Effects of pressure stress work and viscous dissipation in natural convection flows from a horizontal circular cylinder

by

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The thesis entitled “**Effects of pressure stress work and viscous dissipation in natural convection flows from a horizontal circular cylinder”** submitted by **Md Abdus Samad Bhuiyan.** Roll : 100609009P, Registration No.100609009P, Session : October 2006, has been accepted as satisfactory in partial fulfillment of the requirement for the degree of Masters of Philosophy in Mathematics on 12th March 2010.

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Certificate of Research

This is to certify that the work presented in this thesis is carried out by the author under the supervision of **Dr. Md. Mustafa Kamal Chowdhury**, Professor, Department of Mathematics, Bangladesh University of Engineering & Technology, Dhaka. Bangladesh.

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Dedicated to my Beloved Parents

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The Author

Abstract

In this thesis under the title, “Effects of pressure stress work and viscous dissipation in natural convection flows from a horizontal circular cylinder”, we have studied two problems namely effects of pressure stress work and viscous dissipation in natural convection flows from a horizontal circular cylinder and joule heating effects on magnetohydrodynamic (MHD) natural convection flows in presence of pressure stress work and viscous dissipation from a horizontal circular cylinder which belongs to two different chapters.

In chapter two, the steady laminar natural convection flow along the surface of a uniformly heated horizontal circular cylinder, taking into account the effects of viscous dissipation and pressure stress work, has been studied. The results have been obtained by transforming the governing boundary layer equations into a system of non-dimensional equations and by applying implicit finite difference method together with Newton’s linearization approximation. Numerical results for different values of the viscous dissipation parameter, pressure stress work parameter, and Prandtl number have been obtained. The velocity profiles, temperature distributions, skin friction co-efficient and the rate of heat transfer have been presented graphically for the effects of the aforementioned parameters.

In chapter three, the joule heating effects on MHD natural convection flow from a horizontal circular cylinder in the presence of pressure stress work and viscous dissipation has been investigated. The governing boundary layer equations are first transformed into a non-dimensional form and the resulting nonlinear systems of partial differential equations are then solved numerically using finite-difference method. The numerical results of the surface shear stress in terms of skin friction coefficient and the rate of heat transfer, velocity as well as temperature profiles are shown graphically and discussed for a selection of parameters set consisting of joule heating parameter J, magnetic parameter M and the Prandlt number Pr.

Table of Contents

Nomenclature

Symbol	Meaning
Cf_x	local skin friction coefficient
C_p	specific heat
f	dimensionless stream function
g	acceleration due to gravity
Pr	Prandtl number
Gr	local Grashof number
M	magnetic parameter
J	joule heating parameter
Nu_x	local Nusselt number coefficient
T_w	temperature of the surface of cylinder temperature
T_∞	temperature of the ambient fluid
$\bar{u}, \bar{v}$	velocity components
$\bar{x}, \bar{y}$	cartesian coordinates
t_r	temperature ratio
x, y	dimensionless Cartesian coordinates

Greek Symbol

ψ	stream function
ρ	density of the fluid
ν	kinematic viscosity
μ	viscosity of the fluid
θ	dimensionless temperature
λ	viscous dissipation parameter
ε	pressure stress work parameter
β	Co-efficient of thermal expansion
β_0	magnetic field strength

List of Figures

List of Tables

Chapter one

Introduction

1.1: Overview

Fluid dynamics is one of the oldest branches of applied mathematics. It is also the branch in which some of the most significant advances have been made during the last fifty years. These advances have been motivated by exciting development in science and technology and facilitated by growth of computer capabilities and developments of sophisticated mathematical techniques.

An important contribution to the fluid dynamics is the concept of boundary-layer introduced by L. Prandtl. The concept of boundary layer is the consequence of the fact that flows at high Reynolds numbers can be divided into two unequally spaced regions. A very thin layer (called boundary-layer) in the vicinity (of the object) in which the viscous effects dominate, must be taken into account, and for the bulk of the flow region, the viscosity can be neglected and the flow corresponds to the inviscid outer flow.

Although the boundary layer is very thin, it plays a very important role in the fluid dynamics. Boundary-layer theory has become an essential study in analyzing the complex behaviors of real fluids. This concept can be utilized to simplify the Navier-Stokes' equations to such an extent that the viscous effects of flow parameters are evaluated, and these are usable in many practical problems viz. the drag on ships and missiles, the efficiency of compressors and turbines in jet engines, the effectiveness of air intakes for ram and turbojets and so on.

There are three distinct modes of heat transfer, namely conduction, convection and radiation. In reality, the combined effects of these three modes of heat transfer control temperature distribution in a medium. Conduction occurs if energy exchange takes place from the region of high temperature to that of low temperature by the kinetic motion or direct impact of molecules, as in the case of fluid at rest, and by the drift of electrons, as in the case of metals. The radiation energy emitted by a body is transmitted in the space in the form of electromagnetic waves. Energy is emitted from a material due to its temperature level, being larger for a larger temperature, and is then transmitted to another surface, that may be vacuum or a medium, which may absorb, reflect or transmit the radiation depending on the nature and extent of the medium. Considerable effort has been directed to the convective mode of heat transfer. In this mode, relative motion of the fluid provides an additional mechanism for energy transfer. A study of convective heat transfer involves the mechanism of conduction and, sometimes, those of radiation processes as well. This makes the study of convective mode a very complicated one.

The convective mode of heat transfer is divided into two basic processes. If the motion of the fluid arises due to an external agent such as the externally imposed flow of a fluid over a heated object, the process is termed as forced convection. The fluid flow may be the result of a fan, a blower, the wind or the motion of the heated object itself. If the heat transfer to or from a body occurs due to an imposed flow of a fluid at a temperature different from that of the body, problems of forced convection encounters in technology.
If the externally induced flow is provided and flows arising naturally solely due to the effect of the differences in density, caused by temperature or concentration differences in the body force field(such as gravitational field) then these types of flow are called ‘free convection’ or ‘natural convection’ flows. The density difference causes buoyancy effects and these effects act as ‘driving forces’ due to which the flow is generated. Hence free convection is the process of heat transfer, which occurs due to movement of the fluid particles by density differences associated with temperature difference in a fluid.

The viscous dissipation term is always positive and represents a source of heat due to friction between the fluid particles. A variety of expressions are used in the literature for this term like viscous heating, shear stress heating and viscous work. The pressure work is the work that requires pushing fluid into or out of a control volume. When fluid cross a control surface and enters the control volume, it must push’s back the fluid that is already inside the control volume. Since that fluid has a pressure, the entering fluid must do work to move it. For example, rising air expands because as it rises there is less atmospheric pressure compressing it and as it expands becomes cooler. This phenomenon is called adiabatic cooling. The opposite happens when air sinks. This phenomenon is called adiabatic heating. For the pressure work terms the expression adiabatic temperature gradient, adiabatic gradient and adiabatic heating or adiabatic cooling is used. The pressure work term in the energy equation is negative for rising fluid according to the above analysis. The viscous dissipation tends to rise the fluid temperature while the pressure work tends to lower its temperature in the upward flow examined here.

Many natural phenomena and engineering problems are susceptible to viscous dissipation and pressure stress work analysis. It is useful in astrophysics. Geophysicists encounter pressure phenomena in the interactions of conducting fluids fields that are presented in and around heavenly bodies. Engineers employ MHD principles in the design of heat exchangers, pumps and flowmeters, in space vehicle propulsion, control and re-entry, in creating novel power generating systems, and in developing confinement schemes for controlled fusion.

The motion of an electrically conducting fluid, like mercury, under a magnetic field, in general, gives rise to induced electric currents on which mechanical forces are exerted by the magnetic field. On the other hand, the induced electric currents also produce induce magnetic field. Thus there is a two-way interaction between the flow field and the magnetic field; the magnetic field exerts force on the fluid by producing induced currents, and the induced currents change the original magnetic field. Therefore, the hydromagnetic flows (the flows of electrically conducting fluids in the presence of a magnetic field) are more complex than the ordinary hydrodynamic flows. Mathematically also, the hydromagnetic equations have three non-linear terms while in hydrodynamics we have only one. The numbers of governing equations are also increased. The study of hydromagnetic flows is called magnetohydrodynamics (MHD).Two developed branches of physics, namely electromagnetic theory and fluid dynamics interact to produce hydromagnetics and therefore the field of hydromagnetics is much richer than both the parent branches.

Magnetohydrodynamics (MHD) is that branch of continuum mechanics, which deals with the flow of electrically conducting fluids in electric and magnetic fields, probably the largest advance towards understanding of such phenomena comes from the field of Astrophysics. Originally, MHD included only the study of strictly incompressible fluid, but today the terminology is applied to study ionized gases as well. Other names have been suggested, such as magnetofluid-mechanics or magneto-aerodynamics, but original nomenclature has persisted. Many natural phenomena and engineering problems are susceptible to MHD analysis. It is useful in astrophysics.

1.2 Literature Review

The study of viscous dissipation and pressure stress work on natural convection flow is of great importance to the researcher because of their applications in many branches of Science and Engineering. Some of the earlier researchers studied the problem related to the viscous dissipation and pressure stress work on natural convection flow along a both vertical and horizontal flat plate. In almost all natural convection studies, the viscous dissipation and pressure stress terms are neglected in the energy equation. The influence and importance of viscous stress work effects in laminar flows have been examined by Gebhart (1962). But they investigated generally not in a particular case of study. Zakerullah (1972) has been investigated the viscous dissipation and pressure work effects in axisymmetric natural convection flows. Ackroyd (1974) studied the stress work effects in laminar flat plate natural convection flow. Takhar and Soundalgekar (1980) have studied the effects of viscous and Joule heating on the problem posed by Sparrow and Cess (1961), using the series expansion method of Gebhart (1962). Miyamoto *et al.* (1980) has been investigated the effect of axial heat conduction in a vertical flat plate on free convection heat transfer. Joshi and Gebhart (1981) have shown that the effect of pressure stress work and viscous dissipation in some natural convection flows. Pozzi and Lupo (1988) studied the coupling of conduction with laminar natural convection along a flat plate. Effects of pressure stress work and viscous dissipation in natural convection flow along a vertical flat plate with heat conduction has been investigated by Alam *et al.* (2006). Recently, Hye *et al.* (2007) have considered the effects of heat and mass transfer on natural convection flows across an isothermal horizontal circular cylinder with chemical reaction.

Free convection around horizontal cylinders has been extensively investigated, both analytically and numerically. Exclusively attention has so far been paid to viscous dissipation and pressure work noticeable practical interest, as are those relative to cylinders. Free convection heat transfer due to the simultaneous action of buoyancy and induced magnetic forces is very important in some practical problems.

The aforementioned analysis did not consider the effects of pressure stress work and viscous dissipation in natural convection flows from a horizontal circular cylinder. Our present work is to incorporate the idea of natural convection flows from horizontal circular cylinder. It is found that the free convection heat transfer to liquid metals may be significantly affected by the presence of a magnetic field with viscous dissipation and pressure work, but that other fields experience very small effects.

Natural convection flow from a horizontal cylinder due to thermal buoyancy was analyzed by a number of researchers (1961, 1962) under diverse surface boundary conditions (isothermal, uniform heat flux and mixed boundary conditions) using different mathematical technique. The conjugate heat transfer process formed by the interaction between the conduction inside the solid and the convection flow along the solid surface has a significant importance in many practical applications. In fact, conduction within the tube wall is significantly influenced by the convection in the surrounding fluid. Consequently, the conduction in the solid body and the convection in the fluid should have to determine simultaneously. Wilks (1976) studied the MHD free convection about a semi-infinite vertical plate in a strong cross field. He observed that both the velocity profiles and temperature profiles shifted down for increasing value of magnetic parameter and that are rise up for increasing value of Joule heating parameter. Miyamoto (1980) analyzed the effects of axial heat conduction in a vertical flat plate on free convection heat transfer. Miyamoto observed that a mixed-problem study of the natural convection has to be performed for an accurate analysis of the thermo-fluid-dynamic (TFD) field if the convective heat transfer depends strongly on the thermal boundary conditions. Takhar and Soundalgekar (1980) investigated dissipation effects on MHD free convection flow past a semi-infinite vertical plate. Pozzi and Lupo (1988) investigated the entire TFD field resulting from the coupling of natural convection along and conduction inside a heated flat plate by means of two expansions, regular series and asymptotic expansions. Alam *et al.* (1997) studied the viscous and Joule heating effects on MHD free convection flow with variable plate temperature. A considerable amount of research has been done in this field.

