LARGE DEFORMATION ANALYSIS OF
COMPOSITE SHELLS OF REVOLUTION

by

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ABSTRACT

The linear and nonlinear analyses of shells of revolution with discontinuities in slope and curvature of the meridian have been carried out in this thesis. All possible boundary conditions and discontinuities in thickness along the meridian have been taken into account. Specifically, extensive numerical results have been obtained for spherical, ellipsoidal, conical, and plate head pressure vessels based on a linear and a nonlinear theory. Numerical solutions are also obtained for general composite shells and for critical pressures of different types of spherical shells.

Reissner's nonlinear theory of axisymmetric deformations of shells of revolution has been used in this analysis. Explicit equations applicable at the apex of a closed shell have been derived for both the linear and nonlinear theories. The basic concept of multisegment integration as developed by Kalnins and Lestini has been utilized for obtaining the solutions of the governing linear and nonlinear differential equations.

In the course of this investigation a new method of multisegment integration has been developed for solving linear and nonlinear boundary value problems which can not be
solved by direct integration. The new method, which requires much less computational work than the method of Kalnins and Lestinig, has been employed to solve a number of different problems.

A computer program has been developed and presented in the appendix which solves both linear and nonlinear axi-symmetric shells whose meridians are composed of straight, circular, and elliptic elements. The shell may have discontinuities in thickness and may be subjected to any practical boundary conditions. The same program solves for critical pressures of the above mentioned shells when instability is a problem.
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NOTATION

\( b_1, b_{m+1} = (m, 1) \) matrices, contain prescribed variables at the boundary.

\( C = \frac{Eh}{(1 - \nu^2)S/R} \) : extensional rigidity

\( \overline{C} = \frac{(1 - \nu^2)\xi_e}{R} \)

\( D = \frac{Eh^3}{12(1 - \nu^2)} \) : bending rigidity

\( \overline{D} = \frac{1}{12(1 - \nu^2) \frac{P}{T^2}} \)

\( E = \) Young's modulus

\( H = \) horizontal stress resultant

\( \overline{H} = \frac{H}{PR} \) : nondimensional horizontal stress resultant

\( h = \) shell thickness

\( I = (m, m) \) unit matrix

\( k_\theta, k_\xi = \) changes of curvature of the middle surface of shell

\( \overline{k_\theta} = k_\theta \xi_e \) : nondimensional value of \( k_\theta \)

\( \overline{k_\xi} = k_\xi \xi_e \) : nondimensional value of \( k_\xi \)

\( L = \frac{R}{TP} \)

\( M = \) number of segments

\( m = \) order of system of differential equations

\( M_\xi = \) meridional couple resultant

\( M_\theta = \) circumferential couple resultant

\( \overline{M_\xi} = \frac{M_\xi}{PRh} \) : nondimensional value of \( M_\xi \)

\( \overline{M_\theta} = \frac{M_\theta}{PRh} \) : nondimensional value of \( M_\theta \)

\( N_\xi = \) meridional stress resultant
\( N_\theta \) = circumferential stress resultant  
\( \bar{N}_S = N_S / PR \) : nondimensional value of \( N_S \)  
\( \bar{N}_\theta = N_\theta / PR \) : nondimensional value of \( N_\theta \)  
\( P \) = outward normal pressure  
\( \bar{P} = P/E \) : nondimensional value of \( P \)  
\( p_V \) = vertical component of surface load  
\( p_H \) = horizontal component of surface load  
\( Q \) = transverse shear stress resultant  
\( R \) = radius of cylindrical part for pressure vessels, or radius of base circle of shells of revolution  
\( \bar{R} = \xi e / R \)  
\( R_S, R_\theta \) = principal radii of curvature of middle surface of shell  
\( r_o \) = distance of a point on undeformed middle surface from axis of symmetry  
\( r = r_o + u \) : radial distance of a point on deformed middle surface from axis of symmetry  
\( \bar{r}_o = r_o / \xi e \) : nondimensional value of \( r_o \)  
\( S_i \) = denotes \( i \)th segment  
\( T_1, T_{M+1} = (m, m) \) matrices, given by boundary conditions  
\( \bar{T} = R/h \)  
\( u \) = radial (horizontal) displacement  
\( \bar{u} = uEh / PR^2 \) : nondimensional horizontal displacement  
\( V \) = vertical stress resultant  
\( \bar{V} = V/PR \) : nondimensional vertical stress resultant
\[ w = \text{axial (vertical) displacement} \]
\[ \tilde{w} = \frac{wEH}{PR^2} : \text{nondimensional axial displacement} \]
\[ x = \text{independent variable, or a parameter of conical head} \]
\[ x_i = \text{end point of segment} \]
\[ y(x) = (m, 1) \text{ matrix, contains } m \text{ variables} \]
\[ z_0 = \text{axial distance of a point on undeformed middle surface of shell} \]
\[ z = z_0 + w : \text{axial distance of a point on deformed middle surface} \]
\[ \alpha = \text{parameter of meridian of deformed shell, defined in Eq. (1c), or semi-apex angle of conical shell} \]
\[ \alpha_0 = \text{value of } \alpha \text{ corresponding to undeformed shell} \]
\[ \tilde{\beta} = \beta \]
\[ \beta = \text{angle of rotation of normal after deformation} \]
\[ \varepsilon_{\theta}, \varepsilon_{\xi} = \text{middle surface strains} \]
\[ \tilde{\varepsilon}_{\theta} = \varepsilon_{\theta} \frac{Eh}{PR^2} : \text{nondimensional value of } \varepsilon_{\theta} \]
\[ \tilde{\varepsilon}_{\xi} = \varepsilon_{\xi} \frac{Eh}{PR^2} : \text{nondimensional value of } \varepsilon_{\xi} \]
\[ \zeta = \text{normal distance of a point in the shell from middle surface} \]
\[ \bar{\eta} = \frac{x}{R} : \text{parameter of conical head} \]
\[ \xi = \text{parameter of shell meridian, or distance measured along meridian} \]
\[ \bar{\xi} = \frac{\xi}{\xi_e} : \text{nondimensional meridional distance from apex} \]
\[ \xi_e = \text{total meridional length of head for pressure vessels, or total meridional length of composite shell} \]
\[ \phi_0 = \text{angle between normal and axis of symmetry before deformation (meridional angle)} \]
\( \phi = \phi_0 - \beta \): angle between normal and axis of symmetry after deformation

\( \nu = \text{Poisson's ratio} \)

\( \sigma_{ai} = \frac{N_i}{h} + \frac{6M_i}{h^2} \): meridional stress at the extreme inner fibre

\( \sigma_{ao} = \frac{N_o}{h} - \frac{6M_o}{h^2} \): meridional stress at the extreme outer fibre

\( \sigma_{ci} = \frac{N_i}{h} + \frac{6M_i}{h^2} \): circumferential stress at the extreme inner fibre

\( \sigma_{co} = \frac{N_o}{h} - \frac{6M_o}{h^2} \): circumferential stress at the extreme outer fibre

\( \bar{\sigma}_{ai} = \frac{\sigma_{ai}}{E} \): nondimensional value of \( \sigma_{ai} \)

\( \bar{\sigma}_{ao} = \frac{\sigma_{ao}}{E} \): nondimensional value of \( \sigma_{ao} \)

\( \bar{\sigma}_{ci} = \frac{\sigma_{ci}}{E} \): nondimensional value of \( \sigma_{ci} \)

\( \bar{\sigma}_{co} = \frac{\sigma_{co}}{E} \): nondimensional value of \( \sigma_{co} \)

\( \sigma_{o} = \sigma_{co}/(PR/h) \): external surface stress index

\( \sigma_{i} = \sigma_{ci}/(PR/h) \): internal surface stress index

(\ldots\,)' = \text{derivative with respect to } \xi \text{ or } \bar{\xi}
CHAPTER 1

INTRODUCTION

(i) PRELIMINARY

With the passage of time, shell structures are being utilized more and more. As can be seen from the present level of interest in this particular field, the peak has yet to be reached. Although numerous shell structures are now being used in practice, sophisticated shell analysis had been very limited in the past. Just a few years back shell analysis consisted only of linear theory and even that was restricted by wide simplification. As interest in shell structures increased, more sophisticated mathematical analysis of shells were sought. Nonlinear shell analysis, which takes into account finite shell deformation under loading as well as nonlinear stress-strain relations, is currently in its infancy. Though the tempo is very high in this field and though innumerable papers can be seen on this subject, the field is widely unexplored. The reason behind it is that a problem of this type requires the integration of a rather complicated system of simultaneous nonlinear differential equations. Consequently, with the advent of large high speed
digital computers, the authors of several recent papers have focussed their attention on the methods of numerical integration of thin shell equations.

Shell structures are characteristically different from other structures in the sense that large deformation takes place in many shell structures under loading. This sometimes necessitates the consideration of large deformation in the formulation of the problems to obtain reasonable information of the structure. Thus, shell analysis ultimately passes to the domain of nonlinear mathematical analysis. The nonlinearity is introduced into the governing equations of elasticity in three ways: (a) through the strain-displacement relations, (b) through the equations of equilibrium of a volume element of the body, and (c) through the stress-strain relations. In (a) and (b) the retention of the nonlinear terms is conditioned by geometric considerations, i.e., the necessity of taking into account the angles of rotation in determining the changes of dimension in the line elements and in formulating the conditions of equilibrium of a volume element. On the other hand, nonlinear terms appear in the third set of equations (c) if the material doesn't behave in a linearly elastic fashion.

Hence there are two types of nonlinearity: geometrical
and physical. In the problems of shell structures, the angles of rotation can be large, but the strains can be quite elastic. An example of this type of problem is afforded by the bending of a thin steel strip. It is well known that strips of good steel can straighten out without traces of residual deformation after having their ends brought together. This bears witness to the fact that in these strips, even for large displacement and angle of rotation, the stresses do not exceed the yield point. Thus, many shell structures belong to a class of problems which are physically linear but geometrically nonlinear.

(ii) RESUMÉ OF NONLINEAR SHELL ANALYSIS

That linear shell analysis fails to give proper information about the shell stresses and deformation in many problems can be seen in recent papers on the nonlinear shell analysis (1 - 16)*. For this reason the use of nonlinear theory has become rather widely accepted as a plausible basis for predictions of elastic strengths of thin shells of various geometries. Most of the papers currently found in the literature are concerned with the analysis of shells of revolution.

* Number in parenthesis refer to references in the bibliography.
The basic concept of finite deflection analysis, due to Donnell (17), has been employed by numerous investigators to establish collapse loads of cylindrical shells subjected to various loadings. Finite deflection analysis has also been successful in offering reasonable predictions of the elastic buckling loads of shallow spherical caps subject to uniformly distributed external pressure. Kaplan and Fung (18) have presented a perturbation solution to the nonlinear equations that agrees quite well with results of their experiments for very shallow clamped edged shells. Archer (19) extended these results to a greater range of shells. As can be seen from recent papers, very extensive work has been done in this field (1-4, 6, 7, 12, 16, 18). Ball (22) has considered the problems of arbitrarily loaded shells of revolution and obtained solution for a clamped shallow spherical shell uniformly loaded over one-half of its surface. A number of papers based on the nonlinear analysis of stiffened shells, multilayered shells and sandwich shells can also be found in the current literature (39-41, 43-50). Thus, finite deflection studies are available for cylindrical, spherical, as well as other types of shells subject to variety of loadings and boundary conditions. In all cases the predictions of these theories are in better agreement with experimental evidence than those of the classical investigations based upon infinitesimal deformations. Unfortunately, the finite deflection analysis of shells with geometrical discontinuity has not yet been carried out. Until now there has been no attempt at a nonlinear analysis
of shell structures with geometrical discontinuity though it constitutes a quite common feature of shells.

(iii) OBJECT OF THIS INVESTIGATION

The purpose of this investigation is to provide some insight into the nonlinear analysis of shells of revolution with discontinuities in slope and curvature of the meridian. Specifically, an attempt has been made to obtain numerical solution of the following common pressure vessel problems based on both the linear and nonlinear theories of shells:

(a) spherical head pressure vessel (Fig. 1a)
(b) ellipsoidal head pressure vessel (Fig. 1b)
(c) conical head pressure vessel (Fig. 1c)
(d) flat end pressure vessel (Fig. 1d).

A partial parameter study was made in each of the above mentioned cases. The computer program developed for computational purpose is perfectly general to accept, in addition to the pressure vessel problems, problems of any shell of revolution whose meridians are composed of straight, circular, and elliptical elements (see Fig. 2). The program handles all possible boundary conditions and discontinuities in shell thickness.

Solutions to some stability problems have been obtained
to verify that the method employed in this work can be extended to determine critical loads for the types of shells considered.

In the problems considered there are very large rotations at the discontinuities of the meridian and at thickness changes in the shell. The rotation at a discontinuity makes it essential to seek solutions of these problems based on the nonlinear theory of shells.

In the course of this investigation, a new method of multisegment integration has been developed for solving boundary value problems of a system of nonlinear ordinary differential equations. The method of multisegment integration is used to solve those boundary value problems of ordinary differential equations which can not be solved by direct integration, because direct integration loses all of its accuracy in the process of subtraction of almost equal numbers in evaluating the unknown boundary values. The method developed here involves much less computational work than that of multisegment integration developed by Kalnins and Lestingi (2). The new method has been applied to a number of problems to ascertain its soundness and accuracy.

(iv) RESUMÉ OF WORK IN PRESSURE VESSEL PROBLEMS

Until now almost all work done in the field of pressure
vessels is based on the linear theory. Timoshenko and Woinowsky-Krieger (20) report an approximate linear analysis of a pressure vessel with hemispherical head which, in the light of the deficiencies of linear analysis at high loading intensities, fails to give accurate information of the true nature of the stresses and deformations. The general case of a spherical head has been mentioned by Flugge (66) where he discusses the solutions of two particular problems: (a) a hemispherical end cap and (b) a 90°-spherical end cap; both the solutions are based on approximate linear theory.

A program was instituted in 1946 by the Design Division of the Pressure Vessel Research Committee of the Welding Research Council of the Engineering Foundation, New York, which consisted of both analytical and experimental investigations intended to benefit engineers engaged in the design and manufacture of pressure vessels. Under this program, Watts and Lang carried out purely analytical analyses of pressure vessels for three special end caps: (a) conical head (28), (b) flat head (29), and (c) hemispherical head (30). They employed the linear bending theory of Refs. (20, 31). In effect they used the existing solutions of the approximate bending theories of two different geometries and imposed the conditions of continuity at the junction in terms
of radial displacement and rotation components to determine the transverse shear and bending moment developed at the junction necessary to maintain continuity. These papers contain extensive tables of transverse shear, bending moment, and stresses at the junction for variation of different parameters for all three problems. Unfortunately, they did not compare their results with experimental results and thus the deviation of their results from experimental results, even within the range of validity of linear analysis, can not be seen readily.

A finite-difference approximation to the Love-Meissner linear bending equations for the ellipsoidal shell has been used by Kraus, Bilodeau and Langer (35) to calculate stresses in thin-walled pressure vessels with ellipsoidal heads. These results are reported in the form of stress indexes and stress intensity indexes, which are obtained by dividing all stresses in the head and cylinder by the hoop stress at the inner surface of the cylinder far from the junction. Comparisons are made with available experimental results (51, 52) and other analytical work in the literature. They have shown that, within the limitation of the theory, their results compare favourably in certain cases, while they differ appreciably in others. One of the interesting points of this finite-difference solution is that it predicts almost the same peak value of the stresses as the experimental results.
All these results of linear analysis point out the profound effect of the joint in terms of the stresses and deformations. The nature of the problem suggests that there is considerable deformation near the joint of the cylindrical part and its head, and any reasonable estimation of the stresses would have to be sought through a nonlinear formulation of the problem. This nonlinear formulation of the problem permits a more accurate study of stresses as well as the buckling strength of shell structure.

(v) METHODS OF SOLVING NONLINEAR DIFFERENTIAL EQUATIONS

The problem we are concerned with in this thesis requires the solution of a system of nonlinear ordinary differential equations with geometrical discontinuities. Unfortunately, the development of modern mathematics has provided the applied scientists hardly with any general method for solving nonlinear ordinary and nonlinear partial differential equations. Indeed, at a time when nonlinear physical models are quite common and applied scientists eventually need methods of obtaining solutions, analytical mathematical methods are mostly restricted to questions of existence, uniqueness and stability. The situation has been brightened considerably, however, with the development of the modern digital computer and with the simultaneous revitalization and growth of the study of numerical methods.
Though there are quite a number of approximate methods available for solving nonlinear differential equations, there is hardly any method proved to be unique or advantageous over the other methods, leaving aside its applicability to a specific problem. Thus it will be appropriate to look into a short survey of the generally used methods and the reasons for applying the multisegment method of integration in this analysis. The methods most frequently used in solving nonlinear differential equations are:

(1) asymptotic integration (8, 55, 32)
(2) direct numerical integration (38, 37)
(3) finite difference methods
(4) perturbation techniques
(5) Newton's method
(6) method of multisegment integration
(7) method of power series expansion.

In addition to the above mentioned methods there are other methods existing in the literature, namely 'Reversion Method' (36, 42), "variation of Parameter" (27, 55, 36), "Averaging Methods Based on Residuals - (a) Galerkin's Method (36) and (b) Ritz Method" (36), and "the Principle of Harmonic Balance".
Asymptotic integration is not a general method and its scope of application is very limited as can be seen from Refs. (8, 35, 33). Reissner discusses some of the solutions and the limitation of this method in Ref. (33). In the application of this method the solution is expressed in the form of a series where the terms of the series are the inverse powers of the largest parameter in the differential equations (33). Determination of the terms of the series becomes extremely difficult and most of the time the solution contains only the first term approximation. Considering the complexity of the shell equations and remembering that there are geometrical discontinuities at intermediate points, the possibility of obtaining a reasonably good solution by any approximate analytical method is highly unlikely.

Though the direct integration approach has certain advantages, it also has a serious disadvantage, i.e., when the length of the shell is large, a loss of accuracy invariably results. This phenomenon is clearly pointed out in Ref. (23). The loss of accuracy does not result from the accumulative errors in integration, but it is caused by the subtraction of almost equal numbers in the process of determining unknown boundary values. It follows that for every set of geometric and material
parameters of the shell there is a critical length beyond which the solution loses all accuracy.

Finite-difference methods are the most widely used techniques for solving nonlinear differential equations. The advantage of the finite-difference technique over direct integration is that it can avoid the above mentioned loss of accuracy. Here we are concerned with the solution of a large number of nonlinear algebraic equations which would probably have a number of solutions. Most of the time the solutions of nonlinear equations are obtained as the solutions of a sequence of linear equations. It is often difficult to distinguish between instability in the sequence of numerical calculations and the point of instability of the differential equations which correspond to the classical buckling pressures. The different techniques used to solve the nonlinear algebraic equations involved in this approach can be seen in Refs. (1, 3-5). It is usually the case that the finite-difference method is not suitable for application to problems which contain discontinuities or rapidly varying parameters at a point.

The perturbation technique is also a frequently used analytical method for solving nonlinear differential equations (6, 15, 24, 25, 27). Here the functions to be obtained are
expressed in the form of power series in terms of a perturbation parameter and the solutions are obtained as the solutions of a sequence of linear differential equations. The solutions of the linear equations are the terms of the series. There are several drawbacks in this technique. First, there must be a natural or artificially created perturbation parameter which contributes to the nonlinearity of the problem and this parameter must be small enough so that the series is convergent. Secondly, the solutions are obtained as the solutions of a sequence of linear differential equations which constitute a problem in no way simpler than the original problem if the solutions of these latter equations are to be obtained by some numerical method. The linear equations obtained from the original equations retain the same complex form except the nonlinear terms, and the number of terms in the linear differential equations increases with the increasing order of the function of the series.

