FINITE ELEMENT FORMULATION AND MODELING OF RATE DEPENDENT RESPONSE OF NATURAL AND HIGH DAMPING RUBBERS

TANVIR HOSSAIN

DEPARTMENT OF CIVIL ENGINEERING
BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY
DHAKA 1000, BANGLADESH
FINITE ELEMENT FORMULATION AND MODELING OF RATE DEPENDENT RESPONSE OF
NATURAL AND HIGH DAMPING RUBBERS

A THESIS BY
TANVIR HOSSAIN

Approved as to the style and content by:

Dr. A.F.M. Saiful Amin
Associate Professor
Department of Civil Engineering
BUET, Dhaka

Chairman
(Supervisor)

Dr. Syed Ishtiaq Ahmadi
Associate Professor
Department of Civil Engineering
BUET, Dhaka

Member

Dr. Muhammad Zakaria
Professor and Head
Department of Civil Engineering
BUET, Dhaka

Member
(Ex-officio)

Prof. Dr. Sohrabuddin Ahmed
Apartment No. A4
House No. 61B, Road No. 6A
Dhanmondi R.A.
Dhaka-1209

Member
(External)
DECLARATION

It is declared that, except where specific references are made to other investigators, the work embodied in this thesis is the result of investigation carried out by the author under the supervision of Dr. A.F.M. Saiful Amin, Associate Professor of Civil Engineering, BUET.

Neither the thesis nor any part thereof has been submitted or is being concurrently submitted in candidature for any degree at any other institution (Except for own publications).

[Signature]

Author
ACKNOWLEDGEMENT

The author feels extremely privileged to work under Dr. A.F.M. Saiful Amin, Associate Professor, Department of Civil Engineering, BUET. The author takes the opportunity to express his gratitude to Dr. Amin for invaluable suggestions and ardent encouragements in every aspect of supervision of this work. Dr. Amin’s keen interest and encouragement helped the author to understand the subject that he presents in this dissertation.

The author is particularly grateful to Prof. Dr. Yoshiaki Okui, Department of Civil Engineering, Saitama University, Japan, for allowing the author to use the experimental results for verification of FE simulation. The author is also grateful to Mr. Muhammad Abdur Rahman Bhuiyan, Assistant Professor, Department of Civil Engineering, CUET, for his benevolent cooperation. He has put his efforts to assist the author with his thoughtful suggestions.

Last but not the least; the author is thankful to his family members, friends and colleagues, who have always been beside him through thick and thin.
ABSTRACT

Nonlinear rate-dependent responses of natural rubber (NR) and high damping rubber (HDR) are simulated by solving boundary value problems modelled with three dimensional 8-noded brick finite elements. The FE formulation has been obtained from the three-parameter Zener model as proposed recently in Amin et al 2006a by following the concept of nonlinear finite strain continuum mechanics. To this end Bernoulli’s solution technique has been employed to obtain analytical solution of left Cauchy-Green deformation tensor of overstress part. The nonlinear rate-dependent behaviour of NR and HDR are simulated under compression, shear and their combinations.

Explicit expressions for the second Piola-Kirchhoff stress tensor and the Cauchy stress tensor for the equilibrium and over-stress parts are made to formulate the finite element coding. The Lagrangian elasticity tensors for the equilibrium and over-stress parts are also formulated to implement in a general-purpose finite element code. The equilibrium and the over-stress response of NR and HDR under compression and shear have been simulated using the material parameters obtained from experimental observations. The relaxation responses under compression and shear have also been investigated. The simulation results are compared with published experimental findings. The conformity was found encouraging. The results are also in good agreement with the basic properties of NR and HDR. Finally, the FE formulation was utilized to solve the 3D laminated NR and HDR bearings under compression, shear and their combinations.
# CONTENTS

## DECLARATION

## ACKNOWLEDGEMENTS

## ABSTRACT

## CONTENTS

### Chapter 1: INTRODUCTION

1.1 General

1.2 Rubber Microstructure From Chemical Viewpoint

1.3 Constitutive Behavior of Rubbers
   - 1.3.1 Nonlinearity in monotonic response
   - 1.3.2 Rate dependency effect and nonlinearity in viscosity
   - 1.3.3 Practical significance

1.4 Background of the Study

1.5 Objectives

1.6 Contents of the Thesis

### Chapter 2: MECHANICAL BEHAVIOR OF RUBBERS

2.1 General

2.2 Specimens

2.3 Experiments in Compression and Shear
   - 2.3.1 Mullins’ effect
   - 2.3.2 Nonlinearity in monotonic response
   - 2.3.3 Strain-rate dependency
   - 2.3.4 Hysteresis and residual strain
   - 2.3.5 Incompressibility
Chapter 3: NONLINEAR CONTINUUM MECHANICS

3.1 General 25
3.2 Addressing Geometric and Material Nonlinearity in FEM 25
3.3 Nonlinear strain measures 26
3.4 Continuum strain measures 28
3.5 Kinematics 30
   3.5.1 Material and spatial descriptions 30
   3.5.2 Deformation gradient 31
   3.5.3 Strain 33
   3.5.4 Distortional component of the deformation gradient 34
   3.5.5 Velocity and material time derivatives 35
   3.5.6 Velocity gradient 36
   3.5.7 Rate of deformation 37
3.6 Cauchy Stress Tensor 39
   3.6.1 The Kirchhoff stress tensor 40
   3.6.2 The first Piola-Kirchhoff stress tensor 41
   3.6.3 The second Piola-Kirchhoff stress tensor 42
   3.6.4 Deviatoric and pressure components 42
3.7 Principle of Virtual Work 43
3.8 Hyperelasticity 44
   3.8.1 The material or Lagrangian elasticity tensor 45
   3.8.2 The spatial or Eulerian elasticity tensor 46
   3.8.3 Isotropic hyperelasticity material description 47
   3.8.4 Isotropic hyperelasticity spatial description 47

Chapter 4: VISCOELASTICITY

4.1 General 49
4.2 Characteristics of a Viscoelastic Material 49
4.3 Mathematical Models for Linear Viscoelastic Response 50
   4.3.1 The Maxwell spring-dashpot model 51
   4.3.2 The standard linear solid model (Maxwell form) 53
   4.3.3 The Wiechert model 54
   4.3.4 The Kelvin-Voigt model 55
7.4 Numerical Simulation Under Simultaneous Action of Compression and Shear
7.5 Numerical Simulation of Relaxation Response
7.6 Stress and Deformation Patterns

Chapter 8: ANALYSIS OF RUBBER BEARINGS
8.1 General
8.2 Bearing Pad Model
8.3 Analysis of Full Scale Bearing Pad

Chapter 9: CONCLUSION AND RECOMMENDATION
9.1 General
9.2 Verification of Finite Element Formulations
9.3 Finite Element Study on Bearing Pad
9.4 Recommendations for Future Studies

REFERENCES
APPENDIX
1.1 GENERAL

Natural rubber (NR) is obtained in the form of latex from the tree *Hevea Braziliensis*. High degrees of deformability under the action of comparatively small stresses together with incompressibility property make rubber a versatile material. Special fillers, for example, carbon black or silica are usually added during vulcanization process for improving the strength and toughness properties of rubber (Wischt, 1998).

Vulcanized rubbers are widely used in engineering applications e.g. tires, engine mounts, shock-absorbing bushes, seals, tunnel linings and wind shoes (Roeder and Stanton, 1983; Ward, 1985; Mullins, 1987; Castellani et al. 1998). High damping rubber (HDR) are widely used in base isolation devices for protecting buildings and bridges from devastating earthquake is another emerging dimension of engineering applications of rubber (Fujita et al. 1990; Kelly, 1991; Carr et al. 1996; Mori et al. 1996; Dorfmann and Burtscher, 2000).

The isolators covered consist of alternate rubber layers and reinforcing steel plates. They are placed between a superstructure and its substructure to provide both flexibility for decoupling structural systems from ground motion,
and damping capability to reduce displacement at the isolation interface and the transmission of energy from the ground into the structure at the isolation frequency (ISO-22762-1). These base isolation devices when used in bridges and buildings have shown encouraging field level performances by sustaining severe shocks during Loma Prieta (1989), Northridge (1994) and Kobe (1995) earthquakes (Kelly, 1997). Cubes and cylinders are the most common geometries for rubber bearings. Again there are some other variations like trapezoidal or tapered shapes (AASHTO, 2002; Ramberger, 2002; Mattheck and Erb, 1991) and also having V-shaped steel plates (European Commission, 1999). Sometimes lead plugs are used within the bearing pad to increase its stability.

![Figure 1.2 Cubic shape bearing pad with lead plug.](image)

The role of the steel plates is to imply large stiffness under vertical loads, while rubber layers incorporate low horizontal stiffness when the structure is subjected to lateral loads (e.g., earthquakes, wind, etc). Usually the bearings remain under compression due to the gravity loads coming from the superstructures. However, lateral load like wind or an earthquake when strikes, compression and shear deformations act together on these bearings. To estimate the performance of the bearings and thereby finding their optimum design, the engineers usually deal with test data obtain from expensive tests conducted on prototypes or full scale specimens. Again there exists another possibility to develop a reliable numerical procedure like the finite element method for predicting the performance. Nevertheless, the core of such a general numerical procedure depends largely on the constitutive model that is adequate enough for describing the major phenomena of HDR in the relevant deformation range.
1.2 RUBBER MICROSTRUCTURE FROM CHEMICAL VIEWPOINT

From a chemical point of view rubber is a hydrocarbon, described by the chemical formula \((C,H)_n\). \(C,H\) is called an isoprene and natural rubber is built up of regular sequences of isoprenes, which are arranged in cis-configuration, forming long chains of high elasticity as shown in Fig. 1.3. (Treloar, 1975).

![Figure 1.3 Structure formulae of one isoprene molecule (left) and structure of a chain molecule (right).](image)

The chains are linked and lie perfectly regular in the backbone and have freely rotating links at given distance. Natural rubbers have only a few weak crosslinks between the chains. When subjected to an external force the chains disentangle and breakup crosslink causing the material flows and undergoes large displacement until breakage of links or an equilibrium state is reached (Burtscher et al, 1998). When NR is reinforced with additional fillers during vulcanization extra crosslink are established and a coherent network is formed (Treloar, 1975; Gent, 1992). If then subjected to an external load the response is (visco)elastic. When rubber is dynamically loaded a part of the energy is stored in the medium that can be released by unloading or breakage of crosslinks. The remainder of the energy is dissipated by thermal effects. With an increase of crosslinks the network becomes tighter and the motion of the chains becomes more impeded. The network is incapable of dissipating much energy resulting high hardness, low elongation and brittle fracture.

It is important that the filler particles when added should be very small and possess high specific surface area. When filled compounds are strained to a large extent, they show the effect of strain softening. This phenomenon is probably caused by the breakdown of weak chemical bindings between fillers.
and rubber molecules. Unfilled or lightly filled compounds do not show a significant degree of strain softening.

1.3 CONSTITUTIVE BEHAVIOR OF RUBBERS

1.3.1 Nonlinearity in monotonic response

The mechanical behavior of rubbers is dominated by nonlinear rate dependent response (Aklonis et al., 1972). The experimental study of the response of filled rubber was first reported in the early sixties (Mason, 1960; Dannis, 1962) when it was found that the tensile strength of rubber increases with increasing strain rate. The appearance of this property possesses an inherent relation to the presence of carbon black in the rubber matrix. Due to the presence of polymeric network chains, rubber shows a high initial stiffness that decreases with increasing strain, and then the stiffness remains approximately constant and increases at the end as shown in the Fig. 1.4.

![Figure 1.4 An idealized nonlinearity response of rubber (Wiraguna, 2003)](image)

The high initial stiffness results as a consequence of the reinforcing filler; the increase in stiffness at the end arises from the finite extensibility of the chains and possibly also strains crystallization (Burtscher et al, 1998). In the Fig. 1.4 the first part represents the high initial stiffness while the third part represents the strain hardening.
1.3.2 Rate dependency effect and nonlinearity in viscosity

During the vulcanization process a large amount of filler (about 30%) including carbon black, silica, oils, and some other particles are added to the HDR to facilitate the absorption of energy through hysteresis process (Kelly, 1997; Yoshida et al., 2004). As a result HDR is developed to exhibit a strong non-linear rate-dependent response under monotonic loadings and to show significant hysteresis effects or energy dissipation under cyclic loads.

Figure 1.5 Different stretch rates (left) and corresponding responses of HDR (right), showing that response during loading is dependent on stretch rates, whereas during unloading response is nearly independent of rates.

Figure 1.5 illustrates the effect of rate dependency of HDR. L1, L2 and L3 represent the same stretch at different rates and the corresponding responses are represented by R1, R2 and R3. It is obvious that the responses increase with the increase in the stretch rates and vice versa. It is also evident that the response during unloading is nearly independent of rate. The area under each loop is the amount of energy dissipated during each loading cycle. This phenomenon is better known as hysteresis loop and an indication of the presence of viscosity. The relaxation process of HDR shows very fast stress decay during the first few seconds followed by a very slow rate in the long-term range. The presence of fillers in HDR plays a significant influence on the appearance of such relaxation phenomena (Ward, 1985; Mason, 1960; Meinecke and Taftaf, 1987; Wischt, 1998). Figure 1.6 illustrates the relaxation phenomena of HDR. The stress reaches to the equilibrium at an asymptotic sense.
Figure 1.6 (a) Different stretch and stresses of HDR (b) Multi step relaxation test (left) and corresponding responses of HDR (right).

1.3.3 Practical significance

The laminated bearing pad usually remains under compression due to the gravity loads coming from the superstructure. However, lateral load induces shear stresses in the bearing pad during earthquake resulting complex state as compression and shear stresses act together. Again the steel and rubber are placed in alternate layers; the responses in each layer can differ. The bearing pad will also be in a state of inhomogeneous strain field and stress field can be affected due to inhomogeneous rate effects. Different types of materials with completely different materials property are used in the bearing
pad. Constitutive behavior of more than one material (e.g. rubber, steel, lead etc.) affects the system behavior. The boundary interface between the steel and rubber also add complexity to the bearing pad as a whole. All these effects have to be taken into account in order to get the proper response of the bearing pad. Although much work is known on the constitutive behavior of steel and lead, we know a little about rubber and its viscosity effect. Study (Amin, 2001; Amin et al., 2006b) shows that in HDR the variation between compressive equilibrium stress and instantaneous stress is quite large (about 84%) indicating the importance of rate dependency on its behavior. To this end (Amin et al., 2006a) presents the constitutive modeling of rubber based on the nonlinear dependence of viscosity. The contribution of this dissertation may help one to properly model the rate dependent behavior of rubber which is very essential for the complete understanding of the behavior of bearing pad.

1.4 BACKGROUND OF THE STUDY

Based on the above discussion on the chemical composition of rubber, its mechanical behavior and the available constitutive model to represent this behavior, it appears that:

a) Addition of fillers like carbon black, silica, oils etc during the process of vulcanization of rubbers has a marked effect on its engineering properties like initial stiffness, strain hardening, relaxation etc.

b) The HDR exhibit a strong nonlinear rate-dependent response under monotonic loadings and show significant hysteresis effects or energy dissipation during cyclic loads.

c) Under compression and shear deformations, HDR is expected to exhibit a high stiffness under low strains, also it dissipate the energy through hysteresis process, all these properties makes HDR a unique material in engineering applications.

d) The constitutive models of rubber need to be founded on finite strain theories in consistence with the natural laws of thermodynamics.
e) A suitable hyperelastic model (Amin, 2001; Amin et al, 2002) is required to reproduce the elastic part of the total response.

f) A solution technique along with the formulation and computer coding of relevant finite strain theories are required to incorporate the viscoelastic response (Huber and Tsakmakis, 2000a; Amin et al., 2006a) in a general purpose finite element code.

1.5 OBJECTIVES

The present research aims at the following specific objectives:

a. To solve the rate of left Cauchy-Green deformation tensor model as proposed in Amin et al (2002) and Amin et al (2006a) to obtain the left Cauchy-Green deformation tensor for the over-stress part by using the available solution techniques such as Bernoulli’s and Riccati’s solution.

b. To formulate the second Piola-Kirchhoff stress tensor and the Cauchy stress tensor for the equilibrium part and the over-stress part and obtain the total response by adding these two stresses.

c. To formulate the Lagrangian elasticity tensor for the equilibrium part and the over-stress part of the viscoelasticity model.

d. To implement the stress tensor and the elasticity tensor formulation of the viscoelasticity model in a general-purpose finite element code.

e. To simulate the monotonic responses of HDR and NR under compression and shear for different strain-rates and verify the suitability of the developed finite element procedure by comparing the simulation results with the available numerical solution and the experimental data.
f. To simulate the relaxation behavior of HDR and NR under homogeneous and heterogeneous states of compression and simple shear using the developed FEM model and compare these behavior with the available experimental data.

g. To model the full scale 3D Elastomeric seismic-protection isolators as stated in the international standard (ISO 22762-1) and analyze these model under different modes of deformation using the developed finite element procedure and compare the simulation results with the available experimental results.

1.6 CONTENTS OF THE THESIS

In order to formulate the rate dependent behavior of rubber, it is very important to know the exact mechanical behavior. To this end recent experimental results of rubber carried out under compression and shear are summarized in Chapter 2 of this dissertation. The formulation is based on finite strain theory, a complete understanding of the theory that is capable of dealing with material as well as geometric nonlinearity is necessary. A brief description on nonlinear continuum mechanics thus presented in Chapter 3. The formulation is based on Maxwell three parameter models. Other than the Maxwell model, there are several mathematical models that can be used to simulate the viscosity; these are presented in Chapter 4 of this dissertation. Again the formulation needs to be founded on finite strain theories in consistence with the natural laws of thermodynamics. To this end, a brief description of the derivation of the constitutive model following the concept of Zener model is presented in Chapter 5. The formulation for the finite element analysis of rate dependent behavior of rubber is presented in Chapter 6. Here the expression for the Cauchy stress tensor and Lagrangian elasticity tensor for the equilibrium and the overstress part is summarized in a step by step manner. The verification of the proposed formulation is presented in Chapter 7 and in Chapter 8. Here the analytical solution is compared with the experimental results to verify the suitability of the proposed formulation.
2.1 GENERAL

Mechanical behavior of rubbers is dominated by non-linear rate dependent response (Aklonis et al., 1972) that includes other inelastic behavior such as Mullin's effect (Mullin's, 1969) and hysteresis (Gent, 1962a, b). Furthermore, incompressibility is another characteristic feature seen in rubber. The present chapter summarizes information on constitutive behavior of rubber as revealed from recent experiments conducted in compression (Amin, 2001) and shear (Wiraguna, 2003) regimes. These results motivated the later chapters of this dissertation towards obtaining a finite element procedure.

