A STUDY ON

PRIME AND SEMI-PRIME RINGS AND MODULES

M. Phil. Thesis

SUBMITTED BY

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ABSTRACT

Let R be a ring. Then a proper ideal P in a ring R is called a prime ideal of R if for any ideals I, J of R with $IJ \subset P,$ then either $I \subset P$ or $J \subset P.$ A ring R is called a prime ring if 0 is a prime ideal. Let M be a right R-module and $S = End_R(M),$ its endomorphism ring. A submodule X of M is called a fully invariant submodule of M if for any $f \in S,$ we have $f(X) \subset X.$ Let M be a right R-module and $P,$ a fully invariant proper submodule of $M.$ Then P is called a prime submodule of M if for any ideal I of S, and any fully invariant submodule X of $M,$ $I(X) \subset P$ implies $I(M) \subset P$ or $X \subset P.$ A fully invariant submodule X of a right R-module M is called a semi-prime submodule if it is an intersection of prime submodules. An ideal P in a ring R called a semi-prime ideal if it is an intersection of prime ideals. A ring R is called a semi-prime ring if 0 is a semi-prime ideal.

This study describes some properties of prime and semi-prime ideals in associative rings modifying the results on prime and semi-prime Goldie modules investigated in [15]. The structures of prime and semi-prime rings are also available in this study. Finally, some properties of prime and semi-prime submodules as a generalization of prime and semi-prime ideals in associated rings are also investigated.

CONTENTS

CHAPTER I

INTRODUCTION

Ring theory is a subject of central importance in algebra. Historically, some of the major discoveries in ring theory have helped shape the course of development of modern abstract algebra. In view of these basic connections between ring theory and other branches of mathematics, it is perhaps no exaggeration to say that a course in ring theory is an indispensable part of education for any fledgling algebraist.

Background and present state of the problem

Modern ring theory begins when Wedderburn in 1907 proved his celebrated classification theorem for finite dimensional semi-simple algebras over fields. Twenty years later, E. Noether and E. Artin introduced the ascending chain condition and descending chain condition as substitutes for finite dimensionality. Dauns [1] first introduced and investigated the notion of prime modules. In the literature on module theory, there is another notion of primal modules due to Dauns [2]. Goodearl and Warfield [3] and McCasland and Smith [4] introduced the notion of prime submodules of Noetherian modules. Wisbauer [5] introduced the structure of the category σ[M]. He called it the full subcategory of *Mod-R* whose objects are *M*-generated modules, i.e., modules which are isomorphic to submodules of M-generated modules. Later, Beider and Wisbauer [6] introduced the notion of semi-prime and strongly semi-prime modules and rings. Lu [7] and Goodearl [8] introduced the notion of prime submodules of Noetherian modules. In 2004, Behboodi and Koohy [9] defined weakly prime submodules. Ameri [10] and Gaur et al. [11] introduced the structure of prime submodules in multiplication modules over commutative rings.

Recently in 2007, Sanh et al. [14] introduced a new notion of prime and semi-prime submodules over associative rings. A basic tool in the study of Noetherian rings and modules is the Goldie dimension of modules. In 2007, Sanh et al. [15] introduced the new notion of Goldie modules. In 2008, Ahmed et al. [16] investigated some properties of semi-prime modules.

In this thesis, Chapter I deals with the early history of prime and semi-prime rings and modules. All the essential basic definitions, examples and their properties are given in Chapter II. Chapter III, deals with the basic properties of prime and semi-prime rings. In this Chapter, we describe some properties of prime and semi-prime ideals in associative arbitrary rings modifying the results on prime and semi-prime modules investigated in [15]. In Chapter IV, some properties of prime and semi-prime submodules as a generalization of prime and semi-prime ideals in associative rings are investigated.

CHAPTER II

BASIC KNOWLEDGE

Overview

The subject of our study is ring theory. Throughout this thesis, all rings are associative with identity and all modules are unitary right *R*-modules. Ring admit a valuable and natural representation theory, analogous to the permutation representation theory for groups. As we shall see, each ring admits a vast horde of representation as an endomorphism ring of an abelian group. Each of these representations is called a module. A substantial amount of information about a ring can be learned from a study of the class of modules it admits. Modules actually serve as a generalization of both vector spaces and abelian groups, and their basis behaviour is quite similar to that of the more special systems. In this chapter, we introduce the fundamental tools of this study. Section 2.1 reviews the basic facts about rings, subrings, commutative division ring, center of a ring, integral domain, ring homomorphisms and other notions. It also introduces some of the notation and the examples that will be needed later. We denote by *R* an arbitrary ring and by *Mod-R*, the category of all right *R*-modules. The notation M_R indicates a right *R*-module *M* which, when $1 \in R$, is assumed to be unity. The set *Hom(M, N)* denotes the set of right *R*-module homomorphisms between two right *R*-modules M and N and if further emphasis is needed, the notation $Hom_R(M, N)$ is used. The kernel of any $f \in Hom_R(M, N)$ is denoted by *Ker (f)* and the *image* of f by *Im (f)*. In particular, $End_R(M)$ denotes the ring of endomorphisms of a right *R*-module *M.*

2.1. Preliminaries

Before dealing with deeper results on the structure of rings with the help of module theory, we provide first some essential elementary definitions, examples and properties.

Definition 2.1.1 A ring *R* is called *commutative* if the multiplication operation is commutative.

Definition 2.1.2 Let *R* be a ring with unity. Then *R* then *R* is called a *division ring* if every non-zero element in *R* has multiplicative inverse in *R*.

Definition 2.1.3 The ring *R* is said to be a *ring with unity* if $\exists$ *a* multiplicative identity denoted by *1* in *R* such that $a.\ 1 = 1.\ a = a,\ \forall a \in R$. A ring *R* is said to be a ring without zero divisors if it is not possible to find two non-zero elements of *R* whose product is zero i. e. if $ab = 0 \Rightarrow a = 0$ or $b = 0$. A *field* is a commutative ring with unity in which every non-zero element has its multiplicative inverse.

Definition 2.1.4 The *center of a ring R* is the subset of *R* defined by

$$Cen\ \ R = \{r \in R : rx = xr \quad \forall x \in R\}$$

which is a commutative subring of *R*. Of course, *Cen R* is commutative if and only if *R* is equal to its center. We may say that an element $r \in R$ is central in case $r \in CenR.$ Note that if $A \in CenR,$ then the subring of *R* generated by *A* is also in the center of *R*.

Definition 2.1.5 An element *a* of a ring *R* is called a *left zero divisor* if $ab = 0$ for some non-zero $b \in R$, *right zero divisor* if $ba = 0$ for some non-zero $b \in R$ and *zero divisor* if it is a left or right zero divisor.

Definition 2.1.6 An *integral domain* is a commutative ring with unity and without zero divisors. A commutative ring *R* is called an *integral domain* if $\forall\ x, y \in R$, we have $xy = 0$.

Example 2.1.7 The ring $(I,+,\cdot)$ is an integral domain where *I* is the set of all irrational numbers. Also the rings $(Z,+,\cdot)$, $(Q,+,\cdot)$, $(C,+,\cdot)$, $(R,+,\cdot)$ all are examples of integral domains.

Definition 2.1.8 Let R be a ring and I be an ideal of *R*. Then I is called a *principal ideal* of R if I is generated by a single element. If *I* is generated by *a*, then we write $I = (a)$. Let *R* be a ring and *M* be an ideal of *R* such that $M \neq R$. Then M is called *maximal* if for any ideal *A* of *R* such that $M \subset A \subset R,$ then either $M = A$ or $A = R.$

Let I be an ideal. Then the set $R/I = \{a + I : a \in R\}$ is a ring for the operations in R/I defined by the following ways:

$$(I + a) + (I + b) = I + a + b \text{ and } (I + a)(I + b) = I + ab \ \forall a, b \in R.$$

This ring is called the q*uotient ring.*

Definition 2.1.9 A mapping f from a ring R into a ring R' is called a ring *homomorphism* if $\forall a, b \in R$ (i) $f(a+b) = f(a) + f(b)$ and (ii) $f(ab) = f(a)f(b)$

Let f be a *homomorphism* from a ring R into a ring R', then

$$\text{Im } f = \{x' \in R' : f(x) = x', \text{ for some } x \in R\}$$

Let f be a homomorphism from a ring R into a ring R'. Then

$$Kerf = \{x \in R : f(x) = 0'\}$$

where $0'$ is the additive identity of R'. f is said to be an *isomorphism* if it is a one-one and onto.

Definition 2.1.10 Let M be a right R-module. A homomorphism $f : M \to M$ is called an *endomorphism.* The abelian group $\text{Hom}_R(M, M)$ becomes a ring if we use the composition of maps as multiplication. This ring is called the *endomorphism ring* of M, and we denoted by $\text{End}_R(M)$.

2.2 Different kinds of submodules

Definition 2.2.1 Let R be a ring with identity and M an abelian group. Then M is called a *right R-module* if there exists $R \times M \to M, (r, m) \to mr$ satisfying the following conditions:

(i) $\forall\ m, m' \in M$ and $\forall\ r \in R \Rightarrow (m + m')r = mr + m'r$;

(ii) $\forall\ m \in M$ and $\forall\ r, r' \in R \Rightarrow m(r + r') = mr + mr'$;

(iii) $\forall\ m \in M$ and $\forall\ r, r' \in R \Rightarrow m(r + r') = (mr)r'$;

(iv) $\forall\ m \in M$ and $1 \in R \Rightarrow m.1 = m$.

Similarly, we can define *left R- modules* by operating to the left side of M. We write M_R (respectively, M_R) to indicate that M is a right (respectively, left) R-module. Let M be an R-module. A subset L of M is a submodule of M if L is an additive subgroup and

$\forall\ m \in L, \forall\ r \in R \Rightarrow mr \in L,$ i.e., L is a module under operations inherited from *M*. When *L* is a submodule of *M*, we can define the quotient module (factor module) $\frac{M}{L}$ with the operation $M\ /(L \times R) \to M\ /\ L$ given by

$$\text{(i)}\,(m, r) \to mr \quad \text{(ii)}\,(m + L, r) = (m + L)r = mr + L\,.$$

Let *R*, *S* be two rings and *M* an abelian group. Then *M* is called an *R*-*S*-bimodule if *M* is a left *R*-module, right *S*-module, and if for any m$\in$*M*, *r* $\in$ *R*, *s* $\in$ *S*, we have *r (ms) = (r m) s*. We denote it by ${}_R M_S$.

Let *M* be a module and *m* $\in$ *M*. An element *m* generates a cyclic submodule *mR* of *M*. There is an epimorphism α : X $\to$ m X given by

$\alpha(x) = mx\ \forall\ x \in X$ and Ker (α) = $\{x \in X \mid \alpha(x) = 0\}$ = *Ann(m)*,

is the annihilator of *m*. Hence $m\,X \cong X\,/\,Ann(m)$.

A subset *I* of a ring *R* is a right ideal if (i) $\forall x, y \in I, x + y \in I,$ *and* (ii) $\forall x \in I, \forall r \in R, xr \in I$.

Every ring *R* may be considered as a right *R*-module and every right ideal can be considered as a submodule of R_R.

Let $M \in Mod\text{-}R$ and $A, B \subset_{>} M$. Then $A \cap B \subset_{>} M$ but $A \cup B$ may not be a submodule of *M*. Suppose that $X \subseteq M$. Consider $F = \{A \subset_{>} M \mid X \subseteq A\}$. Since $M \in F$, we have $F \neq \phi$. Then $|X) = \bigcap_{X \subseteq A \subset_{>} M} A$ is a submodule of *M*. This is the smallest submodule of *M* containing *X*.

Then $|X)$ is called the submodule of *M* *generated* by X, and $|X)$ = $\{\sum_{i=1}^{n} x_i r_i \mid x_i \in X,$ $r_i \in R, i = 1,2,\ldots,n; n \in N\}$.

Consider $|B) = \{\sum_{i=1}^{n} x_i r_i \mid x_i \in X,\ r_i \in R, i = 1,2,\ldots,n; n \in N\}$.

Then $|X) = \{\sum_{i=1}^{n} x_i r_i \mid x_i \in X,\ r_i \in R, i = 1,2,\ldots,n; n \in N\}$.

For $x \in X, x = x_1 \in B.$ So, $X \subseteq B$ and so $|X) \subseteq B.$ for any

$\sum_{i=1}^{n} x_i r_i \in B, x_i \in X \subseteq |X) \Rightarrow \sum_{i=1}^{n} x_i r_i \in |X)$ and so $B \subseteq |X)$.

Hence $B = |X)$. If $A, B \subset_{>} M_R$. As we know $A \cup B \not\subset_{>} M_R$.

Consider $|A \cup B) \subset_{>} M_R$.

Then $|A \cup B) = \{\sum_{i=1}^{n} x_i r_i : x_i \in A \cup B, r_i \in R\}$.

Now $\sum_{i=1}^{n} x_i r_i = \sum_{i=1}^{n} a_i r_i + \sum_{i=1}^{n} b_i r_i = a + b$.

Then $|A \cup B) = \{a + b | a \in A \wedge b \in B\} = A + B$.

A submodule A of M_R is called a direct summand of M if there exists a submodule $B \subset_{>} M$ such that $M = A + B$ and $A \cap B = \{0\}$. In this case, we write $M = A \oplus B$ and call M *a* direct sum of A and B or the sum $A + B$ is direct. In general, for $i \in I$, let $A_i \subset_{>} M_R$.

The sum $\sum_{i \in I} A_i \subset_{>} M$ is called a direct sum if for any $j \in I$,

$$A_j \cap \sum_{i \neq j, i \in I} A_i = 0.$$

Definition 2.2.2 Let M and N be R-modules. A map $f : M \to N$ is a *homomorphism* if

(i) $\forall\ m, m' \in M$ and $\forall\ r \in R \Rightarrow f(m + m') = f(m) + f(m)$

(ii) $\forall\ m \in M$ and $\forall\ r \in R \Rightarrow f(mr) = f(m)r$

If $f : M \to N$ is R- linear, we define its kernel as kerf = $\{ m \in M : f(m) = 0 \}$ and its image as Imf = $\{ f(m) : m \in M \}$. Kerf is a submodule of M and *Imf* is a submodule of N. f is called a monomorphism if $f(m) = f(m')$.

Definition 2.2.3 Let $A \subset_{>} M_R$. Since M is an abelian group, $(M/A, +)$ is also an abelian group. We provide a scalar multiplication to make M/A a right R-module: $(M/A) \times R \to M/A, (m + A, r) \to (m + A)r = mr + A$. Then M/A is a right R-module, called the factor module of M by A.

Definition 2.2.4 An R-homomorphism $f : M_R \to N_R$ is called

(i) a *monomorphism* if for any $X \in Mod\text{-}R$ and for any homomorphism $h, g : X \to M,\ foh = fog \Rightarrow h = g.$

(ii) an *epimorphism* if for any $X \in Mod - R,$ and for any homomorphism $h, g : N \to X, hof = gof \Rightarrow h = g$.

(iii) an *isomorphism* if f is a monomorphism and an epimorphism.

Remarks

(i) $f : M_R \to N_R$ is a monomorphism iff f is one-one.

(ii) $f : M_R \to N_R$ is an epimorphism iff f is onto.

Definition 2.2.5 A ring *R* is *semi-simple* (or *completely reducible*) if *R* is semi-simple as a right *R*-module. A right ideal *I* of *R* which is simple as an *R*-module am called a *minimal right ideal*. A semi-simple ring is thus a direct sum of minimal right ideals, and every simple module is isomorphic to a minimal right ideal of *R*. The module *0* is semi-simple as an empty sum of simple submodules but *0* is not a simple module, since it was assumed that for a simple module *R, R* $\neq$ *0*. Every abelian group may be considered as a Z-module; so an abelian group is semi simple if it is a semi simple Z-module. The factor group Z / nZ, $n \neq 0$, is a semi-simple Z-module if and only if *n* is square-free (i.e., *n* is the product of pair-wise distinct prime numbers, $n = p_1, \ldots, p_K$ or $n = \pm 1$). The modules Z_Z and Q_Z are not semi-simple since they have no simple submodules.

Proposition 2.2.6([19], page 23) The following properties of a module *S* are equivalent:

(a) *S* is semi-simple.

(b) *S* is a direct sum of simple modules.

(c) Every submodule of *S* is a direct summand.

Proof. $(a) \Leftrightarrow (b)$ follows from Proposition 7.1([19], page-23) (with *L* = *0*), and also $(a) \Leftrightarrow (c)$ is an immediate consequence of Proposition 7.1.

$(c) \Leftrightarrow (a)$: The sum of all simple submodules of S is a direct summand of S, and in order to show that the complementary summand is zero, it is enough to show that every non-zero submodule L of S contains a simple submodule.

The module L may as well assumed to be cyclic, so by Lemma 6.8[19], it contains a maximal proper submodule M. The submodule M splits S as $S = M \oplus K$, and then $L = M \oplus (K \cap L)$. It follows that $K \cap L \cong L/M$ is a simple submodule of L.