This method is particularly suitable for nonlinear dynamics problems of rigid bodies (25) where a natural perturbation parameter exists and the solutions are periodic. In nonlinear shell analysis this technique is used by Archer (19) to clamped spherical shell under uniform pressure where the nondimensionalised radial displacement at the point of maximum deflection has been used as the perturbation parameter. From this solution it is seen
that the computational work involved in obtaining numerical values is so extensive that it would be desirable to apply some numerical technique from the beginning. The result of this solution is compared with experimental and other results by Reiss (12) where it is shown that the perturbation solution is in serious disagreement with the rest of the results. Unless it is positively proved that the series is convergent, either by mathematical analysis, which is hardly possible, or by comparing with known results as is done in Ref. (15), the result is not reliable. In our problem we have to solve a number of sets of differential equations where no suitable perturbation parameter is obvious which is applicable to all the sets. The convergence of the series under the present circumstances can only be established by comparing with known results, but there exist no such results. Moreover, the solutions of the linear equations will have to be sought in the same way as for the nonlinear equations except that the number of equations will be greatly increased.

Newton's method for solving nonlinear differential equations is the extension of Newton's method for calculating roots of algebraic equations. The approach is to express the solution as the sum of two parts; the first part is a known function and the second part is a correction to the known function.
A governing equation for the correction is obtained by substituting the assumed function into the nonlinear equations and neglecting terms which are nonlinear (6). This method does not require the perturbation parameter to be small, as is necessary in the perturbation technique, but involves the solution of a sequence of linear differential equations as in the latter. These linear equations have variable coefficients and generally cannot be solved in closed form. It is paradoxical that the greatest obstacle in solving nonlinear problems is the inability to solve linear differential equations in closed form.

The multi-segment method of integration is the most recent method developed and used by Kalnins and Lestingi (2) to solve nonlinear differential equations. This method involves:

(a) division of the total interval into a number of segments

(b) initial value integration of a system of first order differential equations over each segment

(c) solution of a system of matrix equations which ensures continuity of the variables at the ends of the segments

(d) repetition of (b) and (c) till convergence is
achieved.

(e) integration of an initial value problem to obtain answers at any desired point within each segment.

The main advantage of this method over the finite-difference method is that the solution is obtained everywhere with uniform accuracy, and the iteration process with respect to the mesh size, which is required with the finite-difference approach, is eliminated. But the feature which makes this method most attractive for the problem at hand is that any discontinuity, either in geometry or in loading, can be easily handled by requiring that the end point of a segment coincides with the location of the discontinuity. As the integration is restricted at the beginning of each segment, the precise effect of the discontinuity is obtained by this method which is the main purpose of our present analysis. Moreover, this method is the most accurate of all numerical methods because the problem is solved in the form of a system of first order differential equations in which no derivatives of geometrical or elastic properties appear and because no further numerical derivatives must be taken to obtain any desired results in the calculation.
CHAPTER 2

SHELL THEORY

(1) INTRODUCTION

The literature on shell theories is not devoid of papers in which some of the effects of finite displacements on the deformation of thin shells are accounted for. The earliest work of some generality is Marguerre's nonlinear theory of shallow shells (53). Donnell (54) developed an approximate theory specifically for cylinders and suggested its extension for a general middle surface. The result, a theory for what might be termed "quasi-shallow shells", has been worked out by a number of authors, notably Mushtari and Galimov (55).

The earliest work of a completely general nature appears to be the papers by Synge and Chien (56) followed by a series of papers by Chien (57, 58). The theory of shells developed by Synge and Chien avoids the use of displacements as unknowns in the equations. The theory is deduced from the three-dimensional theory of elasticity and then, by means of series expansion in powers of a small thickness parameter, approximate theories of thin shells are derived.
Another general formulation of the problem is worked out by Ericksen and Truesdell (59). They developed it as a two-dimensional theory instead of attempting to deduce it from three-dimensional theory of elasticity. They were able to account for transverse shear and normal strains and the rotations associated with couple stresses. The two-dimensional approach to shell theory really evades the question of the approximations involved in the descent from three dimensions, but this seems to be a virtue rather than a defect. Such questions are effectively isolated and shown to belong to the part of the theory in which constitutive relations* are established.

Novozhilov (60) has presented an incomplete treatment of the general large deflection theory of thin shells based on the assumption of small middle surface strains.

Other developments which also employ linear constitutive relations are founded upon the Kirchhoff hypothesis and often contain other approximations. Among these are Reissner's (33, 61) formulation of axisymmetric deformation of shells of revolution and the more general works of Sanders (62) and Leonard (63). Beginning with the three-dimensional field equations Naghdi and Nordgren deduced an exact, complete, and fully general nonlinear theory of elastic shells founded upon the Kirchhoff hypothesis.

* Constitutive relations are those which express stress and couple resultants in terms of strains.
Several nonlinear theories for thin shells have been derived in increasing stages of approximation. In most cases the theories are first approximative theories in the sense that transverse shears and normal strains are neglected. But, as may be seen from the results of these papers, the exact and general equations characterizing the deformation of an elastic shell, even under the Kirchhoff hypothesis, are still fairly complex and discouraging from the point of view of application.

With this in mind and motivated by the extremely simple structure of Reissner's derivations the author has used the theory of axisymmetric shell as presented by Reissner (33). This theory differs from others in using radial and axial components of displacements and stress resultants rather than the customary practice of using normal and tangential components of displacements and stress resultants. The modified definition of displacements and stress resultants is very well suited to the problems we are concerned with in this thesis.

(ii) REISSNER'S THEORY OF AXISYMMETRIC DEFORMATIONS OF SHELLS OF REVOLUTION

The basic equations of Reissner's theory of finite axisymmetric deformations of shells of revolution which form the basis of this analysis are presented here for ready reference.
The equation of the meridian of the shell is written in the parametric form (Fig. 3)

\[ r = r(\xi) , \quad z = z(\xi) . \]  \hfill (1a)

The angle \( \phi \) of the tangent to the meridian curve is given by

\[ \cos \phi = r'/\alpha , \quad \sin \phi = z'/\alpha \]  \hfill (1b)

where primes denote differentiation with respect to \( \xi \) and

where \( \alpha \) is given by

\[ \alpha = [(r')^2 + (z')^2]^{1/2} . \]  \hfill (1c)

The principal radii of curvature of the middle surface of the shell are given by

\[ R_\xi = \alpha/\phi' , \quad R_\phi = r/\sin \phi . \]  \hfill (1d)

With reference to Fig. 4, the equation of the deformed middle surface is written in the form

\[ r = r_0 + u , \quad z = z_0 + w \]  \hfill (2a)

where the subscript \( o \) refers to undeformed middle surface and the quantities \( u \) and \( w \) are, respectively, the radial and axial components of displacements.

The angle enclosed by the tangents to the deformed and undeformed meridian, at the same material point, is given by
\[ \beta = \phi_o - \phi \]  

(2b)

With the above definition of displacements and rotation, the strain components and curvature changes of the deformed middle surface are given by the following equations:

\[ \varepsilon_\xi = (\alpha - \alpha_o)/\alpha_o = (\cos \phi_o/\cos \phi)(1 + u'/r_o') - 1 \]  

(2c)

\[ \varepsilon_\theta = u/r_o \]  

(2d)

\[ k_\xi = - (\phi' - \phi_o')/\alpha_o = \beta'/\alpha_o \]  

(2e)

\[ k_\theta = - (\sin \phi - \sin \phi_o)/r_o \]  

(2f)

The equation containing the axial displacement component \( w \) is introduced as:

\[ w' = \alpha \sin \phi - z_o' \]  

(2g)

With the definition of stress resultants and stress couples as shown in Fig. 4 and Fig. 5, the three equations of equilibrium are written as:

\[ (rV)' + r \alpha_p_v = 0 \]  

(3a)

\[ (rH)' - \alpha N_\theta + r \alpha_p_h = 0 \]  

(3b)

\[ (rM_s)' - \alpha \cos \phi M_\theta + r \alpha(H \sin \phi - v \cos \phi) = 0 \]  

(3c)
Equation (3a) is the condition of force equilibrium in the axial direction, Eq. (3b) is the condition of force equilibrium in the radial direction, while Eq. (3c) is the condition of moment equilibrium about circumferential tangent.

With the assumption that the behaviour is elastic, the relations between strains and stress resultants are given by

\[
\begin{align*}
C \varepsilon_5 &= N_5 - \nu N_\theta, & C \varepsilon_\theta &= N_\theta - \nu N_5 \quad (4a) \\
N_5 &= D(k_5 + \nu k_\theta), & M_\theta &= D(k_\theta + \nu k_5) \quad (4b)
\end{align*}
\]

where \( C = Eh \), \( D = Eh^3/[12(1-\nu^2)] \), and \( h \) is the thickness of the shell. The radial stress resultant \( H \) and axial stress resultant \( V \) are related to \( N_5 \) and transverse shear \( Q \) as follows:

\[
N_5 = H \cos \phi + V \sin \phi, \quad Q = -H \sin \phi + V \cos \phi. \quad (4c)
\]

(iii) DERIVATION OF THE FIELD EQUATIONS

The order of the system of equations (2-4) is six with respect to \( \xi \), and consequently it is possible to reduce Eqs. (2-4) to six first-order differential equations which involve six unknowns. In the following derivation the six fundamental variables are taken as \( u, \beta, w, V, H, M_5 \) and the differential
equations are expressed in terms of these variables. The independent variable $\xi$ is taken as the distance measured along the meridian of the shell so that the differential equations can be used for all possible geometrical shapes of the meridian. With this definition of $\xi$, we have, from Eq. (1c)

$$\alpha_o = \left( (r_o')^2 + (z_o')^2 \right)^{\frac{1}{2}} = 1.$$  

From the geometry of the meridian, which is not yet specified, we know

$$r_o = r_o(\xi) \quad (5a)$$

$$\theta_o = \theta_o(\xi) \quad (5b)$$

The following equations are rewritten from the previous section in such an order that, when evaluated serially, they are in terms of fundamental variables. This is done in order to keep the fundamental set of differential equations as simple as possible. Rewriting Eqs. (2d), (2a), (2b), (2f), (4c), (4b) in that order, we have,

$$\varepsilon_\theta = u/r_o \quad (5c)$$

$$r = r_o + u \quad (5d)$$

$$\phi = \phi_o - \beta \quad (5e)$$
\[ k_\theta = \frac{(\sin \theta_0 - \sin \theta)}{r_0} \quad (5f) \]

\[ N_\xi = H \cos \theta + V \sin \theta \quad (5g) \]

\[ k_\xi = \frac{M_\xi}{D - \nu k_\theta} \quad (5h) \]

\[ M_\theta = D(k_\theta + \nu k_\xi) \quad (5i) \]

Eliminating \( N_\theta \) from Eqs. (4a) it follows that

\[ \varepsilon_\xi = \frac{(1 - \nu^2)}{C} \cdot N_\xi - \nu \varepsilon_\theta \quad (5j) \]

Similarly, eliminating \( N_\xi \) from Eqs. (4a) and rearranging, we obtain

\[ N_\theta = \left[ \frac{C}{(1 - \nu^2)} \right] \left\{ \varepsilon_\theta + \nu \varepsilon_\xi \right\} \quad (5k) \]

Rearrangement of Eq. (2c) and substitution of \( \alpha_0 = 1 \) leads to

\[ \alpha = 1 + \varepsilon_\xi \quad (5l) \]

Elimination of \( z_o' \) from Eq. (2g) by means of Eq. (1b) gives

\[ \frac{dw}{d\xi} = \alpha \sin \theta - \sin \theta_0 \quad (5m) \]

Substituting the values of \( \varepsilon_\xi \) from Eq. (5k) and \( r_o' \) from Eq. (1b) in the Eq. (2c), we have

\[ \frac{du}{d\xi} = \alpha \cos \theta - \cos \theta_0 \quad (5n) \]
From Eq. (2e) we get an expression for $\beta'$ in the form

$$\frac{d\beta}{ds} = k_s$$  \hspace{1cm} (5o)

Expansion of the three equations of equilibrium and elimination of $p_V$, $p_H$, and $r'$ from these equations result in the following expressions for $V'$, $H'$, and $M_s'$.

$$\frac{dV}{ds} = -\alpha \left[ \frac{V \cos \phi}{r} - P \cos \phi \right]$$  \hspace{1cm} (5p)

$$\frac{dH}{ds} = -\alpha \left[ \frac{H \cos \phi - N_\theta}{r} + P \sin \phi \right]$$  \hspace{1cm} (5q)

$$\frac{dM_s}{ds} = \alpha \cos \phi \left( M_\theta - M_s \right)/r - \alpha \left( H \sin \phi - V \cos \phi \right)$$  \hspace{1cm} (5r)

where $P$ is the outward normal pressure.

Eqs. (5) are the nonlinear governing equations of the axisymmetric deformations of shells of revolution expressed in terms of the fundamental variables. It should be noted that this fundamental set of differential and algebraic equations are expressed in such a manner that all the quantities of physical importance are evaluated during the process of solution of these equations.

(iv) EQUATIONS FOR THE APEX

The fundamental set of equations derived in the previous section is singular at the pole (Fig.2). In order to remove this singularity the condition that all the physical quantities must be regular at
at the pole should be imposed. From symmetry at the pole, we have,

\[ u = \beta = 0, \]

and as there is no concentrated load at the pole it follows that

\[ v = 0. \]

In the following derivation it is assumed that \( \xi \) is measured from the pole of the axisymmetric shell.

Since \( \varepsilon_\theta \) and \( \varepsilon_\theta' \) must be regular at \( \xi = 0 \), we have, from Eq. (5c)

\[
\lim_{\xi \to 0} \varepsilon_\theta = \frac{u'}{r_o},
\]

and

\[
\lim_{\xi \to 0} \varepsilon_\theta' = \frac{u''r_o' - u'r_o''}{2(r_o')^2}.
\]

From Eq. (1b) it follows that

\[ r_o' = \cos \phi_o \]

and, therefore,

\[ r_o'' = -\sin \phi_o \cdot \phi_o'. \]

Substitution of the values of \( r_o' \) and \( r_o'' \) into the expressions of \( \varepsilon_\theta \) and \( \varepsilon_\theta' \) give
Similarly, the following equations can be deduced from Eqs. (5) by taking the limit as $\xi \to 0$.

\[
\lim_{\xi \to 0} \phi = \phi_0 \tag{6c}
\]

\[
\lim_{\xi \to 0} \phi' = \phi_0' - \beta' \tag{6d}
\]

\[
\lim_{\xi \to 0} k_\phi = \beta' \tag{6e}
\]

\[
\lim_{\xi \to 0} k_{\phi'} = \frac{1}{2}(\beta'' - \phi_0' \beta' \tan \phi_0) \tag{6f}
\]

\[
\lim_{\xi \to 0} N_\xi = H \cos \phi_0 \tag{6g}
\]

\[
\lim_{\xi \to 0} N_\xi' = H' \cos \phi_0 - H\phi_0' \sin \phi_0 + V' \sin \phi_0 \tag{6h}
\]

\[
\lim_{\xi \to 0} M_\phi = \lim_{\xi \to 0} \left[ D(1 - \nu^2)k_{\phi'} + \nu M_{\xi'} \right] \tag{6i}
\]

\[
\lim_{\xi \to 0} N_\phi = \lim_{\xi \to 0} (C\phi_{\phi'} + \nu N_{\xi'}) \tag{6j}
\]

\[
\lim_{\xi \to 0} \alpha = \left[ 1 + \left( \frac{1 - \nu^2}{C} \right) H \cos \phi_0 - \frac{\nu u'}{\cos \phi_0} \right] \tag{6k}
\]
\[ \lim_{\xi \to 0} \alpha' = \lim_{\xi \to 0} \left[ \frac{(1-v^2)}{C} N_\xi - v e_{\alpha'} \right] \quad (6k) \]

\[ \lim_{\xi \to 0} u' = \frac{1 - v}{C} H \cos^2 \phi_o \quad (6m) \]

\[ \lim_{\xi \to 0} \beta' = \frac{M_{\xi}}{D(1 + v)} \quad (6n) \]

\[ \lim_{\xi \to 0} w' = \frac{1 - v}{C} H \sin \phi_o \cos \phi_o \quad (6o) \]

Substitution of Eq. (6m) in Eq. (6k) gives

\[ \lim_{\xi \to 0} \alpha' = 1 + \frac{(1 - v)}{C} H \cos \phi_o \quad (6p) \]

Now

\[ \lim_{\xi \to 0} \frac{V}{r} = \frac{V'}{\alpha \cos \phi_o} \quad (6q) \]

Substituting Eq. (6q) in Eq. (6r) and solving for \( V' \), we get,

\[ \lim_{\xi \to 0} V' = \frac{1}{2} \alpha \rho \cos \phi_o \quad (6r) \]

By differentiating Eq. (5n) and taking the limit as \( \xi \to 0 \), the expression for \( u'' \) at the pole can be derived as

\[ \lim_{\xi \to 0} u'' = \left( \frac{2}{2 + v} \right) \left[ \frac{1 - v^2}{C} N_\xi \cos \phi_o + \alpha \beta' \sin \phi_o \right] - u' \phi'_o \tan \phi_o \],

hence from Eq. (6j)
Taking the limit of Eq. (5q) and eliminating \( N'_\theta \), we obtain
\[
\lim_{\xi \to 0} N'_\theta = \frac{1}{(2+\nu)} [(1 + 2\nu)N'_{\xi} + \cot \beta \tan \phi_o ]
\]

In order to evaluate \( M'_S \) at the pole, the expression of \( M'_\theta \) in terms of \( M'_S \) has to be derived first. By differentiating Eq. (5q) and taking the limit as \( \xi \to 0 \), we obtain
\[
\lim_{\xi \to 0} H' = \frac{1}{3} [(1 - \nu)\phi' H + \frac{\alpha \cot \beta'}{\cos \phi_o \tan \phi_o} + \frac{\alpha \beta'}{2} \sin \phi_o \tan \phi_o]
\]

which, when substituted in Eq. (6i), gives
\[
\lim_{\xi \to 0} \beta'' = \left( \frac{2}{2+\nu} \right) \left\{ \frac{M'_S}{D} + \frac{\nu}{2} \beta' \phi' \tan \phi_o \right\}
\]

Taking the limit of Eq.(5r) and eliminating \( M'_\theta \) we get the expression for \( M'_S \) as
\[
\lim_{\xi \to 0} M'_S = \left( \frac{1 + 2\nu}{2+\nu} \right) M'_S - \left( \frac{1-\nu^2}{2+\nu} \right) \beta' \phi' \tan \phi_o
\]

Thus the equations (6m, 6n, 6o, 6r, 6s, 6t) form the fundamental set of differential equations applicable only at the pole, where \( \alpha \), and \( \phi' \) appearing in these equations are given by Eq. (6p) and Eq. (6d), respectively. These equations can further be
simplified if it is assumed that the curvature of the undeformed shell is continuous at the pole. In this case $\theta = 0$ at the pole and, therefore, the fundamental set becomes

$$u' = \frac{(1 - \nu)H}{C} \quad (7a)$$

$$v' = \frac{M_S}{D(1 + \nu)} \quad (7b)$$

$$w' = 0 \quad (7c)$$

$$\alpha = 1 + \frac{(1 - \nu)}{C} H \quad (7d)$$

$$v' = \frac{\alpha P}{2} \quad (7e)$$

$$H' = 0 \quad (7f)$$

$$M_S' = 0 \quad (7g)$$

(v) LINEARIZED EQUATIONS OF AXISYMMETRIC SHELLS

The equations of small-deflection theory follow from the foregoing Eqs. (5) by referring the differential equations of equilibrium (5p) to (5r) together with (5g) to the undeformed shell and by omitting all nonlinear terms in the remaining equations of the fundamental set (5). The resulting equations
are recorded below for ready reference.

\[ \epsilon_\theta = \frac{u}{r_o} \quad (8a) \]

\[ k_\theta = \beta \cos \theta_o / r_o \quad (8b) \]

\[ N_\xi = H \cos \theta_o + V \sin \theta_o \quad (8c) \]

\[ \epsilon_\xi = \frac{1 - \nu^2}{C} N_\xi - \nu \epsilon_\theta \quad (8d) \]

\[ k_\xi = \frac{M_\xi}{D} - \nu k_\theta \quad (8e) \]

\[ N_\theta = \left( \frac{C}{1 - \nu^2} \right) (\epsilon_\theta + \nu \epsilon_\xi) \quad (8f) \]

\[ M_\theta = D (k_\theta + \nu k_\xi) \quad (8g) \]

\[ \omega' = \epsilon_\xi \sin \theta_o - \beta \cos \theta_o \quad (8h) \]

\[ u' = \epsilon_\xi \cos \theta_o + \beta \sin \theta_o \quad (8i) \]

\[ \beta' = k_\xi \quad (8j) \]

\[ Y' = - \left( \frac{V \cos \theta_o}{r_o} - P \cos \theta_o \right) \quad (8k) \]

\[ H' = - \left( (H \cos \theta_o - N_\theta)/r_o + P \sin \theta_o \right) \quad (8l) \]
The corresponding linearized equations at the pole are obtained in the same manner as Eqs. (8). Expressions for $u'$, $\beta'$ and $w'$ remain the same, whereas, the three equations for equilibrium reduce to

\[ V' = \frac{p \cos \phi_o}{2} \] (8m)

\[ H' = \frac{1}{3} [(1-\nu)\phi_o H + \frac{CB'}{\cos \phi_o}] \tan \phi_o - \frac{p \sin \phi_o}{2} \] (8n)

\[ M_\phi' = -\frac{1}{3} [(2+\nu)H \sin \phi_o + D(1-\nu)\phi_o^2 \beta' \tan \phi_o] \] (8o)

In case of continuous curvature of the meridian at the apex the linearized equations applicable at the pole remain the same as the Eqs. (7) except that $\alpha$ is replaced in Eq. (7e) by unity.