In this analysis, we have also investigated the joule heating effects on MHD natural convection flow from an isothermal horizontal circular cylinder in presence of viscous dissipation and pressure work.

Detailed derivations of the governing equations for the flow and the method of solutions along with the results and discussions are presented in the next chapters.

Chapter two

Effects of pressure stress work and viscous dissipation in natural convection flows from a horizontal circular cylinder

2.1 Introduction:

Generally in natural convection studies, the viscous dissipation and pressure stress term are neglected in the energy equation. This is a valid approximation at an ambient temperature of 300K. at 1 atm pressure and at terrestrial gravity, for most gases and low and moderate Prandtl number liquids. However, for high gravity such as in gas turbine blade cooling applications. Where the intensity of the body force may be as large as 10^4g, viscous dissipation and pressure stress effects may affect transport even at small downstream distances from the leading edge. Also, the effects on transport may be quite significant at low temperatures for gases and for high Prandtl number liquids. Now, we shall discuss both the effects of viscous dissipation and pressure stress work.

2.2 Governing equations of the flow:

This formulation assumes steady, two dimensional vertical natural convection flow. Here $\bar{x}$ is taken to be in the directions of the flow that is along the surface of the cylinder. The temperature quiescent ambient fluid T_∞, is taken to be constant for large values of $\bar{y}$. However, in the upward and downward direction of flows, the velocity is considered zero from the leading edge due to natural convection flow. The fluid properties are assumed to be constant, as evaluated at some reference temperature. Viscous dissipation and hydrostatic pressure terms have been incorporated in the energy equation.

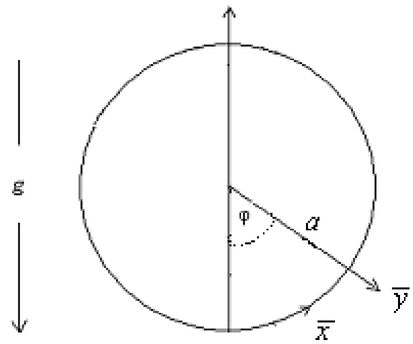

Figure 1: The geometry of the problem

The laminar boundary layer free convection flow is then governed by the following system of equations:

Continuity equation

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0 \tag{2.1}$$

Momentum equation

$$\overline{u}\,\frac{\partial \overline{u}}{\partial \overline{x}} + \overline{v}\,\frac{\partial \overline{u}}{\partial \overline{y}} = \nu\,\frac{\partial^2 \overline{u}}{\partial \overline{y}^2} + \beta g (T - T_\infty)\sin\left(\frac{\overline{x}}{a}\right) \tag{2.2}$$

Energy Equation

$$\overline{u}\,\frac{\partial T}{\partial \overline{x}} + \overline{v}\,\frac{\partial T}{\partial \overline{y}} = \frac{\kappa}{\rho C_P}\frac{\partial^2 T}{\partial \overline{y}^2} + \frac{\mu}{\rho C_P}\left(\frac{\partial \overline{u}}{\partial \overline{y}}\right)^2 + \frac{T\beta}{\rho C_P}\overline{u}\,\frac{\partial P}{\partial \overline{x}} \tag{2.3}$$

The appropriate boundary conditions are

$$\left.\begin{array}{l} \overline{\mathrm{u}} = 0, \overline{\mathrm{v}} = 0, \mathrm{T} = \mathrm{T}_{\mathrm{w}} \;\; at\, \overline{y} = 0 \\ \overline{u} \to 0,\, T \to T_\infty \; as\, \overline{y} \to \infty \end{array}\right\} \tag{2.4}$$

2.3 Transformation of the governing equations:

To solve the equations (2.2) – (2.3) subject to the boundary conditions (2.4), the following transformations for the governing equations are

$$Gr = \frac{G\beta a^3 (T_w - T_\infty)}{\nu^2}, x = \frac{\overline{x}}{a},\; y = Gr^{\frac{1}{4}}\,\frac{\overline{y}}{a},\; u = \frac{a}{\nu}Gr^{\frac{-1}{2}}\overline{u},\, v = \frac{a}{\nu}Gr^{\frac{-1}{4}}\overline{v},\, \theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{2.5}$$

The non dimensional form of the equations (2.1)-(2.3) is as follows:

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.5}$$

Momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}+\theta\sin x \tag{2.6}$$

Energy Equation

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\Pr}\frac{\partial^2 \theta}{\partial y^2}+\varepsilon\left(\frac{\partial u}{\partial y}\right)^2-\frac{u\beta g}{C_P}\left\{\frac{T_\infty+\theta(T_W-T_\infty)}{T_W-T_\infty}\right\} \tag{2.7}$$

The appropriate boundary conditions are

$$\left.\begin{aligned} &\mathrm{u}=0, \mathrm{v}=0,\ \theta=1\ at\ y=0\\ &u\to 0,\ \ \theta\to 0\quad as\ y\to\infty \end{aligned}\right\} \tag{2.8}$$

To solve equation (2.6)-(2.7), subject to the boundary condition (2.8), we assume following transformations

$$\psi = x\,f\left(x,y\right),\ \theta=\theta(x,y) \tag{2.9}$$

Where ψ be the stream function usually defined as

$$u=\partial\psi/\partial y\ \ v=-\partial\psi/\partial x \tag{2.10}$$

Substituting (2.10) into the equations (2.6)-(2.7), the new forms of the dimensionless equations (2.6) and (2.7) are

$$f'''+ff'-f'^2+\theta\frac{\sin x}{x}=x\left[\frac{\partial^2 f}{\partial x\partial y}f'-\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}\right] \tag{2.11}$$

$$\frac{1}{\Pr}\theta''+f\theta'+x^2\lambda f''^2+x\varepsilon f'\left[\frac{T_\infty}{T_W-T_\infty}\right]+\varepsilon x\theta f'=x\left[\frac{\partial\theta}{\partial x}f'-\frac{\partial f}{\partial x}\theta'\right] \tag{2.12}$$

The corresponding boundary conditions reduced the form

$$\left.\begin{aligned} &f(x,0)=f'(x.0)=0,\theta=1\qquad at\qquad y=0\\ &f'(x,\infty)\to 0,\ \theta'(x,\infty)\to 0\qquad as\qquad y\to\infty \end{aligned}\right\} \tag{2.13}$$

In the above equations the primes denote differentiation with respect to y.
Here,
Parameters:

Viscous dissipation parameter $: \lambda = \dfrac{\nu^2 G_r}{C_p a^2 (T_w - T_\infty)}$

Pressure stress parameter $: \varepsilon = \dfrac{g\beta a}{C_p}$

Prandtl number $: Pr = \dfrac{\mu c_p}{\kappa}$

2.4 Results and discussion

In the present problem we have investigated the solutions of the nonsimilar boundary layer equations governing the laminar free convective flow across an isothermal horizontal cylinder with the effects of pressure stress work and viscous dissipation.

Numerical values for the velocity profiles, temperature profiles, local skin friction coefficients and the rate of heat transfer for selected values of viscous dissipation parameter λ (=0.1, 0.3, 0.5, 0.9) for the fluids having Prandtl number *Pr* (=0.72, 0.1, 1.44, 1.74) with Pressure stress work parameter ε (=0.1, 0.3, 0.5, 0.9) are obtained. Here the total surface of the cylinder is isothermal. The numerical solutions start at the lower stagnation point of the cylinder i.e at $x \approx 0.0$, and proceed round the cylinder up to the upper stagnation point $x \approx \pi$. The values of the Prandtl number *Pr* are taken to 0.72 that corresponds physically the air and 1.0 corresponding to electrolyte solutions such as salt water and 1.44, 1.72 are water and Glycerin respectively have been used theoretically.

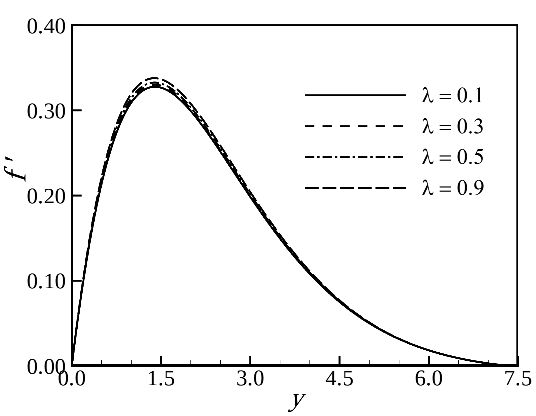

Fig. 2(a). Variation of velocity profiles against *y* for varying of viscous dissipation parameter λ with *Pr*=0.72, and $\varepsilon = 0.5$

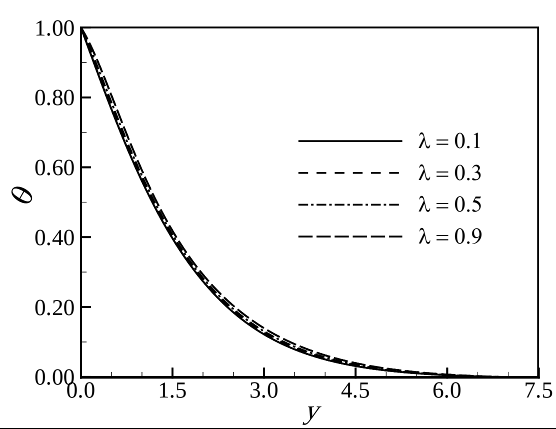

Fig. 2(b) Variation of temperature profiles against *y* for varying of viscous dissipation parameter λ with *Pr*=0.72, and $\varepsilon = 0.5$

From Fig. 2(a), it is observed that velocity increases as the values viscous dissipation parameter λ increases. Near the stagnation point velocity increases significantly along y and becomes maximum and then decreases slowly and finally approaches to zero, the asymptotic value. The maximum values of the velocities are 0.32764966, 0.33015290, 0.33267013 and 0.33774648 for λ=0.1, 0.3, 0.5 and 0.9 respectively which occurs at y = 1.40347467. Here it is observed that the velocity increases by 3.08% as λ increases 0.1 to 0.9. From Fig.2 (b), it is seen that when the values of viscous dissipation parameter λ increases, the temperature also increases.

Figs. 3(a) and 3(b) display results for the velocity and temperature profiles for different values of pressure work parameter ε ($\varepsilon = 0.1$, 0.3 0.5 0.9) plotted against y having Prandtl number *Pr*=0.72, $\lambda = 0.5$. It is observed that, as the pressure work parameter ε increases, the velocity profiles decreases between $0 \le y \le 7.5$ and then increases with very small difference and finally approaches to zero along *y* direction. The temperature profile increases with increasing pressure work parameter ε. The maximum values of the velocity are recorded as 0.34949139, 0.33814515, 0.32764966 and 0.30893520 for $\varepsilon = 0.1$, 0.3 0.5 and 0.9, respectively which occurs at *y* =1.40. It is found that the velocity decreases by 11.60% as the pressure work parameter ε increases from 0.1 to 0.9.