Lemma 2.2.7 Let A be a right ideal of a ring R. Then

$$A \subset_{>}^{\oplus} R_R \Leftrightarrow \exists e \in R, e^2 = e : A = eR.$$

Proof. Assume that $A \subset_{>}^{\oplus} R_R$. Then there exists a right ideal $B \subset_{>} R_R$ such that $R = A \oplus B$. Since $1 \in R = A \oplus B$, there exists $e \in A, f \in B : 1 = e + f$. Then $e = e^2 + ef$ and $e = e^2 + fe$. Then $ef = fe \in A \cap B = 0$, and so $e = e^2$. This shows that e is an idempotent. Similarly, we can show that $f^2 = f$. We first have, $aR \subseteq A$. Let $a \in A$. Then $a = ea + fa \Rightarrow a - ea = fa \in B \cap A = 0$. So $a = ea \in eR$. Hence $A \subseteq eR$. Thus $A = eR$. Conversely, assume that there exists an idempotent $e \in R$ such that $A = eR$. Since $(1-e)^2 = 1 - e - e + e^2 = 1 - e$, and $1 = e + (1-e) \subseteq eR + (1-e)R$, we have $R = eR + (1-e)R$. For each $x \in eR \cap (1-e)R$, we have $x = er = (1-e)s$ for $r, s \in R$. Since $ex = eer = er = x$ and $ex = e(1-e)s = (e - e^2)s = 0$. we have $x = 0$. So $eR \cap (1-e)R = 0$. Thus $R = eR \oplus (1-e)R$.

Definition 2.2.8 Let M be a right R-module and let $A, B \subset_{>} M$. If $A \cap B = 0$, we write $A + B = A \oplus B$. We note that if $x \in A \oplus B$, then $x = a + b$, where $a \in A$ and $b \in B$.

Theorem 2.2.9 If $A \oplus B$ is the internal direct sum and A, B are submodules of M, then show that $A \oplus B \subset_{>} M$.

Proof. Consider $A \times B = \{(a,b) | a \in A \wedge b \in B\}$, we can consider as $A \coprod B$ or $A \prod B$. It is clear that $A \times B$ is a right R-module but $A \times B \not\subset_{>} M$. Define $\varphi : A \times B \to A \oplus B$ by $\varphi(a,b) = a + b$ for all $a \in A$ and $b \in B$. Then φ is an R-homomorphism, because for any $(a,b), (a',b') \in A \times B$ and for any $r \in R$, we have

$\varphi((a,b) + (a',b')) = \varphi(a + a', b + b') = (a + a') + (b + b') = (a + b) + (a' + b') = \varphi(a,b) + \varphi(a',b'),$

and $\varphi((a,b)r) = \varphi(ar, br) = ar + br = (a + b)r = \varphi(a,b)r$.

Also, $\varphi(a,b)=\varphi(a',b')\Rightarrow a+b=a'+b'\Rightarrow a=a'\wedge b=b'\Rightarrow(a,b)=(a',b')$, showing that φ is a monomorphism. For every $y\in A\oplus B$, $y=a+b$ where $a\in A, b\in B$.

Choose $x=(a,b)\in A\times B$. we have $\varphi(x)=y$. then φ is an epimorphism. Thus φ is an isomorphism, i.e., $A\times B\cong A\oplus B$.

Definition 2.2.10 Let $\{A_i, i\in I\}$ be a family of submodules of *M*. If for any $j\in I, A_j\cap\sum_{i\in I, i\neq j}A_i=0$, then $\sum_{i\in I}A_i$ is called the *direct sum* of $\{A_i, i\in I\}$ which is denoted by $\bigoplus_{i\in I}A_i$.

Theorem 2. 2.11([20], Lemma 7.2, page-246) Let *M* be a right *R*-module.

(1) *Z (M)* is a submodule, called the singular submodule of *M*.

(2) *Z (M)*. Soc(R_R) = 0, where soc(R_R) denotes the socal of R_R

(3) If *f*: $M\rightarrow N$ is any *R*-homomorphism, then $f(Z(M))\subseteq Z(N)$.

(4) If $M\subseteq N$, then $Z(M)=M\cap Z(N)$.

Proof. (1) If $m_1, m_2\in Z(M)$, then $ann(m_i)\subseteq_e R_R$ (i = 1, 2) imply that $ann(m_1)\cap ann(m_2)\subseteq_e R_R$. since $ann(m_1+m_2)$ contains the L.H.S, it follows that $m_1+m_2\in Z(M)$. It remains to prove that $ann(m)\subseteq_e R_R\Rightarrow ann(mr)\subseteq_e R_R\forall r\in R$. For this we apply the criterion for essential extensions in (3.27) (1). Give any element $s\in R|ann(mr)$, we have $m(rs)\neq 0$, so from $ann(m)\subseteq_e R_R$, we see that $m(rst)=0$ for some $t\in R$ such that $rst\neq 0$. now we have $0\neq st\in ann(mr)$ which yields the desired conclusion $ann(mr)\subseteq_e R_R$.

(2) For any $m\in Z(M), ann(m)\subseteq_e R_R$, so by Exercise (6.12) (2), $ann(m)\supseteq soc(R_R)$. this shows that $m.soc(R_R)=0$.

(3) Follows from the fact that $ann(m)\subseteq ann(f(m))$ for any $m\in M$.

(4) Directly from definition.

Note that for any $f\in S=\text{End}(M_R)$, we have $f(Z(M))\subset Z(M)$, i.e., *Z (M)* is a fully invariant submodule of *M*.

Example 2.2.13[20]

(a) Any simple ring is nonsingular.

(b) Any semi simple ring *R* is nonsingular because *R* has no proper essential one-sided ideals. More precisely, a ring *R* is semi simple if and only if any right *R*-module *M* is nonsingular.

(c) Let $M \subset N$ be a right *R*-modules. If *N* is nonsingular, then so is *M*, and the converse holds if *M* is essential in *N*. In particular, we see that *M* is nonsingular if and only if its injective hull *E (M)* is nonsingular.

2.3 Noetherian and artinian modules

Definition 2.3.1 A right *R*- module *M* is called *noetherian* if every non empty family of submodules has a maximal element by inclusion. A ring *R* is called right noetherian if R_R is artinian as a right *R*- module.

Proposition 2.3.2 ([19], page 12) A module is noetherian if and only if every strictly ascending chain of submodules is finite.

Proof. Let M be noetherian and $M_1 \subset M_2 \subset ......$ an ascending chain of submodules. The submodule $\bigcup_{i \in i} M_i$ has a finite number of generators, and all of them must lie in some M_{i_0}. It follows that the chain gets stationary at M_{i_0}. Conversely, it is easy to see that the ascending chain condition for submodules implies that every submodule has a finite number of generators.

The `assending chain condition`, i.e. finiteness of all strictly ascending chains, is usually abbreviated as *ACC.*

Proposition 2.3.3 ([19], page 12) Let *L* be a submodule of *M*. Then *M* is noetherian if and only if both *L* and *M / L* are noetherian.

Proof. *M* noetherian obviously implies that *L* is noetherian. It also implies that M/L is noetherian, because the submodules of M/L can be written as M'/L, there $L \subset M' \subset M$.

Suppose conversely that L and M/L are noetherian. If M' is a submodule of M, then $L \cap M'$ is finitely generated as a submodule of L, and $M'/(L \cap M') \cong (L + M')/L$ is finitely generated as a submodule of M/L. It follows from Lemma 3.1(ii) ([19], page-11) that M' is finitely generated. Hence M is noetherian.

The ring R is right noetherian if R_R is a noetherian module, i.e. every right ideal of R is finitely generated.

Proposition 2.3.4 ([19]) If A ring noetherian, then every finitely generated module is noetherian.

Proof. If R_R is noetherian, then every finitely generated free module is noetherian by Prop. 2.3.3, and therefore every finitely generated module is a quotient of a noetherian module and hence noetherian by Prop. 2.3.3.

Definition 2.3.5 A right *R*-module *M* is called *artinian* if every nonempty family of submodule has a minimal element by inclusion. A right *R* is called right- artinian if R_R is artinian as a right *R*-module.

Proposition 2.3.6 For an Artinian ring *R* the following statements are equivalent:

(a) *R* is semisimple;

(b) Every right ideal of *R* is of the form *eR*, where *e* is an idempotent;

(c) Every nonzero ideal in *R* contains a nonzero idempotent;

(d) *R* has no nonzero nilpotent ideals;

(e) *R* has no nonzero nilpotent right ideals.

Proof: $(a) \Rightarrow (b)$. If ℓ is a right ideal of a semisimple ring *R*, then, by theorem 2.2.5[19] and proposition 2.2.4[19] $R = \ell \oplus \ell'$. Let $1 = e + e'$ be a corresponding decomposition of the identity of the ring *R* in a sum of orthogonal idempotents, then by proposition 2.1.1, $\ell = eR$.

$(b) \Rightarrow (c)$ is trivial.

$(c) \Rightarrow (d)$ Follows from the fact that if *e* is a nonzero idempotent, then $e^n = e \neq 0$ for some *n*.

$(d) \Rightarrow (e).$ If $\ell \neq 0$ is a nilpotent right ideal, then $R\ell$ is a two-sided ideal of R and $(R\ell)^n = R\ell^n$ implies that $R\ell$ is nilpotent as well.

$(e) \Rightarrow (a).$ If ℓ is a simple submodule of the right regular module, i.e., a minimal right ideal in the ring *R*, then by hypothesis $\ell^2 \neq 0$ and, by lemma 9.2.8[19] $\ell = eR,$ where *e* is a nonzero idempotent. Therefore, by proposition 2.1.1, there is a decomposition of *R* is Artinian, by proposition 2.2.4, the ring *R* is semisimple.

Definition 2.3.7 A nonempty family of submodules of M_R is said to satisfy the *Ascending Chain Condition* (briefly, *ACC)* if for any chain

$$M_1 \subset M_2 \subset \ldots \subset M_n \subset \ldots$$

of submodules, there exists a positive integer n such that $M_{n+1} = M_n$ for n = 1, 2,.... A right *R*-module *M* is called *noetherian* if and only if every nonempty family of submodules of *M* has a maximal element by inclusion. A ring *R* is called *right* (res., left) *noetherian* if and only if R_R is a noetherian right (res., left) *R*-module. The ring *R* is called a *noetherian ring* if it is both right and left noetherian.

Theorem 2.3.8 ([18], page-127, Prop. 10.9) Let *M* be a right *R*-module and $A \subset_{>} M$. Then the following conditions are equivalent:

(1) *M* is noetherian;

(2) A and M/A are noetherian;

(3) Any ascending chain $A_1 \subset_{>} A_2 \subset_{>} \ldots \subset_{>} A_n \subset_{>} \ldots$ of submodules of M is stationary, i.e., there exists $n \in \mathrm{N}$ such that $A_n = A_{n+1.}$ This condition is called the ascending chain condition or ACC.

(4) Every submodule of *M* is finitely generated.

Proof. $(1) \Rightarrow (3):$ Suppose that every nonempty family of submodules of *M* has a maximal element by inclution. Given an ascending chain

$$A_1 \subset_{>} A_2 \subset_{>} \ldots \subset_{>} A_n \subset_{>} A_{n+1} \subset_{>} \ldots$$

Let $Y = \{A_i | i \in N\}$. By hypothesis, we can find a maximal element of Y by inclusion, say A_k. We can see that for any $n \geq k, A_k \subset_{>} A_n$. But then since A_k is maximal, $A_n \subset_{>} A_k$. Hence for any $n \geq k, A_n = A_k$. This implies that the chain is stationary.

$(3) \Rightarrow (1):$ Let X be a family of submodules of *M* and let

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_n \subseteq \cdots$$

be a chain in X. By assumption, this chain is stationary. So, we can find A_n such that $A_i \subseteq A_n$, for any i. By Zorn's lemma, X has a maximal element. Then *M* is noetherian.

$(3) \Rightarrow (2):$ Let $\quad X_1 \subset_{>} X_2 \subset_{>} \ldots \subset_{>} X_n \subset X_{n+1} \subset_{>} \ldots$

be a chain of submodules in A. Then this chain is also a chain in *M* and hence it must be stationary. So *A* is noetherian. Now let

$$X_1 \subset_{>} X_2 \subset_{>} \ldots \subset_{>} X_n \subset X_{n+1} \subset_{>} \ldots \qquad (*)$$

be a chain of submodules in *M / A*. Then $X_1 = A_1/A, X_2 = A_2/A, \ldots\ldots.$ with $A_1 \subset_{>} A_2 \subset_{>} \ldots \subset_{>} A_n \subset_{>} A_{n+1} \subset_{>} \ldots \subset_{>} M$. since *M* is noetherian, *M* satisfies (3), and so we can find $n_0 \in N$ such that $A_{n_0} = A_{n_0+1}$ Hence the chain $(*)$ is stationary, proving that *M / A* is noetherian.

$(2) \Rightarrow (3):$ Assume that A and *M / A* are noetherian. Let

$$A_1 \subset_{>} A_2 \subset_{>} \ldots\ldots\ldots. \subset_{>} A_n \subset_{>} A_{n+1} \subset_{>} \ldots\ldots..$$

be a chain in *M*. Then

$$A_1 \cap A \subset_{>} A_2 \cap A \subset_{>} \ldots\ldots\ldots. \subset_{>} A_n \cap A \subset_{>} A_{n+1} \cap A \subset_{>} \ldots\ldots.. \subset_{>} A$$

Since *A* is noetherian, by (3), there exists $n_1 \in N$ such that for any $K \geq 0$, we have $A_{n_1+k} \cap A = A_{n_1} \cap A$. Consider $(A_n + A)/A \subset_{>} M/A$, so we have

$$(A_1 + A)/A \subset_{>} (A_2 + A)/A \subset_{>} \ldots\ldots. \subset (A_n + A)/A \subset_{>} \ldots\ldots. \subset_{>} M/A.$$

Since M/A is noetherian, there exists $n_2 \in N$ such that for any $k \geq 0$, we have $(A_{n_2+k} + A)/A = (A_{n_2} + A)/A$. Hence for any $k \geq 0$, we have $A_{n_2+k} + A = A_{n_2} + A$.

Put $n_0 = \max\{n_1, n_2\}$. Then for any $n \geq n_0$, we have $A_{n_0} \cap A = A_{n_0+k} A$ for all $k \geq 0$ and $A_{n_0} + A = A_{n_0+k} + A$ for all $k \geq 0$. Thus for any $k \geq 0$, we have

$$A_{n_0+k} = A_{n_0+k} \cap (A_{n_0+k} + A) = A_{n_0+k} \cap (A_{n_0} + A) = A_{n_0} + (A_{n_0} \cap A) = A_{n_0}.$$

Hence M is noetherian.

$(3)\Rightarrow(4)$: Let $A \subset_{>} M$ and let $0 \neq m_1 \in A$. Then $m_1R \subset_{>} A$. If $m_1R = A$, then we are done. Suppose $m_1R \neq A$. we can find $m_2 \in A/m_1R$ and then $m_1R \subset_{>} m_1R + m_2R \subset_{>} A$. If $m_1R + m_2R = A$, then we are done. Continuing in this way, we have a chain

$$m_1R \subset_{>} m_1R + m_2R \subset_{>} m_1R + m_2R + m_3R \subset_{>} ...$$

in A by (3), this chain is stationary. Thus M is finitely generated.

$(4)\Rightarrow(3)$: Let $A_1 \subset_{>} A_2 \subset_{>} .... \subset_{>} A_n \subset_{>} A_{n+1} \subset_{>} ....$ be a chain in M. Then $\bigcup_{i=1}^{\infty} A_i = \sum_{i=1}^{\infty} A_i =: A$. So $A \subset_{>} M$. By (4), A is finitely generated. Then by the property of finitely generated module, we can find $i_1,........,i_k$ such that $A = A_{i_1} + ... + A_{i_k}$. Let $n = \max\{i_1,.........,i_k\}$. Then $A = A_n$ proving that the above chain is stationary.

Definition 2.3.9 A nonempty family of submodules of M_R is said to satisfy the *DCC* if for any chain $M_1 \supset M_2 \supset .... \supset M_n \supset ....$ of submodules, there exists a positive integer n such that $M_{n+1} = M_n$ for n = 1, 2, …

A right R-module M is called *artinian* if and only if every nonempty family of submodules of M has a minimal element by inclusion. A ring R is called *right artinian* if R_R is an artinian module. The ring R is called an *artinian ring* if it is both right and left artinian.

Theorem 2.3.10 [18] Let M be a right R-module and let A be its submodule. Then the following statements are equivalent:

(a) M is artinian;

(b) A and M/A are artinian;

(c) Any descending chain $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$ of submodules of M is stationary. This condition is called the *descending chain condition or DCC.*

(d) Every factor module of M is finitely co-generated.

Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of right R-modules. Then Y is noetherian (resp. artinian) $\Leftrightarrow$ X and Z are noetherian (resp. artinian).

Corollary 2.3.11 [3] (1) The image of artinian (resp. noetherian) module is also artinian(resp. noetherian).

(2)The finite sum of artinian (resp. noetherian) submodules of *M* is also artinian (resp. noetherian).