2.2 SPECIMENS

Amin (2001) performed experiments in compression on two types of natural rubbers (NR-I and NR-II) and high damping rubber (HDR) whereas Wiraguna (2003) performed experiments in shear on natural rubber (NR-II) and high damping rubber (HDR). The specimen dimensions and related information are presented in Table 2.1.

2.3 EXPERIMENTS IN COMPRESSION AND SHEAR

Uniaxial compression tests were carried out by Amin (2001) to observe the mechanical behavior of NR-I, NR-II and HDR. To investigate the mechanical behavior of rubber under shear deformation, cyclic test of simple shear loading was carried out by Wiraguna (2003). The mechanical behaviors of these rubbers are briefly discussed in the subsequent sub-sections.
**Table: 2.1 Details of the specimens**

<table>
<thead>
<tr>
<th>SPECIMEN DESIGNATION</th>
<th>NR-I</th>
<th>NR-II</th>
<th>HDR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
<td>NR</td>
<td>NR</td>
<td>HDR</td>
</tr>
<tr>
<td>Application</td>
<td>General purpose</td>
<td>Bridge bearing</td>
<td>Bridge bearing</td>
</tr>
<tr>
<td>Manufacturer</td>
<td>Shinoda Rubber Co.</td>
<td>Yokohama Rubber Co.</td>
<td>Yokohama Rubber Co.</td>
</tr>
<tr>
<td>Strength</td>
<td>4.0 MPa*</td>
<td>0.98 MPa**</td>
<td>0.78 MPa**</td>
</tr>
</tbody>
</table>

**MECHANICAL TESTS IN COMPRESSION**

<table>
<thead>
<tr>
<th>Shape</th>
<th>NR-I</th>
<th>NR-II</th>
<th>HDR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>H:50mm, L:50mm, W:50mm</td>
<td>H:41 mm, D:49 mm</td>
<td>H:41 mm, D:49 mm</td>
</tr>
</tbody>
</table>

**MECHANICAL TESTS IN SHEAR**

<table>
<thead>
<tr>
<th>Shape</th>
<th>NR-I</th>
<th>NR-II</th>
<th>HDR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>---</td>
<td>Flat Strip</td>
<td>Flat Strip</td>
</tr>
</tbody>
</table>

H: Height, L: Length, W: Width, D: Diameter, * Tensile strength, ** Shear modulus tested according to JIS K 6301

2.3.1 Mullins' effect

Virgin rubber typically exhibits a softening phenomenon known as Mullin's effect (Mullins, 1969) in the first loading cycle. The stress response depends on the past maximum strain. Mullin's effect was found to be present in all the specimens at the virgin state. The softening has been attributed to breakdown or slippage of weak linkages between the filler and rubber, filler-filler aggregates and breakdown of molecular network chains. The effect is much more pronounced in the vulcanized rubber containing high proportion of reinforcing fillers. Figure 2.1 presents typical stress-stretch responses obtained from pre-loading tests for NR-I. The softening behavior is evident from the figure. The specimens showed a repeatable stress-stretch response after passing 2-3 loading cycles. However, Mullin's softening effect in a specimen recovers slowly with time known as the "healing effect" (Bueche, 1961). Figure 2.2 show the stress history and stress-strain relation from preloading test for HDR. The softening effect in the first loading cycle is
shown from the figure. After 2-3 loading cycles, the same stress-strain behavior was obtained. The removal of Mullin's effect has been confirmed.

Figure 2.1: A typical example of Mullins' effect observed in virgin rubber (NR-I) under compression (a) Stretch history (b) Stress-stretch response.
Figure 2.2: (a) Stretch history (b) Stress history (c) Stress-strain relation of HDR
2.3.2 Nonlinearity in monotonic response

Apart from the Mullins' effect seen in the virgin rubbers, the stress-strain behavior of rubber is nonlinear. Figure 2.3 presents a typical nonlinear response that one can obtain in a monotonic compression test. The stress-strain response can be discussed in three segments. At low stretch the presence of fillers gives a high initial stiffness followed by noticeable large deformability at moderate strain due to the breakdown of rubber-filler bonds. Large-strain hardening at the end part is seen when the free lengths of the molecular network chains get depleted and the material approaches the ultimate deformation limit.

![Figure 2.3: A typical nonlinear response obtained from natural rubber under monotonic compression and after removing Mullins' effect](image)

2.3.3 Strain-rate dependency

Figure 2.4 schematically presents typical rate-dependent responses that can be obtained from a viscoelastic solid. When such a solid is loaded at an infinitely slow rate, the stress-strain curve follows the E-E' path. This behavior is called the equilibrium response. On the other hand, in the case of an infinitely fast loading rate, the stress-strain curve follows the I-I' path.
Such a response is known as the instantaneous response and defines a domain where a viscoelastic effect comes into play.

![Figure 2.4: A schematic representation of responses obtained from a viscoelastic solid.](image)

In experiments, equilibrium response is traced from multi-step relaxation tests (Fig 1.6) while high strain rate monotonic tests (Fig 1.5) are conducted to reach the instantaneous boundary in asymptotic sense. Vulcanized rubber is a typical example of highly viscous solid (Ward 1985), wherein the stress response is highly dependent on the loading rate. Figure 2.5 gives a typical example of rate-dependent response that one can obtain when a rubber is under monotonic compression at several strain rates. Strain-rate effect on the stress-strain response can be seen by comparing responses at different strain rates. A series of cyclic loading tests with different strain rates carried out by Wiraguna et al (2003) to find out the rate dependency and instantaneous state in shear regime. In the tests, the strain rate varies from 0.05/ s to 0.5/s for HDR, and for NR (NR-II) it was from 0.005/s to 0.1/s. The temperature at the time of the tests was 293 K for HDR and 294 K for NR (NR-II). Table 2.2 presents the information of the cyclic test.
Figure 2.5: Viscoelastic effect exhibited by natural rubber at different strain rates in monotonic loading under compression

Table 2.2. Strain and stretch rates in cyclic shear and compression tests

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Stretch rates (/sec)</th>
<th>Strain rates ( /sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>HDR</td>
<td>0.001, 0.005, 0.024, 0.24, 0.88</td>
<td>0.05, 0.25, 0.40, 0.50</td>
</tr>
<tr>
<td>NR (NR-II)</td>
<td>0.001, 0.075, 0.47</td>
<td>0.005, 0.01, 0.05, 0.10</td>
</tr>
</tbody>
</table>

The stress responses of HDR in the cyclic test at different strain rates are shown in Fig. 2.6, while the stress responses of NR (NR-II) at different strain rates in cyclic test are shown in Fig. 2.7. The stress responses of HDR and NR bearing pad at different strain rates are shown in Figs. from 2.8-2.10. Figure 2.8 represents the stress responses of HDR bearing pad at different strain rates, Fig. 2.9 represents the stress responses of NR-BBM bearing pad at different strain rates while Fig. 2.10 represents the stress responses of NR-TFK bearing pad at different strain rates.
Figure 2.6: Stress responses of HDR in cyclic test with different strain rates

Figure 2.7: Stress responses of NR (NR-II) in cyclic test with different strain rates
Figure 2.8: Stress-strain responses of HDR bearing pad at different strain rates
Figure 2.9: Stress-strain responses of NR-BBM bearing pad at different strain rates
Figure 2.10: Stress-strain responses of NR-FTK bearing pad at different strain rates
2.3.4 Hysteresis and residual strain

Apart from the rate dependent effects, rubber also exhibits hysteresis phenomenon and residual strain under cyclic loading. Hysteresis is known as the difference of loading path from unloading where as the residual strain in a cyclic test is the strain measured when the specimen is unloaded to zero load. Filler concentration plays an important role on those behaviors (Ward, 1985). Figure 2.11 illustrates the hysteresis effect and residual strain feature obtained from a HDR specimen subjected to cyclic loading under compression and shear. Here, the term "residual strain" refers to the "set" in the specimen at the end of a cyclic test.

Figure 2.11: Cyclic responses of HDR showing hysteresis and residual strain (a) Under compression (b) Under shear.
The presence of hysteresis and residual strain needs to be discussed from the constitutive view points. Currently experimental evidence shows these effects to be related to material viscosity. The presence of any plastic effects can also be involved of such phenomena.

2.3.5 Incompressibility

The resistance of rubber against shear deformation is very low compared to the resistance against volumetric deformation. This gives a very high value of bulk modulus compared to its shear modulus and the material is considered to be incompressible. Based on this the deformed cross-section of the specimen subjected to uniaxial or biaxial deformation can be predicted to calculate the Cauchy stress of the material. However, Herman et al (1989) indicated the possibility of the existence of voids in the rubber microstructure that might largely affect the bulk modulus.

An experimental setup capable of measuring the deformed cross section of the rubber specimens subjected to a large uniaxial compression was developed by Amin et al (2003). In Figs. 2.12 and 2.13, the mechanical test results have been shown for HDR and NR materials. These experimental evidences justify the near incompressibility feature in the specimens.
Figure 2.12: Mechanical test on HDR subjected to monotonic compression
(a) Applied stretch history (b) Cauchy stress vs. stretch as obtained from Incompressibility assumption and measurement (c) Volume with increasing stretch
Figure 2.13: Mechanical test on NR-II subjected to monotonic compression
(a) Applied stretch history (b) Cauchy stress vs. stretch as obtained from
Incompressibility assumption and measurement (c) Volume with increasing stretch
3.1 GENERAL
The tests carried out on HDR and NR under compression (Amin 2001) and shear (Wiraguna 2003) reveal strong nonlinearity in stress-strain relation (Chapter 2). These stress-strain relations also depend upon the rate of applied strain due to viscosity effect. To describe these responses using a general constitutive law, the theory must be written within the framework of nonlinear continuum mechanics employing finite deformation theory. This chapter presents some basic elements of nonlinear mechanics and their relation in modeling the nonlinear rate-dependent behaviors. These quantities have been used in later chapters for formulating the constitutive model.

3.2 ADDRESSING GEOMETRIC AND MATERIAL NONLINEARITY IN FEM
Two sources of nonlinearity exist in the analysis of solid continua; these are material and geometric nonlinearity. The material nonlinearity occurs when the stress strain behavior given by the constitutive relation is nonlinear, whereas the geometric nonlinearity is important when changes in geometry have a significant effect on the load deformation behavior. Nonlinear and linear continuum mechanics deal with same subjects such as kinematics, stress and equilibrium, and constitutive behavior. In the linear case an assumption is made that the deformation is sufficiently small to enable the effect of changes in the geometrical configuration of the solid to be ignored, whereas in the nonlinear case the magnitude of the deformation is unrestricted. The behavior of a complex component subjected to complex loading can be successfully simulated through numerical analysis, predominantly in the form of finite element method. The finite element method may be summarized as a procedure whereby the continuum behavior described at infinity of points is approximated in terms of a finite number of points, called nodes, located at specific points in the continuum. These nodes
are used to define regions, called finite elements, over which both the geometry and the primary variables in the governing equations are approximated (Bonet and Wood 1997). The governing equations describing the nonlinear behavior of the solid are usually recast in a weak integral form using the principle of virtual work or the principle of stationary total potential energy. The finite element approximations are then introduced into these integral equations yielding a finite set of nonlinear algebraic equations in the primary variable.

The earlier experimental works conducted on HDR and NR by Amin et al (2002),Wiraguna et al (2003), Amin et al (2006a) revealed a strong nonlinearity in monotonic response along with significant strain-rate dependency feature. In this context, the aim of the current work is to describe the observed material and geometrical nonlinearity and thereby to identify the nonlinear material and geometrical constitutive parameters and also the exposition of the nonlinear continuum mechanics necessary to develop the governing equations in continuous and discrete form that can take into account both the material as well as geometrical nonlinearity.

3.3 NONLINEAR STRAIN MEASURES

Structural components or continuum bodies exhibit large strains when undergoing a geometrically nonlinear deformation process. One of the ways, in which these large strains can be measured, a one-dimensional truss element undergoing large displacement and large strain is considered.

A truss member of initial length $L$ and area $A$ is stretched to a final length $l$ and area $a$ is shown in Fig. 3.1. The simplest quantity that can be used to measure the strain in the bar is the engineering strain $\varepsilon_E$ defined as

$$\varepsilon_E = \frac{l - L}{L}$$

(3.1)
Again other measures of strain could be used. For instance, the change in length $\Delta l = l - L$ could be divided by the final length rather than the initial length. In either case, if $l \approx L$ the small strain quantity $\varepsilon = \Delta l / l$ can be recovered. An alternative large strain measure can be obtained by adding up all the small strain increments that take place when the bar is continuously stretched from its original length $L$ to its final length $l$. This integration process leads to the definition of the natural or logarithmic strain $\varepsilon_n$ as,

$$\varepsilon_n = \frac{\int_0^l \frac{dl}{l}}{L} = \ln \frac{l}{L}$$

The above strain definitions can be extrapolated to the deformation of a three-dimensional continuum body. However, this generalization process is complex and computationally costly. A much more generalized strain measures to continuum cases are the Green strain $\varepsilon_G$ and Almansi strain $\varepsilon_A$ (Bonet and Wood, 1997). These are defined as,

$$\varepsilon_G = \frac{l^2 - L^2}{2L^2}$$

$$\varepsilon_A = \frac{l^2 - L^2}{2l^2}$$

It can be shown by using a Taylor series expression that for the case where $l \approx L$, all the above quantities converge to the small strain definition $\Delta l / l$. For instance, when the strain is small, the Green strain converges as,
\[ \varepsilon_G \approx \left( \frac{(l + \Delta l)^2 - l^2}{2l^2} \right) \]
\[ = \frac{1}{2} \left( l^2 + \Delta l^2 + 2\Delta l - l^2 \right) \]
\[ = \frac{\Delta l}{l} \]  

(3.5)

### 3.4 CONTINUUM STRAIN MEASURES

In linear stress-strain analysis the deformation of a continuum body is measured in terms of the small strain tensor \( \varepsilon \). For instance, in a simple two-dimensional case \( \varepsilon \) has components \( \varepsilon_{xx}, \varepsilon_{yy}, \) and \( \varepsilon_{yx} = \varepsilon_{xy} \), which are obtained in terms of the \( x \) and \( y \) components of the displacement of the body as,

\[ \varepsilon_{xx} = \frac{\delta u_x}{\delta x} \]  

(3.6)

\[ \varepsilon_{yy} = \frac{\delta u_y}{\delta y} \]  

(3.7)

\[ \varepsilon_{xy} = \frac{1}{2} \left( \frac{\delta u_x}{\delta y} + \frac{\delta u_y}{\delta x} \right) \]  

(3.8)

These equations are based on the assumption that the displacements \( u_x \) and \( u_y \) are very small, which implies that the initial and final positions of a particle are practically the same. However, when the displacements are large, one must distinguish between the initial and final coordinates of the particles. Capital letters \( X, Y \) are used to designate the initial positions and lower case \( x, y \) are used for the current coordinates.
A small elemental segment \( dX \) initially parallel to the \( x \) axis and deformed to a length \( ds \) as shown in Fig. 3.2.

![Diagram of deformation](image)

Figure 3.2 General deformation of a two-dimensional body

The final length can be evaluated from the displacements as,

\[
ds^2 = \left( dX + \frac{\delta u_x}{\delta X} dX \right)^2 + \left( \frac{\delta u_y}{\delta X} dX \right)^2
\]

Based on the 1-D Green strain Equation (3.3), the \( x \) component of the 2-D Green strain can now be defined as,

\[
E_{xx} = \frac{ds^2 - dX^2}{2dX^2}
\]

\[
= \frac{1}{2} \left[ \left( 1 + \frac{\delta u_x}{\delta X} \right)^2 + \left( \frac{\delta u_y}{\delta X} \right)^2 - 1 \right]
\]

\[
= \frac{\delta u_x}{\delta X} + \frac{1}{2} \left[ \left( \frac{\delta u_x}{\delta X} \right)^2 + \left( \frac{\delta u_y}{\delta X} \right)^2 \right]
\]

Using similar arguments equations for \( E_{xy} \) and shear strains \( E_{xy} = E_{yx} \), are obtained as,
\[
E_{\alpha \beta} = \frac{\partial u_\gamma}{\partial \gamma} + \frac{1}{2} \left[ \left( \frac{\partial u_\gamma}{\partial \gamma} \right)^2 + \left( \frac{\partial u_\gamma}{\partial \gamma} \right)^2 \right] \\
E_{\gamma \delta} = \frac{1}{2} \left( \frac{\partial u_\gamma}{\partial \gamma} + \frac{\partial u_\gamma}{\partial \delta} \right) + \frac{1}{2} \left( \frac{\partial u_\gamma}{\partial \delta} \frac{\partial u_\delta}{\partial \gamma} + \frac{\partial u_\delta}{\partial \gamma} \frac{\partial u_\gamma}{\partial \delta} \right) 
\]

(3.11)  

(3.12)

If the displacements are small, the quadratic terms in the Equations (3.10), (3.11) and (3.12) can be ignored and one recovers Equation (3.6), (3.7) and (3.8).

3.5 KINEMATICS

Kinematics is the study of motion and deformation without reference to the cause (Bonet and Wood, 1997). A proper description of motion is fundamental to finite deformation analysis; however such an emphasis is necessary because infinitesimal deformation analysis implies a host of assumptions that one take for granted and seldom articulated. Consideration of finite deformation enables alternative coordinate systems to be employed, namely, material and spatial descriptions associated with the names of Lagrange and Euler respectively.

3.5.1 Material and spatial descriptions

In finite element analysis a careful consideration has to be made between the coordinate systems that can be chosen to describe the behavior of the body whose motion is under consideration. Relevant quantities such as density can be described in terms of where the body was before deformation or where it is during deformation; the former is called a material description, and the latter is called a spatial description. Alternatively these are often referred to as Lagrangian and Eulerian description. A material description refers to the behavior of a material particle, whereas a spatial description refers to the behavior at a spatial position.
In order to understand the difference between a material and spatial description, a simple scalar quantity such as the current density $\rho$ of the material has been considered.

(a) Material description: the variation of $\rho$ over the body is described with respect to the original coordinate $X$ used to label a material particle in the continuum at time $t = 0$ as,

$$\rho = \rho(X,t)$$  \hspace{1cm} (3.13)

(b) Spatial description: $\rho$ is described with respect to the position in space $x$, currently occupied by a material particle in the continuum at time $t$ as,

$$\rho = \rho(x,t)$$ \hspace{1cm} (3.14)

In Equation (3.13) a change in time $t$ implies that the same material particle $X$ has a different density $\rho$. Consequently interest is focused on the material particle $X$. In Equation (3.14), however, a change in the time $t$ implies that a different density is observed at the same spatial position $x$, now probably occupied by a different particle. Consequently interest is focused on a spatial position $x$.

