Critical Behaviour of the Solution of Hydromagnetic Flows in a Channel with Slip at the Permeable Boundaries

A dissertation submitted in partial fulfillment of the requirements for the award of the degree

of

Master of Philosophy
in Mathematics.

By

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Author’s declaration

I declare that the work in this dissertation was carried out in accordance with the Regulations of Bangladesh University of Engineering and Technology, Dhaka. The work is original except where indicated by special reference in the text and no part of it has been submitted for any other degree or diploma.

The dissertation has not been presented to any other University for examination either in Bangladesh or overseas.

SIGNED: DATE:

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Nomenclature

a	Characteristic half-width of the channel
B_0	Electromagnetic induction
c_n	Continued fraction co-efficient
f	Algebraic function
$F_N(x)$	(N+1)th member of a continued fraction
$U(x)$	Power series
$U_N(x)$	N th partial sum of $U(x)$
$u_N(x)$	Approximant function using $U_N(x)$
x_c	Singular point using $U_N(x)$
$x_{c,N}$	Critical point using $U_N(x)$
ω	Vorticity
H	Hartmann number or magnetic field intensity parameter
k	Permeable parameter
L	Channel characteristic length
$P_M^N(x)$	Pade′ approximant in terms of x
r	Radius of the channel
R	Radius of convergence
Re	Reynolds number
Ψ	Stream function
u	Axial fluid velocity component in x-direction
v	Fluid velocity component in y-direction
x, y	Dimensionless Cartesian co-ordinates
ν	Kinematic viscosity
ξ	Small perturbative parameter

ρ	Density of the fluid
∇	Vector differential operator
η	Similarity variable
μ_e	The magnetic permeability
σ_e	Electrical conductivity of the fluid
p	The fluid pressure
μ	The dynamic viscosity coefficient
	The coefficient of sliding friction
V	Characteristic wall suction velocity
t_ω	Dimensionless wall skin friction
ε	Small dimensionless parameter that characterizes the slow variation in the cross-Section of the diverging channel

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Abstract

In this thesis under the title "Critical behavior of the Solution of Hydromagnetic Flows in a Channel with slip at the permeable boundaries", the effects of externally applied homogeneous magnetic field on the steady flow of conducting viscous fluid in a slowly-varying exponentially and uniform width channel with slip at the permeable boundaries is investigated.

The combined effect of magnetic field and permeable walls slip velocity on the two-dimensional, steady, nonlinear flow of an incompressible conducting viscous fluid in a Channel of uniform width by means of $Hermite - Pade'$ approximation especially high order differential approximate method has been studied. Hydromagnetic flows in a channel with slip at the permeable boundaries is investigated from the semi numerical and semi-analytical point of view. We have obtained the series related to similarity parameters by using algebraic programming language MAPLE. The series is then analyzed by approximate methods to show the dominating singularity behavior of the flow and the critical relationship among the parameters of the solution.

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1.1 Foreword

Magnetohydrodynamics (MHD) is that branch of science, which deals with the motion of highly conducting ionized (electric conductor) fluid in presence of magnetic field. The motion of the conducting fluid across the magnetic field generates electric currents which change the magnetic field and the action of the magnetic field on these currents give rise to mechanical forces, which modify the fluid. It is possible to attain equilibrium in a conducting fluid if the current is parallel to the magnetic field. Then the magnetic forces vanish and the equilibrium of the gas is the same as in the absence of magnetic fields. But most liquids and gases are poor conductors of electricity. In the case when the conductor is either a liquid or a gas, electromagnetic forces will be generated which may be of the same order of magnitude as the hydro dynamical and inertial forces. Thus the equation of motion as well as the other forces will have to take these electromagnetic forces into account. The MHD was originally applied to astrophysical and geophysical problems, where it is still very important but more recently applied to the problem of fusion power where the application is the creation and containment of hot plasmas by electromagnetic forces, since material walls would be destroyed.

The motion of an electrically conducting fluid, like mercury, under a magnetic field, in general, gives rise to induced electric currents on which mechanical forces are exerted by the magnetic field. On the other hand, the induced electric currents also produce induced magnetic field. Thus there is a two-way interaction between the flow field and the magnetic field, the magnetic field exerts force on the fluid by producing induced currents and the induced currents change the original magnetic field. Therefore, the magneto hydrodynamic flows (the flows of electrically conducting fluids in the presence of magnetic field) are more complex than the ordinary hydrodynamic flows.

The study of flow of an electrically conducting fluid has many applications in engineering problems such as magneto hydrodynamics generators, plasma studies, nuclear reactors, geothermal energy extraction, and the boundary layer control in the field of aerodynamics. In the past few years, several simple flow problems associated with classical hydrodynamics have received new attention within the more general context of magneto hydrodynamics. The study of the motion of Newtonian fluids in the presence of a magnetic field has applications in many areas, including the handling of biological fluids and the flow of nuclear fuel slurries, liquid metals and alloys, plasma, mercury amalgams and blood. Another field of application is electromagnetic propulsion system which consists of a power source (such as a nuclear reactor), plasma, and tube through which the plasma is accelerated by electromagnetic forces. The study of such systems, which is closely associated with magneto-chemistry, requires a complete understanding of the equation of state and transport properties such as diffusion, the shear stress- shear rate relationship, thermal conductivity, electrical conductivity, and radiation. Some of these properties will undoubtedly be influenced by the presence of an external magnetic field that sets the plasma in hydro magnetic motion.

In recent years, the flow of fluids through porous media has become an important topic because of the recovery of crude oil from the pores of the reservoir rocks. Also there has been a renewed interest in studying magneto hydrodynamics flow because of physiological flow problems. Many medical diagnostic devices especially those used in diagnosing cardiovascular diseases make use of the interaction of magnetic fields with tissue fluids.

1.2 Literature Survey

Most flow governing equations in engineering practice are non-linear in nature and very few nonlinear problems can be solved exactly. But it is sometimes possible to expand solution in powers of some parameters. For this, series analysis is very essential. When the exact closed form solution of a problem is too complicated then one should try to ascertain the approximate nature of the solution. Khan [1] applied approximant methods to several Fluid Dynamical Problems.

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MHD steady flow in a channel with slip at the permeable boundaries was studied by Makinde and Osalusi [2]. They investigated the hydromagnetic steady flow of a viscous conducting fluid in a channel with slip at the permeable boundaries. Here they constructed analytical solutions for the governing non-linear boundary-value problem using perturbation method together with $Pade'$ approximation technique.

The influence of a transverse uniform magnetic field on the flow of a viscous incompressible electrically conducting fluid between two infinite parallel stationary and insulating plates was presented by Hartmann and Lazarus [3]. A survey of MHD studies in the technological fields can be found in Moreau [4]. Further more, an extensive theoretical work was carried out on the hydromegnatic fluid flow in a channel under various situations e.g. the papers of Hartmann [5], Borkakati and Pop [6], Makinde [7] etc. Meanwhile Beavers and Joseph [8] in their experimental work on boundary condition at naturally permeable wall confirmed the existence of the slip at the interface separating the flow in the channel and permeable boundaries. The importance of slip velocity on ultra –filtration performance has been well illustrated by Singh and Lawrence [9]. Pal et al. [10] investigated the effect on slip on longitudinal, dispersion of treasure particles in a channel bounded by porous media. The problem of laminar flow in channels of slowly varying width permeable boundaries was investigated by Makinde [11]. Similarly, several authors, e.g. Rao and Deshikachar [12], to mention but few, have in one way or the other modeled and studied physiological-type flow in the presence of a magnetic field.

In present work we analyze the critical behavior of the two-dimensional, steady, nonlinear flow of an incompressible conducting viscous fluid in porous channels under the influence of an externally applied homogeneous magnetic field using Hermite-Pade approximation approach especially High order differential approximant method. Our main focus will be to analyze the critical relationship among the parameters of the flow.

The remainder of this introductory chapter begins with an overview of series studied by Walter Rudin [13], then Perturbation theory .The Steady flow of viscous incompressible fluid between two porous parallel plates by Raisinghania [14] and a model of porous-channel flow has also been given. Finally, the objective of the present research and a brief outline of the remaining thesis have been explained.

1.3 Overview of Series

Most flows encountered in nature are non-linear and very few nonlinear problems can be solved exactly. But it is sometimes possible to expand solution in powers of some parameters. When the exact closed form solution of a problem is too complicated then one should try to ascertain the approximate nature of the solution. That is why, the series analysis is very much essential.

1.3.1 Power series as a Taylor series

A power series is written in the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots\ldots, \quad (-R < x < R) \tag{1.3.1}$$

The coefficients of the series can be easily expressed in terms of its sum. For this purpose we perform successive differentiations of (1.3.1) and substitute $x = 0$ into the results. This yields

$$f(0) = a_0$$

$$f'(x) = 1a_1 + 2a_2 x + 3a_3 x^2 + \cdots\cdots, \quad f'(0) = 1a_1$$

$$f''(x) = 1.2a_2 + 2.3a_3\, x + 3.4a_4\, x^2 + \ldots\ldots, \quad f''(0) = 1.2a_2$$

$$f'''(x) = 1.2.3a_3 + 2.3.4a_4\, x + 3.4.5a_5\, x^2 + \ldots\ldots, \quad f'''(0) = 1.2.3a_3 \text{ etc.}$$

Finding $a_0, a_1, a_2, \ldots\ldots$ from these relations and substituting them into (1.3.1) we obtain the expression

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots\ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots\ldots, \quad (-R < x < R) \tag{1.3.2}$$

which is nothing but Taylor's series. Thus, a power series is Taylor's series of its sum.

1.4 Perturbation Theory

Perturbation theory is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ξ. These methods are so powerful that sometimes it is actually advisable to introduce a parameter ξ temporarily into a difficult problem having no small parameter, and then finally to set $\xi = 1$ to recover the original problem. This apparently artificial conversion to a Perturbation problem may be the only way to make progress.

The thematic approach of perturbation theory is to decompose a tough problem into an infinite number of relatively easy ones. Hence, perturbation theory is most useful when the first few steps reveal the important features of the solution and the remaining ones give small corrections.

Sometimes nonlinear problems are solved by expanding the solution in powers of one or several small perturbation parameters. The expansion may contain small or large parameters which appear naturally in the equations, or which may be artificially introduced. Let us consider a problem of the form

$$f(u, x, \xi) = 0 \tag{1.4.1}$$

Where f may be an algebraic function or some non-linear differential operator, and ξ is a parameter. It is seldom possible to solve the problem exactly, but there may exist some particular value of $x = x_0$ for which the solution is known. In this case, for $|x| << 1$, one can seek a series for u in powers of x such that

$$u(x) = \sum_{i=0}^{\infty} a_i(\xi)(x - x_0)^i \text{ as } x \to x_0.$$

Then by substituting this into equation (1.4.1), expanding in powers of x and collecting the terms of $O(x^n)$, we can get the required coefficients of the perturbation series.

Example 1.1 Let us take a quadratic polynomial

$$u^2 - (4 + \xi)u + 3\xi = 0 \tag{1.4.2}$$

The perturbation series for (1.4.2) in powers of ξ may be taken in the form

$$u(\xi) = \sum_{i=0}^{\infty} b_i\, \xi^i \tag{1.4.3}$$

for small ξ.

To obtain the first term in the series, we set $\xi = 0$ in (1.4.2) and solve

$$u^2 - 4u = 0 \tag{1.4.4}$$

This expression is easy to factor and we obtain in *zeroth-order* perturbation theory $u(0) = b_0 = 0,4$.