(3) The finite direct sum of artinian (resp. noetherian) modules of *M* is also artinian (resp. noetherian).

(4) If *R* is semi-simple, then *R* is both left and right artinian (resp. noetherian).

Remarks. (1) If a ring *R* is right artinian, then *R* is right noetherian but the converse is not true. For example, consider Z (northerian), $m\mathrm{Z} \subseteq n\mathrm{Z} \Leftrightarrow n|m$ and $m_1\mathrm{Z} \subseteq m_2\mathrm{Z} \subseteq m_3\mathrm{Z} \subseteq \ldots \Leftrightarrow m_2/m_1, m_3/m_2, \ldots$

The chain $2\mathrm{Z} \supset 2^2\mathrm{Z} \supset 2^3\mathrm{Z} \supset \ldots \supset 2^n\mathrm{Z} \supset \ldots$ is not stationary. So Z is not artinian. Thus Z is noetherian but not artinian.

(2) A right *R*-module *M* is artinian but it need not be noetherian. For example, the Prufer group $\mathrm{Z}_{p\infty,}$ where $Q_p = \{\frac{a}{p^i} | a \in \mathrm{Z}$ and $i \in \mathrm{N} \subset_{>} Q\}$ where $\mathrm{Z} \subseteq Q_p$ and $Q_p / \mathrm{Z} =: \mathrm{Z}_{p\infty}$.

In $\mathrm{Z}_p\,\infty$, there are infinite ascending chains of submodules, so $\mathrm{Z}_p\,\infty$ does not satisfy the *ACC* and consequently, it is not noetherian. The module Z_z is noetherian but not artinian. Indeed, since every ideal of *Z* is principal and therefore finitely generated, it is noetherian. Since the chain $\mathrm{Z} \supset 2\mathrm{Z} \supset 2^2\mathrm{Z} \supset \ldots$ is not stationary, we can conclude that Z_z is not artinian. The Prufer *group* $\mathrm{Z}_p\,\infty := \mathrm{Q}_p/\mathrm{Z}$ is artinian but not noetherian. The ring $R = \begin{pmatrix} Z & Q \\ 0 & Q \end{pmatrix}$ is right noetherian but not left noetherian. On the other hand, the ring $R = \begin{pmatrix} Q & R \\ 0 & R \end{pmatrix}$ is right artinian but not left artinian.

Definition 2.3.12 Let *A* be a submodule of a module *M*. Then *A* is a *direct summand* of *M* if there there exists a submodule *B* of *M* such that $M = A \oplus B$ which is equivalent to saying that

$M = A+B$ and $A \cap B = 0 = \{0\}$. The direct summands of R_R correspond to *idempotent* elements of *R*, i.e., $e \in R$ such that $e^2 = e$.

Definition 2.3.13 Let *M* be a right *R*-module and *X*, a subset of *M*. Then the set $|X)$ is called the *submodule* of *M generated* by *X*, where $|X) = \{ \sum_{1 \le i \le n} x_i r_i : x_i \in X, r_i \in R, I = 1, . . . , n; n \in N\}$, and this is the smallest submodule of *M* containing *X*. A subset *X* of M_R is called a *free set* (or *linearly independent set*) if for any $x_1, x_2, x_3, \ldots, x_k \in X$, and for any $r_1, r_2, \ldots, r_k \in R$, we have $\sum_{i=1}^{k} x_i r_i = 0 \Rightarrow r_i = 0 \forall i \in \{1,2,...,k\}$. A subset X of M_R is called a *basis* of *M* if $M = |X)$ and *X* is a free set. If a module *M* has a basis then *M* is called a *free module.*

Definition 2.3.14 A right *R*-module *M* is said to be *finitely generated* if there exists a finite set of generators for *M*, or equivalently, if there exists an epimorphism $R^n \rightarrow M$ for some $n \in \mathrm{N}$. In particular, *M* is cyclic if it is generated by a single element, or equivalently, if there exists an epimorphism $R \rightarrow M$. It follows that *M* is cyclic if and only if $M \cong R / I$ for some right ideal *I* of *R*. For example,
let *M* be a right *R*-module and $m \in M$. Then m generates a cyclic submodule $m R$ of *M*. There is an epimorphism $f : R \rightarrow mM$ given by $f(r) = mr$ and $Ker (f) = \{r \in R \mid mr = 0\}$, which is a right ideal of *R*. Hence $mR \cong R/ Ker (f)$.

Lemma 2.3.15 ([19], page 11) Let *X* be a submodule of a right *R*-module *M*.

(a) If *M* is finitely generated, then so is M / X.

(b) If *X* and M / X are finitely generated, then so is *M*.

Proof. (i): If $x_1, \ldots, x_n$ generate *M*, then $\overline{x_1}, \ldots, \overline{x_n}$ generate M/X.

(ii): Suppose X is generated by $x_1,\cdots,x_m$ and M/X is generated by $\bar{y}_1,\ldots,\bar{y}_{n,}$ where $y_i \in M$. If $x \in M$, then $\bar{x} = \sum \bar{y}_i a_i$ and hence $x - \sum y_i a_i \in X$, so $x - \sum y_i a_i = \sum x_j b_j$. Thus M is generated by $x_1,\ldots,x_m, y_1,\ldots,y_{n,}$

Example. In $Z_6 = \{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\} = |\bar{1})$, Z_6 is a Z-module. Then

1. $|\bar{1}) = Z_6$, $|\bar{2}) = (\bar{0},\bar{2},\bar{4}) = |\bar{4})$, $|\bar{3}) = \{\bar{0},\bar{3}\}$, $|\bar{2},\bar{3}) = Z_6$, because

   $x\bar{3} + y\bar{2} = \bar{1}$ for some x, y $\in Z$.

2. $\{\bar{2}\}$ is not free because $3 \times \bar{2} = \bar{0}$, $\{\bar{2},\bar{3}\}$ is not free

   because $3 \times \bar{2} + 2 \times \bar{3} = \bar{0}$.

Hence Z_6 is a finitely generated Z-module.

Theorem 2.3.16 The following statements hold:

(1) Every free right *R*-module is isomorphic to $R^{(X)}$ for some set *X*.

(2) Every right *R*-module is an epimorphic image of a free module.

Proof. (1) we have $R^{(X)} = \coprod_{x\in X} R_x = \{(r_x)_{x\in X} \,|(r_x)_{x\in X}$ has finite support$\}$.

Let $e_x = (\delta_{xy})_{y\in X.}$ then $\{e_x | x \in X\}$ is a basis of $R^{(X)}$, and so $R^{(X)}$, is a free module. Let *M* be a free right *R*-module with *X* as its basis. For any $m \in M$, we can write $m = \sum_{i=1}^{k} x_i r_i = \sum_{x\in X} x r_x, r_x = 0$ but a finite number.

Define $M \to R^{(X)}$ by $\varphi(m) = \varphi(\sum_{x\in X} x r_x) = (r_x)_{x\in X}$ for all $m \in M$. It is clear that φ is well-defined. To show that φ is a homomorphism. Let $m = \sum_{x\in X} x r_x$ and $m' = \sum_{x\in X} x r'_x$ be any element in *M*. Since $m + m' = \sum_{x\in X} x(r_x + r'_x)$ and $mr = (\sum_{x\in X} x r_x) r = \sum_{x\in X} x(r_x r)$,

we have $\varphi(m+m') = (r_x + r'_x)_{x\in X} = (r_x)_{x\in X} + (r'_x)_{x\in X} = \varphi(m) + \varphi(m')$, and $\varphi(mr) = (r_x r)_{x\in X} = (r_x)_{x\in X}\, r = \varphi(m) r$. Then φ is an *R*-homomorphism.

Let $m = \sum_{x \in X} x r_x$, and $m' = \sum_{x \in X} x r'_x$ be in M such that $\varphi(m) = \varphi(m')$.

Then $(r_x)_{x \in X} = (r'_x)_{x \in X} \Rightarrow r_x = r'_x \forall x \in X \Rightarrow x r_x = x r'_x \forall x \in X$

$x r_x = x r'_x \ \forall x \in X \Rightarrow m = \sum_{x \in X} x r_x = \sum_{x \in X} x r'_x = m'$. This shows that φ is a monomorphism.

Again, let $(rx)_{x \in X}$ be an arbitrary element in $R^{(X)}$. Consider $m = \sum_{x \in X} x r_x$. Then

$\sum_{x \in X} x r_x$ is a finite sum because $(rx)_{x \in X}$ has finite support. This means that $\varphi(m) = (r_x)_{x \in X}$.

Hence φ is an epimorphism. Thus φ is an isomorphism.

(2) To show that for any $M \in Mod - R$, there exists a free module $F \in Mod - R$ and $\varphi : F \to M$ is an epimorphism. Let X be a generating subset of M. Then $M = |X)$. Consider $F = R^{(X)}$. Then F is free. Define $\varphi : R^{(X)} \to M$ by $\varphi((r_x)_{x \in X}) = \sum_{x \in X} x r_x$. Then it is obvious that φ is an epimorphism.

Definition 2.3.17 A module *M* is *simple* (or irreducible) if $M \neq 0$ and the only submodules of *M* are *0* and *M*. Every simple module *M* is cyclic, in fact it is generated by any non-zero $x \in M$. It is clear that *M* is simple if and only if $M \cong I / J$, where *J* is a maximal right ideal of *I*.

Proposition 2.3.18 ([19], page 9) The following properties of an exact sequence

$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ are equivalent:

(a) The sequence splits.

(b) There exists a homomorphism $\varphi :$ Y $\to$ X such that $\varphi \alpha = 1_X$.

(c) There exists a homomorphism $\psi :$ Z $\to$ Y such that $\beta \psi = 1_Z$

Proof. It is clear that (a) implies (b) and (c). Suppose (b) is satisfied. The maps $\varphi : Y \to X$ and $\beta : Y \to Z$ can be used to define $\mu : Y \to X \oplus Z$ so that the diagram(1) commutes. μ is an isomorphism by Prop. 1.3 [19]. Hence the sequence splits. The proof of $(c) \Rightarrow (a)$ goes dually.

A module Y is said to be generated by a family $(x_i)_I$ of elements of Y if each $x \in Y$ can be written $x = \sum_I x_i a_i$ with all but a finite number of a_i equal to *0*. It it furthermore is true that the coefficients a_i are uniquely determined by x, then the family $(x_i)_I$ is a basis for Y. A module is called free if there exists a basis for it.

Theorem 2.3.19 ([18]) For a left *R*-module the following statements are equivalent:

(a) *M* is semisimple;

(b) *M* is generated by simple modules;

(c) *M* is the sum of some set of simple submodule;

(d) *M* is the sum of its simple submodules;

(e) Every submodule of M is a direct summand;

(f) Every short exact sequence $0 \to L \to M \to N \to 0$ of left *R*-modules splits.

Proof. $(a) \Rightarrow (b)$ Let M be a semisimple left *R*-module with semisimple decomposition $M = \oplus_A T_\alpha$. If $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an exact sequence of *R*-modules, then the sequence splits and both *K* and *N* are semisimple.

Since $\operatorname{Im} f$ is a submodule of *M*. The sequence splits and $N \cong M/\operatorname{Im} f \cong \oplus_\beta T_\beta$. But also $M = (\oplus_{A/B} T_\alpha) \oplus (\oplus_\beta T_\beta)$, so that $K \cong \operatorname{Im} f \cong \oplus_{A/B} T_\alpha$. Every submodule and every factor module of a semisimple module are semisimple. Moreover, every submodule is a direct summand.

$(f) \Rightarrow (e)$ A short exact sequence $0 \to L \to M \to N \to 0$. Let M_1 and M_2 be submodule of a module *M*. so $M = M_1 + M_2$ and $M_1 \cap M_2 = 0$, then *M* is the direct sum of its submodules M_1 and M_2, and we write $M = M_1 \oplus M_2$. Thus $M = M_1 \oplus M_2$. If and only if for each $x \in M$ there exists unique elements $x_1 \in M_1$ and $x_2 \in M_2$ such that $x = x_1 + x_2$.

A submodule M_1 of M is a direct summand of *M* in case there is a submodule M_2 of *M* with M = $M_1 \oplus M_2$; such an M_2 is also a direct summand, and M_1 and M_2 are complementary direct summands.

Also $(b) \Leftrightarrow (c) \Leftrightarrow (d)$ are all trivial. Finally, $(e) \Rightarrow (d)$. Assume that *M* satisfies (e). We claim that every non-zero submodule of *M* has a simple submodule. Inded, Let $x \neq 0$ in *M*.

Thus (2.8) R_x has a maximal submodule, say *H*. By (e), we have $M = H \oplus H'$ for some $H' \le M$. Thus by modularity (2.5), $R_x = R_x \cap M = H \oplus (R_x \cap H')$ and $R_x \cap H' \cong R_x / H$ is simple, so R_x has a simple submodule. Let *N* be the sum of all simple submodules of *M*. Then $M = N \oplus N'$, by (e) for some $N' \le M$. Since $N \cap N' = 0$, N' has no simple submodule. But as we have just seen, this means $N' = 0.$ so $n = M$.

Definition 2.3.20 A module M_R is called a *semi-simple module* if and only if every submodule of *M* is a direct summand, i.e., *M* is semi-simple if and only if for any submodule $X \subset M$, there exists a submodule $Y \subset M$ such that $M = X \oplus Y$.

Definition 2.3.21 A submodule *X* of $M \in$ *Mod-R* is called a *simple submodule(or minimal submodule*) if *X* is a simple module, i.e.,

$X \subset_{>} M$ is minimal $\Leftrightarrow M$ is nonzero and $\forall$ submodule

$X \subset_{>} M, 0 \subset_{>} X \subset_{>} M \Rightarrow X = M$

Definition 2.3.22 Let X be a submodule of a right R-module M. Then *X* is called a *maximal submodule* of *M* or maximum in M if $X \neq M$ and for any submodule $Y \subset_{>} M$, if $X \subset_{>} Y \subset_{>} M$ then $Y = X$ or $Y = M$

Theorem 2.3.23 The following statements hold:

(a) Every finitely generated right *R*-module contains at least one maximal submodule. Therefore, every ring with identity contains at least one maximal right ideal.

(b) For any submodule $X \subset_{>} M$, X is maximal if and only if M / X is a simple module.

(c) *M* is simple if and only if for any $0 \neq m \in M, M = mR.$

Proof. (a) We will prove that every proper submodule of a finitely generated right *R*-module M is contained in a maximal module. Let $M = \sum_{i=1}^{n} x_i R$ and $A \subset_{>} \neq M$. Clearly, $\{X_i \mid i \in \{1, \ldots, n\}\} \not\subset A.$ (If not, we have $A \subseteq M$, a contradiction). Consider

$$Y = \{X \subset_{>} M \mid A \subset X \quad and \quad X \neq M\}$$

We note that $A \subset A$ and $A \neq M$, so $A \in Y$. Thus $Y \neq \phi$. Let

$$A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq A_{n+1} \subseteq \ldots$$

be a chain in Y and $B = \bigcup_{n=1}^{\infty} A_n$. If $B = M$, we have $x_1, \ldots, x_n \in B$, there exists $A_{K_1,\ldots,}A_{K_n}$ such that $x_1 \in A_{K_1}, x_2 \in A_{K_2}, \ldots, x_n \in A_{K_n.}$ Choose $m = \max\{k_1, \ldots, k_n\}$. We have $x_1, \ldots, x_n \in A_m$ implying that $A_m = M$, a contradiction. Then $B \neq M$ and $\bigcup_{n=1}^{\infty} A_n \subset_{>} M$. Hence $B \in F$. By Zorn`s lemma, Y contains a maximal element, say C.

Claim. C is a maximal submodule of M. Since $C \in F, C \neq M$. Suppose that $C \subset_{>} \neq Y \subset_{>} M$. If $Y \neq M$, then $A \subseteq Y$ and $Y \neq M$. It follows that $Y \in F$ contradicting the maximality of *C* in *F* so, *Y* = *M*

Assume that M is simple and $0 \neq m \in M$. Then $0 \neq mR \subset_{>} M$, and hence $M = mR$.

Conversely, assume that $M = mR$ for all $0 \neq m \in M$. Let $X \neq 0$ and $X \subset_{>} M$. Then there exists $x \in M \subseteq X$ and $x \neq 0$. Hence $M = xR \subset_{>} X$, and so $M = X$.

Note: Semisimple module $\Leftrightarrow$ sum (or direct sum) of all simple submodule

$\Leftrightarrow$ Every submodule is a direct summand.

