A STUDY ON PRIME AND SEMI-PRIME GOLDIE RINGS AND MODULES

M. Phil. Thesis

SUBMITTED BY

MD. MOMINUL ISLAM PRODHAN

Student No. 0409093010P, Registration No. 0409093010

Session: April, 2009

DEPARTMENT OF MATHEMATICS

BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY (BUET)

DHAKA-1000

June, 2015

A STUDY ON PRIME AND SEMI-PRIME GOLDIE RINGS AND MODULES

by

Md. Mominul Islam Prodhan

Student No. 0409093010P, Session: April, 2009

Department of Mathematics

Bangladesh University of Engineering and Technology

Dhaka-1000

A dissertation submitted in partial fulfillment of the requirements for the award of the degree

of

MASTER OF PHILOSOPHY

in Mathematics

DEPARTMENT OF MATHEMATICS
BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY
Dhaka-1000

June, 2015

The thesis entitled
A STUDY ON
PRIME AND SEMI-PRIME GOLDIE RINGS AND MODULES

Submitted by

Md. Mominul Islam Prodhan
Student No. 0409093010P, Registration No. 0409093010, Session: April, 2009, a part-time M. Phil. student in Mathematics has been accepted as satisfactory in partial fulfillment of the requirements for the degree of MASTER OF PHILOSOPHY in Mathematics on 16/06/2015

BOARD OF EXAMINERS

____

1. Dr. Khandker Farid Uddin Ahmed Chairman
Professor (Supervisor)
Department of Mathematics, BUET, Dhaka.

____

2. Dr. Md. Manirul Alam Sarker Member
Professor and Head (Ex-Officio)
Department of Mathematics, BUET, Dhaka.

____

3. Dr. Md. Mustafa Kamal Chowdhury Member
Professor
Department of Mathematics, BUET, Dhaka.

____

4. Dr. Md. Abdul Alim Member
Professor
Department of Mathematics, BUET, Dhaka.

____

5. Dr. Salma Nasrin Member
Associate Professor (External)
Department of Mathematics, University of Dhaka.

CANDIDATE’S DECLARATION

It is herby declared that this thesis or any part of it has not been submitted elsewhere for the award of any degree or diploma.

(Md. Mominul Islam Prodhan)

DEDICATED

TO

MY PARENTS

ACKNOWLEDGEMENTS

First of all, I would like to express my sincerest gratitude and appreciation to my supervisor Dr. Khandker Farid Uddin Ahmed, Professor, Department of Mathematics, BUET, Dhaka-1000, for his invaluable suggestion, constant inspiration and supervision that has enabled me to finish this dissertation successfully. I am highly grateful to the honorable members of the Board of Examiners, Prof. Dr. Md. Mustafa Kamal Chowdhury and Prof. Dr. Md. Abdul Alim, for their thoughtful and fruitful comments in preparing and writing this thesis. I am greatly indebted to the Head of the Department, Prof. Dr. Md. Manirul Alam Sarker for providing me with all kinds of generous help and support during the period of my study. I wish to thank the external examiner of the Board of Examiners, Dr. Salma Nasrin, Associate Professor, Department of Mathematics, University of Dhaka, Dhaka-1000.

My special thanks go to all of my respectable teachers and staffs of the Department of Mathematics, BUET, Dhaka, for their kind help and support.

Last, first and always, I am grateful to my parents for their prayers and unstinting support throughout my study period. My special thanks go to my beloved wife Mrs. Parul Akter for her continuous sacrifices and understanding and our daughter Marina Yesmin Mukta for her love, devotion and patience.

Finally, I wish to thank the authority of Udayan Uchcha Madhyamik Bidyalaya, Dhaka University Campus, Dhaka-1000, for giving me permission of pursuing M. Phil. Degree in the Department of Mathematics, BUET.

Above all, I am grateful to Almighty Allah for giving me patience and strength to finish this program.

Md. Mominul Islam Prodhan

ABSTRACT

Let R be a ring. Then the ring R has finite right Goldie dimension if it contains a direct sum of a finite number of nonzero right ideals. Symbolically, we write $G.\dim(R) < \infty.$ A ring R is called a right Goldie ring if it has finite right Goldie dimension and satisfies the ascending chain condition (*ACC)* for right annihilators. A module M is called a Goldie module if it has finite Goldie dimension and if it satisfies the ACC on M-annihilator submodules. In this thesis, we develop some properties of prime and semi-prime submodules over associative endomorphism rings by modifying the properties of prime and semi-prime ideals over associative arbitrary rings. Also, we investigate some properties of prime and semi-prime Goldie modules over associative endomorphism rings.

.

CONTENTS

CHAPTER I

INTRODUCTION

Ring theory is an important part of algebra. Module theory appears as a generalization of theory of vector space over a field. In a vector space, the scalars are treated to be the elements of a field while in a module we will allow the scalars to be the elements of an arbitrary ring. Every field is a ring and every ring may be considered as a module. Goldie's Theorem is a basic structural result in the ring theory, proved by Alfred Goldie in 1950. Alfred Goldie first used the notion of uniform modules to construct a measure of dimension for modules, now known as the uniform dimension or Goldie dimension for modules.

Literature review

Modern ring theory began when Wedderbern (1907) proved his celebrated classification theorem for finite-dimensional semi-simple algebras over fields. Twenty years later Emmy Noether and Emil Artin introduced the Ascending Chain Condition ACC) and Descending Chain Condition (DCC) as substitutes for dimensionality and Artin (1927) proved the analogue of Wedderbern's Theorem for general semi-simple rings. Wedderbern's Theorem for general semi-simple algebras can be extended successfully for rings satisfying the DCC on one-sided ideals. The Wedderbern-Artin Theory is the cornerstone of non-commutative ring theory.

Wedderbern-Artin Theorem: For a ring R, the following conditions are equivalent:

(1) R is a semisimple ring;

(2) Every right ideal I of R is a direct summand of R;

(3) Every left ideal I of R is a direct summand of R;

(4) Every right R-module is semisimple;

(5) Every right R-module is injective;

(6) Every right R-module is projective;

(7) Every cyclic right R-module is injective;

(8) Every cyclic right R-module is projective;

(9) Every simple right R-module is projective;

Prime ideals take an important role in the structure theory of rings and of major researches. In 1928, Krull introduced the notion of prime ideals via product of ideals in both the commutative and non-commutative cases. In the commutative case, there is a close connection between prime ideals and nilpotent elements. In particular, the intersection of all prime ideals equals the set of nilpotent elements. In 1929, Krull proved the existence of minimal primes in commutative case that every prime ideal contains a minimal prime ideal. The concepts of semi-prime ideals in commutative setting were introduced by Krull in 1929 and by Nagata in 1950. Krull proved that a non-commutative ring is semi-prime if and only if it has no nonzero nilpotent elements.

In the viewpoint of module theory, ring structures are generalized by considering as a special case of module structures, so properties are transferred from the category of rings to category of modules. The concepts of prime submodules are generalized from prime ideals.

In 1983, Goodeatl and Warfield and in 1987, McConnell and Robson introduced the notion of prime submodules over a non-commutative ring *R*. They called a left *R*-module *M* a prime module if for any proper submodule *X* of *M*, $ann_R(M) \subset ann_R(X)$.

In 2002, Ameri [10] and Gaur et al. [11] introduced the structure of prime submodules in multiplication modules over commutative rings. Following them, a left R-module *M* is a multiplication module if every submodule *X* is of the form *IM* for some ideal *I* of *R* and *M* is called a weak multiplication module if every prime submodule of *M* is of the form *IM* for some ideal *I* of *R*.

In 2007, Sanh et al. [15] introduced the new notion of prime submodules. They called a fully invariant proper submodule *X* of a right *R*-module *M* a prime submodule if for any ideal *I* of *S* and any fully invariant submodule *X* of *M*, if $I(X)\subset P$ then either $I(M)\subset P$ or $X\subset P$. A right *R*-module *M* is called a prime module if *0* is a prime submodule of *M*.

A fully invariant submodule *X* of a right R-module *M* is called a semi-prime submodule if it is an intersection of prime submodules. A right *R*-module *M* is called a semi-prime module if 0 is a semi-prime submodule of *M*.

We study in this thesis together with the related notion of essential and uniform submodules and applications are made to prime and semi-prime Goldie rings and modules. A non-zero submodule X of a right R-module M is called an essential submodule of M if for any nonzero submodule Y of M, $X \cap Y \neq 0$. A non-zero module M is called uniform if any two non-zero submodules of M have non-zero intersection, i.e. if each non-zero submodule of M is essential in M. A basic tool in the study of Noetherian rings and modules is the Goldie dimension of a module. A right R-module M is said to have finite Goldie dimension if M contains a direct sum of a finite number of nonzero submodules. Equivalently, if M has the finite uniform submodules $U_1, \ldots, U_n$ whose sum is direct and essential in M, then M has finite Goldie dimension. Then the positive integer n is called the Goldie dimension of M and is denoted by G.$dim(M)$ = n. Also M has finite Goldie dimension if M is Noetherian or Artinian.

In 2008, Sanh et. al. [15] introduced a new notion of Goldie modules. Let X be a submodule of a right R-module M and $S = End_R(M)$. Then X is called an M-annihilator if $X = Ker(I) = \bigcap_{f \in I} Ker(f)$, for some $I \subset S$. A right R-module M is called a Goldie module if it has finite Goldie dimension and satisfies the ACC on M-annihilator submodules. Applying this new notion we got many results relating to prime and semi-prime Goldie modules.

In this thesis, Chapter I deals with the early history of prime and semi-prime Goldie rings and modules. All the essential basic definitions, examples and their properties are given in Chapter II. Chapter III deals with the basic properties of prime and semi-prime Goldie rings together with some new properties. Also in this chapter, we describe some properties of prime and semi-prime ideals in associative arbitrary rings by modifying the results on prime and semi-prime modules investigated by Sanh et al. [14]. In Chapter IV,we investigate some properties of prime and semi-prime Goldie modules as generalizations of prime and semi-prime Goldie rings.

CHAPTER II
BASIC KNOWLEDGE

Overview

Throughout this thesis, all rings are associative with identity and all modules are unitary right *R*-modules. A substantial amount of information about a ring can be learned from a study of the class of modules it admits. Modules actually serve as a generalization of both vector spaces and abelian groups, and their basis behaviour is quite similar to that of the more special systems. In this chapter, we introduce the fundamental tools of this study. This chapter reviews the basic facts about rings, subrings, commutative division rings, integral domains, endomorphism rings, ideals and modules, homomorphisms and other notions. It also introduces some of the notations and the examples that will be needed later.

We denote by *R* an arbitrary ring and by *Mod-R*, the category of all right *R*-modules. The notation M_R indicates a right *R*-module *M* which when $1 \in R$ is assumed to be unity, i.e. to have the property that $1.m = m$ for any $m \in M$. The set $Hom_R(M, N)$ denotes the set of all right *R*-module homomorphisms from the right *R*-modules *M* to *N*. In particular, the set $Hom_R(M, M)$ denotes an endomorphism ring of a right *R*-module *M*. It is denoted by $S = End_R(M)$. The kernel of any $f \in Hom_R(M, N)$ is denoted by *Ker(f)* and the image of f by *Im(f)*. A submodule *X* of *M* is indicated by writing $X \leq M$. Also $I \leq R_R$ means that *I* is a right ideal of *R* and $I \leq {}_R R$ is a left ideal of *R*. The notion $I \leq R$ is reserved for ideals, i.e. two-sided ideals. The relation $A \leq_e M$ means that *A* is an essential submodule of *M*. As usual the sets **N**, **Z**, **Q**, **R**, **C** represent the sets of natural numbers, integers, rational numbers, real numbers and complex numbers respectively.

2.1 Preliminaries

Before dealing with deeper results on the structure of rings with the help of module theory, we provide first some essential elementary definitions, examples and properties.

Definition

Let R be a non-empty set with two binary operations addition(+) and multiplication($\bullet$). Then the algebraic structure $(R,+,\bullet)$ is called a ring if the following conditions hold:

(i) $(R,+)$ is an abelian group.
(ii) multiplication is associative, that is, $(a \bullet b) \bullet c = a \bullet (b \bullet c), \forall a,b,c \in R$.
(iii) multiplication is distributive over addition

$a \bullet (b+c) = a \bullet b + a \bullet c$ (left distributive law)

$(a+b) \bullet c = a \bullet c + b \bullet c$ (right distributive law) , $\forall a,b,c \in R$.

Example

(i) The set Z of all integers, is a ring under addition and multiplication. Similarly the sets Q,R,C of rational numbers, real numbers and complex numbers respectively are rings under usual addition and multiplication.

(ii) The set R of all matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,where a,b,c,d being real numbers, with matrix addition and multiplication, is a ring.

Definition

Let R be a ring with identity. Then R is called a *division ring* (or *skew-field*) if every non-zero element in R has a multiplicative inverse. A *field* is a commutative ring with identity in which every non-zero element has its multiplicative inverse.

Example

The sets Q,R,C of rational numbers, real numbers and complex numbers respectively are all a field under addition and multiplication, but the set Z of all integers, is not a field under addition and multiplication, because its every non-zero element except 1 has no multiplicative inverse. Therefore, every field a ring, but the converse is not true.

Definition

A ring R is said to be a *ring with identity* if we can find a multiplicative identity denoted by 1 in R such that $a1 = 1a = a$, for all $a \in R$.

An element x of a ring R is called a *left zero divisor* if $xy = 0$ for some non-zero $y \in R$, *right zero divisor* if $yx = 0$ for some non-zero $y \in R$ and *zero divisor* if it is both a left or a right zero divisor.

Example

(i) The ring $R=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d\in \mathrm{Z}\right\}$ is a ring with zero divisor, because

if $A=\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\neq 0$, $B=\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\neq 0$ then, $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

(ii) The residue classes on Z modulo 6 $\mathrm{Z}_6 = \{\overline{0},\overline{1},\overline{2},\overline{3},\overline{4},\overline{5}\}$ is a ring with zero divisor, because $\overline{2}.\overline{3}=\overline{6}=\overline{0}$, where $\overline{2}\neq\overline{0}$, $\overline{3}\neq\overline{0}$.

Definition

A ring R is said to be a ring without *zero divisor* if $ab=0 \Rightarrow a=0$ or $b=0$, $\forall\ a,b\in R$.

A commutative ring R with identity is called an *integral domain* if it is not possible to find two non-zero elements in R whose product is zero, i.e., $xy=0$, $\forall\ x,y\in R$, where $x\neq 0, y\neq 0$.

Hence an *integral domain* is a *commutative* ring with identity and without zero divisors.

Example

The ring $(\mathbf{I},+,\cdot)$ is an integral domain where $\boldsymbol{I}$ is the set of all irrational numbers. Also, the rings $(\mathbf{Z},+,\cdot)$, $(\mathbf{Q},+,\cdot)$, $(\mathbf{C},+,\cdot)$, $(\mathbf{R},+,\cdot)$ are examples of integral domains.

Definition

Let R, R' be two rings. Then a map $f: R\rightarrow R'$ is called a *ring homomorphism* if $\forall\ r,s\in R$, we have (i) $f(r+s)=f(r)+f(s)$, (ii) $f(rs)=f(r)f(s)$. Then a map f is a called a *monomorphism* if and only if f is one-one, an *epimorphism* if and only if f is onto and an *isomorphism* if and only if f is both one-one and onto.

Let $f: R\rightarrow R'$ be a homomorphism, then the *image* of f is denoted by Im(f) and defined as

$$\mathrm{Im}(f)=\{x'\in R' : f(x)=x',\ \text{for some}\ x\in R\}$$

and *kernel* of f is *denoted by* Ker (f) and defined as

$$\mathrm{Ker}(f)=\{x\in R : f(x)=0\}.$$

Definition

Let R be a ring and I be a nonzero subring of R. Then the set I is called a *right ideal* of R *if* $\forall a,b\in I$ we have $ar\in I$.

Again, I is called a *left ideal* of R if $\forall\ a\in I, \forall\, r\in R$ we have $ra\in I$ and I is called an *ideal* *(i.e., two-sided ideal)* of R *if* $ar\in I$ and $ra\in I$, $\forall\ a\in I, \forall\, r\in R$. Every ideal in a ring R is a submodule of R .

Example

(i) The subring $E = \{\cdots,-4,-2,0,2,4,\cdots\}$ of even integers, is an ideal of the ring of integers $Z = \{\cdots,-2,-1,0,1,2,\cdots\}$.

(ii) Let $\mathrm{R} = \left\{ \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix} : \mathrm{a,b,c,d} \in \mathbf{Z} \right\}$ is a ring. Then $S = \left\{ \begin{bmatrix} \mathrm{a} & 0 \\ \mathrm{b} & 0 \end{bmatrix} : \mathrm{a,b} \in \mathbf{Z} \right\}$ is a left ideal and $T = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a,b \in \mathbf{Z} \right\}$ is a right ideal of R.