3.5.2 Deformation gradient

A key quantity in finite deformation analysis is the deformation gradient $F$, which is involved in all equations relating quantities before deformation to corresponding quantities after deformation. The deformation gradient tensor enables the relative spatial position of two neighboring particles after deformation to be described in terms of their relative material position before deformation. Deformation gradient is the central to the description of deformation and hence strain.
Two material particles \( Q_1 \) and \( Q_2 \) stays in the neighborhood of a material particle \( P \) as shown in the Fig. 3.3. The positions of \( Q_1 \) and \( Q_2 \) relative to \( P \) be given by the elemental vectors \( dX_1 \) and \( dX_2 \) as,

\[
dX_1 = X_{Q_1} - X_p; \quad dX_2 = X_{Q_2} - X_p
\]  

\[\text{(3.15)}\]

![Figure 3.3 General motions in the neighborhood of a particle.](image)

After deformation the material particles \( P, Q_1 \) and \( Q_2 \) have deformed to current spatial positions given by

\[
x_p = \phi(X_p, t); \quad x_{q_1} = \phi(X_{q_1}, t); \quad x_{q_2} = \phi(X_{q_2}, t)
\]  

\[\text{(3.16)}\]

and the corresponding elemental vectors become,

\[
d_{x_1} = x_{q_1} - x_p = \phi(X_p + dX_1, t) - \phi(X_p, t)
\]

\[
d_{x_2} = x_{q_2} - x_p = \phi(X_p + dX_2, t) - \phi(X_p, t)
\]  

\[\text{(3.17)}\]

The deformation gradient tensor \( F \) is defined as,

\[
F = \frac{\delta\phi}{\delta X} = \nabla_0\phi
\]  

\[\text{(3.18)}\]
And the elemental vectors $dx_1$ and $dx_2$ can be obtained in terms of $dX_1$ and $dX_2$ as,

$$dx_1 = FdX_1; \quad dx_2 = FdX_2$$ (3.19a, b)

### 3.5.3 Strain

When the two elemental vectors $dX_1$ and $dX_2$ as shown in Fig. 3.3 deform to $dx_1$ and $dx_2$, it involves both the stretching and changes in the enclosed angle between the two vectors. The spatial scalar product $dx_1 \cdot dx_2$ can be found in terms of the material vectors $dX_1$ and $dX_2$ as,

$$dx_1 \cdot dx_2 = dX_1 \cdot CdX_2$$ (3.20)

where $C$ is termed as the right Cauchy-Green deformation tensor, and is given in terms of the deformation gradient $F$ as,

$$C = F^TF$$ (3.21)

Alternatively the initial material scalar product $dX_1 \cdot dX_2$ can be obtained in terms of the spatial vectors $dx_1$ and $dx_2$ with the helps of left Cauchy-Green or Finger tensor $b$ as,

$$dX_1 \cdot dX_2 = dx_1 \cdot b^{-1}dx_2$$ (3.22)

where $b$ is,

$$b = F^TF$$ (3.23)
The change in scalar product can be found in terms of the material vectors \( dX_1 \) and \( dX_2 \) and the Lagrangian or Green strain tensor \( E \) as,

\[
\frac{1}{2}(dx_1 \cdot dx_2 - dX_1 \cdot dX_2) = dX_1 \cdot EdX_2
\]  

(3.24)

where the material tensor \( E \) is,

\[
E = \frac{1}{2}(C - I)
\]  

(3.25)

Alternatively, the same change in scalar product can be expressed with reference to the spatial elemental vectors \( dx_1 \) and \( dx_2 \) and the Eulerian or Almansi strain tensor \( e \) as,

\[
\frac{1}{2}(dx_1 \cdot dx_2 - dX_1 \cdot dX_2) = dx_1 \cdot edx_2
\]  

(3.26)

where the spatial tensor \( e \) is,

\[
e = \frac{1}{2}(I - b^{-1})
\]  

(3.27)

### 3.5.4 Distortional component of the deformation gradient

In case of incompressible and nearly incompressible materials it is necessary to separate the volumetric component from the distortional components of the deformation. Such a separation must ensure that the distortional component \( \hat{F} \) does not imply any change in volume. As the determinant of the deformation gradient gives the volume ratio, the determinant of \( \hat{F} \) must therefore satisfy,

\[
\det \hat{F} = 1
\]  

(3.28)
The deformation gradient $\mathbf{F}$ can be expressed in terms of the volumetric and distortional components, $J$ and $\mathbf{F}'$, respectively, as,
\[
\mathbf{F} = J^{1/2} \mathbf{F}'
\]  \hspace{1cm} (3.29)

### 3.5.5 Velocity and material time derivatives

**Velocity:** Many nonlinear processes are time dependent; therefore it is necessary to consider velocity and material time derivatives of various quantities. The equation of motion of a usual body is given by
\[
x = \phi(X, t)
\]  \hspace{1cm} (3.30)

From which the velocity of a particle is defined as the time derivative of $\phi$ as,
\[
v(X, t) = \frac{\delta \phi(X, t)}{\delta t}
\]  \hspace{1cm} (3.31)

**Material time derivative:** A general scalar or tensor quantity $g$, expressed in terms of the material coordinates $X$, the time derivative of $g(X, t)$ denoted by $g(X, t)$ is defined as,
\[
\dot{g} = \frac{dg}{dt} = \frac{\delta g(X, t)}{\delta t}
\]  \hspace{1cm} (3.32)

This expression measures the change in $g$ associated with a specific particle initially located at $X$, and it is known as the material time derivative of $g$. 

35
3.5.6 Velocity gradient

Velocity can be expressed as a function of the spatial coordinates as \( v(x,t) \). The derivative of this expression with respect to the spatial coordinates defines the velocity gradient tensor \( I \) as,

\[
I = \frac{\delta v(x,t)}{\delta x} = \nabla v
\]  

(3.33)

This expression gives the relative velocity of a particle currently at point \( q \) with respect to a particle currently at \( p \) as \( dv = ldx \) as shown in the Fig. 3.4

![Figure 3.4 Velocity gradient](image)

The velocity gradient tensor \( I \) enables the time derivative of the deformation gradient to be more explicitly expressed as,

\[
\dot{F} = \frac{\delta v}{\delta X} = \frac{\delta v}{\delta x} \frac{\delta x}{\delta X} = IF
\]  

(3.34)

from which an alternative expression for \( I \) can be written as,

\[
I = \dot{F}F^{-1}
\]  

(3.35)
3.5.7 Rate of deformation

Consider the initial elemental vectors $dX_1$ and $dX_2$, as shown in the Fig. 3.5 and their corresponding pushed forward spatial counterparts $dx_1$ and $dx_2$, given as,

$$dx_1 = FdX_1; \quad dx_2 = FdX_2$$ \hspace{1cm} (3.36a, b)

earlier strain was defined and measured as the change in the scalar product of two arbitrary vectors. Similarly, strain rate can be defined as the rate of change of the scalar product of any pair of vectors. To measure this rate of change, the current scalar product can be expressed in terms of the material vectors $dX_1$ and $dX_2$ and the time-dependent right Cauchy-Green tensor $C$ as,

$$dx_1 \cdot dx_2 = dX_1 \cdot CdX_2$$ \hspace{1cm} (3.37)

![Figure 3.5 Rate of deformation](image)

Differentiating Equation (3.37) with respect to time and using the relationship between the Lagrangian strain tensor $E$ and the right Cauchy-Green tensor as $2E = (C - I)$ gives the current rate of change of the scalar product in terms of the initial elemental vectors as,
\[
\frac{d}{dt}(dx_1 \cdot dx_2) = dX_1 \cdot \dot{C}dX_2 = 2dX_1 \cdot \dot{E}dX_2 \tag{3.38}
\]

where \(\dot{E}\) is the derivative with respect to time of the Lagrangian strain tensor and is also known as the material strain tensor and can be obtain in terms of \(\dot{F}\) as,

\[
\dot{E} = \frac{1}{2} \dot{C} = \frac{1}{2} (F^T \dot{F} + \dot{F}^T F) \tag{3.39}
\]

The material strain rate tensor \(\dot{E}\) gives the current rate of change of the scalar product in terms of the initial elemental vectors. It is often convenient to express the same rate of change in terms of the current spatial vectors. For this purpose the Equations (3.36a, b) can be inverted as,

\[
dX_1 = F^{-1}dx_1; \quad dX_2 = F^{-1}dx_2 \tag{3.40a, b}
\]

Introducing these expressions into Equation (3.38) gives the rate of change of the scalar product in terms of \(dx_1\) and \(dx_2\) as,

\[
\frac{1}{2} \frac{d}{dt}(dx_1 \cdot dx_2) = dx_1 \cdot (F^{-T} \dot{E} F^{-1})dx_2 \tag{3.41}
\]

The tensor in the expression on the right-hand side is the pushed forward spatial counterpart of \(\dot{E}\) and is known as the rate of deformation tensor \(d\) given as,

\[
d = \phi_1[\dot{E}] = F^{-T} \dot{E} F^{-1}; \quad \dot{E} = \phi^{-1}[d] = F^T dF \tag{3.42a, b}
\]

After simple algebra, a more conventional expression of the tensor \(d\) emerges as the symmetric part of \(I\) as,
\[ d = \frac{1}{2} (I + \Gamma) \quad (3.43) \]

3.6 CAUCHY STRESS TENSOR

In order to develop the concept of stress it is necessary to study the action of the forces applied by one region \( R_1 \) of the body on the remaining part \( R_2 \) of the body with which it is in contact shown by Fig. 3.6. For this purpose an element of area \( \Delta a \) normal to \( n \) in the neighborhood of spatial point \( p \) as shown in Fig. 3.6 is considered.

![Figure 3.6 Traction vector](image)

If the resultant force on this area is \( \Delta p \), the traction vector \( t \) corresponding to the normal \( n \) at \( p \) is defined as,

\[ t(n) = \lim_{\Delta a \to 0} \frac{\Delta p}{\Delta a} \quad (3.44) \]

The relationship between \( t \) and \( n \) must be such that satisfies Newton's third law of action and reaction, which is expressed as,

\[ t(-n) = -t(n) \quad (3.45) \]
And hence \( t(n) = \sigma n \), where \( \sigma \) is known as the Cauchy stress tensor, that relates the normal vector \( n \) to the traction vector \( t \) as,

\[
\sigma = \sum_{i,j=1}^{3} \sigma_{ij} e_i \otimes e_j
\]  
\[(3.46)\]

3.6.1 The Kirchhoff Stress Tensor

The internal virtual work done by the stress is expressed as,

\[
\delta W_{\text{int}} = \int \sigma : \delta \mathbf{dV}
\]  
\[(3.47)\]

\( \sigma \) and \( \delta \) in this equation are said to be work conjugate with respect to the current deformed volume in the sense that their product gives work per unit current volume. Expressing the virtual work equation in the material coordinate system an alternative work conjugate pairs of stresses and strain rates will emerge. To achieve this, the spatial virtual work equation is first expressed with respect to the initial volume and area by transforming the integrals for \( dV \) to give,

\[
\int J \sigma : \delta \mathbf{dV} = \int f_0 \cdot \delta \mathbf{dV} + \int t_0 \cdot \delta \mathbf{dA}
\]  
\[(3.48)\]

where \( f_0 = J \mathbf{f} \) is the body force per unit undeformed volume and \( t_0 = \mathbf{t} \left( \frac{da}{dA} \right) \) is the traction vector per unit initial area, where the area ratio can be obtained after some algebra as (Bonet and Wood, 1997),

\[
\frac{da}{dA} = \frac{J}{\sqrt{n \cdot bn}}
\]  
\[(3.49)\]

The internal virtual work given by the left-hand side of Equation (3.48) can be expressed in terms of the Kirchhoff stress tensor \( \tau \) as,
This equation reveals that the Kirchhoff stress tensor $\tau$ is work conjugate to the rate of deformation tensor with respect to the initial volume. However, Equation (3.50b) and the relationship $\rho = \rho_0 / J$ ensure that the work per unit mass is invariant and can be equally written in the current or initial configuration as,

$$\frac{1}{\rho} \sigma : d = \frac{1}{\rho_0} \tau : d \quad (3.51)$$

### 3.6.2 The first Piola-Kirchhoff stress tensor

The transformation resulted in the internal virtual work given in previous section is not entirely satisfactory because of its dependencies on the spatial quantities $\tau$ and $d$. To overcome this lack of consistency, using the symmetry of $\sigma$ together with Equation (3.35) for $I$ in terms of $F$ and the properties of the trace give,

$$\delta W_{\text{int}} = \int \mathbf{J} \sigma : \delta dV$$

$$= \int \mathbf{J} \sigma : (\delta \mathbf{F} F^{-1}) dV$$

$$= \int \mathbf{J} F (JF^{-1} \sigma \mathbf{F}) dV$$

$$= \int \mathbf{J} (\mathbf{F} \sigma \mathbf{F}^{-T}) : \delta \mathbf{F} dV \quad (3.52)$$

It can be observe from this equality that the stress tensor work conjugate to the rate of the deformation gradient $\dot{F}$ is the so called first Piola-Kirchhoff stress tensor given as,

$$\mathbf{P} = J \sigma \mathbf{F}^{-T} \quad (3.53)$$
3.6.3 The second Piola-Kirchhoff stress tensor

The first Piola-Kirchhoff stress tensor $P$ is an unsymmetrical two-point tensor and is not completely related to the material configuration. It is possible to obtain a totally material symmetric stress tensor, known as the second Piola-Kirchhoff stress $S$, by pulling back the spatial element of force $dp$ to give a material force vector $dP$ as,

$$dP = \phi^{-1}[dp] = F^{-1}dp$$ \hspace{1cm} (3.54)

It gives the transformed force in terms of the second Piola-Kirchhoff stress tensor $S$ and the material element of area $dA$ as,

$$dP = SdA; \quad S = JF^{-1}\sigma F^{-T}$$ \hspace{1cm} (3.55a, b)

3.6.4 Deviatoric and pressure components

In many practical applications such as modeling incompressible material, metal plasticity, soil mechanics etc, it is physically relevant to isolate the hydrostatic pressure component $p$ from the deviatoric component $\sigma'$ of the Cauchy stress tensor as,

$$\sigma = \sigma' + pI; \quad p = \frac{1}{3}tr\sigma = \frac{1}{3}\sigma : I$$ \hspace{1cm} (3.56a, b)

where the deviatoric Cauchy stress tensor $\sigma'$ satisfies $tr\sigma' = 0$. Similar decompositions can be established in terms of the first and second Piola-Kirchhoff stress tensors as,

$$P = P' + pJF^{-T}; \quad P' = Js'F^{-T}$$ \hspace{1cm} (3.57a)

$$S = S' + pJC^{-1}; \quad S' = JF^{-1}\sigma'F^{-T}$$ \hspace{1cm} (3.57b)
The tensors $S'$ and $P'$ are often referred to as the true deviatoric components of $S$ and $P$.

### 3.7 PRINCIPLE OF VIRTUAL WORK

Finite element formulations are generally established in terms of a weak form of the differential equations that are under consideration. In the context of solid mechanics this implies the use of the virtual work equation. For this, let $\delta v$ denote an arbitrary virtual velocity from the current position of the body as shown in the Fig. 3.7. The virtual work $\delta w$ per unit volume and time done by the residual force $r$ during this virtual motion is $r \cdot \delta v$, and equilibrium implies,

$$\delta w = r \cdot \delta v = 0 \quad (3.58)$$

The above scalar equation is fully equivalent to the vector equation $r = 0$. The weak statement of the static equilibrium of the body can be expressed as,

$$\delta W = \int \left( \text{div} \sigma + f \right) \cdot \delta v dv = 0 \quad (3.59)$$

![Figure 3.7 Principle of virtual work](image)

A more common and useful expression can be derived to give the divergence of the vector $\sigma \delta v$ as,
\[
\text{div}(\sigma \delta v) = (\text{div}\sigma) \cdot \delta v + \sigma : \nabla \delta v \tag{3.60}
\]

Using this equation together with the Gauss theorem enables Equation (3.59) to be rewritten as,

\[
\int_{\partial V} n \cdot \sigma \delta v \, da - \int_{V} \nabla \delta v \, dv + \int_{V} f \cdot \delta v \, dv = 0 \tag{3.61}
\]

After some simple algebra (Bonet and Wood, 1997) the Equation (3.60) becomes,

\[
\int_{V} \sigma : \delta dv = \int_{\partial V} f \cdot \delta v \, da + \int_{V} t \cdot \delta v \, dv \tag{3.62}
\]

Finally, expressing the virtual velocity gradient in terms of the symmetric virtual rate of deformation $\delta \bar{\mathbf{I}}$ and the antisymmetric virtual spin tensor $\delta \mathbf{w}$ and taking into account the symmetry of $\sigma$ gives the spatial virtual work equation as,

\[
\delta W = \int_{V} \sigma : \delta \bar{\mathbf{I}} \, dv - \int_{\partial V} f \cdot \delta \mathbf{w} \, da - \int_{V} t \cdot \delta \mathbf{w} \, dv = 0 \tag{3.63}
\]

This fundamental scalar equation states the equilibrium of a deformable body and will become the basis for the finite element discretization.