A *second-order* perturbation approximation to the first of these roots consists of writing (1.4.3) as $u_1 = 0 + b_1\,\xi + b_2\,\xi^2 + o(\xi^3)(\xi \to 0)$, substituting this expression into (1.4.2), and neglecting powers of ξ beyond ξ^2. The result is

$$(3 - 4b_1)\xi + \left(b_1{}^2 - b_1 - 4b_2\right)\xi^2 = o\left(\xi^3\right), \qquad \xi \to 0, \tag{1.4.5}$$

Now equating the coefficients of each power of ξ in (1.4.5), give

$$\xi^1 : -4b_1 + 3 = 0; \qquad\qquad \xi^2 : b_1{}^2 - b_1 + 4b_2 = 0$$

and so on. The solutions to the equations are $b_1 = \frac{3}{4},\ b_2 = \frac{3}{64}, \ldots\ldots\ldots.$ Therefore, the perturbation expansion for the root u_1 is

$$u_1 = 0 + \frac{3}{4}\xi + \frac{3}{64}\xi^2 + \ldots\ldots\ldots. \tag{1.4.6}$$

In similar process the other perturbation series is

$$u_2 = 4 - \frac{7}{12}\xi + \frac{7}{72}\xi^2 + o(\xi^3), \qquad \xi \to 0 \quad at\, b_0 = 4\,.$$

Perturbation problems occur in two varieties. One is *regular* perturbation problem and another one is *singular* perturbation problem. In the following subsection, we discuss about them with examples.

1.4.1 Regular and Singular Perturbation Theory

Perturbation series occur in two varieties. A *regular* perturbation problem is defined as one whose perturbation series is a power series in ξ having a nonvanishing radius of convergence. A basic feature of all regular perturbation problems is that the exact solution for small but nonzero $|\xi|$ smoothly approaches the unperturbed or zeroth-order solution as $\xi \to 0$.

A *singular* perturbation problem is defined as one whose perturbation series either does not take the form of a power series or, if it does, the power series has a vanishing radius of convergence. In singular perturbation theory there is sometimes no solution to the unperturbed problem (the exact solution as a function of ξ may cease to exist when $\xi = 0$); when a solution to the unperturbed problem does exist, its qualitative features are distinctly different from those of the exact solution for arbitrarily small but nonzero ξ. In either case, the exact solution for $\xi = 0$ is *fundamentally different in character* from the "neighboring" solutions obtained in the limit $\xi \to 0$. If there is no such abrupt change in character, then we would have to classify the problem as a regular perturbation problem.

When dealing with a singular perturbation problem, one must take care to distinguish between the *zeroth-order* solution (the leading term in the perturbation series) and the solution of the unperturbed problem, since the latter may not even exist. There is no difference between these two in a regular perturbation theory, but in a singular perturbation theory the zeroth-order solution may depend on ξ and may exist only for nonzero ξ.

Example 1.2 The boundary-value problem

$$\xi y'' - y' = 0, \quad y(0) = 0,\ y(1) = 1, \tag{1.4.7}$$

is a singular perturbation problem because the associated unperturbed problem

$$-y' = 0, \quad y(0) = 0,\ y(1) = 1 \tag{1.4.8}$$

has no solution. The solution to this first-order differential equation, y = constant, cannot satisfy both boundary conditions. The solution to (1.4.7) cannot have a regular perturbation expansion of the form $y = \sum_{n=0}^{\infty} y_n(x)\xi^n$ because y_0 does not exist.

Example 1.3 The initial-value problem

$$y'' + (1 - \xi x)y = 0, \quad y(0) = 1,\ y'(0) = 0 \tag{1.4.9}$$

is a regular perturbation problem in ξ over the finite interval $0 \le x \le L$. In fact, the perturbation solution is just

$$y(x) = \cos x + \xi\left(\frac{1}{4}x^2 \sin x + \frac{1}{4}x \cos x - \frac{1}{4}\sin x\right)$$
$$+\xi^2\left(-\frac{1}{32}x^4 \cos x + \frac{5}{48}x^3 \sin x + \frac{7}{16}x^2 \cos x - \frac{7}{16}x \sin x\right) + \ldots\ldots\ldots.., \tag{1.4.10}$$

Which converges for all x and ξ, with increasing rapidity as $\xi \to 0+$ for fixed x.

1.5 Singularities

Singularity of a function is a value of the independent variable or variables for which the function is undefined. Singularities are crucial points of a function, because the expansion of a function into a power series depends on the nature of singularities of the function. For the purpose of this thesis, we are interested to analyze those functions, which have several types of singularities. Practically, one of these singularities dominates the function. Therefore it is important to know about his singular point to analyze the critical behavior of the function around this point.

The convergency of the sequence of partial sums depends crucially on the singularities of the function represented by the series. Several types of singularities may arise in physical (nonlinear) problems. The dominating behavior of the function $u(x)$ represented by a series may be written as

$$u(x) \sim A\left(1-\frac{x}{x_c}\right)^{\alpha} \text{ as } x \to x_c \tag{1.5.1}$$

Where A is a constant and x_c is the critical point with the critical exponent α. If α is a negative integer then the singularity is a pole; otherwise if it is a nonnegative rational number then the singularity is a branch point. We can include the correction terms with the dominating part in (1.5.1) to estimate the degree of accuracy of the critical points. It can be as follows

$$u(x) \sim A\left(1-\frac{x}{x_c}\right)^{\alpha}\left[1+A_1\left(1-\frac{x}{x_c}\right)^{\alpha_1}+A_2\left(1-\frac{x}{x_c}\right)^{\alpha_2}+\ldots\right] \text{ as } x \to x_c \tag{1.5.2}$$

Where $0<\alpha_1<\alpha_2<M$ and A_1, A_2, M are constants. $\alpha_i+\alpha \notin N$ for some i, then the correction terms are called confluent. Sometimes the correction terms can be logarithmic. e.g,

$$u(x) \sim A\left(1-\frac{x}{x_c}\right)^{\alpha}\left\{1+\ln\left|1-\frac{x}{x_c}\right|\right\} \text{as } x \to x_c . \tag{1.5.3}$$

Sometimes the sign of the series coefficients indicate the location of the singularity. If the terms are of the same sign the dominant singular point lie on the positive *x*-axis. If the terms take alternately positive and negative signs then the singular point is on the negative x-axis.

Following are few examples with different types of singularities:

Example 1.4 (Singularities for single variable functions)

1. Singularities that are poles: $u(x) = \frac{1}{2}(2-x)^{-1} + \sin(2x)$.

   Here $u(x)$ is an algebraic function whose singularity is at $x_c = 2$, the critical exponent $\alpha = -1$, which makes the singularity a pole.

2. Algebraic singularities with different exponents:

   $$u(x) = \frac{1}{2}(2-x)^{-1/2} + \left(1-\frac{x}{3}\right)^{-1/3} + \left(1-\frac{x}{2}\right)^{-1/4}.$$

   Here $u(x)$ has several singular points. The singular points are at $x_c = 2, 3, 2$ and the critical exponents are $\alpha = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}$ respectively. In this example the singular points are branch points. Though there are a number of singularities for $u(x)$, only one of these singularities will dominate the local behavior of $u(x)$.

3. Logarithmic singularity:

   $$u(x) = \ln\left(1+\frac{x}{5}\right) + \cos(x).$$

   Here $u(x)$ has a logarithmic singularity at $x_c = -5$.

4. Essential singularity:

   $$u(x) = \exp(1-3x)^{-2}.$$

   Here $u(x)$ has an essential singularity at $x_c = \frac{1}{3}$ with critical exponent α = -2.

5. Algebraic dominant singularity with a secondary logarithmic behavior:

   $$.\,u(x) = \left(1-\frac{x}{3}\right)^{-1/3} + \ln\left(1-\frac{x}{7}\right)$$

   The algebraic dominant singularity of $u(x)$ here is at $x_c = 3$ with critical exponent $\alpha = -\frac{1}{4}$, which makes it a branch point. And a logarithmic singularity at $x_c = 7$.

6. nth root singularity: $u(x) = \left(1 - \frac{x}{2}\right)^{-1/n} + \exp(x)$.

   Here $u(x)$ has a branch point with the critical exponent $\alpha = -\frac{1}{n}$ at $x_c = 2$.

Now in following subsection we discuss about Steady flow of viscous incompressible fluid between two porous parallel plates.

1.6 Steady flow of viscous incompressible fluid between two porous parallel plates:

Consider the steady laminar flow of viscous incompressible fluid between two infinite parallel porous plates separated a distance 2h as shown in figure (1.1). By porous plates we mean that plate's posses very fine holes distributed uniformly over the entire surface of the plates through which fluid can flow freely and continuously. The plate from which the fluid enters the flow is known as the plate with injection and the plate from which fluid leaves the flow reason is known as the plate with suction. Let x be the direction of the main flow, y the direction perpendicular to the flow and the width of the plates parallel to the z direction we take the velocity component w to be zero everywhere and u as a function as y alone. The continuity equation reduces to $\frac{\partial v}{\partial y}$ =0 so that v does not vary with y this implies that the fluid enters the flow region through one plate (say the plate situated at y=-h) as the same constant velocity v_0, say it leaves through the other plate (i.e. the plate situated at y=h) as shown in figure. There is constant velocity component v_0 alone y direction.

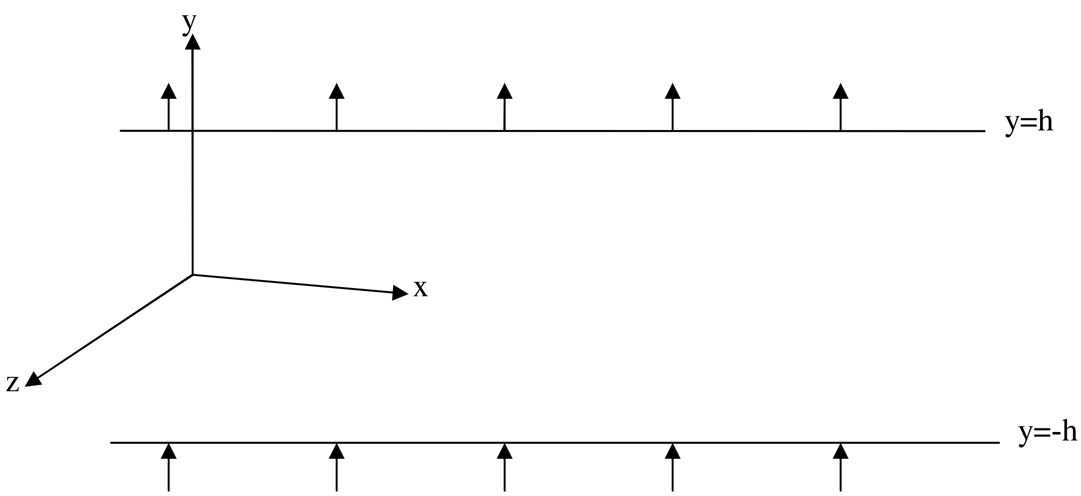

Fig.1.1: Schematic diagram of porous channel flow

For the steady flow in the absence of body forces the Navier- stokes equation for x and y direction are given by

$$\upsilon_0 \frac{du}{dy} = -\frac{1\partial p}{\rho \partial x} + v\frac{d^2u}{dy^2} \tag{1.6.1}$$

and
$$0 = -\frac{1}{2}\frac{\partial \mathrm{p}}{\rho \partial \mathrm{y}}. \tag{1.6.2}$$

(1.6.2) shows that the pressure does not depend on *y*. Hence *p* must be function of *x* alone and so (1.6.1) reduces to

$$\frac{dp}{dx} = \rho\left[v\frac{d^2u}{dy^2} - \upsilon_0 \frac{du}{dy}\right] \tag{1.6.3}$$

Differentiating (1.6.3) w.r.to '*x*', we have

$$\frac{d^2p}{dx^2} = 0 \ or \frac{d}{dx}\left(\frac{dp}{dx}\right) = 0.$$

Integrating,
$$\frac{dp}{dx} = \text{constant} = -P\left(say\right), \tag{1.6.4}$$

where the minus sign has been taken as we expect ρ to decrease as *x* increases. Then (1.6.3) reduces to

$$\frac{d^2u}{dy^2} - \frac{u_0}{v}\frac{du}{dy} = -\frac{p}{\rho v} \tag{1.6.5}$$

Integrating (1.6.5), we have

$$\frac{du}{dy}-\frac{\upsilon_0}{v}u = A-\frac{Py}{\rho v} \tag{1.6.6}$$

which is linear differential equation of first order and first degree.