(vi) LIMITATIONS OF REISSNER'S THEORY

In addition to the approximation of the Kirchhoff hypothesis, Reissner introduced other plausible assumptions and necessary approximations, particularly in the kinematic description of the shell, in order to render the theory more suitable for
practical application. Although these further assumptions are often justified from physical considerations, nevertheless it is desirable to isolate this effect to indicate precisely the position of the approximate theory relative to the exact.

The exact expressions for the strains of the middle surface (Eqs. 2c, 2d, 2e, 2f) under the Kirchhoff hypothesis are deduced by Naghdi and Nordgren (64) and may be written as

\[ \varepsilon_\xi = \frac{1}{2} \left\{ \frac{\alpha^2 - \alpha_o^2}{\alpha_o^2} \right\} \]  
\[ \varepsilon_\theta = \frac{1}{2} \left\{ \frac{r^2 - r_o^2}{r_o^2} \right\} \]  
\[ k_\xi = - \left[ \frac{\alpha \phi' - \alpha_o \phi_o'}{\alpha_o^2} \right] \]  
\[ k_\theta = - \left[ \frac{r \sin \phi - r_o \sin \phi_o}{r_o^2} \right] \]

Expressing Eq. (9a) and Eq. (9c) as

\[ \varepsilon_\xi = \frac{\alpha - \alpha_o}{\alpha_o} \left\{ 1 + \frac{\alpha - \alpha_o}{2\nu_o} \right\} \]
\[ k_\xi = - \frac{1}{\alpha_o} \left[ \phi' \left( 1 + \frac{\alpha - \alpha_o}{\alpha_o} \right) - \phi_o' \right] \]
it is clearly seen that Eqs. (2c) and (2e) are acceptable approximations of Eqs. (9a) and (9c) provided \( \frac{\alpha - \alpha_0}{\alpha_0} \) is very small compared to unity, or in other words \( \varepsilon_\xi << 1 \).

Similarly, if \( \frac{(r - r_0)}{r_0} << 1 \), then Eqs. (2d, 2f) are plausible representations of Eqs. (9b, 9d), and again both subject to the same identical restriction \( \varepsilon_\theta << 1 \). As pointed out by Naghdi and Nordgren, there should have been other additional terms of the forms \( \frac{1}{2}(\phi^{12} - \phi_0^{12})/\alpha_0^{2} \) and \( \frac{1}{2}(\sin^2 \phi - \sin^2 \phi_0)/r_0^{2} \) in Reissner's expressions [Eq. (13) of Ref. (33)] for respectively the tangential and circumferential components of strains. The effect of these additional terms would show up in the stress resultants and couples equations in the form of terms of higher order in \( h/R \).

Furthermore, in the definition of stress resultants and couples [Eqs. (21) and (22) of Ref. (33)] Reissner has omitted terms of the order \( h/R \), compared with unity, and thus restricted the application of the theory to thin shells. The assumption that the middle surface strains are negligible compared to unity is in complete harmony with the initial Kirchhoff's hypothesis, since for small middle surface strains any change in thickness is likely to be negligible. Moreover, dropping of terms of the order of \( h/R \) in the constitutive equations does not introduce any significant error in the solution in most practical shell problems.
(vii) BOUNDARY CONDITIONS OF AXISYMMETRIC SHELLS

The general boundary conditions of a shell on an edge \( \xi_1 = \text{constant} \) are to prescribe, in Sander's (62) notations,

\[
N_{12} + \frac{1}{2}(3R_2^{-1} - R_1^{-1})M_{12} + \frac{1}{2}(N_{11} + N_{22})\phi \quad \text{or} \quad u_1,
\]

\[
Q_1 - \phi_1 N_{11} \quad \text{or} \quad w,
\]

where \( \xi_1 \) and \( \xi_2 \) are the shell coordinates along the principal lines of curvature; \( N \) and \( M \) are the stress and couple resultants; \( \phi \)'s are the rotations about respective axes; \( u \) and \( w \) are the tangential and normal displacement components. When the quantities in (10a) are specialised for axisymmetric deformations of shells of revolution they reduce to prescribing

\[
N_{11} \quad \text{or} \quad u_1,
\]

\[
Q_1 - \phi_1 N_{11} \quad \text{or} \quad w,
\]

and \( M_{11} \) or \( \phi_1 \).

on an edge \( \xi_1 = \text{constant} \). From (10b) it is seen that the boundary conditions consist of the specification of rotational, tangential and normal restraints at the edge. But in most of
the practical cases of shell problems the conditions of the horizontal and vertical restraints are known rather than those of the normal and tangential restraints. So it is concluded that it will be preferable to specify the boundary conditions in terms of the horizontal and vertical restraints from the point of view of practical application. When this is done, the boundary conditions in terms of the notations used in the body of this thesis will be to prescribe

\[ H \text{ or } u, \]

\[ M_\xi \text{ or } \beta, \quad (10c) \]

and \[ V \text{ or } w. \]

on the edge \( \xi = \text{constant}. \)

(viii) NONDIMENSIONALISATION OF THE EQUATIONS

It is always desirable to solve any engineering problem in terms of non-dimensional quantities in order to decrease the number of input physical parameters as well as to increase the applicability of the solution. With this in mind and also to make the variables more or less of the same order of magnitude the displacement components and stress resultants are expressed as ratios of their actual values to those of the circumferential displacement and stress resultant of an unrestrained thin cylindrical shell. The independent variable \( \xi \) is normalised in such a manner that \( \xi_e \), the total length
of the shell meridian in the general case or the total meridional
length of the head in case of pressure vessel problems, corresponds
to unity (Figs. 1, 2). The normalised quantities are defined
mathematically by the following equations:

\[
\bar{w} = \frac{wEh}{PR^2}, \quad \bar{u} = \frac{uEh}{PR^2}, \quad \bar{H} = \frac{H}{PR}, \quad \bar{V} = \frac{V}{PR}, \quad \bar{\beta} = \beta
\]

\[
\bar{M}_S = \frac{M_S}{PRh}, \quad \bar{M}_\theta = \frac{M_\theta}{PRh}, \quad \bar{N}_S = \frac{N_S}{PR}, \quad \bar{N}_\theta = \frac{N_\theta}{PR}
\]

\[
\bar{\varepsilon}_\theta = \frac{\varepsilon_\theta Eh \bar{\xi}_e}{(PR^2)}, \quad \bar{\varepsilon}_x = \frac{\varepsilon_x Eh \bar{\xi}_e}{(PR^2)}, \quad \bar{k}_\theta = k_\theta \bar{\varepsilon}_e \tag{11}
\]

\[
\bar{K}_S = k_S \bar{\varepsilon}_e, \quad \bar{\xi} = \frac{\xi}{\bar{\xi}_e}, \quad \bar{C} = (1 - \nu^2) \bar{\xi}_e / R, \quad \bar{P} = \frac{P}{E}, \quad \bar{T} = \frac{R}{h}
\]

\[
\bar{R} = \frac{\bar{\xi}_e}{R}, \quad \bar{D} = 1/[12(1 - \nu^2) \bar{P} \bar{T}^2 \bar{R}], \quad \bar{L} = \bar{R}/(\bar{P} \bar{T}), \quad \bar{r}_o = r_o / \bar{\xi}_e
\]

where \( R \) is the radius of the cylindrical part in case of pressure
vessel problems or in general \( R = R_\theta \) at \( \bar{\xi}_e \). With the help
of the normalised quantities defined in Eqs. (11) the fundamental
set of differential Eqs. (8) (linear theory) becomes

\[
\bar{\varepsilon}_\theta = \frac{\bar{u}}{\bar{r}_o} \tag{12a}
\]

\[
\bar{k}_\theta = \bar{\beta} \cos \varphi \bar{r}_o / \bar{r}_o \tag{12b}
\]

\[
\bar{N}_S = \bar{H} \cos \varphi \bar{r}_o + \bar{V} \sin \varphi \bar{r}_o \tag{12c}
\]
(12d) \[ \bar{\varepsilon}_x = \bar{C} \bar{N}_x - \nu \bar{\varepsilon}_\theta \]

(12e) \[ \bar{k}_x = \bar{M}_x / \bar{D} - \nu \bar{k}_\theta \]

(12f) \[ \bar{N}_\theta = (\bar{\varepsilon}_\theta + \nu \bar{\varepsilon}_x) / \bar{C} \]

(12g) \[ \bar{M}_\theta = \bar{D} (\bar{k}_\theta + \nu \bar{k}_x) \]

(12h) \[ \bar{w}' = \bar{\varepsilon}_x \sin \phi_o - \bar{\beta} \cos \phi_o \cdot \bar{L} \]

(12i) \[ \bar{u}' = \bar{\varepsilon}_x \cos \phi_o + \bar{\beta} \sin \phi_o \cdot \bar{L} \]

(12j) \[ \bar{\beta}' = \bar{k}_x \]

(12k) \[ \bar{V}' = - (\bar{V} \cos \phi_o / \bar{r}_o - \bar{R} \cos \phi_o) \]

(12l) \[ \bar{H}' = - (\bar{H} \cos \phi_o \bar{N}_\theta / \bar{r}_o + \bar{R} \sin \phi_o) \]

(12m) \[ \bar{M}_x = - \cos \phi_o (\bar{M}_x - \bar{M}_\theta) / \bar{r}_o - \bar{R} \bar{I} (\bar{H} \sin \phi_o - \bar{V} \cos \phi_o) \]

where \( (\ldots)' = \frac{d}{d\bar{\varepsilon}_x} (\ldots) \).

The corresponding nonlinear Equations of the fundamental set

in non-dimensional form are as follows:

(13a) \[ \bar{\varepsilon}_\theta = \bar{u} / \bar{r}_o \]
\( \phi = \phi_o - \beta \) 

(13b)

\( k_\theta = (\sin \phi_o - \sin \phi)/r_0 \) 

(13c)

\( N_\xi = H \cos \phi + \bar{V} \sin \phi \) 

(13d)

\( \varepsilon_\xi = C N_\xi - \nu \varepsilon_\theta \) 

(13e)

\( k_\xi = M_\xi / D - \nu k_\theta \) 

(13f)

\( N_\theta = (\varepsilon_\theta + \nu \varepsilon_\xi)/C \) 

(13g)

\( M_\theta = D(k_\theta + \nu k_\xi) \) 

(13h)

\( \alpha = L + \varepsilon_\xi \) 

(13i)

\( \bar{r} = L \cdot \bar{r}_o + u \) 

(13j)

\( \bar{w}' = \alpha \sin \phi - L \sin \phi_o \) 

(13k)

\( \bar{u}' = \alpha \cos \phi - L \cos \phi_o \) 

(13l)

\( \bar{\beta}' = k_\xi \) 

(13m)

\( \bar{V}' = -\alpha \cos \phi (\bar{V}/\bar{r} - \bar{P} \bar{T}) \) 

(13o)
\[
\begin{align*}
\bar{H}' &= -\bar{\alpha}(\bar{H} \cos \phi - \bar{M}_\theta)/\bar{r} + \bar{P} \bar{T} \sin \phi \quad (13p) \\
\bar{M}_5' &= \bar{\alpha} \cos \phi(\bar{M}_\theta - \bar{M}_\phi)/\bar{r} \\
&\quad - \bar{\alpha} \bar{P} \bar{T}^2(\bar{H} \sin \phi - \bar{V} \cos \phi) \quad (13q)
\end{align*}
\]

The equations at the pole corresponding to the nonlinear set take the following form under normalization.

\[
\begin{align*}
\bar{u}' &= (1-\nu)\bar{R} \bar{H} \cos^2 \phi_0 \quad (14a) \\
\bar{w}' &= (1-\nu) \bar{R} \bar{H} \cos \phi_0 \sin \phi_0 \quad (14b) \\
\bar{\beta}' &= \bar{M}_5/((1-\nu)\bar{D}) \quad (14c) \\
\bar{\alpha} &= \bar{L} + (1-\nu) \bar{R} \bar{H} \cos \phi_0 \quad (14d) \\
\bar{V}' &= \frac{1}{2} \bar{\alpha} \bar{P} \bar{T} \cos \phi_0 \quad (14e) \\
\bar{H}' &= \frac{1}{3}[(1-\nu)\phi_0'\bar{H} + \bar{\alpha} \bar{\beta}'/(\bar{R} \cos \phi_0)] \tan \phi_0 \quad (14f) \\
&\quad - \frac{1}{2} \bar{\alpha} \bar{P} \bar{T} \sin \phi_0 \\
\bar{M}_5' &= \frac{1}{3}[(\bar{\alpha} \bar{P} \bar{T}^2 \bar{H} \sin \phi_0 + \bar{\beta}' \phi_0' \tan \phi_0)/(12 \bar{P} \bar{R} \bar{T}^2)] \quad (14g)
\end{align*}
\]

Eqs. (14) may be simplified in case of continuous
Meridian at the pole as

\[ \bar{u}' = \bar{c} \frac{\bar{u}}{1 + \nu} \]  

(15a)

\[ \bar{w}' = 0 \]  

(15b)

\[ \bar{v}' = \bar{M}_S' / [(1 + \nu) \bar{D}] \]  

(15c)

\[ \bar{v}' = \bar{a} \frac{\bar{v}}{2} \]  

(15d)

\[ \bar{H}' = 0 \]  

(15e)

\[ \bar{M}_S' = 0 \]  

(15f)

Eqs. (14) and (15) may be linearised as before to obtain the corresponding equations at the pole for the linear theory. The nondimensionalised form employed here will make the linear solutions independent of the loading parameter. In addition their values will be comparable to each other and to those of the unrestrained thin cylinder.

It should be noted that some of the nondimensional shell parameters in Eqs. (11) are defined in terms of \( \xi_e \) which will depend on the geometry of the meridian and thus should be derived for each individual case. In some cases there is
no closed form expression for $\xi_e$ and, therefore, $\xi_e$ has
to be evaluated either from a series expression or by numerical
integration. The same is true for the expressions of $\overline{r}_o$ and
$\overline{\phi}_o$ in terms of $\overline{\xi}$. There may not be any closed form
expressions for $\overline{r}_o$ and $\overline{\phi}_o$ and thus numerical integration
has to be applied. The evaluation of shell parameters and the
expressions of $\overline{r}_o$ and $\overline{\phi}_o$ in terms of $\overline{\xi}$ for different
generoies of the meridian are treated in appendix A.
CHAPTER 3

METHOD OF SOLUTION

(i) INTRODUCTION TO MULTISEGMENT INTEGRATION

The fundamental set of linear Eqs. (12) and nonlinear Eqs. (13) together with their corresponding forms at the apex and the boundary conditions (10c) have to be integrated over a finite range of the independent variable $\xi$. But numerical integration of these equations is not possible beyond a very limited range of $\xi$ due to the loss of accuracy in solving for the unknown boundary values, as pointed out by Kalnins (26), and thus we will resort to the multisegment method of integration developed by Kalnins and Lestingi (2).

The multisegment method of integration of a system of $m$ first order ordinary differential equations

$$\frac{dy(x)}{dx} = F(x, y^1(x), y^2(x), \ldots, y^m(x)) \quad (16a)$$

in the interval $(x_1 \leq x \leq x_{M+1})$ consists of (see Fig. 6)

(a) the division of the given interval into $M$ segments
(b) $(m+1)$ initial-value integrations over each segment
(c) solution of a system of matrix equations which ensures continuity of the dependent variables at the nodal points
(d) repetition of (b) and (c) until continuity of the dependent variables at the nodal points is achieved.

In Eqs. (16a) the symbol \( y(x) \) denotes a column matrix whose elements are \( m \) dependent variables, denoted by \( y_j(x) \) \((j = 1, 2, \ldots, m)\); \( F \) represents \( m \) functions arranged in a column matrix form; and \( x \) is the independent variable. It is assumed here for convenience that the first \( m/2 \) elements of \( y(x_1) \) and the last \( m/2 \) elements of \( y(x_{M+1}) \) are prescribed by the boundary conditions.

If at the initial point \( x_i \) of the segment \( S_i \) (see Fig. 6) a set of values \( y(x_i) \) is prescribed for the variables of Eqs. (16a) then the variables at any \( x \) within \( S_i \) can be expressed as

\[
y(x) = f[y_1(x_i), y_2(x_i), \ldots, y_m(x_i)] \tag{16b}
\]

where the function \( f \) is uniquely dependent on \( x \) and the system of equations (16a). From Eqs. (16b) the expressions for the small changes \( \delta y(x) \) can be expressed to a first approximation by the following linear equations:

\[
\delta y(x) = Y_i(x) \delta y(x_i) \tag{16c}
\]

where
Expressing Eqs. (16c) in finite difference form and evaluating them at \( x = x_{i+1} \), we get

\[
y^t(x_{i+1}) - y(x_{i+1}) = Y(x_{i+1})[y^t(x_i) - y(x_i)]
\]

(16e)

where \( y^t \) denotes a trial solution state and \( y \) denotes an iterated solution state based on the condition of continuity of the variables at the nodal points. Eqs. (16e) is rearranged as

\[
Y(x_{i+1})y(x_i) - y(x_{i+1}) = -Z_i(x_{i+1})
\]

(16f)

where \( Z_i(x_{i+1}) = y^t(x_{i+1}) - Y(x_{i+1})y^t(x_i) \).