### 3.8 HYPERELASTICITY

Materials for which the constitutive behavior is only a function of the current state of deformation are generally known as elastic. Under such conditions, any stress measure at a particle $X$ is a function of the current deformation gradient $F$ associated with that particle. The deformation gradient $F$, together with its conjugate first Piola-Kirchhoff stress measure $P$, will be retained in order to define the basic material relationships. Consequently, elasticity can be generally expressed as,
where the direct dependency upon \( X \) allows for the possible inhomogeneity of the material. In the special case when the work done by the stresses during a deformation process is dependent only on the initial state at time \( t_0 \) and the final configuration at time \( t \), the behavior of the material is said to be path-independent and the material is termed hyperelastic (Bonet and Wood, 1997). Because of this and since \( P \) is work conjugate with the rate of deformation gradient \( \dot{F} \), a stored strain energy function or elastic potential \( \psi \) per unit undeformed volume can be established as the work done by the stresses from the initial to the current position as,

\[
\psi(F(X), X) = \int_0^t \mathbf{P}(\mathbf{F}(X), X) : \dot{\mathbf{F}} dt ; \quad \psi = \mathbf{P} : \dot{\mathbf{F}} \tag{3.65}
\]

### 3.8.1 The material or Lagrangian elasticity tensor

The relationship between \( S \) and \( \mathbf{C} = \frac{1}{2} (\mathbf{C} - \mathbf{I}) \), will invariably be nonlinear. Within the framework of a potential Newton-Raphson solution process, this relationship will need to be linearized with respect to an increment \( \mathbf{u} \) in the current configuration. Using the chain rule, a linear relationship between the directional derivative of \( S \) and the linearized strain \( DE[\mathbf{u}] \) can be obtained, initially in a component form as,

\[
DS_{ij}[\mathbf{u}] = \frac{d}{d e} S_{ij} (E_{kl}[\phi + e \mathbf{u}])
\]

\[
= \sum_{k,l=1}^3 \frac{\delta S_{ij}}{\delta E_{kl}} \frac{d}{d e} E_{kl}[\phi + e \mathbf{u}]
\]

\[
= \sum_{k,l=1}^3 \frac{\delta S_{ij}}{\delta E_{kl}} DE_{kl}[\mathbf{u}] \tag{3.66}
\]
This relationship between the directional derivatives of $S$ and $E$ is more concisely expressed as,

$$DS[u] = C : DE[u]$$  \hspace{1cm} (3.67)

where the symmetric fourth-order tensor $C$, known as the Lagrangian or material elasticity tensor, is defined by the partial derivatives as,

$$C_{ijkl} = \frac{\partial C_{ij}}{\partial \varepsilon_{kl}}$$

For convenience these expressions are often abbreviated as,

$$C = \frac{\delta S}{\delta E} = 2 \frac{\delta S}{\delta C} = C_{ijkl}$$  \hspace{1cm} (3.69)

### 3.8.2 The spatial or Eulerian elasticity tensor

To find a spatial equivalent to Equation (3.67) and also to find a relationship between the linearized Cauchy stress and the linearized Almansi strain, an easier route is to interpret Equation (3.67) in a rate form and apply the push forward operation to the resulting equation. This is achieved by linearizing $S$ and $E$ in the direction of $v$, rather than $u$. Using the relationship of $DS[v] = \dot{S}$ and $DE[v] = \dot{E}$ gives,

$$\dot{S} = C : \dot{E}$$  \hspace{1cm} (3.70)

It is now possible to obtain the spatial equivalent of the material linearized constitutive Equation (3.70) as,

$$\sigma^\circ = C : d$$  \hspace{1cm} (3.71)
where \( \varepsilon \) is the Eulerian or spatial elasticity tensor.

### 3.8.3 Isotropic hyperelasticity material description

The hyperelastic constitutive equations are restricted in their application. Isotropy is defined by requiring the constitutive behavior to be identical in any material direction (Bonet and Wood, 1997). This implies that the relationship between \( \psi \) and \( C \) must be independent of the material axes chosen and, consequently, \( \psi \) must only be a function of the invariants of \( C \) as,

\[
\psi(C(X), X) = \psi(I_c, II_c, III_c, X) \tag{3.72}
\]

where the invariants of \( C \) are defined here as,

\[
\begin{align*}
I_c &= tr C = C : I \\
II_c &= tr CC = C : C \\
III_c &= det C = J^2
\end{align*}
\tag{3.73}
\]

As a result of the isotropic restriction, the second Piola-Kirchhoff stress tensor can be rewritten as,

\[
S = 2 \frac{\delta \psi}{\delta C} = 2 \frac{\delta \psi}{\delta I_c} \frac{\delta I_c}{\delta C} + 2 \frac{\delta \psi}{\delta II_c} \frac{\delta II_c}{\delta C} + 2 \frac{\delta \psi}{\delta III_c} \frac{\delta III_c}{\delta C} \tag{3.74}
\]

### 3.8.4 Isotropic hyperelasticity spatial description

In design practices the Cauchy stresses are of the most engineering significance. These can be obtained indirectly from the second Piola-Kirchhoff stresses as,

\[
\sigma = J^{-1}FSF^T \tag{3.75}
\]
Substituting $S$ from Equation (3.74) and noting that the left Cauchy-Green tensor is $B = FF^T$ gives,

$$
\sigma = 2J^{-1}\psi_1 B + 4J^{-1}\psi_2 B^2 + 2J\psi_3 I
$$

(3.76)

In this equation $\psi_1$, $\psi_2$, and $\psi_3$ still involve derivatives with respect to the invariants of the material tensor $C$. Nevertheless it is easy to show that the invariants of $B$ are identical to the invariants of $C$, as the following expressions demonstrate,

$$
\begin{align*}
I_1 &= tr[B] = tr[FF^T] = tr[F'F] = tr[C] = I_C \quad (3.77a) \\
II_2 &= tr[BB] = tr[FF^TFF^T] = tr[F'F^2F'] = tr[CC] = II_C \quad (3.77b) \\
III_3 &= det[B] = det[FF^T] = det[F'F] = det[C] = III_C \quad (3.77c)
\end{align*}
$$

Consequently the terms $\psi_1$, $\psi_2$, and $\psi_3$ in Equation (3.76) are also the derivatives of $\psi$ with respect to the invariants of $B$. 


4.1 GENERAL

The elastic constitutive model works well when time dependent effects can be neglected. However, the experimental results depicted in Chapter 2 shows the cases where time dependent effects cannot be neglected in simulating responses of HDR and NR. The classic material model representing time dependent effects is based on viscoelasticity that incorporates aspects of both fluid behavior and solid behavior together. This chapter presents the physical characteristics for viscoelastic behavior as well as the basic mechanical analogs that can be used to derive constitutive equations.

4.2 CHARACTERISTICS OF A VISCOELASTIC MATERIAL

A viscoelastic material has an elastic component and a viscous component. The viscosity of a viscoelastic material gives the substance a strain rate dependent on time. Purely elastic materials do not dissipate energy in the form of heat when a load is applied, then removed. However, a viscoelastic material loses energy when a load is applied, then removed. Hysteresis is observed in the stress-strain curve, with the area of the loop being equal to the energy lost during the loading cycle. Since viscosity is the resistance to thermally activated plastic deformation, a viscous material will lose energy through a loading cycle. Plastic deformation results in lost energy, which is uncharacteristic of a purely elastic material's reaction to a loading cycle.

Specifically, viscoelasticity is a molecular rearrangement. When a stress is applied to a viscoelastic material like rubber, parts of the long polymer chain change position. This movement or rearrangement is called creep. Polymers remain a solid material even when these parts of their chains are rearranging in order to accompany the stress, and as this occurs, it creates a back stress in the material. When the back stress is the same magnitude as the applied
stress, the material no longer creeps. When the original stress is taken away, the accumulated back stresses will cause the polymer to return to its original form.

\[ \sigma = E \varepsilon \]  (4.1)

Figure 4.1 Stress-Strain Curves for a purely elastic material (a) and a viscoelastic material (b).

The area inside the loop is the amount of heat lost due to the loading and unloading cycle. It is equal to \( \int \sigma d\varepsilon \) where \( \sigma \) is stress and \( \varepsilon \) is strain.

4.3 MATHEMATICAL MODELS FOR LINEAR VISCOELASTIC RESPONSE

Viscoelastic materials such as amorphous polymers, rubbers etc can be modeled in order to determine their stress or strain interactions as well as their temporal dependencies. These models, which include the Maxwell Model, the Kelvin-Voigt Model, the Weichert model and the Standard Linear Solid Model (Roylance, 2001), are used to predict a material's response under different loading conditions. Viscoelastic behavior is comprised of elastic and viscous components modeled as linear combinations of springs and dashpots, respectively.

The elastic component can be modeled as springs of elastic constant \( E \), given the formula:

\[ \sigma = E \varepsilon \]  (4.1)
where $\sigma$ is the stress, $E$ is the elastic modulus of the material, and $\varepsilon$ is the strain that occurs under the given stress.

The viscous components can be modeled as dashpots such that the stress-strain rate relationship can be given as:

$$\sigma = \eta \frac{d\varepsilon}{dt} \quad (4.2)$$

where $\sigma$ is the stress, $\eta$ is the viscosity of the material, and $\frac{d\varepsilon}{dt}$ is the time derivative of strain.

The relationship between stress and strain can be simplified for specific stress rates. For high stress states/short time periods, the time derivative components of the stress-strain relationship dominate. Conversely, for low stress states/longer time periods, the time derivative components are negligible and the dashpot can be effectively removed from the system.

4.3.1 The Maxwell Spring-Dashpot Model

The time dependence of viscoelastic response can be described by ordinary differential equation in time. A convenient way of developing this relation and also to visualize the molecular motions is to employ “spring-dashpot” models. This model uses “Hookean” springs shown in Fig. 4.2 and described by

$$\sigma = k\varepsilon \quad (4.3)$$

where $\sigma$ and $\varepsilon$ are analogous to the spring force and displacement, and the spring constant $k$ is analogous to the Young's modulus $E$, therefore $k$ has the unit of $\text{N/m}^2$. The spring models the instantaneous bond deformation of the material, and its magnitude will be related to the fraction of the mechanical energy stored reversibly as strain energy.
The entropic uncoiling process is fluid like in nature, and can be modeled by a "Newtonian dashpot" also shown in Fig. 4.2.

It produces stress not at a strain but at a strain rate, and is given by

$$\sigma = \eta \dot{\varepsilon}$$

(4.4)

here the overdot denotes time differentiation and $\eta$ is a viscosity with units of $N \cdot s / m^2$. It is more convenient to express the ratio of viscosity to stiffness as

$$\tau = \eta / k$$

(4.5)

The unit of $\tau$ is time, and is a useful measure of the material's viscoelastic response.

Figure 4.2 Hookean spring (left) and Newtonian dashpot (right).

The Maxwell solid as shown in Fig. 4.3 is a mechanical model in which a Hookean spring and a Newtonian dashpot are connected in series.

Figure 4.3 The Maxwell model

The spring should be visualized as representing the elastic or energetic component of the response, while the dashpot represents the conformational or entropic component. The stress on each element in the Maxwell is the same and equal to the imposed stress, while the total strain is the sum of the strain in each element:
Here the subscripts $s$ and $d$ represent the spring and dashpot, respectively.

To find a single equation relating the stress to the strain, it is convenient to differentiate the strain equation and then write the spring and dashpot strain rates in terms of the stress:

$$\dot{\varepsilon} = \dot{\varepsilon}_s + \dot{\varepsilon}_d = \frac{\dot{\sigma}}{k} + \frac{\sigma}{\eta}$$ \hspace{1cm} (4.6)

Multiplying by $k$ and using $\tau = \eta/k$:

$$k \dot{\varepsilon} = \dot{\sigma} + \frac{1}{\tau} \sigma$$ \hspace{1cm} (4.7)

This expression is the constitutive equation for the Maxwell material that relates the stress to the strain (Roylance, 2001).

### 4.3.2 The Standard Linear Solid Model (Maxwell form)

The Maxwell model permits unrestricted flow which most of the polymers do not exhibit. For more typical polymers whose conformational change is eventually limited by the network of entanglements or other types of junction points, more elaborate spring-dashpot models are required.

Placing a spring in parallel with the Maxwell unit gives a very useful model known as the “Standard Linear Solid” (S.L.S) as shown in Fig. 4.4.
The spring has stiffness $k_e$ as it provides "equilibrium" or rubbery stiffness that remains after the stresses in the Maxwell arm have relaxed away as the dashpot extends. In this arrangement the Maxwell arm and the parallel spring $k_e$ experience the same strain, and the total stress $\sigma$ is the sum of stress in each arm:

$$\sigma = \sigma_e + \sigma_m$$  \hspace{1cm} (4.8)

The governing constitutive relation of this model is (Roylance, 2001):

$$\frac{d\varepsilon}{dt} = \frac{k_i}{\eta} \left( \frac{\eta}{k_i} \frac{d\sigma}{dt} + \sigma - k_e \varepsilon \right)$$  \hspace{1cm} (4.9)

Under a constant stress, the modeled material will instantaneously deform to some strain, which is the elastic portion of the strain, and after that it will continue to deform and asymptotically approach a steady state strain, which is also the viscous part of the strain.

4.3.3 The Wiechert Model

The Wiechert model also known as the generalized Maxwell model or the Maxwell-Wiechert model, is the most general form of the models described above (Roylance, 2001).
It takes into account that relaxation does not occur at a single time, but at a distribution of times. Due to molecular segments of different lengths with shorter ones contributing less than longer ones, there is a varying time distribution. The Wiechert model shows this by having as many spring-dashpot Maxwell elements as are necessary to accurately represent the distribution.

The total stress $\sigma$ transmitted by the model is the stress in the isolated spring (of stiffness $k_i$) plus that in each of the Maxwell spring-dashpot arms:

$$\sigma = \sigma_i + \sum_j \sigma_j$$  \hspace{1cm} (4.10)

4.3.4 The Kelvin-Voigt Model

The Kelvin-Voigt model also known as the Voigt model, consists of a Newtonian damper and Hookean elastic spring connected in parallel as shown in Fig. 4.6
It is used to explain stress relaxation behaviors of polymers. The constitutive relation is expressed as a linear first order differential equation as:

\[ \sigma(t) = k\varepsilon(t) + \eta \frac{d\varepsilon}{dt} \] (4.11)

This model represents a solid undergoing a reversible viscoelastic strain. Upon application of a constant stress, the material deforms at a decreasing rate, asymptotically approaching the steady-state strain. When the stress is released, the material gradually relaxes to its undeformed state.
5.1 INTRODUCTION

During the process of vulcanization a large amount of fillers (about 30%) is added in the HDR in the form of carbon black, silica, oils and some other particles (Kelly, 1997; Yoshida et al., 2004). This makes HDR to exhibit a strong nonlinear rate-dependent response under monotonic loadings and to show significant hysteresis effects or energy dissipation during cyclic loads. This phenomenon is also reported by Amin (2001), Amin et al. (2002, 2003) and Wiraguna (2003). All these experimental observation suggest a constitutive model that can represent these nonlinear rate-dependent responses adequately. This chapter introduces a model of this type that follows from the concept of Huber and Tsakmakis (2000a, b). It is based on the multiplicative decomposition of the deformation gradient and the additive split of the free energy as introduced by Lubliner (1985).

5.2 CONSTITUTIVE MODEL

In the finite strain kinematics, the local mapping between the initial and current configuration of a deformable body under motion is described by the deformation gradient tensor \( F \) as,

\[
F = \sum_{a=1}^{3} \lambda_{a} \mathbf{n}_{a} \otimes \mathbf{N}_{a} \tag{5.1}
\]

where \( \lambda_{a} = 1 + \frac{\Delta L_{a}}{L_{a}} \) the stretches in the three principal directions are, \( L_{a} \) are the undeformed lengths of material line elements and \( \Delta L_{a} \) their changes, \( \mathbf{N}_{a} \) and \( \mathbf{n}_{a} \) are the material and spatial vector triads.
The left Cauchy-Green deformation tensor $B$ is obtained as (Bonet and Wood, 1997),

$$B = FF^T = \sum_{\alpha=1}^{d^2} A_{\alpha} n_{\alpha} \otimes n_{\alpha}$$ \hspace{1cm} (5.2)

Figure 5.1. Multiplicative decomposition of the deformation gradient. (After Amin et al. 2006a)

$I_B, II_B$ and $III_B$ are the invariants of $B$ and given as,

$$I_B = tr B$$ \hspace{1cm} (5.3a)

$$II_B = \frac{1}{2} \left( (tr B)^2 - tr(B^2) \right)$$ \hspace{1cm} (5.3b)

$$III_B = det B$$ \hspace{1cm} (5.3c)

The velocity gradient $L$ and the rate of deformation tensor $D$ can be defined as,

$$L = \dot{F} F^{-1}$$ \hspace{1cm} (5.4)

$$D = \frac{1}{2} \left( L + L^T \right)$$ \hspace{1cm} (5.5)

To represent the rate dependent material behavior, the total strain has to be decomposed into elastic and inelastic parts denoted by $e_r$ and $e_i$ as shown in Fig. 5.1.
The corresponding relation in the theory of finite strains can be attained through the multiplicative decomposition of the deformation gradient $F$ into an elastic part $F_e$ and an inelastic part $F_i$ as,

$$ F = F_e F_i $$ \hspace{1cm} (5.6)

Rubber-like materials are assumed to be incompressible, which is also supported by the experimental observations presented in Amin et al. (2003) which leads to,

$$ \det F = \det F_e = \det F_i = 1 $$ \hspace{1cm} (5.7)

The left Cauchy-Green tensors $B_e$ and $B_i$, associated with elastic and inelastic deformations are given as,

$$ B_e = F_e F_e^T, \quad B_i = F_i F_i^T $$ \hspace{1cm} (5.8)

By calculating the material time rate $\dot{B}_e = \dot{F}_e F_e^T + F_e \dot{F}_e^T$ and replacing the rate of the elastic part of deformation gradient using $\dot{F}_e = \frac{d}{dt} \left( FF_e^{-1} \right) = LF_e - F_e \dot{L}_e$, one obtains,

$$ \dot{B}_e = -2F_e \dot{F}_e F_e^T + B_e \dot{L}_e^T + L B_e $$ \hspace{1cm} (5.9)

where the inelastic velocity gradient $\dot{L}_i$ and its symmetric part $\dot{D}_i$ are defined as,

$$ \dot{L}_i = \dot{F}_i F_i^{-1} $$ \hspace{1cm} (5.10)

$$ \dot{D}_i = \frac{1}{2} \left( \dot{L}_i + \dot{L}_i^T \right) $$ \hspace{1cm} (5.11)
For incompressible materials $\det F = 1$, which implies the weighted Cauchy Stress $S = (\det F)\mathbf{T}$ is equal to the Cauchy stress, i.e., $S = \mathbf{T}$. The incompressibility constraint also implies an additive constitutively non-determined contribution $-p\mathbf{I}$ to the stress. As a result one obtains,

$$ S = -p\mathbf{I} + S_e $$

(5.12)

where $p$ is the hydrostatic pressure which is to be determined from the boundary conditions of the problem. From the Zener model, the extra stress $S_e$ is the sum of a rate-independent equilibrium stress $S_e^{(e)}$ and rate-dependent overstress $S_e^{(oc)}$ given as,

$$ S_e = S_e^{(e)} + S_e^{(oc)} $$

(5.13)

To formulate the constitutive relations of the Equation (5.13), it is common practice to evaluate the isothermal form of Clausius Duhem inequality (Coleman and Gurtin, 1967)

$$ -\rho_u \dot{\Psi} + S_e \cdot \mathbf{L} \geq 0 $$

(5.14)

where $\rho_u$ is the mass density of the material in the reference configuration and $\Psi$ is the Helmholtz free energy per unit mass. For rubber like material an additive split of the free energy into the sum of an equilibrium part $W^{(e)}$ and non-equilibrium part $W^{(oe)}$ is proposed as,

$$ \rho_u \Psi = W^{(e)}(I_B, II_B) + W^{(oe)}(I_{B}, II_{B}) $$

(5.15)