Definition

An ideal *S* of a ring *R* is called a *semi-prime* ideal if for any prime ideals I, J of *R* such that $S = I \cap J$.

Example

Let the set $\mathbf{Z}$ of all integers be a ring and *m, n* prime numbers. Then mZ , nZ are both prime ideals and $mZ \cap nZ = m\,nZ$ is a semi-prime ideal of $\mathbf{Z}$.

In the ring $\mathbf{Z} = \{\cdots,-2,-1,0,1,2,\cdots\}$ of all integers, the ideal

$\mathrm{S} = 2\mathbf{Z} = \{\cdots,-6,-4,-2,0,2,4,6,\cdots\}$ and $\mathrm{T} = 3\mathbf{Z} = \{\cdots,-9,-6,-3,0,3,6,9,\cdots\}$ are prime ideals in $\mathbf{Z}$ and $S \cap T = 2\mathbf{Z} \cap 3\mathbf{Z} = 6\mathbf{Z} = \{\ldots,-18,-12,-6,0,6,12,18,\ldots\}$ is a semi prime ideal of $\mathbf{Z}$.

Definition

Let *R* be a ring and *I* be an ideal of *R*. Then *I* is called a *principal ideal* of *R* if *I* is generated by a single element of *R*, if If *I* is generated by *a,* then it is denoted as $I =$ (a), i.e., if $a \in R$, then an ideal of the form $I =$ (a) = $a\mathbf{R} = \{\, ar : r \in \mathbf{R} \,\}$ is called a *principal ideal* of *R*.

Example

In the ring $\mathbf{Z} = \{\cdots,-2,-1,0,1,2,\cdots\}$ of all integers, the ideal

$\mathrm{I} = (5) = 5\mathrm{Z} = \{\cdots,-10,-5,0,5,10,\cdots\}$ is a principal ideal of $\mathbf{Z}$ generated by 5.

Definition

Let *R* be a ring and *M* be an ideal of *R* such that $M \neq R$. Then M is called *maximal* if for any ideal *N* of *R* such that $M \subset N \subset R,$ then either $M = N$ or $N = R,$ i.e., *M* is called a *maximal* if there exists no idesl of *R* which lies between *M* and *R.*

Example

Consider the ring $\mathbf{Z} = \{\cdots,-2,-1,0,1,2,\cdots\}$ of all integers. Choose two ideals

$\mathrm{S} = 6\mathbf{Z} = \{\cdots,-12,-6,0,6,12,\cdots\}$ and $\mathrm{T} = 3\mathbf{Z} = \{\cdots,-6,-3,0,3,6,\cdots\}$ of $\mathbf{Z}$. Here S is not a maximal ideal, because there exists an ideal T which lies between S and $\mathbf{Z}$, i.e., $S \subset T \subset Z$, But if we choose $\mathrm{S} = (5) = 5\mathrm{Z} = \{\cdots,-10,-5,0,5,10,\cdots\}$, then S is a maximal ideal, because there exists no ideal of $\mathbf{Z}$ which lies between S and $\mathbf{Z}$.

Proposition 2.1.1 Every maximal ideal M of a ring R is a prime ideal.

Proof. If I and J are ideals of R not contained in M then $I + M = R$ and $J + M = R$. Now $R = (I + M)(J + M) = IJ + IM + MJ + M^2 \subseteq IJ + M$ and hence $IJ \not\subset M$.

Definition

A *minimal prime ideal* in a ring R is any prime ideal of R that does not properly contain any other prime ideals. For instance, if R is a prime ring, then 0 is the unique minimal prime ideal of R.

Example

(i) In a commutative Artinian ring, every maximal ideal is a minimal prime ideal.

(ii) In an integral domain, the only minimal prime ideal is the zero ideal.

Definition

Let I be an ideal of a ring R. Then the ring $R/I = \{a + I : a \in R\}$ is called a *quotient ring* or *factor ring* of R by I defined by

(i) $(I + a) + (I + b) = I + a + b$ and

(ii) $(I + a)(I + b) = I + ab \quad \forall a, b \in R$.

2.2 Modules and different kind of submodules

Definition

Let R be a ring with identity and M an additive abelian group. Then M is called a *right R-module* if there exists a map $f : M \times R \to M$ defined by $f(mr) = mr$ satisfying the following conditions:

(i) $\forall\ m_1, m_2 \in M$ and $\forall\ r \in R$, we have $(m_1 + m_2)r = m_1 r + m_2 r$;

(ii) $\forall\ m \in M$ and $\forall\ r_1, r_2 \in R$, we have $m(r_1 + r_2) = mr_1 + mr_2$;

(iii) $\forall\ m \in M$ and $\forall\ r_1, r_2 \in R$, we have $m(r_1 r_2) = (mr_1)r_2$;

(iv) $\forall\ m \in M$ and $1 \in R$, we have $m.1 = m$.

Similarly, we can define *left R- modules* by operating to the left side of *M*. If *M* is a right *R*-module, then it is denoted by M_R and if M is a left R-module, then it is denoted by ${}_R M$.

Example

(i) Every ring *R* is an *R*-module over itself. Since *Z* is a ring, so *Z* is a *Z*-module.

(ii) Every additive group is a module over the ring *Z* of intgers. Since *Z, Q, R* are dditive group, so they are *Z*-modules.

(iii) Let *R* be a ring and *I* a left ideal of *R*, then *I* is an *R*-module.

Definition

Let X be a subset of *M*. Then *X* is called a *submodule* of M if (*X*, +) is a subgoup of (*M*, +) and it satisfies the following conditions:

$(i)\ \ \forall\ x, y \in X\ we\ have\ \ x + y \in X \quad (ii)\ \ \forall\ x \in X,\ \forall\ r \in R\ \ we\ \ have\ \ xr \in X \quad \leq M$

Definition

Let *M* be a right R- module and $L \leq M$, a submodule of M. Then the right *R*-module M/L is called a *quotient module* or *factor module* of M by L with the operation $f : M/L \times R \to M/L$ defined by (i) $f(m, r) = mr$ and (ii) $f(m + L, r) = (m + L)r = mr + L$, $\forall\ m \in M$, $\forall\ r \in R.$

Let *R, S* be two rings and *M* an abelian group. Then *M* is called an *R-S-bimodule* if *M* is a left *R*-module, right *S*-module. It is denoted by ${}_R M_S$. If for any m$\in$*M, r* $\in$ *R, s* $\in$ *S,* we have *r (ms) = (r m) s.*

Theorem 2.2.1 (Modular law) If *A, B, C* are submodules of M_R and $B \leq C$ then

$$(A + B) \cap C = (A \cap C) + B.$$

Proof: Let $a + b = c \in (A + B) \cap C,$ where $a \in A,\ b \in B,\ c \in C,$ then it follows from $B \leq C$ that $a = c - b \in (A \cap C),$ thus $a + b = c \in (A \cap C) + B$ and hence $(A + B) \cap C \leq (A \cap C) + C$

Conversely, let $d \in (A \cap C), b \in B.$ Since $B \leq C$, it follows that $d + b \in (A + B) \cap C,$ and thus also that $(A \cap C) + B \leq (A + B) \cap C.$ Hence we get $(A \cap C) + B \leq (A + B) \cap C$

Therefore, $(A + B) \cap C = (A \cap C) + B.$

Definition

Let M and N be R-modules. Then a map $f : M \to N$ is said to be a *homomorphism* if

(i) $\forall\ m, m' \in M$ we have $f(m+m') = f(m) + f(m')$;

(ii) $\forall\ m \in M$ and $\forall\ r \in R$, we have $f(mr) = f(m)r$.

Let $f : M \to N$ be a *homomorphism*. Then image of f is denoted by $\operatorname{Im} f$, and defined as

$\operatorname{Im} f = \{ n \in N : f(m) = n, \textit{ for some } m \in M \}$. $\operatorname{Im} f$ is a submodule of N.

The kernel of f is denoted by $Kerf$ and defined by

$\ker f = \{ m \in M : f(m) = 0 \}$. $\ker f$ is a submodule of M.

An R-homomorphism $f : M \to N$ is called

(i) a *monomorphism* if for any submodule X of M and any homomorphism $h, g : X \to M$, we have $f \,o\, h = f \,o\, g \Rightarrow h = g$.

(ii) an *epimorphism* if for any submodule X of M and any homomorphism $h, g : N \to X$, we have $h \,o\, f = g \,o\, f \Rightarrow h = g$.

(

iii) an *isomorphism* if f is a monomorphism and an epimorphism.

(iv) an *automorphism* if f is an *isomorphism* and $M = N$

Remarks

(i) $f : M \to N$ is a monomorphism if and only if f is one-one.

(ii) $f : M \to N$ is an epimorphism if and only if f is onto.

Definition

Let M and N be two right R-modules. The set $\operatorname{Hom}(M, N)$ denotes the set of all right R-module homomorphisms from$M$ to N. In particular, $\operatorname{Hom}(M, M)$ is the set of all right R-module homomorphisms from M to M. The abelian group $\operatorname{Hom}(M, M)$ becomes a ring if we use the composition of maps for multiplication. This ring is called the *endomorphism ring* of M and it is denoted by $S = End_R(M)$.

Definition

Let M be a right R-module and $X \subseteq M$, a subset of M. Then we say that M is generated by X if $M = |X) = \{ \sum x_i r_i \ / \ x_i \in X,\ r_i \in R,\ i = 1, 2, 3, \cdots \}$. If X is a fiinite subset, then M is finitely generated and we write $M=|X)=\{\sum_{i=1}^{k} x_i r_i / x_i \in X,\ r_i \in R,\ i=1,2,3,\cdots k\}$. A module M is called a self-generator if it generates all of its submodules. If X is a submodule of M and $X=\sum_{f_i \in S} f_i(M)$, then M is called a self-generator, where $S = End_R(M)$.

Definition

Let *M* be a right *R*-module and *X*, a subset of *M*. Then the set $|X)$ is called the *submodule* of *M generated* by *X*, where $|X) = \{\sum_{1\le i\le n} x_i r_i : x_i \in X, r_i \in R,\ i=1,2,\ldots, n;\ n \in N\}$,

A subset *X* of M_R is called a *free set* (or *linearly independent set*) if for any $x_1, x_2, x_3, \ldots, x_k \in X$, and for any $r_1, r_2, \ldots, r_k \in R$, we have $\sum_{i=1}^{k} x_i r_i = 0 \Rightarrow r_i = 0,\ \forall i \in \{1,2,\ldots,k\}$. A subset X of M_R is called a *basis* of *M* if $M = |X)$ and *X* is a free set. If a module *M* has a basis, then *M* is called a *free module.*

Example

In $Z_6 = \{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\} = |\bar{1})$, Z_6 is a Z-module. Then

1. $|\bar{1}) = Z_6$, $|\bar{2}) = \{\bar{0},\bar{2},\bar{4}\} = |\bar{4})$, $|\bar{3}) = \{\bar{0},\bar{3}\}$, $|\bar{2},\bar{3}) = Z_6$, because $x\bar{3} + y\bar{2} = \bar{1}$ for some x, y $\in Z$.
2. $\{\bar{2}\}$ is not free because $3 \times \bar{2} = \bar{0}$, $\{\bar{2},\bar{3}\}$ is not free, because $3 \times \bar{2} + 2\times\bar{3} = \bar{0}$.

Hence Z_6 is a finitely generated Z-module.

Definition

A submodule *A* of M_R is called a *direct summand* of *M* if there exists a submodule $B \le M$ such that $M = A + B$ and $A \cap B = \{0\}$. Then *M* is called a direct sum of *A* and *B* or the sum *A* + *B* is direct.In this case, we write $M = A \oplus B$. In general, the sum $\sum_{i\in I} A_i \le M$ is called a direct sum if for any $j \in I$, we have $A_j \cap \sum_{i\ne j, i\in I} A_i = 0$. If $x \in A \oplus B$ and $x = a + b$, then $a \in A$ and $b \in B$ and *a*, *b* are the the unique elements of *A* and *B* respectly.

Theorem 2.2.2 If $A \oplus B$ is the internal direct sum and A, B are submodules of M, then $A \oplus B \leq M$.

Proof. Consider $A \times B = \{(a,b) | a \in A \wedge b \in B\}$, we can consider as $A \coprod B$ or $A \prod B$. It is clear that $A \times B$ is a right R-module but $A \times B \not\subset_{>} M$. Define $\ddot{\ } : A \times B \to A \oplus B$ by $\ddot{\ }(a,b) = a + b$ for all $a \in A$ and $b \in B$. Then $\ddot{\ }$ is an R-homomorphism, because for any $(a,b),(a',b') \in A \times B$ and for any $r \in R$, we have

$\ddot{\ }((a,b)+(a',b')) = \ddot{\ }(a+a',b+b') = (a+a')+(b+b') = (a+b)+(a'+b') = \ddot{\ }(a,b) + \ddot{\ }(a',b')$,
and $\ddot{\ }((a,b)r) = \ddot{\ }(ar,br) = ar + br = (a+b)r = \ddot{\ }(a,b)r$.
Also, $\ddot{\ }(a,b) = \ddot{\ }(a',b') \Rightarrow a+b = a'+b' \Rightarrow a = a' \wedge b = b' \Rightarrow (a,b) = (a',b')$, showing that $\ddot{\ }$ is a monomorphism. For every $y \in A \oplus B$, $y = a + b$ where $a \in A, b \in B$.

Choose $x = (a,b) \in A \times B$. we have $\ddot{\ }(x) = y$. then $\ddot{\ }$ is an epimorphism. Thus $\ddot{\ }$ is an isomorphism, i.e., $A \times B \cong A \oplus B$.

Definition

A submodule X of a right R-module M is called *essential or large* in M if for any nonzero submodule U of M, $X \cap U \neq 0$. If X is essential in M we denote $\mathrm{X} \leq_e M$. A right ideal I of a ring R is called essential if it is essential in R_R. For any right R-module M, we always have $M \leq_e M$. Any finite intersection of essential submodules of M is again essential in M, but it is not true in general. For example, consider the ring Z of integers. Every nonzero ideal of Z is essential in Z but the intersection of all ideals of Z is 0 which is not essential in Z. Since any two nonzero submodules of Q have nonzero intersection, so Q is an essential extension.

A submodule X of M_R is called *superfluous or coessential or small* in M if for any submodule Y of M, we have $X + Y = M$ implies $Y = M$, or equivalently, $Y \neq M$ implies $X + Y \neq M$. A right ideal I of a ring R is called superfluous in R if it is a superfluous submodule of R_R. Every module has at least one superfluous submodule, namely 0.

Proposition 2.2.3 In Z, every nonzero ideal is essential.

Proof. Let $0 \neq I \subset \mathrm{Z}$. Then $\exists\ m \in \mathrm{Z} : I = m\mathrm{Z}$. For any nonzero ideal $J \subset \mathrm{Z}$, we can find an $n \in \mathrm{Z} : J = n\mathrm{Z}$. Thus $I \cap J = m\mathrm{Z} \cap n\mathrm{Z} = mn\mathrm{Z}$, so $mn \in I \cap J$, and so $I \cap J \neq 0$. Therefore, $I \leq_e \mathrm{Z}$.

Proposition 2.2.4 Let M be a right R-module. Then for any submodule $A \leq M$, $A \leq_e M$ $\Leftrightarrow \forall\, m \in M, m \neq 0, \exists\, r \in R: m\,r \neq 0$ and $m\,r \in A$.

Proof. Assume that $A \leq_e M$. Choose $m \in M, m \neq 0$. Then $mR \neq 0$, and so $A \cap mR \neq 0$. then there exists $0 \neq x \in A \cap mR$.

This means that $0 \neq x \in A$ and there exists $r \in R$ such that $x = mr$. Therefore, $0 \neq x = mr \in A$.

Conversely, let U be a nonzero submodule of M. Choose $0 \neq m \in U$. By hypothesis, there exists $r \in R$ with $mr \neq 0$ and $mr \in A$. But then since $mr \in U$, we have $mr \neq 0$ and $mr \in A \cap U$. Hence $A \leq_e M$.

Proposition 2.2.5 For any $M \in$ Mod-R, let $A \leq B \leq M$. If $A \leq_e M$, then (i) $A \leq_e B$, and (ii) $B \leq_e M$

Proof. (i) Let $U \leq B$ be such that $U \neq 0$. then U is a submodule of M Since $A \leq e\ M, U \cap A \neq 0$. Hence $A \leq_e B$.

(ii) Let $U \leq M$ be such that $U \neq 0$. Then $0 \neq A \cap U \subseteq B \cap U$, because $A \cap U \neq 0$, and so $B \leq_e M$

Proposition 2.2.6 Let A and B be essential submodules in M_R. Then $A \oplus B \leq_e M$ and $A \cap B \leq_e M$.

