0,\ T_f \to T_\infty \ \ as \quad \bar{y} \to \infty, \quad \bar{x} > 0$$

Observe that the equations (3.2) and (3.3) together with the boundary conditions (3.5) are non-linear partial differential equations. In the following sections the solution methods of these equations are discussed in details.

3.3 Transformation of the governing equations

Equations (3.1) to (3.3) may be non-dimensionalzed by using the following dimensionless variables

$$x = \frac{\bar{x}}{l},\ y = \frac{\bar{y}}{l} Gr^{\frac{1}{4}},\ u = \frac{\bar{u} l}{\upsilon} Gr^{-\frac{1}{2}},\ v = \frac{\bar{v} l}{\upsilon} Gr^{-\frac{1}{4}}, \theta = \frac{T_f - T_\infty}{T_b - T_\infty},\ Gr = \frac{g\beta l^3 (T_b - T_\infty)}{\upsilon^2} \tag{3.6}$$

where l is the length of the plate, Gr is the Grashof number, θ is the non dimensional temperature.

Substituting the relations (3.6) into the equations (3.1) to (3.3) then the following non-dimensional equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.7}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + \theta \tag{3.8}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\Pr}(1+\gamma\theta)\frac{\partial^2 \theta}{\partial y^2} + \frac{\gamma}{\Pr}\left(\frac{\partial \theta}{\partial y}\right)^2 + N\left(\frac{\partial u}{\partial y}\right)^2 \tag{3.9}$$

where $\Pr = \frac{\mu\ C_p}{\kappa_\infty}$ is the Prandtl number, $N = \frac{\upsilon^2 Gr}{l^2 C_p\ (T_b - T_\infty)}$ is the dimensionless viscous dissipation parameter and $\gamma = \delta(T_b - T_\infty)$ is the dimensionless thermal conductivity variation parameter. The corresponding boundary conditions (2.12) then take the following form

$$\begin{aligned} &u = 0, v = 0,\ \theta - 1 = (1+\gamma\theta)\ p\frac{\partial \theta}{\partial y} \quad on\ \ y = 0, x > 0 \\ &u \to 0, \theta \to 0 \quad as\ \ y \to \infty, x > 0 \end{aligned} \tag{3.10}$$

where $p = \left(\frac{\kappa_\infty b}{\kappa_s l} \right) Gr^{\frac{1}{4}}$ is the conjugate conduction parameter. The described problem is governed by the coupling parameter p. In actual fact, magnitude of $O(p)$ depends on b/l and $Gr^{1/4}$ being the order of unity. Since l is small, the term b/l becomes greater than one. For air, $\frac{\kappa_\infty}{\kappa_s}$ attains very small values if the plate is highly conductive and reaches the order of 0.1 for materials such as glass. Therefore in different cases p is different but not always a small number. In the present investigation we have considered $p = 1$ which is accepted for b/l of $O(\frac{\kappa_\infty}{\kappa_s})$.

To solve the equations (3.8) and (3.9) subject to the boundary conditions (3.10) the following transformations are introduced (Merkin & Pop 1996)

$$\begin{aligned} \psi &= x^{\frac{4}{5}} (1+x)^{-\frac{1}{20}} f(x,\eta) \\ \eta &= y\, x^{-\frac{1}{5}} (1+x)^{-\frac{1}{20}} \\ \theta &= x^{\frac{1}{5}} (1+x)^{-\frac{1}{5}} h(x,\eta) \end{aligned} \tag{3.11}$$

here η is the similarity variable and ψ is the non-dimensional stream function which satisfies the continuity equation and is related to the velocity components in the usual way as $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$. Moreover, $h\ (x,\eta)$ represents the non-dimensional temperature. The momentum and energy equations (Equation (3.8) and (3.9), respectively) are transformed for the new co-ordinate system. At first, the velocity components are expressed in terms of the new variables for this transformation. Thus the following equations

$$f''' + \frac{16+15x}{20(1+x)} f f'' - \frac{6+5x}{10(1+x)} f'^2 + h = x\left(f' \frac{\partial f'}{\partial x} - f'' \frac{\partial f}{\partial x} \right) \tag{3.12}$$

$$\begin{aligned} \frac{1}{\Pr} h'' + \frac{\gamma}{\Pr} \left(\frac{x}{1+x} \right)^{\frac{1}{5}} h h'' + \frac{\gamma}{\Pr} \left(\frac{x}{1+x} \right)^{\frac{1}{5}} h'^2 + \frac{16+15x}{20(1+x)} f h' \\ - \frac{1}{5(1+x)} f' h + Nx f''^2 = x \left(f' \frac{\partial h}{\partial x} - h' \frac{\partial f}{\partial x} \right) \end{aligned} \tag{3.13}$$

Where, prime denotes partial differentiation with respect to η. The boundary conditions as mentioned in equation (3.10) then take the following form

$$f(x,0) = f'(x,0) = 0$$
$$h'(x,0) = \frac{x^{\frac{1}{5}}(1+x)^{-\frac{1}{5}}h\,(x,0) - 1}{(1+x)^{-\frac{1}{4}} + \gamma x^{\frac{1}{5}}(1+x)^{-\frac{9}{20}}h(x,0)} \tag{3.14}$$
$$f'(x,\infty) \to 0,\; h\,(x,\infty) \to 0$$

The set of equations (3.12) and (3.13) together with the boundary conditions (3.14) are solved by applying implicit finite difference method with Keller box (1978) scheme (see also the appendix). A good description of this method and its application to the boundary layer flow problems are given in the book by Cebeci and Bradshaw (1984). From the process of numerical computation, in practical point of view, it is important to calculate the values of the surface shear stress in terms of the skin friction coefficient. This can be written in the non-dimensional form as (Molla et.al 2005)

$$C_f = \frac{Gr^{-\frac{3}{4}}l^2}{\mu \upsilon}\tau_w \tag{3.15}$$

where $\tau_w[=\mu(\partial\,\bar{u}/\partial\,\bar{y})_{\bar{y}=0}]$ is the shearing stress. Using the new variables described in (2.14), the local skin friction coefficient can be written as

$$C_{fx} = x^{\frac{2}{5}}(1+x)^{-\frac{3}{20}}f''(x,0) \tag{3.16}$$

The numerical values of the surface temperature are obtained from the relation

$$\theta(x,0) = x^{\frac{1}{5}}(1+x)^{-\frac{1}{5}}h(x,0) \tag{3.17}$$

3.4 Results and Discussion

The system of non-linear ordinary differential equations (3.12) and (3.13) together with the boundary condition (3.14) has been solved numerically by employing implicit finite difference method together with Keller-box elimination technique. The values of the Prandtl number *Pr* are considered to be 0.73, 1, 1.73, 2.97 and 4.24 that corresponds to hydrogen, steam, water, methyl chloride and sulfur dioxide respectively. Detailed numerical results of the velocity, temperature, skin friction coefficient and surface temperature profiles for different values of the thermal conductivity variation parameter γ, Prandtl number *Pr* and viscous dissipation parameter *N* are presented graphically.

The velocity and the temperature fields obtained from the solutions of the equations (3.12) and (3.13) are depicted in figures 12(a) to 14(b). Also the local skin friction coefficient and surface temperature distribution fields obtained from the solutions of the equations (3.16) and (3.17) are depicted in figure 15(a) to figure 17(b).

The effect of thermal conductivity variation parameter γ on the velocity and the temperature profiles within the boundary layer with Pr = 1.73 and N = 0.10 are shown in figure 12(a) and figure 12(b), respectively. It is seen from figure 12(a) and figure 12(b), that the velocity and temperature increase within the boundary layer with the increasing values of γ. It means that the velocity boundary layer and the thermal boundary layer thickness increase for large values of γ.

Figure 13(a) and figure 13(b) illustrate the velocity and temperature profiles for different values of Prandtl number Pr with γ = 0.10 and N = 0.10. From figure 13(a), it can be observed that the velocity decreases as well as its position moves toward the interface with the increasing values of Pr. From figure 13(b), it is seen that the temperature profiles shift downward with the increasing values of Pr.

In figure 14(a) and figure 14(b) describe the velocity and temperature profiles for different values of viscous dissipation parameter N with γ= 0.10 and Pr = 1.73. It is seen from figure 14(a) and 14(b) that the velocity and temperature increase within the boundary layer with the increasing values of N. But for velocity boundary layer and for thermal boundary layer the effect is small.

Figure 15(a) and figure 15(b) illustrate the effect of the thermal conductivity variation parameter on the skin friction coefficient C_{fx} and surface temperature distribution $\theta(x,0)$ against x with Pr = 1.73 and N = 0.10. It is seen from figure 15(a) that the skin friction increases monotonically along the upward direction of the plate for a particular value of γ. It is also seen that the local skin friction coefficient increases for the increasing values of γ. From figure 15(b), it can be seen that the surface temperature distribution increases due to the increasing values along the positive x direction for a particular γ.