For isotropic tensor functions the time rate of the free energy is given by Haupt (2000) and can be expressed as,
Equation (5.16) can also be written as,

\[
\rho \dot{\psi} = \frac{\partial W^{(f)}}{\partial \dot{I}_B} \dot{I}_B + \frac{\partial W^{(v)}}{\partial \dot{I}_B} \dot{I}_B + \frac{\partial W^{(0e)}}{\partial \dot{I}_{B_e}} \dot{I}_{B_e} + \frac{\partial W^{(0e)}}{\partial \dot{I}_{B_e}} \dot{I}_{B_e} \tag{5.16}
\]

Equation (5.17) indicates that the free energy is proportional to the time rate of the Cauchy-Green tensors \( B \) and \( B_e \). The velocity gradient can be decomposed into the sum of pure elastic and a mixed part as,

\[
L = F_e F_e^{-1} + F_e \dot{F}_e F_e^{-1} F_e^{-1} = L_e + F_e \dot{L}_e F_e^{-1} \tag{5.18}
\]

The stress power splits into the power of the equilibrium stress and the power of the overstress with respect to the elastic and inelastic deformations as,

\[
S_B \cdot L = S_B^{(e)} \cdot L + S_B^{(0e)} \cdot L_e + (F_e^{-1} S_B^{(0e)} F_e) \cdot \dot{L}_e \tag{5.19}
\]

By calculating the time rates \( \dot{B} = \frac{d}{dt} (FF^T) = LB + BL^T \) and \( \dot{B}_e = \frac{d}{dt} (F_e F_e^T) = L_e B_e + B_e L_e^T \), the total and elastic velocity gradients can be expressed as,

\[
L = \dot{B} B^{-1} - B L^T B^{-1} \quad \text{and} \quad L_e = \dot{B}_e B_e^{-1} - B_e L_e^T B_e^{-1} \tag{5.20}
\]

Considering the isotropic function of \( B \) and after some calculation finally leads to,

\[
S_B^{(e)} \cdot L = \frac{1}{2} (B^{-1} S_B^{(e)} \cdot B \tag{5.21}
\]
A similar augmentation leads to the expression for the elastic power of the over stress as,

$$ S_E^{(OE)} \cdot L_e = \frac{1}{2} (B_e^{-1} S_E^{(OE)}) \cdot \dot{B_e} \quad (5.22) $$

Inserting Equations (5.21) and (5.22) into Equation (5.19) one obtains,

$$ S_e \cdot L = \frac{1}{2} (B^{-1} S_E^{(OE)}) \cdot \dot{B} + \frac{1}{2} (B_e^{-1} S_E^{(OE)}) \cdot \dot{B_e} + (F_e^{-1} S_E^{(OE)} F_e) \cdot \dot{D} $$

$$ (5.23) $$

the stress power contains two terms which are proportional to the time rates of the total and the elastic left Cauchy-Green tensors. Inserting Equation (5.19) and Equation (5.23) into Equation (5.14) and rearranging the terms leads to

$$ \left\{ \left\{ \frac{1}{2} (B^{-1} S_E^{(OE)}) - \left( \frac{\delta W^{(E)}}{\delta I_B} - 1 + \frac{\delta W^{(E)}}{\delta I_B} \left( I_B B^{-1} - B^{-2} \right) \right) \right\} \cdot \dot{B} + (F_e^{-1} S_E^{(OE)} F_e) \cdot \dot{D} 
+ \left\{ \frac{1}{2} (B_e^{-1} S_E^{(OE)}) \right\} \cdot \left( \frac{\delta W^{(OE)}}{\delta I_{B_e}} - 1 + \frac{\delta W^{(OE)}}{\delta I_{B_e}} \left( I_B B^{-1} - B^{-2} \right) \right) \right\} \cdot \dot{B} \geq 0 $$

$$ (5.24) $$

In order to satisfy Equation (5.24) the corresponding factors of proportionality have to vanish which leads to the following stress-strain relations:

$$ S_E^{(OE)} = 2 \frac{\delta W^{(E)}}{\delta I_B} B + 2 \frac{\delta W^{(E)}}{\delta I_B} \left( I_B 1 - B^{-1} \right) $$

$$ (5.25) $$

$$ S_E^{(OE)} = 2 \frac{\delta W^{(OE)}}{\delta I_{B_e}} B_e + 2 \frac{\delta W^{(OE)}}{\delta I_{B_e}} \left( I_B 1 - B_e^{-1} \right) $$

$$ (5.26) $$

As the hydrostatic pressure in Equation (5.12) is constitutively undetermined, the terms which are proportional to the unit tensor in Equations (5.25) and (5.26) can be omitted.
To represent the behavior of rubber, the strain energies $W^{(e)}$ and $W^{(o_e)}$ have to be adequate to represent both the equilibrium and instantaneous responses of the material.

The finite strain viscoelasticity presented above consists of non-linear springs in two parallel branches to describe the equilibrium and instantaneous responses that correspond to infinitely slow and fast rate of deformation. These are rate-independent elastic responses that bound a domain where viscosity effects come into play (Huber and Tsakmakis, 2000a).

An adequate hyperelasticity model formulated on the basis of experimental observations on HDR an NR under compression and shear (Amin et al., 2002, 2006b) is required to represent the non-linear rate independent elastic response. The following two equations show the relation for both the equilibrium and the overstress response, respectively:

\[
W^{(e)}(I_B, I_{B^e}) = C_2^{(e)}(I_B - 3) + C_3^{(e)} N + 1 (I_B - 3)^{N+1} + C_4^{(e)} M + 1 (I_B - 3)^{M+1} + C_2^{(e)} (II_B - 3)
\]

\[
W^{(o_e)}(I_{B^e}, I_{B^2}) = C_2^{(o_e)}(I_{B^e} - 3) + C_3^{(o_e)} N + 1 (I_{B^e} - 3)^{N+1} + C_4^{(o_e)} M + 1 (I_{B^e} - 3)^{M+1} + C_2^{(o_e)} (II_{B^e} - 3)
\]

where $C_2^{(e)}, C_3^{(e)}, C_4^{(e)}, C_2^{(o_e)}$, $M$ and $N$ are the material constants of the equilibrium relation while $C_2^{(o_e)}, C_3^{(o_e)}, C_4^{(o_e)}, C_2^{(o_e)}$ are those of the overstress.
From Huber and Tsakmakis (2000a), the rate of left Cauchy-Green deformation tensor is given as,

$$\dot{B}_e = B_e L^* + \dot{L} B_e - \frac{2}{\eta} B_e (S_e - S_e^{(e)})$$  \hspace{1cm} (5.31)

The (\dot{}) indicates the material time derivative and $\eta$ is the material viscosity represented by the dashpot.

### 5.3 PARAMETER IDENTIFICATION

In order to represent the stress-strain relationship of the rubber like material by the constitutive model, identification of the material parameters is very important. The experimental works carried on HDR and NR with different strain-rates under compression (Amin et al 2002) and shear (Wiraguna 2003) are used.

Figures 5.2 and 5.3 show the equilibrium locus and the monotonic loadings paths form the compression test on HDR and NR (Amin et al 2002). The figures displayed a diminishing trend in the increase of the stress response at higher stretch rates indicating the approach of the instantaneous state. The stress response corresponds to the 0.88/s and 0.65/s stretch rates can be considered as the neighborhood of the instantaneous state for HDR and NR respectively.

Figures 5.4 and 5.5 shows the positive loading paths of simple shear test by Wiraguna (2003) to determine the material parameters. It is seen that the stresses increase with increasing strain rate due to viscosity effect. At higher strain rates, a diminishing trend in the increase of stress response was observed indicating the approach of instantaneous state. A stress response in HDR at 0.25/s and in NR at 0.05/s can be considered as the neighborhood of the instantaneous state.

Based on these experimental data for shear and compression, identification of material parameters was carried out by (Wiraguna et al 2003) using the least square method. These parameters excluding $M$ and $N$ are determined for the
equilibrium and instantaneous state independently. Hence, two sets of material parameters are considered, in one case the parameters $M$ and $N$ are identified from the experimental data at the equilibrium state and the other from those at the instantaneous state.

To determine the material viscosity parameter $\eta$, the results of simple relaxation tests (Amin et al 2002) are used. In these tests, the loading stretch rate for each specimen was maintained at 0.5/s, which is in the neighborhood of instantaneous states for all the materials. Figures 5.6 and 5.7 illustrate the fundamental stress relaxation phenomena of the materials from the instantaneous to the equilibrium state and thus include the entire viscosity domain. To find an adequate value of $\eta$, the numerical trials of the rate-dependent hyperelastic model (Amin et al 2002) was carried out and experimental data were compared with those of numerical results.

The values of the material parameters used in computer coding of the viscoelasticity models are presented in Tables 5.1 and 5.2.
Figure 5.2. Monotonic compression test stretch-stress responses of HDR

Figure 5.3. Monotonic compression test stretch-stress responses of NR
Figure 5.4. Monotonic path of simple shear of HDR at different strain rates

Figure 5.5. Monotonic path of simple shear of NR at different strain rates
Figure 5.6. Stretch history and stretch-stress response in multi step relaxation test of HDR

Figure 5.7. Stretch history and stretch-stress response in multi step relaxation test of NR
Table 5.1. Elasticity parameters for HDR

<table>
<thead>
<tr>
<th>Responses</th>
<th>C2 MPa</th>
<th>C3 MPa</th>
<th>C4 MPa</th>
<th>C5 MPa</th>
<th>M</th>
<th>N</th>
<th>η MPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilibrium</td>
<td>0.145</td>
<td>1.182</td>
<td>-5.297</td>
<td>4.262</td>
<td>0.06</td>
<td>0.27</td>
<td></td>
</tr>
<tr>
<td>Instantaneous</td>
<td>0.166</td>
<td>2.477</td>
<td>-11.689</td>
<td>9.707</td>
<td>0.06</td>
<td>0.27</td>
<td></td>
</tr>
<tr>
<td>Overstress</td>
<td>0.021</td>
<td>1.295</td>
<td>-6.392</td>
<td>5.445</td>
<td>3.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.2. Elasticity parameters for NR

<table>
<thead>
<tr>
<th>Responses</th>
<th>C2 MPa</th>
<th>C3 MPa</th>
<th>C4 MPa</th>
<th>C5 MPa</th>
<th>M</th>
<th>N</th>
<th>η MPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilibrium</td>
<td>0.095</td>
<td>0.019</td>
<td>-0.515</td>
<td>0.754</td>
<td>0.15</td>
<td>1.29</td>
<td></td>
</tr>
<tr>
<td>Instantaneous</td>
<td>0.176</td>
<td>0.043</td>
<td>-0.861</td>
<td>1.056</td>
<td>0.15</td>
<td>1.29</td>
<td></td>
</tr>
<tr>
<td>Overstress</td>
<td>0.081</td>
<td>0.024</td>
<td>-0.346</td>
<td>0.302</td>
<td></td>
<td></td>
<td>3.50</td>
</tr>
</tbody>
</table>
6.1 GENERAL

In order to obtain a proper response of the material under different strain rates, an adequate viscoelastic model is required. In Chapter 4 a finite strain viscoelastic model (Huber and Tsakmakis, 2000a) has been presented along with the improved strain energy function (Amin et al 2002) to represent the elastic response. This chapter is devoted to discuss how the over-stress component of the extra part can be obtained from the rate of left Cauchy-Green deformation tensor (Huber and Tsakmakis, 2000a). Also how the stress component can be derived from strain energy function has been discussed. The improved strain energy function (Amin et al 2002, Wiraguna et al 2003) will be utilized for this purpose.

6.2 DERIVATION OF CAUCHY STRESS TENSOR

For isotropic elastic materials, the strain energy function \( W \) can be expressed as a function of invariants of a deformation tensor \( I \)

\[
W = W(I, II, III)
\]

(6.1)

When the material is incompressible, the third invariant \( III = 1 \), and \( W \) is represented as a function of \( I \) and \( II \) only:

\[
W = W(I, II)
\]

(6.2)

The deformation invariants can be written in terms of the principal stretches \( \lambda_i, (i = 1, 2, 3) \) as,
\begin{align*}
I &= \text{tr} \mathbf{B} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
II &= \frac{1}{2} \left[ \left( \text{tr} \mathbf{B} \right)^2 - \text{tr} \left( \mathbf{B} \mathbf{B} \right) \right] = (\lambda_1 \lambda_2)^2 + (\lambda_2 \lambda_3)^2 + (\lambda_3 \lambda_1)^2 \\
III &= \det \mathbf{B} = (\lambda_1 \lambda_2 \lambda_3)^2
\end{align*}

(6.3)

where the stretch \( \lambda \) is defined as the ratio of current length \( l \) to that of the initial length \( l_0 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{stretch_def.png}
\caption{Definition of stretch, \( \lambda = \frac{l}{l_0} \)}
\end{figure}

From Truesdell and Noll (1992), it follows that the weighted Cauchy stress \( \mathbf{S} \) is decomposed into volumetric part \( -p\mathbf{1} \) and extra part \( \mathbf{S}_E \) as:

\begin{equation}
\mathbf{S} = -p\mathbf{1} + \mathbf{S}_E
\end{equation}

(6.4)

where \( \mathbf{1} \) is the identity tensor, \( p \) is the hydrostatic pressure, and the subscript \( "E" \) denotes the deviatoric part. Following the idea of the Zener model, the extra stress \( \mathbf{S}_E \) can be decomposed into the sum of a rate-independent equilibrium stress \( \mathbf{S}_E^{(p)} \) and a rate-dependent overstress \( \mathbf{S}_E^{(o)} \) as,

\begin{equation}
\mathbf{S}_E = \mathbf{S}_E^{(p)} + \mathbf{S}_E^{(o)}
\end{equation}

(6.5)

Now the total stress is obtained by adding the Equation (6.4) and (6.5) as,

\begin{equation}
\mathbf{S} = -p\mathbf{1} + \mathbf{S}_E^{(p)} + \mathbf{S}_E^{(o)}
\end{equation}

(6.6)
This can also be written in the form of strain energy density function (Huber and Tsakmakis, 2000a) as,

\[
S = -\rho l + 2 \frac{\partial W^{(e)}}{\partial l} B - 2 \frac{\partial W^{(e)}}{\partial l} B^{-1} + 2 \frac{\partial W^{(oe)}}{\partial l} B^{-1} - 2 \frac{\partial W^{(oe)}}{\partial l} B^{-1} \quad (6.7)
\]

For uniaxial compression case, the Cauchy stress tensor \( S \) can be expressed as,

\[
S = \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (6.8)
\]

The deformation gradient tensor \( F \) can be written as,

\[
F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad (6.9)
\]

Considering isotropy and incompressibility we have \( \lambda_2^2 = \lambda_3^2 = \lambda_1^{-1} \), which leads the strain invariants to be,

\[
I_1 = \frac{2}{\lambda_1}, \quad I_2 = \frac{1}{\lambda_1^2} + 2\lambda_1, \quad I_3 = 1 \quad (6.10)
\]

Again by using the Equation (6.8) the left Cauchy-Green deformation tensor \( B = FF^T \) can be obtained as,

\[
B = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \frac{1}{\lambda_1} & 0 \\ 0 & 0 & \frac{1}{\lambda_1} \end{bmatrix} \quad (6.11)
\]
And the inverse of left Cauchy-Green deformation tensor is obtained as,

\[
B^{-1} = \begin{pmatrix}
\frac{1}{\lambda_1^2} & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \lambda_1
\end{pmatrix}
\]  

(6.12)

For uniaxial case, we find that \( S_{22} = S_{33} = 0 \); hence

\[
S_{22} = 0 = -p + S_{22E}
\]

(6.13)

Now the volumetric part is obtained as,

\[
p = S_{22E}
\]

(6.14)

Again the deformation gradient \( F \) can be decomposed into an elastic part \( F_e \) and an inelastic part \( F_i \) as,

\[
F = F_e F_i
\]

(6.15)

Now the pressure component of Equation (6.14) can be explicitly obtained as,

\[
p = \frac{2}{\lambda_e} \frac{\partial W^{(e)}}{\partial I_B} - 2\lambda_e \frac{\partial W^{(e)}}{\partial \lambda_e} + 2\lambda_i \frac{\partial W^{(ie)}}{\partial \lambda_e} - 2\lambda_i \frac{\partial W^{(ie)}}{\partial \lambda_i}
\]

(6.16)

and the extra part as,

\[
S_{11E} = 2\lambda_e^2 \frac{\partial W^{(e)}}{\partial I_B} - 2\lambda_e \frac{\partial W^{(e)}}{\partial \lambda_e} + 2\lambda_i^2 \frac{\partial W^{(ie)}}{\partial \lambda_i} - 2\lambda_i^2 \frac{\partial W^{(ie)}}{\partial \lambda_i}
\]

(6.17)

Now the Cauchy stress can be obtained as,
Using Equation (6.16) and (6.17), Equation (6.18) can be written as,

\[
S_{11} = -p + S_{11E} \tag{6.18}
\]

For simple shear, the shear stress tensor \( \tau \) can be expressed as,

\[
\tau = \begin{pmatrix}
S_{11} & S_{12} & 0 \\
S_{21} & S_{22} & 0 \\
0 & 0 & S_{33}
\end{pmatrix} \tag{6.20}
\]

The deformation gradient tensor \( F \) can be written as,

\[
F = \begin{pmatrix}
1 & \gamma & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \tag{6.21}
\]

The left Cauchy-Green deformation tensor \( B \) is obtained as,
\[
B = \begin{pmatrix}
1 + \gamma^2 & \gamma & 0 \\
\gamma & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  
(6.22)

and the inverse of \( B \) as,

\[
B^{-1} = \begin{pmatrix}
1 & -\gamma & 0 \\
-\gamma & 1 + \gamma^2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  
(6.23)

For simple shear deformation the strain invariants can be written as,

\[
I = trB = 3 + \gamma^2
\]
\[
II = \frac{1}{2} \left[ (trB)^2 - tr(BB) \right] = 3 + \gamma^2
\]  
(6.24)
\[
III = \det B = 1
\]

where \( \gamma \) is denoted as the shear strain. The shear strain can be decomposed into elastic shear strain and inelastic shear strain as,

\[
\gamma = \gamma_e + \gamma_i
\]  
(6.25)