Proof. Let $U \leq M$ be such that $U \neq 0$. Then $U \cap (A \cap B) = (U \cap A) \cap B \neq 0$.

Hence $A \cap B \leq_e M$. We have $A \leq A \oplus B \leq M$ and $A \leq_e M$, implying that $A \oplus B \leq_e M$.

Note: Every nonzero submodule of M is essential in M, i.e., a non-zero submodule A of M is called essential in M if A has nonzero intersection with any non-zero submodule of M.

Lemma 2.2.7 Let L be a submodule of a right R-module$M$. Then L is an essential submodule of M if and only if for any nonzero element $m \in M, \exists\, r \in R: m\,r \neq 0$ and $m\,r \in L$.

Proof: We assume that $L \leq_e M$ and choose m $\in M$, $m \neq$ o. Then $mR \neq$ o, and so $L \cap mR \neq 0$. Then $\exists$ o $\neq x \in L \cap mR$. This means that $0 \neq x \in L$ and $\exists\ r \in R$ such that $x = mr$. Therfore, $0 \neq x = mr \in L$.

Conversery , let Y be a nonzero submodule of M. Choose $0 \neq m \in Y$. By hypothesis, $\exists$ r $\in$ R with $mr \neq 0$ and mr $mr \in L$. But since $mr \in$ Y, so we have $mr \neq 0$ and $mr \in L \cap Y$. Hence $L \leq_e M$.

Definition

A submodule X of M_R is called a *maximal* submodule of M if $X \neq M$ and for any submodule Y of M if $X \leq Y \leq M$, then either $Y = X$ or $Y = M$.

Example

Consider the module $\mathbf{Z} = \{\cdots,-2,-1,0,1,2,\cdots\}$ of all integers. Choose two submodules $S = 6\mathbf{Z} = \{\cdots,-12,-6,0,6,12,\cdots\}$ and $T = 3\mathbf{Z} = \{\cdots,-6,-3,0,3,6,\cdots\}$ of **Z.** Here S is not a maximal submodule, because there exists an submodule T which lies between S and $\mathbf{Z}$, i.e., $S \subset T \subset Z$, But if we choose $S = (5) = \{\cdots,-10,-5,0,5,10,\cdots\}$, then S is a maximal submodule, because there exists no submoduleof $\mathbf{Z}$ which lies between S and **Z.**

Definition

A submodule X of M_R is called a *minimal* (or simple submodule) submodule if $X \neq 0$ and for any submodule Y of M such that $0 \leq Y \leq X$, then $Y = 0$ or $Y = X$.

Theorem 2.2.8 Let N be any proper submodule of M . Then N is maximal in M if and only if $\forall$ m $\in M/_N$, we have $mR + N = M$.

Proof : Suppuse that N is maximal in M . Choose any $m \in M/_N$.

Then $N \subset\!> \neq N + mR \leq M$. By definition , $N + mN = M$.

Conversely , we suppose that $N \leq \neq$ Y $\leq$ M. Then we can find. so, $y \in Y/_N$. So $y \in M/_N$.

By assumption, N+y R = M . Since $N \leq Y$ and $y \in Y$, we have N+ y R $\leq Y$. It shows that M = N+ y R $\leq Y \leq M$. It follows that Y = M or N is maximal in M .

Lemma 2.2.9 Let M be a right R-module and U, X be submodules of M with $U \subset X$. Then X is maximal in M if and only if X/U is maximal in M/U.

Proof : ($\Rightarrow$) Let Y be a submudule of M and $X/U \subset Y/U \subset M/U$. Then X is a submodule of Y. Since X is maximal in M, we have $X = Y$ implying that and $X/U = Y/U$.

Let ($\Leftarrow$) $U \leq X \subset_{>} \neq M$. and Then $X/U \leq Y/U \leq \neq M/U$. Since X/U is maximal in M/U, we have X/U=M/U, showing that X=Y.

Definition

Let M be a right R-module and $m \in M$. Then the submodule of the form $mR = \{mrr \setminus r \in R\}$ is called a *cylic submodule* of M and M is cylic if $M = mR$. A module M is *simple* if $M \neq 0$ and only 0 and M are submodules of M. Every simple module M is cyclic, in fact it is generated by any non-zero $m \in M$.

2.3 Noetherian and Artinian Rings and Modules

Definition

A nonempty family $F$of submodules of M_R is said to satisfy the *Ascending Chain Condition* (briefly, *ACC)* if for any chain $M_1 \subset M_2 \subset \ldots \subset M_n \subset \ldots$

of submodules in F, there exists a positive integer n such that $M_{n+1} = M_n$ for n = 1,2,…

A ring which satisfies *ACC* for right(left) ideals is called a *right (left) noetherian* ring. A ring which is both right noetherian and left noetherian is called a *noetherian* ring.

A module M is called *noetherian* if *ACC* holds for M. An R-module M is *noetherian* if every submodule of M is finitely generated.

Example

(i) Every finitely generated abelian group ring is noetherian. Since *Q [Q =1.Q=\ Q)]*,

Z,R are finitely generated abelian group, *so Q, Z,R* are noetherians.

(ii) Any principal ideal domain (*PID*) is a noetherian ring, because every ideal of *PID* is generated by a single element. The set Zof all integers, is a noetherian ring, because it is a principal ideal domain (PID).

(iii) Every finitely generated abelian group over a noetherian ring is a noetherian module.

Since Z is a noetherian ring, so the module Z_Z is noetherian.

Proposition 2.3.1[19] A module is noetherian if and only if every strictly ascending chain of submodules is finite.

Proof. Let M be noetherian and $M_1 \subset M_2 \subset ......$ an ascending chain of submodules. The submodule $\bigcup_{i \in i} M_i$ has a finite number of generators, and all of them must lie in some M_{i_0}. It follows that the chain gets stationary at M_{i_0}. Conversely, it is easy to see that the ascending chain condition for submodules implies that every submodule has a finite number of generators.

The `assending chain condition`, i.e. finiteness of all strictly ascending chains, is usually abbreviated as *ACC.*

Proposition 2.3.2 [19] Let *L* be a submodule of *M*. Then *M* is noetherian if and only if both *L* and *M / L* are noetherian.

Proof. M is noetherian obviously implies that L is noetherian. It also implies that M/L is noetherian, because the submodules of M/L can be written as M'/L, there $L \subset M' \subset M$. Suppose conversely that L and M/L are noetherian. If M' is a submodule of M, then $L \cap M'$ is finitely generated as a submodule of L, and $M'/(L \cap M') \cong (L + M')/L$ is finitely generated as a submodule of M/L. It follows from Lemma 3.1(ii) ([19], page-11) that M' is finitely generated. Hence M is noetherian.

The ring *R* is right noetherian if R_R is a noetherian module, i.e. every right ideal of *R* is finitely generated.

Proposition 2.3.3 [19] If a ring R is noetherian, then every finitely generated module is noetherian.

Proof. If R_R is noetherian, then every finitely generated free module is noetherian by Prop. 2.3.3, and therefore every finitely generated module is a quotient of a noetherian module and hence noetherian by Prop. 2.3.3.

Theorem 2.3.4 [18] Let M be a right R-module and $A \leq M$. Then the following conditions are equivalent:

(1) M is noetherian;

(2) A and M/A are noetherian;

(3) Any ascending chain $A_1 \leq A_2 \leq \ldots \subset_{>} A_n \leq \ldots$ of submodules of M is stationary, i.e., there exists $n \in \mathrm{N}$ such that $A_n = A_{n+1.}$ This condition is called the ascending chain condition or ACC.

(4) Every submodule of M is finitely generated.

Proof. $(1) \Rightarrow (3):$ Suppose that every nonempty family of submodules of M has a maximal element by inclution. Given an ascending chain

$$A_1 \leq A_2 \leq \ldots \leq A_n \leq A_{n+1} \leq \ldots$$

Let $\mathrm{Y} = \{A_i | \, i \in \mathrm{N}\}$. By hypothesis, we can find a maximal element of Y by inclusion, say A_k. We can see that for any $n \geq k, A_k \leq A_n$. But then since A_k is maximal, $A_n \leq A_k$. Hence for any $n \geq k$, $A_n = A_k$. This implies that the chain is stationary.

$(3) \Rightarrow (1):$ Let X be a family of submodules of M and let $A_1 \subseteq A_2 \subseteq \ldots A_n \subseteq \ldots$ be a chain in X. By assumption, this chain is stationary. So, we can find A_n such that $A_i \subseteq A_n$, for any i. By Zorn's lemma, X has a maximal element. Then M is noetherian.

$(3) \Rightarrow (2):$ Let $\quad X_1 \leq X_2 \leq \ldots \leq X_n \subset X_{n+1} \leq \ldots$

be a chain of submodules in A. Then this chain is also a chain in M and hence it must be stationary. So A is noetherian. Now let

$$X_1 \leq X_2 \leq \ldots \leq X_n \subset X_{n+1} \leq \ldots \qquad (*)$$

be a chain of submodules in M / A. Then $X_1 = A_1/A, X_2 = A_2/A, \ldots\ldots.$ with $A_1 \leq A_2 \leq \ldots \leq A_n \leq A_{n+1} \subset_{>} \ldots \leq M$. since M is noetherian, M satisfies (3), and so we can find $n_0 \in \mathrm{N}$ such that $A_{n_0} = A_{n_0+1}$ Hence the chain $(*)$ is stationary, proving that M / A is noetherian.

$(2) \Rightarrow (3):$ Assume that A and M/A are noetherian.

Let $A_1 \leq A_2 \leq \ldots\ldots\ldots. \leq A_n \leq A_{n+1} \leq \ldots\ldots.$

be a chain in M. Then $A_1 \cap A \leq A_2 \cap A \leq \ldots\ldots\ldots. \leq A_n \cap A \leq A_{n+1} \cap A \leq \ldots\ldots. \leq A$

Since A is noetherian, by (3), there exists $n_1 \in \mathrm{N}$ such that for any $K \geq 0$, we have $A_{n_1+k} \cap A = A_{n_1} \cap A$. Consider $(A_n + A)/A \subset_{>} M/A$, so we have

$$(A_1 + A)/A \leq (A_2 + A)/A \leq .... \subset (A_n + A)/A \leq .... \leq M/A.$$

Since M/A is noetherian, there exists $n_2 \in \mathrm{N}$ such that for any $k \geq 0$, we have $(A_{n_2+k} + A)/A = (A_{n_2} + A)/A$. Hence for any $k \geq 0$, we have $A_{n_2+k} + A = A_{n_2} + A$.

Put $n_0 = \max\{n_1, n_2\}$. Then for any $n \geq n_0$, we have $A_{n_0} \cap A = A_{n_0+k} A$ for all $k \geq 0$ and $A_{n_0} + A = A_{n_0+k} + A$ for all $k \geq 0$. Thus for any $k \geq 0$, we have

$$A_{n_0+k} = A_{n_0+k} \cap (A_{n_0+k} + A) = A_{n_0+k} \cap (A_{n_0} + A) = A_{n_0} + (A_{n_0} \cap A) = A_{n_0}.$$

Hence M is noetherian.

$(3) \Rightarrow (4)$: Let $A \leq M$ and let $0 \neq m_1 \in A$. Then $m_1 R \leq A$. If $m_1 R = A$, then we are done. Suppose $m_1 R \neq A$. we can find $m_2 \in A/m_1 R$ and then $m_1 R \subset_{>} m_1 R + m_2 R \leq A$. If $m_1 R + m_2 R = A$, then we are done. Continuing in this way, we have a chain

$$m_1 R \subset_{>} m_1 R + m_2 R \leq m_1 R + m_2 R + m_3 R \leq ...$$

in A by (3), this chain is stationary. Thus M is finitely generated.

$(4) \Rightarrow (3)$: Let $A_1 \leq A_2 \leq .... \subset_{>} A_n \leq A_{n+1} \leq ....$ be a chain in M. Then $\bigcup_{i=1}^{\infty} A_i = \sum_{i=1}^{\infty} A_i =: A$. So $A \subset_{>} M$. By (4), A is finitely generated. Then by the property of finitely generated module, we can find $i_1, ..., i_k$ such that $A = A_{i_1} + ... + A_{i_k}$. Let $n = \max\{i_1, ..., i_k\}$. Then $A = A_n$ proving that the above chain is stationary.

Definition

A nonempty family F of submodules of M_R is said to satisfy the *DCC* if for any chain $M_1 \supset M_2 \supset .... \supset M_n \supset ....$ of submodules in F, there exists a positive integer n such that $M_{n+1} = M_n$ for n = 1, 2, …

A ring which satisfies the DDC(descending chain condition) for right(left) ideals is called a *right(left) artinian* ring. A ring which is both *right artinian* and *left artinian* is called an *artinian* ring.

Example

(i) Every finite ring is Artinian.

(ii) A module which has only finitely many submodules is artinian. In particular, finite abelian groups are artinian as module over $\mathbb{Z}$.

If a ring R is right artinian, then R is right noetherian but the converse is not true. For example, consider Z (northerian), $m\mathrm{Z} \subseteq n\mathrm{Z} \Leftrightarrow n \mid m$ and

$m_1\mathrm{Z} \subseteq m_2\mathrm{Z} \subseteq m_3\mathrm{Z} \subseteq \ldots \Leftrightarrow m_2/m_1, m_3/m_2, \ldots$

The chain $2\mathrm{Z} \supset 2^2\mathrm{Z} \supset 2^3\mathrm{Z} \supset \ldots \supset 2^n\mathrm{Z} \supset \ldots$ is not stationary. So Z is not artinian. Thus Z

is noetherian but not artinian. we can conclude that the module Z_Z is noetherian but Z_Z is not artinian.

Theorem 2.3.5 [18] Let M be a right R-module and let A be its submodule. Then the following statements are equivalent:

(a) M is artinian;

(b) A and M/A are artinian;

(c) Any descending chain $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$ of submodules of M is stationary. This condition is called the *descending chain condition or DCC.*

(d) Every factor module of M is finitely co-generated.

(e) Let $0 \to L \to M \to N \to 0$ be an exact sequence of right R-modules. Then M is noetherian (resp. artinian) $\Leftrightarrow$ L and N are noetherian (resp. artinian).

Corollary 2.3.6 [3] (1) The image of artinian (resp. noetherian) module is also artinian(resp. noetherian).

(2)The finite sum of artinian (resp. noetherian) submodules of M is also artinian (resp. noetherian).

(3) The finite direct sum of artinian (resp. noetherian) modules of M is also artinian (resp. noetherian).

(4) If R is semi-simple, then R is both left and right artinian (resp. noetherian

2.4 Exact Sequences, Injective and Projective modules

Definition

Let $\{A_i, i \in I\}$ be a collection of right R-modules. For each $i \in I$, let $f_i : A_i \to A_{i+1}$ be an R-homomorphism. Then a sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \ldots\ldots\ldots\ldots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \ldots\ldots.$$

is called an *exact sequence* at A_n if $\operatorname{Im}(f_{n-1}) = \ker(f_n)$. The sequence is called an *exact sequence* if it is exact at each A_n.

An special exact sequence of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a *short exact sequence.*

Remarks

(i) If the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact, then *f* is a monomorphism, *g* is an epimorphism and $\operatorname{Im}(f) = \ker(g)$.

(ii) Let $X \leq M \in Mod - R$, then the inclusion map $\S : X \to M$ defined by $\S(x) = x$ for any $x \in X$ is called the embedding homomorphism. Then the sequence $0 \to X \xrightarrow{\S} M \xrightarrow{\neg} M/X \to 0$ is exact and the map $\neg$ defined by $\neg(m) = m + X$ for any $m \in M$ is called the natural or canonical homomorphism.

Definition

A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called *split exact* if $\operatorname{Im}(f) \leq^{\oplus} B$, (i.e., there exists $B' \leq B : B = \operatorname{Im}(f) \oplus B'$).

Proposition 2.4.1 [19] The following properties of an exact sequence

0 $\to$ X $\xrightarrow{\sim}$ Y $\longrightarrow$ Z $\to$ 0 are equivalent:

(a) The sequence splits.

(b) There exists a homomorphism ¨ : Y $\to$ X such that ¨ ~ = 1_X .

(c) There exists a homomorphism · : Z $\to$ Y such that · = 1_Z

Proof. It is clear that (a) implies (b) and (c). Suppose (b) is satisfied. The maps ¨ $: Y \to X$ and $: Y \to Z$ can be used to define « $: Y \to X \oplus Z$ so that the diagram(1) commutes. « is an isomorphism by Prop. 1.3 [19]. Hence the sequence splits. The proof of $(c) \Rightarrow (a)$ goes dually.

A module Y is said to be generated by a family $(x_i)_I$ of elements of Y if each $x \in Y$ can be written $x = \sum_I x_i a_i$ with all but a finite number of a_i equal to *0*. It it furthermore is true that the coefficients a_i are uniquely determined by x, then the family $(x_i)_I$ is a basis for Y. A module is called free if there exists a basis for it.