Figure 16(a) and figure 16(b) deal with the effect of Prandtl number Pr on the local skin friction coefficient and surface temperature distribution against x with γ = 0.10 and N = 0.10. It can be observed from figure 16(a) that the skin friction coefficient decreases monotonically for a particular value of Pr. It can be noted that the skin friction coefficient decreases for the increasing values of *Pr*. From figure 16(b), it can be seen that the surface temperature distribution decreases due to the increasing values of *Pr*. Also along the positive *x* direction this increases for a particular value of *Pr*.

The variation of the local skin friction coefficient C_{fx} and surface temperature distribution $\theta(x,0)$ for different values of *N* with $\gamma = 0.10$ and $Pr = 1.73$ at different positions are illustrated in figures 17(a) and 17(b), respectively. It can also be noted from figure 17(a) that the skin friction coefficient increases monotonically for a particular value of *N*. Again figure 17(b) shows that the surface temperature $\theta(x,0)$ increases for increasing values of *N*.

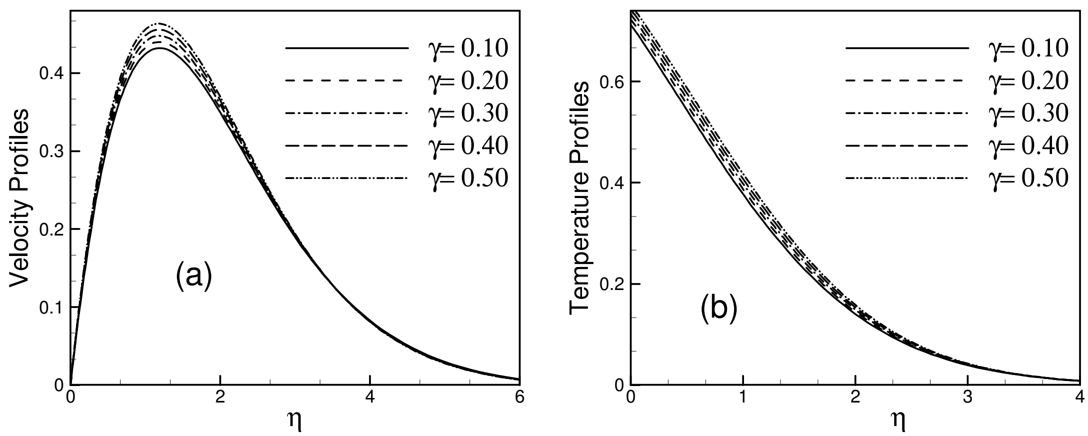

Figure 12(a) Velocity and (b) Temperature profiles against η for different values of γ with $Pr = 1.73$ and $N = 0.10$.

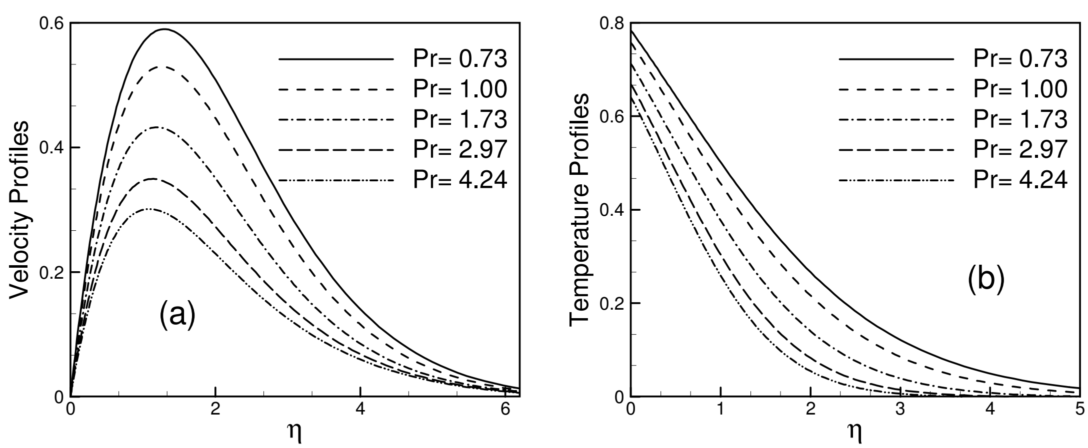

Figure 13(a) Velocity and (b) Temperature profiles against η for different values of *Pr* with $\gamma = 0.10$ and $N = 0.10$.

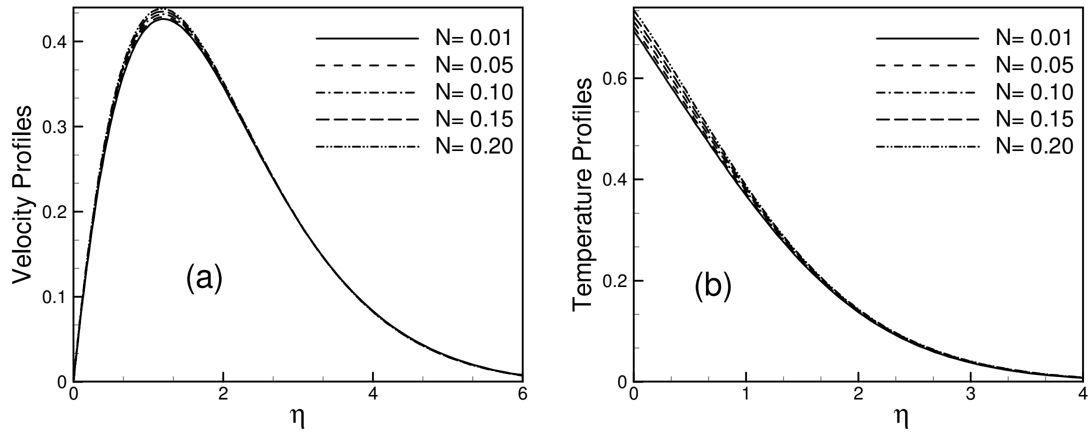

Figure 14(a) Velocity and (b) Temperature profiles against η for different values of N with γ = 0.10 and Pr =1.73.

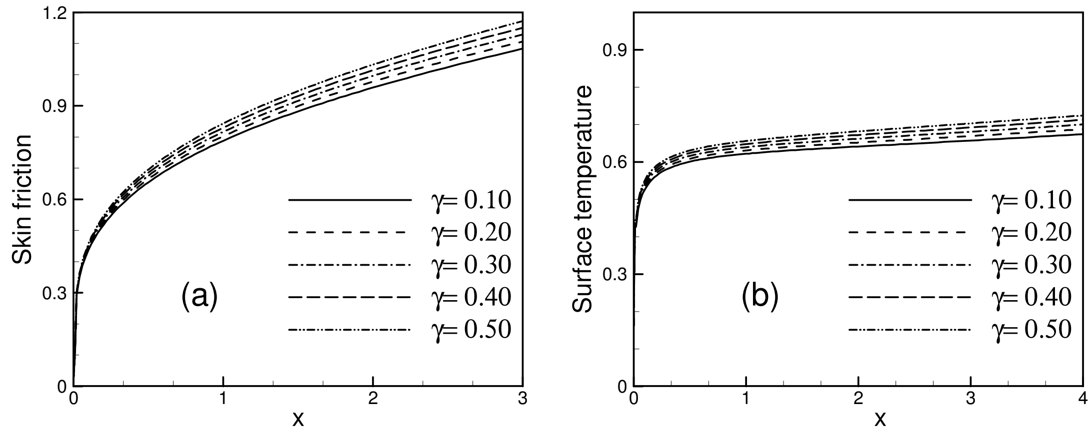

Figure 15(a) Local skin friction coefficient and (b) Surface temperature distribution against x for different values of γ with Pr = 1.73 and N = 0.10.