Definition 2.3.24 A submodule *A* of a right *R*-module *M* is called *essential or large* in *M* if for any nonzero submodule *U* of *M*, $A \cap U \neq 0$. If *A* is essential in *M* we denote $A \subset_{>}^{*} M$. A right ideal *I* of a ring *R* is called essential if it is essential in R_R. For any right *R*-module *M*, we always have $M \subset_{>}^{*} M$. Any finite intersection of essential submodules of *M* is again essential in *M*, but it is not true in general. For example, consider the ring Z of integers. Every nonzero ideal of Z is essential in Z but the intersection of all ideals of Z is 0 which is not essential in Z. Since any two nonzero submodules of *Q* have nonzero intersection, *Q* is an essential extension of Z. *A* monomorphism $\kappa : U \rightarrow M$ is said to be essential if $\mathrm{Im}(\kappa) \subset_{>}^{*} M$.

$$A \subset^*_> M \Leftrightarrow 0 \neq U \subset_> M, A \cap U \neq 0.$$

$$\Leftrightarrow \forall U \subset_> M, U \neq 0 \Rightarrow A \cap U \neq 0$$

$$\Leftrightarrow \forall U \subset_> M, A \cap U = 0 \Rightarrow U = 0.$$

Proposition 2.3.25 In Z, every nonzero ideal is essential.

Proof. Let $0 \neq I \subset Z$. Then $\exists\ m \in Z : I = mZ$. For any nonzero ideal $J \subset Z$, we can find an $n \in Z : J = nZ$. Thus $I \cap J = mZ \cap nZ = mnZ$, so $m\,n \in I \cap J$, and so $I \cap J \neq 0$. Therefore, $I \subset^*_> Z$.

Proposition 2.3.26 Let M be a right R-module. Then for any submodule $A \subset_> M$, $A \subset^*_> M$ $\Leftrightarrow \forall\ m \in M, m \neq 0, \exists\, r \in R: m\,r \neq 0$ and $m\,r \in A$.

Proof. Assume that $A \subset^*_> M$. Choose $m \in M$, $m \neq 0$. Then $mR \neq 0$, and so $A \cap mR \neq 0$. then there exists $0 \neq x \in A \cap mR$.

This means that $0 \neq x \in A$ and there exists $r \in R$ such that $x = mr$. Therefore, $0 \neq x = mr \in A$.

Conversely, let U be a nonzero submodule of M. Choose $0 \neq m \in U$. By hypothesis, there exists $r \in R$ with $mr \neq 0$ and $mr \in A$. But then since $mr \in U$, we have $mr \neq 0$ and $mr \in A \cap U$. Hence $A \subset^*_> M$.

Proposition 2.3.27 For any $M \in$ Mod-R, let $A \subset_> B \subset_> M$. If $A \subset^*_> M$, then (i) $A \subset^*_> B$, and (ii) $B \subset^*_> M$.

Proof. (i) Let $U \subset_> B$ be such that $U \neq 0$. then U is a submodule of M Since $A \subset^*_> M, U \cap A \neq 0$. Hence $A \subset^*_> B$.

(ii) Let $U \subset_> M$ be such that $U \neq 0$. Then $0 \neq A \cap U \subseteq B \cap U$, because $A \cap U \neq 0$, and so $B \subset^*_> M$.

Proposition 2.3.28 Let A and B be essential submodules in M_R. Then $A \oplus B \subset^*_> M$ and $A \cap B \subset^*_> M$.

Proof. Let $U \subset_{>} M$ be such that $U \neq 0$. Then $U \cap (A \cap B) = (U \cap A) \cap B \neq 0$.

Hence $A \cap B \subset_{>}^{*} M$. We have $A \subset_{>} A \oplus B \subset_{>} M$ and $A \subset_{>}^{*} M$, implying that $A \oplus B \subset_{>}^{*} M$.

Note: Every $\neq 0$ submodule of *M* is essential is *M*, i.e., a non-zero submodule *A* of *M* is called essential in *M* if *A* has non-zero intersection with any non-zero submodule of *M*.

Definition 2.3.29 A submodule *A* of M_R is called *superfluous or coessential or small* in *M* if for any submodule *U* of *M, A + U = M* implies *U = M*, or equivalently, $U \neq M$ implies $X + U \neq M$. A right ideal *I* of a ring *R* is called superfluous in *R* if it is a superfluous submodule of R_R. Every module has at least one superfluous submodule, namely 0.

The sum of a finite number of superfluous submodules of *M* is again a superfluous in *M,* but we are not sure about the arbitrary sum. For example, take Q_Z as a *Z*-module. In Q_Z, every cyclic submodule is superfluous but the sum $\sum_{q \in Q} qZ = Q$ which is not superfluous in *Q*. An epimorphism $\delta : M \to N$ is said to be superfluous if $\text{Ker}(\delta) \subset_{*}^{0} M$. If *A* is superfluous in *M* we denote by

$$A \subset_{*}^{o} M, i.e., A \subset_{*}^{o} M \Leftrightarrow \forall U \subset_{>} M, A + U = M \Rightarrow U = M.$$

2.4 Radical and socle of modules

Definition 2.4.1 Let *M* be a right *R*-module. Then the sum of all superfluous submodules of *M* is called the *radical* of *M* and is denoted by *rad(M).* It is also the intersection of all maximal submodules of *M*. The radical of a module plays a very important role in studying the structure of modules or rings. If *M* is finitely generated, then *rad(M)* is superfluous in *M*. The first important property is that *rad(M / rad(M)) = 0*. $Rad(M) = \sum_{\subset_{>}^{*}} X$

The following theorem gives one of the properties of radical of a module.

If there is no maximal submodule of *M,* then *rad(M) = M*. For example, $rad(Q_Z) = Q$. For a ring *R,* we have $rad(R_R) = rad({}_R R)$, and by this fact we can define the Jacobson radical of a ring *R* by $J(R) = rad(R_R) = rad({}_R R)$ as a two-sided ideal of *R*. There are many kinds of radicals of a ring and we use the terminology Jacobson radical to mention the intersection of all maximal right ideals. The Jacobson radical of a ring *R* can be described in a more tangible way as follows:

Proposition 2.4.2 ([18], page 120) Let *M* be a left *R*-module. Then

$rad\,(M) = \bigcap \{K \leq M \mid K$ is maximal in $M\}$

$= \sum \{L \leq M \mid L$ is superfluous in $M\}$.

Proof: Since $K \leq M$ is maximal in *M* if and only if M/K is simple, the first equality is immediate from the definition of the reject in *M* of a class.

For the second equality, Let $L << M$. If K is a maximal submodule of *M,* and if $L \not\subset$ K, then $K + L = M$; but the since $L << M$, we have $K = M$, a contradiction. We infer that every superfluous submodule of *M* is contained in *Rad M.* On the otherhand, Let $x \in M$. If $N \leq M$ with $R_x + N = M$, then either $n = M$ or there is a maximal submodule *K* of *M* with $N \leq K$ and $x \notin K$. If $x \in RadM$, then the latter cannot occur, thus $x \in RadM$ forces $R_x << M$ and the second equality is proved.

Definition 2.4.3 Let *M* be a right *R*-module. Then the sum of all simple (minimal) submodule of *M* is called the socle of *M* and is denoted by *soc(M).* We also define the socal of a module *M* as the intersection of all essential submodules of *M.*

If $sos(M) = 0$, then *M* does not contain any simple submodules. For example, we have soc(Z_Z) = 0. Especially, $soc(M) = M$ if and only if *M* is semisimple.

Lemma 2.4.4 For any $m \in M, m \in rad(M) \Leftrightarrow mR \subset_{>}^{0} M$.

Proof. Suppose that $m \in rad(M)$. Then $m \in \sum_{X \subset_{>}^{0}} X$

$\Rightarrow m = x_1 + x_2 + \ldots + x_n, x_i \in X_i \subset_{>}^{0} M \;\; \forall i \in I$

$\Rightarrow m \in X_1 + X_2 + \ldots + X_n \subset_{>}^{0} M$

$\Rightarrow mR \subseteq X_1 + X_2 + \ldots + X_n \subset_{>}^{0} M \;\; \Rightarrow mR \subset_{>}^{0} M$.

Conversely, assume that $mR \subset_{>}^{0} M$. Then $mR \in rad(M)$ and so $m \in rad(M)$.

Lemma 2.4.5 For any $m \in M, m \in rad(M) \Leftrightarrow$ there exists a maximal submodule $A \subset_{>} M$ such that $m \notin A$.

Proof. Assume that $m \notin rad(M)$. By Lemma 2.4.4, $mR \not\subset_{>}^{0} M$. Then we can find a submodule $U \subset_{>} M$ such that $m \notin U$ and $mR + U = M$. Consider the family $F = \{U \subset_{>} M \mid m \notin U \text{ and } mR + U = M\}$.

Let $U_1 \subset_{>} .... \subset_{>} U_n \subset_{>} ....$ be any chain in *F*. Put $V = \bigcup_{i=1}^{\infty} U_n$. Then $V \subset_{>} M$ and $m \notin V$. We can see that $mR + V = M$. This shows that $V \in F$. By Zorn's lemma, *F* contains a maximal element, say *B*.

Claim. *B* is a maximal submodule of *M*. Suppose that *B* is not a maximal submodule of *M*. Then there exists a submodule $C \subset_{>} M$ such that $B \subset_{>} \neq C \subset_{>} \neq M$. If $m \in C$, then $M = mR + B \subseteq C$, and so $C = M$, which is contradiction. Hence $m \notin C$ and $mR + C = M = mR + B$. this means that $C \in F$, contradicting the maximality of *F*. Therefore, *B* is a maximal submodule of *M* and $m \notin B$. conversely, if $A \subset_{>}^{\max} M$ and $m \notin A$, then $A \subset mR + A = M \Rightarrow mR \subset_{>}^{0} M \Rightarrow m \notin rad(M)$.

Theorem 2.4.6 $\sum_{X \subset_{>}^{0} M} X = \bigcap_{A \subset_{>}^{\max} M} A.$

Proof. $m \in rad(M) \Leftrightarrow \forall$ maximal submodule $A \subset_{>} M, m \in A \Leftrightarrow m \in \bigcap_{A \subset_{>}^{\max} M} A$.

Theorem 2.4.7 $soc(M) = \bigcap_{Y \subset_{>}^{*} M} Y = \sum_{X \subset_{>}^{simple} M} X.$

Proof. Let $X \subset_{>} M$ be any simple submodule of M and $Y \subset_{>}^{*} M$. Then $X \cap Y \neq 0$ and $X \cap Y \subseteq Y.(0 \neq X \cap Y = X)$. Since X is simple, $X \cap Y = X$, and so $X \subset_{>} Y$. Then $\sum_{X \subset_{>} M, xsimple} X \subset_{>} Y \subset_{>}^{*} M \Rightarrow \sum_{X \subset_{>} M, Xsimple} X \subset_{>} \bigcap_{Y \subset_{>}^{*} M} Y$

Then $\sum_{X \subset_{>} M, x simple} X$ is the largest semisimple submodule. To show that $\bigcap_{Y \subset^*_{>} M} Y$ is semisimple, let

$U \subset_{>} \bigcap_{Y \subset^*_{>} M} Y$ =:K.

Claim. For any submodule U of M, there exists a submodule V of M such that $U \oplus V \subset^*_{>} M$.

Let $F = \{B \subset_{>} M | B \cap U = 0\}$. Since $0 \in F$, so $F \neq \phi$.

Consider $B_1 \subset_{>} B_2 \subset_{>} ......$ any chain in F. Take $A = \bigcup_{i=1}^{\infty} B_i$. Then $A \cap U = 0$, and so $A \in F$.

By Zorn's lemma, F has a maximal element, say V. Clearly, $U \oplus V = M$. Next, to show that $U \oplus V \subset^*_{>} M$. Suppose $U \oplus V \not\subset^*_{>} M$. Then there exists a submodule $X \neq 0$ of M such that $(U + V) \cap X = 0$ Then $U \oplus V \oplus X = M$ and $V \subset V \oplus X$, and $(V \oplus X) \cap U = 0$, a contradiction.

Let U be a submodule of $\bigcap_{Y \subset^*_{>} M} Y$.Then there exists a submodule V of M such that $U \oplus V \subset^*_{>} M$. Then $\bigcap_{Y \subset^*_{>} M} Y \subseteq U \oplus V$. Using modular law,

we have $\bigcap_{Y \subset^*_{>} M} Y = \bigcap_{Y \subset^*_{>} M} Y \subseteq U \oplus V .= U \oplus (\bigcap_{Y \subset^*_{>} M} Y \cap V)$. Hence $\bigcap_{Y \subset^*_{>} M} Y$ is semisimple.

2.5 Exact Sequences, Injective and Projective module

Definition 2.5.1 Let $\{A_i, i \in I\}$ be a collection of right R-modules. For each $i \in I$, let $f_i : A_i \to A_{i+1}$ be an R-homomorphism. Then a sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots\cdots\cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \cdots\cdots$$

is called an *exact sequence* at A_n if $\operatorname{Im}(f_{n-1}) = \ker(f_n)$. The sequence is called an *exact sequence* if it is exact at each A_n.

An special exact sequence of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a *short exact sequence.*

Remarks.

(i) If the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact, then f is a monomorphism, g is an epimorphism and $\mathrm{Im}(f) = \ker(g)$.

(ii) Let $X \subset_{>} M \in Mod-R$. Consider the inclusion map $\iota : X \to M$ defined by $\iota(x) = x$ for any $x \in X$.

(iii) Then the sequence $0 \to X \xrightarrow{\iota} M \xrightarrow{\nu} M/X \to 0$ is exact, where ν is the canonical map.

Definition 2.5.2 A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called *split exact* if $\mathrm{Im}(f) \subset_{>}^{\oplus} B$, (i.e., there exists $B' \subset_{>} B : B = \mathrm{Im}(f) \oplus B'$).

Theorem 2.5.3 Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of right R-modules. Then the following statements are equivalent:

(1) The given sequence splits;

(2) There exists a homomorphism $f' : B \to A : f'f = 1_A$;

(3) There exists a homomorphism $g' : C \to B : gg' = 1_C$.

Proof. $(1) \Rightarrow (2)$. Assume that $\mathrm{Im}(f)$ is a direct summand of B. Then there exists a submodule $B' \subset_{>} B$ such that $B = \mathrm{Im}(f) \oplus B'$. We will define a homomorphism $f : B \to A$. to do this let $b \in B$. Then there exists $y \in \mathrm{Im}(f)$ and $b' \in B'$ such that $b = y + b'$ which is the unique decomposition. Since f is a monomorphism, there is a unique $a \in A$ such that $y = f(a)$. Let $f'(b) = a$. It is clear that f' is a map. We now show that f' is a homomorphism. To do this, let $b_1, b_2 \in B$ and $r \in R$. Then $b_1 = y_1 + b_1'$ and $b_2 = y_2 + b_2'$, where $y_1, y_2 \in \mathrm{Im}(f)$ and $b_1', b_2' \in B'$. Thus $b_1 + b_2 = (y_1 + b_1') + (y_2 + b_2') = (y_1 + y_2) + (b_1' + b_2')$. Then there exists $a_1, a_2 \in A$ Such that $y_1 = f(a_1)$ and $y_2 = f(a_2)$. Then $y_1 + y_2 = f(a_1) + f(a_2) = f(a_1 + a_2)$ and so $f'(b_1 + b_2) = a_1 + a_2 = f'(b_1) + f'(b_2)$. For $r \in R$, if $b_1 = y_1 + b_1'$, then $b_1 r = y_1 r + b_1' r$ and $b_1 r = f(a_1) r = f(a_1 r)$. Hence $f'(b_1 r) = a_1 r = f'(b_1 r)$. To show $ff' = 1_A$, let $a \in A$ be such that $b = f(a)$. Then $f'(b) = a$ and so $ff'(a) = a$. Thus $ff' = 1_A$.

$(2) \Rightarrow (1)$. Assume that there is a homomorphism $f' : B \to A$ such that $ff' = 1_A$. Let $B' = Ker(f') \subset_{>} B$. Then $\mathrm{Im}(f) \oplus B' \subset_{>} B$. For each $b \in B$, we have $f'(b) \in A$ and so $ff'(b) \in \mathrm{Im}(f)$. Then $f'(ff'(b)) = f'f(f'(b)) = f'(b)$.

Hence $ff'(b) - b \in Ker(f') = B'$. Then there exists $b' \in B'$ such that $ff'(b) - b = b'$ and so $b = ff'(b) - b' \in \mathrm{Im}(f) + B'$. Thus $B \subseteq \mathrm{Im}(f) + B'$ and then $B = \mathrm{Im}(f) + B'$. To prove $\mathrm{Im}(f) \cap B' = 0$ let $b \in \mathrm{Im}(f) \cap B'$. Then $b \in \mathrm{Im}(f)$ and $b \in B'$. Thus there is a $a \in A$ such that $b = f(a)$ and $f'(b) = 0$. Then $a = f'f(a) = f'(b) = 0$. This implies that $b = 0$. Therefore, $B = \mathrm{Im}(f) \oplus B'$.

Properties 2.5.4 For a right R-module M, the following conditions are true:

(1) Let X be a maximal submodule of M and let $m \in M - X$ If $X \subset_{>} \neq X + mR \subset_{>} M$, then $X + mR = M$.

(2) Let $X \subset_{>} M$. Suppose that for any $m \in M - X$, $X + mR = M$. If $X \subset_{>} \neq Y \subset_{>} M$ and $Y \neq X$, then there exists $X \subset_{>} \neq X + mR \subset_{>} Y \subset_{>} M$. Hence $Y = M$. Thus X is maximal.