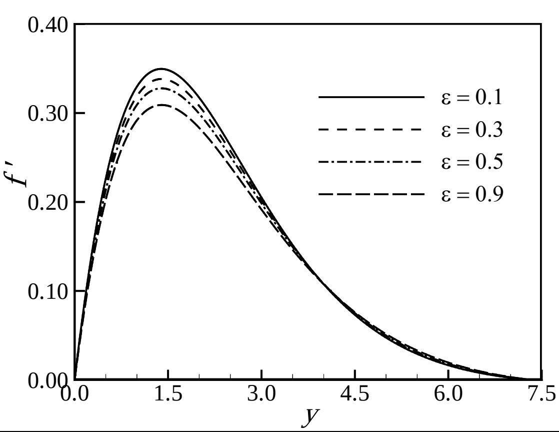

Fig. 3(a). Variation of velocity profiles against *y* for varying of pressure stress work parameter ε with *Pr*=0.72, and $\lambda = 0.5$

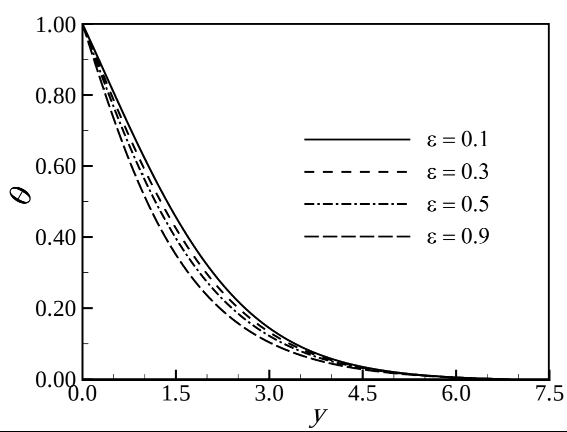

Fig. 3(b). Variation of temperature profiles against *y* for varying of pressure stress work parameter ε with *Pr*=0.72, and $\lambda = 0.5$

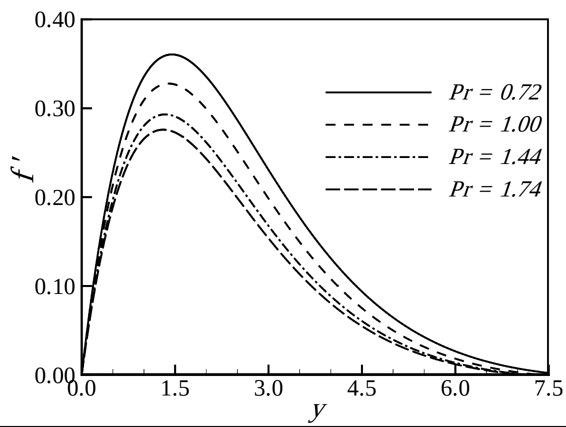

Fig. 4(a). Variation of velocity profiles against *y* for varying of Prandtl number *Pr* with $\lambda = 0.5$ and $\varepsilon = 0.5$

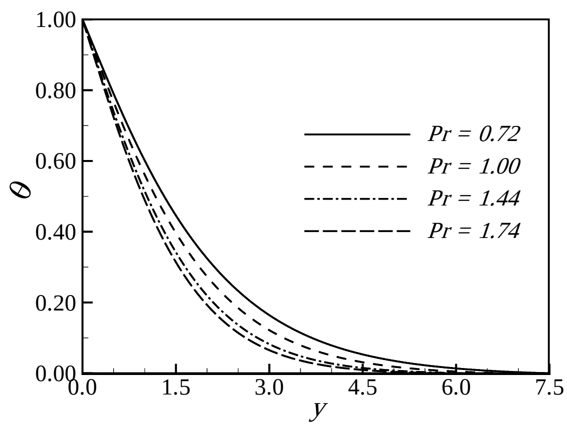

Fig. 4(b). Variation of temperature profiles against *y* for varying of Prandtl number *Pr* with $\lambda = 0.5$ and $\varepsilon = 0.5$

Figs. 4(a) and 4(b) indicate the effects of the Prandtl number *Pr* with $\lambda = 0.5$ and $\varepsilon = 0.5$ on the velocity profiles and the temperature profiles. From Fig. 4(a) it is observed that the increasing values of Prandtl number *Pr* leads to the decrease in the velocity profiles. The maximum values of the velocity are 0.360475529, 0.32498678, 0.29275551 and 0.27574108 for *Pr* = 0.72, 1.00, 1.44 and 1.74 respectively which occur at *y* = 1.45, 1.56, 1.38, 1.30 for first, second, third and forth maximum value respectively. Here it is depicted that the velocity decreases by 23.50% as *Pr* increases from 0.72 to 1.44. Again from Fig. 4(b) it is observed that the temperature profiles decreases with the increasing values of Prandtl number.

It can easily be seen that the effect of viscous dissipation parameter λ leads to an increase in the local skin friction coefficient Cf_x and a decrease in the rate of heat transfer Nu_x which are shown in fig. 5(a) and 5(b) respectively. This phenomenon can easily be understood from the fact that the viscous dissipation parameter slightly increases the skin friction in presence of stress work parameter. Owing to increasing the viscous dissipation parameter in presence of stress work parameter, the fluid temperature within the boundary layer slightly increases and the associated thermal boundary layer becomes thicker. For increasing fluid temperature, the temperature difference between fluid and surface decreases and the corresponding rate of heat transfer decreases.

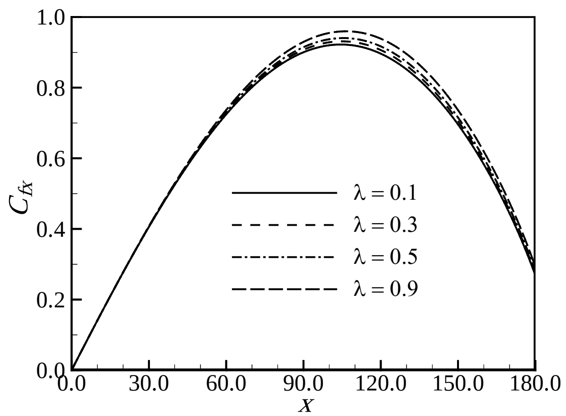

Fig.5 (a). Variation of skin friction against x for varying of viscous dissipation parameter λ with *Pr*=0.72, and $\varepsilon = 0.5$

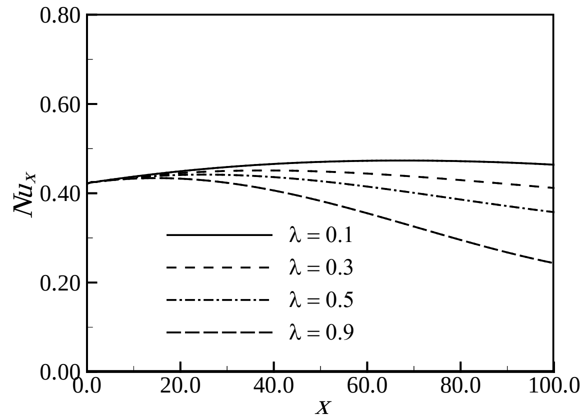

Fig.5 (b). Variation of heat transfer against x for varying of viscous dissipation parameter λ with *Pr*=0.72, and $\varepsilon = 0.5$

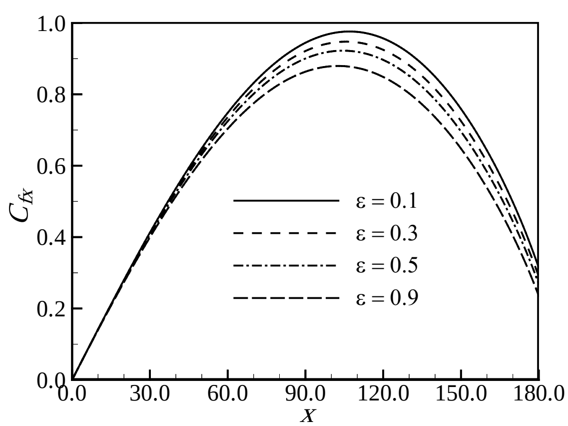

Fig. 6(a). Variation of skin friction against *x* for varying of pressure stress work parameter ε with *Pr*=0.72, and $\lambda = 0.5$

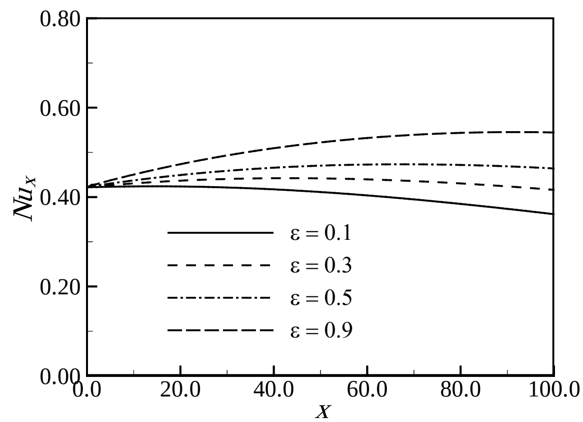

Fig. 6(b). Variation of heat transfer against *x* for varying of pressure stress work parameter ε with *Pr*=0.72, and $\lambda = 0.5$

The variation of reduced local skin friction coefficient and the local rate of heat transfer for different values of stress work parameter ε ($\varepsilon = 0.1$, 0.3 0.5 0.9) are illustrated in Figs. 6(a) and 6(b) while Pr = 0.72 and $\lambda = 0.5$. From the fig.6 (a) it can be easily seen that the stress work parameter leads to a decrease in the local skin friction coefficient Cf_x and an increase in the local Nusselt number Nu_x. This are expected, since the stress work mechanism decreases fluid temperature, the temperature difference between fluid and surface increases and the regarding rate of heat transfer increases. In order to decreasing temperature, the viscosity of the fluid decreases and the corresponding local skin friction Cf_x decreases by 9.90%.

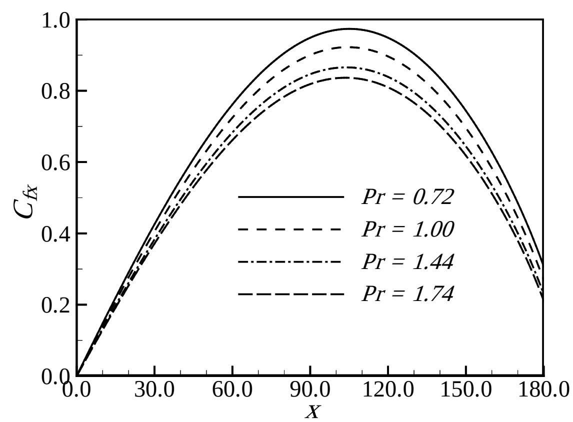

Fig. 7(a). Variation of skin friction against *x* for varying of Prandtl number *Pr* with $\lambda = 0.5$ and $\varepsilon = 0.5$

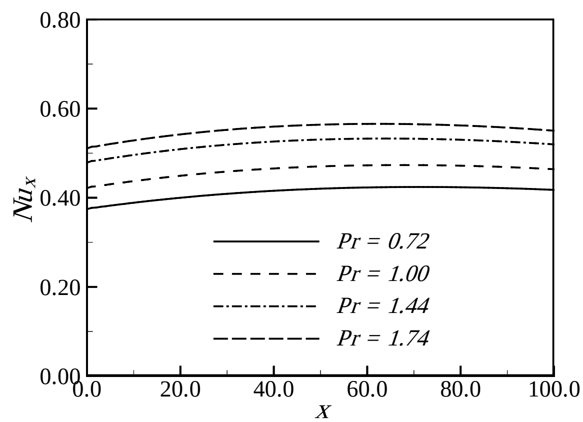

Fig. 7(b). Variation of heat transfer against *x* for varying of Prandtl number *Pr* with $\lambda = 0.5$ and $\varepsilon = 0.5$

The local skin friction coefficient and the local rate of heat transfer for different values of Prandtl nimber *Pr* (*Pr* = 0.72, 1.00, 1.44 and 1.74) are illustrated in Figs.7(a) and 7(b) while $\varepsilon = 0.5$ and $\lambda = 0.5$. From these two figures it can be easily seen that for the higher Prandtl number local skin friction coefficients Cf_x decrease and there is an increase in the local Nusselt number Nu_x. This is expected because the higher Prandtl number has a lower skin friction.

In order to verify the accuracy of the present work, the numerical values of skin friction coefficient Cf_x for $\lambda = 0.0$, $\varepsilon = 0.0$ and *Pr* = 1.00 in different position of x are compared with those reported by Merkin (1976), Nazar et al.(2002) and Hye et al. (2007) as presented in table 2. 1. The results are found to be in excellent agreement.

Table 2.1:

Compare the numerical values of Cf_x for different values of *x* while *Pr*=1.0 $\lambda = 0.0$ and $\varepsilon = 0.0$

x	Merkin (1976)	Nazar et al.(2002)	Hye et al. (2007)	Present(2010)
0.0	0.0000	0.0000	0.0000	0.0000
$\pi/6$	0.4151	0.4148	0.4145	0.4139
$\pi/3$	0.7558	0.7542	0.7539	0.7528
$\pi/2$	0.9579	0.9545	0.9541	0.9526
$2\pi/3$	0.9756	0.9698	0.9696	0.9678
$5\pi/6$	0.7822	0.7740	0.7739	0.7718
π	0.3391	0.3265	0.3264	0.3239

2.5 Conclusion

The effects of pressure stress work and viscous dissipation in natural convection flows from a horizontal circular cylinder has been investigated numerically. The governing boundary layer equations of motions are transformed into a non-dimensional form and the resulting non-linear systems of partial differential equations are reduced to local non-similarity boundary layer equations, which are solved numerically by using implicit finite difference method together with the Keller- box scheme. From the present investigation the following conclusions may be drawn:

- With effect of viscous dissipation parameter λ the local skin friction coefficient Cf_x slightly increases and the rate of heat transfer Nu_x decreases.
- An increase in values of viscous dissipation parameter λ slightly increases velocity profiles and temperature distributions.
- For increasing values of stress work parameter ε the skin friction coefficient decreases but Nusselt number increases.
- With the effect of stress work parameter both the velocity and temperature distributions decreases significantly the thickness of the thermal boundary layer.
- An increasing value of Prandtl number Pr leads to decrease in the velocity and the temperature distributions.