Following the similar procedures as in the case of uniaxial compressive stress, the shear stress is obtained as,

\[
\sigma_{12} = 2\gamma_e \left( \frac{\delta W^{(e)}}{\delta I_1} + \frac{\delta W^{(e)}}{\delta I_2} \right) + 2\gamma_i \left( \frac{\delta W^{(ie)}}{\delta I_1} + \frac{\delta W^{(ie)}}{\delta I_2} \right)
\]  
(6.26)
6.2.1 Derivation of inelastic left Cauchy-Green deformation tensor using strain energy density based on first invariant \((I)\)

The rate of left Cauchy-Green deformation tensor given by Huber and Tsakmakis (2000a) can be expressed as,

\[
\dot{B}_e = B_e L^T + LB_e - \frac{2}{\eta} B_e \left( S_e - S_e^{(b)} \right)
\]

(6.27)

The \((\cdot)\) indicates the material time derivative, \(\eta\) is the material viscosity and \(L\) is the velocity gradient expressed as,

\[
L = \dot{F} F^{-1}
\]

(6.28)

The explicit expression of the rate of \(B_e\) can be obtained by substituting the strain energy density function (Amin et al, 2002) based on first invariants in Equation (6.27) as,

\[
\dot{B}_e = B_e L^T + LB_e - \frac{4}{\eta} B_e \left\{ C_s^{(OE)} B_e + C_3^{(OE)} \left( I_{\eta_e} - 3 \right)^W B_e + C_4^{(OE)} \left( I_{\eta_e} - 3 \right)^M B_e \right\}
\]

(6.29)

The solution of the above Equation (6.29) is as follows:

\[
\dot{B}_e = B_e L^T + LB_e - \frac{4}{\eta} B_e \left\{ C_s^{(OE)} + C_3^{(OE)} \left( I_{\eta_e} - 3 \right)^W + C_4^{(OE)} \left( I_{\eta_e} - 3 \right)^M \right\}
\]

\[
= B_e L^T + LB_e - \frac{4}{\eta} B_e C
\]

where \(C = C_s^{(OE)} + C_3^{(OE)} \left( I_{\eta_e} - 3 \right)^W + C_4^{(OE)} \left( I_{\eta_e} - 3 \right)^M\)
\[
\begin{align*}
\dot{B}_e - B_e \dot{L} - LB_e &= -\frac{4}{\eta} CB^2_e \\
\dot{B}_e - B_e (L' + L) &= -\frac{4}{\eta} CB^2_e \\
\dot{B}_e + P(t)B_e &= Q(t)B_e^2
\end{align*}
\]

where \(P(t) = -\left(L' + L\right)\), \(Q(t) = -\frac{4}{\eta} C\) and taking \(B_e = y\) one can get,

\[
\frac{dy}{dt} + P(t) \cdot y = Q(t) \cdot y^2
\]  

(6.30)

The above equation is of the typical form of Bernoulli’s equation (Pennisi, 1972) which is given as,

\[
\frac{dy}{dt} + P(t) \cdot y = Q(t) \cdot y^n
\]  

(6.31)

where \(n\) is any real number and the functions \(P\) and \(Q\) are continuous over the intervals on which the solutions of equation are sought. Suppose that \(n\) is neither zero nor unity, one may write the Equation (6.31) as,

\[
y^{-n} \frac{dy}{dt} + P(t) \cdot y^{-n+1} = Q(t)
\]  

(6.32)

By means of the substitution \(z = y^{n-1}\), Equation (6.32) may be written as,

\[
\frac{dz}{dt} + (1-n) \cdot P(t) \cdot z = (1-n) \cdot Q(t)
\]  

(6.33)

This is linear equation of the first order. The general solution of Equation (6.31) can be found from the general solution of the linear Equation (6.33), which is given as,
\[ y^{n+1} = \exp\left[ - \int (1-n) \cdot P(t) \cdot dt \right] \exp\left( \int (1-n) \cdot Q(t) \cdot dt + c \right) \]  

(6.34)

where \( c \) is the constant of integration, the value of which can be obtained from the boundary conditions.

Now the solution of Equation (6.30) can be found from Equation (6.34) by putting \( n = 2 \) as,

\[ y^{2+1} = \exp\left[ - \int (1-2) \cdot P(t) \cdot dt \right] \exp\left( \int (1-2) \cdot Q(t) \cdot dt + c \right) \]

Now recalling \( y = B \), the above equation can be written as,

\[ B^e = \frac{1}{c \cdot e^{rt} - Qt} \]  

(6.35)

At \( t = 0 \), \( B^e = B \) when applied to Equation (6.35) one can get, \( c = \frac{1}{B} \).

Finally by putting the value of \( c, P \) and \( Q \) in Equation (6.35), the expression for left Cauchy-Green deformation tensor for the overstress part can be obtained as,

\[ B^e = \frac{1}{\frac{1}{e^{-(1+1)t}} + \frac{4C}{\eta} \cdot t} \]

Or more explicitly as,

\[ B^e = \frac{1}{\frac{1}{e^{-(1+1)t}} + \frac{4\left[C_{5}^{(0e)} + C_{3}^{(0e)}(I_{n_e} - 3)^{n_e} + C_{4}^{(0e)}(I_{n_e} - 3)^{n_e} \right]}{\eta} \cdot t} \]  

(6.36)
6.2.2 Derivation of inelastic left Cauchy-Green deformation tensor using strain energy density based on first and second invariants ($I$ and $II$)

The rate of left Cauchy-Green deformation tensor given by Huber and Tsakmakis (2000a) can be explicitly expressed by using the strain density function proposed by Wiraguna (2003) as,

$$
\dot{B}_e = B_e L^T + LB_e - \frac{4}{\eta} B_e \left\{ C_2^{(OE)} B_e + C_3^{(OE)} (I_{p_e} - 3)^N B_e + C_4^{(OE)} (I_{p_e} - 3)^M B_e - C_2^{(OE)} B_e^{-1} \right\}
$$

(6.37)

Rearranging the terms the Equation (6.37) is reduced to,

$$
\dot{B}_e = \frac{4}{\eta} C_2^{(OE)} + B_e (L + L^T) - \frac{4}{\eta} B_e \left\{ C_3^{(OE)} (I_{p_e} - 3)^N + C_4^{(OE)} (I_{p_e} - 3)^M \right\}
$$

$$
\frac{dB_e}{dt} = P(t) + Q(t) \cdot B_e + R(t) \cdot B_e^2
$$

where,

$$
P(t) = \frac{4}{\eta} C_2^{(OE)}
$$

$$
Q(t) = (L + L^T)
$$

$$
R(t) = -\frac{4}{\eta} \left\{ C_3^{(OE)} (I_{p_e} - 3)^N + C_4^{(OE)} (I_{p_e} - 3)^M \right\}
$$

A differential equation of the first order and degree one of the form,

$$
\frac{dB_e}{dt} = P(t) + Q(t) \cdot B_e + R(t) \cdot B_e^2
$$

(6.38)
is called a Riccati's equation (Pennisi, 1972). Owing to the term $B^2$, Equation (6.38) is not linear in $B$. We shall assume that the functions $P, Q$ and $R$ are continuous over the intervals on which the solution of the Equation (6.38) are sought.

If one knows a particular solution of Equation (6.38), then it can be reduced to a linear equation of Bernoulli's type. Thus Equation (6.38) is solvable by a sequence of integrations, whenever a particular solution is known.

The general solution of Equation (6.38) is given as,

$$B_c = B_{c_0} + \frac{1}{z} \tag{6.39}$$

where $B_{c_0}$ is the particular solution and $z$ is given as (Pennisi, 1972),

$$z = \exp\left[-\int (Q + 2RB_{c_0}) \cdot dt\right] \left[c - \exp\left(\int (Q + 2RB_{c_0}) \cdot dt\right) \cdot R \cdot dt\right] \tag{6.40}$$

In Equation (6.40) $c$ be the constant of integration and can be found by applying the boundary conditions.

The particular solution of Equation (6.38) can be found by fractional integration on the right hand side by using the two roots of

$$B_c = \frac{-Q + \sqrt{Q^2 - 4RP}}{2R} \text{ and } \frac{-Q - \sqrt{Q^2 - 4RP}}{2R}$$

If the particular solution is negligibly small, then the solution of Equation (6.38) is turned in the typical form of Bernoulli's solution.

### 6.2.3 Derivation of Cauchy stress tensor using strain energy density based on first invariant ($I$)

The improved strain energy based on first invariants as proposed by Amin et al. (2002) can be additively split into equilibrium and overstress part as,
\[ W^{(E)}(I_n) = C_5^{(E)}(I_n - 3) + \frac{C_3^{(E)}}{N+1}(I_n - 3)^{N+1} + \frac{C_4^{(E)}}{M+1}(I_n - 3)^{M+1} \] (6.41)

\[ W^{(OE)}(I_n) = C_5^{(OE)}(I_n - 3) + \frac{C_3^{(OE)}}{N+1}(I_n - 3)^{N+1} + \frac{C_4^{(OE)}}{M+1}(I_n - 3)^{M+1} \] (6.42)

where \( C_5^{(E)}, C_3^{(E)}, C_4^{(E)} \), and \( M, N \) are the material parameters for the equilibrium response as determined by Amin et al (2002), and \( C_5^{(OE)}, C_3^{(OE)} \) and \( C_4^{(OE)} \) are the material parameters for the overstress response.

For uniaxial compression the Cauchy stress can be expressed as,

\[ S = -\rho I + S_E \] (6.43)

where \( \rho \) is the hydrostatic pressure which needs to be determined from the boundary conditions, and \( S_E \) is the extra part of the stress which can be further divided into sum of equilibrium stress \( S_E^{(E)} \) and the overstress \( S_E^{(OE)} \) as,

\[ S_E = S_E^{(E)} + S_E^{(OE)} \] (6.44)

Now for the equilibrium stress, the strain invariant can be expressed (Bonet and Wood 1997) as,

\[ I = I_n = tr B = I_C = tr C \] (6.45)

The strain energy density function for the equilibrium stress can be expressed as,

\[ \hat{W}_E^{(E)}(C) = C_5^{(E)}(tr C - 3) + \frac{C_3^{(E)}}{N+1}(tr C - 3)^{N+1} + \frac{C_4^{(E)}}{M+1}(tr C - 3)^{M+1} \] (6.46)
The second Piola-Kirchhoff stress for the equilibrium part $\hat{S}^{(e)}_t$ can be obtained as (Bonet and Wood 1997)

$$\hat{S}^{(e)}_t = 2 \frac{\delta \hat{S}^{(e)}_t(C)}{\delta C}$$

$$= 2 \frac{\delta}{\delta C} \left[ C^{(e)}_s \left( tr \dot{C} - 3 \right) + \frac{C^{(e)}_3}{N + 1} \left( tr \dot{C} - 3 \right)^{N+1} + \frac{C^{(e)}_4}{M + 1} \left( tr \dot{C} - 3 \right)^{M+1} \right]$$

$$= \left[ \hat{S}^{(e)}_{t1} + \hat{S}^{(e)}_{t2} + \hat{S}^{(e)}_{t3} \right]$$

The three terms of the Equation (6.49) can be obtained as,

$$\hat{S}^{(e)}_{t1} = 2 \frac{\delta}{\delta C} \left[ C^{(e)}_s \left( tr \dot{C} - 3 \right) \right]$$

$$= 2C^{(e)}_s \frac{\delta}{\delta C} \left( tr \dot{C} - 3 \right)$$

$$= 2C^{(e)}_s \frac{\delta}{\delta C} tr \dot{C}$$

$$= 2C^{(e)}_s \left( \text{III}_C^{\frac{1}{2}} C : \mathbf{I} \right)$$

$$= 2C^{(e)}_s \text{III}_C^{\frac{1}{2}} \left( \mathbf{I} - \frac{1}{3} I_c C^{-1} \right)$$

(6.50)

Similarly,

$$\hat{S}^{(e)}_{t2} = 2 \frac{\delta}{\delta C} \left( \frac{C^{(e)}_3}{N + 1} \left( tr \dot{C} - 3 \right)^{N+1} \right)$$

$$= 2C^{(e)}_3 \left( tr \dot{C} - 3 \right)^{N} \frac{\delta}{\delta C} \left( tr \dot{C} - 3 \right)$$

$$= 2C^{(e)}_3 \left( \text{III}_C^{\frac{1}{2}} C : \mathbf{I} - 3 \right)^N \cdot \text{III}_C^{\frac{1}{2}} \left( \mathbf{I} - \frac{1}{3} I_c C^{-1} \right)$$

$$= 2C^{(e)}_3 \left( I - 3 \right)^N \cdot \text{III}_C^{\frac{1}{2}} \left( \mathbf{I} - \frac{1}{3} I_c C^{-1} \right)$$

(6.51)
and

\[ \hat{S}_b^{(e)} = 2C_1^{(e)}(I_h - 3)^M \cdot III_c^{\frac{3}{2}} \left( I - \frac{1}{3} I_c C^{-1} \right) \] (6.52)

Now the Cauchy stress for equilibrium part \( S_h^{(e)} \) can be obtained as follows (Bonet and Wood 1997),

\[ S_h^{(e)} = J^{-1} \hat{F} \hat{S}_b^{(e)} F^T \] (6.53)

Substituting the value of \( S_b^{(e)} \) from Equation (6.49) one get,

\[ S_h^{(e)} = J^{-1} \hat{F} \left( \hat{S}_1^{(e)} + \hat{S}_2^{(e)} + \hat{S}_3^{(e)} \right) F^T \]

\[ = J^{-1} \hat{F} \hat{S}_1^{(e)} F^T + J^{-1} \hat{F} \hat{S}_2^{(e)} F^T + J^{-1} \hat{F} \hat{S}_3^{(e)} F^T \]

\[ = \hat{S}_1^{(e)} + \hat{S}_2^{(e)} + \hat{S}_3^{(e)} \] (6.54)

The three terms of right hand side of Equation (6.54) can be derived as follows (Bonet and Wood 1997)

\[ \hat{S}_1^{(e)} = J^{-1} \hat{F} \left[ 2C_s^{(e)} III_c^{\frac{3}{2}} \left( I - \frac{1}{3} I_c C^{-1} \right) \right] F^T \]

\[ = 2C_s^{(e)} J^{-\frac{3}{2}} \left( I - \frac{1}{3} I_c C^{-1} \right) F^T \quad \text{[} III = \det C = J^3 \text{]} \]

\[ = 2C_s^{(e)} J^{-\frac{3}{2}} \left( B - \frac{1}{3} I_h I \right) \] (6.55)

and,

\[ \hat{S}_3^{(e)} = J^{-1} \hat{F} \left[ 2C_s^{(e)} (I_h - 3)^N III_c^{\frac{3}{2}} \left( I - \frac{1}{3} I_c C^{-1} \right) \right] F^T \]
\[ S_{ii}^{(e)} = (n-\frac{1}{3}) \left( 2J - \frac{1}{3}I \right) \left( \frac{c_{ii}^{(e)}}{J} + C_3^{(e)}(1 - 3)N + C_4^{(e)}(1 - 3)M \right) \] (6.57)

Now the Cauchy stress tensor for the equilibrium part is obtained as,

\[ S_{ii}^{(e)} = \left( B - \frac{1}{3} I \right) \left( 2J^{-\frac{3}{2}} \right) \left( C_5^{(e)} + C_3^{(e)}(I - 3)^N + C_4^{(e)}(I - 3)^M \right) \] (6.58)

Now following the same procedure as in the case of equilibrium stress, the Cauchy stress tensor for the overstress part can be obtained as,

\[ S_{ii}^{(OE)} = \left( B - \frac{1}{3} I \right) \left( 2J^{-\frac{3}{2}} \right) \left( C_5^{(UE)} + C_3^{(OE)}(I - 3)^N + C_4^{(UE)}(I - 3)^M \right) \] (6.59)

Substituting the Equations (6.58) and (6.59) in Equation (6.44) and by adding the pressure term the total Cauchy stress is obtained as,

\[ S = \left( B - \frac{1}{3} I \right) \left( 2J^{-\frac{3}{2}} \right) \left( C_5^{(e)} + C_3^{(e)}(I - 3)^N + C_4^{(e)}(I - 3)^M \right) + \left( B - \frac{1}{3} I \right) \left( 2J^{-\frac{3}{2}} \right) \left( C_5^{(UE)} + C_3^{(OE)}(I - 3)^N + C_4^{(UE)}(I - 3)^M \right) - pI \] (6.60)
6.2.4 Derivation of Lagrangian elasticity tensor based on first invariant 

The Lagrangian elasticity tensor can be split into deviatoric and pressure component as,

\[ C = 2 \frac{\partial \mathbf{S}}{\partial \mathbf{C}} = \hat{\mathbf{C}}_E + C_p \]  

(6.61)

Again the deviatoric part can be split into the equilibrium part and the overstress part as,

\[ \hat{\mathbf{C}} = \hat{\mathbf{C}}_E^{(e)} + \hat{\mathbf{C}}_E^{(o)} \]  

(6.62)

The equilibrium part of the Lagrangian elasticity tensor can be evaluated (Bonet and Wood 1997) as,

\[ \hat{\mathbf{C}}_E^{(e)} = 2 \frac{\partial \mathbf{S}_E^{(e)}}{\partial \mathbf{C}} = 2 \frac{\delta}{\delta \mathbf{C}} \left( \hat{\mathbf{S}}_1^{(e)} + \hat{\mathbf{S}}_2^{(e)} + \hat{\mathbf{S}}_3^{(e)} \right) = \hat{\mathbf{C}}_1^{(e)} + \hat{\mathbf{C}}_2^{(e)} + \hat{\mathbf{C}}_3^{(e)} \]  

(6.63)

The three terms of the right hand side of Equation (6.63) can be evaluated as (Bonet and Wood, 1997),

\[ \hat{\mathbf{C}}_1^{(e)} = 2 \frac{\partial \hat{\mathbf{S}}_1^{(e)}}{\partial \mathbf{C}} = 4 C_5^{(e)} III^{-7/2} \left[ \frac{1}{3} I_c III - \frac{1}{3} I \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right] \]

\[ \hat{\mathbf{C}}_2^{(e)} = 2 \frac{\partial \hat{\mathbf{S}}_2^{(e)}}{\partial \mathbf{C}} = 4 C_3^{(e)} (I_n - 3)^W III^{-7/2} \left[ \frac{1}{3} I_c III - \frac{1}{3} I \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right] \]

\[ \hat{\mathbf{C}}_3^{(e)} = 2 \frac{\partial \hat{\mathbf{S}}_3^{(e)}}{\partial \mathbf{C}} = 4 C_4^{(e)} (I_n - 3)^M III^{-7/2} \left[ \frac{1}{3} I_c III - \frac{1}{3} I \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right] \]
Now the Lagrangian elasticity tensor for the equilibrium part is obtained as,

$$\hat{C}_E^{(E)} = 4J^{-3/2} \left[ C_3^{(E)}(I_n - 3)^N + C_4^{(E)}(I_n - 3)^M \right] \times \left[ \frac{1}{3} I_c \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right]$$