Theorem 2.4.2 Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of right *R*-modules. Then the following statements are equivalent:

(a) The given sequence splits;

(b) There exists a homomorphism $f' : B \to A : ff' = 1_A$;

(c) There exists a homomorphism $g' : C \to B : gg' = 1_C$.

Proof. $(a) \Rightarrow (b)$ Suppose that $\mathrm{Im}(f)$ is a direct summand of *B*. Then there exists a submodule $B' \subset_> B$ such that $B = \mathrm{Im}(f) \oplus B'$. We will define a homomorphism $f : B \to A$. to do this let $b \in B.$ Then there exists $y \in \mathrm{Im}(f)$ and $b' \in B'$ such that $b = y + b'$ which is the unique decomposition. Since *f* is a monomorphism, there is a unique $a \in A$ *such* that $y = f(a)$. Let $f'(b) = a.$ It is clear that f' is a map. We now show that f' is a homomorphism. To do this, let $b_1, b_2 \in B$ and $r \in R.$ Then $b_1 = y_1 + b_1'$ and $b_2 = y_2 + b_2'$, where $y_1, y_2 \in \mathrm{Im}(f)$ and $b_1', b_2' \in B'.$ Thus $b_1 + b_2 = (y_1 + b_1') + (y_2 + b_2') = (y_1 + y_2) + (b_1' + b_2').$ Then there exists $a_1, a_2 \in A$ Such that $y_1 = f(a_1)$ and $y_2 = f(a_2).$ Then

$y_1 + y_2 = f(a_1) + f(a_2) = f(a_1 + a_2)$ and so

$f'(b_1 + b_2) = a_1 + a_2 = f'(b_1) + f'(b_2).$ For $r \in R,$ if $b_1 = y_1 + b_1'$, then $b_1 r = y_1 r + b_1' r$ and $b_1 r = f(a_1) r = f(a_1 r).$ Hence $f'(b_1 r) = a_1 r = f'(b_1 r).$ To show $ff' = 1_A,$ let $a \in A$ be such that $b = f(a)$. Then $f'(b) = a$ and so $ff'(a) = a.$ Thus $ff' = 1_A.$

$(b) \Rightarrow (a)$ Assume that there is a homomorphism $f' : B \to A$ such that $ff' = 1_A.$ Let $B' = Ker(f') \subset_> B.$ Then $\mathrm{Im}(f) \oplus B' \subset_> B.$ For each $b \in B,$ we have $f'(b) \in A$ and so $ff'(b) \in \mathrm{Im}(f).$ Then $f'(ff'(b)) = f'f(f'(b)) = f'(b).$

Hence $ff'(b) - b \in Ker(f') = B'.$ Then there exists $b' \in B'$ such that $ff'(b) - b = b'$ and so $b = ff'(b) - b' \in \mathrm{Im}(f) + B'.$ Thus $B \subseteq \mathrm{Im}(f) + B'$ and then $B = \mathrm{Im}(f) + B'.$ To prove $\mathrm{Im}(f) \cap B' = 0$ let $b \in \mathrm{Im}(f) \cap B'.$ Then $b \in \mathrm{Im}(f)$ and $b \in B'.$ Thus there is a $a \in A$ such that $b = f(a)$ and $f'(b) = 0.$ Then $a = f f'(a) = f'(b) = 0.$ This implies that $b = 0$. Therefore, $B = \mathrm{Im}(f) \oplus B'.$

Definition

A right R-module E is called an *injective module* if for any right R-modules L and M, any monomorphism $f : L \to M$ and any homomorphism $g : L \to E,$ there exists a homomorphism $h : M \to E$ such that $h - f = g$

$$
\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, "h"] \\
& E\ (M) &
\end{tikzcd}
$$

Figure 1

If the above condition is true only for a special module E, then E is called M-injective module. Thus, a right R-module E is said to be injective if and only if it is M-injective for any right R-module M. A right R-module M is called quasi-injective if M is M-injective.

Example :
$$
\begin{tikzcd}
0 \arrow[r] & 2Z \arrow[r] \arrow[d] & Z \arrow[dl] \\
& \boldsymbol{Q} &
\end{tikzcd}
$$

Figure 2

Q and Q/Z are injective Z-modules.

Theorem 2.4.3 Let M be any right R-module. Then the following statements are equivalent:
(1) M is injective;
(2) Any exact sequence of the form $0 \to M \to A \to B \to 0$ splits.
Proof. $(1) \Rightarrow (2).$ Assume that M is injective. Consider the exact sequence

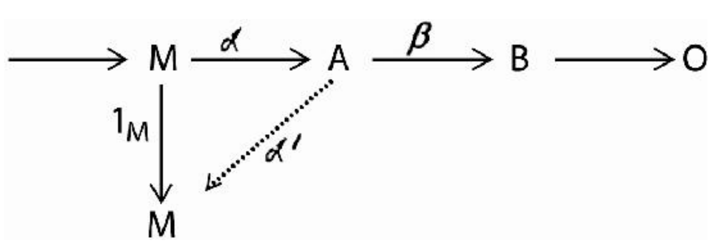

Figure 3

Since M is injective, there exists $\alpha' : A \to M$ such that $\alpha'\alpha = 1_M$. so we get the sequence is splits.

$(2) \Rightarrow (1)$. Let

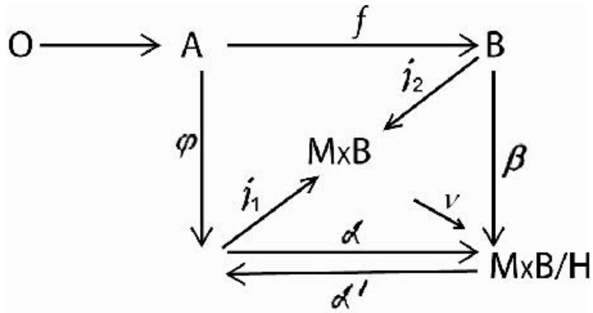

Figure 4

Define $i_1 : M \to M \times B$ and $i_2 : B \to M \times B$ by $i_1(m) = (m,0) \forall m \in M$ and $i_2(b) = (0,b)$ $\forall b \in B$. Let $H = \{(\varphi(a), -f(a)) | a \in A\} \subseteq M \times B$. Consider $(M \times B)/H$ and define $\alpha = \nu i_1$ and $\beta = \nu i_2$.

For every $a \in A$, $\alpha\varphi(a) = \nu i_1 \varphi(a) = \nu((\varphi(a),0)) = (\varphi(a),0) + H$

and $\beta f(a) = \nu i_2 f(a) = \nu((0, f(a)) = (0, f(a)) + H$.

Since $(\varphi(a),0) - (0, f(a)) = (\varphi(a) - f(a)) \in H$, we have $(\varphi(a),0) + H = (0, f(a)) + H$. We also have $\alpha\varphi = \beta f$. To show that α is a monomorphism. Let $m \in Ker(\alpha)$. Then $\alpha(m) = 0 \Rightarrow \nu i_1(m) = 0 \Rightarrow \nu((m,0)) = 0$, i.e., $(m,0) + H = 0 + H$ and so $(m,0) \in H$. Then there exists $a \in A$ such that $(m,0) = (\varphi(a) - f(a))$ which implies that $\varphi(a) = m$ and $f(a) = 0$. Since f is a monomorphism, $a = 0$ and we have $m = \varphi(0) = 0$, i.e. $Ker(\alpha) = 0$. Hence α is a monomorphism. Consider an exact sequence

$$0 \to M \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} (M \times B)/H \to ((M \times B)/H)/\mathrm{Im}(\alpha) \to 0$$

Then by hypothesis, there exists $\alpha' : (M \times B)/H \to M$ such that $\alpha'\alpha = 1_M$. Chose $\bar{\varphi} = \alpha'\beta$.

Then $\bar{\varphi} : B \to M$ and $\bar{\varphi} f = \alpha'\beta f = \alpha'\alpha\varphi = 1_M \varphi = \varphi$. Therefore, M is injective.

Projective module: A right R-module P is called a *projective module* if for any right R-modules M and N, for any epimorphism $f : M \to N$ and any homomorphism $g : P \to N$, there exists a homomorphism $h : P \to M$ such that f - $h = g$. Then P is called *M*-projective module. Thus a right *R*-module *M* is called quasi-projective if *M* is *M*-projective.

$$\begin{tikzcd} & P\ (M) \arrow[dl, "h"'] \arrow[d, "g"] & \\ M \arrow[r, "f"] & N \arrow[r] & O \end{tikzcd}$$

Figure 5

Example

(i) Every free mudule is projective but the converse is not true. Consider the ring *R*=Z/6Z which can be composed as $R = \overline{2} \oplus \overline{3}$. The ideals $\overline{2}$ and $\overline{3}$ are projective mudules but they are not free.

(ii) For every $n \in \mathrm{N}$, $\mathrm{Z}_n = \mathrm{Z}/n\mathrm{Z}$ is quasi- projective but not *Z*- projective.

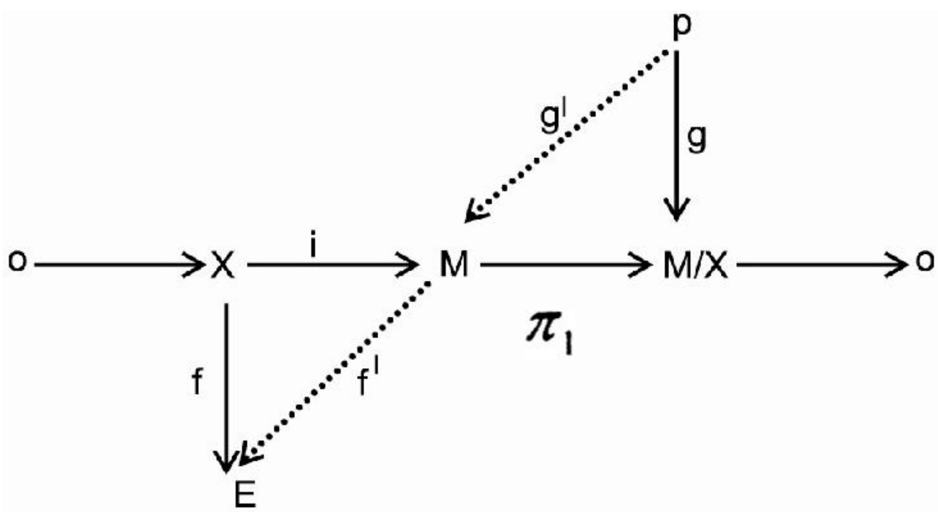

Figure 6

(i) ®$_1$ o g' = g projective. (ii) f' o $i = f$ injective.

(iii) A right R-module M is called quasi-projective if M is M-projective, so $P = M$

(iv) A right R-module M is called quasi-injective if M is M-injective, so $E = M$.

Proposition 2.4.4 Let M be any right R-module. Then the following statements are equivalent:

(a) M is projective.

(b) Any exact sequence of the form $0 \to X \to Y \to M \to 0$ splits.

Proof. $(1) \Rightarrow (2)$. Assume that M is projective. Consider the exact sequence

$0 \to X \xrightarrow{f} Y \xrightarrow{g} M \to 0$.

Since M is projective, there exists a homomorphism $g' : M \to Y$ such that $g'g = 1_M$. so have the sequence is splits.

$(2) \Rightarrow (1)$. Assume that every exact sequence of the form $0 \to X \to Y \to M \to 0$ splits. Let

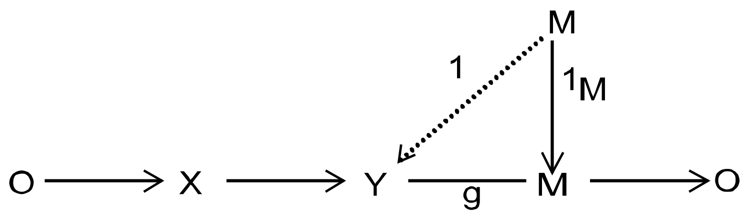

Figure 7

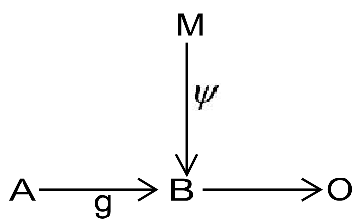

Figure 8

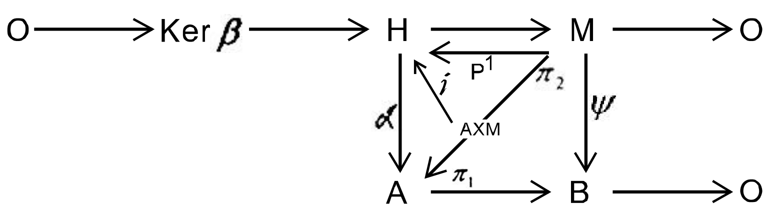

Figure 9

Define $H = \{(a,m) | g(a) = \cdot\ (m)\}$. Then $H \subseteq A \times M$. To show that $H \subset_{>} A \times M$. Let $(a,m),(a',m')H$. Then $g(a) = \cdot\ (m)$ and $g(a') = \cdot\ (a')$.

(i) $g(a+a') = g(a) + g(a') = \cdot\ (m) + \cdot\ (m') = \cdot\ (m+m')$. Thus $(a+a', m+m') \in H$.

(ii) Let $a \in A$ and $r \in R$. Then $g(a) = \cdot\ (m) \Rightarrow g(a)r = \cdot\ (m)r \Rightarrow g(ar) = \cdot\ (m)r$.

Then $(a,m)r = (ar,mr) \in H$. Therefore, $H \subset_{>} A \times M$. Let $§ : H \to A \times M$ be the embedding map. Put $\sim \; = ®_1§$ and $\; = ®_2§$. We first note that $g\sim \; = \cdot \;$ such that for any $x \in H$, we have $x = (a,m)$ with $g(a) = \cdot \; (m)$ and

$g\sim (x) = g(\sim (a,m)) = g(®_1§(a,m)) = g(®_1(a,m)) = g(a)$ and

$\cdot \; (x) = \cdot \; (\; (a,m)) = \cdot \; (®_2§(a,m)) = \cdot \; (®_2(a,m)) = \cdot \; (m)$. Hence $g\sim (x) = \cdot \; (x) \forall x \in H$ and so $g\sim \; = \cdot \;$. To show that is an epimorphism. Let $m \in M$. Then $\cdot \; (m) \in B$. Since g is an epimorphism, there is $a \in A$ such that $\cdot \; (m) = g(a)$. So $(a,m) \in H$ and

$(a,m) = ®_2§(a,m) = ®_2(a,m) = m$. Hence is an epimorphism. By assumption, the exact sequence splits. Then there exists $' : M \to H$ such that $' = 1_M$. Choose $\bar{\cdot} \; = \sim \; '$. Then

$\bar{\cdot} : M \to A$ and so $g\bar{\cdot} \; = g\sim \; ' = \cdot \; ' = \cdot \; 1_M = \cdot$. Therefore, M is projective.

Proposition 2.4.5 Every free right R-module is projective.

Proof. Let F be a free right R-module and Let X be its basis. Then $F = \bigoplus_{x \in X} xR$. For $x \in X$, we have $\cdot \; (x) \in B$. we can find $a \in A$ such that $\cdot \; (x) = g(a)$ and we see that we can find many $a \in A$ like that but we choose one and we denote it by a_x.

$$\begin{tikzcd} & F \arrow[d, "\psi"] \arrow[dl, dotted, "\bar{\psi}"'] & \\ A \arrow[r, "g"'] & B \arrow[r] & 0 \end{tikzcd}$$

Figure 10

Put $\bar{\cdot} \; (x) = a_x$. For $f \in F$, $f = \sum_{i=1}^{n} x_i r_i$ and $\bar{\cdot} \; (f) = \sum_{i=1}^{n} a_{x_i} r_i \in A$. Then $\bar{\cdot}$ is an R-homomorphism and $g\bar{\cdot} \; = \cdot$. This shows that F is projective.

Note: M is injective $\Leftrightarrow \forall X, Y \in Mod - R$.

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\iota"] \arrow[d, "\varphi"'] & Y \arrow[dl, dotted, "\overline{\varphi}"] \\
& M &
\end{tikzcd}
$$

$\Leftrightarrow$ Figure 11

$$
\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\iota"] \arrow[d, "\varphi"'] & R_R \arrow[dl, dotted, "\overline{\varphi}"] \\
& M &
\end{tikzcd}
$$

Figure 12

Proposition 2.4.6 Every projective module is isomorphic to a direct summand of a free module, and conversely, any direct summand of a free module is projective.

Proof. Let P be a projective right R-module. By the previous lemma, there exists a free module F such that $¨ : F \to P$ is an epimorphism. Consider the exact sequence $£ : 0 \to \ker(¨) \xrightarrow{§} F \xrightarrow{¨} P$.