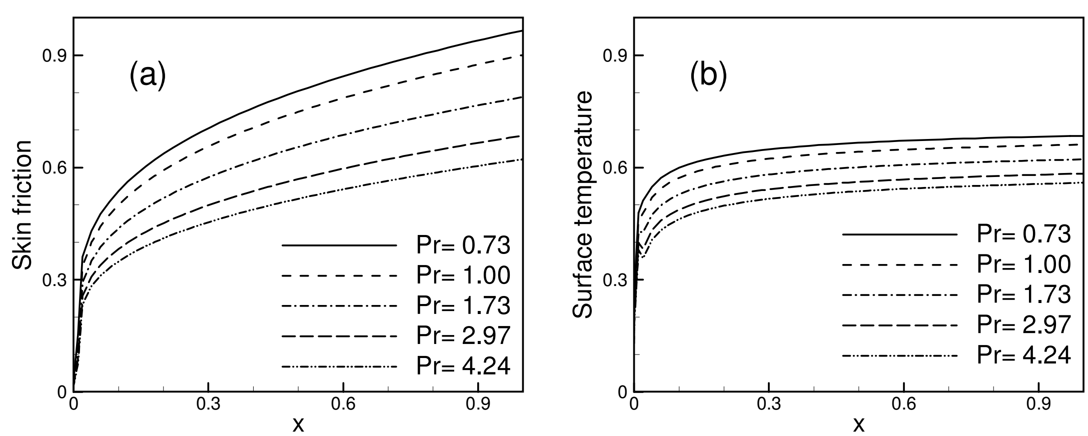

Figure 16(a) Local skin friction coefficient and (b) Surface temperature distribution against x for different values of Pr with γ= 0.10 and N = 0.10.

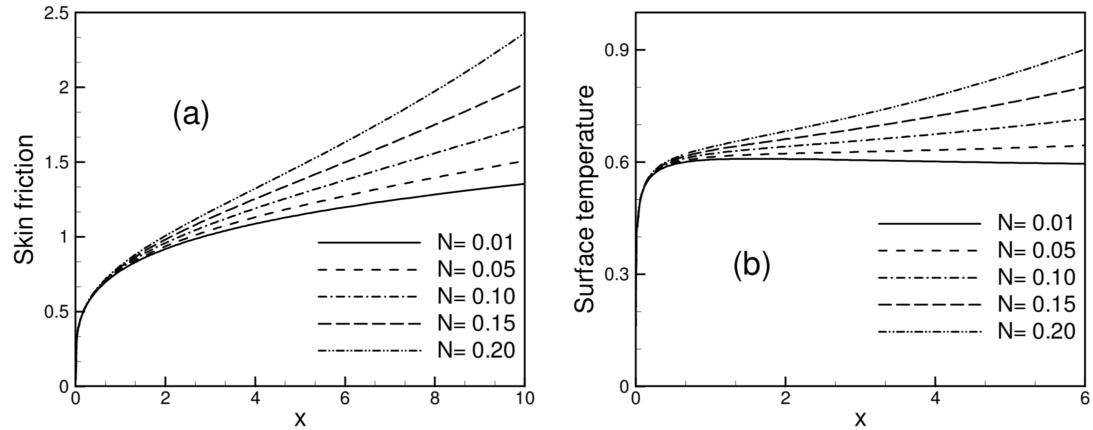

Figure 17(a) Local skin friction coefficient and (b) Surface temperature distribution against x for different values of N with $\gamma = 0.10$ and $Pr = 1.73$.

3.5 Comparison of the Results

Table 3.1 and 3.2 depict the comparisons of the present numerical results of the skin friction coefficient C_{fx} and the surface temperature $\theta(x,0)$ with those obtained by Pozzi and Lopo(1988) and Merkin and Pop(1996) respectively. Here, the thermal conductivity variation parameter γ and viscous dissipation parameter N are ignored (i.e. $\gamma = 0$ and N = 0) and the Prandtl number $Pr = 0.733$ with $x^{\frac{1}{5}} = \xi$ is chosen. It is clearly seen that there is an excellent agreement among the present results with the solutions Pozzi and Lopo(1988) and Merkin and Pop(1996).

Table 3.1: Comparison of the present numerical results of skin friction coefficient C_{fx} with Prandtl number $Pr = 0.733$, $\gamma = 0$ and $N = 0$ against x.

C_{fx}			
$x^{\frac{1}{5}} = \xi$	Pozzi and Lupo (1988)	Merkin and Pop (1996)	Present work
0.7	0.430	0.430	0.423
0.8	0.530	0.530	0.528
0.9	0.635	0.635	0.633
1.0	0.741	0.745	0.748
1.1	0.829	0.859	0.857
1.2	0.817	0.972	0.972

Table 3.2: Comparison of the present numerical results of surface temperature $\theta(x,0)$ with Prandtl number Pr = 0.733, γ= 0 and N = 0 against x.

$\theta(x,0)$			
$x^{\frac{1}{5}} = \xi$	Pozzi and Lupo (1988)	Merkin and Pop (1996)	Present work
0.7	0.651	0.651	0.651
0.8	0.684	0.686	0.687
0.9	0.708	0.715	0.716
1.0	0.717	0.741	0.742
1.1	0.699	0.762	0.763
1.2	0.640	0.781	0.781

3.6 Summary and Conclusion of this Chapter

The effects of the temperature dependent thermal conductivity and viscous dissipation on free convective flow along a vertical flat plate have been studied in this chapter. From the present investigation the following conclusions may be drawn

- The velocity within the boundary layer increases for decreasing values of Pr and for increasing values of γ and N.
- The temperature within the boundary layer increases for increasing values of γ and N and decreases for increasing values of Pr.
- The local skin friction coefficient decreases for the increasing values of Pr and increases for increasing values of γ and N.
- An increase in the values of γ and N leads to an increase in surface temperature. On the other hand, this decreases for increasing values of Pr.

Chapter 4

4.1 Conclusion

The effects of temperature dependent thermal conductivity on MHD free convection flow along a vertical flat plate have been investigated.

The effect of thermal conductivity variation due to temperature and conduction on MHD free convection flow along a vertical flat plate with heat generation and joule heating have been studied numerically and presented graphically. Also the effects of temperature dependent thermal conductivity and viscous dissipation on free convection flow along a vertical flat plate have been investigated. The coupled effects of natural convection and conduction required that the temperature and the heat flux be continuous at the interface. From the present investigation the following conclusions may be drawn

- The velocity within the boundary layer increases for decreasing values of the magnetic parameter M, the Prandtl number Pr and for increasing values of the thermal conductivity variation parameter γ, the heat generation parameter Q, the joule heating parameter J and the viscous dissipation parameter N.
- The temperature within the boundary layer increases for the increasing values of magnetic parameter M, the thermal conductivity variation parameter γ, the heat generation parameter Q, the joule heating parameter J and the viscous dissipation parameter N and for decreasing values of the Prandtl number Pr.
- The local skin friction coefficient decreases for the increasing values of the magnetic parameter M, and the Prandtl number Pr and increases for increasing values of the thermal conductivity variation parameter γ, the heat generation parameter Q, the joule heating parameter J and the viscous dissipation parameter N.
- An increase of the values of the magnetic parameter M, the thermal conductivity variation parameter γ, the heat generation parameter Q, the joule heating parameter J and the viscous dissipation parameter N leads to an increase in the surface temperature. On the other hand, the surface temperature decreases for the increasing values of the Prandtl number Pr.
- The presence of a magnetic field normal to the flow in an electrically conducting fluid introduces a Lorenz force, which acts against the flow. This resistive force tends to slow down the flow and hence the fluid velocity decreases with the increase of the magnetic parameter. Since there is a friction between magnetic field and fluid

flow produces heat, as a result the temperature profiles increase with the increase of the magnetic parameter and also surface temperature increase with the increase of the magnetic parameter. Since the velocity decreases for the increasing value of magnetic parameter, so skin friction reduces for increasing value of magnetic parameter.

4.2 Extension of this work

The present work can be extended in different ways. Some of those are:

- Temperature dependent thermal conductivity has been considered in the present study. For further extension temperature dependent viscosity of the fluid can be considered.
- Inclusion of Viscous dissipation and Joule heating effects may be another extension.
- The problem can be extended considering the Radiation heat transfer effects.
- Forced convection may be studied with the same geometry.
- Mixed convection may be studied with the same geometry.
- Forced and Mixed convection may be studied with the same geometry.
- Critical behavior of the flow may be studied.