In order to determine the coefficients \( Y(x) \) in Eqs. (16f) the jth column of \( Y(x) \) can be regarded as a set of new variables, which is a solution of an initial value problem governed within each segment by a linear system of first order differential equations, which is obtained from Eqs. (16a) by
differentiation with respect to \( y_j(x_i) \) in the form

\[
\frac{d}{dx} \left[ \frac{\partial y(x)}{\partial y_j(x_i)} \right] = \frac{\partial}{\partial y_j(x_i)} \left\{ F[x, y^1(x), y^2(x), \ldots, y^m(x)] \right\} . \tag{16g}
\]

Thus the columns of the matrix \( Y_i(x) \) are defined as the solutions of \( m \) initial value problems governed in \( S_i \) by (16g) (with \( j = 1, 2, \ldots, m \)) with the initial values, in view of Eqs. (16c), specified by

\[
Y_i(x_i) = I \tag{16h}
\]

where \( I \) denotes the \((m, m)\) unit matrix. To obtain the iterated solution \( y(x_i) \) Eqs. (16f) are rewritten as a partitioned matrix product of the form

\[
\begin{bmatrix}
y^1(x_{i+1}) \\
y^2(x_{i+1})
\end{bmatrix}
= \begin{bmatrix}
y^1_1(x_{i+1}) & y^2_1(x_{i+1}) \\
y^3_1(x_{i+1}) & y^4_1(x_{i+1})
\end{bmatrix}
\begin{bmatrix}
y^1(x_i) \\
y^2(x_i)
\end{bmatrix}
+ \begin{bmatrix}
z^1_i(x_{i+1}) \\
z^2_i(x_{i+1})
\end{bmatrix}
\]

so that the known boundary conditions are separated from the unknowns and, therefore, turns into a pair of equations given by

\[
\begin{align*}
y^1_1(x_{i+1})y^1_1(x_i) + y^2_1(x_{i+1})y^2_1(x_i) - y^2_1(x_i) &= -z^1_i(x_{i+1}) \\
y^3_1(x_{i+1})y^1_1(x_i) + y^4_1(x_{i+1})y^2_1(x_i) - y^2_1(x_i) &= -z^2_i(x_{i+1})
\end{align*}
\]

\( (16l) \)
The result is a simultaneous system of \( 2M \) linear matrix equations, in which the known coefficients \( Y_i^j(x_{i+1}) \) and \( Z_i^j(x_{i+1}) \) are \((m/2, m/2)\) and \((m/2, 1)\) matrices, respectively, and the unknown, \( y_j^i(x_i) \) are \((m/2, 1)\) matrices. Since \( y_1^1(x_1) \) and \( y_2^1(x_{M+1}) \) are known, there are exactly \( 2M \) unknowns: \( y_1^i(x_i) \), with \( i = 2, 3, \ldots, M+1 \), and \( y_2^i(x_i) \), with \( i = 1, 2, \ldots, M \).

By means of Gaussian elimination, the system of equations (16i) is first brought to the form

\[
\begin{align*}
E_i y_2^i(x_i) - y_1^i(x_{i+1}) &= A_i \\
C_i y_1^i(x_{i+1}) - y_2^i(x_{i+1}) &= B_i
\end{align*}
\]

for \( i = 1, 2, \ldots, M \). Using the notations \( Z_i^j \) and \( Y_i^j \) in place of the symbols \( Z_i^j(x_{i+1}) \) and \( Y_i^j(x_{i+1}) \), the \((m/2, m/2)\) matrices \( E_i \) and \( C_i \) in the Eqs. (16j) are defined by

\[
E_i = Y_i^2, \quad C_i = Y_i^4
\]

and

\[
E_i = Y_i^2 + Y_i^1 C_i^{-1}
\]

\[
C_i = (Y_i^4 + Y_i^3 C_i^{-1}) E_i^{-1}
\]

for \( i = 2, 3, \ldots, M \).
The \((m/2, 1)\) matrices \(A_i\) and \(B_i\) are given by

\[
A_1 = -z_1 \ Y_1 \ y_1(x_1)
\]
\[
B_1 = -z_1 \ Y_2 \ y_1(x_1) - y_1 E_1^{-1} A_1
\]

and

\[
A_i = -z_1 \ Y_1 \ c_{i-1}^{-1} B_{i-1}
\]
\[
B_i = -z_1 \ Y_3 \ c_{i-1}^{-1} B_{i-1} - (y_1^4 + y_1^3 c_{i-1}^{-1}) E_1^{-1} A_i
\]

for \(i = 2, 3, \ldots, M\).

Then the unknowns of \((16i)\) are obtained by

\[
y_1(x_{M+1}) = c_M^{-1} \left[b_M - y_2(x_{M+1})\right]
\]
\[
y_2(x_M) = E_M^{-1} [y_1(x_{M+1}) + A_M]
\]

and

\[
y_1(x_{M-i+1}) = c_{M-i}^{-1} [y_2(x_{M-i+1}) + B_{M-i}]
\]
\[
y_2(x_{M-i}) = E_{M-i}^{-1} [y_1(x_{M-i+1}) + A_{M-i}]
\]

for \(i = 1, 2, \ldots, M-1\).

Assuming \(y(x_1)\) as the next trial solution \(y^r(x_1)\), the process is repeated until the integration results of Eqs.\((16a)\) at \(x_{i+1}\), as obtained from the integrations in segment \(S_i\) with the initial values \(y(x_i)\), match with the elements of \(y(x_{i+1})\) as obtained from \((16f)\) and also with the boundary conditions at \(x_{M+1}\).
(ii) DERIVATION OF ADDITIONAL EQUATIONS

In the multisegment integration technique for a set of ordinary differential equations it has already been noted that in addition to the integration of the given set of equations we have to integrate another \( m \) sets of equations represented by (16g). Thus, in order to apply the method of multisegment integration, differential equations corresponding to Eqs. (16g) for the \( m^2 \) additional variables as represented in Eqs. (16d) have to be derived. These differential equations are obtained by differentiating Eqs. (12) for the linear case and Eqs. (13) for the nonlinear case with respect to each of the fundamental variables. As the variables in any column of (16d) have the same form, we derive here the system of equations (16g) for the variables of any one column of (16d) where the new variables are identified from the fundamental variables by the subscript \( a \).

From the nonlinear equations (13), we have, by differentiation in succession,

\[
\bar{\theta}_{\theta a} = \frac{u}{a} \frac{1}{r_o} \quad (17a)
\]

\[
\bar{\phi}_a = - \bar{\beta}_a \quad (17b)
\]

\[
\bar{v}_{\theta a} = \bar{\beta}_a \cos \bar{\phi}/r_o \quad (17c)
\]

\[
\bar{N}_{\phi} = (\bar{H}_a - \bar{V}_a \bar{\beta}_a) \cos \bar{\phi} + (\bar{H}_a \bar{\beta}_a + \bar{V}_a) \sin \bar{\phi} \quad (17d)
\]
\[\varepsilon_{\xi a} = C \nabla_{\xi a} - \nu \varepsilon_{\theta a}\] (17e)

\[\kappa_{\xi a} = \frac{\nabla_{\xi a}}{D} - \nu \kappa_{\theta a}\] (17f)

\[N_{\theta a} = (\varepsilon_{\theta a} + \nu \varepsilon_{\xi a})/C\] (17g)

\[M_{\theta a} = D (\kappa_{\theta a} + \nu \kappa_{\xi a})\] (17h)

\[\alpha_a = \varepsilon_{\xi a}\] (17i)

\[r_a = \alpha_a\] (17j)

\[u_{a} = \alpha_a \cos \phi + \beta_a \alpha \sin \phi\] (17k)

\[w_{a} = \alpha_a \sin \phi - \alpha \beta_a \cos \phi\] (17l)

\[\beta_{a} = \kappa_{\xi a}\] (17m)

\[V_{a} = - (\alpha_a \cos \phi + \beta_a \sin \phi)(\sqrt{V/V_a} - \bar{P} \bar{T})\]

\[\alpha \cos \phi (\frac{V}{V_a} - \frac{V}{V_a}/r^2)\] (17n)

\[\bar{H}_{a} = - \alpha_a ((\bar{H} \cos \phi - \nabla_{\theta})/r + \bar{P} \bar{T} \sin \phi) - \alpha (\bar{H} \cos \phi + \beta_a \bar{H} \sin \phi - \nabla_{\theta})/r - \bar{P} \bar{T} \beta_a \cos \phi\] (17o)

\[M_{\xi a} = (\alpha_a \cos \phi + \beta_a \alpha \sin \phi)((\bar{M}_{\theta} - \nabla_{\xi})/r + \bar{P} \bar{T}^2 \bar{V})\]

\[\alpha (\cos \phi [\bar{P} \bar{T}^2 \bar{V} + (\bar{M}_{\theta} - \nabla_{\xi})/r - \bar{u} (\bar{M}_{\theta} - \nabla_{\xi})/r]) -\]
At the pole the corresponding equations are obtained from (14) as

\[ \vec{u} = (1 - \nu) R H \cos^2 \phi_0 \] \hspace{1cm} (17q)

\[ \vec{w} = (1 - \nu) R H \cos \phi_0 \sin \phi_0 \] \hspace{1cm} (17r)

\[ \vec{\beta}_a = \frac{M_{\xi a}}{(1 - \nu) D} \] \hspace{1cm} (17s)

\[ \vec{\alpha}_a = (1 - \nu) R H \cos \phi_0 \] \hspace{1cm} (17t)

\[ \vec{V} = \frac{1}{2} \vec{P} T \cos \phi_0 \vec{\alpha}_a \] \hspace{1cm} (17u)

\[ \vec{H}_a' = \frac{1}{3} [(1 - \nu)(\phi' \vec{H}_a - \vec{\beta}_a' \vec{H}) + (\vec{\alpha}_a' \vec{\beta}_a + \vec{\alpha}_a \vec{\beta}_a')/(R \cos \phi_0)] \tan \phi_0 \]

\[ - \frac{1}{2} \vec{\alpha}_a \vec{P} T \sin \phi_0 \] \hspace{1cm} (17v)

\[ \vec{M}_{\xi}' = \frac{1}{3} [\vec{P} T^2 \sin \phi_0 (\vec{\alpha}_a' \vec{H} + \vec{\alpha}_a \vec{H}_a') + (\vec{\beta}_a' \phi' - \vec{\beta}_a' \vec{\beta}_a') \tan \phi_0] \]

\[ (12 \vec{P} T^2 \vec{R}^{-2}) \] \hspace{1cm} (17w)

Equations (17a-17p) which take the form (17q-17w) at \( \xi = 0 \), have to be integrated as initial value problems.
times in each segment with the initial values given by (16h). It should be noted that the equations (17) contain not only the variables (16d) but also the variables of the fundamental set. Thus, Eqs. (17) can not be integrated unless the fundamental set of equations is integrated first and the values of the fundamental variables are stored for use in Eqs. (17). It should be further pointed out that one point integration formula can not be used for the integration of Eqs. (17) since this formula needs evaluation of the derivatives at intermediate points where the variables are never evaluated.

The corresponding equations for the linear theory are given by the homogeneous form of Eqs. (12) and thus readily obtainable by dropping the load terms in Eqs. (12).

(iii) TREATMENT OF BOUNDARY CONDITIONS

In the introduction of the multisegment method of integration it was assumed that the first \( m/2 \) elements of \( y(x) \) at \( x_1 \) and the last \( m/2 \) elements of \( y(x) \) at \( x_{M+1} \) were prescribed as boundary conditions. But, in general, the boundary conditions are given as

\[
\begin{align*}
T_1 y(x_1) &= b_1 \quad \text{at} \quad x_1 \\
\text{and} \\
T_{M+1} y(x_{M+1}) &= b_{M+1} \quad \text{at} \quad x_{M+1}
\end{align*}
\] (18a)
in which any \( m/2 \) elements of \( b_1 \) and any \( m/2 \) elements of \( b_{M+1} \) will be specified as boundary conditions. The symbols \( T_1 \) and \( T_{M+1} \) represent nonsingular \((m, m)\) matrices which are known from the specification of the boundary conditions at the ends of the interval.

By rearranging the rows of \( T_1 \) and \( T_{M+1} \) in a special order, equations (18a) can always be stated in a manner such that the prescribed elements of \( b_1 \) and \( b_{M+1} \) become respectively the first and the last \( m/2 \) elements of \( b_1 \) and \( b_{M+1} \). When this is achieved, evaluation of (16f) at \( i = 1 \) and \( i = M \), and then elimination of \( y(x_1) \) and \( y(x_{M+1}) \) by means of (18a) yield

\[
V_1(x_2) T_1^{-1} b_1 - y(x_2) = - Z_1(x_2) \quad (18b)
\]

\[
T_{M+1} Y_M(x_{M+1}) y(x_{M+1}) - b_{M+1} = - T_{M+1} Z_M(x_{M+1}) \quad (18c)
\]

We can now retain the form and notation of (16f) if we regard that the coefficient matrices \( Y_1(x_2) \), \( Y_M(x_{M+1}) \), \( Z_M(x_{M+1}) \), occurring in (16f), represent \( V_1(x_2) T_1^{-1} \), \( T_{M+1} Y_M(x_{M+1}) \), and \( T_{M+1} Z_M(x_{M+1}) \), respectively. In so doing, the solution of (16f) will not yield \( y(x_1) \) and \( y(x_{M+1}) \) but rather the transformed variables \( b_1 \) and \( b_{M+1} \). When \( y(x_1) \) and \( y(x_{M+1}) \) is desired they can be obtained by the inversion of the matrix equations (18c).
It should be noted here that with reference to the boundary conditions (10c) stated in terms of the fundamental variables the matrices $T_1$ and $T_{M+1}$ are both unit matrices of order 6. The construction of $T_1$ and $T_{M+1}$, in accordance with any possible statement of (10c) so that equations (18a) are in order, is treated in Appendix A.

(iv) A NEW METHOD OF MULTISEGMENT INTEGRATION

The multisegment method of integration developed here is, like the method of Kalnins and Lestingi (2), applicable to any system of $m$ first order ordinary differential equations (16a) in the interval $(x_1 \leq x \leq x_{M+1})$ with $m/2$ boundary conditions stated at $x_1$ and other $m/2$ boundary conditions stated at $x_{M+1}$. The philosophy of this method is identical to that of the method of Kalnins and Lestingi except that the $m$ times initial value integration of (16g) is avoided which results in considerable saving in numerical computation. In this case we can reverse the functional relationship of (16b) and, instead of (16b), write

$$y(x_i) = f(y^1(x), y^2(x), \ldots, y^m(x))$$

(19a)

where the symbol $y(x)$ represents, as before, the values of the variables at some point in segment $S_i$, corresponding to
the initial values $y(x_i)$ at the beginning of $S_1$. A first order approximation for the small changes of the variables in (19a) is given by

$$
\delta y^j(x_i) = \frac{\delta y^j(x_i)}{\delta y^1(x)} \delta y^1(x) + \frac{\delta y^j(x_i)}{\delta y^2(x)} \delta y^2(x) + \ldots +
$$

$$
\frac{\delta y^j(x_i)}{\delta y^m(x)} \delta y^m(x)
$$

(19b)

$$
= \sum_{k=1,2}^{k=m} \frac{\delta y^j(x_i)}{\delta y^k(x)} \delta y^k(x)
$$

for $j = 1, 2, 3, \ldots, m$.

The small changes $\delta y(x_i)$ and $\delta y(x)$ in (19b) are expressed in finite-difference forms with the following meanings:

$$
[y^j(x_i)]_k = y^j(x_i) - [y^j(x_i)]_k
$$

(19c)

where the $y^j(x_i)$ are assumed to be the exact values of $y^j(x)$ at $x_i$ which would satisfy the differential equations (16a) and the boundary conditions; and $[y^j(x_i)]_k$ are the assumed values of $y^j(x)$ at $x_i$. The subscript $[\ldots]_k$ in (19c) is used to differentiate $\delta y^j(x_i)$ for different assumed values of
Similarly, \[ k \delta y^k(x) \] is given by the expression
\[ k y^k(x) - y^k(x) \] (19d)

where \( y^k(x) \) are the exact values of the variables at \( x \) in \( S_i \) corresponding to \( y^i(x) \); and \( y^k(x) \) are the values at \( x \) corresponding to the assumed values \( y^i(x) \).

It is now assumed that equations (19b) are evaluated (m+1) times for (m+1) different values of \( y^i(x) \) which, with the help of notation of (19c, d), can be written as

\[ \sum_{k=1}^{m} \frac{\partial y^i(x)}{\partial y^k(x)} \delta y^k(x) = 0 \] (19e)

where \( j = 1, 2, ..., m \), \( \ell = 1, 2, ..., m+1 \).

The \( (m \times m) \) unknown elements \[ \frac{\partial y^i(x)}{\partial y^k(x)} \] in equations (19e) are obtained by solving (19e) for \( j = 1, 2, ..., m \) and \( \ell = 2, 3, ..., m \) which will result in

\[ \frac{\partial y^j(x)}{\partial y^k(x)} = \left( \frac{\text{Det } A_k}{\text{Det } A} \right) \] \( j, k = 1, 2, ..., m \) (19f)
where $\text{Det } A = \begin{bmatrix}
[\delta y_1^1(x)]_2 & [\delta y_2^2(x)]_2 & \cdots & [\delta y_m^m(x)]_2 \\
[\delta y_1^1(x)]_3 & [\delta y_2^2(x)]_3 & \cdots & [\delta y_m^m(x)]_3 \\
\vdots & \vdots & \ddots & \vdots \\
[\delta y_1^1(x)]_{m+1} & \cdots & \cdots & [\delta y_m^m(x)]_{m+1}
\end{bmatrix}$

and $\text{Det } A_k$ is equivalent to $\text{Det } A$ with its $k$th column being replaced by

$\begin{bmatrix}
[\delta y_1^j(x_i)]_2 \\
[\delta y_1^j(x_i)]_3 \\
\vdots \\
[\delta y_1^j(x_i)]_{m+1}
\end{bmatrix}$

Substitution of (19f) in (19e) for $\ell = 1$ will produce

$$[\delta y_1^j(x_i)]_1 = \sum_{k=1}^{m} \left\{ \frac{\text{Det } A_k}{\text{Det } A} [\delta y_k^j(x)]_1 \right\}$$

(19g)

where $j = 1, 2, \ldots, m$. 
As Det. A is independent of the subscript \( k \) equations (19g) can be rearranged as

\[
\text{Det. } A \left[ \delta y^j(x_i) \right]_1 = \text{Det. } A_1 [\delta y^1(x)]_1 + \text{Det. } A_2 [\delta y^2(x)]_1 + \ldots \\
+ \text{Det. } A_m [\delta y^m(x)]_1 \tag{19h}
\]

When the values of the determinants \( A \) and \( A_k \) are substituted in equations (19h) it can be shown that (19h) becomes equivalent to the determinant equations

\[
\begin{bmatrix}
[\delta y^j(x_i)]_1 & [\delta y^1(x)]_1 & [\delta y^2(x)]_1 & \ldots \ldots & [\delta y^m(x)]_1 \\
[\delta y^j(x_i)]_2 & [\delta y^1(x)]_2 & [\delta y^2(x)]_2 & \ldots \ldots & [\delta y^m(x)]_2 \\
& \vdots & \vdots & \ddots & \vdots \\
[\delta y^j(x_i)]_{m+1} & [\delta y^1(x)]_{m+1} & [\delta y^2(x)]_{m+1} & \ldots \ldots & [\delta y^m(x)]_{m+1}
\end{bmatrix} = 0
\]

(19i)

where \( j = 1, 2, \ldots, m \).