Integrating factor $= e^{-\int (\upsilon_0/v)dy} = e^{-(\upsilon_0 y/v)}$

Hence the solution of (1.6.6) is given by

$$ue^{-(\upsilon_0 y/v)} = \int\left(A-\frac{Py}{\rho v}\right)e^{-(\upsilon_0 y/v)}dy + B$$

$$=\left(A-\frac{Py}{\rho v}\right)\left(-\frac{v}{\upsilon_0}e^{-(\upsilon_0 y/v)}\right)-\left(-\frac{P}{\rho v}\right)\left(\frac{v^2}{\upsilon_0^2}e^{-(\upsilon_0 y/v)}\right)+B$$

$$u=-\frac{v}{\upsilon_0}\left(A-\frac{Py}{\rho v}\right)+\frac{Pv}{\rho\upsilon_0^2}+Be^{(\upsilon_0 y/v)}$$

Or $$u=C+\frac{P}{\rho\upsilon_0}y+Be^{(\upsilon_0 y/v)} \tag{1.6.7}$$

where $C=-(Av/\upsilon_0)$. Here B and C are constants of integration to be determined. Let the plate situated at y = − h be at rest and the plate at y = h be moving with a constant velocity U. Then, B and C will be determined from the boundary conditions.

$$u=0 \ at \ y=-h; \ \ u=U \ \ at \ \ y=h \tag{1.6.8}$$

Using (1.6.8), (1.6.7) gives.

$$\left.\begin{aligned} 0 &= C-\frac{Ph}{\rho\upsilon_0}+Be^{-(\upsilon_0 h/v)} \\ U &= C+\frac{Ph}{\rho\upsilon_0}+Be^{(\upsilon_0 h/v)} \end{aligned}\right\} \tag{1.6.9}$$

Solving (1.6.9) for B and C and substituting the values so obtained in (1.6.7), we have.

$$u=\left(U-\frac{2Ph}{\rho\upsilon_0}\right)\frac{e^{(\upsilon_0 y/v)}-e^{-(\upsilon_0 h/v)}}{2\sinh(\upsilon_0 h/v)}+\frac{p}{\rho\upsilon_0}(y+h) \tag{1.6.10}$$

Let $$\mathrm{Re}=\frac{\upsilon_0 h}{v} \text{ and } \eta=\frac{y}{h}. \tag{1.6.11}$$

Then (1.6.10) reduce to

$$u=\left(U-\frac{2Ph^2}{\mu \mathrm{Re}}\right)\frac{e^{\eta \mathrm{Re}}-e^{-\mathrm{Re}}}{2\sinh \mathrm{Re}}+\frac{Ph^2}{\mu \mathrm{Re}}(1+\eta) \tag{1.6.12}$$

which gives the velocity distribution in terms of non-dimensional quantities Re (Reynold's number) and η. Notice that plates are situated at $\eta = \pm 1$.

We now consider two particular cases.

Case I. Plane Couette Flow.

In this case there is no pressure gradient *i.e.* P = 0. Then (1.6.12) reduces to.

$$u = \frac{1}{2} U \left(e^{\eta \mathrm{Re}} - e^{-\mathrm{Re}} \right) \cos ech\ \mathrm{Re}. \tag{1.6.13}$$

The shearing stress at any point is given by

$$\sigma_{xy} = \mu \frac{du}{dy} = \frac{\mu \mathrm{Re}\ U e^{\eta \mathrm{Re}}}{2h \sinh \mathrm{Re}} \tag{1.6.14}$$

Hence the skin friction at the plate $\eta = \pm 1$ are given by

$$\left[\sigma_{yx} \right]_{\eta=1} = \frac{\mu \mathrm{Re}\ U}{2h} \frac{e^{\mathrm{Re}}}{\sinh \mathrm{Re}} \tag{1.6.15}$$

and

$$\left[\sigma_{xy} \right]_{\eta=-1} = \frac{\mu \mathrm{Re}\ U}{2h} \frac{e^{\mathrm{Re}}}{\sinh \mathrm{Re}} \tag{1.6.16}$$

Cash II. Plane Poiseuille flow.

In the case both the plates are taken at rest. Hence the velocity distribution may be deduced from (1.6.12) be writing U = 0. Thus we obtain

$$u = \frac{Ph^2}{\mu \mathrm{Re}} \left(1 + \eta - \frac{e^{\eta \mathrm{Re}} - e^{-\mathrm{Re}}}{\sinh \mathrm{Re}} \right) \tag{1.6.17}$$

It can be easily established that the maximum velocity occurs when

$$\eta = \frac{1}{\mathrm{Re}} \log \frac{\sinh \mathrm{Re}}{\mathrm{Re}} \tag{1.6.18}$$

and the skin friction is given by

$$\sigma_{yx} = -\frac{Ph^2}{\mathrm{Re}} \left(\frac{\mathrm{Re}\ e^{\eta \mathrm{Re}}}{\sinh \mathrm{Re}} - 1 \right) \tag{1.6.19}$$

In the following subsection, a model of porous-channel flow is given.

1.7 Model of porous-channel flow:

We consider here the steady two-dimensional flow of an incompressible viscous fluid driven through a channel by uniform suction at parallel rigid porous walls (Figure 1.2). Some mathematician like Berman, Durlofsky & Brady and Zaturska et al. has worked on this problem in various ways. For asymmetric, unsteady, three-dimensional perturbation flows, the solution leads to a pitchfork bifurcation. We apply the multivariable algebraic approximant method to this problem to compute this structurally unstable bifurcation.

Consider the two-dimensional infinite channel with walls at $y = \pm h$. Let the velocity components be u and v for $-\infty < \mathrm{x} < \infty$ and $-h < y < h$. Then by expressing the velocity components in terms of a stream function f,

$$u = xf', \quad u = -f, \tag{1.7.1}$$

and using a similarity transformation , the problem can be expressed as a fourth-order nonlinear ordinary differential equation.

$$\frac{d^4 f}{dy^4} = \mathrm{Re}\left(\frac{df}{dy}\frac{d^2 f}{dy^2} - f\frac{d^3 f}{dy^3}\right), \; -1 < y < 1, \tag{1.7.2}$$

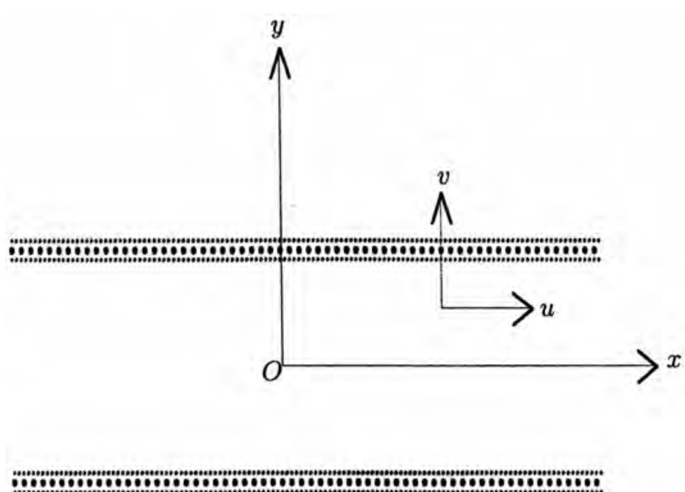

Figure: 1.2: A schematic diagram of the porous channel flow.

With the boundary conditions

$$f = \pm 1, \; f' = 0, \; at\; y = \pm 1 \tag{1.7.3}$$

The dimensionless Reynolds number is $\mathrm{Re}=\dfrac{Vh}{v}$, where $V>0$ the suction velocity on the walls is, h is the half-width of the channel and v is the kinematic viscosity of the fluid. We can use MAPLE to compute the perturbation series for the stream function $f=f(y,\,\mathrm{Re})$ for low Reynolds number:

$$f(y,\,\mathrm{Re})=\frac{y}{2}\left(y^2-3\right)+\frac{y}{280}\left(2-3y^2+y^6\right)\mathrm{Re}+\ldots$$

Zaturska et al. traced the evolution of the stable solution as Re increases slowly. They remarked that the solution is stable for $-\infty<\mathrm{Re}<6.001$ and turns to a pair of asymmetric stable solutions as Re increases further. These include steady solutions for $6.001<\mathrm{Re}<12.963$ and chaotic solutions for $\mathrm{Re}=(\approx 20)$. As in Brady, the conclusion is that the first bifurcation occurs at $\mathrm{Re}\cong 6.0$. This bifurcation is a pitchfork. We refer the reader to Zaturska et al. for a detailed discussion of the nature of the solutions beyond the pitchfork bifurcation.

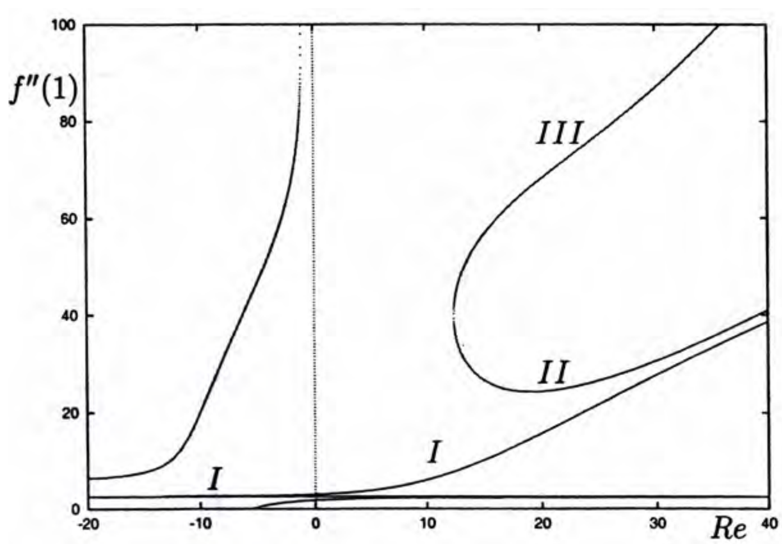

Figure 1.3: Approximate diagram for the type I, II and III symmetric solutions by using the Drazin-Tourigny method. The other curves are spurious.

Drazin & Tourigny [25] considered this series but, as shown in Figure 1.3, their method was unable to reveal the pitchfork bifurcation. The method of high-order differential approximants is equally ineffective in this respect. See Table (1.1).

$$f(-1,\mathrm{Re}) = 1-\varepsilon,\ f(1,\mathrm{Re}) = -1,\ f'(\pm 1,\ \mathrm{Re}) = 0, \tag{1.7.4}$$

where ε is some small parameter. Drazin & Tourigny obtained an approximate bifurcation diagram showing the imperfect pitchfork for $\varepsilon = 10^{-2}$. For this, they were obliged to use the partial sums of the power series about the point Re= 8, which they computed in floating-point arithmetic by the Lyness algorithm.