Since P is projective, £ splits. Then $F = IM(§) \oplus F'$ for some $F' \leq= \ker(¨) \oplus F'$. Thus $P \cong F/\ker(¨) \cong F' \leq^{\oplus} F$.

CHAPTER III

PRIME AND SEMIPRIME GOLDIE RINGS

Overview

In trying to understand the ideal theory of a commutative ring, it is important to first understand the prime ideals. We recall that a proper ideal P in a commutative ring R is prime if, whenever two elements a and b in R such that if $ab \in P$, it follows that $a \in P$ or $b \in P$. Equivalently, P is a prime ideal if and only if the factor ring R/P is a domain. The terminology comes from algebraic number theory.

In the non-commutative setting, we define an integral domain just as we do in the commutative case (as a nonzero ring in which the product of any two nonzero elements is nonzero) but it turns out not to be a good idea to concentrate our attention on ideals P such that R/P is a domain. In fact, many non-commutative rings have no factor rings which are domains, e.g., a matrix ring over a field. Thus a more relaxed definition for the concept of a prime ideal in the non-commutative case is desirable. The key is to change the commutative definition by replacing products of elements with products of ideals, which was first proposed by Krull in 1928 [24].

In the commutative case, there is a close connection between prime ideals and nilpotent elements. In particular, the intersection of all prime ideals equals the set of nilpotent elements. The non-commutative analog of this theory is presented in the opening sections of this chapter. We then see how prime ideals arise as annihilators, which is responsible for much of their significance.

3.1 Prime and Semi-prime Rings

Definition

A proper ideal P of a ring R is said to be a *prime ideal* of R if for any ideals I,J of R, if $IJ \subseteq P$ then either $I \subseteq P$ or $J \subseteq P$. An ideal I of a ring R is called a *strongly prime* ideal if for any $x, y \in R$ with $x\,y \in I$, then either $x \in I$ or $y \in I$.

Example: Consider the set $\mathbf{Z} = \{\cdots,-2,-1,0,1,2,\cdots\}$ of all integers, is a ring,

$\text{P} = 2\mathbf{Z} = \{\cdots,-6,-4,-2,0,2,4,6,\cdots\}$, $\text{I} = 3\mathbf{Z} = \{\cdots,-9,-6,-3,0,3,6,9,\cdots\}$ and

$\text{J} = 4\mathbf{Z} = \{\cdots,-12,-8,-4, 0, 4, 8, 12,\cdots\}$ are ideals of Z..

Now $IJ = 12Z = \{\cdots,-24,-12, 0, 12, 24,\cdots\} \subset P$. Then $J \subset P$, hence P is a prime ideal of Z. Therefore, if p_1 , $p_2 ... p_n$ are prime numbers , then $p_1Z, p_2Z ... p_nZ$ are prime ideals of Z.

Definition

A *prime ring* is a ring in which 0 ={0} is a prime ideal or equivalently, a ring *R* is called a *prime ring* if there are no nonzero ideals *I* and *J* of *R* such that *I J = 0.*

Example : Consider the set $\mathbf{Z} = \{\cdots,-2,-1,0,1,2,\cdots\}$ of all integers, is a ring and two nonzero ideals *I, J* of Z, where, $\text{I} = 3\mathbf{Z} = \{\cdots,-9,-6,-3,0,3,6,9,\cdots\}$,

$$\text{J} = 4\mathbf{Z} = \{\cdots,-12,-8,-4, 0, 4, 8, 12,\cdots\}$$

Now $IJ = 12Z = \{\cdots,-24,-12, 0, 12, 24,\cdots\} \neq 0$. Hence Z.is a *prime ring* .

Goodearl and Warfield [3] introduced the following properties for checking the primeness of an ideal *P* over an arbitrary ring *R*.

Proposition 3.1.1 For a proper ideal *P* in a ring *R*, the following conditions are equivalent:

(a) *P* is a prime ideal.

(b) If *I* and *J* are any ideals of *R* such that $I \supset P$ and $J \supset P$, then $IJ \not\subset P$.

(c) R / P is a prime ring.

(d) If *I* and *J* are any right ideals of *R* such that $IJ \subseteq P$, then either $I \subseteq P$ *or* $J \subseteq P$.

(e) If *I* and *J* are any left ideals of *R* such that $IJ \subseteq P$, then either $I \subseteq P$ *or* $J \subseteq P$.

(f) If $x, y \in R$ with $xRy \subseteq P$, then either $x \in P$ or $y \in P$.

Proof. $(a) \Rightarrow (b)$: Follows the definition of prime ideal.

$(b) \Rightarrow (c)$: Let I and J be ideals in R/P, where P is a prime ideal of R. Then there exists ideals $I_1 \supseteq P$ and $J_1 \supseteq P$ in R such that $I = I_1 / P$ and $J = J_1/P$. Suppose that $IJ = 0$.

Then $I_1 J_1 \subseteq P$. Since P is a prime ideal of R, it follows that either $I_1 \subseteq P$ or $J_1 \subseteq P$ and so either $I = 0$ or $J = 0$.

$(c) \Rightarrow (a)$: Let R/P be a prime ring and I and J be ideals of R satisfying $IJ \subseteq P$ then $(I+P)/P$ and $(J+P)/P$ are ideals in R/P whose product is equal to zero. Since R/P is a prime ring, we have $(I+P)/P = 0$ or $(J+P)/P = 0$. Hence $I \subseteq P$ or $J \subseteq P$.

$(a) \Rightarrow (d)$: Since I and J are right ideals of R, $(RI)(RJ) = RIJ \subseteq P$. Thus either $RI \subseteq P$ or $RJ \subseteq P$ and so $I \subseteq P$ or $J \subseteq P$.

$(a) \Rightarrow (e)$: Since I and J are left ideals of R, $(IR)(JR) = IJR \subseteq P$. Thus either $IR \subseteq P$ or $JR \subseteq P$ and so $I \subseteq P$ or $J \subseteq P$.

$(d) \Rightarrow (f)$: Since $(xR)(yR) \subseteq P$, either $xR \subseteq P$ or $yR \subseteq P$ and so $x \in P$ or $y \in P$.

$(f) \Rightarrow (a)$: For any ideals $I \not\subset P$ or $J \not\subset P$, choose elements $a \in I - P$ and $b \in J - P$. Then $aRb \not\subset P$ whence $IJ \not\subset P$.

$(a) \Rightarrow (b)$: For any ideals I and J of R, their multiplication is not contained in P.

By induction hypothesis, it follows immediately that if P is a prime ideal in a ring R and $I_1, I_2, \ldots, I_n$ are right ideals of R such that $I_1 I_2 \cdots I_n \in P$ then some $I_i \subset P$.

Sanh et al.[14] modified the above structure of prime ideals over an arbitrary ring as follows:

Corollary 3.1.2 For a proper ideal P in a ring R, the following conditions are equivalent:

(a) P is a prime ideal;

(b) If I and J are any ideals of R such that $I \supset P$ and $J \supset P$, then $IJ \not\subset P$.

(c) If I and J are any right ideals of R such that $IJ \subset P$, either $I \subset P$ or $J \subset P$;

(d) If I and J are any left ideals of R such that $IJ \subset P$, either $I \subset P$ or $J \subset P$;

(e) If $x, y \in R$ with $xRy \subset P$, either $x \in P$ or $y \in P$;

(f) For any $a \in R$ and any ideal I of R such that $aI \subset P$, either $aR \subset P$ or $I \subset P$;

(g) R/P is a prime ring.

Definition

A *minimal prime ideal* in a ring *R* is any prime ideal of *R* that does not properly contain any other prime ideals. For instance, if *R* is a prime ring, then *0* is the unique minimal prime ideal of *R*.

Example

(i) In a commutative artinian ring, every maximal ideal is a minimal prime ideal.

(ii) In an integral domain, the only minimal prime ideal is the zero ideal.

Proposition 3.1.3 [3] Any prime ideal *P* in a ring *R* contains a minimal prime ideal.

Proof. Let X be the set of those prime ideals of *R* which are contained in *P*. We may use Zorn's Lemma going downward in X provided we show that any nonempty chain $\mathrm{Y} \subseteq \mathrm{X}$ has a lower bound of X.

The set $Q = \cap \mathrm{Y}$ is an ideal of *R*, and it is clear that $Q \subseteq P.$ we claim that *Q* is a prime ideal. Thus consider any $x, y \in R$ such that $xRy \subseteq Q$ but $x \notin Q.$ Then $x \notin P_1$ for some $P_1 \in \mathrm{Y}.$ For any $P_2 \in \mathrm{Y}$ such that $P_2 \subseteq P_1$ we have $x \notin P_2$ and $xRy \subseteq Q \subseteq P_2,$ whence $y \in P_2.$ In particular, $y \in P_1.$ If $P_2 \in \mathrm{Y}$ and $P_2 \not\subset P_1,$ then $P_1 \subset P_2,$ and so $y \in P_2.$ Hence, $y \in P_2$ for all elements P_2 of Y, and so $y \in Q,$ which proves that *Q* is a prime ideal.

Now $Q \in \mathrm{X},$ and *Q* is a lower bound for $\mathrm{Y}.$ Thus, by Zorn's Lemma, we can get a prime ideal $P_2 \in \mathrm{X}$ that is minimal among the ideals in $\mathrm{X}.$ Since any prime ideal contained in P_2 is in X, we conclude that P_2 is a minimal prime ideal of *R*.

Definition

A *semi-prime* ideal in a ring *R* is an intersection of prime ideals. In $\mathrm{Z},$ the intersection of any finite list $p_1 Z, \ldots, p_n Z$ of prime ideals is the ideal $p_1 p_2 \cdots p_n Z$, where $p_1, \ldots, p_n$ are distinct prime integers, Hence the nonzero semiprime ideals of Z consist of the ideals kZ, where *k* is any square-free positive integer including $k = 1.$ A *semiprime ring* is any ring in which *0* is a semi-prime ideal.

Example

Cnosider the ring $\mathbf{Z} = \{\cdots,-2,-1,0,1,2,\cdots\}$ of all integers.

Then $2\mathbf{Z} = \{\cdots,-6,-4,-2,0,2,4,6,\cdots\}$ and

$3\mathbf{Z} = \{\cdots,-9,-6,-3,0,3,6,9,\cdots\}$ are two prime ideals in $\mathbf{Z}$ and

$2\mathbf{Z} \cap 3\mathbf{Z} = 6\mathbf{Z} = \{\ldots,-18,-12,-6,0,6,12,18,\ldots\}$ is a semi-prime ideal of $\mathbf{Z}$.

Goodearl and Warfield [3] introduced the following properties for checking the semi-primeness of an ideal over an arbitrary ring R.

Corollary 3.1.4 For an ideal *I* in a ring *R*, the following conditions are equivalent:

(a) *I* is a semiprime ideal.

(b) If *J* is any ideal of *R* such that $J^2 \subseteq I$, then $J \subseteq I$.

(c) If *J* is any right ideal of *R* such that $J^2 \subseteq I$, then $J \subseteq I$.

(d) If *J* is any left ideal of *R* such that $J^2 \subseteq I$, then $J \subseteq I$.

Proof. $(a) \Rightarrow (d)$: For any $x \in J$, we have $xRx \subseteq J^2 \subseteq I$, whence $x \in I$ by theorem 3.7 [3]. Thus $J \subseteq I$.

$(c) \Rightarrow (b)$: If $J \not\subset I$, then $I + J$ properly contains I. But since

$$(I+J)^2 = I^2 + IJ + JI + J^2 \subseteq I,$$

we have a contradiction to (c). Thus $J \subseteq I$.

$(b) \Rightarrow (a)$: Given any $x \in R$ such that $xRx \subseteq I$, we have $(RxR)^2 = RxRx \subseteq I$ and so $RxR \subseteq I$, where $x \in I$. By Theorem 3.7[3] is semi-prime.

$(a) \Leftrightarrow (c)$: By symmetry.

Definition

A right, left or two-sided ideal *I* of a ring *R* is called a *nil ideal* if and only if $\forall\, a \in I$, $\exists$ $n \in \mathbf{N}$ such that $a^n = 0$, *nilpotent ideal* if and only if $\exists\, n \in \mathbf{N}$ such that $I^n = 0$. More generally, *I* is called a nil ideal if each of its elements is nilpotent. The sum of all nil ideals of a ring *R* is called the *nil radical* of *R* and is denoted by *N(R).* The *prime radical P*(*R*) of a ring R is the intersection of all the prime ideals of R. Hence we can conclude that $P(R) \subset N(R)$.

Example

Consider the ring $Z_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{8}\}$, integer modilo 8 and the ideal

$I = 2Z_8 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$. I is a nil ideal of Z_8, because

$\bar{0}^1 = \bar{0}$, $\bar{2}^3 = \bar{0}$, $\overline{4^2} = \bar{0}$, $\overline{6^3} = \bar{0}$.

Again, I is a nilpotent ideal of Z_8, because $I^3 = 0$ *i. e.,* $\bar{0}^3 = \bar{0}$, $\bar{2}^3 = \bar{0}$, $\overline{4^3} = \bar{0}$, $\overline{6^3} = \bar{0}$.

Proposition 3.1.5 [13] A ring R is semiprime if and only if it contains no nonzero nilpotent elements.

Wisbauer [21] introduced that in a semi-prime ring *R*, the intersection of prime ideals is zero. That implies, *R* is a semi-prime ring if and only if *0* is a semi-prime ideal. If *R* is a semi-prime ring, then *P(R) = 0.*

Lemma 3.1.6 For a ring *R* with identity, the following conditions are equivalent:

(a) *R* is a semi-prime ring (*i.e., P(R) = 0)*;

(b) *0* is the only nilpotent ideal in *R*;

(c) For ideals *I, J* in *R* with *I J = 0* implies $J \cap I = 0$.

Proof. $(a) \Rightarrow (b)$. Let *R* is prime ring if and only if *0* is prime ideal. *R* is semi-prime ring if and only if *0* is a semi-prime ideal. *R* is semi-prime ring *P(R) = 0.* In noetherian rings, all nil one-sided ideals are nilpotent. If *R* is the non zero ring, it has no prime ideals, and so *P(R) = R.* If *R* is nonzero, at has at least one maximal ideal. A ring is semi-prime if and only if *P(R) = 0.* In any case, *P(R)* is the smallest semi-prime ideal of *R*, and because *P(R)* is semi-prime, it contains all nilpotent one-sided ideals of *R.* Since all nilpotent (left) ideals of *R* are contained in *P(R).*

$(b) \Rightarrow (c)$. If *AB = 0* then $(A \cap B)^2 \subset AB = 0$ and $A \cap B = 0$.

$(c) \Rightarrow (b)$. If *AA = 0* then also $A \cap A = A = 0$.

$(b) \Rightarrow (a)$. Let $0 \neq a \in R$. Then $(Ra)^2 \neq 0$ and with $a = a_0$ there exists $0 \neq a_1 \in a_0 R a_0$.

Then also $(Ra_1)^2 \neq 0$ and we find $0 \neq a_2 \in a_1Ra_1$, and so on. Hence *a* is not strongly nilpotent and $a \notin P(R)$. Therefore *P(R) =0.*

Let *R* be a semiprime ring and *I,J* right ideals of *R* such that *I J = 0*. Then $(JI)^2 = 0$ and $(J \cap I)^2 = 0$. So that *JI = 0* and $J \cap I = 0$.

and $a \notin P(R)$. Therefore *P(R) =0.*

Proof: If R/P is semi-prime and right Artinian, then it is semi-simple by Proposition 10.24[20]. Since R/P is in fact prime, it can have only one simple component. Therefore, R/P is simple. In other words, *P* is a maximal ideal.

In commutative ring theory, it is well-known that *R* is Artinian if and only if *R* is Noetherian

Definition

Let *X* be a subset of a ring *R*. Then the *right annihilator* of *X* in *R* is

$$r_R(X) = \{r \in R \mid xr = 0 \ \forall \ x \in X\}$$

and the *left annihilator* of *X* in *R* is given by

$$l_R(X) = \{r \in R \mid rx = 0 \ \forall \ x \in X\}.$$

Definition

The *singular right ideal* of a ring *R* is defined by

$Z(R) = \{x \in R \mid xK = 0$, for some essential right ideal *K* of *R*$\}$. If $Z(R) = 0$, then the ring *R* is called a *right non-singular ring*. Singular left ideals are defined similarly.

Lemma 3.1.7 [3] Let *R* be a commutative ring. Then the right singular ideal *Z(R)* of *R* is zero if and only if *R* is semi-prime.