Elements of equations (19i) are replaced by their values as defined by (19c, d).
Equations (19j) can be explicitly expressed as

\[
- y^j(x_1) C(x) - \sum_{k=1}^{k=m} [y^k(x) \delta_{kj}(x)] + b_j(x) = 0
\]

where \( b_j(x) = \det \left[ \begin{array}{cccc}
y^j(x_1) & y^1(x) & y^2(x) & \ldots & y^m(x) \\
y^j(x_1) & y^1(x) & y^2(x) & \ldots & y^m(x) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
y^j(x_1) & y^1(x) & y^2(x) & \ldots & y^m(x) \\
y^j(x_1) & y^1(x) & y^2(x) & \ldots & y^m(x) \\
\end{array} \right] \]

\[
\text{Det. } C(x) \quad \text{is equivalent to } \det b_j(x) \quad \text{where the elements of}
\]
the first column of $b_j(x)$ are replaced by unity; and

$$\det a_{kj}(x)$$ is equivalent to $\det b_j(x)$ with its elements of $k$th column are replaced by a unit column vector. It should be noted here that in coming from equation (19j) to (19k) there should have been additional terms in (19k) which vanish due to the fact that two or more columns are identical in the determinants.

In equations (19k) all the determinants $a_{kj}$, $b_j$ and $C'$ are known from the $(m+1)$ initial value integrations of (16a) in the segment $S_i$. Evaluating (19k) at $x_{i+1}$, we get

$$y^j(x_i) C(x_{i+1}) + \sum_{k=1}^{k=m} y^k(x_{i+1}) a_{kj}(x_{i+1}) = b_j(x_{i+1}) \quad (19'i)$$

where $j = 1, 2, ..., m$.

Equations (19'i) can be expressed in matrix notations as

$$Z_i + Y_i y(x_{i+1}) = y(x_i) \quad (19'm)$$

where the $(m \times m)$ matrix $Y_i$ is given by

$$Y_i = \left[ \begin{array}{c} a_{kj}(x_{i+1}) \\ - C(x_{i+1}) \end{array} \right]$$
and the \((m, 1)\) matrix \(Z_i\) is defined as

\[
Z_i = \begin{bmatrix}
\frac{b_i(x_i+1)}{C(x_i+1)}
\end{bmatrix}
\]

It is now assumed that the first \(m/2\) elements of \(y(x_i)\) and the last \(m/2\) elements of \(y(x_{M+1})\) are prescribed by the boundary conditions; and, in order to separate the known from the unknown elements, Eq. (19m) is partitioned in the form

\[
\begin{bmatrix}
Z_i^1 \\
Z_i^2
\end{bmatrix} + \begin{bmatrix}
y_i^1 & y_i^2 \\
y_i^3 & y_i^4
\end{bmatrix} \begin{bmatrix}
y_i(x_i+1) \\
y_i(x_{i+1})
\end{bmatrix} = \begin{bmatrix}
y_i(x_i) \\
y_i(x_{i+1})
\end{bmatrix}
\]

which results into the following pair of equations

\[
\begin{align*}
Z_i^1 + y_i^1 y_i(x_i+1) + y_i^2 y_i(x_{i+1}) &= y_i(x_i) \\
Z_i^2 + y_i^3 y_i(x_i+1) + y_i^4 y_i(x_{i+1}) &= y_i(x_{i+1})
\end{align*}
\]

The system of equations (19n) is first brought to the form

\[
\begin{align*}
y_i(x_i+1) &= A_i + B_i y_i(x_{i+1}) \\
y_i(x_{i+1}) &= Z_i^2 + y_i^3 y_i(x_i+1) + y_i^4 y_i(x_{i+1})
\end{align*}
\]
where the matrices $A_i$ and $B_i$ are defined by

$$A_i = [y_1^{i-1}]^{-1} [y_1(x_i) - Z_1],$$

$$B_i = - [y_1^{i-1}]^{-1} y_1^2,$$

and for $i = 2, 3, \ldots, M$

$$A_i = [y_1^{i-1} - B_{i-1} y_1^3]^{-1} [A_{i-1} + B_{i-1} Z_1^2 - Z_1],$$

$$B_i = [y_1^{i-1} - B_{i-1} y_1^3] [B_{i-1} y_1^4 - y_1^2].$$

The unknowns are then solved by evaluating equations (190) for $i = M$ to 1 in sequence. In the next iteration the worst of the $(m+1)$ initial guesses of $y(x_i)$ is replaced by its new improved value obtained from (190) and the process is continued until continuity of solutions are achieved at the nodal points.

(v) ADVANTAGE OF THE NEW METHOD

Comparing the new method with the method of Kalnins and Lestinig it can be readily seen that, in the first pass of iteration, $(m+1)$ initial value integrations in each segment are necessary in both the methods. In addition the new method requires the evaluation of the determinants $a_{k,j}, b_j$ and $C$. 
for each segment. But, from the second iteration onward we need only one initial value integration of (16a) for the new method and, thus, avoid completely the additional \( m \) initial value integrations of (16g). It is therefore expected that the computational work in integration in the new method will be approximately \( \frac{1}{m} \) th time of the other method. Of course, some additional work will be required to evaluate the determinants which, in comparison of the computational work of the \( m \) initial value integrations, will be negligible. Moreover, the system of equations (16g) which are derived from (16a) by differentiation will be in general much more complicated than (16a) depending upon the amount of nonlinearity present in the latter system of equations. Another fact which should be taken into consideration is that single point integration formula can be used in this method which was not possible in the other method as pointed out earlier. It should also be noted that the sequence of matrix operations needed in the new method to solve the unknowns from the matrix equations would involve lesser amounts of work than the previous method.
(i) ACCURACY OF THE METHOD

It is always desirable that the solutions obtained by any technique should be compared with the available results in the literature in order to ascertain that the method employed is reliable, that no error is committed in formulating the problem, and that no mistake is made in the programming. With this in mind a number of problems of different kinds are solved and the results are compared with those obtained by other techniques.

(a) Uniformly Loaded Circular Plate with Clamped Edge

This particular problem was solved because an exact analytical solution based on linear theory is available in closed form. Both the results of the analytical solution and the multisegment integration are given in Table 1. The upper values in each row of this table are the results of multisegment integration based on the linear theory whereas the lower values are the results computed from the analytical solution of Ref. (20). It is seen from this table that the results of multisegment integration are correct up to eight digits. It should
be mentioned that the results of multisegment integration are based on the linear shell theory presented in this thesis whereas the analytical solution is based on the bending theory of plates. Both the theories are equivalent in this case as they are founded on the same hypothesis and approximations.

(b) Pressurized Spherical Shells with Clamped Edge

Exact and different approximate solutions of this problem based on linear bending theory of shells are available in Refs. (20, 67). The exact analytical solution of this problem is in the form of a hypergeometrical series, and the limitations of application of this series solution to different problems are pointed out by Flugge (65). Based on this series solution the axial resultant couple \( M_\phi \) and the circumferential stress resultant \( N_\theta \) for a clamped-edge spherical shell are presented in Ref. (20) in graphical form. These curves of \( M_\phi \) and \( N_\theta \) are reproduced in Figs. 8a and 8b, respectively, for ready comparison with our results given in Fig. 8c. Our numerical results are also recorded in Table 2 in order to see the exact magnitudes of different physical quantities at the nodal points. Comparing the values of \( M_\phi \) and \( N_\theta \) of Figs. 8a and 8b with those of Fig. 8c and also of Table 2 we can see that the results obtained here are exactly the same as the analytical solution. It should be noted that the
actual values of $N_\theta$ in Fig. 8b is given by the difference of the two curves for $N_\theta$.

Problems of clamped hemispherical shells for two different cases were also solved and the results are compared in Table 3 with other exact and approximate analytical solutions taken from Ref. (67, p. 284). These tabular results show that our results compare quite favourably with the different analytical solutions. The slight difference in the results may be attributed to the fact that the angular meridional measure of Ref. (67) can not be expressed in whole numbers as distance along the meridian which was employed in our computations.

(c) Pressure Vessel with Ellipsoidal Head

The stresses in thin-walled pressure vessels with ellipsoidal heads are worked out by Kraus, Bilodeau and Langer (35) based on the linear theory. One of their problems was solved by the Author for the same values of parameters as in Figs. 2 and 3 of Ref. (35). The Author obtained both the linear and nonlinear results for this problem which are presented in Figs. 30 and 31 in the form of indexes as used in Ref. (35). Comparison of linear results of Figs. 30 and 31 with the corresponding results of Figs. 2 and 3 of Ref. (35) shows that there is hardly any difference between these two results which can be noticed from graphical values.
It should be mentioned here that the results of Figs. 30 and 31 and those of Ref. (35) are based on different theories of shells which differ in their degree of approximations. Moreover, these two numerical results are obtained by using two different numerical techniques and, therefore, it is expected that there should be some difference between the two results. However, the difference is so negligible that it can only be detected by comparing their exact numerical values obtained under the two cases.

The above results and their comparison with other results prove amply that the method employed here is extremely accurate for any practical purpose and that no error is committed either in formulating the problem or in the programming. Actually the accuracy of the method of multisegment integration is self ascertaining. Once the values of the fundamental variables at the nodal points are known from the multisegment method of integration, the fundamental set of differential equations can be integrated over each segment of the meridian as initial value problems. If the values of the fundamental variables at the end of segment $S_i$, as obtained from the initial value integration of the fundamental set of differential equations, match up to six or seven digits with their respective initial values for the segment $S_{i+1}$, for $i = 1, 2, \ldots, M$, and also with the given boundary conditions at $x_{M+1}$, then
it can be concluded that the results are correct up to six or seven digits of their numerical figures.

(ii) ACCURACY OF THE NEW METHOD OF MULTISEGMENT INTEGRATION

Regarding the new method of multisegment integration it was decided that the method should be applied, first, to simple problems for which analytical solution is known so that the soundness of the philosophy on which this method is based can be verified and that the error committed in programming the procedure of numerical computation can be easily detected. Thus, in steps, the method was applied to a fourth order linear differential equation, to a second order nonlinear differential equation and, finally, to different problems of shell analysis based on the nonlinear theory.

(a) The Linear Differential Equation: \( EI \frac{d^4 w}{dx^4} = q \)

The solution of this fourth order linear differential equation was obtained by the new method for \( w \) and its derivatives for \( q/EI = 0.1 \). The range of the independent variable \( x \) was divided into two segments. The values of the four dependent variables at the three nodal points at every sweep of the computer program are presented in Table (4). From this table it is seen
that the rate of convergence of the method is quite fast and that the exact values of the variables are obtained in the fourth pass of the program. The exact solution of this problem obtained by the new method of multisegment integration demonstrates sufficiently that the method is founded on sound basis.

(b) The Nonlinear Equation: \[ \frac{d^2y}{dx^2} + \frac{(1-\mu)}{\mu y} \left( \frac{dy}{dx} \right)^2 = 0 \]

An attempt was made to apply the method to nonlinear differential equations in order to check its rate of convergence and general behaviour concerning the accuracy of the results. The above nonlinear equation was solved for \( \mu = 2 \) and for \( x \) from zero to one. The total range of the independent variable was divided into three segments and the values for both \( y \) and \( \frac{dy}{dx} \) were obtained at the nodal points. The results are presented in Table (5) where the instantaneous values of \( y \) and \( \frac{dy}{dx} \) are recorded at every pass of the program and their exact values are recorded at the bottom of table. From this table it is seen that the results obtained at the fourth pass are correct upto three places of every number and those at the fifth pass represent the exact solution.

(c) Fixed-Edged Hemispherical Shell with Open Top
When the mathematical foundation and numerical procedure of the method was verified with the simple differential equations, the method was applied to nonlinear problems of shells whose meridians were several multiples of their critical length as defined in Ref. (67). The method was first applied to a fixed-edge hemispherical shell with open top whose meridional length was approximately three times the critical length of the shell. The meridional length of the shell was divided into three segments and then the problem was solved by both the new method and the method of Kalnin's. The results are presented in Table (6) where the top values of every row are the results obtained by the new method while the bottom values are the results of Kalnin's method. It can be seen from Table (6) that the results of the two methods are identical up to three decimal places for the stress resultants and meridional bending moment. Thus, within the limitations of a numerical technique, the results of the new method are fairly accurate. It should be noted that there is some discrepancy in the values of axial displacement. But the values of axial displacement are so small in this problem that a better agreement is quite unlikely for a numerical technique.
(d) Both Ends Fixed Pressurized Cylindrical Shell

The solution of this problem was prompted by the fact that Kraus (67, p. 426) cited this problem to show the failure of the direct integration method. The results of this problem as obtained by the new method and the numerical values of the transverse shear and the axial bending moment at the fixed ends as given by Kraus are presented in Table (7). The accuracy of the results are best ascertained here by the condition of symmetry. We can see from Table (7) that our results maintain symmetry quite well and the values of bending moment and transverse shear at the fixed ends compare favourably with those given by Kraus (67, p. 426). It should be mentioned here that our results are based on nonlinear theory whereas those of Kraus are based on linear theory. But the two results are almost the same, because, there is hardly any difference between the solutions of linear and nonlinear theories at such a low value of loading. As a matter of interest, according to Kraus, when this problem was attempted by the method of direct integration, the values obtained for the transverse shear at the two fixed ends were 323.86 and 1846.20 instead of being equal. The values for the bending moment at the two ends were 560.37 and 4141.20 which, due to symmetry, should have the same magnitude.
(e) Cylindrical Shell Restrained at One End

This problem is exactly the same as (d) except that one end of the cylindrical shell is free from restraints. The shell meridian was divided into three segments so that the length of each segment was smaller than the critical length of the shell. The results of this problem are presented in Table (8) along with the numerical values of transverse shear and bending moment at the fixed end computed from the analytical solution of linear theory (20, p.475). Our results compare quite well with the analytical solution of linear theory at this low value of loading. It is quite interesting to note that the maximum values of the transverse shear and bending moment in this case are greater than their corresponding values in the case of both ends fixed cylindrical shell.

(iii) RESULTS

(a) Hemispherical Head Pressure Vessels.

Of all the types of pressure vessel heads in general use the hemispherical head pressure vessel is the most superior one as it avoids high local stresses. Because of this fact the hemispherical head pressure vessel is studied here more extensively than any
other type. It has already been mentioned in the introduction that most of our present knowledge of pressure vessel problems is based on linear theory as, until now, the more accurate theory of shells, i.e., the nonlinear theory was not amenable to either analytical or numerical solutions. Thus, both the linear and nonlinear solutions are obtained and presented in graphical forms in all the cases so that the difference between these two results can be readily noticeable. It should be noted here that in all the graphs presented, the linear solution may be considered as equivalent to the nonlinear solution at zero loading.

In Figs. 9, 10, 11, and 12 the nondimensional meridional bending moment \( \bar{M}_\xi \) for the hemispherical head pressure vessels are plotted against the meridional length of the shell for \( R/h \) equal to 500, 300, 100 and 25, respectively. In these figures and in the following figures on pressure vessels the zero value of the nondimensional meridional length \( \xi \) represents the apex whereas the value \( \xi = 1 \) represents the head-cylinder juncture. From Figs. 9, 10, and 11 it is observed;

(1) that the peak values of the meridional bending moment based on both the linear and nonlinear theories have almost the same magnitude and are identical in distribution in the head and the cylindrical part for all three \( R/h \) ratios,
(2) that the bending effects make a significant contribution to the results only within a short distance from the juncture,

(3) that the region over which the bending effects make a significant contribution to the solution decreases as the thickness of the shell is reduced,

(4) that the linear solution fails completely to predict the actual bending moment at higher loadings,

(5) that the distribution of bending moment as given by nonlinear theory differs substantially from that of linear theory at higher loadings,

(6) that the distance of peak values from the juncture decreases slightly with increasing loads,

(7) that the peak values of the bending moment as predicted by linear theory remain almost the same for all three R/h ratios, and

(8) that the bending moment is almost zero at the juncture in all the cases.

The first observation indicates that the effects of edge bending moment and transverse shear force are nearly the same in hemispherical and cylindrical shells. This conclusion gives credit to the approach of Ref. (20) of applying the linear solution of cylindrical shell under edge loadings to the
hemispherical head. From observations 4 and 6 we can see that the linear solution overestimates the meridional stresses substantially at higher loadings and that the position of maximum stresses as given by linear theory is further away from the joint than its actual position. Observation 7 indicates that, as far as linear theory is concerned, the actual bending moment $M_\phi$ in the hemispherical head pressure vessel is directly proportional to the product $Rh$. It should be mentioned here that the circumferential bending moment in a hemispherical head pressure vessel is approximately \( \sqrt{v} \) times the meridional bending moment and, thus, no curves for the former are presented.

Fig. 13 presents the distribution of the circumferential stress resultant $\bar{N}_\theta$ for both the linear and nonlinear solutions. It shows that the distribution given by the nonlinear solution differs substantially from that of the linear solution. In the cylindrical part linear solution overestimates $\bar{N}_\theta$ while in the hemispherical part it underestimates $\bar{N}_\theta$. But the nonlinear solutions indicate that the maximum value of $\bar{N}_\theta$ will never exceed the membrane hoop stress resultant of the cylindrical part.

Fig. 14 shows the distribution of the nondimensional circumferential stresses at the inner and outer fibres of the shell. The circumferential stresses are the most significant
of all the stresses in a hemispherical head pressure vessel. From the figure it is seen that the circumferential stress at the outer fibre in the cylindrical part is the highest of all the stresses. At relatively higher loading the linear solution predicts substantially higher values of these stresses in the cylindrical part than their actual values as given by the nonlinear solution. It is also seen that, at higher loadings, the stresses do not exceed the membrane stress of the circular cylinder, whereas, at lower loadings, the maximum value that these stresses can attain is approximately 1.035 times the membrane stress. In the hemispherical cap the linear solution tends to predict lower values of stresses than their values predicted by the nonlinear theory. The maximum value of the circumferential stress occurs at the junction which is 1.5 times greater than the membrane solution.

The distribution of the meridional stresses at the inner and outer fibres of the hemispherical head pressure vessel is given in Fig. 15. The distribution of these stresses and their peak values are almost identical in the two parts. The maximum value occurs at the outer fibre in the cylindrical part and at the inner fibre in the hemispherical part. The difference between the solutions of the two theories increases with the increase
in load. At high loadings the maximum stresses as estimated by the linear theory are substantially larger than their true values. Both in the cylindrical and hemispherical parts the axial stresses always remain lower than the circumferential stresses.

The 1943 API-ASME Code or the 1950 ASME Code for Unfired Pressure Vessels is based entirely on the maximum membrane stresses in the respective parts of the vessel where no allowance for corrosion or joint efficiency is made. The allowable stress, selected from a table, is sufficiently low to ensure a safe design.

Our results in Figs. 14 and 15 show that the stresses at any point in the cylindrical part at higher loadings do not exceed the circumferential membrane stress of the cylinder. The present calculations based on nonlinear theories thus indicate that the code design procedure for the cylindrical part need not be modified when the closure is a hemispherical cap.

But the maximum circumferential stresses in the hemispherical cap is 1.5 times the membrane stress. Also the axial stress is substantially higher than the membrane stress.
Consequently, the code design procedure for the hemispherical cap is questionable. The distribution of stresses in the hemispherical cap suggests that the head designed on the basis of membrane stresses should have a gradually thickened edge towards the junction.

The maximum circumferential stress in a hemispherical head pressure vessel for different R/h ratios is plotted against loadings in Fig. 16. The corresponding curves for maximum axial stress are given in Fig. 17. From these two figures it is noted that the over estimation of the maximum stresses by the linear theory increases with the increase in loadings and also with the increase in R/h ratios. For a particular value of R/h ratio the maximum stress is approximately linear for lower values of loadings.