We now proceed to demonstrate the power of the multivariable method for

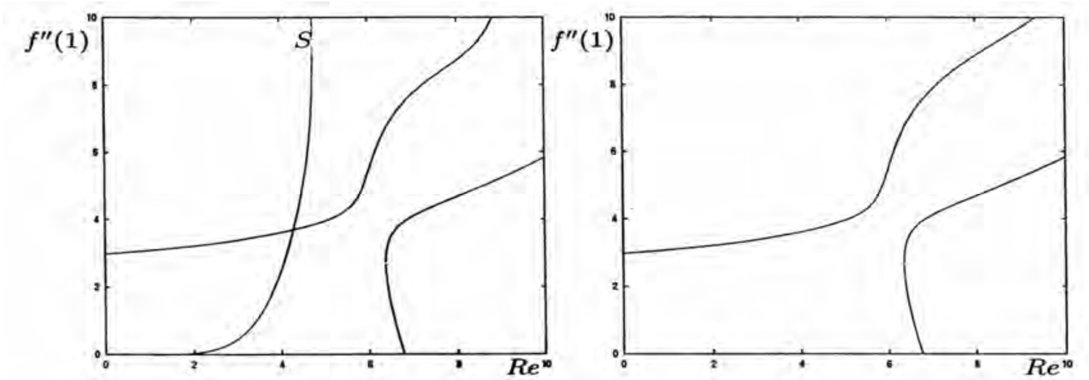

Figure 1.4: Approximation of the imperfect pitchfork bifurcation obtained by using multivariable cubic approximants with deg $A_N^{(k)} = 13$ and 15. Here $\varepsilon = 0.01$. Note that curve S is spurious.

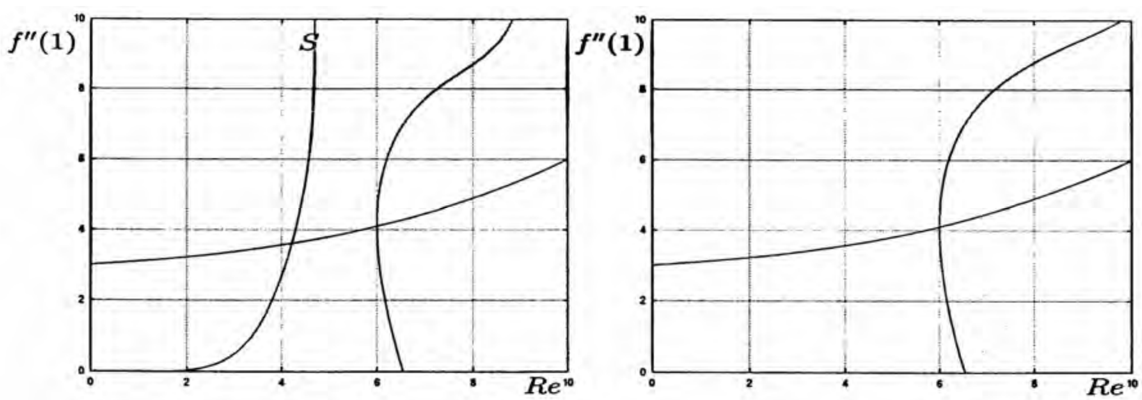

Figure: 1.5: Approximation of the pitchfork obtained by using multivariable cubic approximants with deg $A_N^{(k)} = 13$ and 15. Here $\varepsilon = 0$. Note that curve S is spurious.

Table 1.1: Estimates of the critical Reynolds number Re_c, the radius of convergence R and the critical exponent β obtained by the high-order homogeneous differential approximant method for the porous channel symmetric flow.

d	N	Re_c, N	$\mathrm{R_N}$	β_N
3	9	5.64130447 ± 10.229969i	11.6823198	1.089079524 − 1.421250 i
4	14	6.76058878 ± 10.407116i	12.4102232	$0.089079524 + 0.421250^{-1}$i
5	20	6.71504321 ± 10.365590i	12.3505980	$0.495508774 + 0.421250^{-1}$i
6	27	6.71766289 ± 10.363206i	12.3500218	$0.499627654 + 0.71610^{-3}$i
7	35	6.71769105 ± 10.363149i	12.3499897	$0.500012420 + 0.86210^{-6}$i
8	44	6.71769136 ± 10.363150i	12.3499904	$0.499999975 + 0.17910^{-6}$i

this problem. We compute the double power series

$$f(y,\mathrm{Re};\varepsilon)=\frac{1}{2}y\left(y^2-3y\right)-\frac{1}{12}\varepsilon\left(3y^3-16y+6\right)+$$

$$\frac{1}{1120}\mathrm{Re}\ (\varepsilon-2)(y-1)^2(y+1)^2\left(y^3\varepsilon-2y^3+2y\varepsilon-4y-35\varepsilon\right)+\ldots,$$

with the boundary conditions (1.7.4). Using multivariable cubic approximants, we compute the approximate bifurcation diagrams shown in Figure 1.4. These correspond to the case $\varepsilon=10^{-2}$; the imperfect pitchfork can be seen.

Now, having computed the Pade-Hermite polynomials, we are at liberty to set $\varepsilon = 0$. The resulting diagrams are shown in Figure 1.5. The (perfect) pitchfork is strikingly revealed.

1.8 Overview of the Work

This thesis is concerned with the study of computer based Perturbation method together with an approximation technique which is $Pade'$ approximation technique. Many researchers have studied the application of this approximation technique in fluid dynamical problems. The rest of the thesis has been arranged in the following three chapters.

In Chapter 2, we have reviewed the $Hermite-Pade'$ class of approximation techniques to determine the coefficients of the approximant. We have discussed several of these kinds of approximants with some examples.

In chapter 3, we have studied the critical behavior of hydro magnetic flows in a channel with slip at the permeable boundaries. Mathematical formulation of the problem has been given in this chapter. Perturbation expansion and internal flow separation have also been shown in this chapter. Makinde [2] analyzed the magnetic effect in the steady flow of a viscous fluid in a Channel with slip at the permeable boundaries. Analytical solutions for the governing nonlinear boundary value problem using perturbation expansion together with $Pade'$ approximants are discussed by Makinde. We extend the work by the comparison of our method with Makinde [2], study the effect of magnetic intensity and the critical relation among the parameters of the flow. This thesis ends with the results and discussion on this study and finally drawing conclusions.

Finally in chapter 4, we have summarized our work and give some ideas for future work.

This thesis is based on the study of the application of computer based approximation techniques to reveal the local behavior of a perturbation series around its singular point and the critical relationship among the perturbation parameters.

The approximation methods [1,15,17,18,19,20,22] are widely used to approximate functions in many areas of applied mathematics. Approximant methods are the techniques for summing power series. A function is said to be approximant for a given series if its Taylor series expansion reproduces the first few terms of the series.

$Pade'$ Approximants was studied by Bender and Orszag [15]. After these, it is given a brief description of continued fractions and convergence of $Pade'$ approximants studied by Bender and Orszag [15]. . Algebraic and Differential approximants [15] are some useful generalizations of Pade′ approximants. Khan [21] analyzed singularity behavior by summing power series. Khan [1] also introduced a new model of Differential approximant for single independent variable, called High-order differential approximant (HODA), for the summation of power series. The method is a special type of $Hermite-Pade'$ class and it is one of the best methods of singularity analysis for the problems of single independent variable.

The reminder of this Chapter is organized as follows:

We study the $Hermite-Pade'$ class of approximants and then the development of some approximants in this class such as Algebraic and Differential Approximants. Drazin-Tourigney method is one kind of Algebraic approximant and High-order differential approximants and High-order partial differential approximants [22] is an extension of Differential Approximants.

2.1 Hermite-Pade' Approximants

In 1893, $Hermite$ and $Pade'$ introduced $Hermite-Pade'$ class. The entire one variable approximants that were used or discussed throughout this thesis paper belong to the $Hermite-Pade'$ class. In its most general form, this class is concerned with the simultaneous approximation of several independent series. Firstly we describe the $Hermite-Pade'$ class from its point of view.

Let $d \in \mathrm{N}$ and let the $d+1$ power series $U_0(x), U_1(x), \ldots, U_d(x)$

are given. We say that the $(d+1)$ tuple of polynomials

$$P_N^{[0]}, P_N^{[1]}, \ldots, P_N^{[d]}$$

where $$\deg P_N^{[0]} + \deg P_N^{[1]} + \ldots + \deg P_N^{[d]} + d = N, \tag{2.1.1}$$

is a $Hermite-Pade'$ form of these series if

$$\sum_{i=0}^{d} P_N^{[i]}(x) U_i(x) = O\left(x^N\right) \text{ as } x \to 0. \tag{2.1.2}$$

Here $U_0(x), U_1(x), \ldots, U_d(x)$ may be independent series or different form of a unique series. We need to find the polynomials $P_N^{[i]}$ that satisfy the equations (2.1.1) and (2.1.2). These polynomials are completely determined by their coefficients. So, the total number of unknowns in equation (2.1.2) is

$$\sum_{i=0}^{d} \deg P_N^{[i]} + d + 1 = N + 1\,. \tag{2.1.3}$$

Expanding the left hand side of equation (2.1.2) in powers of *x* and equating the first *N* equations of the system equal to zero, we get a system of linear homogeneous equations. To calculate the coefficients of the $Hermite-Pade'$ polynomials we require some sort of normalization, such as

$$P_N^{[i]}(0) = 1 \text{ for some } 0 \le i \le d\,. \tag{2.1.4}$$

It is important to emphasize that the only input required for the calculation of the $Hermite-Pade'$ polynomials are the first *N* coefficients of the series $U_0, \ldots, U_d$. The equation (2.1.3) simply ensures that the coefficient matrix associated with the system is square. One way to construct the $Hermite-Pade'$ polynomials is to solve the system of linear equations by any standard method such as Gaussian elimination or Gauss-Jordan elimination.

2.2 Pade′ Approximants

For over a century approximant methods have been used most frequently in many areas of applied mathematics. Approximate Methods are the techniques for summing power series. A function is said to be approximant for a given series if its Taylor series expansion reproduces the first few terms of the series. The partial sum of a series is the simplest approximant, which is a very good approximant, if the function has no singularities. When the series converges rapidly, such approximants can provide good approximations for the series.

The convergents in the continued fraction expansion of a power series are rational approximants. In fact, these are particular type of *Pade′* approximants that have the property that the numerator and denominator are of the same degree. In general, such approximants are more accurate than the partial sum of the power series studied by Baker and Graves-Morris [16].

The idea of *Pade′* summation is to replace a power series $\sum a_n x^n$ by a sequence of rational functions (a rational function is a ratio of two polynomials) of the form

$$P_M^N(x) = \frac{\sum_{n=0}^{N} A_n x^n}{\sum_{n=0}^{M} B_n x^n} \qquad (2.2.1)$$

where B_0=1 is chosen without loss of generality. The remaining $(M+N+1)$ coefficients $A_0, A_1, ..., A_N, B_1, B_2, ..., B_M$, are chosen so that the first $(M+N+1)$ terms in the Taylor series expansion of $P_M^N(x)$ match the first $(M+N+1)$ terms of the power series $\sum_{n=0}^{\infty} a_n x^n$. The resulting rational function $P_M^N(x)$ is called a *Pade′* approximant.