Theorem 2.5.5 A submodule $X \subset_{>} M$ is maximal if and only if $M - X$ is simple.

Proof. Suppose that X is a maximal submodule of M. Let $0 \neq U \subset_{>} M - X$. Then U is of the form $U = V - X$ for some $X \subset V \subset M$. Since X is maximal, $V = X$ or $V = M$. Then $U = 0$ or $U = M - X$. Hence $M - X$ is simple.

Conversely, assume that $M - X$ is simple. Let $X \subset_{>} Y \subset_{>} M$. Then $Y - X \subset_{>} M - X$. Since $M - X$ is simple, we must have $Y - X = 0$ or $Y - X = M - X$. It follows that $Y = X$ or $Y = M$. Hence X is maximal.

Theorem 2.5.6 Let M be a right R-module. Then M is semisimple if and only if $M = \sum_{i \in I} M_i$, where each M_i is simple for any $i \in I$.

Proof. Assume that M *is* semisimple. Let $X=\sum_{i\in I}M_i,$ where M_i a simple is submodule of M, for all $i\in I.$ Then X is a direct summand of $M.$ This means that $M=X\oplus Y$ for some submodule Y of M. If $Y\neq 0,$ let $0\neq y\in Y.$ Then yR *is* cyclic where $yR\subset_{>} Y\subset_{>} M$ and then yR is finitely generated. So yR contains a maximal submodule A, say. We see that both of yR *and* A is a direct summand of M. By the modular law, A *is* a direct summand of yR. Hence $yR=A\oplus B$ and so $B\cong yR/A$ is simple. Then $B\subset_{>} yR\subset_{>} Y.$ Then B is simple but $B\not\subset_{>} X,$ a contradiction. Hence $M=X\oplus 0=X=\sum_{i\in I}M_i,$ where each M_i is simple for all $i\in I.$

Conversely, assume that $M=\sum_{i\in I}M_i,$ where each M_i is simple for any $i\in I.$ Let X be any submodule of $M.$ We must show that X *is* a submodule of $M.$ If $X = 0,$ it is obvious that $M = 0 + M.$ So, suppose that $X\neq 0.$ Let M_i be a submodule of M for all $i\in I.$ Then $M_i\subset_{>} X$ or $M_i\cap X=0$ for all $i\in I.$ If $M_i\cap X\neq 0$ for all $i\in I,$ then $0\neq M_i\cap X\subset_{>} M_i$ for all $i\in I.$ Since M_i is simple, $M_i\cap X=M_i$ for all $i\in I.$ Then $M_i\subset_{>} X$ for all $i\in I.$ Let $F=\{M_i | M_i\cap X=0\}$ and $G=\{M_i | M_i\subset_{>} X\}$. Then we have

$$M=\sum_{i\in I}M_i=\sum_{M_i\in F}M_i\oplus\sum_{M_i\in G}M_i\subseteq X\oplus\sum_{M_i\in F_i}M_i\subseteq M$$

Thus $M=X\oplus\sum_{i\in I}M_i$ and so X is a direct summand of M. Therefore M is semisimple.

Definition 2.5.7 For a pair of sets A and B, a map $f:A\to B$ is called *injective* if and only if it has a left inverse, which means that there is a map $f':B\to A$ such that $f'of=1_A,$ the identity map of A. Dually, for a pair of sets C and D, a map $g:C\to D$ is called *surjective* if and only if it has a right inverse. This means that there exists a map $g':D\to C$ such that $gog'=1_D,$ the identity map of D. We now extend this notion to modules. Let $f:A\to B$ be an R-homomorphism of right R-modules A and B. If there exists an R-homomorphism $f':B\to A$ such that $f'of=1_A,$ then f is a *monomorphism*. Suppose that $f:A\to B$ is a *monomorphism* of right R-modules. Then there does not always exists an R-homomorphism

$f' : B \to A$ such that $f' o f = 1_A$. If B is semisimple, then there exists f' for all right R-modules A. For all right R-modules B, if such homomorphism f' exists, then we call A an *injective module.*

In the categorical viewpoint, a right R-module M is called an *injective module* if for any right R-module A and B, any monomorphism $f : A \to B$ and any homomorphism

$\varphi : A \to M$, there exists a homomorphism $\bar{\varphi} : B \to M$ such that $\bar{\varphi}\ o\ f = \varphi$.

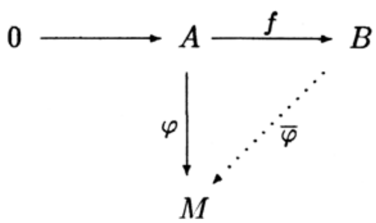


Figure 1

If the above condition is true only for a special module B, then M is called an B-injective module. Thus, a right R-module M is said to be injective if and only if it is B-injective for any right R-module B. A right R-module B is called quasi-injective if B is B-injective.

Theorem 2.5.8 Let M be any right R-module. Then the following statements are equivalent:

(1) M is injective;

(2) Any exact sequence of the form $0 \to M \to A \to B \to 0$ splits.

Proof. $(1) \Rightarrow (2)$. Assume that M is injective. Consider the exact sequence

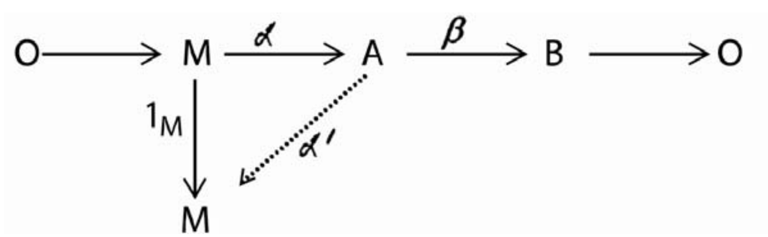


Figure 2

Since M is injective, there exists $\alpha' : A \to M$ such that $\alpha'\alpha = 1_M$. so we get the sequence is splits.

$(2) \Rightarrow (1)$. Let

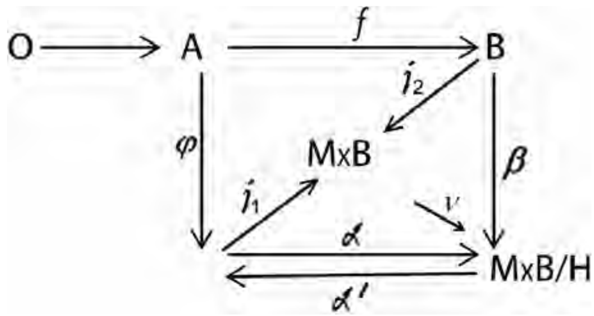


Figure 3

Define $\iota_1 : M \to M \times B$ and $\iota_2 : B \to M \times B$ by $\iota_1(m) = (m,0) \forall m \in M$ and $\iota_2(b) = (0,b)$ $\forall b \in B$. Let $H = \{(\varphi(a), -f(a)) | a \in A\} \subset_{>} M \times B$. Consider $(M \times B)/H$ and define $\alpha = \nu\iota_1$ and $\beta = \nu\iota_2$.

For every $a \in A$,

$$\alpha\varphi(a) = \nu\iota_1\varphi(a) = \nu((\varphi(a),0)) = (\varphi(a),0) + H$$

and $$\beta f(a) = \nu\iota_2 f(a) = \nu((0, f(a)) = (0, f(a)) + H.$$

Since $(\varphi(a),0) - (0, f(a)) = (\varphi(a) - f(a)) \in H$, we have $(\varphi(a),0) + H = (0, f(a)) + H$. We also have $\alpha\varphi = \beta f$. To show that α is a monomorphism. Let $m \in Ker(\alpha)$. Then $\alpha(m) = 0 \Rightarrow \nu\iota_1(m) = 0 \Rightarrow \nu((m,0)) = 0$, i.e., $(m,0) + H = 0 + H$ and so $(m,0) \in H$. Then there exists $a \in A$ such that $(m,0) = (\varphi(a) - f(a))$ which implies that $\varphi(a) = m$ and $f(a) = 0$. Since f is a monomorphism, $a = 0$ and we have $m = \varphi(0) = 0$, i.e. $Ker(\alpha) = 0$. Hence α is a monomorphism. Consider an exact sequence

$$0 \to M \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} (M \times B)/H \to ((M \times B)/H)/\mathrm{Im}(\alpha) \to 0$$

Then by hypothesis, there exists $\alpha' : (M \times B)/H \to M$ such that $\alpha'\alpha = 1_M$. Chose $\bar{\varphi} = \alpha'\beta$.

Then $\bar{\varphi} : B \to M$ and $\bar{\varphi} f = \alpha'\beta f = \alpha'\alpha\varphi = 1_M \varphi = \varphi$. Therefore, M is injective.

Definition 2.5.9 Consider an R-homomorphism $g : C \to D$ of right R-modules. If there is an R-homomorphism $g' : D \to C$ such that $gog' = 1_D$, then g is an epimorphism. In general, such a homomorphism does not always exist. If it exists for all modules C, then D is a free module. When it exists for any module C, we call D a *projective module.*

In categorical viewpoint, right *R*-module *M* is said to be B *projective* if for any epimorphism $g: B \to C$ and any homomorphism $\psi : M \to C$, there exists a homomorphism $\bar{\psi} : M \to B$ such that $g \circ \bar{\psi} = \psi$.

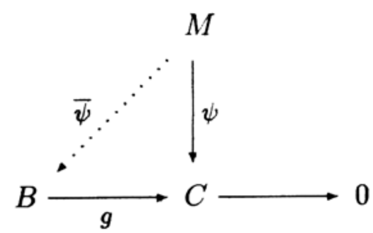


Figure 4

If *M* is *B*-projective for any right *R*-module *B*, then *M* is called a *projective module.* If *B* is *B*-projective, then *B* is called *quasi-projective.*
Every free module is projective but the converse is not true. Consider the ring $R = Z / 6Z$ which can be decomposed as $R = (\bar{2}) \oplus (\bar{3})$. The ideal$(\bar{2})$ and $(\bar{3})$ are projective modules but they are not free. For every $n \in N$, $Z_n = Z / nZ$ is quasi-projective but not *Z*- projective. Thus *Z*-modules Q / Z and Z_{p^∞} are not quasi-projective.

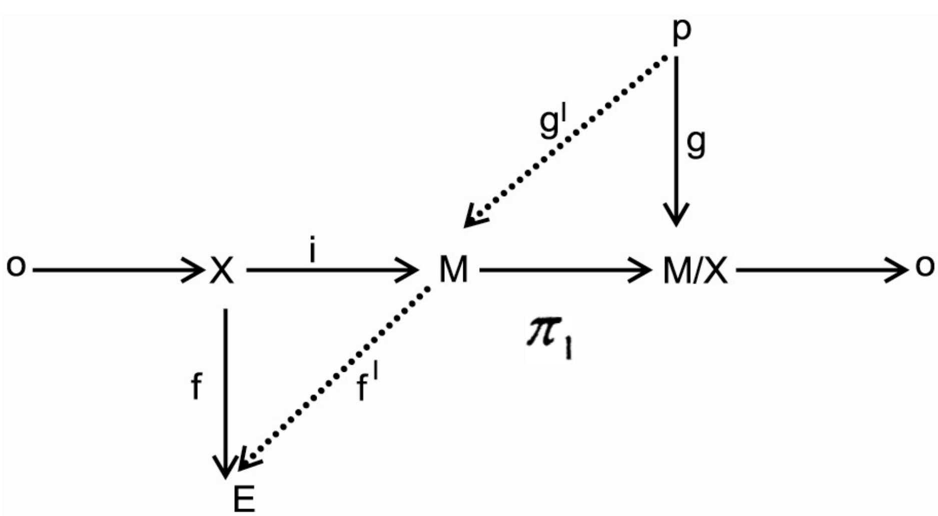


Figure 5

(i) $\pi_1 \circ g' = g$ projective. (ii) $f' \circ i = f$ injective.

(iii) A right R-module M is called quasi-projective if M is M-projective, so $P = M$

(iv) A right R-module M is called quasi-injective if M is M-injective, so $E = M$.

Proposition 2.5.10 Let M be any right R-module. Then the following statements are equivalent:

(a) M is projective.

(b) Any exact sequence of the form $0 \to X \to Y \to M \to 0$ splits.

Proof. (1) $\Rightarrow$ (2). Assume that M is projective. Consider the exact sequence

$0 \to X \xrightarrow{f} Y \xrightarrow{g} M \to 0$.

Since M is projective, there exists a homomorphism $g' : M \to Y$ such that $g'g = 1_M$. so have the sequence is splits.

(2) $\Rightarrow$ (1). Assume that every exact sequence of the form $0 \to X \to Y \to M \to 0$ splits. Let

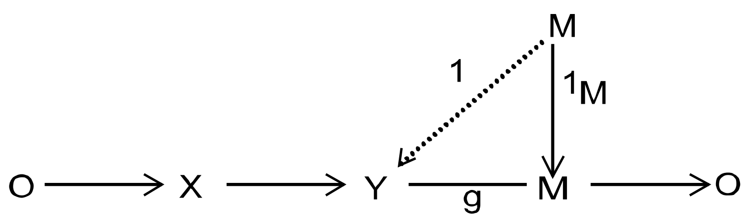


Figure 6

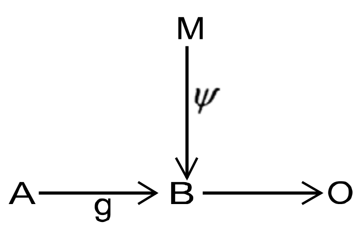


Figure 7

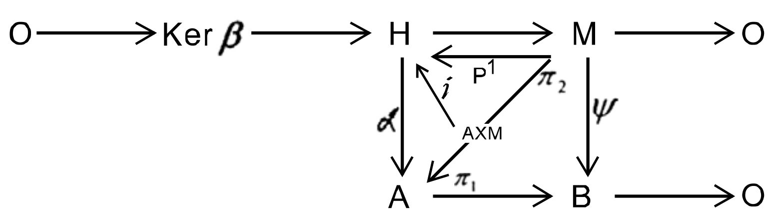


Figure 8

Define $H = \{(a,m) | g(a) = \psi(m)\}$. Then $H \subseteq A \times M$. To show that $H \subset_{>} A \times M$. Let $(a,m),(a',m')H$. Then $g(a) = \psi(m)$ and $g(a') = \psi(a')$.

(i) $g(a+a') = g(a) + g(a') = \psi(m) + \psi(m') = \psi(m+m')$. Thus $(a+a', m+m') \in H$.

(ii) Let $a \in A$ and $r \in R$. Then $g(a) = \psi(m) \Rightarrow g(a)r = \psi(m)r \Rightarrow g(ar) = \psi(m)r$.

Then $(a,m)r = (ar, mr) \in H$. Therefore, $H \subset_{>} A \times M$. Let $\iota : H \to A \times M$ be the embedding map. Put $\alpha = \pi_1 \iota$ and $\beta = \pi_2 \iota$. We first note that $g\alpha = \psi\beta$ such that for any $x \in H$, we have $x = (a,m)$ with $g(a) = \psi(m)$ and

$g\alpha(x) = g(\alpha(a,m)) = g(\pi_1 \iota(a,m)) = g(\pi_1(a,m)) = g(a)$ and

$\psi\beta(x) = \psi(\beta(a,m)) = \psi(\pi_2 \iota(a,m)) = \psi(\pi_2(a,m)) = \psi(m)$. Hence $g\alpha(x) = \psi\beta(x) \forall x \in H$ and so $g\alpha = \psi\beta$. To show that β is an epimorphism. Let $m \in M$. Then $\psi(m) \in B$. Since g is an epimorphism, there is $a \in A$ such that $\psi(m) = g(a)$. So $(a,m) \in H$ and

$\beta(a,m) = \pi_2 \iota(a,m) = \pi_2(a,m) = m$. Hence β is an epimorphism. By assumption, the exact sequence splits. Then there exists $\beta' : M \to H$ such that $\beta\beta' = 1_M$. Choose $\bar{\psi} = \alpha\beta'$. Then $\bar{\psi} : M \to A$ and so $g\bar{\psi} = g\alpha\beta' = \psi\beta\beta' = \psi 1_M = \psi$. Therefore, M is projective.

Proposition 2.5.11 Every free right R-module is projective.

Proof. Let F be a free right R-module and Let X be its basis. Then $F = \bigoplus_{x \in X} xR$. For $x \in X$, we have $\psi(x) \in B$. we can find $a \in A$ such that $\psi(x) = g(a)$ and we see that we can find many $a \in A$ like that but we choose one and we denote it by a_x.

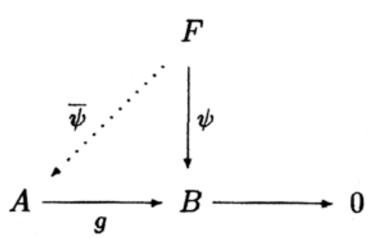
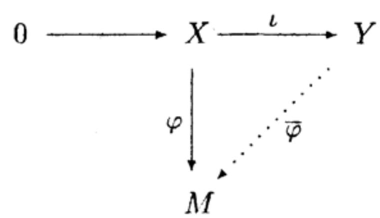


Figure 9

Put $\overline{\psi}(x) = a_x$. For $f \in F$, $f = \sum_{i=1}^{n} x_i r_i$ and $\overline{\psi}(f) = \sum_{i=1}^{n} a_{x_i} r_i \in A$. Then $\overline{\psi}$ is an R-homomorphism and $g\overline{\psi} = \psi$. This shows that F is projective.