Chapter three

Joule heating effects on MHD natural convection flows in presence of pressure stress work and viscous dissipation from a horizontal circular cylinder

3.1 Introduction

In the traditional area of thesis research, it has been seen that heat transfer in a fluid flow with the viscous dissipation and pressure stress work assumed in a vertical plate. In this case it is usual to prescribe with the viscous dissipation and pressure stress work a considerable amount of research has done in order to understand the heat transfer characteristics and temperature profile over a wide range of flow configurations and fluid properties. However in many real engineering systems with the viscous dissipation cannot be neglected and able to significantly affect the fluid flow and the heat transfer characteristics of the fluid in the surface of the wall. In order to take account of physical reality, there has been a tendency to move away from considering mathematical problems in which the surface is considered to be isothermal. Thus the velocity and temperature profile in the fluid should be determined simultaneously.

In this thesis, the MHD-conjugate free convection flow from an isothermal horizontal circular cylinder with Joule heating effects in presence of viscous dissipation and pressure stress work has been investigated. The governing boundary layer equations are transformed into a non dimensional form and the resulting non linear partial differential equations are solved numerically using the implicit finite difference method together with the Keller box technique. The temperature distributions, velocity profiles, skin friction coefficients and the heat transfer rates are presented graphically.

3.2 Governing equations of the flow:

Let us consider a steady natural convection flow of a viscous incompressible fluid from an isothermal horizontal circular cylinder of radius a placed in a fluid of uniform temperature. A uniform magnetic field having strength B_0 is acting normal to the cylinder surface. The effects of pressure stress work, viscous dissipation and joule heating in the flow region and conduction from surface considered in the present study. Under the balance laws of mass, momentum and energy and with the help of Boussinesq approximation for the body force term in the momentum equation, the equations governing this boundary-layer natural convection flow can be written as:

Continuity equation

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{3.1}$$

Momentum equation

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \nu \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + g\beta(T_w - T_\infty)\sin\left(\frac{\bar{x}}{a}\right) - \frac{\sigma B_0^{\,2}\bar{u}}{\rho} \tag{3.2}$$

Energy Equation

$$\bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \frac{K}{\rho C_P}\frac{\partial^2 T}{\partial \bar{y}^2} + \frac{\mu}{\rho C_P}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 + \frac{T\beta}{\rho C_P}\bar{u}\frac{\partial P}{\partial \bar{x}} + \frac{\sigma B_0^{\,2}\bar{u}}{\rho} \tag{3.3}$$

The appropriate boundary conditions are

$$\left.\begin{aligned} &\bar{u} = 0, \bar{v} = 0, T = T_w \ \ at\ \bar{y} = 0 \\ &\bar{u} \to 0,\ T \to T_\infty \ as\ \bar{y} \to \infty \end{aligned}\right\} \tag{3.4}$$

3.3 Transformation of the governing equations

To solve the equations (3.2) – (3.3) subject to the boundary conditions (3.4), the following transformations for the governing equations are,

$$Gr = \frac{G\beta a^3(T_w - T_\infty)}{\nu^2}, x = \frac{\bar{x}}{a},\ y = Gr^{\frac{1}{4}}\frac{\bar{y}}{a},\ u = \frac{a}{\nu}Gr^{\frac{-1}{2}}\bar{u},\ v = \frac{a}{\nu}Gr^{\frac{-1}{4}}\bar{v},\ \theta = \frac{T - T_\infty}{T_w - T_\infty}$$

The non dimensional form of the equations (3.1)-(3.3) is as follows:

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.5}$$

Momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + Mu = \frac{\partial^2 u}{\partial y^2} + \theta \sin x \tag{3.6}$$

Energy Equation

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\Pr}\frac{\partial^2 \theta}{\partial y^2} + \varepsilon\left(\frac{\partial u}{\partial y}\right)^2 + \frac{u\beta g}{C_P}\left\{\frac{T_\infty + \theta(T_W - T_\infty)}{T_W - T_\infty}\right\} + Ju^2 \tag{3.7}$$

The appropriate boundary conditions are

$$\left.\begin{array}{l} u = 0, v = 0,\ \theta = 1\ at\ y = 0 \\ u \to 0,\ \ \theta \to 0 \quad as\ y \to \infty \end{array}\right\} \tag{3.8}$$

Where $M = (\sigma a^2 B_0{}^2)/(\nu \rho Gr^{1/2})$ is the magnetic parameter, $J = (\sigma \nu B_0{}^2 Gr^{1/2})/\{\rho c_p (T_w - T_\infty)\}$ is the joule heating parameter and $Pr = \mu c_p / \kappa$ is the Prandtl number.

To solve equation (3.5)-(3.7), subject to the boundary condition (3.8), we assume following transformations

$$\psi = x\, f(x, y),\ \theta = \theta(x, y) \tag{3.9}$$

Where ψ is the stream function usually defined as

$$u = \partial\psi / \partial y\ \ v = -\partial\psi / \partial x \tag{3.10}$$

Substituting (3.10) into the equations (3.5)-(3.7), the new forms of the dimensionless equations (3.6) and (3.7) are

$$f''' + ff'' - f'^2 - Mf' + \theta \frac{\sin x}{x} = x\left(f' \frac{\partial f'}{\partial x} - f'' \frac{\partial f}{\partial x} \right) \tag{3.11}$$

$$\frac{1}{\Pr}\theta'' + f\theta' + \lambda x^2 f''^2 + \varepsilon f'\left[\frac{T_\infty}{T_W - T_\infty} \right] + \varepsilon \theta f' + Jx^2 f'^2 = x\left(f' \frac{\partial \theta}{\partial x} - \theta' \frac{\partial f}{\partial x} \right) \tag{3.12}$$

In the above equations primes denote differentiation with respect to *y*.

The corresponding boundary conditions reduced the form

$$\left.\begin{array}{lll} f(x,0) = f'(x.0) = 0, \theta = 1 & at & y = 0 \\ f'(x,\infty) \to 0,\ \theta'(x,\infty) \to 0 & as & y \to \infty \end{array}\right\} \tag{2.13}$$

Parameters:

Viscous dissipation parameter $: \lambda = \dfrac{\nu^2 G_r}{C_p a^2 (T_w - T_\infty)}$

Pressure stress parameter $: \varepsilon = \dfrac{g\beta a}{C_p}$

Prandtl number $: \Pr = \dfrac{\upsilon C_p}{\kappa}$

Magnetic parameter $: M = \dfrac{\sigma a^2 B_0{}^2}{\nu \rho Gr^{1/2}}$

Joule heating parameter $: J = \dfrac{\sigma \nu B_0{}^2 Gr^{1/2}}{\rho c_p (T_w - T_\infty)}$

3.4 Method of Solution:

To get the solutions of the parabolic differential equations (3.11) and (3.12) along with the boundary condition (3.13), we shall employ implicit finite difference method together with Keller- box elimination technique.

3.5 Results and discussion

Joule heating effects on magnetohydrodynamic natural convection flow in presence of pressure stress work and viscous dissipation from a horizontal cylinder has been investigated. The velocity profiles, temperature distributions, local skin-friction and the local rate of heat transfer obtained by the finite difference method for various values of the governing parameters. The aims of the figures are to display how the profiles vary with *x*, the scaled stream wise coordinate. From Fig. 8(a), it is observed that the velocity increases as the values of the Joule heating parameter *J* increase. The velocity increases significantly along y and becomes maximum and then decreases slowly and finally approaches to zero, the asymptotic value. The maximum values of the velocity are 0.32931258, 0.33958319 and 0.36033431 for *J* = 0.1, 0.3, 0.5 and 0.9 respectively which occur at y= 1.80 for first, second maximum values, at y = 1.45 for third and fourth maximum values. Here it is observed that the velocity increase by 15.23% as *J* increases from 0.1 to 0.9. From Fig. 8(b), it is seen that when the values of joule heating parameter *J* increase, the temperature also increases.

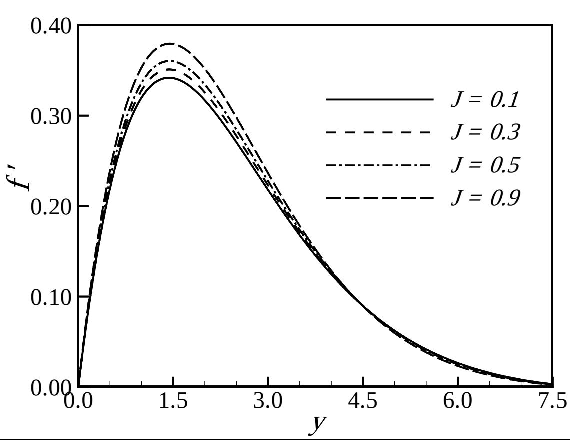

Fig.8 (a). Variation of velocity profile against *y* for varying of *J* with *M* = 0.1, $\lambda = 0.5$, $\varepsilon = 0.5$ and *Pr* =1.0.

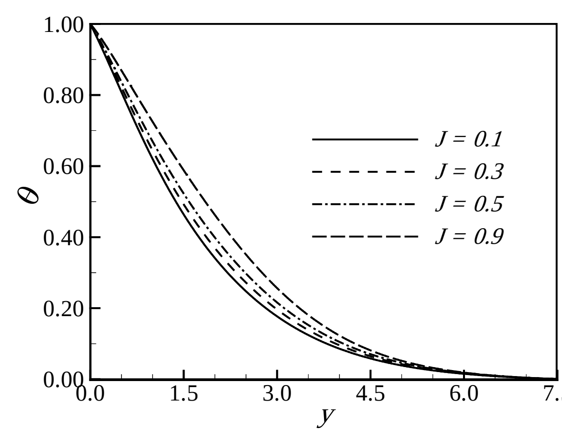

Fig.8 (b). Variation of temperature against *y* for varying of *J* with *M* = 0.1, $\lambda = 0.5$, $\varepsilon = 0.5$ and *Pr*=1.0.

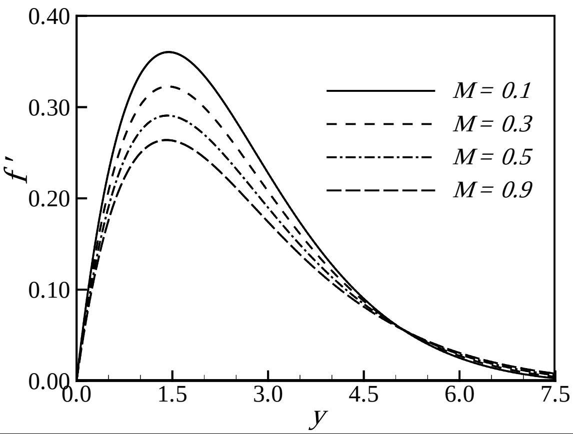

Fig.9 (a). Variation of velocity profile against *y* for varying of *M* with *J* = 0.1, $\lambda = 0.5$, $\varepsilon = 0.5$ and *Pr*=1.0.

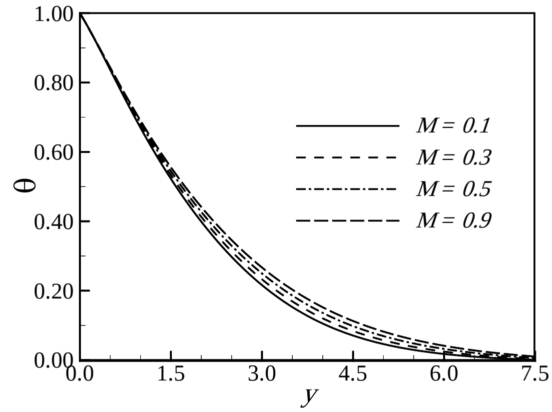

Fig.9 (b). Variation of temperature against *y* for varying of *M* with *J* = 0.1, $\lambda = 0.5$, $\varepsilon = 0.5$ and *Pr*=1.0.

Figs. 9(a) and 9(b) display results for the velocity and temperature profiles for different values of magnetic parameter *M* (*M* = 0.1, 0.3, 0.5, 0.9) having Prandtl number *Pr* = 1.0, *J* = 0.1 $\lambda = 0.5$, $\varepsilon = 0.5$. It is observed that, as the magnetic parameter *M* increases, the velocity profile decreases between $0 \le y \le 5$ and then increases with very small difference and finally approaches to zero along y direction. The temperature profile increases with increasing magnetic parameter *M*. The maximum values of the velocity are recorded as 0.36033401, 0.32258730, 0.29037867 and 0.263928080 for *M* = 0.1, 0.3, 0.5 and 0.9 respectively which occur at y = 1.43 for 1st, 2nd, 3rd and 4th maximum values. It is found that the velocity decreases by 26.75% as the magnetic parameter *M* increases from 0.1 to 0.9.