The construction of $P_M^N(x)$ is very useful. If $\sum a_n x^n$ is a power series representation of the function *f*(*x*), then in many instances $P_M^N(x) \to f(x)$ as *N, M*→∞, even if $\sum a_n x^n$ is a divergent series. Usually we consider only the convergence of the *Pade′* sequences P_0^J, P_1^{1+J}, P_2^{2+J}, P_3^{3+J},......... having *N*=*M*+*J* with *J* fixed and *M*→∞. The special sequence *J*=0 is called the *diagonal* sequence.

The full power series representation of a function need not be known to construct a *Pade′* approximant- just the first *(M + N + 1)* terms. Since *Pade′* approximants involve only algebraic operations, they are more convenient for computational purposes. In fact, the general *Pade′* approximant can be expressed in terms of determinants.

Computation of $P_1^0(x)$: To compute $P_1^0(x)$ we expand this approximant in a Taylor series

$$P_1^0(x) = A_0/(1+B_1\,x) = A_0 - A_0\,B_1\,x + O(x^2)\ (x \to 0).$$

Comparing this series with the first two terms in the power series representation of $f(x) = \sum_{n=0}^{\infty} a_n\,x^n$ gives two equations $a_0 = A_0,\ a_1 = -A_0\,B_1$. Thus, $P_1^0(x) = a_0/(1 - x\,a_1/a_0)$.

Example 2.1 *Pade′* approximations for the function $f(x) = (1-2x)^{-2} + \ln(1-x)$.

We convert the above function *f(x)* in the polynomial form using a computer symbolic algebra language, MAPLE as follows:

$$g(x) = 1 + 3x + \frac{23}{2}x^2 + \frac{95}{3}x^3 + \frac{319}{4}x^4 + \frac{959}{5}x^5 + \frac{2687}{6}x^6 + \ldots\ldots\ldots\ldots$$

Now we reform the above series into several diagonal *Pade′* approximants of order *N*= *M*+*M* as follows:

$$g(x) = \frac{\sum_{i=0}^{M} a_i\,x^i}{\sum_{i=0}^{M} c_i\,x^i} \tag{2.2.2}$$

This method fails when we evaluate near the zeros of the denominator of the fraction. In that case we get the singular points. Because, we know that singularity of a function is a value of the independent variable or variables for which the function is undefined. We equate the denominator of equation (2.2.2) to zero for different values of *N* and observe that how rapidly it converges to our desired value, which has been given in the following table:

Table 2.1

Convergence of singularity by $Pade'$ approximant for the function in the example 2.1:

d	2	3	4
x_c	0.4304862724	0.4818596590	0.4962878690

There are several variations on the $Pade'$ method of summing power series. One such method consists of recasting the series into *continued-fraction* instead of rational-fraction form. In the following subsection, we describe on continued fractions with examples and how to evaluate continued fractions.

2.3. Continued Fractions

Continued fraction is very useful to analyze the dynamical systems, notably in connection with renormalization. We will discuss the basic concepts of continued fractions.

A continued fraction is an infinite sequence of fractions whose (N+1)th member $F_N(x)$ has the form

$$F_N(x) = \cfrac{c_0}{1+\cfrac{c_1 x}{1+\cfrac{c_2 x}{1+\ddots \cfrac{c_{N-1}x}{1+c_N x}}}} \tag{2.3.1}$$

The coefficients c_n are determined by expanding the terminated continued fraction $F_N(x)$ in a Taylor series and comparing the coefficients with those of the power series to be summed. This procedure closely resembles $Pade'$ summation because here also only algebraic operations are required.

Example 2.2 Let $x=\frac{67}{59}$**,** then $\frac{67}{59}=1+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{2}}}}}$

Again, let $f(x)=e^{-x}$, then

$$e^{-x}=1+\cfrac{x}{-1+\cfrac{x}{-2+\cfrac{x}{-3+\cfrac{x}{2-\frac{x}{5}}}}}$$

2.4 Algebraic Approximants

Algebraic approximant is a special type of $Hermite-Pade'$ approximants. In the $Hermite-Pade'$ class we take

$$d\geq 1, U_0=1, U_1=U, \ldots, U_d=U^d.$$

Let $U(x)$ represent power series of a function and $U_N(x)$ is the partial sum of that series. An Algebraic approximant $u_N(x)$ of $U(x)$ can be defined as the solution of the equation:

$$P_N^{[0]}(x)+P_N^{[1]}(x)U_N(x)+P_N^{[2]}(x)U_N^2(x)+\ldots+P_N^{[d]}(x)U_N^d(x)=0 \qquad (2.4.1)$$

Where d represent the degree of the partial sum $U_N(x)$. The Algebraic approximant $u_N(x)$, is in general a multivalued function with d branches.

The solution of the equation (2.4.1) with $d\geq 1$ gives us the coefficients of the polynomials $P_N^{[i]}(x)$. The discriminant of this equation approximates the singularity of $U(x)$.

Here, $$\sum_{i=0}^{d} P_N^{[i]}(x)U_N^{\,i}(x) = O(x^N) \qquad (2.4.2)$$

And $$\sum_{i=0}^{d} \deg P_N^{[i]}(x) + d = N \qquad (2.4.3)$$

And the total number of unknowns in (2.4.1) is

$$\sum_{i=0}^{d} \deg P_N^{[i]}(x) + d + 1 = N + 1\,. \qquad (2.4.4)$$

In order to determine the coefficients of the polynomials $P_N^{[i]}$ one can set $P_N^{[0]}(0) = 1$ for normalization.

Example 2.3 Consider

$$u(x) = (1-3x)^{\frac{1}{4}} + \cos x\,.$$

Let $d = 2$ and $\deg P_8^{[0]} = \deg P_8^{[1]} = \deg P_8^{[2]} = 2$ to apply the Algebraic approximation method on the power series of the given function. After we set the normalization condition $P_8^{[0]}(0) = 1$, we get the polynomials

$$P_8^{[1]}(x) = 1 + \frac{3526971629\ \ 7856619747\ \ 4510399}{1536406837\ \ 3667731790\ \ 339225}x - \frac{3429921755\ \ 3657798313\ \ 4255624}{1382766153\ \ 6300958611\ \ 3053025}$$

$$P_8^{[1]}(x) = \frac{3042270620\ 0192196360\ 5233312}{4609220512\ 1003195371\ 017675} - \frac{8430222733\ 2183876227\ 061592}{3072813674\ 7335463580\ 67845}x$$
$$+ \frac{27400787512067103620478184 6}{1382766153630095861130530 25}x^2$$

$$P_8^{[2]}(x) = -\frac{7490659888304408198640212 8}{4609022051210031953710176 75} + \frac{1190278021467603104801681 24}{1536406837366773179033922 5}x$$
$$- \frac{611570570835452228303483 71}{1382766153630095861130530 25}x^2$$

Here the discriminant gives us the singularity at x$_c$ =0.2157409001. If we increase the degree of the polynomial coefficients it may give us a better approximation. So, again let $\deg P_{11}^{[0]} = \deg P_{11}^{[1]} = \deg P_{11}^{[2]} = 3$ and $d = 2$, following the same procedure we get the singularity at x$_c$=0.2537984673.

Again taking $d=2$ and $\deg P_{14}^{[0]} = \deg P_{14}^{[1]} = \deg P_{14}^{[2]} = 4$ the singularity is calculated at x_c=0.2965143468.The table below shows the comparative results of the convergence of the Algebraic approximation method to the singular point.

Table 2.2: The approximation of x_c by Algebraic approximants for the function in Example 2.3

deg $P_N^{[i]}$	d	x_c
3	2	0.2157409001
3	2	0.2537984673
4	2	0.2965143468

Note that $d=3$ may be more accurate for this problem.

2.5 Drazin -Tourigney Approximants

Drazin and Tourigney in [16] implemented the idea $d = O\left(\sqrt{N}\right)$ as $N \to \infty$.Their method is simply a particular kind of Algebraic approximant, satisfying the equation (2.4.1). In this method they considered

$$\deg P_N^{[i]} = d - i \tag{2.5.1}$$

and

$$N = \frac{1}{2}\left(d^2 + 3d - 2\right). \tag{2.5.2}$$

2.6 Differential Approximants

Differential approximants is an important member of the $Hermite-Pade'$ class. It is obtained by taking

$$d \geq 2,\quad U_0 = 1, U_1 = U, U_2 = DU \text{ and } U_d = D^{d-1}U\ ,$$

where $D \equiv \dfrac{d}{dx}$, a differential approximant $u_N(x)$ of the series $U(x)$ can be defined as the solution of the differential equation

$$P_N^{[0]} + P_N^{[1]}U_N + P_N^{[2]}DU_N + \ldots + P_N^{[d]}D^{d-1}U_N = 0\,. \tag{2.6.1}$$

Here (2.6.1) is homogeneous linear differential equation of order $(d-1)$ with polynomial coefficients. There are $(d-1)$ linearly independent solutions, but only one of them has the same first few Taylor coefficients as the given series $U(x)$. When $d>2$, the usual method for solving such an equation is to construct a series solution.

Differential approximants are used chiefly for series analysis. They are powerful tools for locating the singularities of a series and for identifying their nature.

The singularities of $U(x)$ are located at the zeroes of the leading polynomial $P_N^{[d]}(x)$. Hence, the zeroes of $P_N^{[d]}(x)$ may provide approximations of the singularities of the function $u(x)$.

Example 2.4 Consider $u(x)=\dfrac{e^x(1+\sin x)}{\sqrt{1-\frac{1}{3}x}}$.

Taking $d=4$ for (2.4.3) and applying (2.1) and (2.2), we obtain the singular point at $x_c=3.000563091$. In a similar procedure taking $d=5$ gives us more accurate result, i.e. $x_c=2.999999734$.

The table below shows a comparative result.

Table2.3: The approximation of x_c by Differential Approximant for the function in Example 2.4

N	d	x_c
15	3	2.991301923
21	4	3.000563091
28	5	2.999999734

2.7 High-Order Differential Approximants

Khan [1] introduced an extension of differential approximant, which he mentioned as High-order differential approximant. When the function has a countable infinity of branches, then the fixed low-order differential approximants may not be useful. So, for these cases he considered d increase with N. It lead to a particular kind of differential approximant $u_N(x)$, satisfying equation (2.5.2). Here

$$N = \frac{1}{2}d(d+3) \text{ and } \deg P_N^{[i]} = i. \tag{2.7.1}$$

From (2.1.3) he deduced that there are $\frac{1}{2}\left(d^2 + 3d + 2\right)$

unknown parameters in the definition of the $Hermite - Pade'$ form. In order to determine those parameters, we use the N equations

$$P_N^{[0]}(x) + \sum_{i=1}^{d} P_N^{[i]}(x) D^{i-1} U_N(x) = O\left(x^N\right) \text{ as } x \to 0.$$

In addition one can normalize by setting $P_N^{[0]}(0) = 1$. Then there remains as many equations as unknowns. One of the roots, say $x_{c,N}$, of the coefficient of the highest derivative, i.e. $P_N^{[d]}\left(x_{c,N}\right) = 0$, gives an approximation of the dominant singularity x_c of the series U. If the singularity is of algebraic type, then the exponent α may be approximated by

$$\alpha_N = d - 2 - \frac{P_N^{[d-1]}\left(x_{c,N}\right)}{DP_N^{[d]}\left(x_{c,N}\right)}. \tag{2.7.2}$$

2.8 High-Order Partial Differential Approximants

Consider the function $f(x,y)$ of two independent variables, represented by its power series

$$U(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij} x^i y^j \quad (x,y) \to (0,0) \tag{2.8.1}$$

and the partial sum

$$U_N(x,y) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} c_{ij} x^i y^j \tag{2.8.2}$$

By using that partial sum, we try to construct the following $(2d+1)$ polynomials

$$P_{[0,0]},\ P_{[1,0]},\ P_{[0,1]}\ldots\ldots\ldots\ldots, P_{[d,0]}, P_{[0,d]} \tag{2.8.3}$$

in x and y such that

$$P_{[0,0]}U_N + P_{[1,0]}\frac{\partial U_N}{\partial x} + P_{[0,1]}\frac{\partial U_N}{\partial y} + \ldots\ldots + P_{[d,0]}\frac{\partial^d U_N}{\partial x^d} + P_{[0,d]}\frac{\partial^d U_N}{\partial y^d} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} e_{ij}x^i y^j \tag{2.8.4}$$

Where $$e_{ij} = 0 \quad \text{for } i + j < N = 3d - 1 \tag{2.8.5}$$

By equating the coefficients of the variables and their powers from (2.9.5), one can obtain a total of

$$N_e = \frac{3d(3d-1)}{2} \tag{2.8.6}$$

equations to determine the unknown coefficients of the polynomials in (2.9.4), we impose the normalization condition

$$P_{[0,0]} = 1,\ or\, P_{[d,0]} = 1 \quad or\, P_{[0,d]} = 1 \quad \text{for } (x, y) = (0,0). \tag{2.8.7}$$

Thus the remaining unknowns

$$N_u = \frac{1}{3}d(d^2 + 6d + 11)$$

must be found by the use of N_e equations.