Proof: Suppose that *R* is a semi-prime ring. Let $z \in Z(R)$. We will show that $z = 0$. Set $I = zR \cap r_R(z)$. We have $zR \,.\, r_R(z) = 0$. In fact, for any $t \in R$ and any $t_1 \in r_R(z)$, we have $t_1 z = 0$. So $z\,tt_1 = tt_1 z = t.0 = 0$, showing that for any $t \in R$, $zR.r_R(z) = 0$. We have $I^2 \subseteq I = zR \cap r_R(z) = 0$. So $I^2 = 0$. Since *R* is a semi-prime ring, *0* is a semi-prime ideal. It follows that $I = 0$. But $r_R(z)$ is an essential right ideal of *R*. This implies that $zR = 0$. Thus $z = 0$.

Conversely, suppose that $Z(R) = 0$. Let *a* be an element of *R* such that $a^2 = 0$. We will show that $a = 0$ from which it follows that *R* has no non-zero nilpotent element. Let $0 \neq x \in R$.

Then we need to consider two cases: (i) $ax = 0 \Rightarrow x \in r_R(a)$; (ii) $ax \neq 0 \Rightarrow a(ax) = a^2 x = 0 \Rightarrow ax \in r_R(a)$. Hence $x R \cap r_R(a) \neq 0$. Therefore $r_R(a)$ is an essential right ideal of R. This implies that $a \in Z(R)$. Thus $a = 0$. This completes the proof.

Singular and nilpoten ideals play a vital role in ring theory. The following theorem sets up a relation between singulaar and nilpotent ideals.

Theorem 3.1.8 Let R be a ring with the *ACC* for right annihilators. Then the right singular ideal $Z(R)$ of R is nilpotent.

Proof: We write Z rather than $Z(R)$ for the right singular ideal of R. Since $Z \supseteq Z^2 \supseteq Z^3 \supseteq \cdots$, we have $r_R(Z) \subseteq r_R(Z^2) \subseteq r_R(Z^3) \subseteq \cdots$. So that there exists a positive integer n such that $r_R(Z^n) = r_R(Z^{n+1})$. Suppose that $Z^{n+1} \neq 0$. We obtain a contradiction. There is an element $a \in Z$ such that $Z^n a \neq 0$. Choose such an element a with $r_R(a)$ large enough. Take any $b \in Z$, then $r_R(b)$ is an essential right ideal of R whence $r_R(b) \cap aR \neq 0$. Thus there exists an element $r \in R$ such that $ar \neq 0$ and $ar \in r_R(b)$. We have $ba \in Z$ and $r_R(a) \subseteq r_R(ba)$. But $ar \neq 0$ and $bar = 0$. Therefore, $r_R(a)$ is strictly contained in $r_R(ba)$. It follows from the choice of a that $Z^n ba = 0$. But b is an arbitrary element of Z. Hence $Z^{n+1}a = 0$ and so $Z^n a = 0$. This completes the proof of the theorem.

Theorem 3.1.9 Let R be a semi-prime ring with the *ACC* for right annihilators. Then R has no non-zero nil one-sided ideals.

Proof: Let I be a nonzero one-sided ideal of R and let $0 \neq a \in I$ with $r_R(a)$ as large as possible. Since R is semi-prime, there is an element $x \in R$ such that $axa \neq 0$. Thus axa is a nonzero element of I such that $r_R(a) \subseteq r_R(axa)$. So that $r_R(a) = r_R(axa)$. We have $ax \neq 0$, i.e. $x \notin r_R(a)$. Thus $x \notin r_R(axa)$. So that $(ax)^2 \neq 0$. Hence $xax \notin r_R(a)$ implying that $(ax)^3 \neq 0$. Therefore ax and hence also xa is not nilpotent and $ax \in I$ or $xa \in I$.

Corollary 3.1.10 Let R be a right Noetherian ring. Then each nil one-sided ideal of R is nilpotent.

Proof: Let S be the sum of all the nilpotent right ideals of R. The S is an ideal. Since R is right Noetherian, S is the sum of a finite number of nilpotent right ideals and hence S is nilpotent. It follows that the quotient R/S has no nonzero nilpotent right ideals. Let I be a nil one-sided ideal of R. Then the image of I in R/S is zero. Hence $I \subseteq S$.

3.2 Prime and Semi-prime Goldie Rings

Definition

A ring R has *finite right* Goldie *dimension* if it contains a direct sum of finite number of nonzero right ideals. Symbolically, we write $G.\dim(R) < \infty$. A ring R is called a *right* Goldie *ring* if $G.\dim(R) < \infty$ and satisfies the *ACC* for right annihilators. Also, every noetherian ring is a *Goldie ring.*

Example

Since Q, Z are noetherian rings, so Q, Z are *Goldie rings*.

Definition

An element $c \in R$ is called *right regular* (respectively, *left regular*) if for any $r \in R$, $cr = 0$ implies $r = 0$ (respectively, $rc = 0$ implies $r = 0$). If $cr = 0 = rc$, then c is called a *regular element*. Every non-zero element of an integral domain is regular.

Theorem 3.2.1 Let R be a ring with finite right Goldie dimension and let c be a right regular element of R. Then cR is an essential right ideal of R.

Proof. Suppose that cR is not essential in R. Then there exists a nonzero right ideal I of R such that $I \cap cR = 0$. Since $I \neq 0$, we have $cI \neq 0$ and $cI \subset cR$ with $I \cap cI = 0$. So the sum $I + cI$ is direct. Consider $(I + cI) \cap c^2 I$. Take any $x \in (I + cI) \cap c^2 I$. Then $x = c^2 t = u + cv$ where $t, u, v \in I$. This implies that $u = c(ct - v) \in I \cap cR = 0$. So $u = 0$. Also, $c^2 t = cv$. Then $v = ct \in I \cap cI = 0$. So $x = 0$. This shows that the sum $I + cI + c^2 I$ is direct. By induction, the sum $I + cI + c^2 I + c^3 I + \cdots$ is direct. Since R has finite right Goldie dimension, $\sum_{n=0}^{\infty} c^n I = 0$ for some n and since c is right regular, we have $I = 0$, a contradiction. Thus cR is an essential right ideal of R.

Theorem 3.2.2 Let R be a semi-prime right Goldie ring and let I be an essential right ideal of R. Then I contains a regular element of R.

Proof: First we show that R contains a right regular element. By Theorem 3.1.9, I is not nil. Let a be a non-nilpotent element of I such that $r_R(a)$ is as large as possible. We have $r_R(a) \subseteq r_R(a^2)$ where a^2 is a non-nilpotent element of I. By the choice of a, we have $r_R(a) = r_R(a^2)$. If $r_R(a) = 0$, we stop. If not, we have $r_R(a) \cap I \neq 0$. Let b be a non-nilpotent element of $r_R(a) \cap I$ such that $r_R(b)$ is as large as possible. Then $r_R(b) = r_R(b^2)$. Let $ar = bs$ for some $r, s \in R$. Since $ab = 0$, we have $a^2 r = 0$. Therefore, $ar = 0$. Hence the sum $aR + bR$ is direct. The same argument shows that $r_R(a+b) = r_R(a) \cap r_R(b)$. If $r_R(a) \cap r_R(b) = 0$, we stop. Otherwise, let c be a non-nilpotent element of $r_R(a+b) \cap I$ with $r_R(c)$ as large as possible. Then $r_R(c) = r_R(c^2)$ and the sum $aR + bR + cR$ is direct because $ab = ac = bc = 0$. Thus $r_R(a+b+c) = r_R(a) \cap r_R(b) \cap r_R(c)$. Since R has finite right Goldie dimension, this process must stop after a finite number of steps. Then there exist elements $a_1, a_2, \ldots, a_n$ in I such that $r_R(a_1 + a_2 + \cdots + a_n) = 0$.

Chatters and Hajarnavis [13] established the following Lemma over non-singular ring .

Lemma 3.2.3 Let R be a right non-singular ring with finite right Goldie dimension. Then the right regular elements of R are regular.

Proof. Let c be a right regular element of R. Then by Lemma 3.2.1, $cR \leq_e R$. But $l(c) = l(cR)$. Take any $x \in l(c)$. Then $xc = 0 = xcR$. So that $x \in l(cR)$ implying that $l_R(c) \subset l_R(cR)$. Again, take any $t \in l(cR)$. Then $t(cR) = 0 \Rightarrow tc = 0 \Rightarrow t \in l(c)$. Therefore, we have $l_R(cR) \subset l_R(c)$. Suppose that $l(cR) \neq 0$. Then there is a $t \in l(cR)$ with $t \neq 0$ such that $t(cR) = 0$. Since $cR \leq_e R$, we have $t \in Z(R_R) = 0$ because R is a right non-singular ring. So $t = 0$, a contradiction. Thus $l(cR) = 0$ and so $l(c) = 0$. This means that c is left regular and consequently, c is regular.

Corollary 3.2.4 Let R be a semi-prime right Goldie ring. Then right regular elements of R are regular.

Proof. Let R be a right Goldie ring. Then it satisfies the ACC for right annihilators. By Theorem 3.1.8, the right singular ideal $Z(R)$ of R is nilpotent. Since R is semi-prime, by

Lemma 3.1.6, 0 is the only nilpotent ideal, i.e. $Z(R)=0$. This implies that R is right non-singular. Let c be a right regular element in R, i.e. $r_R(c)=0$. Since R has finite right Goldie dimension, by Lemma 3.25, $l_R(c)=0$. This completes the proof.

Corollary 3.2.5 [13] Let M be a right R-module and $m \in M$ with $m \neq 0$. If X is an essential submodule of M, then there is an essential right ideal Y of R such that $0 \neq mY \subset X$.

Lemma 3.2.6 Let R be a right non-singular ring with finite right Goldie dimension. Then R satisfies ACC and DCC for right annihilators.

Proof. Let A and B be right annihilators in R with $A \subseteq B$. Suppose that $A \leq_e B$. Let $b \in B$. Then by Corollary 3.2.5, there exists an essential right ideal L of R such that $bL \subseteq A$. This implies that $l_R(A)bL = 0$. Since R is right non-singular, we have $l_R(A)b \in Z(R_R) = 0$. So $l_R(A)b = 0$ and thus $b \in r_R(l_R(A)) = A$. Therefore, $A = B$.

Suppose that $A \subset B$ and A is not essential in B. Then there exists a non-zero right ideal $C \subset R$ such that $C \subseteq B$, $A \cap C = 0$ and $A \oplus C \leq_e B$. If $A \oplus C = B$, then we are to finish. If not, there exists a non-zero right ideal $C' \subset$ R such that $A \oplus C \oplus C' \leq_e B$.

Consider a strictly ascending chain of right annihilators of R:

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \ldots$$

where $A_2 = A_1 \oplus A_2'$, $A_3 = A_1 \oplus A_2' \oplus A_3'$, …, $A_n = A_1 \oplus A_2' \oplus \ldots \oplus A_n'$, . . .

But this contradicts the hypothesis that R has finite right Goldie dimension. So the chain must be stationary. Therefore, $A_n = A_{n+1}$ for some $n \in \mathrm{N}$, i.e. R has the ACC for right annihilators.

Finally, consider a strictly descending chain of right annihilators of R:

$$A_1 \supset A_2 \supset \ldots \supset A_n \supset A_{n+1} \supset \ldots$$

where $A_1 = A_2 \oplus A_1'$, $A_2 = A_3 \oplus A_2' \oplus A_1'$, . . . , $A_n = A_{n+1} \oplus A_n' \oplus \cdots \oplus A_2' \oplus A_1' \oplus \cdots$

But this contradicts the hypothesis that *R* has finite right Goldie dimension. So the chain must be stationary. Therefore, $A_n = A_{n+1}$ for some $n \in \mathbb{N}$. Thus *R* has the *DCC* for right annihilators. This completes the proof.

Corollary 3.2.7 A semi-prime right Goldie ring has the *DCC* for right annihilators.

Proof. Let *R* be a semi-prime right Goldie ring. Then *R* has the *ACC* for right annihilators. By Theorem 3.1.8, the right singular ideal *Z*(*R*) of *R* is nilpotent. Since *R* is semi-prime, by Lemma 3.1.6, 0 is the only nilpotent ideal, i.e. $Z(R) = 0.$ This implies that *R* is right non-singular. Thus, *R* has *DCC* for right annihilators.

Theorem 3.2.8 Let *R* be a semiprime ring satisfying ACC on right annihilators and let $J \subset I$ be right ideals in *R* such that $r_R(I) \subseteq r_R(J)$. Then there exists $x \in I$ such that $xI \neq 0$ and $xI \cap J = 0.$ In particular, J cannot be essential in *I*.

Proof. Since *R* satisfies ACC on right annihilators, it satisfies DCC on left annihilators. Therefore, there exists a left annihilator *A* minimal with respect to $ann_l(I) \subset A \subseteq Aann_l(J).$ Then $AI \neq 0$ and so $AIAI \neq 0,$ because *R* is semiprime. Take any $a \in A$ and $y \in I$ such that $AyaI \neq 0.$ Then $yaI \neq 0$ and so $ya \neq 0.$ It suffices to show that $yaI \cap J = 0$ and the desired element $x \in I$ is chosen to be $ya.$ Suppose that yay' be a nonzero element in *J* for some $y' \in I.$ Put $A' = ann_l(y') \cap A,$ then A' is a left annihilator with $ann_l(I) \subseteq A' \subseteq A \subseteq Aann_l(J).$ But if $Ayay' \subseteq AJ = 0,$ then $Aya \subseteq A'$ and if $AyaI \neq 0,$ then $Aya \not\subset ann_l(I).$ So, $ann_l(I) \neq A'.$ By the minimality of *A*, we have $A' = A,$ that is, $A \subseteq ann_l(y').$ But then $Ay' = 0$ which contradicts the fact that $yaI \neq 0.$ This completes the proof of the theorem.

Theorem 3.2.9 Let *R* be a right non-singular ring with the ACC for right annihilators and let *c* be an element of *R* such that *cR* is an essential right ideal of *R*. Then *c* is right regular.

Proof. Let $x \in R$ and let *A* and *B* be right annihilators of *R* such that *A* is an essential *R*-submodule of *B*. We show that $xA \leq_e xB.$ Let $b \in B$ with $xb \neq 0.$ Then there is an essential right ideal *L* of *R* such that $bL \subset A$ or $xbL \subset xA$. For any $y \in xbL,$ we have $y = xbl,$ where $l \in L,$ and so $y \in xA$ since $bl \in bL \subset A.$ If $xbL = 0,$ then $xb \in Z(R),$ because *R* is right non-singular. This implies that $xb = 0,$ a contradiction. Thus, $xbL \neq 0.$ We see that $0 \neq xbL \subset xbR \cap xA.$ It follows that $xA \leq_e xB.$ Since $A \leq_e B$ and $xA \leq_e xB,$ we have $c(cR) \leq_e cR \leq_e R.$ So $c^2R \leq_e R.$ By induction, $c^kR \leq_e R$ for each positive integer k. Since R has the ACC for right annihilators, the chain $r_R(c) \subseteq r_R(c^2) \subseteq r_R(c^3) \subseteq \cdots$ stabilizes. So $r_R(c^n) = r_R(c^{n+1})$ for some n. This implies that $c^nR \cap r_R(c) = 0.$ If $c^nR \cap r_R(c) \neq 0,$ for any $t \in c^nR \cap r_R(c),$ we have $t = c^nu$ for $u \in R$ and $t \in r_R(c).$ This implies that $ct = 0.$ So that $c(c^nu) = c^{n+1}u = 0.$ It follows that $u \in r_R(c^{n+1}) = r_R(c^n).$ Therefore, $c^{n+1}u = 0 = c^nu = t,$ a contradiction. Thus, $r_R(c) = 0.$ Hence, c is right regular.