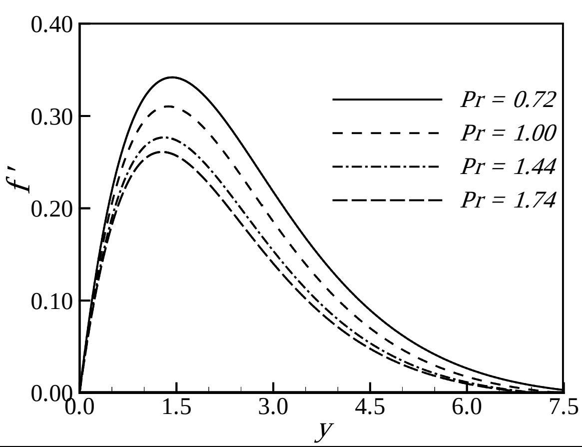

Fig.10 (a). Variation of velocity profile against *y* for varying of *Pr* with *M*= 0.1, $\lambda = 0.5$, $\varepsilon = 0.5$ and *J*=0.1.

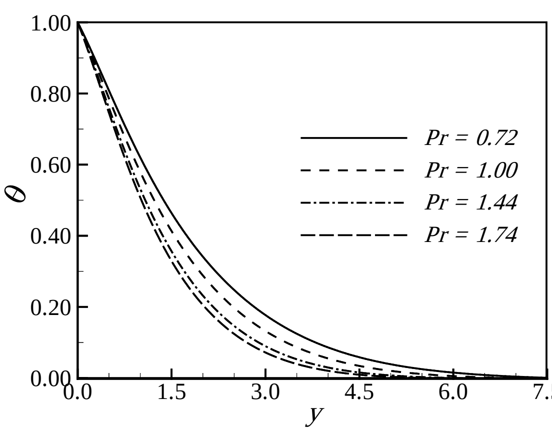

Fig.10 (b). Variation of temperature profile against *y* for varying of *Pr* with *M* =0.1, $\lambda = 0.5$, $\varepsilon = 0.5$ and *J*=0.1.

Figs. 10(a) and 10(b) indicate the effects of the Prandtl number *Pr* with M = 1.0, $\lambda = 0.5$ J = 0.1 and $\varepsilon = 0.5$ on the velocity profiles and the temperature profiles. From Fig. 10(a) it is observed that the increasing values of Prandtl number *Pr* leads to the decrease in the velocity profiles. The maximum values of the velocity are 0.34181852, 0.31031986, 0.27675142 and 0.26104378 for Pr = 0.72, 1.0, 1.44 and 1.74 respectively which occur at y = 1.43, y = 1.36, y = 1.30 and y = 1.26 for the first, second, third and forth maximum value. Here it is depicted that the velocity decreases by 23.63% as *Pr* increases from 0.72 to 1.74. Again from Fig. 10(b) it is observed that the temperature profiles decreases with the increasing values of Prandtl number *Pr*.

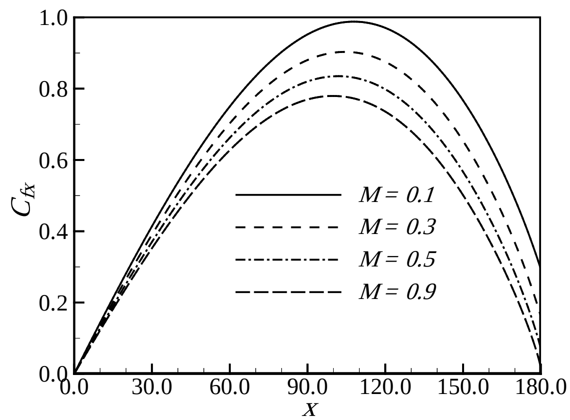

Fig. 11(a). Variation of skin friction against *x* for varying of *M* with J = 0.1, $\lambda = 0.5$, $\varepsilon = 0.5$ and *Pr*=1.0.

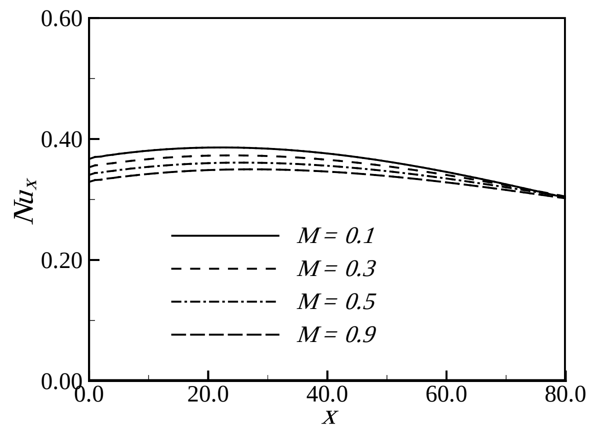

Fig.11 (b). Variation of heat transfer against *x* for varying of *M* with *J*= 0.1, $\lambda = 0.5$, $\varepsilon = 0.5$ and *Pr*=1.0.

It can easily be seen that the effect of the magnetic parameter *M* leads to a decrease in the local skin friction coefficient Cf_x and the local Nusselt number Nu_x in Fig. 11(a) and 11(b). This phenomenon can easily be understood from the fact that the magnetic parameter *M* opposes the flow, therefore decreases the velocity gradient and hence the local skin friction coefficient Cf_x decreases. Owing to increasing values of *M* in the presence of viscous dissipation and pressure stress work, the fluid temperature with in the boundary layer increases and the associate thermal boundary layer becomes thicker. For increasing fluid temperature, the temperature difference between fluid and surface decreases and the corresponding rate of heat transfer decreases.

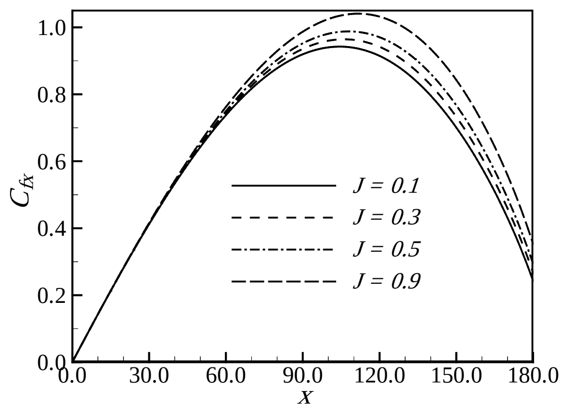

Fig.12 (a). Variation of skin friction against *x* for varying of *J* with *M* = 0.1, $\lambda = 0.5$, $\varepsilon = 0.5$ and *Pr*=1.0

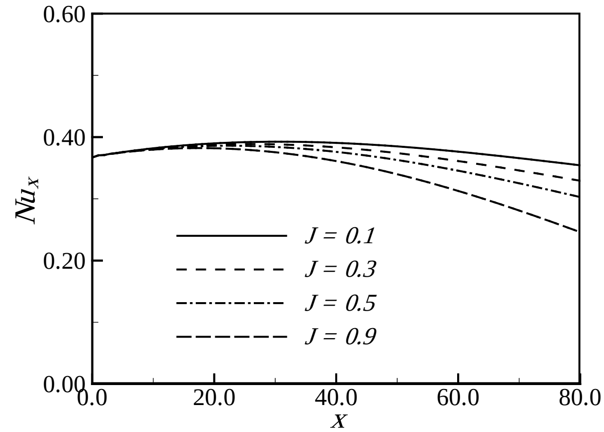

Fig. 12(b). Variation of heat transfer against *x* for varying of *J* with *M* = 0.1, $\lambda = 0.5$, $\varepsilon = 0.5$ and *Pr*=1.0.

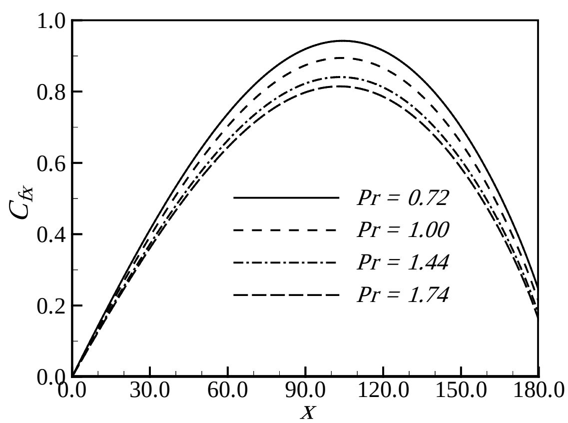

Fig.13 (a). Variation of skin friction against *x* for varying of *Pr* with *M* = 0.1, $\lambda = 0.5$, $\varepsilon = 0.5$, and *J*=0.1

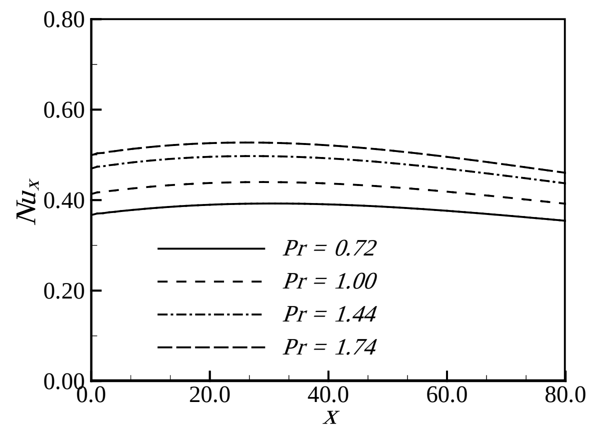

Fig.13 (b). Variation of heat transfer against *x* for varying of *Pr* with *M* = 0.1, $\lambda = 0.5$, $\varepsilon = 0.5$, and *J*=0.1

The variation of the reduced local skin friction coefficient and the local rate of heat transfer for different values of the joule heating parameter *J* (*J* = 0.1, 0.3, 0.5, 0.9) are illustrated in Figs. 12(a) and 12(b) while with *M* = 1.0, $\lambda = 0.5$ and $\varepsilon = 0.5$ and Prandtl number *Pr* = 1.0. From the figures it can be seen that the increase of the joule heating parameter *J* leads to an increase in the local skin-friction coefficient Cf_x and a decrease in the local Nusselt number Nu_x. These are expected, since the joule heating mechanism in presence of viscous dissipation and pressure stress work creates a layer of hot fluid near the surface, and finally the resultant temperature of the fluid exceeds the surface temperature. For this reason the rate of heat transfer from the surface decreases. Owing to the enhanced temperature, the viscosity of the fluid increases and the corresponding local skin-friction coefficient increases.

In order to verify the accuracy of the present work, the numerical values of the local Nusselt number Nu_x for M = 0.0, J = 0, $\lambda = 0.0$, $\varepsilon = 0.0$ and Pr = 1.00 in different position of x are compared with those reported by Merkin (1976), Nazar et al.(2002) and Hye et al. (2007) as presented in table 3.1. The results are found to be in excellent agreement.

Table 3.1

Compare the Numerical values of Nu_x for different values of x while Pr=1.0, J=0 .0, M = 0.0 $\lambda = 0.0$ and $\varepsilon = 0.0$				
x	Merkin (1976)	Nazar et al.(2002)	Hye et al. (2007)	Present(2010)
0.0	0.4214	0.4214	0.4241	0.4216
$\pi/6$	0.4161	0.4161	0.4161	0.4163
$\pi/3$	0.4007	0.4005	0.4005	0.4006
$\pi/2$	0.3745	0.3741	0.3741	0.3741
$2\pi/3$	0.3364	0.3355	0.3355	0.3355
$5\pi/6$	0.2825	0.2811	0.2811	0.2811
π	0.1945	0.1916	0.1916	0.1912

3.6 Conclusion

We have studied the joule heating effects on magneto-hydrodynamic (MHD) natural convection flow cylinder in presence of viscous dissipation and pressure stress work from a horizontal circular. The transformed non-similar boundary layer governing equations of the flow together with the boundary conditions were solved numerically using implicit finite difference method together with Keller box scheme. The coupled effect of natural convection that the temperature and the rate of heat transfer is continuous at the surface. From the present investigation, the following conclusions may be drawn:

- The local skin friction coefficients and the rate of heat transfer along the surface of the cylinder decrease for the increasing value of magnetic parameter *M*.