It would be helpful to write the system of linear equations $e_{i,j} = 0$ into the matrix form with the $N_u \times 1$ unknown matrix $\underline{x}$.

Thus the non-homogeneous system of N_e linear equations with N_u unknowns can be written in matrix form as

$$A\underline{x} = \underline{b}$$

where A is $N_e \times N_u$ matrix and $\underline{b}$ is the non-zero column matrix of order $N_e \times 1$. Thus system will be solvable if

$$N_e \le N_u. \tag{2.8.8}$$

However, the system may be consistent or inconsistent. If the system is consistent , then the system can be solved by converting the augmented matrix $[A|\underline{b}]$ to row echelon or reduced row echelon form by using the Gaussian elimination or Gauss-Jordan elimination. It is to note that, there will exist some free variables. Naturally the values of the free variables in the multivariable approximant methods can be chosen at random. For all the calculation reported in the remainder of this chapter, we have in fact set all the free variables to either zero or one. There is no particular reason to pick up these particular numbers. We might for instance seek a solution such that the polynomials in (2.9.3) have as few high-order terms as possible. Our experience suggests that the accuracy of the method does not depend critically on the particular choice made.

Once the polynomials (2.9.3) have been found, it is more practical to find the singular points by solving either of the polynomials coefficients of the highest derivatives

$P_{[d,0]}(x, y) = 0$ or $P_{[0,d]}(x, y) = 0$ or both simultaneously

2.9 Discussion

$Hermite - Pade'$ class is constructed over the technique of truncated continued fraction. The polynomial coefficients were constructed by taking successive truncated continued fractions. In this chapter we had an overall study about the $Hermite - Pade'$ class of approximation methods. Examples show the performance of Algebraic approximant and Differential approximant explicitly. We must mention that Drazin -Tourigney method is an improved Algebraic approximation technique. High-order differential approximants is modified Differential approximant whose performance is almost in every case convincing. High-order partial differential approximants [22] is a multivariable differential approximants method is applied to determine critical relation between the solution parameter.

In Chapter 3 we analyze the Critical Behavior of the Hydro magnetic Flows in a Channel with slip at the permeable boundaries and the Critical relationship among the flow parameters.

CRITICAL BEHAVIOR OF THE SOLUTION OF HYDROMAGNETIC FLOWS IN A CHANNEL WITH SLIP AT THE PERMEABLE BOUNDARIES

3.1 Introduction

In this chapter, we have investigated the flows through a slowly varying exponentially symmetrical channel with slip at the permeable boundaries under the influence of an externally applied homogeneous magnetic field. It is assumed that the fluid has small electrical conductivity and the electromagnetic force produced is very small. Here we have determined numerically the effect of the externally applied homogeneous magnetic field and different values of the permeable parameter (k) on the internal flow separation development as flow Reynolds number increases using perturbation method together with Differential approximation technique. For this, at first we have formulated our problem mathematically and obtained the solution series for stream function (F) and vorticity (G) using perturbation method. Then we extend the solution series using a computer symbolic algebra Language MAPLE in order to examine the effect of internal forces and the Hartmann number or the magnetic field intensity parameter (H) on the flow structure as well as internal flow separation development. This chapter begins with the mathematical formulation of the problem and finally we draw conclusions.

3.2 Mathematical Formulation

Consider the flow of fluid with small electrical conductivity and the electromagnetic force produced is very small under the effect of an external applied homogeneous magnetic field. Let the fluid is flowing through a slowly varying exponentially symmetrical channel with slip at the permeable boundaries as shown in figure 3.1.

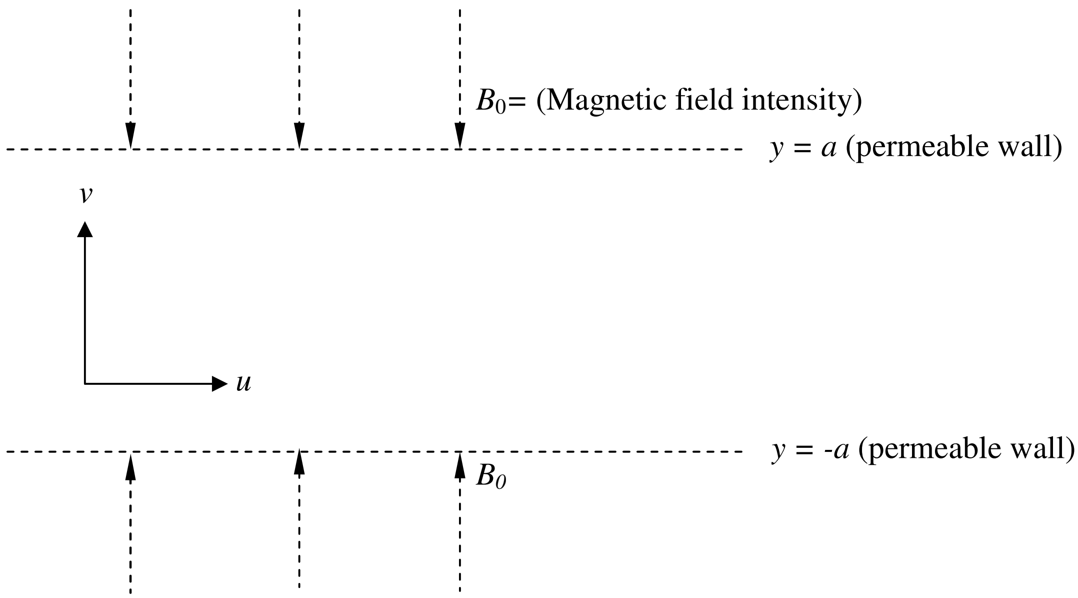

Fig.3.1: Schematic diagram of the problem.

Consider a cartesian coordinate system (x,y) where x is the axis of the channel which passes through the centre of the channel and y is the distance measured in the normal section such that y = a is the channel's half width. Denote the increasing velocity components in x and y directions by u and v respectively. Then, the continuity and Navier-Stokes equations governing the flow are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u-\frac{\sigma_e B_0^2 u}{\rho} \tag{3.2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v \tag{3.3}$$

where B_0 = (μ_e H) the electromagnetic induction,

μ_e = the magnetic permeability,

H = the intensity of magnetic field,

σ_e = the conductivity of the fluid,

p = the fluid pressure,

ρ = the fluid density.

In order to complete the formulation of the problem, the boundary conditions have to be specified. These can be as follows:

$$\frac{\partial u}{\partial y} = 0,\ v = 0, \text{ on } \quad y = 0 \tag{3.4}$$

$$\text{and } \mu \frac{\partial u}{\partial y} = -\beta u,\ v = V \ \text{ on } y = a \tag{3.5}$$

The boundary conditions is well known Beavers and Joseph slip condition

μ = the dynamic viscosity coefficient

β = the coefficient of sliding friction

and V = characteristic wall suction velocity

The following dimensionless variables are introduced into eqs. (3.1-3.5):

$$x = \frac{x'}{a},\ y = \frac{y'}{a}, P = \frac{ap'}{\rho v V},\ u = \frac{u'}{V},\ v = \frac{v'}{V}$$

$$\text{Re} = \frac{Va}{v},\ H = \sqrt{\frac{\sigma B_0^2 a}{\rho V}},\ k = \frac{\mu}{a\beta} \tag{3.6}$$

and we obtain (neglecting the prime clarity)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.7}$$

$$\text{Re}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - H^2 u \tag{3.8}$$

$$\text{Re}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{3.8}$$

$$\frac{\partial u}{\partial y} = 0, \quad v = 0 \quad on \quad y = 0 \tag{3.10}$$

$$u = -k\frac{\partial u}{\partial y}, \quad v = 1 \ \ on \ \ y = 1 \tag{3.11}$$

Where

Re = Reynolds number

k = permeable parameter.

H = Hartmann number.

Eliminating the pressure p, using dimensionless variables and introducing stream function ψ and vorticity ω in equations 3.8-3.9 in the following manner:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad \text{and} \quad \omega = \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \tag{3.12}$$

$$\nabla^2 \omega = \mathrm{Re}\left(\frac{\partial(\omega, \psi)}{\partial(x, y)} - H^2 \frac{\partial^2 \psi}{\partial y^2} \right), \quad \omega = -\nabla^2 \psi \tag{3.13}$$

and the appropriate boundary conditions are:

$$\frac{\partial \psi}{\partial x} = 0, \ \frac{\partial^2 \psi}{\partial y^2} = 0 \ \text{ on } \ y = 0 \tag{3.14}$$

$$\text{and} \ \frac{\partial \psi}{\partial x} = -1, \ \frac{\partial \psi}{\partial y} = -k \frac{\partial^2 \psi}{\partial x^2} \ \text{ on } \ y = 1 \tag{3.15}$$

Now using the following similarity variables

$$\psi = xF(y) \ \text{ and } \ \omega = xG(y) \tag{3.16}$$

The dimensionless governing equations together with the appropriate boundary conditions can be written as:

$$\frac{\partial^2 G}{\partial y^2} = \mathrm{Re}\left[G \frac{dF}{dy} - F \frac{dG}{dy} + H^2 G \right], \quad G = -\frac{d^2 F}{dy^2} \tag{3.17}$$

$$\frac{d^2 F}{dy^2} = 0, F = 0, \ on \ \ y = 0 \tag{3.18}$$

$$\frac{dF}{dy} = -k \frac{d^2 F}{dy^2}, \ F = -1 \ \text{ on } \ y = 1 \tag{3.19}$$

Equations 3.17-3.19 above is form a nonlinear boundary-value 4th order parameter dependent ordinary differential equation.