Appendix

Implicit Finite Difference Method

To get the solutions of the transformed governing equations (2.19), (2.20), (3.12) and (3.13) along with the boundary conditions (2.21) and (3.14) we employed implicit finite difference method together with Keller box elimination technique, which is well documented and widely used by Keller (1978) and Cebeci (1984)

To apply the aforementioned method, we first convert equations (2.19), (2.20), (3.12) and (3.13) into the following system of first order equations with dependent variables $u(\xi,\eta), v(\xi,\eta), p(\xi,\eta)$ and $g(\xi,\eta)$ as

$$f' = u,\ u' = v = f'',\ h' = g' = p, h'' = p', f''' = v' \tag{A1}$$

$$v' + p_1 f\, v - p_2 u^2 - p_4 u + g = \xi(u\frac{\partial u}{\partial \xi} - v\frac{\partial f}{\partial \xi}) \tag{A2}$$

$$\frac{1}{\Pr} p' + p_1 f\ p - p_3 u\ g + \frac{p_5}{\Pr} g\ p' + \frac{p_5}{\Pr} p^2 + p_6 v^2 + p_7 u^2 + p_8 g = \xi\ (u\ \frac{\partial g}{\partial \xi} - p\ \frac{\partial f}{\partial \xi}) \tag{A3}$$

where $\xi = x$, $h = g$ and

$$p_1 = \frac{16+15x}{20(1+x)},\ p_2 = \frac{6+5x}{10(1+x)},\ p_3 = \frac{1}{5(1+x)},\ \ p_4 = M\, x^{\frac{2}{5}}\, (1+x)^{\frac{1}{10}}, p_5 = \left(\frac{x}{1+x}\right)^{\frac{1}{5}} \gamma,$$

$$p_6 = Nx, p_7 = Jx^{\frac{7}{5}}(1+x)^{\frac{1}{10}}, p_8 = Qx^{\frac{2}{5}}(1+x)^{\frac{1}{10}}$$

And the boundary conditions are

$$f(\xi,0) = 0,\ \ u(\xi,0) = 0$$

$$p(\xi,0) = \frac{\xi^{\frac{1}{5}}(1+\xi)^{-\frac{1}{5}} g\ \ (\xi,0) - 1}{(1+\xi)^{-\frac{1}{4}} + \gamma \xi^{\frac{1}{5}}(1+\xi)^{-\frac{9}{20}} g(\xi,0)} \tag{A4}$$

$$u(\xi,0) = 0,\ \ g\ \ (\xi,0) = 0$$

Now consider the net rectangle on the *(ξ, η)* plane shown in the figure A1 and denote the net points by

$$\xi^0 = 0,\ \ \xi^n = \xi^{n-1} + k_n,\ \ n = 1,2,\cdots,N$$
$$\eta_0 = 0,\ \ \eta_j = \eta_{j-1} + h_{j,}\quad j = 1,2,\cdots,J$$

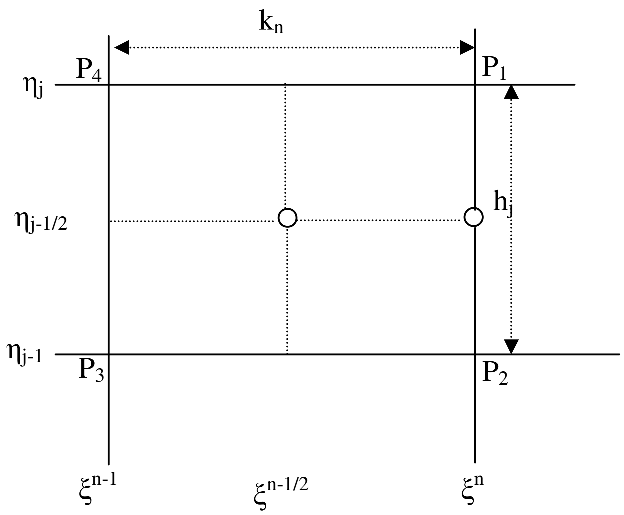

Figure: A-1 Net rectangle for difference approximations for the Box scheme.

here n and j are just sequence of numbers on the (ξ,η) plane, k_n and h_j are the variable mesh widths.

Approximate the quantities f, u, v, p at the points (ξ^n, η_j) of the net by $f_j^n, u_j^n, v_j^n, p_j^n$

which call net function. We also employ the nation P_j^n for the quantities midway between net points shown in Figure: A-1 and for any net function as

$$\xi^{n-1/2} = \frac{1}{2}(\xi^n + \xi^{n-1}) \tag{A5}$$

$$\eta_{j-1/2} = \frac{1}{2}(\eta_j + \eta_{j-1}) \tag{A6}$$

$$g_j^{\ n-1/2} = \frac{1}{2}(g_j^{\ n} + g_j^{\ n-1}) \tag{A7}$$

$$g_{j-1/2}^n = \frac{1}{2}(g_j^n + g_{j-1}^n) \tag{A8}$$

The finite difference approximations according to box method to the three first order ordinary differential equations (A1) are written for the mid point $(\xi^n, \eta_{j-1/2})$ of the segment P_1P_2 shown in the Figure:A-1 and the finite difference approximations to the two first order differential equations (A2) and (A3) are written for the mid-point $(\xi^{n-1/2}, \eta_{j-1/2})$ of the rectangle $P_1P_2P_3P_4$. This procedure yields.

$$\frac{f_j^n - f_{j-1}^n}{h_j} = u_{j-1/2}^n = \frac{u_{j-1}^n + u_j^n}{2} \tag{A9}$$

$$\frac{u_j^n - u_{j-1}^n}{h_j} = v_{j-1/2}^n = \frac{v_{j-1}^n + v_j^n}{2} \tag{A10}$$

$$\frac{g_j^n - g_{j-1}^n}{h_j} = p_{j-1/2}^n = \frac{p_{j-1}^n + p_j^n}{2} \tag{A11}$$

$$\begin{aligned} &\frac{1}{2}\left(\frac{v_j^n - v_{j-1}^n}{h_j} + \frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) + p_1 (fv)_{j-1/2}^{n-1/2} - p_2 \left(u^2\right)_{j-1/2}^{n-1/2} - p_4 (u)_{j-1/2}^{n-1/2} \\ &+ g_{j-1/2}^{n-1/2} = \xi_{j-1/2}^{n-1/2} \left(u_{j-1/2}^{n-1/2} \frac{u_{j-1/2}^n - u_{j-1/2}^{n-1}}{k_n} - v_{j-1/2}^{n-1/2} \frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right) \end{aligned} \tag{A12}$$

$$\begin{aligned} &\frac{1}{2P_r}\left(\frac{p_j^n - p_{j-1}^n}{h_j} + \frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right) + p_1 (fp)_{j-1/2}^{n-1/2} - p_3 (ug)_{j-1/2}^{n-1/2} + \\ &+ \frac{p_5}{2\,\mathrm{Pr}} g_{j-1/2}^{n-1/2} \left(\frac{p_j^n - p_{j-1}^n}{h_j} + \frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right) + \frac{p_5}{\mathrm{Pr}} \left(P^2\right)_{j-1/2}^{n-1/2} \\ &+ p_6 \left(v^2\right)_{j-1/2}^{n-1/2} + p_7 \left(u^2\right)_{j-1/2}^{n-1/2} + p_8 g_{j-1/2}^{n-1/2} \\ &= \xi_{j-1/2}^{n-1/2} \left(u_{j-1/2}^{n-1/2} \frac{g_{j-1/2}^n - g_{j-1/2}^{n-1}}{k_n} - p_{j-1/2}^{n-1/2} \frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right) \end{aligned} \tag{A13}$$

Now from the equation (A12) we have

$$\begin{aligned} &\Rightarrow \frac{1}{2}\left(\frac{v_j^n - v_{j-1}^n}{h_j}\right) + \frac{1}{2}\left(\frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) + \frac{1}{2} p_1 \left\{(fv)_{j-1/2}^n + (fv)_{j-1/2}^{n-1}\right\} \\ &- \frac{1}{2} p_2 \left\{(u^2)_{j-1/2}^n + (u^2)_{j-1/2}^{n-1}\right\} - \frac{1}{2} p_4 \left\{u_{j-1/2}^n + u_{j-1/2}^{n-1}\right\} \\ &+ \frac{1}{2}\left\{g_{j-1/2}^n + g_{j-1/2}^{n-1}\right\} = \frac{1}{2k_n} \xi_{j-1/2}^{n-1/2} (u_{j-1/2}^n + u_{j-1/2}^{n-1})(u_{j-1/2}^n - u_{j-1/2}^{n-1}) \\ &- \frac{1}{2k_n} \xi_{j-1/2}^{n-1/2} (v_{j-1/2}^n + v_{j-1/2}^{n-1})(f_{j-1/2}^n - f_{j-1/2}^{n-1}) \\ &\Rightarrow h_j^{-1} (v_j^n - v_{j-1}^n) + p_1 (fv)_{j-1/2}^n - p_2 (u^2)_{j-1/2}^n - p_4 (u)_{j-1/2}^n + g_{j-1/2}^n \\ &= \alpha_n [\{(u^2)_{j-1/2}^n - (u)_{j-1/2}^n (u)_{j-1/2}^{n-1} + (u)_{j-1/2}^n (u)_{j-1/2}^{n-1}\} \\ &- \{v_{j-1/2}^n f_{j-1/2}^n - v_{j-1/2}^n f_{j-1/2}^{n-1} + f_{j-1/2}^n v_{j-1/2}^{n-1}\}] - [h_j^{-1} (v_j^{n-1} - v_{j-1}^{n-1}) - p_1 (fv)_{j-1/2}^{n-1} - \\ &p_2 (u^2)_{j-1/2}^{n-1} - p_4 (u)_{j-1/2}^{n-1} + g_{j-1/2}^{n-1}] + \alpha_n [-(u^2)_{j-1/2}^{n-1} + (fv)_{j-1/2}^{n-1}] \end{aligned}$$