(6.64)

Similarly, for over stress part it can be obtained as,

$$\hat{C}_E^{(OE)} = 4J^{-3/2} \left[ C_3^{(OE)}(I_n - 3)^N + C_4^{(OE)}(I_n - 3)^M \right] \times \left[ \frac{1}{3} I_c \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right]$$

(6.65)

The total Lagrangian elasticity tensor is obtained by adding the Equation (6.64) and (6.65) with the pressure term as,

$$\hat{C} = 4J^{-3/2} \left[ C_3^{(E)} + C_3^{(OE)}(I_n - 3)^N + C_4^{(E)}(I_n - 3)^M + C_3^{(OE)}(I_n - 3)^N + C_4^{(OE)}(I_n - 3)^M \right] \times \left[ \frac{1}{3} I_c \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right] + C_p$$

(6.66)

6.2.5 Derivation of Cauchy stress tensor using strain energy density function based on first and second invariants \((I, II)\)

The strain energy density function based on first and second invariants proposed by Wiraguna (2003) can be additively split into equilibrium and over stress part as,

$$W^{(E)}(I_n, II_n) = C_3^{(E)}(I_n - 3) + \frac{C_3^{(E)}}{N+1}(I_n - 3)^{N+1} + \frac{C_4^{(E)}}{M+1}(I_n - 3)^{M+1} + C_2^{(n)}(II_n - 3)$$

(6.67)
The strain energy function for the equilibrium part can be expressed in terms of $\nu C$ as,

$$ W^{(e)}(I_{r}, II_{r}) = C_{s}^{(e)}(I - 3) + \frac{C_{3}^{(e)}}{N + 1} (I - 3)^{N - 1} + \frac{C_{4}^{(e)}}{M + 1} (I - 3)^{M - 1} + C_{2}^{(e)}(I - 3) \quad (6.68) $$

The second Piola-Kirchhoff stress for the equilibrium part can be obtained as (Bonet and Wood, 1997),

$$ S^{(e)}(C) = 2 \frac{\delta W^{(e)}(C)}{\delta C} = \left[ C_{s}^{(e)} (I - 3) + \frac{C_{3}^{(e)}}{N + 1} (I - 3)^{N - 1} + \frac{C_{4}^{(e)}}{M + 1} (I - 3)^{M - 1} \right] + C_{2}^{(e)} \left[ \frac{1}{2} (I - 3) - I r(CC) \right] \quad (6.69) $$

Now,

$$ \hat{S}_{1}^{(e)} = 2 \frac{\delta}{\delta C} \left[ C_{s}^{(e)} (I - 3) \right] = 2C_{s}^{(e)} \left[ I r(C) - 3 \right] = 2C_{s}^{(e)} \left[ \frac{1}{3} I r(C) - 3 \right] \quad (6.70) $$

$$ \hat{S}_{2}^{(e)} = 2 \frac{\delta}{\delta C} \left[ \frac{1}{3} I r(C) - 3 \right] = 2 \frac{\delta}{\delta C} \left[ \frac{1}{3} I r(C) - 3 \right] \quad (6.71) $$
Similarly,

\[ \mathbf{\dot{S}}^{(e)}_s = 2C_2^{(e)}(I_n - 3)^{\nu} III_c^{\gamma/2} \left( \mathbf{I} - \frac{1}{3} I_c C^{-1} \right) \quad (6.73) \]

and

\[ \mathbf{\dot{S}}^{(e)}_i = 2 \frac{\delta}{\delta C} \left[ C_2^{(e)}(II_n - 3) \right] \]

\[ = 2C_2^{(e)} \delta \frac{\delta}{\delta C} \left[ \frac{1}{2} \left( tr \mathbf{\hat{C}} \right)^3 - tr(CC) - 3 \right] \]

\[ = 2C_2^{(e)} tr \mathbf{\hat{C}} \delta \frac{\delta}{\delta C} (tr \mathbf{\hat{C}}) \]

\[ = 2C_2^{(e)} tr \mathbf{\hat{C}} III_c^{\gamma/2} \left( \mathbf{I} - \frac{1}{3} I_c C^{-1} \right) \quad (6.74) \]

Now following the same procedure as in the case of deriving the Cauchy stress tensor for the strain energy based on first invariant, the Cauchy stress tensor for the equilibrium part of the strain energy based on first and second invariants can be obtained as,

\[ \mathbf{S}^{(e)} = 2C_2^{(e)} \mathbf{J}^{-\nu/2} \left( \mathbf{B} - \frac{1}{3} I_n \mathbf{I} \right) + 2C_3^{(e)} \mathbf{J}^{-\nu/2} (I_n - 3)^{\nu} \left( \mathbf{B} - \frac{1}{3} I_n \mathbf{I} \right) + 2C_4^{(e)} \mathbf{J}^{-\nu/2} (I_n - 3)^{\nu} \left( \mathbf{B} I_n - \frac{1}{3} I_n I_n \mathbf{I} \right) \]

Similarly, for the overstress part it can be obtained as,
By adding Equation (6.75) and (6.76) together with the pressure, one can obtain the total Cauchy stress as,

\[
S = 2J^{-\frac{3}{2}} \left( B_e - \frac{1}{3} I_{b_e} I \right) + 2C_3^{(OE)} J^{-\frac{5}{2}} \left( I_{b_e} - 3 \right) \left( B_e - \frac{1}{3} I_{b_e} I \right) + 2C_4^{(OE)} J^{-\frac{5}{2}} \left( I_{b_e} - 3 \right) \left( B_e - \frac{1}{3} I_{b_e} I \right)
\]

(6.76)

6.2.6 Derivation of Lagrangian elasticity tensor based on first and second invariants \((I, II)\)

The Lagrangian elasticity tensor can be split into deviatoric and pressure component as (Bonet and Wood, 1997),

\[
C = 2 \frac{\delta S}{\delta C} = \dot{C} + C_p
\]

(6.78)

Again the deviatoric component can be split into its equilibrium and overstress component as,

\[
\dot{C} = \dot{C}_E^{(e)} + \dot{C}_E^{(OE)}
\]

(6.79)

Now the equilibrium part is obtained as follows,
The first three terms of Equation (6.80) can be obtained by following the similar procedure as in the case of elasticity tensor based on first invariants. The fourth term of the above equation is obtained as,

\[
\dot{C}_4^{(E)} = 2 \frac{\delta \tilde{S}_4^{(E)}}{\delta C} = 4C_2^{(E)}(trC)III \dot{\gamma}_S \times
\left[ \frac{1}{3} I_c III - \frac{1}{3} I \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right]
\]

(6.81)

Now, the total elasticity component of the equilibrium part is as follows,

\[
\dot{C}_\varepsilon^{(E)} = 4J^{-\frac{1}{2}} \left[ \frac{1}{3} I_c III - \frac{1}{3} I \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right] \times\left[ C_5^{(E)} + C_4^{(E)}(I_B - 3) + C_6^{(E)}(I_B - 3) + C_7^{(E)}(trC) \right]
\]

(6.82)

Similarly the elasticity component for the overstress part is obtained as,

\[
\dot{C}_\varepsilon^{(OE)} = 4J^{-\frac{1}{2}} \left[ \frac{1}{3} I_c III - \frac{1}{3} I \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right] \times\left[ C_5^{(OE)} + C_4^{(OE)}(I_B - 3) + C_6^{(OE)}(I_B - 3) + C_7^{(OE)}(trC) \right]
\]

(6.83)

Adding Equation (6.82) with (6.83) together with the pressure term, the total Lagrangian elasticity tensor is as follows,

\[
\dot{C} = \left[ C_5^{(E)} + C_3^{(E)}(I_B - 3) + C_4^{(E)}(I_B - 3) + C_2^{(E)}(trC) + C_5^{(OE)} + C_3^{(OE)}(I_B - 3) + C_4^{(OE)}(I_B - 3) + C_2^{(OE)}(trC) \right] +
4J^{-\frac{1}{2}} \left[ \frac{1}{3} I_c III - \frac{1}{3} I \otimes C^{-1} - \frac{1}{3} C^{-1} \otimes I + \frac{1}{9} I_c C^{-1} \otimes C^{-1} \right] + C_p
\]

(6.84)
Chapter 7

Finite Element Verification of the Formulation

7.1 GENERAL

The experimental investigation carried out on different specimen of rubbers reveals a strong non-linear rate dependency and are presented in Chapter 2. In Chapter 4 the continuum mechanics needed to incorporate the rate dependency in a constitutive relation has been discussed. The available constitutive model on viscoelasticity and the proposed formulation to incorporate the rate dependency in the constitutive relation based on the concept of Huber and Tsakmakis (2000a) and Lubliner (1985) are presented in Chapter 5 and 6 respectively. This chapter is devoted to compare the experimental results with the finite element simulation and to illustrate the capability of the proposed finite element formulation. For FEM coding, a general-purpose finite element program FEAP has been used. FEAP has been developed by Taylor (2000) at University of California at Berkeley and partially documented in Zeinkiewicz and Taylor (1996). Several subroutines have been developed in FEAP for this purpose.

7.2 MODELING OF RUBBERS

Different 3D models are used for the purpose of finite element simulation. Figures 7.1 to 7.3 represent the different simulation models used in FEAP program to simulate compression, shear, combined action of compression and shear and relaxation phenomena. The actual geometries of the test specimens were used to eliminate the possibility of the interference in simulation due to the difference between shape of the FE models and the actual test specimens. Figure 7.1 represents the 3D model using 216 eight-noded brick elements for compression test simulation. This model is also used to simulate the relaxation phenomena. Figure 7.2 represents 3D model using
100 eight-noded brick elements for shear test simulation. This model is also used to simulate the relaxation phenomena under shear strain. Figure 7.3 represents 3D model using 100 eight-noded brick elements for simulation under combined action of compression and shear.

Figure 7.1 3D simulation model using 216 brick elements for compression and relaxation test
Figure 7.2 3D Simulation model using 100 brick elements for shear and relaxation test

Figure 7.3 3D Simulation model using 100 brick elements for combined action of compression and shear
NUMERICAL SIMULATION UNDER MONOTONIC COMPRESSION 
AND SHEAR

From Tuesdell and Noll (1992), the Cauchy stress can be decomposed into volumetric part \((-pI)\) and extra part \(S_E\) as:

\[
S = -pI + S_E
\]  

(7.1)

The deviatoric part of the Cauchy stress tensor, \(S_E\) can be decomposed into equilibrium part \(S_E^{(e)}\) and the overstress part \(S_E^{(oe)}\) as,

\[
S_E = S_E^{(e)} + S_E^{(oe)}
\]  

(7.2)

with

\[
S_E^{(e)} = 2 \frac{\delta W^{(e)}}{\delta I_B} B - 2 \frac{\delta W^{(e)}}{\delta I_B} B^{-1}
\]  

(7.3)

\[
S_E^{(oe)} = 2 \frac{\delta W^{(oe)}}{\delta I_{n_e}} B_e - 2 \frac{\delta W^{(oe)}}{\delta I_{n_e}} B_e^{-1}
\]  

(7.4)

Substituting the strain energy density function as stated in Equations (5.29) and (5.30) into Equations (7.1)-(7.4) one can have the explicit expressions for \(S\) as,

\[
S = -pI + 2\left[C^{(e)}_3 + C^{(e)}_3 \left(I_B - 3\right)^N + C^{(e)}_4 \left(I_B - 3\right)^M\right] B - 2C^{(e)}_2 B^{-1} + 2\left[C^{(oe)}_3 + C^{(oe)}_3 \left(I_{n_e} - 3\right)^N + C^{(oe)}_4 \left(I_{n_e} - 3\right)^M\right] B_e - 2C^{(e)}_2 B_e^{-1}
\]  

(7.5)

A subroutine has been developed using the above equation in FEAP to simulate the experimental results with the constitutive relations. In Figs. 7.4 to 7.7 the simulation results under monotonic compression along with other
results for HDR have been shown. It has been found that at higher stretch rate the proposed simulation is not in good agreement with the experimental results for HDR, whereas at low stretch rates the proposed simulation is in good agreement with the experimental results. It has also been found that the similar trend is also true for NR under monotonic compression. In Figs. 7.12 to 7.15 the simulation results of monotonic shear (up to 200% shear strain) have been given for HDR and the simulation results are in good agreement with the experimental results at lower strain rates, but at higher strain rates the accuracy decreases. This may be due to the effect of strain rates. The simulation results of monotonic shear for NR under different strain rates have been given in Figs. 7.16 to 7.19. It has been found here that irrespective of the strain rates the simulation results are in good agreement with the experimental results.

7.4 NUMERICAL SIMULATION UNDER SIMULTANEOUS ACTION OF COMPRESSION AND SHEAR

Usually the structural elements remain under compression due to the gravity loads coming from the superstructures. However, compression and shear deformations act together on these elements when a lateral load like wind or earthquake strikes. No such experimental results are available in this regard. The strain energy density function proposed by Wiraguna (2003) has been used for this purpose. In Figs. 7.20 and 7.21 the simulation results of the combined action of compression and shear of HDR under different shear strain rates are given. From Fig. 7.20 it has been found that the shear stress is less under combined action of compression and shear. This difference is more pronounced at higher strain rates, whereas with the decrease in strain rates this difference also decreases as it was evident from Fig. 7.21.
7.5 NUMERICAL SIMULATION OF RELAXATION RESPONSE

Non-linear dependence of viscosity as proposed by Amin et al (2006a) has been used for the simulation of relaxation phenomena of HDR. Relaxation phenomena are typical properties of the viscoelastic materials. At constant strain the viscoelastic material loses stress as the strain energy loses in the form of heat energy. The expression for the non-linear viscosity can be expressed as,

\[ \eta = \frac{\eta_0 \|B\|^{\varphi}}{\|S_E^{(OE)}\|^{\delta}} \]  

(7.6)

where \( \|B\| \) and \( \|S_E^{(OE)}\| \) are the magnitudes of the left Cauchy-Green deformation tensor \( B \) and the overstress \( S_E^{(OE)} \), and are given as,

\[ \|B\| = \left[ \text{Trace}[B] \cdot \text{Transpose}[B] \right]^{\frac{1}{5}} \]  

(7.7)

\[ \|S_E^{(OE)}\| = \left[ \text{Trace}[S_E^{(OE)}] \cdot \text{Transpose}[S_E^{(OE)}] \right]^{\frac{1}{5}} \]  

(7.8)

and \( \eta_0, \delta, \varphi \) are the viscosity parameters that are given in Table 7.1.

A subroutine has been developed to simulate the relaxation phenomena of HDR in FEAP. Using this model numerical simulation of relaxation of HDR is carried out under compression and shear and presented in Fig. 7.22. Under compression the simulation is carried out at 0.5 stretch levels and up to 120 seconds. It has been found that the simulation result is in good agreement with the experimental one. Under shear the simulation is carried out at 100% shear strain and up to 120 seconds, it has been found that the simulation result gives less stress than that of the experimental one.

Table 7.1 Viscosity parameters

<table>
<thead>
<tr>
<th>Specimen</th>
<th>( \eta_0 ) (MPa)</th>
<th>( \delta )</th>
<th>( \varphi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>HDR</td>
<td>1.63</td>
<td>1.46</td>
<td>2.29</td>
</tr>
</tbody>
</table>

96
A typical flow chart is presented here for the calculation of viscoelastic response of NR and HDR.

1. Input material parameters and Bulk Modulus
2. Calculate the Jacobian determinant
3. Calculate the pressure constant
4. Calculate the left Cauchy-Green deformation tensor from the material geometry
5. Calculate the Cauchy stress tensor for the equilibrium part
6. Calculate the Cauchy elastic modulus for the equilibrium part
7. Calculate the velocity gradient
8. Calculate the left Cauchy-Green deformation tensor for the overstress part
9. Calculate the Cauchy stress tensor for the overstress part
10. Calculate the Cauchy elastic modulus for the overstress part
11. Calculate the total stress by adding the equilibrium stress with the overstress
Figure 7.4 Cauchy stress-stretch relations of HDR at 0.88 per second stretch rates
Figure 7.5 Cauchy stress-stretch relations of HDR at 0.24 per second stretch rates
Figure 7.6 Cauchy stress-stretch relations of HDR at 0.005 per second stretch rates
Figure 7.7 Cauchy stress-stretch relations of HDR at 0.001 per second stretch rates.
Figure 7.8 Cauchy stress-stretch relations of NR at 0.65 per second stretch rates.
Figure 7.9 Cauchy stress-stretch relations of NR at 0.225 per second stretch rates
Figure 7.10 Cauchy stress-stretch relations of NR at 0.072 per second stretch rates
Figure 7.11 Cauchy stress-stretch relations of NR at 0.001 per second stretch rates
Figure 7.12 Shear stress-strain relations of HDR at 0.5 per second strain rates
Figure 7.13 Shear stress-strain relations of HDR at 0.4 per second strain rates
Figure 7.14 Shear stress-strain relations of HDR at 0.25 per second strain rates
Figure 7.15 Shear stress-strain relations of HDR at 0.05 per second strain rates
Figure 7.16 Shear stress-strain relations of NR at 0.1 per second strain rates
Figure 7.17 Shear stress-strain relations of NR at 0.01 per second strain rates
Figure 7.18 Shear stress-strain relations of NR at 0.05 per second strain rates
Figure 7.19 Shear stress-strain relations of NR at 0.005 per second strain rates
Figure 7.20 Shear stress-strain relations of HDR under simultaneous action of compression and shear (25% compression and 100% shear) at 0.5 per second strain rate.
Figure 7.21 Shear stress-strain relations of HDR under simultaneous action of compression and shear (25% compression and 100% shear) at 0.05 per second strain rate.
Figure 7.22 Relaxation phenomena of HDR under compression and shear (a) at 0.5 stretch (b) at 1.00 shear strain
The numerical simulation representation under monotonic compression, monotonic shear and combined action of compression and shear has been given in Figs. 7.4 to 7.22. The graphical representation of stress distribution and deformation patterns is given in Figs. 7.23 to 7.25. From these graphs, the graphical representation of stress and deformation patterns can be seen in an illustrated way. In Fig. 7.23 the deformation-2 ($\lambda_2$) and the stress-22 ($s_{22}$) for 3D model under compression has been shown. It is clear that the uniformity of stress distribution has been attained here. In Fig. 7.24 the stress-12 ($s_{12}$) and the deformation-1 ($\lambda_1$) has been given for 3D model under monotonic shear. It is found here that the stress-12 ($s_{12}$) is symmetric about the diagonal lines. The deformation-1 ($\lambda_1$) has been found homogeneous. In Fig. 7.25 the deformation-1 ($\lambda_1$) and the stress-12 ($s_{12}$) has been given for 3D model under combined action of compression and shear. It is found here that the stress-12 ($s_{12}$) is symmetric about the diagonal lines and shows the increasing trends towards the left-top corner and right-bottom corner, which may be due to the coupling effect of compression and shear. Again the deformation-1 ($\lambda_1$) has been found in homogenous pattern. However, Figs. 7.24 and 7.25 show the geometric nonlinearity, the finite deformation model taking care of.
Figure 7.23 Deformation pattern and stress distribution for 3D model under compression (a) deformation-2 at 0.5 stretch (b) stress-22 distribution at 0.5 per second stretch rates
Figure 7.24 Stress and deformation pattern for 3D-shear model (a) stress-12 at 100% strain (b) deformation-1 at 100% strain
Figure 7.25 Deformation and stress patterns for combined action of compression and shear (a) deformation-1 at 60% shear and 15% compression (b) stress-12 at 25% compression and 80% shear
Elastomeric seismic isolators are used to provide buildings or bridges with protection from earthquake damage. The bearing pad remains under compression due to the gravity load coming from its superstructure. However, when an earthquake strikes the horizontal force comes into play. The isolators covered consist of alternate rubber layers and reinforcing steel plates. They are placed between a superstructure and its substructure to provide for flexibility for decoupling structural systems from ground motion, and damping capability to reduce displacement at the isolation interface and the transmission of energy from the ground into the structure at the isolation frequency (ISO 22762-1). Both the natural rubbers (NR) and high damping rubbers (HDR) are used in these bearings. However HDR shows better performance than NR in damping capability. In this chapter the simulation results of NR and HDR bearing pads found using the developed FEM code are compared with the experimental results. Thus is possible to verify the applicability of the developed FEM code in the analysis of bearing pads.