Note: M is injective $\Leftrightarrow \forall X, Y \in Mod - R.$



$\Leftrightarrow$ Figure 10

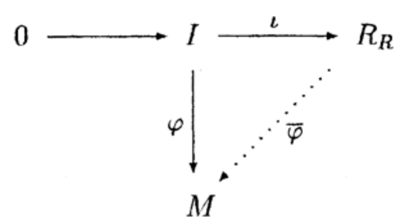


Figure 11

Proposition 2.5.12 Every projective module is isomorphic to a direct summand of a free module, and conversely, any direct summand of a free module is projective.

Proof. Let P be a projective right R-module. By the previous lemma, there exists a free module F such that $\varphi : F \to P$ is an epimorphism. Consider the exact sequence $\varepsilon : 0 \to \ker(\varphi) \xrightarrow{\iota} F \xrightarrow{\varphi P \to 0}$

Since P is projective, ε splits. Then $F = IM(\iota) \oplus F'$ for some $F' \subset_{>} = \ker(\varphi) \oplus F'$. Thus $P \cong F / \ker(\varphi) \cong F' \subset_{>}^{\oplus} F$.

Definition 2.5.13 An element c $\in R$ is called right regular (resp. left regular) if for any r$\in R$, *cr = 0* $\Rightarrow$ *r = 0* (resp. *rc = 0* $\Rightarrow$ *r = 0)*. If *cr = 0 = rc*, then *c* is called a regular element. For example, every non-zero element of an integral domain is regular and if *F* is a field, then any element of the set $M_n(F)$ is regular if and only if its determinant values is zero. Elements which are regular on one side need not be regular.

Proposition 2.5.14 ([19] Prop. 12.1, page 39) The following properties of a ring R are equivalent:

(1) R is regular.

(2) Every principal right ideal of R is generated by an idempotent element.

(3) Every finitely generated right ideal of R is generated by an idempotent element.

(4) Every left R-module is flat.

Proof. $(1) \Rightarrow (2)$: For any $a \in R$, aR is a principal right ideal of R. Choose x such that $a = axa$. Then ax is an idempotent and $aR = axR$.

$(2) \Rightarrow (1)$: Given $a \in R$, chose an idempotent e such that $eR = aR$. Then $e = ax$ for some x, and $a = ea = axa$.

$(1) \Rightarrow (3)$: It clearly suffices to show that if e and f are idempotents, then $eR + fR$ is a principal right ideal. We have $eR + fR = eR + (f - ef)R$, and if $x \in R$ is chosen so that $f - ef = (f - ef)x(f - ef)$, then $f' = (f - ef)x$ is an idempotent with $ef' = 0$, and $eR + fR = eR + f'R$. Now $eR + f'R = (e + f' - f'e)R$, because $e = (e + f' - f'e)e$ and $f' = (e + f' - f'e)f'$. Thus $eR + fR$ is principal.

$(3) \Rightarrow (4)$: The condition (3) means that every finitely generated right ideal is a direct summand of R, and Prop. 10.6 [19] then immediately gives that all left R-modules are flat.

$(4) \Rightarrow (1)$: If I is any right ideal and J is any left ideal of R, then the flatness of R/J implies that the canonical map $I \otimes (R/J) \to R/J$ is a monomorphism. But $I \otimes (R/J) \cong I/IJ$, by Example 8.1[19], so this means that the canonical map $I/IJ \to R/J$ is a monomorphism, i.e. $I \cap J = IJ$. Choosing in particular $I = aR$ and $J = Ra$, we find that $aR \cap Ra = aRa$, and hence $a \in aRa$.

CHAPTER III

PRIME AND SEMI-PRIME RINGS

Definition 3.1

A *prime ideal* in a ring R is any proper ideal P of R such that, whenever I and J are ideals of R with $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$. An ideal I of a ring R is called *strongly prime* if for any $x, y \in R$ with $x\,y \in I$, then either $x \in I$ or $y \in I$. A *prime ring* is a ring in which 0 is a prime ideal or equivalently, a ring R is called a prime ring if there are no nonzero two-sided ideals I and J of R such that $I\,J = 0$.

A *minimal prime ideal* in a ring R is any prime ideal of R that does not properly contain any other prime ideals.

For instance, if R is a prime ring, then 0 is the unique minimal prime ideal of R.

A *semi-prime* ideal in a ring R is any ideal of R which is an intersection of prime ideals. A *semi-prime ring* is any ring in which 0 is a semi-prime ideal. An ideal P in a ring R is semi-prime if and only if R / P is a semi-prime ring. The intersection of any finite list $p_1 Z$,é , $p_k Z$ of prime ideals, where p_1,é , p_k are distinct prime integers, is the ideal p_1é $p_k Z$. Hence the nonzero semi-prime ideals of Z consist of the ideals nZ, where n is any square-free positive integer including $n = 1$.

Example 3.2

(i) In a commutative artinian ring, every maximal ideal is a minimal prime ideal.

(ii) In an integral domain, the only minimal prime ideal is the zero ideal.

Proposition 3.3 The ring R is semi-prime if and only if it contains no nonzero nilpotent elements.

Proposition 3.4([3], page 49) any prime ideal P in a ring R contains a minimal prime ideal.

Proof. Let X be the set of those prime ideals of R which are contained in P. We may use Zornós Lemma going downward in X provided we show that any nonempty chain $Y \subseteq X$ has a lower bound of X.

The set $Q = \cap \mathrm{Y}$ is an ideal of *R*, and it is clear that $Q \subseteq P.$ we claim that *Q* is a prime ideal. Thus consider any $x, y \in R$ such that $xRy \subseteq Q$ but $x \notin Q.$ Then $x \notin P'$ for some $P' \in \mathrm{Y}.$ For any $P'' \in \mathrm{Y}$ such that $P'' \subseteq P'$ we have $x \notin P''$ and $xRy \subseteq Q \subseteq P'',$ whence $y \in P''.$ In particular, $y \in P'.$ If $P\,P'' \in \mathrm{Y}$ and $P'' \not\subset P',$ then $P' \subset P'',$ and so $y \in P''.$ Hence, $y \in P''$ for all elements P'' of Y, and so $y \in Q,$ which proves that *Q* is a prime ideal.

Now $Q \in \mathrm{X},$ and *Q* is a lower bound for Y. Thus, by Zornȏs Lemma, we can get a prime ideal $P'' \in \mathrm{X}$ that is minimal among the ideals in X. Since any prime ideal contained in P'' is in X, we conclude that P'' is a minimal prime ideal of *R*.

Proposition 3.5([3], page 51) If *R* is a commutative ring, then the following are true:

(a) The intersection of all prime ideals of *R* is precisely the set of nilpotent element of *R*.

(b) For every ideal *I* of *R*, the intersection of all of the prime ideals of *R* containing I is the set of elements $r \in R$ such that $r^n \in I$ for some positive integer *n*.

(c) The ring *R* is semi-prime if and only if it contains no nonzero nilpotent elements.

Proof. (a) If *r* is a nilpotent element of *R*, then *r* must be contained in every prime ideal, since if *P* is a prime ideal, then R/P has no nonzero nilpotent elements. Hence, all nilpotent elements are in the intersection of the prime ideals. Conversely, if *r* is not nilpotent, then, letting $X = \{r^n | n \in \mathrm{N}\}$, we can apply Lemma 3.5([3], page 51) to obtain a prime ideal *P* of *R* such that $r \notin P,$ and so *r* is not in the intersection of the prime ideals.

Clearly, (b) follows from (a) by passing to the factor ring R/I and (c) is a special case of (a).

Definition 3.6 A right, left or two-sided ideal *I* of a ring *R* is called a *nil ideal* if and only if $\forall\, a \in I,\ \exists\ \ n \in N$ such that $I^n = 0,$ *nilpotent ideal* if and only if $\exists\, n \in N$ such that $x^n = 0$. More generally, *I* is called a nil ideal if each of its elements is nilpotent. The sum of all nil ideals of a ring *R* is called the nil radical of *R* and is denoted by *N(R)*. The prime radical *P(R*) of a ring *R* is the intersection of all the prime ideals of *R*. Hence we can conclude that $P(R) \subset N(R).$

Let *R* be a semi-prime ring and *I,J* right ideals of *R* such that $IJ = 0$. Then $(JI)^2 = 0$ and $(J \cap I)^2 = 0$. So that $JI = 0$ and $J \cap I = 0$. Moreover, we have the following lemma.

Corollary 3.7 [17]

(a) Every one-sided or two-sided nilpotent ideal is a nil ideal.

(b) The sum of two nilpotent right, left or two-sided ideals is again nilpotent.

(c) If R_R is Noetherian then every two-sided nil ideal is nilpotent.

Proof. (a) Clear.

(b) Let $A \subset_{>} R_R$, $B \subset_{>} R_R$ and $A^m = 0, B^n = 0.$ We assert that $(A+B)^{m+n} = 0.$ Let $a_i \in A, b_i \in B, i = 1,2,3,\ldots, m+n,$ then by binomial theorem $\prod_{i=1}^{m+n}(a_i + b_i)$ is a sum of products of $m + n$ factors of which either at least m factors are from A or at least n factors are from B. Since A and B are right ideals the assertion follows.

(c) Let N be a two sided nil ideal of R. Since R_R is Noetherian, among the nilpotent right ideals contained in N, there is a maximal one. Let A be one such and suppose we have $A^n = 0.$ By (b), A is indeed the largest nilpotent right ideal contained in N. Since for $x \in R$, $x A$ is also a nilpotent right ideal contained in N, A is in fact a two-sided ideal. If for an element $b \in N$ we have: $(bR)^K \subset_{>} A,$ then it follows that $(bR)^{Kn} = 0,$ thus $bR \subset_{>} A.$

Definition 3.8 A ring R is a *prime ring* if for any two elements a and b of R, if a $r\ b = 0$ for all r in R, then either $a = 0$ or $b = 0$.

Example 3.9

(a) Any domain.

(b) A matrix ring over an integral domain. In particular, the ring of 2 x 2 integer matrices is a prime ring.

Lemma 3.10 A non-zero central element of a prime ring R is not a zero divisor in R. In particular, the center $Z(R)$ is an integral domain.

Proof: Let $0 \neq a \in Z(R)$, and $ab = 0$. Then $aRb = Rab = 0.$ implies that $b = 0$ because R is a prime ring. Thus the result follows.

Definition 3.11 A ring R is called a reduced ring if R contains no nonzero nilpotent element.

Lemma 3.12 For any semi-prime ring $R, Z(R)$ is reduced.

Proof: Let $a \in Z(R)$ be such that $a^2 = 0.$ Then $aRa = Ra^2 = 0$ implies that $a = 0$.

Note: Let *P* be a prime ideal, *I* a left ideal and *J* a right ideal of a ring *R*. Then $IJ \subseteq P$ does not imply that $I \subseteq P$ or $J \subseteq P.$ The following statement in the mixed case turns out to be valid([20], Prop. 10.2, page 165) $P \underset{\neq}{\subset} R$ is prime if and only if for any right ideal *I* and any left ideal $J, IJ \subseteq P$ implies that either $I \subseteq P$ or the óifô part follows directly from Prop. 10.2[20]. For the óonly ifô part, let us assume $IJ \subseteq P,$ where *I* is a right ideal and *J* is a left ideal. Then RI and JR are ideals, with $(RI)(JR) = R(IJ)R \subseteq RPR \subseteq P.$ Therefore, we have either $RI \subseteq P$ or $JR \subset P.$ This implies that either $I \subseteq P$ or $J \subseteq P.$

Lemma 3.13 A ring R is a domain if and only if R is prime and reduced.

Proof: First assume that R is a domain. Then $a^n = 0$ for some $n \in$ which implies that $a = 0$ and so R is reduced. Also for all $a, b \in R,$ $aRb = 0 \Rightarrow ab = 0 \Rightarrow a = 0$ or $b = 0,$ so R is a prime ring.

Conversely, assume that R is prime and reduced. Let $a, b \in R$ be such that $ab = 0$. Then for any $r \in R,$ we have $(bra)^2 = (bra)(bra) = br(ab)ra = 0,$ so that $bra = 0.$ This means that $bRa = 0,$ implying that $b = 0$ or $a = 0$, because *R* is a prime ring. This completes the proof.

Proposition 3.14 In a right Artinian ring R, every prime ideal P is maximal. Equivalently, R is prime if and only if it is simple.

Proof: If R/P is semi-prime and right Artinian, then it is semi-simple by Proposition 10.24[20]. Since R/P is in fact prime, it can have only one simple component. Therefore, R/P is simple. In other words, P is a maximal ideal.

In commutative ring theory, it is well-known that *R* is Artinian if and only if *R* is Noetherian and every prime ideal of *R* is maximal by Corollary 23.12[20].

Proposition 3.15 For a ring *R*, the following conditions are equivalent:

(i) All proper ideals are prime;

(ii) The ideals of R are linearly ordered by inclusion;

(iii) All ideals $I \subseteq R$ are idempotent.

Proof: (1) $\Rightarrow$ (2): Let I and J be two proper ideals of R. Then by (i), $I \cap J$ is prime. So that $IJ \subseteq I \cap J$ implies that either $I \subseteq I \cap J$ or $J \subseteq I \cap J$. Thus, we have either $I \subseteq J$ or $J \subseteq I$. This follows (ii).

To show (iii), let us assume that $I \neq R$. By (1), I^2 is a prime ideal. Since $I.I \subseteq I^2$, we must $I \subseteq I^2$. Hence $I = I^2$.

(2) $\Rightarrow$(1): Let P be an ideal of R such that $P \neq R$, and let $I \supseteq P$ and $J \supseteq P$ be two ideals of R such that $IJ \subseteq P$. We must show that $I \subseteq P$ or $J \subseteq P$. By (ii), we may assume that $I \subseteq J$. By(iii), we have $I = I^2 \subseteq IJ \subseteq P$. Thus $I = P$. If $J \subseteq I$, then by (iii), $J = J^2 \subseteq IJ \subseteq P$. Thus $J = P$.

Proposition 3.16 Let $R = End_K(V)$ where V is a vector space over a division ring K. Then R satisfies the properties of the above proposition. In particular, every non-zero homomorphic image of R is a prime ring.

Proof. If $\dim_K(V) < \infty$, R is a simple ring. Therefore, it suffices to treat the case when V is infinite dimensional. By Exercise 3.16[20], the ideals of R are linearly ordered by inclusion. Next consider any ideal $I \neq 0, R$. By Exercise 3.16[20], there exists an infinite cardinal $\beta < \dim_K(V)$ such that $I = \{f \in R : \dim_K f(V) < \beta\}$. For any $f \in I$, let $f' \in R$ be such that f' is the identity on *f(V)*, and zero on a direct complement of *f(V)*. Then $f' \in I$, and $f = f f'$. Therefore, $f \in I^2$, and we have proved that $I = I^2$.

Example 3.17 For any integer *n > 0,*

(i) $R = \begin{pmatrix} Z & nZ \\ Z & Z \end{pmatrix}$ is a prime ring, but $R' = \begin{pmatrix} Z & nZ \\ 0 & Z \end{pmatrix}$ is not.

(ii) R is not isomorphic to the prime ring $P = M_2(Z)$ if *n > 1*.

Discussions: (i) In R', there is a nilpotent ideal $\begin{pmatrix} 0 & nZ \\ 0 & 0 \end{pmatrix} \neq 0$, so R' is not semi-prime. Let alone prime. To show R a prime ring, consider it as a subring of $P = \mathrm{M}_2(\mathrm{Z})$.

Note that $nP \subseteq R$. If $a, b \in R$ are such that $aRb = 0$, then $naPb \subseteq aRb = 0$, and hence $aPb = 0$. Since P is a prime ring, by Theorem 10.20[20], we can conclude that $a = 0$ or $b = 0$.

(ii) Assume that $n > 1$. We show that $R \cong P$. By Proposition 3.1[20], the ideals of P are of the form $\mathrm{M}_2(k\mathrm{Z}) = k\mathrm{M}_2(\mathrm{Z}) = kP$, where $k \in \mathrm{Z}$. Now R has an ideal $\mathrm{M}_2(n\mathrm{Z})$. Since this ideal of R is not of the form kR for any integer k, it follows that $R \cong P$.

In [3], we have the notion of a prime ideal.

Definition 3.18 A proper ideal P in a ring R is a prime ideal if for any ideals I and J of R with $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$. A prime ring is a ring in which 0 is a prime ideal. Note that a prime ring must be non-zero.

Proposition 3.19 ([3], page 48) For a proper ideal P in a ring R, the following conditions are equivalent:

(a) P is a prime ideal.