$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n) + p_1 (fv)_{j-1/2}^n - p_2 (u^2)_{j-1/2}^n - p_4 (u)_{j-1/2}^n + g_{j-1/2}^n$$
$$- \alpha_n \left[\left\{(u^2)_{j-1/2}^n - (fv)_{j-1/2}^n + v_{j-1/2}^n f_{j-1/2}^{n-1} - f_{j-1/2}^n v_{j-1/2}^{n-1}\right\}\right]$$
$$= \alpha_n [(fv)_{j-1/2}^{n-1} + (u^2)_{j-1/2}^{n-1}] - L_{j-1/2}^{n-1}$$

$$where\ L_{j-1/2}^{n-1} = h_j^{-1}(v_j^{n-1} - v_{j-1}^{n-1}) - p_1 (fv)_{j-1/2}^{n-1} - p_2 (u^2)_{j-1/2}^{n-1} - p_4 (u)_{j-1/2}^{n-1} + g_{j-1/2}^{n-1}$$

$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n) + \{p_1 + \alpha_n\}(fv)_{j-1/2}^n - \{p_2 + \alpha_n\}(u^2)_{j-1/2}^n$$
$$- p_4 (u)_{j-1/2}^n + g_{j-1/2}^n + \alpha_n (f_{j-1/2}^n v_{j-1/2}^{n-1} - v_{j-1/2}^n f_{j-1/2}^{n-1})$$
$$= \alpha_n \left\{(fv)_{j-1/2}^{n-1} - (u^2)_{j-1/2}^{n-1}\right\} - L_{j-1/2}^{n-1}$$

$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n) + \{p_1 + \alpha_n\}(fv)_{j-1/2}^n - \{p_2 + \alpha_n\}(u^2)_{j-1/2}^n$$
$$- p_4 (u)_{j-1/2}^n + g_{j-1/2}^n + \alpha_n (f_{j-1/2}^n v_{j-1/2}^{n-1} - v_{j-1/2}^n f_{j-1/2}^{n-1})$$
$$= R_{j-1/2}^{n-1} \tag{A14}$$

$$where\ R_{j-1/2}^{n-1} = \alpha_n \left\{(fv)_{j-1/2}^{n-1} - (u^2)_{j-1/2}^{n-1}\right\} - L_{j-1/2}^{n-1}$$

where $\alpha_n = \dfrac{1}{k_n} \xi_{j-1/2}^{n-1/2}$

Again from the equation (A13) then

$$\Rightarrow \frac{1}{P_r} h_j^{-1}(p_j^n - p_{j-1}^n + p_j^{n-1} - p_{j-1}^{n-1}) + \frac{p_1}{2}[(fp)_{j-1/2}^n + (fp)_{j-1/2}^{n-1}] - \frac{p_3}{2}[(ug)_{j-1/2}^n + (ug)_{j-1/2}^{n-1}]$$
$$+ \frac{p_5}{4\Pr}[(g_{j-1/2}^n + g_{j-1/2}^{n-1})(p_j^n - p_{j-1}^n + p_j^{n-1} - p_{j-1}^{n-1})]h_j^{-1} + \frac{p_5}{2\Pr}[(p^2)_{j-1/2}^n + (p^2)_{j-1/2}^{n-1}]$$
$$= \frac{1}{2k_n} \xi_{j-1/2}^{n-1/2}[(u_{j-1/2}^n + u_{j-1/2}^{n-1})(g_{j-1/2}^n + g_{j-1/2}^{n-1}) - (p_{j-1/2}^n + p_{j-1/2}^{n-1})(f_{j-1/2}^n + f_{j-1/2}^{n-1})]$$
$$\Rightarrow \frac{1}{P_r} h_j^{-1}(p_j^n - p_{j-1}^n) + (p_1 + \alpha_n)(fp)_{j-1/2}^n - (p_3 + \alpha_n)(ug)_{j-1/2}^n$$
$$+ \frac{p_5}{2P_r} h_j^{-1} g_{j-1/2}^n (p_j^n - p_{j-1}^n + p_j^{n-1} - p_{j-1}^{n-1}) + \frac{p_5}{2P_r} h_j^{-1} g_{j-1/2}^{n-1}(p_j^n - p_{j-1}^n)$$
$$+ \frac{p_5}{P_r}(p^2)_{j-1/2}^n + p_6 (v^2)_{j-1/2}^n + p_7 (u^2)_{j-1/2}^n + p_8 g_{j-1/2}^n +$$
$$+ \alpha_n [u_{j-1/2}^n g_{j-1/2}^{n-1} - g_{j-1/2}^n u_{j-1/2}^{n-1} - p_{j-1/2}^n f_{j-1/2}^{n-1} + f_{j-1/2}^n p_{j-1/2}^{n-1}]$$
$$= \alpha_n [(fp)_{j-1/2}^{n-1} - (ug)_{j-1/2}^{n-1}] - M_{j-1/2}^{n-1}$$

$$\Rightarrow \frac{1}{P_r} h_j^{-1} (p_j^n - p_{j-1}^n) + (p_1 + \alpha_n)(fp)_{j-1/2}^n - (p_3 + \alpha_n)(ug)_{j-1/2}^n$$
$$+ \frac{p_5}{2P_r} h_j^{-1} g_{j-1/2}^n (p_j^n - p_{j-1}^n + p_j^{n-1} - p_{j-1}^{n-1}) + \frac{p_5}{2P_r} h_j^{-1} g_{j-1/2}^{n-1} (p_j^n - p_{j-1}^n)$$
$$+ \frac{p_5}{P_r} \left(p^2\right)_{j-\frac{1}{2}}^n + p_6 \left(v^2\right)_{j-1/2}^n + p_7 \left(u^2\right)_{j-1/2}^n + p_8 g_{j-1/2}^n \tag{A15}$$
$$+ \alpha_n [u_{j-1/2}^n g_{j-1/2}^{n-1} - g_{j-1/2}^n u_{j-1/2}^{n-1} - p_{j-1/2}^n f_{j-1/2}^{n-1} + f_{j-1/2}^n p_{j-1/2}^{n-1}]$$
$$= T_{j-1/2}^{n-1}$$

where $T_{j-1/2}^{n-1} = - M_{j-1/2}^{n-1} + \alpha_n \left\{ (fp)_{j-1/2}^{n-1} - (ug)_{j-1/2}^{n-1} \right\}$

$$\Rightarrow \frac{1}{\Pr} \{ P_j{}^i + \delta P_j{}^i - P^i{}_{j-1} - \delta P^i{}_{j-1} \} + \frac{(P_1 + \alpha_n)}{2} \{ (fp)_j^i + \delta (fp)_j^i + (fp)_{j-1}^i + \delta (fp)_{j-1}^i \}$$
$$\frac{(P_3 + \alpha_n)}{2} \{ (ug)_j^i + \delta (ug)_j^i + (ug)_{j-1}^i + \delta (ug)_{j-1}^i \} + \frac{P_5 h^{-1}}{4\Pr} \{ g_j^i + \delta g_{j-1}^i + g_j^i + \delta g_{j-1}^i \}$$
$$\{ P_j^n - P_{j-1}^n + P_j^{n-1} - P_{j-1}^{n-1} \} + \frac{P_5 h^{-1}}{2\Pr} g_{j-\frac{1}{2}}^{n-1} \{ P_j{}^i + \delta P_j{}^i - P_{j-1}{}^i + \delta P_{j-1}{}^i \} +$$
$$\frac{P_5}{2\Pr} \{ (P^2)_j^i + \delta (P^2)_j^i + (P^2)_{j-1}^i + \delta (P^2)_{j-1}^i \} +$$
$$\frac{p_6}{2} \left\{ \left(v^2\right)_j^i + \delta \left(v^2\right)_j^i + \left(v^2\right)_{j-1}^i + \delta \left(v^2\right)_{j-1}^i \right\}$$
$$+ \frac{p_7}{2} \left\{ \left(u^2\right)_j^i + \delta \left(u^2\right)_j^i + \left(u^2\right)_{j-1}^i + \delta \left(u^2\right)_{j-1}^i \right\} + \frac{p_8}{2} \left\{ g_j^i + \delta g_j^i + g_{j-1}^i + \delta g_{j-1}^i \right\}$$
$$+ \frac{\alpha_n}{2} [\{ u_j^i + \delta u_j^i + u_{j-1}^i + \delta u_{j-1}^i \} g_{j-\frac{1}{2}}^{n-1} - \{ g_j^i + \delta g_j^i + g_{j-1}^i + \delta g_{j-1}^i \} u_{j-\frac{1}{2}}^{n-1}$$
$$- (P_j^i + \delta P_j^i + P_{j-1}^i + \delta P_{j-1}^i) f_{j-\frac{1}{2}}^{n-1} + (f_j^i + \delta f_j^i + f_{j-1}^i + \delta f_{j-1}^i) p_{j-\frac{1}{2}}^{n-1}$$
$$= T_{j-\frac{1}{2}}^{n-1}$$