Theorem 3.2.10 Let R be a prime ring with the ACC and DCC for right annihilators, I an essential right ideal of R and let $a \in R.$ Then $a + I$ contains a regular element of R, where $a + I = \{a + x \mid x \in I\}.$

Proof. Let $x \in I$ with $r_R(a + x) = 0$ as small as possible. Put $c = a + x.$ Let B be a right ideal of R with $B \cap cR = 0.$ Let $b \in B \cap I,$ then $c + b = (a + x) + b = a + (x + b) \in a + I$ because $x + b \in I.$ Since $cR \cap bR = 0,$ we have $r_R(c + b) = r_R(c) \cap r_R(b).$

Take any $t \in r_R(c + b),$ then $(c + b)t = 0 = ct + bt \Rightarrow ct = 0$ and $bt = 0 \Rightarrow t \in r_R(c)$ and $t \in r_R(b)$ so that $t \in r_R(c) \cap r_R(b).$ Therefore, $r_R(c + b) \subset r_R(c) \cap r_R(b).$ Again, take any $u \in r_R(c) \cap r_R(b),$ then $u \in r_R(c)$ and $u \in r_R(b) \Rightarrow cu = 0$ and $bu = 0 \Rightarrow (c + b)u = 0$ and $u \in r_R(c + b).$ Therefore, $r_R(c) \cap r_R(b) \subset r_R(c + b).$

Since $r_R(c + b) \subset r_R(c).$ By the choice of c, we have $r_R(c + b) = r_R(c).$ Hence $r_R(c) \subseteq r_R(b).$ This implies that $b.r_R(c) = 0$ for any $b \in B \cap I.$ Therefore, $(B \cap I)\ r_R(c) = 0.$ Since R is a prime ring, we have either $r_R(c) = 0$ or $B \cap I = 0.$ If $B \cap I = 0,$ then $B = 0$ because I is essential in R. It follows that $cR \leq_e R.$ By Theorem 3..2.9 and Theorem 3.2.8, we can conclude that $r_R(c) = 0.$ So that $a + I$ contains a right regular element. Take any $d \in a + I$ with $l_R(d)$ as small as possible. Then there is a right R-submodule Y of I such that $Y \cap dR = 0$ and $y \oplus dR \leq_e R.$ Let A be a left ideal of R with $A \cap Rd = 0$ and let $y \in A \cap Y.$ We have $Rd \cap Ry = 0.$ So $l_R(d + y) = l_R(d) \cap l_R(y).$ Also, we have $d + y \in a + I$ and since

$dR \cap yR = 0,$ we have $r_R(d+y) = r_R(d) \cap r_R(y) = 0$ because $r_R(d) = 0.$ Therefore, by the choice of d, we have $l_R(d+y) = l_R(d).$ So $l_R(d) \subseteq l_R(y),$ i.e. $l_R(d).y = 0$ for any $y \in A \cap Y.$ Hence $l_R(d)(A \cap Y) = 0.$ So $l_R(d)YA = 0$ because if A, Y are right ideals of R, then $YA \subset A \cap Y$. Therefore, $Al_R(d)Y = 0$ because R is a semi-prime ring, $l_R(d)Y$ is an ideal and A is a left ideal of R. This implies that $Al_R(d)(dR \oplus Y) = 0$ because $Y \subseteq dR \oplus Y.$ Since $dR \oplus Y \leq_e R,$ we have $Al_R(d) \in Z(R) = 0,$ i.e. R is right nonsingular and is prime. So either $A = 0$ or $l_R(d) = 0.$ If $A = 0,$ then $Rd \leq_e R.$ By Theorem 3..2.3 and Theorem 3.2.9, $l_R(d) = 0,$ i.e., d is left regular. Thus the result follows.

Theorem 3.2.11 [13] Let R be a semi-prime ring with the ACC (equivalently, DCC) for annihilator ideals. Then R has only a finite number of minimal prime ideals. If $P_1, \ldots, P_n$ are the minimal prime ideals of R, then $P_1 \cap \cdots \cap P_n = 0.$ Also, a prime ideal is minimal if and only if it is an annihilator ideal.

CHAPTER IV

PRIME AND SEMI-PRIME GOLDIE MODULES

Overview

Prime submodules and prime modules appear in many contexts. By an adaptation of basic properties of prime ideals, we introduced the notion of prime submoduules and prime modules and studied their structures. In this thesis, we investigate some properties of prime and semi-prime submodules over non-commutative rings. Sanh et al. [14] introduced the notion of prime and semi-prime submodules of a given right *R*-module. Throughout the work, all rings are associative with identity and all modules are unitary right *R*- modules. $S = End_R(M)$ denotes an endomorphisms ring of right *R*-module *M*.

4.1 Prime and Semi-Prime Submodules

Definition

A submodule *X* of *M* is called a *fully invariant submodule* of *M* if for any $f \in S$, we have $f(X) \subset X$.

Let *M* be a right *R*-module and *X* be a fully invariant proper submodule of *M*. Then *X* is called a *prime submodule* if for any ideal *I* of *S* and and any fully invariant submodule *U* of *M*, $I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$. Especially, an ideal *P* of a ring *R* is a *prime ideal* if for any ideals I, J or R, $IJ \subset P$ implies $I \subset P$ or $J \subset P$. A right *R*-module *M* is called a *prime module* if 0 is a prime submodule of *M*.

A fully invariant submodule *X* of *M* is called *strongly prime* if for any $f \in S$ and any $m \in M$, $f(m) \in X$ implies $f(M) \subset X$ or $m \in X$. Especially, an ideal *I* of a ring *R* is strongly prime if for any $a, b \in R$ with $ab \in I$ implies $a \in I$ or $b \in I$.

Example

(i) Let $Z_4 = \{0,1,2,3\}$ be the additive group of integers modulo 4. Then $X = <2>$ is a prime submodule of the Z-module Z_4. If *M* is simple, then 0 is a prime submodule.

(ii) Every simple module is prime

Sanh et at.[14] investigated the following theorem as some characterizations of prime submodules over endomorphism rings similar to the Proposition 3.1.1 for prime ideals over arbitrary rings. We will use it as a tool for checking the primeness.

Theorem 4.1.1 Let X be a proper fully invariant submodule of M and $S = End_R(M)$, its endomorphism ring. Then the following conditions are equivalent:

(1) X is a prime submodule of M;

(2) For any right ideal I of S, any submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$;

(3) For any ¨ $\in S$ and fully invariant submodule U of M, if ¨ $(U) \subset X$, then either ¨ $(M) \subset X$ or $U \subset X$;

(4) For any left ideal I of S and subset A of M, if $IS(A) \subset X$, then either $I(M) \subset X$ or $A \subset X$;

(5) For any ¨ $\in S$ and for any $m \in M$, if ¨ $(S(m)) \subset X$, then either ¨ $(M) \subset X$ or $m \in M$. Moreover, if M is quasi-projective, then the above conditions are equivalent to:

(6) M/X is a prime module.

After investigating the above new Theorem for modules, Sanh et at. found that Proposition 3.1.1 may be developed as follows:

Corollary 4.1.2 For a proper ideal P in a ring R, the following conditions are equivalent:

(a) P is a prime ideal.

(b) If I and J are any ideals of R properly containing P, then $I J \not\subset P$.

(c) R / P is a prime ring.

(d) If I and J are any right ideals of R such that $I J \subset P$, then either $I \subset P$ *or* $J \subset P$.

(e) If I and J are any left ideals of R such that $I J \subset P$, then either $I \subset P$ *or* $J \subset P$.

(f) If $x, y \in R$ with $x R y \subset P$, then either $x \in P$ or $y \in P$.

(g) For any $x \in R$ and any ideal I of R such that $x I \subset P$, then either $x R \subset P$ or $I \subset P$.

Definition

A prime submodule X of a right R-module M is called a *minimal prime submodule* if it is minimal in the class of prime submodules of M.

As generalizations of prime ideals, the following results are investigated-

Corollary 4.1.3 [14] If P is a prime submodule of a right R-module M, then P contains a minimal prime submodule of M.

Lemma 4.1.4 [14] Let M be a right R-module and $S = End_R(M)$. Suppose that X is a fully invariant submodule of M. Then the set $I_X = \{f \in S : f(M) \subseteq X\}$ is a two-sided ideal of S.

Proposition 4.1.5 [16] Let M be a right R-module which is a self-generator. Then we have the following:

(1) If X is a minimal prime submodule of M, then I_X is a minimal prime ideal of S.

(2) If P is a minimal prime ideal of S, then $X := P(M)$ is a minimal prime submodule of M and $I_X = P$.

Lemma 4.1.6 [14] Let M be a right R-module, $S = End_R(M)$ and X a fully invariant submodule of M. If X is a prime submodule of M, then I_X is a prime ideal of S. Conversely, if M is a self-generator and if I_X is a prime ideal of S, then X is a prime submodule of M.

Lemma 4.1.7 [14] Let M be a prime module. Then its endomorphism ring S is a prime ring. Conversely, if M is a self-generator and if S is a prime ring, then M is a prime module.

Definition

A fully invariant submodule is called a *semi-prime submodule* if it is an intersection of prime submodules. A right R-module M is called a *semi-prime module* if 0 is a semi-prime submodule of M. Consequently, a ring R is called semi-prime ring if R_R is a semi-prime module. Every semi-simple module is semi-prime.

Definition

Let M be a right R-module and $X \subseteq M$, a subset of M. Then we say that M is generated by X if $M = |X) = \{\sum x_i r_i \,/\, x_i \in X,\ r_i \in R,\ i = 1, 2, 3, \cdots\}$. If X is a fiinite subset, then M is finitely generated and we write $M = |X) = \{\sum_{i=1}^{k} x_i r_i / x_i \in X,\ r_i \in R,\ i = 1, 2, 3, \cdots k\}$. A module M is called a self-generator if it generates all of its submodules. If X is a submodule of M and $X = \sum_{f_i \in S} f_i(M)$, then M is called a self-generator, where $S = End_R(M)$.

According to Ahmed et al.[16] the new structure of the Corollary 3.8 [3] for right R-modules over endomorphism rings is as follows:

Theorem 4.1.8 Let M be a right R-module which is a self-generator. Let X be a fully invariant submodule of M and $S = End_R(M)$.Then the following conditions are equivalent:

(1) X is a semi-prime submodule of M;

(2) If J is any ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;

(3) If J is any ideal of S such that $J(M) \underset{\neq}{\supset} X$, then $J^2(M) \not\subset X$;

(4) If J is any right ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;

(5) If J is any left ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$.

Using the above results,we investigate the following results for prime and semi-prime submodules.

Proposition 4.1.9 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a Noetherian module, then there exist only finitely many minimal prime submodules.

Proof. If M is a Noetherian module, then S is a right Noetherian ring. Indeed, suppose that we have an ascending chain of right ideal of S, say $I_1 \subset I_2 \subset \ldots$ Then we have $I_1(M) \subset I_2(M) \subset \ldots$ is ascending chain of submodules of M. Since M is a Noetherian module, there is an integer n such that $I_n(M) = I_k(M)$, for all $k > n$. Then we have $I_n = \mathrm{Hom}(M, I_n(M)) =$

Hom$(M, I_k(M)) = I_K$. Thus the chain $I_1 \subset I_2 \subset \ldots$ is stationary, so S is a right Noetherian ring. By Corollary 3.1.3, S has only finitely many minimal prime ideals $P_1, \ldots, P_t$. By Lemma 4.1.6, $P_1(M), \ldots, P_t(M)$ are the only minimal prime submodules of M.

Lemma 4.1.10 Let M be a quasi-projective, finitely generated right R-module which is a self-generator and X, a minimal submodule of M. Then I_X is a minimal right ideal of S.

Proof. Let J be a right ideal of S such that $0 \neq J \subset I_X$. Then $J(M)$ is a nonzero submodule of M and $J(M) \subset X$. Thus $J(M) = X$ and it follows that $J = I_X$.

Proposition 4.1.11 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Let X be a minimal submodule of M. Then either $I_X^2 = 0$ or $X = f(M)$ for some idempotent $f \in I_X$.

Proof. Since X is a minimal submodule of M, I_X is a minimal right ideal of S, by Lemma 4.1.10. Suppose that $I_X^2 \neq 0$. Then there is $g \in I_X$ such that $g\, I_X \neq 0$. Since $g\, I_X$ is a right ideal of S and $g\, I_X \subset I_X$, we have $g\, I_X = I_X$. Then there exists $f \in I_X$ such that $g f = g$. Then set $I = \{h \in I_X : g h = 0\}$ is a right ideal of S and I is properly contained in I_X since $f \notin I$. By the minimality of I_X, we must have $I = 0$. We have $f^2 - f \in I_X$ and $g(f^2 - f) = 0$, so $f^2 = f$. Since $f(M) \subset X$ and $f(M) \neq 0$, we have $f(M) = X$.

Corollary 4.1.12 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Let X be a minimal submodule of M. If M is a semi-prime module, then $X = f(M)$ for some idempotent $f \in I_X$.

Proof. Since M is a semi-prime module, $I_X^2 \neq 0$. Thus $X = f(M)$ for some idempotent $f \in I_X$, by Proposition 4.1.11.

Definition

The singular submodule of a right R-module M is denoted by $Z(M)$ and defined as

$Z(M) = \{m \in M \mid mK = 0$ for some essential right ideal K of $R\}$ Or, equivalently,

$Z(M) = \{m \in M \mid r_R(m)$ is an essential right ideal of $R\}$, where $r_R(m) = \{r \in R \mid mr = 0\}$.

A right R-module M is called a *non-singular module* if $Z(M) = 0$ and a *singular module* if $Z(M) = M$.

For any $m, m' \in Z(M)$, we have $r_R(m) \leq_e R$ and $r_R(m') \leq_e R$. Since $r_R(m+m') \supseteq r_R(m) \cap r_R(m')$, we have $r_R(m) \cap r_R(m') \leq_e R$. So $r_R(m+m') \leq_e R$. Thus $m+m' \in Z(M)$.

Proposition 4.1.13 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then $Z(S)(M) \subset Z(M)$ where $Z(S)$ is a singular ideal of S and $Z(M)$ is a singular submodule of M.

Proof. Let $f \in Z(S)$ and $x \in M$. We will show that $f(x) \in Z(M)$. Since $f \in Z(S)$, there exists an essential right ideal K of S such that $fK = 0$. Then $fK(M) = 0$.

From K is an essential right ideal of S, we have $K(M)$ is an essential submodule of M, and so $x^{-1} K(M)$ is an essential right ideal of R. We have $f(x)(x^{-1} K(M)) = f(x(x^{-1} K(M))) \subset fK(M) = 0$, proving that $f(x) \in Z(M)$.

Corollary 4.1.14 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a non-singular module, then S is a right non-singular ring.

We investigate the following Proposition for semi-prime submodule which is similar to Theorem 3.2.11 for ideals over arbitrary rings.

Proposition 4.1.15 Let M be a right R-module which is a self-generator. If M is a semi-prime module with the *ACC* for M-annihilators, then M has only a finite number of minimal prime submodules. If $P_1, \ldots, P_n$ are minimal prime submodules of M, then $P_1 \cap, \ldots, \cap P_n = 0$. Also a prime submodule P of M is minimal if and only if I_P is an annihilator ideal of S.

Proof. Since M is a semi-prime module, S is a semi-prime ring. If satisfies the *ACC* for M-annihilators, then S satisfies the *ACC* for right annihilators. By Lemma 3.4[3], S has only a finite number of minimal prime ideals. Therefore M has only a finite number of minimal prime submodules, by Lemma 4.1.10 If $P_1, \ldots, P_n$ are minimal prime submodules of M, then

$I_{P_{1,}}, \ldots, I_{P_n}$ are minimal prime ideals of *S*. Thus $I_{P_1} \cap, \ldots, \cap I_{P_n} = 0$, by Theorem 3.2.1. But $I_{P_{1,}} \cap, \ldots, \cap I_{P_n} = I_{P_1 \cap, \ldots, \cap P_{n,}}$, we have $P_1 \cap, \ldots, \cap P_n = 0$. Finally, a prime submodule *P* of *M* is minimal if and only if I_P is a minimal prime ideal of *S*. It is equivalent to saying that I_P *is* an annihilator ideal of *S*, by Lemma 4.1.10.

4.2 Prime and Semi-prime Goldie Modules

Annihilators

Let *M* be a right *R*-module and let $X \subseteq M$, a subset of *M*. Then the annihilator of *X* is the set $r_R(X) = \{r \in R \mid xr = 0 \ \forall \ x \in X\}$ which is a right ideal of *R*.

Proof. (i) $0 \in r_R(X)$ implies that $r_R(X) \neq$ ¤.

(ii) For any $r, r_1, r_2 \in r_R(X)$ and for any $x \in X$, we have $x(r_1 + r_2) = xr_1 + xr_2 = 0$ so that $r_1 + r_2 \in r_R(X)$.

(iii) For any ª $\in R$, we have $x(r$ª$) = (xr)$ª $= 0.$ª $= 0$ so that rª $\in r_R(X)$.

Moreover, if *X* is a submodule of *M*, then $r_R(X)$ is a two-sided ideal of *R* because then $x($ª$r) = (x$ª$)r = 0$ for xª $\in X$.

According to S. Ebrahimi Atani and S. Khojasteh G. Ghaleh:

Let *R* be a ring and *X* a submodule of an *R*-module *M*. Then the set

$(X : M) = \{r \in R : Mr \subseteq X\}$ is a two-sided ideal of *R*.

According to Sanh et al. [15],

Let *M* be a right *R*-module, $S = End_R(M)$ endomorphism ring and $I \subset S$, a subset of *S*. Then a submodule *X* of *M* is called an *M*-annihilator if

$$X = r_M(I) = Ker(I) = \bigcap_{f \in I} Ker(f) = \{m \in M : f(m) = 0 \ \forall \ f \in I\}$$

For any $m, m' \in r_M(I)$, we have $f(m) = f(m') = 0 \Rightarrow f(m + m') = 0 \Rightarrow m + m' \in r_M(I)$ and for any $m \in r_M(I)$ and any $r \in R$, $f(m) = f(m') = 0 \Rightarrow f(m + m') = 0 \Rightarrow m + m' \in r_M(I)$.