3.3 Method of Solution

Now we seek the solution in the form of power series in Re:

$$F = \sum_{i=0}^{\infty} \mathrm{Re}^i F_i \ \text{ and } \ G = \sum_{i=0}^{\infty} \mathrm{Re}^i G_i \tag{3.20}$$

Now we substitute equation (3.20) into equations (3.17)-(3.19) and collect the coefficients of like powers of Re and obtain as follows:

Zeroth order:

$$\frac{d^2G_0}{dy^2}=0,\ G_0=-\frac{d^2F_0}{dy^2} \tag{3.21}$$

$$\frac{d^2F_0}{dy^2}=0,\ F_0=0 \text{ on } y=0 \tag{3.22}$$

$$\frac{dF_0}{dy}=-k\frac{d^2F_0}{dy^2},\ F_0=-1 \ on \ y=1 \tag{3.23}$$

Higher order : $(n\geq 1)$

$$\frac{\partial^2G_n}{\partial y^2}=\mathrm{Re}\left[\sum_{i=0}^{\infty}\left(G_i\frac{dF_{(n-i-1)}}{dy}-F_i\frac{dG_{(n-i-1)}}{dy}\right)+H^2G_{n-1}\right],\ G_n=-\frac{d^2F_n}{dy^2} \tag{3.24}$$

$$\frac{d^2F_n}{dy^2}=0,\ F_n=0 \ on \ y=0 \tag{3.25}$$

$$\frac{dF_n}{dy}=-k\frac{d^2F_n}{dy^2},\ F_n=0 \text{ on } y=1 \tag{3.26}$$

It is difficult to find many terms of the solution series manually. So we have written a MAPLE program that calculates successively the coefficients of the solution series. It consists of the following segments:

(i) Declaration of arrays for the solution series coefficients; F = array (0……..20), G = array(0……20).

(ii) Input the leading order term and their derivatives i.e. F_0, G_0

(iii) Using a MAPLE loop procedure, iterate to solve equations (3.24)-(3.26) for the higher order terms i.e. F_n, G_n, n = 1,2,3,…… .

(v) Compare the skin friction , the axial pressure gradient and centerline axial velocity coefficients.

some of the solution for stream-function and vorticity obtained are given as follows:

$$F=\frac{y^3-3(1-2k)y}{2(1+3k)}+\frac{(y^2-1)y}{280(1+3k)^3}(y^4+3y^4k+42h^2y^2k+7H^2y^2+63y^2k^2H+y^2$$
$$+3y^2k-18k-7h^2-2-27h^2k-147k^2H^2)\,\mathrm{Re}\ +O(\mathrm{Re}^2) \tag{3.27}$$

$$G=\frac{3y}{(1+3k)}+\frac{\mathrm{Re}\,y}{140(1+3k)^3}(70H^2y^2+213y^2-42H^2-9-336H^2k+63y^4k$$

$$+420y^2kH^2-63k+630y^2k^2H^2-630k^2H^2)+O(\mathrm{Re}^2) \qquad (3.28)$$

Dimensionless wall skin friction t_{ω} in terms of stream function can be written as

$$t_{\omega}=-\mu xV\frac{d^2F}{dy^2} \text{ on } y=1 \qquad (3.29)$$

Where μ= the dynamic viscosity. Using (3.27),we obtain the expression for t_{ω}

$$t_{\omega}=\mu xV\left[\frac{3}{(1+3k)}+\frac{\mathrm{Re}(84H^2k+28H^2+12)}{140(1+3k)^3}+\frac{\mathrm{Re}^2}{323400(1+3k)^5}(9240H^2-453k\right.$$

$$-1848H^4+3152-27720k^2+11088H^2k-133056k^2H^2-249480H^2k^3$$

$$\left.-166320H^4k^3-127512H^4k^2-29568H^4k)+O(\mathrm{Re}^3)\right] \qquad (3.30)$$

Let $$\frac{\partial p}{\partial x}=xA \qquad (3.31)$$

using (3.12)and(3.16),we get

$$A=\frac{d^3F}{dy^3}-\mathrm{Re}\left[\left(\frac{dF}{dy}\right)^2-F\frac{d^2F}{dy^2}+H^2\frac{dF}{dy}\right] \qquad (3.32)$$

And explicity as

$$A=\frac{3}{(1+3k)}-\frac{3\mathrm{Re}}{140(1+3k)^3}(315H^2k^3+315k^3+105k^2H^2$$

$$-210k^2-42kH^2-21k-14H^2+27)+O(\mathrm{Re}^2) \qquad (3.33)$$

Fig.3.2(a), Fig.3.2(b), Fig. 3.3(a) and Fig. 3.3(b) show the fluid velocity profile.

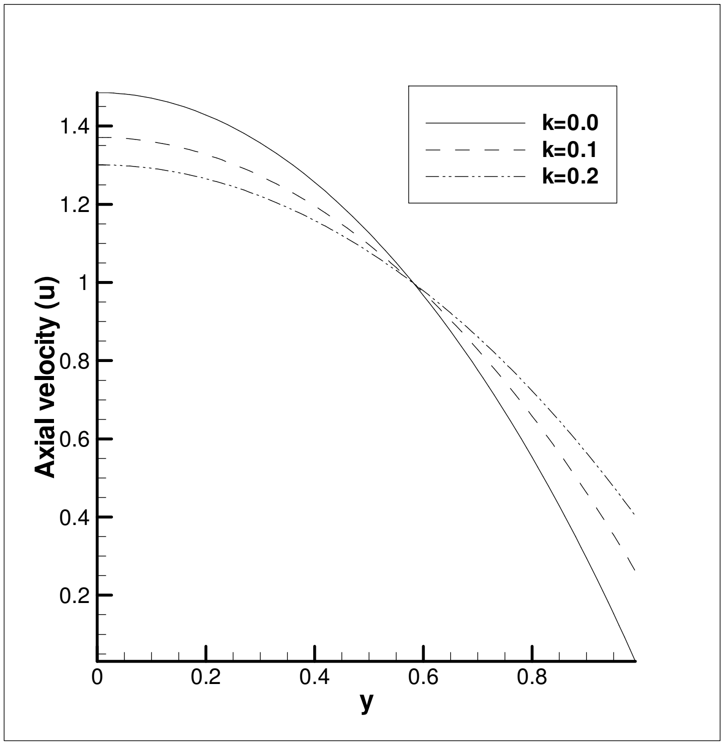

Fig. 3.2(a): Axial velocity profiles for different values of k, Re=1.0

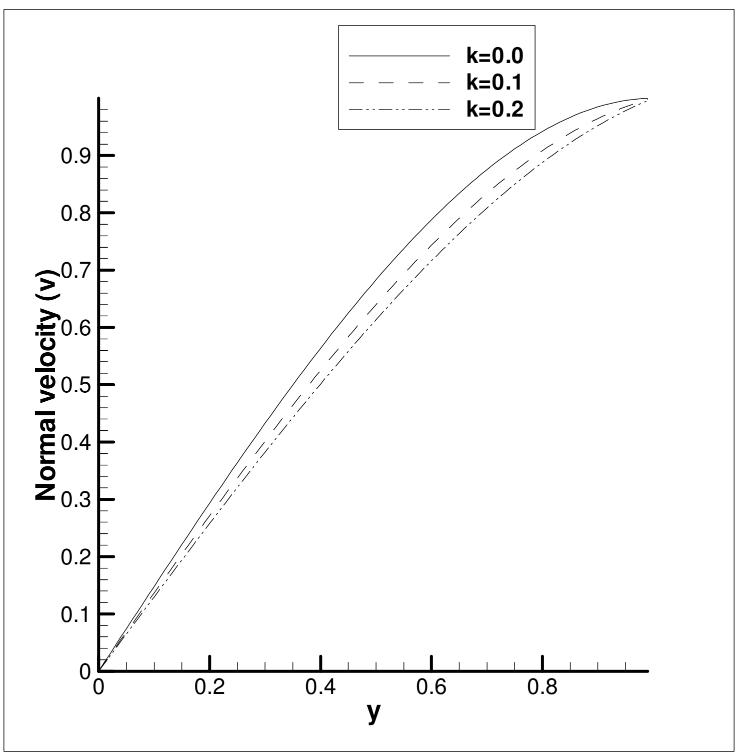

Fig.3.2(b). Normal velocity profiles for different values of k, Re=1.0

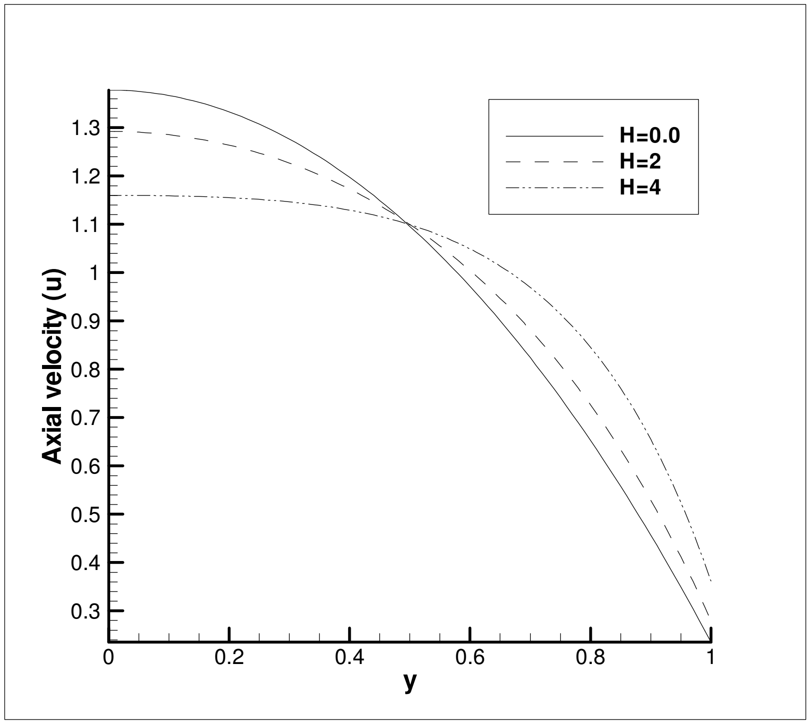

Fig.3.3(a). Axial velocity profiles for different values of H, Re=1.0

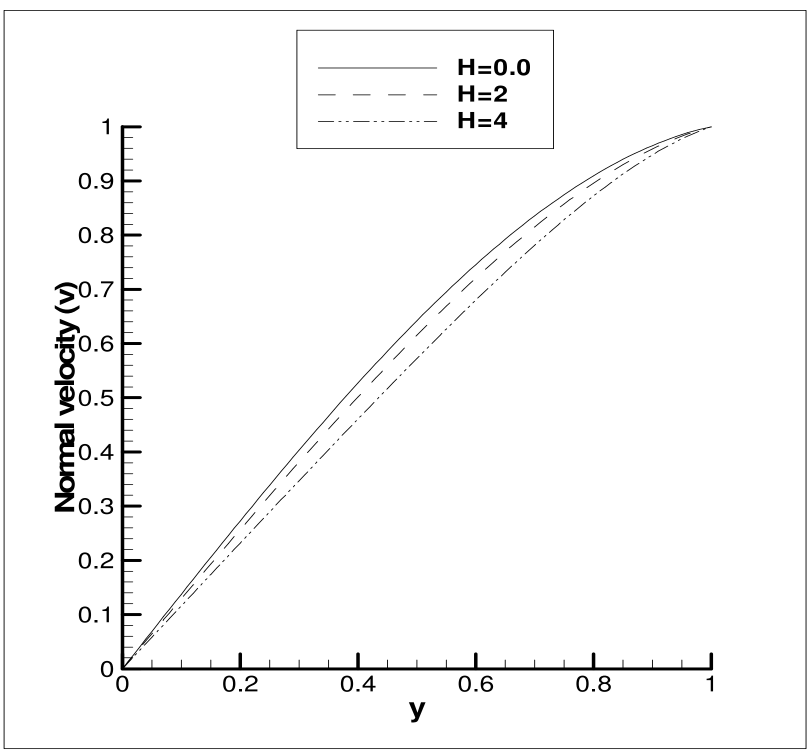

Fig.3.3(b). Normal velocity profiles for different values of H, Re=1.0

The wall skin friction with respect to flow Reynolds number is shown in fig.3.4 and 3.5.