• An increase in values of *M* leads to decrease the velocity distribution but slightly increase the temperature distribution.

- For increasing values of Joule heating parameter *J*, the skin-friction coefficient increases but the Nusselt number decreases significantly within the boundary layer.
- With the effect of Joule heating parameter *J*, both the velocity and temperature distributions increase significantly the thickness of the thermal boundary layer.
- An increasing value of Prandtl number *Pr* leads to decrease in the velocity and the temperature distributions.

Appendix

Numerical Solution with Newton's Method

A.1 Finite difference method

To get the solutions of the differential equations (2.11) and (2.12) along with the boundary condition (2.13) , we shall employ a most practical, an efficient and accurate solution technique, known as implicit finite difference method together with Keller-box elimination technique.

To apply the aforementioned method, we first convert the equations (2.11) and (2.12) into the following system of first order differential equations with dependent variables $u(\xi,\eta)$, $v(\xi,\eta)$ and $p(\xi,\eta)$ where (x, y) denoted as (ξ,η) along with the boundary condition (2.13) as

$$f^{/} = u \tag{A.1}$$

$$u^{/} = v \tag{A.2}$$

$$\theta^{/} = p \tag{A.3}$$

Equations (2.11) and (2.13) transform to

$$v^{/} + p_1 fv - p_2 u^2 + p_3\theta = \xi\left(u\frac{\partial u}{\partial \xi} - v\frac{\partial f}{\partial \xi}\right) \tag{A.4}$$

$$\frac{1}{\Pr} p^{/} + p_1 fp + p_6 v^2 + p_7 u + p_8 u\theta + = \xi\left(u\frac{\partial \theta}{\partial \xi} - p\frac{\partial f}{\partial \xi}\right) \tag{A.5}$$

Where

$$p_1 = 1,\ p_2 = 1,\ p_3 = \frac{\sin\xi}{\xi},\ p_6 = x^2\lambda,\ p_7 = x\varepsilon T_r,\ p_8 = x\varepsilon, \tag{A.6}$$

The boundary conditions are:

$$\begin{aligned} f^{/} = u = 0,\ \theta = 1 \quad at \quad y = 0 \\ f^{/} = u = 0,\ \theta = 0 \quad at \quad y = \infty \end{aligned} \tag{A.7}$$

We now consider the net rectangle on the (•, •) plane shown in the figure (A.1) and denote the net points by

$$\left.\begin{array}{l}\xi^0 = 0, \quad \xi^n = \xi^{n-1} + k_n \;\; where \;\; n = 1,2,\dots N \\ \eta_0 = 0, \quad \eta_j = \eta_{j-1} + h_j \;\; where \;\; j = 1,2,\dots J\end{array}\right\} \qquad \text{(A.8)}$$

Here 'n' and 'j' are just sequence of numbers on the *(• , •)* plane , k_n and h_j are the variable mesh widths.

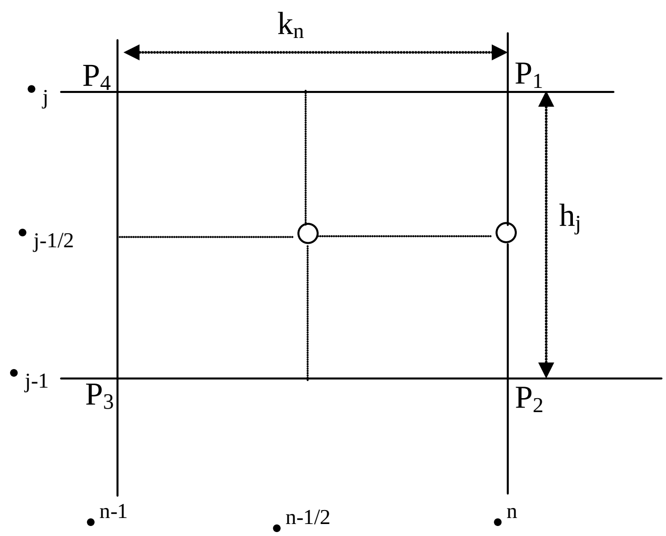

Figure A.1**:** Net rectangle of the difference approximation for the Box scheme.

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We approximate the quantities (f, u, v, p) at the points ($•^n$, $•_j$) of the net by($f_j^n, u_j^n \;\; v_j^n, \;\; p_j^n$) which we call net function . We also employ the notation g_j^n for the quantities midway between net points shown in figure (A.1) and for any net function as

$$\xi^{n-1/2} = \frac{1}{2}(\xi^n + \xi^{n-1}) \qquad \text{(A.9a)}$$

$$\eta_{j-1/2} = \frac{1}{2}(\eta_j - \eta_{j-1}) \qquad \text{(A.9b)}$$

$$\theta_j^{\;n-1/2} = \frac{1}{2}(\theta_j^{\;n} + \theta_j^{\;n-1}) \qquad \text{(A.9c)}$$

$$\theta_{j-1/2}^n = \frac{1}{2}(\theta_j^{\;n} + \theta_{j-1}^n) \qquad \text{(A.9d)}$$

Now we write the difference equations that are to approximate the three first order ordinary differential equations (2.1)-(2.3) according to Box method by considering one mesh rectangle. We start by writing the finite difference approximation of the above three equations using central difference quotients and average about the mid-point $(\xi^n, \eta_{j-1/2})$ of the segment P_1P_2 shown in the figure (A.1) and the finite difference approximations to the two first order differential equations (2.4)-(2.5) are written for the mid point $(\xi^{n-1/2}, \eta_{j-1/2})$ of the rectangle $P_1P_2P_3P_4$. This procedure yields.

$$h_j^{-1}(f_j^n - f_{j-1}^n) = u_{j-1/2}^n = \frac{u_{j-1}^n + u_j^n}{2} \tag{A.10}$$

$$h_j^{-1}(u_j^n - u_{j-1}^n) = v_{j-1/2}^n = \frac{v_{j-1}^n + v_j^n}{2} \tag{A.11}$$

$$h_j^{-1}(\theta_j^n - \theta_{j-1}^n) = p_{j-1/2}^n = \frac{p_{j-1}^n + p_j^n}{2} \tag{A.12}$$

$$\begin{aligned}&\frac{1}{2}\left[\frac{v_j^n - v_{j-1}^n}{h_j} + \frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right] + (p_1 fv)_{j-1/2}^{n-1/2} - (p_2 u^2)_{j-1/2}^{n-1/2} + (p_3\theta)_{j-1/2}^{n-1/2} \\ &= \xi_{j-1/2}^{n-1/2}\left[u_{j-1/2}^{n-1/2}\left\{\frac{u_{j-1/2}^n - u_{j-1/2}^{n-1}}{k_n}\right\} - v_{j-1/2}^{n-1/2}\left\{\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right\}\right]\end{aligned} \tag{A.13}$$

$$\begin{aligned}&\frac{1}{2P_r}\left[\frac{p'^n_j - p'^n_{j-1}}{h_j} + \frac{p'^{n-1}_j - p'^{n-1}_{j-1}}{h_j}\right] + (p_1 fp)_{j-1/2}^{n-1/2} + \frac{p_6}{2}\left[v^{2\,n}_{j-1/2} + v^{2\,n-1}_{j-1/2}\right] + \frac{p_7}{2}\left[u_{j-1/2}^n\right. \\ &+ \frac{p_8}{2}\left[(u\theta)_{j-1/2}^n + (u\theta)_{j-1/2}^{n-1}\right] \\ &= \xi_{j-1/2}^{n-1/2}\left[u_{j-1/2}^{n-1/2}\left\{\frac{\theta_{j-1/2}^n - \theta_{j-1/2}^{n-1}}{k_n}\right\} - p_{j-1/2}^{n-1/2}\left\{\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right\}\right]\end{aligned} \tag{A.14}$$

Now from the equation (A.13) we get

$$\begin{aligned}&\frac{1}{2}\left(\frac{v_j^n - v_{j-1}^n}{h_j}\right) + \frac{1}{2}\left(\frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) + \frac{1}{2}\left\{(p_1 fv)_{j-1/2}^n + (p_1 fv)_{j-1/2}^{n-1}\right\} \\ &- \frac{1}{2}\left\{(p_2 u^2)_{j-1/2}^n + (p_2 u^2)_{j-1/2}^{n-1}\right\} + \frac{1}{2}\left\{(p_3\theta)_{j-1/2}^n + (p_3\theta)_{j-1/2}^{n-1}\right\} \\ &= \frac{1}{2k_n}\xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^n + u_{j-1/2}^{n-1}\right)\left(u_{j-1/2}^n - u_{j-1/2}^{n-1}\right) \\ &- \frac{1}{2k_n}\xi_{j-1/2}^{n-1/2}\left(v_{j-1/2}^n + v_{j-1/2}^{n-1}\right)\left(f_{j-1/2}^n - f_{j-1/2}^{n-1}\right)\end{aligned}$$

$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n) + h_j^{-1}(v_j^{n-1} - v_{j-1}^{n-1}) + (p_1)_{j-1/2}^n (fv)_{j-1/2}^n + (p_1)_{j-1/2}^{n-1} (fv)_{j-1/2}^{n-1}$$
$$-(p_2)_{j-1/2}^n (u^2)_{j-1/2}^n - (p_2)_{j-1/2}^{n-1} (u^2)_{j-1/2}^{n-1} + (p_3)_{j-1/2}^n (\theta)_{j-1/2}^n + (p_3)_{j-1/2}^{n-1} \theta_{j-1/2}^{n-1}$$
$$= \alpha_n \{(u^2)_{j-1/2}^n - (u^2)_{j-1/2}^{n-1} - (fv)_{j-1/2}^n + v_{j-1/2}^n f_{j-1/2}^{n-1} - v_{j-1/2}^{n-1} f_{j-1/2}^n + (fv)_{j-1/2}^{n-1}\},$$
$$Where \;\; \alpha_n = \frac{\xi_{j-1/2}^{n-1/2}}{k_n}$$

$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n) + \{(p_1)_{j-1/2}^n + \alpha_n\}(fv)_{j-1/2}^n - \{(p_2)_{j-1/2}^n + \alpha_n\}(u^2)_{j-1/2}^n$$
$$+(p_3)_{j-1/2}^n (\theta)_{j-1/2}^n = \alpha_n[-(u^2)_{j-1/2}^{n-1} + v_{j-1/2}^n f_{j-1/2}^{n-1} - v_{j-1/2}^{n-1} f_{j-1/2}^n$$
$$+(fv)_{j-1/2}^{n-1}] - (p_1)_{j-1/2}^{n-1} (fv)_{j-1/2}^{n-1} + (p_2)_{j-1/2}^{n-1} (u^2)_{j-1/2}^{n-1} - h_j^{-1}(v_j^{n-1} - v_{j-1}^{n-1})$$
$$-(p_3)_{j-1/2}^{n-1} \theta_{j-1/2}^{n-1}$$

$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n) + \{(p_1)_{j-1/2}^n + \alpha_n\}(fv)_{j-1/2}^n - \{(p_2)_{j-1/2}^n + \alpha_n\}(u^2)_{j-1/2}^n$$
$$+(p_3)_{j-1/2}^n (\theta)_{j-1/2}^n + \alpha_n (v_{j-1/2}^{n-1} f_{j-1/2}^n - v_{j-1/2}^n f_{j-1/2}^{n-1})$$
$$= \alpha_n \{(fv)_{j-1/2}^{n-1} - (u^2)_{j-1/2}^{n-1}\} - (p_1)_{j-1/2}^{n-1} (fv)_{j-1/2}^{n-1} + (p_2)_{j-1/2}^{n-1} (u^2)_{j-1/2}^{n-1}$$
$$- h_j^{-1}(v_j^{n-1} - v_{j-1}^{n-1}) - (p_3)_{j-1/2}^{n-1} (\theta)_{j-1/2}^{n-1}$$
$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n) + \{(p_1)_{j-1/2}^n + \alpha_n\}(fv)_{j-1/2}^n - \{(p_2)_{j-1/2}^n + \alpha_n\}(u^2)_{j-1/2}^n$$
$$+(p_3)_{j-1/2}^n (\theta)_{j-1/2}^n + \alpha_n (v_{j-1/2}^{n-1} f_{j-1/2}^n - v_{j-1/2}^n f_{j-1/2}^{n-1})$$
$$= -L_{j-1/2}^{n-1} + \alpha_n \{(fv)_{j-1/2}^{n-1} - (u^2)_{j-1/2}^{n-1}\}$$
$$L_{j-1/2}^{n-1} = (p_1)_{j-1/2}^{n-1} (fv)_{j-1/2}^{n-1} - (p_2)_{j-1/2}^{n-1} (u^2)_{j-1/2}^{n-1} + h_j^{-1}(v_j^{n-1} - v_{j-1}^{n-1})$$
$$+(p_3)_{j-1/2}^{n-1} (\theta)_{j-1/2}^{n-1}$$

$$\Rightarrow h_j^{-1}(v_j^n - v_{j-1}^n) + \{(p_1)_{j-1/2}^n + \alpha_n\}(fv)_{j-1/2}^n - \{(p_2)_{j-1/2}^n + \alpha_n\}(u^2)_{j-1/2}^n$$
$$+(p_3)_{j-1/2}^n (\theta)_{j-1/2}^n + \alpha_n (v_{j-1/2}^{n-1} f_{j-1/2}^n - v_{j-1/2}^n f_{j-1/2}^{n-1}) = R_{j-1/2}^{n-1} \qquad \text{(A.15)}$$
$$R_{j-1/2}^{n-1} = -L_{j-1/2}^n + \alpha_n \{(fv)_{j-1/2}^{n-1} - (u^2)_{j-1/2}^{n-1}\}$$