In Fig. 18a the percentage reduction of maximum circumferential stress as given by the linear theory is plotted against R/h ratios for different stress levels. The percentage reduction of maximum axial stress is given in Fig. 18b. When the maximum stress is known from the linear results for a particular vessel the corresponding value of this maximum stress as given by the nonlinear theory can be found out from these curves. Figures 18a and 18b indicate that the percentage reduction in
the linear results increases with increasing $R/h$ ratios and also with increasing stress levels. The percentage reduction is more in axial stress than in circumferential stress. It should be pointed out that the locations of maximum axial and circumferential stresses are functions of $R/h$ ratios and loadings. While using curves in Fig. 18a it should be remembered that the maximum circumferential stress can never be less than the circumferential membrane stress in the cylinder.

It should be kept in mind that the results as presented here are based on a theory which is valid only for the elastic deformations of the shells. Thus, if for a particular material the stress level in the shells at a certain loading exceeds the yield point stress, the results are not valid for that material. As for example, it is observed that for steel in general, the range of elastic deformations is limited by relative elongations of the order of magnitude $10 \times 10^{-4}$ to $75 \times 10^{-4}$. Therefore, the results of figures 14 to 18b which show the nondimensional stresses; i.e., the relative elongations of hemispherical head pressure vessels are within the range of its validity and far below the upper limit if the material is steel of good quality.

In order to simplify calculations, sometimes the Poisson's ratio is taken as zero regardless of its probable value.
Figures 19 to 21 have been drawn to show the effect of varying Poisson's ratio over the entire practical range from 0 to 0.5. Figure 19 shows the effect of Poisson's ratio on the meridional bending moment based on both the linear and non-linear theories. It can be seen from this figure that the value of $v$ has a quite large effect on the bending moment. This is also true for the axial stress in Fig. 20 and the circumferential stress in Fig. 21.

Fig. 22 shows the plotting of maximum bending moment against loadings for different $R/h$ ratios. The linear result is shown by the solid line. This figure indicates that for low values of $R/h$ ratios and low loadings the prediction of linear theory is moderately accurate. But the linear theory fails completely to give a reasonable estimation of the bending moment at higher values of loadings and $R/h$ ratios.

(b) Spherical Head Pressure Vessels.

In order to study the behaviour of general spherical head pressure vessels, the problem of a particular head with a meridional angle of 45 degrees at the junction was solved. Both the linear and nonlinear results were obtained for different $R/h$ ratios and loadings. Figure 23 shows the distribution of meridional bending moment along the meridian for $R/h = 600$. 
It shows that the distribution of bending moment is completely different from that of hemispherical head vessels. Moreover, the magnitude of the maximum bending moment is so high that it can not be compared with that of the hemispherical head. The distribution in the cylindrical part is different from that in the spherical part. The region over which the bending moment has significant influence increases with decreasing values of R/h ratios.

Figure 24 shows the variation of \( \bar{M}_S \) with loadings for different R/h ratios. In this figure the linear solution for \( \bar{M}_S \) at a particular R/h ratio is given by the nonlinear solution at zero loading for that ratio. It is thus seen that the linear solution for \( \bar{M}_S \) is not independent of R/h ratios as it is in case of hemispherical head. Figure 24 also shows that the prediction of linear theory about the maximum bending moment is completely unreliable for thin shells.

The linear and nonlinear solutions for the stresses are presented in Figs. 25 and 26 respectively. Figure 25 shows that the maximum axial stress as given by the linear theory is about 27 times its corresponding membrane solution. Whereas, the nonlinear solution (Fig. 26) shows that the maximum axial stress at a loading of \( P/E = 0.1 \times 10^{-5} \) is about 22.7 times
its membrane solution. For the circumferential stress the linear solution predicts that its maximum value is approximately 10 times of its membrane solution, while nonlinear solution predicts that the maximum value is 8.3 times the membrane solution. These figures show that a zone with high compressive stress develops on both the sides of the junction. The perturbation in the stresses is extremely high from their membrane values and the region of perturbation widens with decreasing values of $R/l_h$ ratios. These results indicate clearly that it is not a good practice to use a spherical cap for pressure vessels. If for some reasons or others the use of spherical head can not be avoided, then the edges of both the parts should be gradually stiffened towards the junction.

Figure 27 shows the difference between the linear and nonlinear solutions for the maximum axial stress at increasing load levels and for different $R/l_h$ ratios. We can see from this figure that the over-estimation of the maximum axial stress by the linear theory increases with the increasing values of $R/l_h$ ratios and load levels. The same thing would have been observed if similar curves for other quantities were presented.

(c) Ellipsoidal Head Pressure Vessels.

Problems of ellipsoidal head pressure vessels were
worked out for various $R/h$ and minor to major axes ratios. Results of one such problem are presented in Figs. 28 and 29. Figure 28 shows the distribution of bending moments $\bar{M}_5$ and $\bar{M}_\theta$ along the meridian. The distribution is to some extent identical to that of the hemispherical head vessels. It is noted that the magnitude of maximum bending moment in the ellipsoidal part is greater than that of the cylindrical part. As usual the linear theory is too much conservative in predicting the bending moments. This conservativeness of the linear theory will be increasing with increasing values of loads and $R/h$ ratios.

Figure 29 shows the distribution of circumferential stress resultant $\bar{N}_\theta$. It indicates that, like the spherical head pressure vessel, a compressive zone is set up in the ellipsoidal head near the junction. The maximum value of $\bar{N}_\theta$, which occurs in the cylindrical part, is about 1.06 times the membrane value. But, as the nonlinear solution indicates, the actual value of maximum $\bar{N}_\theta$ will be very near to its membrane solution at higher loadings.

The stresses at the extreme fibres in an ellipsoidal head pressure vessel with $R/h = 38$ are shown in Figs. 30 and 31. The stresses in these figures are normalised by dividing with the circumferential membrane stress in a cylinder. It is observed from these figures that the most critical stress in this
pressure vessel is the axial stress at the inner fibre of the ellipsoidal part at $\xi = 0.915$. The maximum circumferential stress occurs at the apex of the ellipsoidal head which is about 11 percent greater than the circumferential membrane stress of the cylinder. It should be mentioned here that the stresses in an ellipsoidal head vessel have been observed to be quite sensitive to its ratio of minor to major axes.

(d) Plate End Pressure Vessels

Results of plate end pressure vessels are shown in Figs. 32 to 37. Figure 32 shows the deformed shape of the vessel under load. The deformed configuration of the cylindrical part is drawn in a different scale from that of the plate head. The deformed shape indicates that the plate end takes an inflated shape under load and behaves more or less like a plate with built-in edges. A very highly compressive region is formed in the cylindrical part near the junction which indicates the usefulness of a reinforcing ring at this place. The large amount of deformation as seen from Fig. 32 implies that the linear theory of shells is entirely unreliable in analysing stresses in a plate end pressure vessel.

Figures 33 and 34 show the distribution of meridional and circumferential bending moments along the meridian. In these
figures it is noted that considerable amount of bending moments are developed both at the junction and in the plate end. However, even at low loading intensity, the difference between the results of linear and nonlinear theories is excessively high. The same thing is observed in Fig. 35 where the meridional bending moments at the junction and at the centre of the plate end are plotted against $R/h$. Figure 35 shows that the rate of increase of meridional bending moment with increasing values of $R/h$ is more at the junction than at the centre of the end plate. At about $R/h = 90$ the bending moment at the centre of the end plate starts decreasing with further increase in $R/h$, while bending moment at the junction starts decreasing no sooner than $R/h = 140$.

The linear and nonlinear solutions for the circumferential stress resultant are plotted against meridional distance in Fig. 36. It has already been pointed out and confirmed by this result that a very highly compressive region is developed in cylindrical part of the vessel near the junction. The magnitude of the compressive stress is about 12 times the membrane solution of the circular cylinder. As seen from this figure, the solution of linear theory does not have any bearing on the actual state of stress in the end plate. This is attributed to the fact that the linear theory did not take into account
the enormous deformation undergone by the end plate. It is interesting to note that the distribution of $N_\theta$ in the plate end as given by the nonlinear theory is qualitatively somewhat like that of a build-in edges spherical shell. Figure 36 also indicates that the magnitude of $N_\theta$ decreases everywhere with the increase in loadings except at the centre portion of the plate where it increases with loadings. So, at high loading intensity, the centre portion of the plate end becomes as critical as the junction of the vessel.

Figure 37 shows the axial displacement at the centre of the end plate for different values of $R/h$. It just points out the fact that the linear theory becomes inadequate in analysing thin plate end pressure vessels.

(e) Conical Head Pressure Vessels.

Results for only a particular problem of conical head pressure vessels are presented here in Figs 38 to 41. For this problem the values of the semi-apex angle of the conical part, and $R/h$ are 45 degrees and 100, respectively. It has already been mentioned that, in case of conical head pressure vessels, the meridian is assumed to be tipped at the apex by a circular arc meeting tangentially with the conical part. In this
case the parameter \( \tilde{\tau} \), which determines the size of the spherical tip, has a value of 0.1.

The deformed shape of the conical head pressure vessel based on both the linear and nonlinear theories is shown in Fig. 38. This figure indicates that the edge of the conical part moves inward under pressure while the cylindrical parts expand outward as a whole. This discontinuity at the junction leads to the formation of a highly compressive zone there shared by both the conical and cylindrical parts. Concentrating our attention to the linear theory and comparing the elastic curve of the meridian of this conical head pressure vessel with that of the plate end pressure vessel, we can see that the increase in semi-apex angle of the conical head will shift the compressive zone towards the cylindrical part. But the elastic curve of the nonlinear theory shows that the compressive zone will always be shared by both the parts at the junction. The elastic curve of the meridian as given by the nonlinear theory is more smooth than that of the linear theory and, thus, the actual state of stress will be much lower than that given by the linear theory.

The distribution of meridional bending moment for the conical head pressure vessel is shown in Fig. 39. As seen from this figure the meridional bending moment developed at the junction
is enormously high and produces a bending stress which is, according to linear theory, about 5 times the circumferential membrane stress in the cylinder. Therefore, the maximum stress in this pressure vessel is the meridional stress at the inner surface of either the cylindrical or conical part at the junction. Although the meridional bending stress at the junction as predicted by the linear theory is much higher than the actual stress as indicated by the nonlinear results it still remains to be the maximum of all the stresses. It should be noted that the overestimation of bending moment by the linear theory is more at the points of maxima in the cylinder and the cone than at the junction, and that the points of maxima in the cylinder and cone as given by the nonlinear theory move towards the junction with the increase in loads. The most interesting observation in Fig. 39 is that the amount of bending moment developed in the spherical tip of this vessel is practically zero. Had there been no spherical tip the bending moment at the apex of the cone would definitely be much greater than that at the junction. This is a clear indication of the fact that the best possible way of avoiding the stress concentration at the junction is to use a spherical ring there.

Figure 40 shows that the distribution of the circumferential bending moment has approximately the same qualitative
nature as the meridional bending moment. But the contribution of maximum circumferential moment to the stresses is about $\sqrt{v}$ times the contribution of the maximum meridional moment.

Figure 41 shows that the membrane state of circumferential stress resultant $\bar{N}_\theta$ in the cone and cylinder has been disturbed by the formation of a highly compressive zone at the junction. The maximum positive value of $\bar{N}_\theta$ occurs in the cone rather than in the cylinder. The magnitude of maximum compressive value of $\bar{N}_\theta$ is about 1.72 times its membrane value in the cylindrical part. It should be noted that the actual value of maxima of $\bar{N}_\theta$ at the operating pressure will be much lower than those given by the linear theory.

(f) A pressurized Composite Shell with Clamped Edge:

Solution was also obtained for a composite shell made up of an inverted conical frustum, a cylindrical part, and a spherical part (Fig. 7). The values of $R/h$ for the conical frustum, cylindrical part, and spherical part were taken as 20, 30, and 40, respectively. The results for various quantities are shown in Figs. 44 to 46. Figure 44 shows that the meridional bending moment is the dominating contributor to stresses in the shell. The plotting of extreme-fibre meridional stresses in Fig. 46 bears testimony to this fact. Figure 45 shows that the junctions are under high compression circumferentially, while the middle portions of the cylindrical and conical part are under high tensions. The tensile regions
are developed because of the fact that the lengths of the cylindrical and conical portion are short compared to their respective thicknesses. As this shell is relatively thick, the linear and nonlinear results differ very little from each other.

(g) Problems of Stability

The method developed in this thesis for obtaining solutions of nonlinear equations of shells has also been applied to problems of stability to determine the critical loads. The buckling phenomenon is interpreted by the so-called "classical criterion" of buckling. According to this criterion, a given state of equilibrium becomes unstable when there are equilibrium positions infinitesimally near to that state of equilibrium for the same external load. These unstable states of equilibrium are the "bifurcation points" on the path of equilibrium configurations. Accordingly, the critical value of the external load corresponds to the first bifurcation point on the path of equilibrium configurations. The bifurcation points appear automatically if the equilibrium configurations based on the principle of stationary potential energy are determined corresponding to all values of load parameter (34). Thus, the nonlinear differential equations of shells, which embody the principle of minimum potential energy, are solved for increasing values of load parameter till the first unstable state of equilibrium
is reached. The unstable state of equilibrium is signalled by the failure of the method of solution to achieve convergence for the next increasing step of load parameter, however small the step size may be. The onset of the first bifurcation point is hinted by a substantial increase in the displacements and stresses of the shell for very small increase in the load parameter. Right at the bifurcation point any increase in the load parameter, however small, produces enormous deformations and, thus, the numerical technique fails to converge to any solution.

The bifurcation point on the path of equilibrium configurations can further be confirmed by obtaining solutions in the post-buckled region of this path. The technique of obtaining solutions of the nonlinear equations in the post buckled zone is pointed out by Thurston (4) while solving the stability problems of shallow spherical shells.

It should be mentioned here that, while solving problems of stability for shallow spherical shells by perturbation technique, Archer (19) interpreted the bifurcation point as the one where the derivative of external load parameter with respect to the maximum axial deflection becomes zero. Archer (19) also pointed out that the center deflection to pressure relation used by a number of authors to interpret buckling must be generalised
by interpreting buckling from a maximum deflection (in general, away from the center) to pressure relation in order to reveal the buckling in the cases where the deflection modes get more involved. This contention is borne out by the deflection to pressure relation for various points in the shells near the bifurcation point as shown in Figs. 42 and 43. From the figures it is observed that the zero derivative of pressure with respect to deflection is better defined on the maximum deflection to pressure relation curve than on the centre deflection curve.

Critical pressures are obtained for three different spherical shells - a shallow spherical shell with a semi-opening angle of 3.6 degrees; a spherical shell with an opening angle of 90 degrees, and a hemispherical shell. The solutions for these problems at the critical pressures are given in tables 9 to 11 and the pressure to axial deflection relations for various points of the shells are presented in Figs. 42 and 43.

It should be pointed out here that all the research works in the field of stability problems of spherical shells are confined to shallow spherical shells. This is due to the fact that the nonlinear equations of shells could be solved only when the simplifications pertaining to the shallowness of the shell were made. The simplified equations were then solved by different methods mentioned in the introduction. Thus, the solution for only
the shallow spherical shell can be compared with results obtained by other authors.

The critical pressure for the shallow spherical shell as given in Table (9) corresponds to a value of 1.061 for $\gamma/\mu^4$ of Ref. (4) or $P_{cr}$ of Ref. (3), and the shell parameters correspond to $\mu^2 = 52.2$ of Ref. (4) or $\lambda = 7.0$ of Ref. (3). The values of critical pressure for this problem as given by Ref. (4) and Ref. (3) are 1.062 and 1.068, respectively. This comparison shows that our results are extremely accurate.

Archer's perturbation solution (19) gives a value of 0.765 for the critical pressure of the above problem. This shows that the perturbation solution is subject to serious correction.

The value of critical pressure for the hemispherical shell as obtained by the present method is almost the same as the classical buckling pressure of hemispherical shell. For the 90 degrees spherical shell the ratio of the critical pressure as obtained here to that of the classical buckling pressure of hemisphere is about 1.125. This shows that the classical theory gives quite good approximation of the critical pressures for spherical shells except for extremely shallow shells. It should be mentioned here that the experimental values of critical pressures for spherical shells are much lower than the values obtained theoretically. This discrepancy is due to the fact that the
idealization made in the theory is never attainable in experimental shells. The effects of initial imperfections and variation of thickness contribute tremendously to the buckling pressure as can be observed from the scattered experimental values in Refs. (3, 4, 12, 19). The problem of thin spherical shells were investigated by Huang (3) on the assumption of unsymmetrical buckling. The results of this investigation are comparatively lower than those of the symmetrical buckling but still much higher than the experimental values.

From our investigations it is concluded that the present method can be used to determine buckling pressure of any composite axisymmetric shell with all possible boundary conditions. Thus the present knowledge of theoretical investigation of buckling pressure for specific problems is extended to include all problems of shells of revolution.
CHAPTER 5

SUMMARY AND CONCLUSIONS

Although it was believed and confirmed by the results presented in this thesis that the linear theory is unable to predict actual state of stresses and deformations in shells of revolution made of parts of different geometries, yet no work has been done to obtain solution of these problems based on the more advanced theory of shells capable of accounting the shortcomings of linear theory. Thus, analysis of composite shells based on the nonlinear theory has been achieved here with specific attention to problems of pressure vessels. The nonlinear theory of axisymmetric shells as developed by Reissner (33) has been used in this thesis. The basic concept of multisegment integration developed by Kalnins and Lestini (2) has been employed to obtain the solutions of the nonlinear equations of shells.

The basic equations of shells pertaining to the apex have been derived and recorded here for both the linear and nonlinear theories. The equations for the apex have further been specialised for shells whose meridian has a continuous slope at the apex. These equations which are founded on the finiteness of the physical quantities of the shell at the apex can be used for shells with no central hole, and, thus the usual approximations as made in Ref. (23) is not at all necessary.
In course of this investigation, a new method of multi-segment integration has been developed for solving the nonlinear equations of shells. This new method requires only one initial value integration in each segment for second and onward passes of iteration instead of the \((m+1)^*\) initial value integrations of the method of Kalnins and Lestingi. The new method has been applied to several problems in order to evaluate its soundness and ascertain its accuracy.

A computer program has been developed for computational purposes which solves all problems of shells of revolution, either single or composite, with meridional variation in thickness and for all possible boundary conditions. The output of the computer program consists of the linear solution and the nonlinear solutions for various desired loadings. In case of stability problems, the same program will solve for the critical pressure of any shell of revolution with any specified boundary conditions provided that the number of loading steps is specified to be a large one.

Solutions are obtained for various types of commonly used pressure vessels for the whole range of working pressure and some of the results of these problems are presented here. Solutions are also obtained for a general composite shell and for various stability problems of spherical shells.

Based upon the results of various problems presented here

* \(m\) is the number of dependent variables in the system of differential equations.
the following conclusions are made:

(1) The linear theory of shells is, in general, very conservative in predicting the state of stresses and deformations in pressure vessels, and the degree of conservativeness increases with decreasing thickness of the shell.

(2) The prediction of linear theory becomes obsolete in case of thin flat end pressure vessels.

(3) Any discontinuity in geometry of the meridian induces enormous bending stresses in the shell. If the change in geometry is also associated with the discontinuity of slope, then the maximum values of the bending moments occur at the junction. Under this circumstance the extreme-fibre meridional stresses become usually the maximum of all the stresses of the shell.

(4) If the included angle of a junction is less than 180 degrees then a circumferentially compressive zone is developed there under load.

(5) As far as the uniformity of stress distribution is concerned the hemispherical head pressure vessel appears to be the most suitable of all pressure vessels in general use, followed by the hemi-ellipsoidal head pressure vessels.