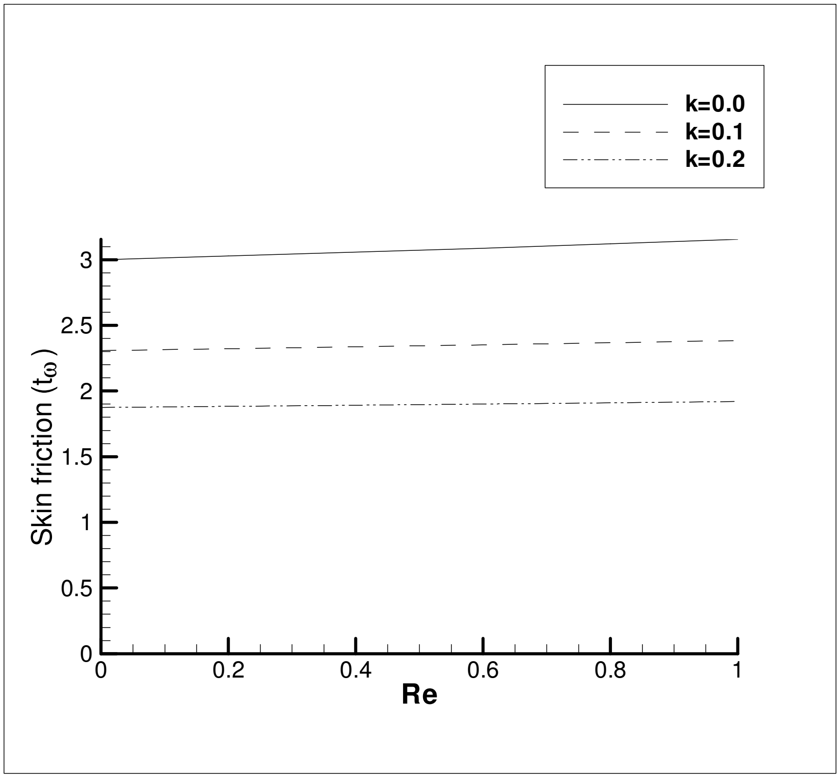

Fig.3.4.Wall skin Friction for different values of k, H=0.5

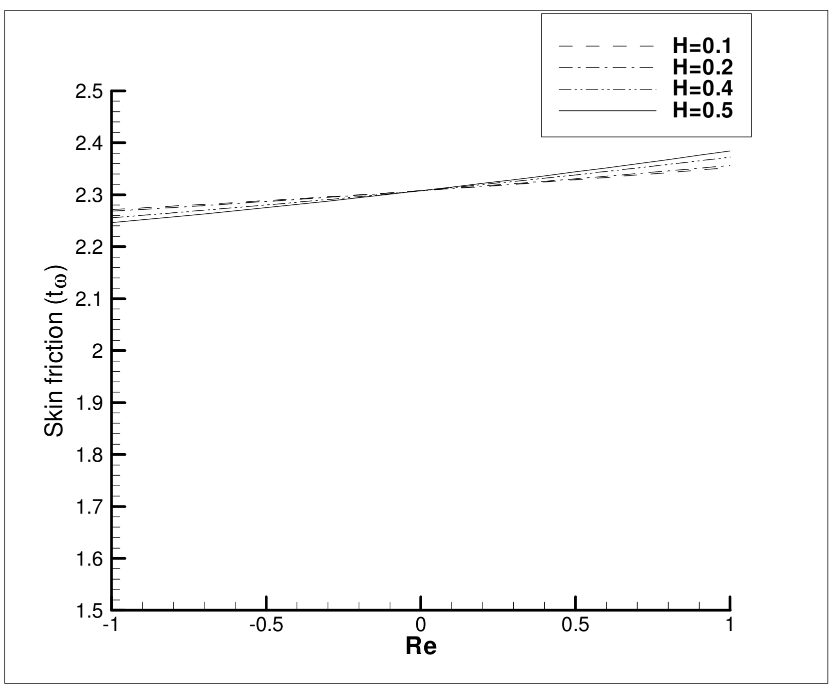

Fig.3.5.Wall skin Friction for different values of H, k=0.1

Although the computational complexity increases rapidly, we managed to compute the first 52 terms for A in terms of single parameter Re for k=0,0.1,0.2,0.3,0.4,0.5 and $Ha = 0,1,2,3$. These series are then analyzed by higher order approximate methods to determine the critical relationship among the parameters.

Table 3.1 Estimates of Critical Reynolds number Re at $H = 0$ and $k = 0,0.1,0.2,0.3,0.4,0,0.5$ using High-order differential approximants [1]

d	N	k=0	K=0.1	K=0.2	k=0.3	k=0.4	k=0.5
3	12	6.71147± 10.6106I	6.45401± 10.40286I	6.6976± 10.51365I	7.0259± 10.58459I	7.44391± 10.6095I	7.8023± 10.60238I
4	18	6.69061± 10.38553I	6.43281± 10.16998I	6.6888± 10.26651I	7.02596± 10.35030I	7.36363± 10.41630I	7.70902± 10.44889I
5	25	6.71335± 10.36280I	6.47221± 10.14552I	6.6606± 10.25218I	7.04851± 10.35272I	7.39568± 10.37394I	7.75231± 10.38573I
6	33		6.45540± 10.14255I	6.69277± 10.20238I	7.04220± 10.32766I	7.40985± 10.36480I	7.76659± 1037653I.
7	42		6.45862± 10.14624I	6.69126± 10.24815I	7.04478± 10.32512I	7.41399± 10.36438I	7.76646± 10.37753I
8	52		6.46055± 10.14434I	6.69151± 10.24787I	7.04445± 10.32503I	7.41322± 10.36408I	7.76640± 10.37716I

Table 3.2 Estimates of Critical Reynolds number Re at $H=1$ and $k=0,0.1,0.2,0.3,0.4,0,0.5$ using High-order differential approximants [1]

d	N	k=0	K=0.1	K=0.2	k=0.3	k=0.4	k=0.5
3	12	1.73246 ± 9.61216I	1.1879 ± 9.1432I	.91484 ± 9.21336I	.79384 ± 9.37011I	.74936 ± 9.52167I	.74673 ± 9.65392I
4	18	1.54655 ± 9.26680I	0.98970 ± 8.87023I	.69274 ± 8.8939I	.54655 ± 9.01751I	.48656 ± 9.13888I	.47680 ± 9.24103I
5	25	1.57833 ± 9.23398I	1.02318 ± 8.84272I	.71598 ± 8.89355I	.58535 ± 9.00824I	.53243 ± 9.13996I	.49744 ± 9.26073I
6	33		1.02938 ± 8.83589I	.72796 ± 8.88320I	.58706 ± 9.00719I	.52993 ± 9.13718I	.50385 ± 9.26400I
7	42		1.03231 ± 8.87270I	.72595 ± 8.88174I	.58151 ± 9.00607I	.52126 ± 9.13780I	.50430 ± 9.25738I
8	52		1.03040 ± 8.83109I	.72491 ± 8.88001I	.58236 ± 9.00533I	.52347 ± 9.13716I	.50387 ± 9.25682I

Table 3.3 Estimates of Critical Reynolds number Re at $H = 2$ and $k = 0,0.1,0.2,0.3,0.4,0,0.5$ using High order differential approximants [1]

d	N	k=0	K=0.1	K=0.2	k=0.3	k=0.4	k=0.5
3	12	-3.20081±4.50826I	-3.2591±3.78172I	-3.38747±3.29734I	-3.52100±2.99690I	-3.6393±2.7728I	-3.7390±2.5922I
4	18	-3.20081±4.50826I	-3.2350±3.75949I	-3.32858±3.27939I	-3.47750±2.97729I	-3.57152±2.69230I	-3.64761±2. 50628I
5	25	-3.19907±4.51133I	-3.2035±3.72496I	-3.33004±3.22819I	-3.443±2.9270I	-3.57653 ±2.70703I	-3.64547±2.50628I
6	33		-3.2042±3.72133I	-3.32254±3.22838I	-3.47002±2.93565I	-3.57370±2.69624I	-3.64298±2.51114I
7	42		-3.1992±3.71338I	-3.33822±3.23693I	-3.46608±2.92063I	-3.57500±2.70620I	-3.66073±2.51156I
8	52		-3.2003±3.71080I	-3.33830±3.24589I	-3.46539±2.92963I	-3.57348±2.70341I	-3.66200±2.52747I

3.4 Results and Discussion

In this thesis, we have studied the Critical Behavior of the solution of the flows through a slowly varying exponentially symmetrical channel with slip at the permeable boundaries under the influence of an externally applied homogeneous magnetic field. A number of interesting features have been brought to our attention, the foremost of which are different problems including the critical relationship among the parameters of the flow.

From the present investigation the following conclusions can be drawn:

- For different value of H, Re increases uniformly as k increases. An increase in the positive value of flow Re represents an increase in the fluid suction while an increase in the negative value of Re represents an increase in the fluid injection
- A parabolic axial velocity profile (in figure 3.2(a) and 3.3(a)) is observed with maximum value at the channel centerline and minimum value at the walls. A general decrease in the magnitude of both axial and normal velocity profiles are noticed with an increase in both wall slip(k) and magnetic field intensity (H).The occurrence of negative axial velocity near the channel walls due to slip indicates the possibility of flow reversal near the wall.
- The magnitude of the wall skin friction increase with suction and decreases with injection .A general decrease in wall skin friction is observed with an increase in wall slip and a decrease in magnetic field intensity.

3.5 Conclusion

We investigate the combined effects of wall slip and magnetic field on the steady flow of conducting viscous incompressible fluid in a channel with permeable boundaries .Our results reveled that the fluid velocity is reduced by both magnetic field and wall slip. The presence of flow reversal near the wall due to wall slip. Wall skin friction increases with suction and decreases with injection. Both the wall slip and the magnetic field have great influence on wall skin friction.

4.1 Conclusion

We have described the Approximant methods which is applied to analyze fluid dynamical problems. Then the Critical Behavior of Hydro magnetic Flows in a Channel with slip at the permeable boundaries has been studied using various types of Approximants methods. The critical relationship among the flow parameters has also been analyzed.

The results of analyzing the Hydro magnetic Flows in Channel with slip at the permeable boundaries show that the critical Reynolds number changes uniformly due to the effect of magnetic parameter Hartmann number and slip parameter in numerically. Then the critical relationship represents the significant variation in the corresponding solution parameter by the effect of magnetic intensity.

We try to provide a basis for guidance about what method of summing power series should be chosen for many problems in fluid dynamics to show the critical behavior of the flow.

4.2 Future Work

There are some ideas to form the basis of future work:

- Critical Behavior of the solution of Hydro magnetic Flows in Channel with slip at the permeable boundaries has been analyzed using High-order differential approximant. Therefore, further research in this regard could be carried out series in more terms and the series in terms of parameter H and then analyzing them by using multi variable approximants method.
- Application of Approximation methods to more physical models.
- Application of Approximation methods in other fields that include perturbation series and their performance in these fields.

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