$$\Rightarrow \delta P_j^i[\frac{h_j^{-1}}{\Pr}+\frac{P_1+\alpha_n}{2}f_j^i+\frac{P_5 h_j^{-1}}{2\Pr}g_{j-\frac{1}{2}}^{n-1}+\frac{P_5}{\Pr}P_j^i-\frac{\alpha_n}{2}f_{j-\frac{1}{2}}^{n-1}]$$

$$+\delta P_{j-1}^i[-\frac{h_j^{-1}}{\Pr}+\frac{P_1+\alpha_n}{2}f_{j-1}^i-\frac{P_5 h_j^{-1}}{2\Pr}g_{j-\frac{1}{2}}^{n-1}+\frac{P_5}{\Pr}P_{j-1}^i-\frac{\alpha_n}{2}f_{j-\frac{1}{2}}^{n-1}]$$

$$+\delta f_j^i[\frac{P_1+\alpha_n}{2}p_j^i+\frac{\alpha_n}{2}P_{j-\frac{1}{2}}^{n-1}]$$

$$+\delta f_{j-1}^i[\frac{P_1+\alpha_n}{2}p_{j-1}^i+\frac{\alpha_n}{2}P_{j-\frac{1}{2}}^{n-1}]$$

$$+\delta u_j^i[-\frac{P_3+\alpha_n}{2}g_j^i+\frac{\alpha_n}{2}g_{j-\frac{1}{2}}^{n-1}+\frac{p_7}{2}u_j^i]$$

$$+\delta u_{j-1}^i[-\frac{P_3+\alpha_n}{2}g_{j-1}^i+\frac{\alpha_n}{2}g_{j-\frac{1}{2}}^{n-1}+\frac{p_7}{2}u_{j-1}^i]$$

$$+\delta g_j^i[-\frac{P_3+\alpha_n}{2}u_j^i+\frac{P_5}{4pr}\{P_j^n-P_{j-1}^n+P_j^{n-1}-P_{j-1}^{n-1}\}+\frac{p_8}{2}-\frac{\alpha_n}{2}u_{j-\frac{1}{2}}^{n-1}]$$

$$+\delta g_{j-1}^i[-\frac{P_3+\alpha_n}{2}u_{j-1}^i+\frac{P_5}{4pr}\{P_j^n-P_{j-1}^n+P_j^{n-1}-P_{j-1}^{n-1}\}+\frac{p_8}{2}-\frac{\alpha_n}{2}u_{j-\frac{1}{2}}^{n-1}]$$

$$+\delta v_j^i[p_6]v_j^i+\delta v_{j-1}^i[p_6]v_{j-1}^i$$

$$=T_{j-1/2}^{n-1}-\frac{h_j^{-1}}{pr}\left(p_j^i-p_{j-1}^i\right)-\frac{p_1+\alpha_n}{2}\left\{(fp)_j^i+(fp)_{j-1}^i\right\}+\frac{p_3+\alpha_n}{2}\left\{(ug)_j^i+(ug)_{j-1}^i\right\}$$

$$-\frac{p_5 h_j^{-1}}{4pr}\left(g_j^i+g_{j-1}^i\right)\left(p_j^n-p_{j-1}^n+p_j^{n-1}-p_{j-1}^{n-1}\right)-\frac{p_5 h_j^{-1}}{2pr}\left(p_j^i-p_{j-1}^i\right)g_{j-1/2}^{n-1}$$

$$-\frac{p_5}{2pr}\left\{\left(p^2\right)_j^i+\left(p^2\right)_{j-1}^i\right\}-\frac{\alpha_n}{2}\left(u_j^i+u_{j-1}^i\right)g_{j-1/2}^{n-1}-\frac{p_7}{2}\left\{\left(u^2\right)_j^i+\left(u^2\right)_{j-1}^i\right\}$$

$$-\frac{p_8}{2}\left\{g_j^i+g_{j-1}^i\right\}-\frac{\alpha_n}{2}\left(f_j^i+f_{j-1}^i\right)p_{j-1/2}^{n-1}+\frac{\alpha_n}{2}\left(g_j^i-g_{j-1}^i\right)u_{j-1/2}^{n-1}$$

$$+\frac{\alpha_n}{2}\left(p_j^i+p_{j-1}^i\right)f_{j-1/2}^{n-1}-\frac{p_6}{2}\left\{\left(v^2\right)_j^i+\left(v^2\right)_{j-1}^i\right\}$$

The boundary condition becomes

$$f_0^n=0,\quad u_0^n=0,\ p_0^n(\xi,0)=\xi^{\frac{1}{5}}(1+\xi)^{\frac{-1}{5}}g_0^n-1/(1+\xi)^{\frac{-1}{4}}+\gamma\xi^{1/5}(1+\xi)^{\frac{-9}{20}}g_0^n \tag{A16}$$
$$u_j^n=0,\quad g_j^n=0$$

If assume $f_j^{n-1}, u_j^{n-1}, v_j^{n-1}, g_j^{n-1}, p_j^{n-1}$, to be known for $0 \le j \le J$, equations (A5) to (A15) from a system of $5J+5$ non linear equations for the solutions of the $5J+5$ unknowns $(f_j^n, u_j^n, v_j^n, g_j^n, p_j^n)$, $j = 0,1,2,3,\ldots,J$.

These non-linear systems of algebraic equations are to be non-linearized by Newton's Quassy linearization method. We define the iterates $(f_j^n, u_j^n, v_j^n, g_j^n, p_j^n)$, $i = 0,1, 2, 3,\ldots,N$

With initial values equal those at the previous x-station. For the higher iterates Thus the following form

$$f_j^{(i+1)} = f_j^i + \delta f_j^i \tag{A17}$$

$$u_j^{(i+1)} = u_j^i + \delta u f_j^i \tag{A18}$$

$$v_j^{(i+1)} = v_j^i + \delta v_j^i \tag{A19}$$

$$g_j^{(i+1)} = g_j^i + \delta g_j^i \tag{A20}$$

$$p_j^{(i+1)} = p_j^i + \delta p_j^i \tag{A21}$$

Now by substituting the right hand sides of the above equations in place of f_j^n, u_j^n, v_j^n and g_j^n dropping the terms that are quadratic in $\delta f_j^i, \delta u_j^i, \delta v_j^i$, and δP_j^i then the equations (A9), (A10) and (A14) in the following form

$$f_j^{(i)} + \delta f_j^{(i)} - f_{j-1}^{(i)} - \delta f_{j-1}^{(i)} = \frac{h_j}{2}\left\{u_j^{(i)} + \delta u_j^{(i)} + u_{j-1}^{(i)} + \delta u_{j-1}^{(i)}\right\}$$

$$\delta f_j^{(i)} - \delta f_{j-1}^{(i)} - \frac{h_j}{2}(\delta u_j^{(i)} + \delta u_{j-1}^{(i)}) = (r_1)_j \tag{A22}$$

$$\delta u_j^{(i)} - \delta u_{j-1}^{(i)} - \frac{h_j}{2}(\delta v_j^{(i)} + \delta v_{j-1}^{(i)}) = (r_4)_j \tag{A23}$$

$$\delta g_j^{(i)} - \delta g_{j-1}^{(i)} - \frac{h_j}{2}(\delta p_j^{(i)} + \delta p_{j-1}^{(i)}) = (r_5)_j \tag{A24}$$

where, $(r_1)_j = f_{j-1}^{(i)} - f_j^{(i)} + h_j u_{j-1/2}^{(i)}$

$$(r_4)_j = u_{j-1}^{(i)} - u_j^{(i)} + h_j v_{j-1/2}^{(i)}$$
$$(r_5)_j = g_{j-1}^{(i)} - g_j^{(i)} + h_j p_{j-1/2}^{(i)}$$

$$\Rightarrow \frac{1}{2}\left(\frac{v_j^n - v_{j-1}^n}{h_j} + \frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) + P_1(fv)_{j-\frac{1}{2}}^{n-\frac{1}{2}} - P_2(u^2)_{j-\frac{1}{2}}^{n-\frac{1}{2}} - P_4 u_{j-\frac{1}{2}}^{n-\frac{1}{2}} + g_{j-\frac{1}{2}}^{n-\frac{1}{2}}$$