Since $p_i = p_i(\xi)$, so we may write the above equation as

$$h_j^{-1}(v_j^n - v_{j-1}^n) + \{(p_1)^n + \alpha_n\}(fv)_{j-1/2}^n - \{(p_2)^n + \alpha_n\}(u^2)_{j-1/2}^n$$
$$+(p_3)^n (\theta)_{j-1/2}^n + \alpha_n (v_{j-1/2}^{n-1} f_{j-1/2}^n - v_{j-1/2}^n f_{j-1/2}^{n-1}) = R_{j-1/2}^{n-1} \qquad \text{(A.16)}$$

Again from the equation (A.14) we get

$$\frac{1}{2P_r}\left[\frac{p'^n_j - p'^n_{j-1}}{h_j} + \frac{p'^{n-1}_j - p'^{n-1}_{j-1}}{h_j}\right] + (p_1 fp)^{n-1/2}_{j-1/2} + \frac{p_6}{2}\left[v^2{}^n_{j-1/2} + v^2{}^{n-1}_{j-1/2}\right] + \frac{p_7}{2}\left[u^n_{j-1/2} + u^{n-1}_{j-1/2}\right]$$

$$+\frac{p_8}{2}\left[(u\theta)^n_{j-1/2} + (u\theta)^{n-1}_{j-1/2}\right] + \frac{p_9}{2}\left[u^2{}^n_{j-1/2} + u^2{}^{n-1}_{j-1/2}\right]$$

$$=\xi^{n-1/2}_{j-1/2}\left[u^{n-1/2}_{j-1/2}\left\{\frac{\theta^n_{j-1/2} - \theta^{n-1}_{j-1/2}}{k_n}\right\} - p^{n-1/2}_{j-1/2}\left\{\frac{f^n_{j-1/2} - f^{n-1}_{j-1/2}}{k_n}\right\}\right]$$

$$\Rightarrow \frac{1}{P_r} h^{-1}_j (p^n_j - p^n_{j-1}) + \frac{1}{P_r} h^{-1}_j (p^{n-1}_j - p^{n-1}_{j-1}) + (p_1)^{n-1/2}_{j-1/2}(fp)^n_{j-1/2} + (p_1)^{n-1/2}_{j-1/2}(fp)^{n-1}_{j-1/2}$$

$$+ p_6(v^2)^n_{j-1/2} + p_7(u)^n_{j-1/2} + (p_8 - \alpha_i)(u\theta)^n_{j-1/2} + p_9(u^2)^n_{j-1/2} + \alpha_n[u^n_{j-1/2}\theta^{n-1}_{j-1/2}$$

$$- u^{n-1}_{j-1/2}\theta^n_{j-1/2} - v^n_{j-1/2} f^{n-1}_{j-1/2} + v^{n-1}_{j-1/2} f^n_{j-1/2}] = M^{n-1}_{j-1/2} + \alpha_n\left\{(fv)^{n-1}_{j-1/2} - (u\theta)^{n-1}_{j-1/2})\right\}$$

Where, $M^{n-1}_{j-1/2} = \frac{1}{P_r} h^{-1}_j (p^{n-1}_j - p^{n-1}_{j-1}) + (p_1)^{n-1/2}_{j-1/2}(fp)^{n-1}_{j-1/2} + p_6(v^2)^{n-1}_{j-1/2} + p_7(u)^{n-1}_{j-1/2} +$

$p_8(u\theta)^{n-1}_{j-1/2}$

$$\Rightarrow \frac{1}{P_r} h^{-1}_j (p^n_j - p^n_{j-1}) + + (p_1)^n_{j-1/2}(fp)^n_{j-1/2} + + p_6(v^2)^n_{j-1/2} + p_7(u)^n_{j-1/2}$$

$$+ (p_8 - \alpha_i)(u\theta)^n_{j-1/2} + p_9(u^2)^n_{j-1/2} + \alpha_n[u^n_{j-1/2}\theta^{n-1}_{j-1/2}$$

$$- u^{n-1}_{j-1/2}\theta^n_{j-1/2} - v^n_{j-1/2} f^{n-1}_{j-1/2} + v^{n-1}_{j-1/2} f^n_{j-1/2}] = T^{n-1}_{j-1/2} + \alpha_n\left\{(fv)^{n-1}_{j-1/2} - (u\theta)^{n-1}_{j-1/2})\right\} \qquad \text{(A.17)}$$

Where

$$T^{n-1}_{j-1/2} = -M^{n-1}_{j-1/2} + \alpha_n\left\{(fp)^{n-1}_{j-1/2} - (u\theta)^{n-1}_{j-1/2}\right\}$$

The boundary conditions become

$$f^n_0 = 0 \qquad \theta^n_0 = 1 \qquad \text{(A.18)}$$
$$u^n_J = 0 \qquad \theta^n_J = 0$$

If we assume $f^{n-1}_j, u^{n-1}_j, v^{n-1}_j, \theta^{n-1}_j, p^{n-1}_j$ to be known for $0 \le j \le J$, equations (A.10) to (A.12) and (A.15) – (A.18) form a system of 5J + 5 non linear equations for the solutions of the 5J + 5 unknowns ($f^n_j, u^n_j, v^n_j, \theta^n_j, p^n_j$) , j = 0,1,2 …J. These non linear systems of algebraic equations are to be linearized by Newton's Quassy linearization method. We define the iterates [$f^n_j, u^n_j, v^n_j, \theta^n_j, p^n_j$] , i = 0,1,2 …N with initial values equal those at the previous x-station, which are usually the best initial guess availables. For the higher iterates we set:

$$f^{(i+1)}_j = f^{(i)}_j + \delta f^{(i)}_j \qquad \text{(A.19)}$$

$$u^{(i+1)}_j = u^{(i)}_j + \delta u^{(i)}_j \qquad \text{(A.20)}$$

$$v^{(i+1)}_j = v^{(i)}_j + \delta v^{(i)}_j \qquad \text{(A.21)}$$

$$\theta^{(i+1)}_j = \theta^{(i)}_j + \delta \theta^{(i)}_j \qquad \text{(A.22)}$$

$$p_j^{(i+1)} = p_j^{(i)} + \delta\, p_j^{(i)} \tag{A.23}$$

Now we substitute the right hand sides of the above equations in place of f_j^n, u_j^n, v_j^n, θ_j^n and p_j^n in equations (A.10) to (A.17) and (A.18) and omitting the terms that are quadratic in $\delta\, f_j^i$, $\delta\, u_j^i$, δv_j^i, $\delta \theta_j^{\,i}$ and $\delta\, p_j^i$ we get the equations (A.10) to (A.12) in the following form:

$$\delta\, f_j^{(i)} - \delta\, f_{j-1}^{(i)} - \frac{h_j}{2}\,(\delta u_j^{(i)} + \delta u_{j-1}^{(i)}) = (r_1)_j \tag{A.24}$$

Where $(r_1)_j = f_{j-1}^{(i)} - f_j^{(i)} + h_j u_{j-1/2}^{(i)}$ (A.25)

$$\delta u_j^{(i)} - \delta u_{j-1}^{(i)} - \frac{h_j}{2}\,(\delta v_j^{(i)} + \delta v_{j-1}^{(i)}) = (r_4)_j \tag{A.26}$$

$$(r_4)_j = u_{j-1}^{(i)} - u_j^{(i)} + h_j v_{j-1/2}^{(i)} \tag{A.27}$$

$$\delta \theta_j^{(i)} - \delta \theta_{j-1}^{(i)} - \frac{h_j}{2}\,(\delta\, p_j^{(i)} + \delta\, p_{j-1}^{(i)}) = (r_5)_j \tag{A.28}$$

Where $(r_5)_j = \theta_{j-1}^{(i)} - \theta_j^{(i)} + h_j\, p_{j-1/2}^{(i)}$ (A.29)

$$h_j^{-1}\,(v_j^{(i)} + \delta v_j^{(i)} - v_{j-1}^{(i)} - \delta v_{j-1}^{(i)}) + \left\{(p_1)_{j-1/2}^n + \alpha_n\right\}\left\{(fv)_{j-1/2}^{(i)} + \delta\,(fv)_{j-1/2}^{(i)}\right\}$$
$$-\left\{(p_2)_{j-1/2}^n + \alpha_n\right\}\left\{(u^2)_{j-1/2}^{(i)} + \delta(u^2)_{j-1/2}^{(i)}\right\} + (p_3)_{j-1/2}^n\left\{(\theta)_{j-1/2}^{(i)} + \delta(\theta)_{j-1/2}^{(i)}\right\}$$
$$+\alpha_n\,(f_{j-1/2}^{(i)} + \delta\, f_{j-1/2}^{(i)})\,v_{j-1/2}^{n-1} - \alpha_n (v_{j-1/2}^{(i)} + \delta v_{j-1/2}^{(i)})\,f_{j-1/2}^{n-1}$$
$$= R_{j-1/2}^{n-1}$$
$$\Rightarrow h_j^{-1}(v_j^{(i)} + \delta v_j^{(i)} - v_{j-1}^{(i)} - \delta v_{j-1}^{(i)}) + \left\{(p_1)_{j-1/2}^n + \alpha_n\right\}$$
$$\left\{(fv)_{j-1/2}^{(i)} + \frac{1}{2}(f_j^{(i)}\delta(v)_j^{(i)} + v_j^{(i)}\delta(f)_j^{(i)} + f_{j-1}^{(i)}\delta(v)_{j-1}^{(i)} + v_{j-1}^{(i)}\delta(f)_{j-1}^{(i)}\right\}$$
$$-\left\{(p_2)_{j-1/2}^n + \alpha_n\right\}\left\{(u^2)_{j-1/2}^{(i)} + \frac{1}{2}\{\delta(u^2)_j^{(i)} + \delta(u^2)_{j-1}^{(i)}\right\}$$
$$+(p_3)_{j-1/2}^n\left\{(\theta)_{j-1/2}^{(i)} + \frac{1}{2}(\delta(\theta)_j^{(i)} + \delta(\theta)_{j-1}^{(i)})\right\}$$
$$+\alpha_n\left[\left\{v_{j-1/2}^{n-1}(f_{j-1/2}^{(i)} + \frac{1}{2}(\delta\, f_j^{(i)} + \delta\, f_{j-1}^{(i)}))\right\} - \left\{f_{j-1/2}^{n-1}(v_{j-1/2}^{(i)} + \frac{1}{2}(\delta v_j^{(i)} + \delta_{j-1}^{i}))\right\}\right] = R_{j-1/2}^{n-1}$$
$$\Rightarrow (s_1)_j\,\delta v_j^{(i)} + (s_2)_j\,\delta v_{j-1}^{(i)} + (s_3)_j\,\delta\, f_j^{(i)} + (s_4)_j\,\delta\, f_{j-1}^{(i)} + (s_5)_j\,\delta u_j^{(i)}$$
$$+(s_6)_j\,\delta u_{j-1}^{(i)} + (s_7)_j\,\delta\theta_j^{(i)} + (s_8)_j\,\delta\theta_{j-1}^{(i)} + (s_9)_j\,\delta\, p_j^i + (s_{10})_j\,\delta\, p_{j-1}^i \tag{A.30}$$
$$= (r_2)_j$$