(6) In designing pressure vessels with discontinuity of slope of the meridian care has to be taken of the extreme stress concentration at the junction.

(7) The best possible way of avoiding the stress concentration at the junction is to use a spherical ring there so that no discontinuity of slope takes place.
Further extension of this work consists mainly of the application of the computer program to obtain results for different problems and present those results in a manner suitable for design purposes. For example, curves of maximum stress versus pressure, as presented here for hemispherical head pressure vessels, should be prepared for other types of pressure vessels for different ratios of head to cylinder thickness and for different values of head parameters. Given the operating pressure and allowable stress of the material, these curves can be used for finding the thicknesses of the respective parts of the vessel.

As already mentioned, the present literature is almost devoid of works on stability problems of ellipsoidal, conical, deep spherical, and all types of composite shells. The computer program can be used to determine the critical pressures of these problems and prepare curves of critical pressure versus shell characteristics.

If it is desired, the program can be modified to deal with all types of meridionally varying loads and material properties. This can be easily achieved by changing a few cards in the program.
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Fig. 1a Spherical head pressure vessels showing the head characteristic $\varphi_e, \xi_e$ is the meridional distance from apex to junction.

Fig. 1b Ellipsoidal head pressure vessel showing the head characteristic $Z = A/B, \xi_e$ is the meridional distance from apex to junction.

Fig. 1c Conical head pressure vessels showing the head characteristics $\alpha$ and $x, \xi_e$ is the distance from O to J measured along the meridian.
Fig. 1d Plate end pressure vessel. $S_e = R$.

Fig. 2. A composite shell consisting of a cylindrical part, a spherical part, a conical frustum, and an ellipsoidal top. $R$ is the radius at the base circle. $S_e$ is the total meridional distance from apex to base circle.
Fig. 3 Middle surface of shell

Fig. 4 Side view of element of shell in deformed and undeformed states.

Fig. 5 Element of shell showing stress resultants and couples.
Fig. 6. Division for multisegment integration

Fig. 7. Built-in edges composite shell; results are presented in Figs. 44, 45, and 46.
Fig. 6a, Built-in edges spherical shell (Linear Theory)

Meridional bending moments $M_\phi$ inch lbs/inch

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Fig. 6b, Built-in edges spherical shell (Linear Theory)

Membrane hoop force $N_\theta = 45.0$ lbs per inch

Hoop forces due to bending $N_\theta$ lbs/inch

---

Fig. 6c, Built-in edges spherical shell (Linear Theory)

Horizontal stress
resultant $H$ (lbs/in.)
Circumferential stress
resultant $N_\theta$ (lbs/in.)
Circumferential bending moment $M_\theta$ (in lb./in.)
Meridional bending moment $N_\phi$ (in lb./in.)

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110.
Fig. 9 Meridional bending moment of a hemispherical head pressure vessel
Fig. 10 Meridional bending moment of a hemispherical head pressure vessel.
Fig. 11 Meridional bending moment of a hemispherical head pressure vessel.
Fig. 12 Meridional bending moment of a hemispherical head pressure vessel.
Fig. 13 Circumferential stress resultant of a hemispherical head pressure vessel.
Fig. 14 Extreme-fibre circumferential stresses of a hemispherical head pressure vessel.
Fig. 15 Extreme-fibre meridional stresses of a hemispherical head pressure vessel.
Fig. 16  Maximum circumferential stress of hemispherical head pressure vessels.
Fig. 17 Maximum meridional stress of hemispherical head pressure vessels for different values of R/h.
Fig. 18a Percentage reduction in maximum circumferential stress as obtained from linear theory for hemispherical head pressure vessels.
Fig. 13b  Percentage reduction in maximum meridional stress as obtained from linear theory for hemispherical head pressure vessels.
Fig. 19 Effect of Poisson's ratio on meridional bending moment of hemispherical head pressure vessel.
Fig. 20 Effect of Poisson's ratio on the extreme-fibre meridional stresses of a hemispherical head pressure vessel.
Fig. 21 Effect of Poisson's ratio on the extreme-fibre circumferential stresses of a hemispherical head pressure vessel.
Fig. 22 Maximum meridional bending moment of hemispherical head pressure vessel for different values of R/h.
Fig. 2: Critical bending moment of a spherical pressure vessel

- Linear
- Nonlinear

\[ \frac{P}{E} = 0.1 \times 10^{-5} \]
\[ \frac{P}{E} = 0.225 \times 10^{-5} \]

\( R/h = 600 \)
\( \phi_e = 45 \) degrees
\( \nu = 0.3 \)
Fig. 24 Maximum meridional bending of spherical head pressure vessels for different values of $R/h$. 
Fig. 25 Extreme-fibre stresses of a spherical head pressure vessel based on linear theory.
Fig. 26 Extreme-fibre stresses of a spherical head pressure vessel.
Fig. 27 Maximum meridional surface stress of spherical head pressure vessels for different values of $R/h$. 
Fig. 28 Meridional and circumferential bending moments of an ellipsoidal head pressure vessel.
Extreme-fiber circumferential stress indexes $\sigma_1$ and $\sigma_0$

$\sigma_0$ = external surface stress index = actual stress/(PR/h)

$\sigma_1$ = internal surface stress index = actual stress/(PR/h)

Fig. 30 Ellipsoidal head pressure vessel. Circumferential stress indexes
Fig. 31 Meridional stress indexes of an ellipsoidal head pressure vessel.

- Linear
- Nonlinear

$R/h = 38$
$B/A = 0.5$
$\nu = 0.3$
$P/E = 0.5 \times 10^{-4}$
Fig. 32 Deformed shape of a plate end pressure vessel. Deformation in the cylindrical part is 40 times more magnified than the deformation in the plate head.
Fig. 33 Meridional bending moment of a plate end pressure vessel.
Fig. 34  Circumferential bending moment of a plate end pressure vessel.
Figure 35 Variation of meridional bending moment of a plate end pressure vessel with $R/h$. 

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Linear

Nonlinear

$P/E = 0.1 \times 10^{-6}$

$\nu = 0.3$

$M_{\xi}$ at $\xi = 1.0$

$M_{\xi}$ at $\xi = 0.0$
Fig. 36 Circumferential stress resultant of a plate end pressure vessel.
Fig. 37 Variation of axial displacement at the apex of a plate end pressure vessel with $R/h$. 

$P/E = 0.1 \times 10^6$, $\nu = 0.3$
Fig. 38 Deformed shape of a conical head pressure vessel.
Fig. 39 Meridional bending moment of a conical head pressure vessel.
Fig. 40 Circumferential bending moment of a conical head pressure vessel.
Fig. 41 Circumferential stress resultant of a conical head pressure vessel.
Fig. 42 Axial displacements at different points of a shallow built-in edges spherical shell at and near the critical pressure.
Fig. 43 Axial displacements at different points of a built-in edges hemispherical shell at and near the critical pressure.
Fig. 44 Meridional and circumferential bending moments of a built-in edges composite shell.
Fig. 45 Stress resultants $N_5$ and $N_\theta$ of a composite shell
Fig. 46 Extreme-fibre meridional stresses of a composite shell.
### TABLE (1)

**TRANSVERSELY LOADED FIXED-EDGED CIRCULAR PLATE**

Results are Based on Linear Bending Theory.

Radius = 31.4 inches, Modulus of elasticity $E = 30 \times 10^6$ lbs/in.$^2$,

Poisson's ratio $v = 0.3$, thickness = 0.2 in., Loading $P = 10$ lbs./in.$^2$

<table>
<thead>
<tr>
<th>Distance From Centre (r in.)</th>
<th>Transverse Displacement (in.)</th>
<th>Transverse Shear (lb./in.)</th>
<th>Axial Bending Moment (in.lbf./in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>31.40</td>
<td>0.000000000E + 00, 0.000000000E + 00</td>
<td>0.157000000E + 03, 0.157000000E + 03</td>
<td>0.123245000E + 04, 0.123245000E + 04</td>
</tr>
<tr>
<td>25.00</td>
<td>-.92629553E + 00, -.92629554E + 00</td>
<td>0.125000000E + 03, 0.125000000E + 03</td>
<td>0.487970000E + 03, 0.487970000E + 03</td>
</tr>
<tr>
<td>18.00</td>
<td>-.31152644E + 01, -.31152645E + 01</td>
<td>0.900000000E + 02, 0.900000000E + 02</td>
<td>-.132842500E + 03, -.132842500E + 03</td>
</tr>
<tr>
<td>12.56</td>
<td>-.48765040E + 01, -.48765041E + 01</td>
<td>0.628000000E + 02, 0.628000000E + 02</td>
<td>-.47572570E + 03, -.47572575E + 03</td>
</tr>
<tr>
<td>6.28</td>
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<td>0.314000000E + 02, 0.314000000E + 02</td>
<td>-.71975080E + 03, -.71975081E + 03</td>
</tr>
<tr>
<td>4.00</td>
<td>-.66886592E + 01, -.66886595E + 01</td>
<td>0.200000000E + 02, 0.200000000E + 02</td>
<td>-.76809250E + 03, -.76809250E + 03</td>
</tr>
<tr>
<td>1.50</td>
<td>-.68796381E + 01, -.68796380E + 01</td>
<td>0.750000000E + 01, 0.750000000E + 01</td>
<td>-.79645187E + 03, -.79645187E + 03</td>
</tr>
<tr>
<td>1.20</td>
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<td>-.79812250E + 03, -.79812250E + 03</td>
</tr>
<tr>
<td>1.04</td>
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<td>0.520000000E + 01, 0.520000000E + 01</td>
<td>-.79886170E + 03, -.79886175E + 03</td>
</tr>
<tr>
<td>1.00</td>
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<td>0.500000000E + 01, 0.500000000E + 01</td>
<td>-.79903000E + 03, -.79903000E + 03</td>
</tr>
<tr>
<td>0.00</td>
<td>-.69111451E + 01, -.69111454E + 01</td>
<td>0.000000000E + 00, 0.000000000E + 00</td>
<td>-.80109256E + 03, -.80109250E + 03</td>
</tr>
<tr>
<td>Distance From Apex ((\xi) in.)</td>
<td>Radial Displacement ((u) in.)</td>
<td>Axial Displacement ((w) in.)</td>
<td>Axial Moment ((M_u) in.lbs./in.)</td>
</tr>
<tr>
<td>---------------------------------</td>
<td>--------------------------------</td>
<td>-------------------------------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td>.54977871E + 02</td>
<td>.000000000E + 00</td>
<td>0.000000000E + 00</td>
<td>-.3755207E + 02</td>
</tr>
<tr>
<td>.47123890E + 02</td>
<td>-.32087883E - 05</td>
<td>0.11259890E - 04</td>
<td>-.57789233E + 01</td>
</tr>
<tr>
<td>.39269908E + 02</td>
<td>-.86469679E - 05</td>
<td>0.28968583E - 04</td>
<td>0.66807802E + 01</td>
</tr>
<tr>
<td>.31415927E + 02</td>
<td>-.10737765E - 04</td>
<td>0.42823245E - 04</td>
<td>0.81374440E + 01</td>
</tr>
<tr>
<td>.23561945E + 02</td>
<td>-.96219696E - 05</td>
<td>0.50587557E - 04</td>
<td>0.50514046E + 01</td>
</tr>
<tr>
<td>.15707963E + 02</td>
<td>-.68151724E - 05</td>
<td>0.53754439E - 04</td>
<td>0.23893947E + 01</td>
</tr>
<tr>
<td>.78539816E + 01</td>
<td>-.34502278E - 05</td>
<td>0.54573225E - 04</td>
<td>0.37852867E + 00</td>
</tr>
<tr>
<td>.00000000E + 00</td>
<td>0.00000000E + 00</td>
<td>.54659520E - 04</td>
<td>-.29175167E + 00</td>
</tr>
<tr>
<td>$\theta$(deg.)</td>
<td>2$N_\theta$/PR</td>
<td>$M_\theta$/Pr$^2$</td>
<td></td>
</tr>
<tr>
<td>---------------</td>
<td>----------------</td>
<td>-----------------</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Exact</td>
<td>Geckeler</td>
<td>Asymptotic</td>
</tr>
<tr>
<td>90</td>
<td>0.300</td>
<td>0.300</td>
<td>0.300</td>
</tr>
<tr>
<td>95</td>
<td>0.371</td>
<td>0.369</td>
<td>0.373</td>
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<tr>
<td>80</td>
<td>0.514</td>
<td>0.514</td>
<td>0.516</td>
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<tr>
<td>70</td>
<td>0.802</td>
<td>0.807</td>
<td>0.802</td>
</tr>
<tr>
<td>60</td>
<td>0.970</td>
<td>0.973</td>
<td>0.968</td>
</tr>
<tr>
<td>50</td>
<td>1.028</td>
<td>1.027</td>
<td>1.026</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\theta$(deg.)</th>
<th>2$N_\theta$/PR</th>
<th>$M_\theta$/Pr$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exact</td>
<td>Geckeler</td>
</tr>
<tr>
<td>90</td>
<td>0.300</td>
<td>0.300</td>
</tr>
<tr>
<td>95</td>
<td>0.695</td>
<td>0.695</td>
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<tr>
<td>80</td>
<td>0.988</td>
<td>0.988</td>
</tr>
<tr>
<td>70</td>
<td>1.010</td>
<td>1.009</td>
</tr>
<tr>
<td>60</td>
<td>0.999</td>
<td>0.999</td>
</tr>
</tbody>
</table>
TABLE (4)

SOLUTION OF THE LINEAR EQUATION \( EI \frac{d^4 w}{dx^4} = q \)

Case: \( q/EI = 0.1 \)

Boundary Conditions: \( w = \frac{dw}{dx} = 0 \) at \( x = 0 \), \( \frac{d^2 w}{dx^2} = \frac{d^3 w}{dx^3} = 0 \) at \( x = 2 \).

<table>
<thead>
<tr>
<th>Number of Pass = 1</th>
<th>x</th>
<th>w</th>
<th>( \frac{dw}{dx} )</th>
<th>( \frac{d^2 w}{dx^2} )</th>
<th>( \frac{d^3 w}{dx^3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of Pass = 2</th>
<th>x</th>
<th>w</th>
<th>( \frac{dw}{dx} )</th>
<th>( \frac{d^2 w}{dx^2} )</th>
<th>( \frac{d^3 w}{dx^3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.000</td>
<td>0.000</td>
<td>3.468</td>
<td>6.453</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.163</td>
<td>0.052</td>
<td>0.202</td>
<td>0.254</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>-0.094</td>
<td>4.376</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of Pass = 3</th>
<th>x</th>
<th>w</th>
<th>( \frac{dw}{dx} )</th>
<th>( \frac{d^2 w}{dx^2} )</th>
<th>( \frac{d^3 w}{dx^3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.042</td>
<td>-0.064</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.015</td>
<td>0.027</td>
<td>0.029</td>
<td>0.036</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.074</td>
<td>0.029</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of Pass = 4 (Exact Values)</th>
<th>x</th>
<th>w</th>
<th>( \frac{dw}{dx} )</th>
<th>( \frac{d^2 w}{dx^2} )</th>
<th>( \frac{d^3 w}{dx^3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.00</td>
<td>0.00</td>
<td>0.20</td>
<td>-0.20</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.07083</td>
<td>0.11667</td>
<td>0.05</td>
<td>-0.10</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.20</td>
<td>1.13333</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
TABLE (5)

SOLUTION OF NONLINEAR EQUATION: \( \frac{d^2}{dx^2} y + \frac{(1-u)}{\mu y} \left( \frac{dy}{dx} \right)^2 = 0 \).

Case: \( \mu = 2 \)

Boundary conditions: \( y = 4 \) at \( x = 1 \), \( \frac{dy}{dx} = 0 \) at \( x = 0 \).

<table>
<thead>
<tr>
<th>NUMBER OF PASS = 1</th>
<th>x</th>
<th>( \frac{dy}{dx} )</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>1.000</td>
<td>2.000</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>1.222</td>
<td>2.222</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.476</td>
<td>2.476</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>5.612</td>
<td>4.000</td>
</tr>
<tr>
<td>NUMBER OF PASS = 2</td>
<td>x</td>
<td>( \frac{dy}{dx} )</td>
<td>y</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>1.000</td>
<td>2.192</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>1.045</td>
<td>2.396</td>
</tr>
<tr>
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<td>0.5</td>
<td>0.469</td>
<td>2.622</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>5.044</td>
<td>4.000</td>
</tr>
<tr>
<td>NUMBER OF PASS = 3</td>
<td>x</td>
<td>( \frac{dy}{dx} )</td>
<td>y</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>1.000</td>
<td>2.926</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>1.028</td>
<td>3.129</td>
</tr>
<tr>
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<td>0.5</td>
<td>1.069</td>
<td>3.444</td>
</tr>
<tr>
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<td>1.0</td>
<td>1.157</td>
<td>4.000</td>
</tr>
<tr>
<td>NUMBER OF PASS = 4</td>
<td>x</td>
<td>( \frac{dy}{dx} )</td>
<td>y</td>
</tr>
<tr>
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<td>0.0</td>
<td>1.000</td>
<td>2.916</td>
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<tr>
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<td>0.2</td>
<td>1.033</td>
<td>3.119</td>
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<td>0.5</td>
<td>1.084</td>
<td>3.437</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.169</td>
<td>4.000</td>
</tr>
<tr>
<td>NUMBER OF PASS = 5</td>
<td>x</td>
<td>( \frac{dy}{dx} )</td>
<td>y</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>1.00000</td>
<td>2.91450</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>1.03417</td>
<td>3.11785</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>1.08560</td>
<td>3.43578</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.17135</td>
<td>4.00000</td>
</tr>
<tr>
<td>EXACT SOLUTION</td>
<td>x</td>
<td>( \frac{dy}{dx} )</td>
<td>y</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>1.00000</td>
<td>2.91421</td>
</tr>
<tr>
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<td>0.2</td>
<td>1.03431</td>
<td>3.11765</td>
</tr>
<tr>
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<td>0.5</td>
<td>1.08578</td>
<td>3.43566</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.17157</td>
<td>4.00000</td>
</tr>
</tbody>
</table>
TABLE (6)

FIXED-EDGES HEMISPHERICAL SHELL WITH OPEN TOP

Poisson's ratio $v = 0.3$, Radius $R = 20$ inch., Modulus of elasticity $E = 30 \times 10^6$ PSI

Pressure $P = 1.0$ PSI

<table>
<thead>
<tr>
<th>Distance From Apex (inch)</th>
<th>Horizontal Displacement (inch)</th>
<th>Vertical Displacement (inch)</th>
<th>Vertical Stress Resultant (lbs./inch)</th>
<th>Horizontal Stress Resultant (lbs./inch)</th>
<th>Meridional Moment (inch lbs./inch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>31.415</td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
<td>0.29192E+01</td>
<td>-1.2623E+01</td>
<td>0.98366E+00</td>
</tr>
<tr>
<td></td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
<td>0.29190E+01</td>
<td>-1.2628E+01</td>
<td>0.98317E+00</td>
</tr>
<tr>
<td>28.0</td>
<td>0.52834E-04</td>
<td>-3.7993E-05</td>
<td>0.26692E+01</td>
<td>0.54609E+00</td>
<td>-1.5465E+00</td>
</tr>
<tr>
<td></td>
<td>0.52734E-04</td>
<td>-4.2957E-05</td>
<td>0.26690E+01</td>
<td>0.54447E+00</td>
<td>-1.5387E+00</td>
</tr>
<tr>
<td>24.0</td>
<td>0.54996E-04</td>
<td>-1.8460E-05</td>
<td>0.17234E+01</td>
<td>0.67020E+00</td>
<td>0.87066E+02</td>
</tr>
<tr>
<td></td>
<td>0.55092E-04</td>
<td>-2.4609E-05</td>
<td>0.17233E+01</td>
<td>0.66920E+00</td>
<td>0.90833E+02</td>
</tr>
<tr>
<td>20.0</td>
<td>0.56775E-04</td>
<td>-4.5284E-06</td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
</tr>
<tr>
<td></td>
<td>0.56035E-04</td>
<td>-5.1281E-06</td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
</tr>
</tbody>
</table>
TABLE (7)

BOTH EDGES BUILT-IN PRESSURIZED CYLINDRICAL SHELL

RESULTS ARE BASED ON NONLINEAR THEORY

Radius $R = 20$ inch., Poisson's ratio $\nu = 0.3$, Modulus of elasticity $E = 30 \times 10^6$ psi

Pressure $P = 1$ Psi, Length $L = 30$ inch.