For any $m \in r_M(I)$ and any $f \in I \subseteq S$, $f(m) = 0$, i.e. $m \in Ker(f)$ for any $f \in I$. This implies that $m \in \bigcap_{f \in I} Ker(f) = Ker(I)$. So $r_M(I) \subseteq Ker(I)$. Also, $m \in \bigcap_{f \in I} Ker(f)$ implies that $m \in Ker(f)$ for any $f \in I \subseteq S$. So that $f(m) = 0$ for any $f \in I$. Thus $m \in r_M(I)$.
Therefore, $Ker(I) = \bigcap_{f \in I} Ker(f) \subseteq r_M(I)$ showing that $r_M(I) = Ker(I)$.

Definition

We denote $l_S(.)$ and $r_M(.)$ to be a subset of *M* in *S* and the right annihilator of a subset of *S* in *M*, respectively. A submodule *K* of *M* is said to be *essentia*l in *M* if for any nonzero submodule *L* of *M*, we have $K \cap L \neq 0$. In this case, *M* is an essential extension of *K*. It is easy to show that the intersection of a finite number of essential submodules of *M* is again essential in *M* and any submodule containing an essential submodule is essential.

A nonzero right *R*-module *M* is said to be *uniform* if any two nonzero submodules have nonzero intersection, i.e. if each nonzero submodule of *M* is essential in *M*.

A module *M* has *finite Goldie dimension* if it does contain a direct sum of a finite number of nonzero submodules. A module *M* has finite Goldie dimension if it is Noetherian orArtinian. If every nonzero submodule of a module *M* is esstetial in *M*, then *M* has finite Goldie dimension. A right *R*-module *M* is called a *Goldie module* if *M* has finite Goldie dimension and *M* satisfies the ACC for *M*-annihilator submodules.

An Artinian ring (Noetherian ring) with unity is always a Goldie ring, because an Artinian ring is Noetherian and a Noetherian ring is always a Goldie ring and consequently, a Goldie module but the converse is not true.

Example

(i) The set ***Q*** of rational numbers has finite Goldie dimension as a ***Z***-module, because for any $0 \neq q \in Q$, we have $qZ \leq_e Q$.

(ii) The set ***Z*** of all integers has finite Goldie dimension as a ***Z***-module, because for any $0 \neq m \in \mathrm{Z}$ we have $mZ \leq_e \mathrm{Z}$.

Also, Q and Z are both noetherian Z-modules, because every nonzero submodule of them is finitely generated. Since every noetherian module is a Goldie module, so Q and Z are both Goldie Z-modules.

If V is a vector space, then $G.\dim(V) < \infty$ if and only if V has finite dimension in the usual sense of linear algebra, and in these circumstances, they are equal.

We investigate the following properties for Goldie modules over associative arbitrary and endomorphism rings.

Lemma 4.2.1 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a Goldie module, then S is a right Goldie ring.

Proof. Let M be a Goldie module. Then M has finite Goldie dimension and satisfies the ACC for M-annihilators. Thus S has ACC for right annihilators. Since M is a quasi-projective, finitely generated, self-generator and has finite Goldie dimension, we must have S has finite Goldie dimension. Hence S is right Goldie ring.

We develop the following Proposition for Goldie modules over endomorphism rings which is similar to Theorem 3.2.1 for ideals over arbitrary rings.

Proposition 4.2.2 Let M be a right R-module with finite Goldie dimension and let $f \in S = End_R(M)$ be a monomorphism. Then $f(M)$ is an essential submodule of M.

Proof. Suppose that $f(M)$ is not an essential submodule of M. Then there exists a nonzero submodule X of M such that $f(M) \cap X = 0$. Since X is nonzero, we have $f(X)$ is a nonzero submodule of $f(M)$ and $X \cap f(X) = 0$. So the sum $X + f(X)$ is direct. Consider $(X + f(X)) \cap f^2(X)$ and take any $x \in (X + f(X)) \cap f^2(X)$. Then $x = y + f(u) = f^2(v)$ where $y, u, v \in X$. So that $y = f^2(v) - f(u) = f(f(v) - u) \in X \cap f(M) = 0$ implying that $y = 0$. Also, $f^2(v) = f(u)$ implies that $u = f(v) \in X \cap f(X) = 0$. So that $u = 0$ and consequently, $x = 0$. So that the sum $X + f(X) + f^2(X)$ is direct. By induction hypothesis,

$\sum_{n=0}^{\infty} f^n(X)$ is direct for some *n*, which is a contradiction. Thus, $f(M)$ is an essential submodule of *M*.

Lemma 4.2.3: For any $m \in Z(M)$ and any $r \in R,\ mr \in Z(M)$.

Proof. Let $\Omega = \{ I \subseteq R_R \mid I \leq_e R\}$. Consider the following cases:

(i) For any $I, J \in \Omega,\ I \cap J \in \Omega.$

(ii) If $I \subseteq J \subseteq R_R$ and $I \in \Omega$, then $J \in \Omega$.

Also, for $I \subseteq R_R$ and $r \in R,$ define $r^{-1}(I) = \{a \in R \mid ra \in I\}$ which is a right ideal of *R*.

(iii) If $I \in \Omega$ and $r \in R,$ then $r^{-1}(I) \in \Omega,$ i.e. if $I \leq_e R_R$ and $r \in R,$ then $r^{-1}(I) \leq_e R_R$.

Define a mapping $f_r : R \to R, x \mapsto rx.$ Then, we have

$$f_r^{-1}(I) = \{x \mid f_r(x) \in I\} = \{x \mid rx \in I\} = r^{-1}I.$$

We show that for any $I \leq_e R,\ f_r^{-1}(I) \leq_e R.$ We know that $f : C \to B$ and $A \leq_e B,$ then $f^{-1}(A) \leq_e C.$ So that if $f_r : R \to R$ and $I \leq_e R$ then $f_r^{-1}(I) \leq_e R.$

Now we show that $m \in Z(M)$ if and only if there exists $I \leq_e R$ such that $mI = 0.$

To do this, first assume that $m \in Z(M).$ Then $mr_R(m) = 0.$ Choose $I = r_R(m).$ Then $I \leq_e R.$ Conversely, assume that there exists $I \leq_e R$ with $mI = 0.$ Then for any $r \in I,$ we have $mr = 0.$ This implies that $r \in r_R(m).$ Thus, $I \subseteq r_R(m) \subseteq R.$ Since $I \leq_e R$ and $r_R(m) \leq_e R,$ we have $m \in Z(M).$

Finally, we show that for any $m \in Z(M)$ and any $r \in R,\ mr \in Z(M).$ Let $m \in Z(M),\ r \in R$ and $I = r_R(m) \leq_e R.$ Consider $J = r^{-1}(I) = \{a \in R \mid ra \in I\}.$ Take any $a \in J.$ Then $ra \in r_R(m) \Rightarrow m(ra) = (mr)a = 0$ and so $mrJ = 0.$ Hence $mr \in Z(M).$

Lemma 4.2.4 Let *I* be a right ideal of a ring *R* and *M* a right *R*-module. If $X \leq_e M$, an essential submodule of *M*, then *I* is an essential right ideal of *R*, where

$$I = \{r \in R \mid Mr \subset X\}.$$

Proof. Assume that $X \leq_e M$. Since $M0 \subset X$, then $0 \in I$. Thus, $I \neq$ ¤. For any $s_1, s_2 \in I$, we have $Ms_1 \subset X$ and $Ms_2 \subset X$. Then $M(s_1 + s_2) = M(s_1) + M(s_2) \subset X$ because $X \leq M$. Thus $s_1 + s_2 \in I$. For any $s \in I,\ r \in R,$ we have $Ms \subset X$. This implies that

$M(sr) = (Ms)r \subset Xr \subset X$. Thus, $sr \in I$. We also can show that I is a left ideal of R. For any $s \in I,\ r \in R,$ we have $Ms \subset X$. Since $M(rs) = (Mr)s \subset Xs \subset X$. So $rs \in I$.

Suppose that I is not essential in R. Then there exists $0 \neq J \subset R_R$ such that $I \cap J = 0$. We want to show that $MI \cap MJ = 0$. First we show that $MJ \neq 0$. If $MJ = 0$, then $0 \subset X$. This means that $J \subset I$, which is a contradiction because $I \cap J = 0$. So $MJ \neq 0$.

To show that $MI \cap MJ = 0$. If $MI \cap MJ \neq 0$, take any nonzero element $x \in MI \cap MJ$. Then $x = m_1 i = m_2 j$ for any $m_1, m_2 \in M$ and $0 \neq i \in I, 0 \neq j \in J$. If $j \in I$, then we have a contradiction because $I \cap J = 0$. So $j \notin I$, $m_2 j \notin X$ and so $MJ \not\subset X$. But $m_2 j \in M$. Since $X \leq_e M$, there exists a $t \in R$ such that $m_2 jt \in X$ for any $m_2 \in M$. This shows that $M(jt) \subset X$ and so $jt \in I$. Since $J \subset R_R$, we have $jt \in J$. Thus, $I \cap J \neq 0$, which is a contradiction. Hence $MI \cap MJ = 0$.

So $MJ \neq 0$, $MJ \subset M$ and $MJ \not\subset X$. Let $K = \{r \in R \mid MJr \subset X\}$. Then K is a two-sided ideal of R. Since $X \leq_e M$, $MJ \not\subset X$ and $K \neq 0$, so $0 \neq MJK \subset X$. This implies that $JK \subset I$, $JK \subset J, JK \neq 0$ and so $I \cap J \neq 0$, which is a contradiction. Thus, I is an essential right ideal of R.

Lemma 4.2.5 Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If X is an essential submodule of M, then $I_X = \{f \in S \mid f(M) \subset X\}$ is an essential right ideal of $S = End_R(M)$.

Proof. Since M is a self-generator and $X \neq 0$, we have $I_X \neq 0$. Let J be a right ideal of S such that such that $I_X \cap J = 0$. By 18.4 [21] we have

$I_X = Hom(M, I_X(M)) = Hom(M, X)$ and $J = Hom(M, J(M))$. It would imply that

$$0 = I_X \cap J = Hom(M, X) \cap Hom(M, J(M)) = Hom(M, X \cap J(M)).$$

It follows that $X \cap J(M) = 0$ because M is a self-generator and hence $J(M) = 0$ proving that $J = 0$. This shows that I_X is an essential right ideal of S.

Definition

An element $f \in S$ is called left regular if $l_S(f) = 0$, where $l_S(f) = \{g \in S : gf = 0\}$ and is called right regular if and only if $f : M \to M$ is monomorphism, where $S = End_R(M)$.

With this definition, Lemma 3.2.3 may be modified over associative endomorphism rings as follows:

Lemma 4.2.6 Let *M* be a non-singular right *R*-module with finite Goldie dimension. Then every one to one endomorphism of *M* is left regular in *S*.

Proof. Let $f \in S$ be a monomorphism. By Proposition 4.2.2, $f(M)$ is essential in *M*. Take any $g \in l_S(f)$. Then $gf = 0$ and hence $gf(M) = 0$. For each $m \in M$, let $I_m = \{r \in R \mid mr \in f(M)\}$. Then I_m is an essential right ideal of *R* and $mI_m \subset f(M)$. It follows that $g(m)I_m \subset gf(M) = 0$. Therefore, $g(m) \in Z(M) = 0$. It would imply that $g = 0$, showing that $l_S(f) = 0$. Hence f is left regular in *S*.

Lemma 4.2.7 Let *M* be a quasi-projective, finitely generated right *R*-module which is a self-generator. If *M* is a semi-prime Goldie module, then the left annihilator of every essential right ideal of a ring *S* is zero.

Proof. Since *M* is a semi-prime Goldie module, *S* is a semi-prime right Goldie ring ry Lemma 4.2.1. Then the singular ideal *Z(S)* of *S* is nilpotent since *S* satisfies the *ACC* for right annihilators. Since *S* is semi-prime, we have *Z(S) = 0*. It implies that the left annihilator of every essential right ideal of *S* is zero.

CONCLUSION

In 1928, Krull investigated some properties of prime and semi-prime ideals over commutative rings. Goodearl and Warfield developed the commutative definitions by replacing products of elements with products of ideals. Also they investigated some characterizations of prime and semi-prime rings.

In 2009, Sanh et al. introduced a notion of prime and semi-prime submodules and investigated some properties of prime and semi-prime Goldie rings and modules.

In this thesis, we have developed the properties of prime and semi-prime submodules over associative endomorphism rings by modifying the properties of prime and semi-prime ideals over associative arbitrary rings.

Finally, we have investigated some characterizations of prime and semi-prime Goldie modules over endomorphism rings as generalizations of prime and semi-prime ideals over associative arbitrary rings.

REFERENCES

[1] J. Dauns, *Prime Modules*, J. Reine Angew. Math., Vol. 298 (1978), 156-181.

[2] J. Dauns, *PrimalModules*, *Comm. Algebra*, Vol. 25, No 8 (1997), 2409-2435.

[3] K. R. Goodearl and R. B. Warfield, *An Introduction to Non-commutative Noetherian Rings*, London Mathematical Society Student Texts, Vol. 61, Cambridge University Press, Cambridge, 2004

[4] R. L. McCasland and P. F. Smith, *Prime Submodules of Noetherian Modules*, Rocky Mountain J. Math. Vol. 23, No. 3 (1993), 1041-1062.

[5] R. Wisbauer, *On Prime Modules and Rings*, Comm. Algebra, Vol. 11, No. 20 (1983), 2249-2265.

[6] K. I. Beider and R. Wisbauer, *Strongly Semi-prime Modules and Rings*, Comm. Moscow Math. Soc., Russian Math. Surveys, Vol. 48, No. 1 (1993), 163-200.

[7] C. P. Lu, *Prime Submodules of Modules*, Comment. Mat. Univ. St. Pauli, Vol. 33, No. 1 (1984), 61-69.

[8] K. R. Goodearl, *Ring Theory: Nonsingular Rings and Modules*, Marcel Dekker Inc., New York and Basel, 1976.

[9] M. Behboodi and H. Koohy, *Weakly prime modules*, Vietnam J. Math., Vol. 32, No. 2 (2004), 185-196.

[10] R. Ameri, *On the Prime Submodules of Multiplication Modules*, Int. J. Math. Math. Sci., Vol. 27 (2003), 1715-1724.

[11] A. Gaur, A. K. Maloo and A. Parkash, *Prime Submodules in Multiplication Modules*, Int. J. Algebra, Vol. 1, No. 8 (2007), 375-380.

[12] A. Gaur and A. K. Maloo, Minimal Prime Submodules, International Journal of Algebra, Vol. 2, No. 20 (2008), 953-956.

[13] A. W. Chatters and C. R. Hajarnavis, *Rings with Chain Conditions*, Pitman Advanced Publishing Program, 1980.

[14] N. V. Sanh, N. A. Vu, K. F. U. Ahmed, S. Asawasamrit and L. P. Thao, *Primeness in Module Category,* Asian-European Journal of Mathematics, Vol. 3, No. 1 (2010), 145-154.

[15] N. V. Sanh, K. F. U. Ahmed, S. Asawasamrit and L. P. Thao, *On Prime and Semi-prime Goldie Modules,* Asian-European Journal of Mathematics, Vol. 4, No. 2 (2011), 321-334.

[16] K. F. U. Ahmed, L. P. Thao and N. V. Sanh, *On Semi-prime Modules with Chain Conditions,* East-West Journal of Mathematics, Vol. 15, No. 2 (2013), 135-151.

[17] F. Kasch, *Modules and Rings,* London Mathematical Society Monograph, No.17, Academic Press, London - New York - Paris,1982.

[18] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules,* Graduate Texts in Math., No.13, Springer-Verlag, New York - Heidelberg - Berlin, 1992.

[19] B. Stenstrom, *Rings of Quotients,* Springer-Verlag, Berline-Heidelberg- New York, 1975.

[20] T. Y. Lam, A First Course in Noncommutative Rings, Graduate Texts in Mathematics, Vol. 131, Springer-Verlag, Berline-Heidelberg-New York, 1991.

[21] R. Wisbauer, *Foundations of Module and Ring Theory,* Gordon and Breach, Tokyo,1991.

[22] T. Y. Lam, *Lectures on Modules and Rings,* Graduate Texts in Mathematics, Vol. 189, Springer-Verlag, Berlin-Heidelberg-New York, 1999.

[23] T. Y. Lam, Exercises in Modules and Rings, Problem Books in Mathematics, Springer Science + Business Media, LLC, 2007.

[24] W.Kull, Primidealketten in Allgemeinen Ringbereichen S-8, Heidelberg. Akad. Wiss, Vol. 7, 1928.