8.2 BEARING PAD MODEL

Two types of bearing pads are generally in use, one for bridges and the other for buildings. The requirements of isolators for bridges and for buildings are quite different. The main differences to be noted between isolators for bridges and buildings are that, isolators for bridges are mainly rectangular in shape while those for buildings are circular in shape; isolators for bridges are designed to be used for both rotation and horizontal displacement, while those for buildings are designed for horizontal displacement only. Isolators for bridges are designed to withstand dynamic loads caused by vehicles on a daily basis as well as earthquakes, while isolators for buildings are mainly designed to withstand dynamic loads caused by earthquakes only. Isolators
for bridges are designed to perform on a daily basis to accommodate length changes of bridges caused by temperature changes as well as during earthquakes; while those for buildings are designed to perform only during earthquakes (ISO 22762-1). Based on this the International Organization for Standardization proposed some standard test specimen. The test specimen used in this study is square in shape having a dimension of 240mm x 240mm, and contains 6 layers of rubber with 5mm thickness each. The inner steel plate thickness is 2mm each. Figure 8.1 shows the geometric model of rubber bearing pad used in this study.

Figure: 8.1 3D bearing pad with applied displacement.
A laminated rubber bearing consists of alternate layers of rubber and steel. The steel plates are provided to incorporate the vertical stiffness in the bearing pads and the rubber layers are provided to incorporate flexibility. While in service, the bearing pads usually remain under compression for the load coming from the superstructure. Keeping this fact in mind, in this study all the specimens are subjected to prior compression before test. In FEM analysis the model specimens were also put into compression before applied shear. The FEM analysis results are then compared with experimental results to find out the suitability of the proposed formulation. Figure 8.2 to 8.6 represent the comparison between the FEM solution and the experimental results. Figure 8.7 represent the relaxation simulation of HDR and NR bearing pad at 1.00 shear strain. Figure 8.2 to 8.4 represent the distribution of shear stress-12 for HDR at different strain rates. It appears that the distribution holds good for lower strain rates than for higher strain rates. Figures 8.5 and 8.6 represent the distribution of shear stress-12 for NR bearing pad at different strain rates. It is clear that in case of NR the rate-dependency does not significantly affect the stress response. The stress distribution patterns are also shown in these figures. Here it is found that the shear stress-12 distribution is symmetric about the diagonal line. And the stress is concentrated at the top left and bottom right corners. These may be due to the coupling effects of compression and shear.
Figure 8.2 Simulation of HDR bearing pad at 5.5/sec strain rates (a) Stress-strain response (b) stress patterns
Figure 8.3 Simulation of HDR bearing pad at 1.5/sec strain rates (a) Stress-strain response (b) stress patterns
Figure 8.4 Simulation of HDR bearing pad at 0.5/sec strain rates (a) Stress-strain response (b) stress patterns
Figure 8.5 Simulation of NR bearing pad at 5.5/sec strain rates (a) Stress-strain response (b) stress patterns
Figure 8.6 Simulation of NR bearing pad at 1.5/sec strain rates (a) Stress-strain response (b) stress patterns
Figure 8.7 Simulation of relaxation response of bearing pad at 1.00 shear strain (a) HDR (b) NR
Chapter 9

Conclusion and Recommendation

9.1 GENERAL

Non-linear rate dependent response of Natural Rubber (NR) and High Damping Rubbers are simulated here. The total response is decomposed into two parts, the equilibrium stress and the viscosity induced overstress. The equilibrium response is simulated using the hyperelasticity model as presented by Amin et al. (2006a) and the viscosity induced overstress is simulated by using the finite element formulations obtained by solving the boundary value problems. Bernoulli’s solution technique has been used here in obtaining the formulations for finite element analysis.

The present research work has been carried out in two steps, first the formulations for finite element analysis and second the verification of the formulations. The expression for the second Piola-Kirchhoff stress tensor and the Cauchy stress tensor for the equilibrium stress and overstress have been formulated and then added to obtain the total response. These expressions then implemented in a general purpose finite element code. The responses of NR and HDR under compression and shear have been simulated using the material parameters obtained from experimental observations (Amin et al 2006a and Wiraguna et al 2003).

9.2 VERIFICATION OF FORMULATIONS FOR FINITE ELEMENT ANALYSIS

The numerical simulation of NR and HDR under uniaxial compression and simple shear deformation are compared with the experimental observations to find out the adequacy of the proposed formulations. All the simulations have been carried out for 3D model. Under uniaxial compression the FEM simulation have been
compared with the constitutive relation and experimental observations. It has been found that for HDR and NR the response is in good accord with the experimental one. However, with the increasing stretch rates the accuracy of the FEM simulation is less than the response at lower stretch rates for HDR. But in the case of NR the FEM result is in good agreement with the experimental results despite the stretch rate effects. The geometric nonlinearity has been well explained in all these simulations. From the numerical simulation it has been seen that the stress is uniformly distributed throughout the thickness of the specimen at a level of uniaxial compression. This conforms to the adopted experimental condition.

In order to simulate the responses of NR and HDR under shear, 3D finite element analysis have been carried out and the results were compared with those obtained from the constitutive relations and experimental observations. From the observations of numerical simulations it has been found that the results are in good agreement with the results obtained from the constitutive relations and experiments. The geometric nonlinearity has also been well experienced here.

For simulation of relaxation response, finite element analysis has been carried out for HDR under compression and shear. The simulation results then compared with the available experimental results. It has been found that the simulation results are in good agreement with the experiments. For the first few seconds the stress relaxes quickly then there is a gradual decrease in stress for a long period of time. This trend has also been found in the experiments.

9.3 FINITE ELEMENT STUDY ON BEARING PAD

The proposed formulations then applied to analyze the full scale rubber bearing pads of NR and HDR. The finite element analysis of the bearing pad has been carried out on the 3D model as proposed in International Standard Organization
(ISO 22762-1). The simulations were carried out under compression and shear and then added together to obtain the combined response. The simulation response then compared with the available experimental results to find out the applicability of the proposed formulations in analyzing the full scale bearing pad. From the comparison it has been found that the simulation results are in good agreement with the experiments for both in the HDR and NR under different strain rates. This confirms the applicability of the proposed formulation in analyzing the bearing pad.

9.4 RECOMMENDATIONS FOR FUTURE STUDIES

The finite element formulations for the overstress part have been obtained by solving the boundary value problems by using the available solution techniques such as Bernoulli's principle. This solution technique may limit the applicability of the formulation in a general case. It has been found that the proposed formulation works well under compression and shear. To make it a general formula it needs to work well to other forms of deformation such as under tension or torsion or their combinations. So the simulation under tension or torsion or their combinations are proposed for future studies.

NR and HDR are being extensively used now all over the world in constructing bearing pad. These bearing pads while in service subjected to variable temperature. The experiments carried out on rubbers are all in a fixed temperature, but in service condition the temperature may varied, and the constitutive relation may not hold good to properly represent the response. Thus the temperature may be included as a parameter in the constitutive modeling of rubbers in future studies.
American Association for State Highway and Transportation Officials,
Washington, DC.

Aklonis, J. J., Macnight, W. J. and Shen, M., 1972. Introduction to Polymer
Viscoelasticity, John Wiley & Sons, Canada.

Amin, A.F.M.S., Lion, A., Sekita, S., Okui, Y., 2006a. Nonlinear dependence
of viscosity in modeling the rate-dependent response of natural and high
damping rubbers in compression and shear: Experimental identification
and numerical verification, Int. J. Plasticity, 22, 1610-1657.

damping Rubbers in compression and shear, J. Engrg. Mech, ASCE,
Vol. 132, 54-64.

Deformation in Natural and High damping Rubbers in Large Deformation

Amin, A.F.M.S., Alam, M.S., and Okui, Y., 2002. An Improved hyperelasticity
Relation of Natural and High Damping Rubbers in Compression:
Experiments, Parameter Identification and Numerical Verification,
Mechanics of Materials, 34, 75-95

Amin, A.F.M.S., 2001. Constitutive Modeling for Strain-Rate Dependency of
Natural and High Damping Rubbers. Ph.D. Thesis, Department of Civil
and Environmental Engineering, Saitama University, Japan.


Bertscher, S., Dorfmann, A., Bergmeister, K., 1998. "Mechanical Aspects of
High Damping rubber", 2nd Int. PhD Symposium in Civil Engineering,
Budapest.


Ramberger, G., 2002. Structural bearings and expansion joints for bridges, structural engineering documents 6., International Association for Bridge and Structural Engineering, Zurich, Switzerland.


subroutine stnh3f(d,detf,bb,sig,aa,estore)

* F E A P * * A Finite Element Analysis Program

Copyright (c) 1984-1999: Robert L. Taylor

Finite Deformation Elasticity Neo-Hookean Model

INPUT variables

d(IOO) Material parameters
d(21) Bulk modulus
d(22) Shear modulus
detf Jacobian determinant at t \( t_n+1 \)
bb(6) Left Cauchy-Green tensor

OUTPUT variables

sig(6) CAUCHY stress tensor
aa(6,6) CAUCHY (spatial) elastic moduli
estore Stored energy density

implicit none

integer i
real*8 detf, press, logj, uppj, mu, mu2, estore, mu3, mu4,
sigg3, sigg4
real*8 aminshear, aminbulk, mu5, mu6, mu7, \( f(6), inf(6), velo(6) \)
real*8 d(*), sig(6), aa(6,6), bb(6), detbb, inbb(6), velotr(6)
real*8 cons1(6), cons2, dt, def, fdt(6), be(6), bdote(6), ovs(6)
real*8 cig(6), eta, detbe, inbe(6), pre1, cons3, dd(6,6), ee(6,6)
save

Compute pressure and its derivative
aminbulk=10000.0
logj = log(abs(detf))
c press = d(21)*logj/detf
c uppj = d(21)/detf
press = aminbulk*logj/detf
uppj = aminbulk/detf
c Set CAUCHY stresses and elastic tangent moduli
c mu = d(22)/detf
mu5= 2*4.262/detf
mu6= -2*0.145/detf
mu3= 2*1.182/detf
mu4= 2*(-5.297)/detf
mu= mu5+mu3*(bb(1)+bb(2)+bb(3)-3)**0.27+
1mu4*(bb(1)+bb(2)+bb(3)-3)**0.06
mu7 = mu6
detbb=bb(1)*
(detbb=bb(1)*
bb(2)*bb(3)-bb(5)*bb(5)-bb(4)*bb(4)*bb(3)-bb(6)*
bb(5))+bb(6)**2/detbb
inbb(1)=(bb(2)*bb(3)-bb(5)**2)/detbb
inbb(2)=(bb(1)*bb(3)-bb(6)**2)/detbb
inbb(3)=(bb(1)*bb(2)-bb(4)**2)/detbb
inbb(4)=(bb(6)*bb(5)-bb(4)*bb(3))/detbb
inbb(5)=(bb(4)*bb(6)-bb(1)*bb(5))/detbb
inbb(6)=(bb(4)*bb(5)-bb(2)*bb(6))/detbb
do i = 1,3
c sig(i ) = mu*bb(i ) - mu + press
c sig(i ) = mu*bb(i ) + mu6*inbb(i ) + press -mu -mu6
c sig(i+3) = (mu+sigg3+sigg4)*bb(i )-mu+press
c sig(i+3) = (mu+sigg3+sigg4)*bb(i+3)
ee(i ,i ) =2*mu - 2.d0*press + uppj - 2*mu7
nee(i+3,i+3) = mu - press - mu6
c aa(i,i ) = 4.d0*mu5- 2.d0*press + uppj
c aa(i+3,i+3) = mu5 - press
end do
c Generation of over stress
do i = 1,6
c f(i)=bb(i)**0.5
c end do
f(1)=bb(1)**0.5
f(2)=bb(2)**0.5
f(3)=bb(3)**0.5
f(4)=bb(4)**0.5
\[ f(5) = bb(5) I^{0.5} \]
\[ f(6) = bb(6) I^{0.5} \]

\[ \text{def} = f(1)*(f(2)*f(3)-f(5)*2)+f(4)*(f(5)*f(6)-f(4)*f(5)) + \\
   f(6)*(f(4)*f(5)-f(2)*f(6)) \]

\[ \text{inf}(1) = (f(2)*f(3)-f(5)*2)/\text{def} \]
\[ \text{inf}(2) = (f(1)*f(3)-f(6)*2)/\text{def} \]
\[ \text{inf}(3) = (f(1)*f(2)-f(4)*2)/\text{def} \]
\[ \text{inf}(4) = (f(5)*f(6)-f(4)*f(3))/\text{def} \]
\[ \text{inf}(5) = (f(4)*f(6)-f(1)*f(5))/\text{def} \]
\[ \text{inf}(6) = (f(4)*f(5)-f(2)*f(6))/\text{def} \]

\[ \eta = 3.00 \]

\[ \Delta t = 0.00002272727 \]
\[ \text{do i = 1,6} \]
\[ \text{fdt}(i) = f(i)/1.d0*\Delta t \]
\[ \text{end do} \]

\[ \text{do i = 1,6} \]
\[ \text{velo}(i) = \text{fdt}(i)*\text{inf}(i) \]
\[ \text{end do} \]

\[ \text{do i = 1,6} \]
\[ \text{cons}(i) = \text{velo}(i) + \text{velo}(i) \]
\[ \text{end do} \]

\[ \text{cons}2 = (9.707-4.262)+(2.477-1.182)*(bb(1)+bb(2)+bb(3)-3)**0.27 + \\
   (-11.689+5.297)*(bb(1)+bb(2)+bb(3)-3)**0.06 \]

\[ \text{do i = 1,6} \]
\[ \text{be}(i) = 1/((1/bb(i))*\exp((-1)*\text{cons}(i)*\Delta t)+(4/(\eta))*\text{cons}2*\Delta t) \]
\[ \text{end do} \]

\[ \text{detbe} = \text{be}(1)*\text{be}(2)*\text{be}(3)-\text{be}(5)*\text{be}(5)-\text{be}(4)*\text{be}(4)*\text{be}(3)-\text{be}(6)* \\
   \text{be}(5)) + \text{be}(6)*\text{be}(4)*\text{be}(5)-\text{be}(6)*\text{be}(2)) \]
\[ \text{inbe}(1) = (\text{be}(2)*\text{be}(3)-\text{be}(5)*2)/\text{detbe} \]
\[ \text{inbe}(2) = (\text{be}(1)*\text{be}(3)-\text{be}(6)*2)/\text{detbe} \]
\[ \text{inbe}(3) = (\text{be}(1)*\text{be}(2)-\text{be}(4)*2)/\text{detbe} \]
\[ \text{inbe}(4) = (\text{be}(6)*\text{be}(5)-\text{be}(4)*\text{be}(3))/\text{detbe} \]
\[ \text{inbe}(5) = (\text{be}(4)*\text{be}(6)-\text{be}(1)*\text{be}(5))/\text{detbe} \]
\[ \text{inbe}(6) = (\text{be}(4)*\text{be}(5)-\text{be}(2)*\text{be}(6))/\text{detbe} \]

\[ \text{prel} = \text{press} \]

139
cons3=cons2

do i=1,3
    ovs(i)=((9.707-4.262)+(2.477-1.182)*(bb(1)+bb(2)+bb(3)-3)**0.27
        +(-11.689+5.297)*(bb(1)+bb(2)+bb(3)-3)**0.06)*2*be(i)-
        1(0.166-0.145)*2*inbe(i)
    ovs(i+3)=((9.707-4.262)+(2.477-1.182)*(bb(1)+bb(2)+bb(3)-3)**0.27
        +(-11.689+5.297)*(bb(1)+bb(2)+bb(3)-3)**0.06)*2*be(i+3)-
        1(0.166-0.145)*2*inbe(i+3)

    end do

    do i=1,3
        dd(i,i)=2*cons3 -2*prel- 2*(0.166-0.145)
        dd(i+3,i+3) = cons3 -prel - (0.166-0.145)+uppj
    end do

    do i=1,3
        aa(i,i ) =dd(i ,i )+ee(i,i)
        aa(i+3,i+3) =dd(i+3, i+3)+ee(i+3,i+3)
    end do

c Add volumetric correction to aa

    aa(1,2) = aa(1,2) + uppj
    aa(2,1) = aa(1,2)
    aa(1,3) = aa(1,3) + uppj
    aa(3,1) = aa(1,3)
    aa(2,3) = aa(2,3) + uppj
    aa(3,2) = aa(2,3)

c Compute stored energy

estore = aminbulk*logj*logj*0.5d0
    + mu*(0.5d0*(bb(1) + bb(2) + bb(3)) - 3.0d0 - logj)

end