$$= x_{j-\frac{1}{2}}^{n-\frac{1}{2}}\left[u_{j-\frac{1}{2}}^{n-\frac{1}{2}}\left\{\frac{u_{j-\frac{1}{2}}^{n} - u_{n-\frac{1}{2}}^{n-1}}{k_n}\right\} - v_{j-\frac{1}{2}}^{n-\frac{1}{2}}\left\{\frac{f_{j-\frac{1}{2}}^{n} - f_{n-\frac{1}{2}}^{n-1}}{k_n}\right\}\right]$$

$$\Rightarrow \frac{1}{2}[v_j^n - v_{j-1}^n + v_j^{n-1} - v_{j-1}^{n-1}]h_j^{-1} + P_1\frac{1}{2}[(fv)^n_{j-\frac{1}{2}} + (fv)^{n-1}_{j-\frac{1}{2}}] - \frac{P_2}{2}[(u^2)^n_{j-\frac{1}{2}} + (u^2)^{n-1}_{j-\frac{1}{2}}$$

$$-\frac{P_4}{2}[u^n_{j-\frac{1}{2}} + u^{n-1}_{j-\frac{1}{2}}] + \frac{1}{2}[g^n_{j-\frac{1}{2}} + g^{n-1}_{j-\frac{1}{2}}] = \frac{x^{n-\frac{1}{2}}_{j-\frac{1}{2}}}{2k_n}[\{u^n_{j-\frac{1}{2}} + u^{n-1}_{j-\frac{1}{2}}\}\{u^n_{j-\frac{1}{2}} - u^{n-1}_{j-\frac{1}{2}}\}$$

$$-\{v^n_{j-\frac{1}{2}} + v^{n-1}_{j-\frac{1}{2}}\}\{f^n_{j-\frac{1}{2}} + f^{n-1}_{j-\frac{1}{2}}\}]$$

$$\Rightarrow [v_j^n - v_{j-1}^n]h_j^{-1} + P_1(fv)^n_{j-\frac{1}{2}} - P_2(u^2)^n_{j-\frac{1}{2}} - P_4 u^n_{j-\frac{1}{2}} + g^n_{j-\frac{1}{2}} = \alpha_n[\{(u^2)^n_{j-\frac{1}{2}} - u^n_{j-\frac{1}{2}}u^{n-1}_{j-\frac{1}{2}}$$

$$+u^n_{j-\frac{1}{2}}u^{n-1}_{j-\frac{1}{2}}\} - \{v^n_{j-\frac{1}{2}}f^n_{j-\frac{1}{2}} - v^n_{j-\frac{1}{2}}f^{n-1}_{j-\frac{1}{2}} + v^{n-1}_{j-\frac{1}{2}}f^n_{j-\frac{1}{2}}\}] - [\{v_j^{n-1} - v_{j-1}^{n-1}\}h_j^{-1} - P_1(fv)^{n-1}_{j-\frac{1}{2}}$$

$$-P_2(u^2)^{n-1}_{j-\frac{1}{2}} - P_4 u^{n-1}_{j-\frac{1}{2}} + g^{n-1}_{j-\frac{1}{2}}] + \alpha_n[\{-(u^2)^{n-1}_{j-\frac{1}{2}} + (fv)^{n-1}_{j-\frac{1}{2}}\}]$$

$$\Rightarrow [v_j^n - v_{j-1}^n]h_j^{-1} + P_1(fv)^n_{j-\frac{1}{2}} - P_2(u^2)^n_{j-\frac{1}{2}} - P_4 u^n_{j-\frac{1}{2}} + g^n_{j-\frac{1}{2}} - \alpha_n[\{(u^2)^n_{j-\frac{1}{2}} - (fv)^n_{j-\frac{1}{2}}$$

$$+v^n_{j-\frac{1}{2}}f^{n-1}_{j-\frac{1}{2}} + v^{n-1}_{j-\frac{1}{2}}f^n_{j-\frac{1}{2}}\}] = \alpha_n[\{(u^2)^{n-1}_{j-\frac{1}{2}} - (fv)^{n-1}_{j-\frac{1}{2}}\}] - L^{n-1}_{j-\frac{1}{2}}$$

where $L^{n-1}_{j-\frac{1}{2}} = [v_j^{n-1} - v_{j-1}^{n-1}]h_j^{-1} + P_1(fv)^{n-1}_{j-\frac{1}{2}} - P_2(u^2)^{n-1}_{j-\frac{1}{2}} - P_4 u^{n-1}_{j-\frac{1}{2}} + g^{n-1}_{j-\frac{1}{2}}$

$$\Rightarrow [v_j^n - v_{j-1}^n]h_j^{-1} + (P_1+\alpha_n)(fv)^n_{j-\frac{1}{2}} - (P_2+\alpha_n)(u^2)^n_{j-\frac{1}{2}} - P_4 u^n_{j-\frac{1}{2}} + g^n_{j-\frac{1}{2}}$$

$$+\alpha_n[v^{n-1}_{j-\frac{1}{2}}f^n_{j-\frac{1}{2}} - v^n_{j-\frac{1}{2}}f^{n-1}_{j-\frac{1}{2}}] = \alpha_n[\{-(u^2)^{n-1}_{j-\frac{1}{2}} + (fv)^{n-1}_{j-\frac{1}{2}}\}] - L^{n-1}_{j-\frac{1}{2}}$$

$$\Rightarrow [v_j^n - v_{j-1}^n]h_j^{-1} + \frac{(P_1+\alpha_n)}{2}\{(fv)_j^n + (fv)_{j-1}^n\} - \frac{(P_2+\alpha_n)}{2}\{(u^2)_j^n + (u^2)_{j-1}^n$$

$$-\frac{P_4}{2}\{u_j^n + u_{j-1}^n\} + \frac{1}{2}\{g_j^n + g_{j-1}^n\} + \frac{\alpha_n}{2}[\{f_j^n - f_{j-1}^n\}v^{n-1}_{j-\frac{1}{2}} - \{v_j^n + v_{j-1}^n\}f^{n-1}_{j-\frac{1}{2}}]$$

$$= R^{n-1}_{j-\frac{1}{2}}$$

$$h_j^{-1}\left(v_j^{(i)}+\delta v_j^{(i)}-v_{j-1}^{(i)}-\delta v_{j-1}^{(i)}\right)+\left(\frac{P_1+\alpha_n}{2}\right)\left\{(fv)_j^{(i)}+\delta(fv)_j^{(i)}+(fv)_{j-1}^{(i)}+\delta(fv)_{j-1}^{(i)}\right\}$$
$$-\left(\frac{P_2+\alpha_n}{2}\right)\left\{(u^2)_j^{(i)}+\delta(u^2)_j^{(i)}+(u^2)_{j-1}^{(i)}+\delta(u^2)_{j-1}^{(i)}\right\}-\frac{P_4}{2}\left\{(u)_j^{(i)}+\delta(u)_j^{(i)}+(u)_{j-1}^{(i)}+\delta(u)_{j-1}^{(i)}\right\}$$
$$+\frac{1}{2}\left\{g_j^{(i)}+\delta g_j^{(i)}+g_{j-1}^{(i)}+\delta g_{j-1}^{(i)}\right\}+\frac{\alpha_n}{2}\left\{f_j^{(i)}+\delta f_j^{(i)}+f_{j-1}^{(i)}+\delta f_{j-1}^{(i)}\right\}v_{j-1/2}^{n-1}$$
$$-\frac{\alpha_n}{2}\left(v_j^{(i)}+\delta v_j^{(i)}+v_{j-1}^{(i)}+\delta v_{j-1}^{(i)}\right)f_{j-1/2}^{n-1}=R_{j-1/2}^{n-1}$$
$$\Rightarrow \delta v_j^{(i)}\left[h_j^{-1}+\frac{P_1+\alpha_n}{2}f_j^{(i)}-\frac{\alpha_n}{2}f_{j-1/2}^{n-1}\right]$$
$$+\delta v_{j-1}^{(i)}\left[-h_j^{-1}+\frac{(P_1+\alpha_n)}{2}f_j^{(i)}-\frac{\alpha_n}{2}f_{j-1/2}^{n-1}\right]$$
$$+\delta f_j^{(i)}\left[\frac{(P_1+\alpha_n)}{2}v_j^{(i)}+\frac{\alpha_n}{2}v_{j-1/2}^{n-1}\right]$$
$$+\delta f_{j-1}^{(i)}\left[\frac{(P_1+\alpha_n)}{2}v_{j-1}^{(i)}+\frac{\alpha_n}{2}v_{j-1/2}^{n-1}\right]$$
$$+\delta u_j^{(i)}\left[-(P_1+\alpha_n)u_j^{(i)}-\frac{P_4}{2}\right]$$
$$+\delta u_{j-1}^{(i)}\left[-(P_1+\alpha_n)u_{j-1}^{(i)}-\frac{P_4}{2}\right]$$
$$+\delta g_j^{(i)}[1/2]+\delta g_{j-1}^{(i)}[1/2]$$
$$=(r_2)_j$$