Where $(s_1)_j = (h_j^{-1} + \dfrac{(p_1)_{j-1/2}^n + \alpha_n}{2}\, f_j^{(i)} - \dfrac{1}{2}\alpha_n\, f_{j-1/2}^{n-1})$ (A.31)

$$(s_2)_j = (-h_j^{-1} + \frac{(p_1)_{j-1/2}^n + \alpha_n}{2} f_{j-1}^{(i)} - \frac{1}{2}\alpha_n f_{j-1/2}^{n-1}) \quad (A.32)$$

$$(s_3)_j = (\frac{(p_1)_{j-1/2}^n + \alpha_n}{2} v_j^{(i)} + \frac{1}{2}\alpha_n v_{j-1/2}^{n-1} \quad (A.33)$$

$$(s_4)_j = (\frac{(p_1)_{j-1/2}^n + \alpha_n}{2} v_{j-1}^{(i)} + \frac{1}{2}\alpha_n v_{j-1/2}^{n-1}) \quad (A.34)$$

$$(s_5)_j = -\{(p_2)_{j-1/2}^n + \alpha_n\}\, u_j^{(i)} \quad (A.35)$$

$$(s_6)_j = -\{(p_2)_{j-1/2}^n + \alpha_n\}\, u_{j-1}^{(i)} \;) \quad (A.36)$$

$$(s_7)_j = (p_3)^n / 2 \quad (A.37)$$

$$(s_8)_j = (p_3)^n / 2 \quad (A.38)$$

$$(s_9)_j = 0 \quad (A.39)$$

$$(s_{10})_j = 0 \quad (A.40)$$

$$\begin{aligned}(r_2)_j &= R_{j-1/2}^{n-1} - \left\{h_j^{-1}(v_j^{(i)} - v_{j-1}^{(i)}) + ((p_1)_{j-1/2}^n + \alpha_n)(fv)_{j-1/2}^{(i)}\right\} \\ &+ ((p_2)_{j-1/2}^n + \alpha_n)(u^2)_{j-1/2}^{(i)} - \alpha_n\{f_{j-1/2}^{(i)} v_{j-1/2}^{n-1} - f_{j-1/2}^{n-1} v_{j-1/2}^{(i)}\} \\ &- \left\{(p_3)_{j-1/2}^n \theta_{j-1/2}^{(i)}\right\}\end{aligned} \quad (A.41)$$

Here the coefficients $(s_9)_j$ and $(s_{10})_j$, which are zero in this case, are included here for the generality.

Similarly by using the equations (A.19) to (A.23) in the equation (A.17) we get the following form:

$$\begin{aligned}&(t_1)_j\, \delta p_j^{(i)} + (t_2)_j\, \delta p_{j-1}^{(i)} + (t_3)_j\, \delta f_j^{(i)} + (t_4)_j\, \delta f_{j-1}^{(i)} + (t_5)_j\, \delta u_j^{(i)} \\ &+ (t_6)_j\, \delta u_{j-1}^{(i)} + (t_7)_j\, \delta \theta_j^{(i)} + (t_8)_j\, \delta \theta_{j-1}^{(i)} + (t_9)_j\, \delta v_j^{(i)} + (t_{10})_j\, \delta v_{j-1}^{(i)} = (r_3)_j\end{aligned} \quad (A.42)$$

Where $(t_1)_j = \frac{1}{P_r} h_j^{-1} + \frac{(p_1)_{j-1/2}^{n-1/2} + \alpha_n}{2} f_j^{(i)} - \frac{1}{2}\alpha_n f_{j-1/2}^{n-1}$ (A.43)

$$(t_2)_j = -\frac{1}{P_r} h_j^{-1} + \frac{(p_1)_{j-1/2}^{n-1/2} + \alpha_n}{2} f_{j-1}^{(i)} - \frac{1}{2}\alpha_n f_{j-1/2}^{n-1} \quad (A.44)$$

$$(t_3)_j = \frac{(p_1)_{j-1/2}^{n-1/2} + \alpha_n}{2} p_j^{(i)} + \frac{1}{2}\alpha_n p_{j-1/2}^{n-1} + \frac{1}{2}\alpha_n v_{j-1/2}^{n-1} + \frac{1}{2}\alpha_n v_{j-1/2}^{n} \quad (A.45)$$

$$(t_4)_j = \frac{(p_1)_{j-1/2}^{n-1/2} + \alpha_n}{2} p_{j-1}^{(i)} + \frac{1}{2}\alpha_n v_{j-1/2}^{n} + \frac{1}{2}\alpha_n v_{j-1/2}^{n-1} + \frac{1}{2}\alpha_n p_{j-1/2}^{n-1} \quad (A.46)$$

$$(t_5)_j = -\frac{p_8 + \alpha_n}{2}\theta_j^{(i)} + \frac{1}{2}\alpha_n \theta_{j-1/2}^{n-1} + \frac{p_7}{2} \quad (A.47)$$

$$(t_6)_j = -\frac{\alpha_n}{2}\theta_{j-1}^{(i)} + \frac{1}{2}\alpha_n\,\theta_{j-1/2}^{n-1} + p_9\,u_{j-1/2}^{(i)} + \frac{p_7}{2} \tag{A.48}$$

$$(t_7)_j = -\frac{p_8-\alpha_n}{2}u_j^{(i)} - \frac{1}{2}\alpha_n u_{j-1/2}^{n-1} \tag{A.49}$$

$$(t_8)_j = -\frac{p_8-\alpha_n}{2}u_{j-1}^{(i)} - \frac{1}{2}\alpha_n u_{j-1/2}^{n-1} \tag{A.50}$$

$$(t_9)_j = -\frac{\alpha_n}{2}f_j^{(i)} + \frac{1}{2}\alpha_n\,f_{j-1/2}^{n-1} + p_4\,v_j^{(i)} \tag{A.51}$$

$$(t_{10})_j = -\frac{\alpha_n}{2}f_j^{(i)} + \frac{1}{2}\alpha_n\,f_{j-1/2}^{n-1} + p_4\,v_j^{(i)} \tag{A.52}$$

$$\begin{aligned}(r_3)_j &= T_{j-1/2}^{n-1} - \frac{1}{P_r}h_j^{-1}(p_j^{(i)} - p_{j-1}^{(i)}) - \frac{(p_1)_{j-1/2}^{n-1/2} + \alpha_n}{2}(fp)_{j-1/2}^{(i)}\\ &+ p_4(v^2)_{j-1/2}^{(i)} + p_7\,u_{j-1/2}^{n} + (p_8-\alpha_n)(u\theta)_{j-1/2}^{(i)} + (fv)_{j-1/2}^{(i)}\\ &+\alpha_n u_{j-1/2}^{n-1}\,\theta_{j-1/2}^{(i-1)} - \alpha_n u_{j-1/2}^{n-1}\,\theta_{j-1/2}^{(i-1)} + \alpha_n\,f_{j-1/2}^{n-1}v_{j-1/2}^{n-1} + \alpha_n\,f_{j-1/2}^{n}v_{j-1/2}^{n-1}\end{aligned} \tag{A.53}$$

The boundary conditions (A.18) become

$$\begin{aligned}&\delta f_0^n = 0,\ \delta u_0^n = 0,\ \delta\theta_0^n = 1\\ &\delta u_J^n = 0,\ \delta\theta_J^n = 0\end{aligned} \tag{A.54}$$

Which just express the requirement for the boundary conditions to remain during the iteration process.

Now the system of linear equations (A.24) - (A.30), (A.41), (A.42) and (A.53) together with the boundary conditions (A.54) can be written in a block matrix from a coefficient matrix, which are solved by modified ‘Keller Box’ methods

The solutions of the above equations together with the boundary conditions enable us to calculate the skin friction Cf_x and the rate of heat transfer Nu_x at the surface in the boundary layer from the following relations:

$$Cf_x = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\rho v^2}{a^2}\xi\,Gr^{\frac{3}{4}}\,f''(\xi,0) \tag{A.55}$$

$$Nu_x = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = -k\frac{T_w - T_\infty}{a}Gr^{\frac{1}{4}}\,\theta'(\xi,0) \tag{A.56}$$

A.2. The important dimensionless parameters related to the problem:

The governing equations of the fluid flow are discussed at the previous sections in this chapter. These equations contain a number of variables. It is difficult to study the effect of each variable on the process. Moreover these equations are nonlinear. There is no general method to find the solution of these nonlinear equations. In order to bring out the essential features of flow, it is necessary to find important dimensionless parameters, which characterize the flow. These parameters are very useful in the analysis of experimental results. Some non-dimensional parameters related to our problems are discussed below:

Reynolds number R_e

Reynolds number is the most important non-dimensional parameter of the fluid dynamics of a viscous fluid. It represents the ratio of the inertia force and the viscous force. It is denoted by $\boldsymbol{R_e}$.

$$\boldsymbol{R_e} = \frac{\text{Inertia force}}{\text{Viscous force}}$$

$$= \frac{\text{Mass} \times \text{Acceleration}}{\text{Shear stress} \times \text{Cross sectional area}}$$

$$= \frac{\text{Volume} \times \text{Density} \times (\text{Velocity/Time})}{\text{Shear stress} \times \text{Cross sectional area}}$$

$$= \frac{\text{Cross sectional area} \times \text{Linear dimension} \times \text{Density} \times (\text{Velocity/Time})}{\text{Shear stress} \times \text{Cross sectional area}}$$

$$= \frac{\rho V^2 L^2}{\mu V L} = \frac{V L}{\nu}$$

where, V, L, ρ and μ denote the characteristic velocity, the characteristic length, the density and the coefficient of viscosity of the fluid flow respectively. Here $\nu = \dfrac{\mu}{\rho}$ is the kinematic viscosity. This result implies that viscous forces are dominant for small Reynolds numbers and inertia forces are dominant for large Reynolds numbers. The Reynolds number is used as the criterion to determine the change from laminar to turbulent flow.

Prandtl number *Pr*

Prandtl gave an important number known as Prandtl number. The Prandtl number is a dimensionless parameter of a convective system that characterizes the regime of convection. It is the ratio of viscous force to the thermal force and is defined as follows:

$$Pr = \frac{\text{Viscous force}}{\text{Thermal force}}$$

$$= \frac{\text{Kinematic viscosity}}{\text{Thermal diffusivity}}$$

$$= \frac{\mu / \rho}{\kappa / \rho C_p} = \frac{\nu}{\alpha}$$

The Prandtl number is large when thermal conductivity is small and viscosity is large, and small when viscosity is small and thermal conductivity is large. For small value of ν, a thin region will be affected by viscosity, which is known as boundary layer region. For the small value of $\kappa / \rho C_p$, a thin region will be affected by heat conduction which is known as the thermal boundary layer. Prandtl number also gives the relative importance of viscous dissipation to the thermal dissipation. Thus it represents the relative importance of momentum and energy transport by the diffusion process. Usually for gases $\Pr \cong 1$, the transfer of momentum and energy by the diffusion process is comparable. For oils, $\Pr \rangle\rangle 1$, hence the momentum diffusion is much greater than the energy diffusion; but for liquid metals, $\Pr \langle\langle 1$ and the solution is reversed.

Grashof number *Gr*

The Grashof number gives the relative importance of buoyancy force to the viscous forces and is defined as

$$Gr = \frac{g \beta L^3 \left(T - T_w\right)}{\nu^2}$$

where g is the acceleration due to gravity, L the characteristic length of the problem, β the coefficient of volume expansion, and $T - T_w$ is the excess temperature of the fluid over the reference temperature T_0. This number is of great importance and plays a similar role in free convection as does the Reynolds number in forced convection. A critical value of the Grashof number is used to indicate transition from laminar to turbulent flow in free convection.

Joule heating parameter *J:*

In electronics and in physics more broadly, Joule heating or Ohmic heating refers to the increase in temperature of a conductor as a result of resistance to an electrical current flowing through it.
At an atomic level, Joule heating is the result of moving electrons colliding with atoms in a conductor, where upon momentum is transferred to the atom, increasing its kinetic energy. Joule heating is named for James Prescott Joule, the first to articulate what is now Joule's law, relating the amount of heat released from an electrical resistor to its resistance and the charge passed through it. In our problem we used a dimensionless parameter $J = \dfrac{\sigma H_0^2 \nu d^{\frac{1}{2}}}{\rho c_p (T_w - T_\infty)}$, which is Joule heating parameter.

Extension of this work

Some proposals related to this problem are given below:

- Effect of atmospheric pressure on the fluid flow can be shown.
- It can be considered time dependent flow of the fluid.

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