(b) If I and J are any ideals of R such that $I \supset P$ and $J \supset P$, then $IJ \not\subset P$.

(c) R/P is a prime ring.

(d) If I and J are any right ideals of R such that $IJ \subseteq P$, then either $I \subseteq P$ *or* $J \subseteq P$.

(e) If I and J are any left ideals of R such that $IJ \subseteq P$, then either $I \subseteq P$ *or* $J \subseteq P$.

(f) If $x, y \in R$ with $xRy \subseteq P$, then either $x \in P$ or $y \in P$.

Proof. $(a) \Rightarrow (c)$: Let I and J be ideals in R/P, where P is a prime ideal of R. Then there exists ideal $I' \supseteq P$ and $J' \supseteq P$ such that $I = I'/P$ and $J = J'/P$. suppose $IJ = 0$, then $I'J' \subseteq P$. since P is a prime ideal of R, it follows that

either $I' \subseteq P$ or $J' \subseteq P$ and so either $I = 0$ or $J = 0$.

$(c) \Rightarrow (a)$: Let R/P be a prime ring and I and J be ideals of R satisfying $IJ \subseteq P$ then $(I+P)/P$ and $(J+P)/P$ are ideals in R/P, whose product is equal to zero, since R/P is a prime ring.

We have $(I+P)/P = 0$ or $(J+P)/P = 0.$ Hence $I \subseteq P$ or $J \subseteq P.$

$(a) \Rightarrow (d)$: Since I and J are right ideal of $R,$ $(RI)(RJ) = RIJ \subseteq P.$

Thus $RI \subseteq P$ or $RJ \subseteq P$, and so $I \subseteq P$ or $J \subseteq P.$

$(a) \Rightarrow (e)$: By symmetry.

$(d) \Rightarrow (f)$: Since $(xR)(yR) \subseteq P,$ either $xR \subseteq P$ or $yR \subseteq P$ and so $x \in P$ or $y \in P.$

$(a) \Rightarrow (b)$: I and J are any ideals of R then multiplication not containing $P.$

In [14], Sanh et al. modified the above structure of prime ideals.

Corollary 3.20 ([14], Corollary 1.3) For a proper ideal P in a ring R, the following conditions are equivalent:

(a) P is a prime ideal;

(b) If I and J are any ideals of R such that $I \supset P$ and $J \supset P,$ then $IJ \not\subset P.$

(c) If I and J are any right ideals of R such that $IJ \subset P,$ either $I \subset P$ or $J \subset P;$

(d) If I and J are any left ideals of R such that $IJ \subset P,$ either $I \subset P$ or $J \subset P;$

(e) If $x, y \in R$ with $xRy \subset P,$ either $x \in P$ or $y \in P;$

(f) For any $a \in R$ and any ideal I of R such that $aI \subset P,$ either $aR \subset P$ or $I \subset P;$

(g) R/P is a prime ring.

In this thesis, we prove some properties of prime and semi-prime rings.

Proposition 3. 21 Every maximal ideal M of a ring R is a prime ideal.

Proof. If I and J are ideals of R not contained in M then $I + M = R$ and $J + M = R.$ Now $R = (I+M)(J+M) = IJ + IM + MJ + M^2 \subseteq IJ + M$ and hence $IJ \not\subset M$.

Theorem 3. 22 Let R be a commutative ring.

(a) A maximal ideal is prime.

(b) An ideal P is prime if and only if R / P is integral.

(c) An ideal M is maximal if and only if R / M is a field.

Proof. (a) Let R be a commutative ring and M be an ideal of R such that $M \neq R$. Take any ideal A of R such that $M \subset A \subset R$. Let M be a maximal ideal and $x \in R, x \notin M$. Then $M + Rx = R$. So $M + Rx$ is a proper ideal $\neq M$, and $M + Rx$ must be R, since M is maximal.

(b) Let R be a commutative ring with unity and P a prime ideal. Clearly R/P is a ring. Let $a+P, b+P \in R/P, \forall a,b \in R$. Now

$$(a+P)(b+P) = ab+P = ba+P = (b+P)(a+P).$$

Thus R/P is commutative. Since $1 \in R$, so $1+P \in R/P$. Let $r+P \in R/P$. Now $(1+P)(r+P) = 1.r+P = r+P$ and $(r+P)(1+P) = r.1+P = r+P$. Thus $1+P$ is a unity of R/P. Let $(a+P)(b+P) = P$ additive identity of R/P.

Now $ab+P = P \Rightarrow ab \in p \Rightarrow a \in P$ or $b \in P$. This means that P is a prime ideal.

Again, since $a+P = P$ or $b+P = P$, we can conclude that R/P has no zero divisors.

Hence R/P is a integral domain.

Conversely, let R/P is an integral domain. Then P is an ideal. Let $ab \in P$. Then $ab+P = P \Rightarrow (a+P)(b+P) = P \Rightarrow a+P = P$ or $b+P = P$. So $a \in P$ or $b \in P$. Hence P is a prime ideal of R.

(c) Let R be a Commutative ring with unity. Let M be a maximal ideal of R, we shall prove that R/M is a field: we have R/M is a ring. Let $M+a, M+b \in R/M; a,b \in R$

Now $(M+a)(M+b) = M+ab = M+ba = (M+b)(M+a)$, since R is commutative. Therefore R/M is commutative. Since $1 \in R \therefore M+1 \in R/M$.

Let $M+r \in R/M$. Now $(M+r)(M+1) = M+r.1 = M+r$

$(M+1)(M+r) = M+1.r = M+1$

$\therefore M+1$ is the unitary of R/M.

Let $a \in R$ and $a \notin M \Rightarrow M+a \neq M$

$\therefore M+a$ is a non-zero element of R/M. Since M is a maximal ideal of R, so $M \subset (M,a) = R,\ 1 \in (M,a)\ \therefore 1 \in R$

$\therefore 1 = m + ar$ for some $1 \in R, m \in R$

Now $(M+a)(M+r) = M + ar = M - m + m + ar = M + I$ identity of R/M.

$M+r$ is the multiplicative inverse of $M+a$. Hence R/M is a field.

Conversely, Let R/M is a field, we shall to show that M is a maximal ideal of R. clearly M is a ideal. Since R/M is a field, so it has at least two element, therefore $R \neq M$. Thus $M + a \neq M$. $\therefore M+a$ is a non-zero element of R/M. So $(M+a)$ has a multiplicative inverse, say $M+r$. Since R/M is a field.

$\therefore (M+a)(M+r) = M+1$ unity of R/M

$\Rightarrow M + ar = M + 1$
$\Rightarrow M = M + 1 - ar$
$\Rightarrow 1 - ar \in M$
$\Rightarrow \exists m \in M$

Such that $1 - ar = m \Rightarrow 1 = m + ar \Rightarrow 1 \in (m,a)$

Let $b \in R, b = b.1 \in (M,a)$ therefore $R \subset (M,a), R = (M,a)$

Hence M is a maximal ideal of R.

Corollary 3.23[3] For an ideal I in a ring R, the following conditions are equivalent:

(a) I is a semi-prime ideal.

(b) If J is any ideal of R such that $J^2 \subseteq I$, then $J \subseteq I$.

(c) If J is any right ideal of R such that $J^2 \subseteq I$, then $J \subseteq I$.

(d) If J is any left ideal of R such that $J^2 \subseteq I$, then $J \subseteq I$.

Proof. $(a) \Rightarrow (d)$: For any $x \in J$, we have $xRx \subseteq J^2 \subseteq I$, whence $x \in I$ by theorem 3.7[3]. Thus $J \subseteq I$.

$(c) \Rightarrow (b)$: If $J \not\subset I$, then $I + J$ properly contains I. But since

$(I+J)^2 = I^2 + IJ + JI + J^2 \subseteq I$. we have a contradiction to (c). Thus $J \subseteq I$.

$(b) \Rightarrow (a)$: Given any $x \in R$ such that $xRx \subseteq I$, we have $(RxR)^2 = RxRx \subseteq I$ and so $RxR \subseteq I$, where $x \in I$. By Theorem 3.7[3] is semi-prime.

$(a) \Leftrightarrow (c)$: By symmetry.

Note: *R* is prime ring if and only if *0* is prime ideal. *R* is semi-prime ring if and only if *0* is semi-prime ideal. *R* is semi-prime ring *P(R) = 0*

Lemma 3. 24 [21] for a ring *R* with identity, the following conditions are equivalent:

(a) *R* is a semi-prime ring (*i.e., P(R) = 0);*

(b) *0* is the only nilpotent ideal in *R;*

(c) For ideals *I, J* in *R* with *I J = 0* implies $J \cap I = 0$.

Proof. $(a) \Rightarrow (b)$.Let *R* is prime ring if and only if *0* is prime ideal. *R* is semi-prime ring if and only if *0* is a semi-prime ideal. *R* is semi-prime ring *P(R) = 0.* In noetherian rings, all nil one-sided ideals are nilpotent. If *R* is the non zero ring, it has no prime ideals, and so *P(R) = R.* If *R* is nonzero, at has at least one maximal ideal. A ring is semi-prime if and only if *P(R) = 0.* In any case, *P(R)* is the smallest semi-prime ideal of *R*, and because *P(R)* is semi-prime, it contains all nilpotent one-sided ideals of *R*. Since all nilpotent (left) ideals of *R* are contained in *P(R).*

$(b) \Rightarrow (c)$.If *AB = 0* then $(A \cap B)^2 \subset AB = 0$ and $A \cap B = 0$.

$(c) \Rightarrow (b)$.If *AA = 0* then also $A \cap A = A = 0$.

$(b) \Rightarrow (a)$.Let $0 \neq a \in R$. Then $(Ra)^2 \neq 0$ and with $a = a_0$ there exists $0 \neq a_1 \in a_0Ra_0$. Then also $(Ra_1)^2 \neq 0$ and we find $0 \neq a_2 \in a_1Ra_1$, and so on. Hence a is not strongly nilpotent and $a \notin P(R)$. Therefore *P(R) =0.*

Definition 3.25 The *singular submodule* of a right *R*-module *M* is defined by

$Z(M) = \{m \in M: mK = 0$ for some essential right ideal *K* of *R*}.

The singular right ideal of a ring *R* is defined by $Z(R_R) = \{x \in R \mid xK = 0$ for some essential right ideal *K* of *R*}. In other words, $x \in Z(R_R)$ if and only if $r_R(x)$ is an essential right ideal of R, where $r_R(x)$ is the right annihilator of x in R. If $Z(R_R) = 0$ then the ring R is called a *right non-singular ring*. Singular left ideals are defined similarly.

The singular submodule of a right R-module M is defined by $Z(M) = \{m \in M | mK = 0$ for some essential right ideal K of $R\}$. It is equivalent to saying that $Z(M) = \{m \in M | r_R(m)$ is an essential right ideal of R $\}$, where $r_R(m) = \{r \in R | mr = 0\}$. A right R-module M is called a non-singular module if $Z(M) = 0$ and a singular module if $Z(M) = M$.

Lemma 3.26 Let R be a commutative ring. Then the right singular ideal $Z(R)$ of R is zero if and only if R is semi-prime.

Proof: Suppose that R is a semi-prime ring. Let $z \in Z(R)$. We will show that $z = 0$. Set $I = zR \cap r_R(z)$. We have $zR \cap r_R(z) = 0$. In fact, for any $t \in R$ and any $t' \in r_R(z)$, we have zt' $= 0$. So $z\,tt' = tzt' = 0$, showing that $zR.\ r_R(z) = 0$. We have $I^2 \subseteq I = zR \cap r_R(z) = 0$. So $I^2 = 0$. Since R is a semi-prime ring, 0 is a semi-prime ideal. It follows that $I = 0$. But $r_R(z)$ is an essential right ideal of R. This implies that $zR = 0$. Thus $z = 0$.

Conversely, suppose that $Z(R) = 0$. Let a be an element of R such that $a^2 = 0$. We will show that $a = 0$, from which it follows that R has no non-zero nilpotent element. Let $0 \neq x \in R$. Then we need to consider two cases: (i) $ax = 0 \Rightarrow r_R(a)$; (ii) $ax \neq 0 \Rightarrow a(ax) = a^2x = 0 \Rightarrow$ $ax \in r_R(a)$. Hence $xR \cap r_R(a) \neq 0$. Therefore $r_R(a)$ is an essential right ideal of R. This implies that $a \in Z(R)$. Thus $a = 0$.

Definition 3.27 Let X be a subset of a right R-module M. The *right annihilator* of X is the set $r_R(X) = \{r \in R\colon x\,r = 0$ for all $x \in X\}$ is a right ideal of R. If X is a submodule of M, then $r_R(X)$ is a two-sided ideal of R. Annihilators of subsets of left R-modules are defined analogously, and are left ideals of R. If $M = R$, then the *right annihilator* of $X \subset R$ is

$$r_R(X) = \{ r \in R : x\,r = 0 \text{ for all } x \in X \}$$

As well as a *left annihilator* of X is

$$l_R(X) = \{ r \in R : r\,x = 0 \text{ for all } x \in X \}.$$

A right annihilator is a right ideal of R which is of the form $r_R(X)$ (or simply $r(X)$) for some X and a left annihilator is a left ideal of the form $l(X)$. An element c of a ring R is called *right regular* if $r_R(c) = 0$, *left regular* if $1_R(c) = 0$ and regular if

$$r_R(c) = l_R(c) = 0.$$

For example, every nonzero element of an integral domain is regular. Let $M = M_R$ and $m \in M$. Then $r_R(m) = \{r \in R : mr = 0\} \subset R_R$.

Definition 3.28 A ring R has finite right Goldie dimension if it contains a direct sum of finite number of nonzero right ideals. Symbolically, we write $G.\dim(R) < \infty$. A ring R is called a right *Goldie ring* if it has finite right Goldie dimension and satisfies the *ACC* for right annihilators.

Theorem 3.29 Let R be a ring with the *ACC* for right annihilators. Then the right singular ideal $Z(R)$ of R is nilpotent.

Theorem 3.30 Let R be a semi-prime ring with the *ACC* for right annihilators. Then R has no non-zero nil one-sided ideals.

Corollary 3.31 Let R be a right Noetherian ring. Then each nil one-sided ideal of R is nilpotent.

Defination 3.32 An element $c \in R$ is called right regular (resp. left regular) if for any $r \in R$, $cr = 0 \Rightarrow r = 0$ (resp. $rc = 0 \Rightarrow r = 0$). If $cr = 0 = rc$, then c is called a regular element. For example, every non-zero element of an integral domain is regular and if F is a field, then any element of the set $M_n(F)$ is regular if and only if its determinant value is non-zero. Elements which are regular on one side need not be regular.

Theorem 3.33 Let R be a semi-prime right Goldie ring and let I be an essential right ideal of R. Then I contains a regular element of R.

Lemma 3.34 Let R be a ring with finite right Goldie dimension and let c be a right regular element of R. Then $cR \subset^{*}_{>} R$.

Proof. Let I be a right ideal of R with $I \cap cR = 0$. Since $cI \subset cR$, we can write $I \cap cI = 0$ and so the sum $I + cI$ is direct. Consider $(I + cI) \cap c^2 I$. Take any $x \in (I + cI) \cap c^2 I$. Then $x = c^2 t = u + cv$ where $t, u, v \in I$. This implies that $u = c(ct - v) \in I \cap cR = 0$. Also, $c^2 t = cv$. Then $v = ct \in I \cap cI = 0$. So $x = 0$. This shows that the sum $I + cI + c^2 I$. is direct. By induction, the sum $I + cI + c^2 I + c^3 I$ is direct. Since R has finite right Goldie dimension, $c^n \mathrm{I} = 0$ for some n and since c is right regular, we have $I = 0$. Thus cR is an essential right ideal of R.

Definition 3.35 The singular right ideal of a ring R is defined by $Z(R_R) = \{x \in R \mid xK = 0$ for some essential right ideal K of $R\}$. In other words, $x \in Z(R_R)$ if and only if $r_R(x)$ is an essential right ideal of R, where $r_R(x)$ is the right annihilator of x in R. If $Z(R_R) = 0$ then the ring R is called a *right non-singular ring*. Singular left ideals are defined similarly.

The singular submodule of a right R-module M is defined by $Z(M) = \{m \in M \mid mK = 0$ for some essential right ideal K of $R\}$. It is equivalent to saying that $Z(M) = \{m \in M \mid r_R(m)$ is an essential right ideal of R $\}$, where $\mathrm{r}_R(\mathrm{m}) = \{r \in R \mid mr = 0\}$. A right R-module M is called a non-singular module if $Z(M) = 0$ and a singular module if $Z(M) = M$.

Lemma 3.36 Let R be a right non-singular ring with finite right Goldie dimension. Then the right regular elements of R are regular.

Proof: Let c be a right regular element of R. Then by Lemma 3.17, $cR \subset^{*}_{>} R$. But $l(c) = l(cR)$. Suppose that $l(cR) \neq 0$. Then there is a $t \in l(cR)$ with $t \neq 0$ such that $t(cR) = 0$. Since $cR \subset^{*}_{>} R$, we have $t \in Z(R_R) = 0$ because R is a right non-singular ring. So $t = 0$, a contradiction. Thus $l(cR) = 0$ and so $l(c) = 0$. This means that c is left regular and consequently, c is regular.