$$\Rightarrow (s_1)_j\,\delta v_j^{(i)}+(s_2)_j\,\delta v_{j-1}^{(i)}+(s_3)_j\,\delta f_j^{(i)}+(s_4)_j\,\delta f_{j-1}^{(i)}+(s_5)_j\,\delta u_j^{(i)}$$
$$+(s_6)_j\,\delta u_{j-1}^{(i)}+(s_7)_j\,\delta g_j^{(i)}+(s_8)_j\,\delta g_{j-1}^{(i)}+(s_9)_j.0+(s_1)_j.0=(r_2)_j \tag{A25}$$

where, $(s_1)_j=h_j^{-1}+\frac{P_1+\alpha_n}{2}f_j^{(i)}-\frac{\alpha_n}{2}f_{j-1/2}^{n-1}$

$$(s_2)_j=-h_j^{-1}+\frac{P_1+\alpha_n}{2}f_j^{(i)}-\frac{\alpha_n}{2}f_{j-1/2}^{n-1}$$

$$(s_3)_j=\frac{(P_1+\alpha_n)}{2}v_j^{(i)}+\frac{\alpha_n}{2}v_{j-1/2}^{n-1}$$

$$(s_4)_j=\frac{(P_1+\alpha_n)}{2}v_{j-1}^{(i)}+\frac{\alpha_n}{2}v_{j-1/2}^{n-1}$$

$$(s_5)_j=-(P_1+\alpha_n)u_j^{(i)}-\frac{P_4}{2}$$

$$(s_6)_j=-(P_1+\alpha_n)\quad u_{j-1}^{(i)}-\frac{P_4}{2}$$

$$(s_7)_j=[1/2]$$

$(s_8)_j = [1/2]$

$(s_9)_j = 0$

$(s_{10})_j = 0$

$$(r_2)_j = R^{n-1}_{j-1/2} - h_j^{-1}\left\{v_j^{(i)} - v_{j-1}^{(i)}\right\} - \frac{(P_1+\alpha_n)}{2}\left\{(fv)_j^{(i)} + (fv)_{j-1}^{i}\right\}$$
$$+\frac{(P_1+\alpha_n)}{2}\left\{(u^2)_j^{(i)} + (u^2)_{j-1}^{(i)}\right\} + \frac{P_4}{2}\left\{u_j^{(i)} + u_{j-1}^{(i)}\right\} - \frac{1}{2}\left(g_j^{(i)} + g_{j-1/2}^{(i)}\right) \quad \text{(A26)}$$
$$-\frac{\alpha_n}{2}\left(f_j^{(i)} + f_{j-1}^{(i)}\right)v^{n-1}_{j-1/2} + \frac{\alpha_n}{2}\left(v_j^{(i)} + v_{j-1}^{(i)}\right)f^{n-1}_{j-1/2}$$

Here the coefficients $(s_9)_j$ and $(s_{10})_j$, which is zero in this case, are included here for the generality.

Similarly by using the equations (A17) to (A21) then the equation (A15) in the following form

$$(t_1)_j\,\delta p_j^{(i)} + (t_2)_j\,\delta p_{j-1}^{(i)} + (t_3)_j\,\delta f_j^{(i)} + (t_4)_j\,\delta f_{j-1}^{(i)} + (t_5)_j\,\delta u_j^{(i)}$$
$$+(t_6)_j\,\delta u_{j-1}^{(i)} + (t_7)_j\,\delta g_j^{(i)} + (t_8)_j\,\delta g_{j-1}^{(i)} + (t_9)_j\,\delta v_j^{(i)} + (t_{10})_j\,\delta v_{j-1}^{(i)} = (r_3)_j \quad \text{(A27)}$$

where, $(t_1)_j = \frac{1}{\Pr}h_j^{-1} + \frac{(p_1+\alpha_n)}{2}f_j^{(i)} + \frac{p_5}{2\Pr}h_j^{-1}g^{n-1}_{j-1/2} + \frac{p_5}{\Pr} - \frac{\alpha_n}{2}f^{n-1}_{j-1/2}$

$(t_2)_j = -\frac{1}{\Pr}h_j^{-1} + \frac{(p_1+\alpha_n)}{2}f_{j-1}^{(i)} - \frac{p_5}{2\Pr}h_j^{-1}g^{n-1}_{j-1/2} + \frac{p_5}{\Pr} - \frac{\alpha_n}{2}f^{n-1}_{j-1/2}$

$(t_3)_j = \frac{(p_1+\alpha_n)}{2}p_j^{(i)} + \frac{\alpha_n}{2}p^{n-1}_{j-1/2}$

$(t_4)_j = \frac{(p_1+\alpha_n)}{2}p_{j-1}^{(i)} + \frac{\alpha_n}{2}p^{n-1}_{j-1/2}$

$(t_5)_j = -\frac{(p_3+\alpha_n)}{2}g_j^{(i)} + \frac{\alpha_n}{2}g^{n-1}_{j-1/2}$

$(t_6)_j = -\frac{(p_3+\alpha_n)}{2}g_{j-1}^{(i)} + \frac{\alpha_n}{2}g^{n-1}_{j-1/2}$

$(t_7)_j = -\frac{(p_3+\alpha_n)}{2}u_j^{(i)} + \frac{p_5}{4\Pr}h_j^{-1}\{P_j^n - P_{j-1}^n + P_j^{n-1} - P_{j-1}^{n-1}\} - \frac{\alpha_n}{2}u^{n-1}_{j-1/2}$

$(t_8)_j = -\frac{(p_3+\alpha_n)}{2}u_{j-1}^{(i)} + \frac{p_5}{4\Pr}h_j^{-1}\{P_j^n - P_{j-1}^n + P_j^{n-1} - P_{j-1}^{n-1}\} - \frac{\alpha_n}{2}u^{n-1}_{j-1/2}$

$(t_9)_j = 0$

$(t_{10})_j = 0$

$$
\begin{aligned}
(r_3)_j =& T^{n-1}_{j-1/2} - \frac{1}{P_r} h_j^{-1}\left(p_j^{(i)} - p_{j-1}^{(i)}\right) - \frac{(P_1+\alpha_n)}{2}\left\{(fp)_j^{(i)} + (fp)_{j-1}^{(i)}\right\} \\
&+ \frac{(P_3+\alpha_n)}{2}\left\{(ug)_j^{(i)} + (ug)_{j-1}^{(i)}\right\} - \frac{P_5}{2\Pr} h_j^{-1}(g_j^{(i)} - g_{j-1}^{(i)})\{P_j^n - P_{j-1}^n + P_j^{n-1} - P_{j-1}^{n-1}\} \\
&- \frac{P_5}{4\Pr} h_j^{-1}(p_j^{(i)} - p_{j-1}^{(i)}) g^{n-1}_{j-1/2} - \frac{P_5}{4\Pr}\left\{(p^2)_j^{(i)} + (p^2)_{j-1}^{(i)}\right\} - \frac{\alpha_n}{2}(u_j^{(i)} + u_{j-1}^{(i)}) g^{n-1}_{j-1/2} \\
&+ \frac{\alpha_n}{2}(g_j^{(i)} - g_{j-1}^{(i)}) u^{n-1}_{j-1/2} + \frac{\alpha_n}{2}(p_j^{(i)} + p_{j-1}^{(i)}) f^{n-1}_{j-1/2} - \frac{\alpha_n}{2}(f_j^{(i)} + f_{j-1}^{(i)}) p^{n-1}_{j-1/2}
\end{aligned}
\tag{A28}
$$

The boundary conditions (A16) becomes

$$
\begin{aligned}
&\delta f_0^n = 0, \quad \delta u_0^n = 0, \quad \delta p_0^n(\xi,0) = \delta\left[\xi^{\frac{1}{5}}(1+\xi)^{\frac{-1}{5}} g_0^n - 1/(1+\xi)^{\frac{-1}{4}} + \gamma\xi^{1/5}(1+\xi)^{\frac{-9}{20}} g_0^n\right] \\
&\delta u_j^n = 0, \quad \delta g_j^n = 0
\end{aligned}
\tag{A29}
$$

Which just express the requirement for the boundary conditions to remain during the iteration process. Now the system of linear equations (A22), (A23), (A24),(A25) and (A27) together with the boundary conditions (A29) can written in a black matrix form a coefficient matrix. The whole procedure, namely reduction to first order followed by central difference approximations, Newton's Quasi linearization method and the block Thomas algorithm, is well known as Keller-box method.

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