Lemma 3.37 Let M be a right R-module and $m \in M$ with $m \neq 0$. If X is an essential submodule of M, then there is an essential right ideal Y of R such that $0 \neq m\,Y \subset X$.

Lemma 3.38 Let R be a right non-singular ring with finite right Goldie dimension. Then R satisfies the *ACC* and the *DCC* for right annihilators.

Proof. Let A, B be right annihilators of R with $A \subseteq B$. Suppose that $A \subset_{>}^{*} B$. Let $b \in B$. Then by Lemma 3.20, there exists an essential right ideal L of R such that $b\,L \subseteq A$. This implies that $l_R(A)b\,L = 0$. Since R is right non-singular, we have $l_R(A)b \in Z(R_R) = 0$. So $l_R(A)\mathrm{b} = 0$ and thus $\mathrm{b} \in r_R(l_R(A)) = A$. Therefore, $A = B$.

Suppose that $A \subseteq B$ and A is not essential in B.

Then there exists a non-zero right ideal $C \subset R$ such that $C \subseteq B$, $A \cap C = 0$ and $A \oplus C \subset_{>}^{*} \mathrm{B}$. If $A \oplus C = \mathrm{B}$, then we are done. Suppose that $A \oplus C \neq \mathrm{B}$. Then there exists a non-zero right ideal $C' \subset \mathrm{R}$ such that $A \oplus C \oplus C' \subset_{>}^{*} \mathrm{B}$.

Consider a strictly ascending chain of right annihilators of R:

$A_1 \subset A_2 \subset, \text{é} ., \subset A_n \subset A_{n+1} \subset \text{é}$

Where $A_2 = A_1 \oplus A_2'$, $A_3 = A_1 \oplus A_2'$, é , $A_n = A_1 \oplus A_2' \oplus \text{é} \oplus A_n'$, . . .

But this contradicts the hypothesis that R has finite right Goldie dimension. So the chain must be stationary. Therefore, $A_n = A_{n+1}$ for some $n \in \mathrm{N}$. Thus R has the *ACC* for right annihilators.

Finally, consider a strictly descending chain of right annihilators of R:

$A_1 \supset A_2 \supset \text{é} \supset A_n \supset A_{n+1} \supset \text{é}$

Where $A_1 = A_2 \oplus A_1'$, $A_2 = A_3 \oplus A_2' \oplus A_1'$, . . , $A_n = A_{n+1} \oplus A_n' \oplus \text{é} . \oplus A_2' \oplus A_1'$, é

But this contradicts the hypothesis that R has finite right Goldie dimension. So the chain must be stationary. Therefore, $A_n = A_{n+1}$ for some $n \in \mathrm{N}$. Thus R has the *DCC* for right annihilators.

Theorem 3.39 A semi-prime right Goldie ring has the *DCC* for right annihilators.

Proof. Let R be a right Goldie ring. Then R has the *ACC* for right annihilators. By Theorem 3.29, $Z(R_R) = 0$, i.e., R is right non-singular. Thus, by Lemma 3.38, R has the *DCC* for right annihilators.

CHAPTER IV

PRIME AND SEMI-PRIME MODULES

In [14], Sanh et al. introduced the notion of prime and semi-prime submodules of a given right *R*-module. In this thesis, we investigate some properties of prime and semi-prime submodules.

Throughout the work, all rings are associative with identity and all modules are unitary right *R*- modules. Let *M* be a right *R*-module and $S=End_R(M)$, its endomorphism ring. Recall that in Ahmed et al. [14] investigated the following results:

Theorem 4.1 Let X be a proper fully invariant submodule of M. Then the following conditions are equivalent:

(1) X is a prime submodule of M;

(2) For any right ideal I of S, any submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$;

(3) For any $\varphi \in S$ and fully invariant submodule U of M, if $\varphi(U) \subset X$, then either $\varphi(M) \subset X$ or $U \subset X$;

(4) For any left ideal I of S and subset A of M, if $IS(A) \subset X$, then either $I(M) \subset X$ or $A \subset X$;

(5) For any $\varphi \in S$ and for any $m \in M$, if $\varphi(S(m)) \subset X$, then either $\varphi(M) \subset X$ or $m \in M$. Moreover, if M is quasi-projective, then the above conditions are equivalent to:

(6) M/X is a prime module.

Proposition 4.2 Let *M* be a right *R*-module which is a self-generator. Then we have the following:

(1) If *X* is a minimal prime submodule of *M*, then I_X is a minimal prime ideal of *S*.

(2) If *P* is a minimal prime ideal of *S*, then $X := P(M)$ is a minimal prime submodule of *M* and $I_X = P$.

Theorem 4.3 Let *M* be a right R-module which is a self-generator. Let *X* be a fully invariant sub module of *M.* Then the following conditions are equivalent:

(1) *X* is a semi-prime submodule of *M;*

(2) If *J* is any ideal of *S* such that $J^2(M) \subset X$, then $J(M) \subset X$;

(3) If *J* is any ideal of *S* such that $J(M) \underset{\neq}{\supset} X$, then $J^2(M) \not\subset X$;

(4) If *J* is any right ideal of *S* such that $J^2(M) \subset X$, then $J(M) \subset X$;

(5) If *J* is any left ideal of *S* such that $J^2(M) \subset X$, then $J(M) \subset X$.

Now, we have more properties about prime and semi-prime submodules.

Definition 4.4 Let *M* be a right *R*-module and *X*, a subset of *M*. Then the set $|X)$ is called the *submodule* of *M generated* by *X,* where $|X) = \{\sum_{1\le i\le n} x_i r_i : x_i \in X, r_i \in R, i = 1, \ldots, n; n \in N\}$, and this is the smallest submodule of *M* containing *X*. A subset *X* of M_R is called a *free set* (or *linearly independent set*) if for any $x_1, x_2, x_3, \ldots, x_k \in X$, and for any $r_1, r_2, \ldots, r_k \in R$, we have $\sum_{i=1}^{k} x_i r_i = 0 \Rightarrow r_i = 0 \forall i \in \{1,2,\ldots,k\}$. A subset X of M_R is called a *basis* of *M* if $M = |X)$ and *X* is a free set. If a module *M* has a basis then *M* is called a *free module.*

Definition 4.5 A right *R*-module *M* is called a self-generator if it generates all of its submodules. A right *R*-module *M* is said to be *finitely generated* if there exists a finite set of generators for *M*, or equivalently, if there exists an epimorphism $R^n \to M$ for some $n \in \mathrm{N}$. In particular, *M* is cyclic if it is generated by a single element, or equivalently, if there exists an epimorphism $R \to M$. It follows that *M* is cyclic if and only if $M \cong R/I$ for some right ideal *I* of *R*. For example, Let *M* be a right *R*-module and $m \in M$. Then m generates a cyclic submodule mR of *M*. There is an epimorphism $f : R \to mM$ given by $f(r) = mr$ and $Ker(f) = \{r \in R \mid mr = 0\}$, which is a right ideal of *R*. Hence $mR \cong R/Ker(f)$.

Proposition 4.6 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a Noetherian module, then there exists only finitely many minimal prime submodules.

Proof. If M is a Noetherian module, then S is a right Noetherian ring. Indeed, suppose that we have an ascending chain of right ideal of S, say $I_1 \subset I_2 \subset \ldots$ Then we have $I_1(M) \subset I_2(M) \subset \ldots$ is ascending chain of submodules of M. Since M is a Noetherian module, there is an integer n such that $I_n(M) = I_k(M)$, for all $k > n$. Then we have I_n = Hom$(M, I_n(M))$ = Hom$(M, I_k(M)) = I_K$. Thus the chain $I_1 \subset I_2 \subset \ldots$ is stationary, so S is a right Noetherian ring. By Theorem 3.4 [3], S has only finitely many minimal prime ideals $P_1, \ldots, P_t$. By Proposition 4.2, $P_1(M), \ldots, P_t(M)$ are the only minimal prime submodules of M.

Lemma 4.7 Let M be a quasi-projective, finitely generated right R-module which is a self-generator and X, a minimal submodule of M. Then I_X is a minimal right ideal of S.

Proof. Let J be a right ideal of S such that $0 \neq J \subset I_X$. Then $J(M)$ is a nonzero submodule of M and $J(M) \subset X$. Thus $J(M) = X$ and it follows that $J = I_X$.

Proposition 4.8 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Let X be a minimal submodule of M. Then either $I_X^2 = 0$ or $X = f(M)$ for some idempotent $f \in I_X$.

Proof. Since X is a minimal submodule of M, I_X is a minimal right ideal of S, by Lemma 4. 1. 7. Suppose that $I_X^2 \neq 0$. Then there is $g \in I_X$ such that $g\, I_X \neq 0$. Since $g\, I_X$ is a right ideal of S and $g\, I_X \subset I_X$, we have $g\, I_X = I_X$. Then there exists $f \in I_X$ such that $g f = g$. Then set $I = \{h \in I_X : g h = 0\}$ is a right ideal of S and I is properly contained in I_X since $f \notin I$. By the minimality of I_X, we must have $I = 0$. We have $f^2 - f \in I_X$ and $g(f^2 - f) = 0$, so $f^2 = f$. Since $f(M) \subset X$ and $f(M) \neq 0$, we have $f(M) = X$.

Corollary 4.9 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Let X be a minimal submodule of M. If M is a semi-prime module, then $X = f(M)$ for some idempotent $f \in I_X$.

Proof. Since M is a semi-prime module, $I_X^2 \neq 0$. Thus $X = f(M)$ for some idempotent $f \in I_X$, by Proposition 4.8.

Proposition 4.10 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then $Z(S)(M) \subset Z(M)$ where $Z(S)$ is a singular ideal of S and $Z(M)$ is a singular submodule of M.

Proof. Let $f \in Z(S)$ and $x \in M$. We will show that $f(x) \in Z(M)$. Since $f \in Z(S)$, there exists an essential right ideal K of S such that $fK = 0$. Then $fK(M) = 0$.

From K is an essential right ideal of S, we have $K(M)$ is an essential submodule of M, and so $x^{-1}K(M)$ is an essential right ideal of R. We have $f(x)(x^{-1}K(M)) = f(x(x^{-1}K(M))) \subset fK(M) = 0$, proving that $f(x) \in Z(M)$.

Corollary 4.11 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a nonsingular module, then S is a right nonsingular ring.

Proposition 4.12 Let M be a right R-module which is a self-generator. If M is a semi-prime module with the ACC for M-annihilators, then M has only a finite number of minimal prime submodules. If $P_1, \ldots, P_n$ are minimal prime submodules of M, then $P_1 \cap, \ldots, \cap P_n = 0$. Also a prime submodule P of M is minimal if and only if I_P is an annihilator ideal of S.

Proof. Since M is a semi-prime module, S is a semi-prime ring. If satisfies the ACC for M-annihilators, then S satisfies the ACC for right annihilators. By Lemma 3.4[3], S has only a finite number of minimal prime ideals. Therefore M has only a finite number of minimal prime submodules, by Proposition 4.2. If $P_1, \ldots, P_n$ are minimal prime submodules of M, then $I_{P_1}, \ldots, I_{P_n}$ are minimal prime ideals of S. Thus $I_{P_1} \cap, \ldots, \cap I_{P_n} = 0$, by Lemma 3.4[3]. But $I_{P_1} \cap, \ldots, \cap I_{P_n} = I_{P_1 \cap, \ldots, \cap P_n}$, we have $P_1 \cap, \ldots, \cap P_n = 0$. Finally, a prime submodule P of M is minimal if and only if I_P is a minimal prime ideal of S. It is equivalent to saying that I_P *is* an annihilator ideal of S, by Lemma 3.4[3].

Proposition 4.13 Let M be a quasi-projective right R-module and X, a fully invariant submodule of M. Then the following are equivalent:

(1) X is a semi-prime submodule of M.

(2) M / X is a semi-prime module.

Proof. (1) $\Rightarrow$ (2). We write $X = \bigcap_{P_i \subset M, P_i\, prime} P_i$. Then by P_i/X is a prime submodule of M/X. So $\bigcap_{P_i \subset_{>} M, P_i\, prime} (P_i/X) = (\bigcap_{P_i \subset_{>} M, P_i\, prime} P_i)/X = X/X = 0$. Thus *0* is a semi-prime submodule of M/X, proving that M/X is semi-prime.

(2) $\Rightarrow$ (1) . Since M/X is semi-prime, *0* is a semi-prime submodule of M/X .

We can write $0 = \bigcap_{Q_i \subset M/X, Q_i\, prime} Q_{i.}$ Then $X = \nu^{-1}(0) = \nu^{-1}(\bigcap_{Q_i \subset_{>} M/X, Q_i\, prime} Q_i) =$ $\bigcap_{Q_i \subset_{>} M/X, Q_i\, prime} \nu^{-1}(Q_i)$. Since Q_i is a prime submodule of M/X, $\nu^{-1}(Q_i)$ is a prime submodule of M. Therefore X is a semi-prime submodule of M.

Lemma 4.14 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a semi-prime Goldie module, then the left annihilator of every essential right ideal of a ring S is zero.

Proof. Since M is a semi-prime Goldie module, S is a semi-prime right Goldie ring [15]. Then the singular ideal *Z(S)* of S is nilpotent since S satisfies the *ACC* for right annihilators. Since S is semi-prime, we have $Z(S) = 0$. It implies that the left annihilator of every essential right ideal of S is zero.

Theorem 4.15 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a semi-prime Goldie module, then for any $f \in S$, the following conditions are equivalent:

(1) f is regular;

(2) f is right regular;

(3) $f(M)$ is an essential submodule of M.

Proof. Obviously (1) $\Rightarrow$ (2), By Proposition 3.12 [15], we have (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Since M is a semi-prime Goldie module, S is a semi-prime right Goldie ring [15]. We first show that fS is an essential right ideal of S. Let P be a right ideal of S such that $fS \cap P = 0$. Then $0 = fS \cap P = Hom(M, f(M) \cap P(M))$.

Since M is a self-generator, $f(M) \cap P(M) = 0$. From $f(M)$ is essential in M, $P(M) = 0$ and so $P = 0$. Thus fS is an essential right ideal of S. By Lemma 4.14, $l_S(f) = l_S(fS) = 0$. Now, put $I = r_S(f)$ and we wish to show that $I = 0$. Choose a right ideal J of S maximal with respect to the property that $I \cap J = 0$. Then $I + J$ is an essential right ideal of S. We claim that fJ is essential in fS. Let fg be a nonzero element of fS. Put $K := g^{-1}(I + J) = \{h \in S: gh \in I + J\}$ then K is an essential right ideal of S, so $l_S(K) = 0$ by Lemma 4.14. Thus, $fgK \neq 0$. But $fgK \subset f(I + J) = fJ$. Therefore fJ is essential in fS. Since $J \cap r_S(f) = 0$, we have $J \cong fJ$. Thus $dim(J) = dim(fJ) = dim(fS) = dim(Ss)$. Hence J is essential in S, so $I = 0$.

CONCLUSION

Sanh et al. [14] introduced the new notion of prime and semi-prime submodules and prime and semi-prime Goldie modules. We can say that this new approach is non-trivial, creative and well-posed. In [14], many results have been investigated that are unparallel.

As an extension of our work, we first give the notions of rings of fractions [13]:

Let R be a ring. Then the *right quotient ring* (*right ring of fractions*) of R, if it exists, is a ring Q satisfying the following properties:

(a) R is a subring of Q ;

(b) Each regular element of R is a unit of Q ;

(c) Each element $q \in Q$ is of the form ac^{-1} for some $a, c \in R$ with c regular, i.e., $qc \in R$ for some regular $c \in R$.

In this case, R is said to be a *right order* in Q.

Let X be a multiplicatively closed subset of a ring R, i.e., for any $x, y \in X$, we have $xy \in X$ and $1 \in X$. Then RX^{-1} exists if and only if X satisfies:

(a) (Right permutable) For any $a \in R$ and $x \in X$, we can find $b \in R$ and $y \in X$ such that $ay = xb$;

(b) (Right reversible) For any $a \in R$, if $xa = 0$ for some $x \in X$, then $ay = 0$ for some $y \in X$.

If $X \subset R$ satisfies (a) and (b), then X is called a *right denominator set*.

The following definition is given in [19].

Let X be a right denominator set in the ring R. For each right R-module M, the *module of fractions* of M with respect to X is defined by MX^{-1} with the canonical structure as a right RX^{-1}-module.

The following results are given in [13].

Theorem 1

Let R be a ring. Then R has a right quotient ring Q which is semi-simple artinian if and only if R is semi-prime right Goldie ring.

Theorem 2

Let R be a ring. Then R has a right quotient ring Q which is simple artinian if and only if R is prime right Goldie ring.

We hope that the notions of prime and semi-prime modules and Goldie modules given in ([14],[15]) along with the notion of modules of fractions given in [19] will be strong tools in generalizing the above theorem (Theorem 1 and Theorem 2) from rings to modules.

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