<table>
<thead>
<tr>
<th>Meridional Distance (inch)</th>
<th>Horizontal Displacement (inch)</th>
<th>Rotation of Normal (radian)</th>
<th>Axial Displacement (inch)</th>
<th>Axial Stress Resultant (lb./inch.)</th>
<th>Transverse Shear (lb./inch.)</th>
<th>Axial Moment (inch.lbs./inch.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3000E+02</td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
<td>0.46894E+01</td>
<td>-0.32332E+01</td>
<td>0.56248E+01</td>
</tr>
<tr>
<td>0.2000E+02</td>
<td>0.12870E-04</td>
<td>-0.11913E-06</td>
<td>-0.25357E-06</td>
<td>0.46894E+01</td>
<td>0.18574E+00</td>
<td>-0.36652E+00</td>
</tr>
<tr>
<td>0.1000E+02</td>
<td>0.12876E-04</td>
<td>0.11800E-06</td>
<td>0.24657E-06</td>
<td>0.46893E+01</td>
<td>-0.18377E+01</td>
<td>-0.36747E+00</td>
</tr>
<tr>
<td>0.0000E+00</td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
<td>0.00000E+00</td>
<td>0.46893E+01</td>
<td>0.32342E+01</td>
<td>0.56238E+01</td>
</tr>
</tbody>
</table>

* as given by Kraus
TABLE (8)

**ONE END FIXED AND OTHER END FREE CYLINDRICAL SHELL**

RESULTS ARE BASED ON NONLINEAR THEORY

Radius $R = 20$ inch, Poisson's ratio $\nu = 0.3$, Modulus of elasticity $E = 30 \times 10^6$ Psi

Pressure $P = 1.0$ Psi, Length $L = 30$ inch.

<table>
<thead>
<tr>
<th>Meridional Distance (inch)</th>
<th>Radial Displacement (inch)</th>
<th>Rotation of Normal (radian)</th>
<th>Axial Displacement (inch)</th>
<th>Axial Stress Resultant (lb./inch)</th>
<th>Transverse Shear (lb./inch)</th>
<th>Axial Moment (inch.lb./inch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.30000E+02$</td>
<td>$0.00000E+00$</td>
<td>$0.00000E+00$</td>
<td>$0.00000E+00$</td>
<td>$-0.40496E-04$</td>
<td>$-0.34788E+01$</td>
<td>$0.60520E+01$</td>
</tr>
<tr>
<td>$0.20000E+02$</td>
<td>$0.13872E-04$</td>
<td>$-0.11755E-06$</td>
<td>$0.12649E-05$</td>
<td>$-0.26119E-04$</td>
<td>$0.18760E+00$</td>
<td>$-0.42005E+00$</td>
</tr>
<tr>
<td>$0.10000E+02$</td>
<td>$0.13103E-04$</td>
<td>$0.86076E-07$</td>
<td>$0.33009E-05$</td>
<td>$-0.26921E-04$</td>
<td>$0.13201E-00$</td>
<td>$0.77594E-02$</td>
</tr>
<tr>
<td>$0.00000E+00$</td>
<td>$0.16332E-04$</td>
<td>$-0.10502E-05$</td>
<td>$0.19155E-04$</td>
<td>$0.00000E+00$</td>
<td>$0.00000E+00$</td>
<td>$0.00000E+00$</td>
</tr>
</tbody>
</table>

* analytical solution based on linear theory.
TABLE (9)

SOLUTION AT THE CRITICAL LOAD OF A SHALLOW BUILT-IN EDGES SPHERICAL SHELL

SHELL PARAMETERS:

Radius to thickness ratio \( (R/h) = 4000 \)

Semi-opening angle = 3.6 degrees

Poisson's ratio \( (v) = 0.3 \)

SOLUTION:

Critical pressure \( \left( \frac{P}{E} \right) = 0.80312 \times 10^{-7} \) (as obtained by the present method)

<table>
<thead>
<tr>
<th>Meridional Angle in Degrees</th>
<th>Horizontal Displacement ( (\bar{u}) )</th>
<th>Rotation ( (\bar{\beta}) )</th>
<th>Axial Displacement ( (\bar{w}) )</th>
<th>Axial Stress Resultant ( (\bar{V}) )</th>
<th>Horizontal Stress Resultant ( (\bar{H}) )</th>
<th>Meridional Bending Moment ( (M_\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.6</td>
<td>0.000000000E+00</td>
<td>0.000000000E+00</td>
<td>0.000000000E+00</td>
<td>0.31395333E-01</td>
<td>0.55910981E+00</td>
<td>0.52837016E-01</td>
</tr>
<tr>
<td>1.8</td>
<td>0.27800168E-01</td>
<td>-0.72871733E-02</td>
<td>-0.10682911E+01</td>
<td>0.15700951E-01</td>
<td>0.54670639E+00</td>
<td>-0.13839261E+00</td>
</tr>
<tr>
<td>0.0</td>
<td>0.000000000E+00</td>
<td>0.000000000E+00</td>
<td>-0.19456085E+00</td>
<td>0.000000000E+00</td>
<td>0.22993659E+00</td>
<td>0.48579070E-01</td>
</tr>
</tbody>
</table>
TABLE (10)

SOLUTION AT THE CRITICAL PRESSURE FOR A BUILT-IN EDGES SPHERICAL SHELL WITH AN OPENING ANGLE OF 90 DEGREES

SHELL PARAMETERS:
- Radius to thickness ratio \( (R/h) = 25 \)
- Opening angle = 90 degrees
- Poisson's ratio \( (\nu) = 0.3 \)

SOLUTION: Critical Pressure \( \left( \frac{P}{E} \right) = 0.21826 \times 10^{-2} \) (as obtained by the present method)

<table>
<thead>
<tr>
<th>Meridional Angle in Degrees</th>
<th>Horizontal Displacement ( (u) )</th>
<th>Rotation ( (\beta) )</th>
<th>Axial Displacement ( (w) )</th>
<th>Vertical Stress Resultant ( (\bar{V}) )</th>
<th>Horizontal Stress Resultant ( (\bar{H}) )</th>
<th>Meridional Stress Resultant ( (\bar{R}_S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>45.0</td>
<td>0.000000000E+00</td>
<td>0.000000000E+00</td>
<td>0.000000000E+00</td>
<td>0.35355404E+00</td>
<td>0.42597006E+00</td>
<td>0.46199435E-01</td>
</tr>
<tr>
<td>22.5</td>
<td>0.32787789E+00</td>
<td>-0.85882954E-01</td>
<td>-0.98953840E+00</td>
<td>0.18239674E+00</td>
<td>0.49460654E+00</td>
<td>-0.14249477E+00</td>
</tr>
<tr>
<td>0.0</td>
<td>0.000000000E+00</td>
<td>0.000000000E+00</td>
<td>-0.20403974E+00</td>
<td>0.000000000E+00</td>
<td>0.22073491E+00</td>
<td>0.52197297E-01</td>
</tr>
</tbody>
</table>
TABLE (11)

SOLUTION FOR A BUILT-IN EDGES HEMISPHERICAL SHELL AT THE CRITICAL PRESSURE

SHELL PARAMETERS: Radius to thickness ratio \( \frac{R}{h} = 25 \)

Poisson's ratio \( \nu = 0.3 \)

SOLUTION: Critical pressure \( \left( \frac{P}{E} \right) = 0.19625 \times 10^{-2} \) (as obtained by the present method)

Critical pressure by classical theory \( \left( \frac{P}{E} \right) = 0.194 \times 10^{-2} \)

<table>
<thead>
<tr>
<th>Meridional Angle in Degrees</th>
<th>Horizontal Displacement ( (\vec{u}) )</th>
<th>Rotation ( (\beta) )</th>
<th>Axial Displacement ( (\vec{v}) )</th>
<th>Vertical Stress Resultant ( (\vec{V}) )</th>
<th>Horizontal Stress Resultant ( (\vec{H}) )</th>
<th>Meridional Moment ( (M_{\phi}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>90.0</td>
<td>0.00000000E+00</td>
<td>0.00000000E+00</td>
<td>0.00000000E+00</td>
<td>0.50000000E+00</td>
<td>-23233451E-02</td>
<td>0.98903208E-01</td>
</tr>
<tr>
<td>54.0</td>
<td>0.11487533E+00</td>
<td>-80841206E-01</td>
<td>-12137628E+00</td>
<td>0.40169532E+00</td>
<td>0.26606119E+00</td>
<td>0.61013469E-01</td>
</tr>
<tr>
<td>27.0</td>
<td>0.26337574E+00</td>
<td>-29453730E-01</td>
<td>-66512322E+00</td>
<td>0.22054445E+00</td>
<td>0.44856301E+00</td>
<td>-61941112E-01</td>
</tr>
<tr>
<td>0.0</td>
<td>0.00000000E+00</td>
<td>0.00000000E+00</td>
<td>-26515334E+00</td>
<td>0.00000000E+00</td>
<td>0.36563538E+00</td>
<td>0.15637581E-01</td>
</tr>
</tbody>
</table>
APPENDIX A

PROGRAMMING FEATURES

(i) GENERAL

The computer program developed for computational purposes and presented in Appendix B is perfectly general for application to any problem of shells of revolution whose meridian is composed of segments consisting of line elements, circular elements, and elliptic elements. The shell may be open or closed at the apex, and may have variation of thickness from segment to segment. All possible boundary conditions are incorporated in the program. In case of a shell closed at the apex it is assumed that the curvature of the meridian is continuous at this point. Thus shells having discontinuity of curvature at the apex should be assumed to be tipped by a small circular arc meeting tangentially at the joint. This is done because for all practical purposes a shell would not have a perfectly pointed apex due to obvious difficulty in manufacturing; and even if it has a pointed apex, it should be avoided for the reason that this will produce enormous stress concentration at the apex. Regarding application to pressure vessels, the program will solve the following problems.
(a) spherical head pressure vessels.
(b) flat end pressure vessels.
(c) conical head pressure vessels.
(c) ellipsoidal head pressure vessels.

The program first prints out the results based on the linear theory of shells which is followed by the print-out of nonlinear results for the same loading. From here on, if desired, the program will produce nonlinear results for increasing loading steps up to the number of steps as directed. In part A of the program the necessary information required for the solution of a problem is read in. Part B of the program deals with the problem of adjusting the given boundary conditions with regard to the solutions of the matrix equations. In part C, which applies only to pressure vessel problems, the constant $\bar{R}$, called 'RC' in the program, is determined in accordance with the type of pressure vessel. Part D of the program concerns with the calculation of the normalised constants involving shell parameters, material constants, and loading. These normalised constants vary from segment to segment of the shell meridian. Under the Part E of the program the output of the results is handled. The remaining portion of the program deals with integration of different systems of differential equations and the solution of matrix equations.
The integration technique used in the program is a predictor-corrector method. The predictor and the corrector are respectively given by formulas (19.16) and (19.17) of Ref. (66). To secure the six starting values necessary for the application of this pair of predictor and corrector the six point formulas (19.10 - 19.14) of Ref. (66) are being used. All these formulas have an error $O(h^7)$ where $h$ is the distance between two consecutive computational points, and thus they are highly sophisticated formulas.

Each segment of the shell is divided into twenty-one computational points. As the length of each segment of the shell is restricted by the values of the shell parameters for that segment it was assumed that twenty-one computational points would adequately cover a segment and, therefore, no variation of computational points were allowed for different segments. The results printed out are the normalised radial displacement $\tilde{u}$, axial displacement $\tilde{w}$, circumferential moment $\tilde{M}_\theta$, axial moment $\tilde{M}_\xi$, circumferential stress resultant $\tilde{N}_\theta$, axial stress resultant $\tilde{N}_\xi$, circumferential stress at inner surface $\tilde{\sigma}_{ci}$, circumferential stress at outer surface $\tilde{\sigma}_{co}$, axial stress at inner surface $\tilde{\sigma}_{ci}$, and axial stress at outer surface $\tilde{\sigma}_{ao}$; and they are in that order columnwise.
(ii) CONSIDERATION OF DIFFERENT GEOMETRICAL SHAPES OF THE HEAD

It has already been pointed out that the constant \( \bar{R} \) and the explicit relationships of the normalised independent variable \( \bar{S} \) with the normalised radial distance \( \bar{r}_o \) and the meridional angle \( \phi_o \) are necessary in Eqs. (12, 13, 14, 15). These quantities are dependent on the geometrical shape of the meridian of the shell and, therefore, have to be determined separately for each geometrical shape.

(a) Spherical Head Pressure Vessels

In case of a spherical head pressure vessel the meridional length of the head \( \xi_e \) is given by

\[
\xi_e = \phi_e \frac{R}{\sin \phi_e} \tag{A-1}
\]

where \( R \) is the radius of the cylinder and \( \phi_e \) is the meridional angle of spherical part at the junction.

The constant \( \bar{R} \) is then given by

\[
\bar{R} = \frac{R}{\xi_e} = \frac{\phi_e}{\sin \phi_e} \tag{A-2}
\]

which, in case of a hemispherical head, becomes

\[
\bar{R} = \frac{\pi}{2} \tag{A-3}
\]
The whole range of spherical head pressure vessels is thus covered by $\frac{\pi}{2} \leq \phi_o \leq 0$. The expressions for $\phi_o$ and $r_o$ in terms of $\bar{\phi}$ can easily be derived as

\begin{align*}
\phi_o &= \bar{\phi} \cdot \phi_e \quad (A-4) \\
and \\
r_o &= \sin \phi_o / \phi_e \quad (A-5)
\end{align*}

(b) Flat End Pressure Vessel

For a flat end pressure vessel the three quantities $\bar{R}$, $\phi_o$, and $r_o$ have the following expressions.

\begin{align*}
\bar{R} &= 1.0 \quad (A-6) \\
\phi_o &= 0.0 \quad (A-7) \\
r_o &= \bar{\phi} \quad (A-8)
\end{align*}

(c) Conical Head Pressure Vessels

In this case it was thought to be proper to replace the apex of the conical head by a small circular arc meeting tangentially with the conical part so that the geometrical model represents practical cases as closely as possible. The radius at the junction of conical and spherical parts is expressed as a fraction of the radius of the cylindrical part.
Thus, by specifying the value of the fraction, the size of the spherical top can be reduced to any desired value. With this arrangement the constant \( \bar{R} \) becomes

\[
\bar{R} = \frac{(1 - \eta)}{\sin \alpha} + \left( \frac{\pi/2 - \alpha}{\cos \alpha} \right) \cdot \eta \quad (A-9)
\]

where \( \eta \) is the ratio of the radius at the junction of conical and spherical parts to the radius of the cylindrical part, and \( \alpha \) is the semi-angle of the conical part.

The expressions for \( \phi_o \) and \( r_o \) are

\[
\phi_o = \frac{\pi}{2} - \alpha \quad (A-10)
\]

\[
r_o = \eta/\bar{R} + \left[ \bar{s} - (\pi/2 - \alpha)/(\bar{R} \cos \alpha) \right] \sin \alpha \quad (A-11)
\]

for \( 1 \leq \bar{s} \leq [\eta(\pi/2 - \alpha)/(\bar{R} \cos \alpha)] \)

and

\[
\phi_o = \bar{s} \bar{R} \cos \alpha/\eta \quad (A-12)
\]

\[
r_o = \eta \sin \phi_o/(\bar{R} \cos \alpha) \quad (A-13)
\]

for \( [\eta(\pi/2 - \alpha)/(\bar{R} \cos \alpha)] \leq \bar{s} \leq 0 \).

The input quantities \( \eta \) and \( \alpha \) are designated by 'XL' and 'ALP', respectively, in the program.
(d) Ellipsoidal Head Pressure Vessels

For ellipsoidal head pressure vessels there is no closed form expression for the meridional length of the head. The integral expression for $S_e$ is

$$S_e = A \int_0^{\pi/2} (1 - Z^2 \sin^2 \theta)^{1/2} d\theta$$  \hspace{1cm} (A-14)

where $Z^2 = 1 - (B/A)^2$

$B$ = length of the minor axis

and $A$ = length of the major axis and, therefore, equal to the radius $R$ of the cylinder.

From Eq. (A-14) the constant $\bar{R}$ yields

$$\bar{R} = \frac{S_e}{R} = \int_0^{\pi/2} (1 - Z^2 \sin^2 \theta)^{1/2} d\theta$$

$$= \frac{\pi}{2} \left[ 1 - \left( \frac{1}{2} \right)^2 \frac{2}{1} - \left( \frac{1}{2} \cdot \frac{3}{4} \right)^2 \frac{4}{3} - \left( \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \right)^2 \frac{6}{5} - \ldots \right]$$  \hspace{1cm} (A-15)

The meridional angle $\theta_o$ is related to the independent variable $S_e$ by the differential equation
This differential equation can not be integrated in a closed form. The explicit relationship between $\phi_o$ and $\xi$ should thus be obtained by numerical integration of equation (A-16). Once $\phi_o$ is known from (A-16) the radial distance $\rho_o$ is given by

\[ \frac{d\phi_o}{d\xi} = \frac{R}{z^2} \left[ \sin^2 \phi_o + z^2 \cos^2 \phi_o \right]^{3/2} \] 

(A-16)

The quantity $z^2$ is called 'ER' in the program. It should be noted that the quantities $\rho_o$ and $\phi_o$ remain constant within the cylindrical part for all the pressure vessel problems and their values are

\[ \rho_o = \frac{1}{R} \] 

(A-18)

\[ \phi_o = \frac{\pi}{2} \] 

(A-19)

(e) General Case of Shells of Revolution

To deal with the shells with general composite meridian constituted of line elements, circular elements, and elliptic elements, the length of the shell meridian has to be determined
separately for individual cases. The constant $\bar{R}$, defined as $\bar{S}_e/R - R$ being the radius of the shell at the base, is then directly read in by the program. Other informations needed are the values of the meridional angle of each segment $S_i$ at the nodal point $i$ and the type of each segment $S_i$.

(1) Line Element.

If the segment $S_i$ is a line element, then the meridional angle $\theta_o$ remains constant over the segment $S_i$ and its value is

$$\theta_o = (\theta_o)_i$$  \hspace{1cm} (A-20)

The normalised radial distance $\bar{r}_o$ is given by the expression

$$\bar{r}_o = (r_o)_i - [(\bar{S})_i - \bar{S}] \cos (\theta_o)_i$$  \hspace{1cm} (A-21)

where the subscript $i$ refers to the nodal point $i$.

(2) Circular Element.

If the segment $S_i$ is a circular element then the quantities $\bar{r}_o$ and $\theta_o$ over the segment $S_i$ are given by

$$\theta_o = (\theta_o)_i - [(\bar{S})_i - \bar{S}] \sin (\theta_o)_i / (r_o)_i$$  \hspace{1cm} (A-22)

and

$$\bar{r}_o = (r_o)_i \cdot \sin \theta_o / \sin (\theta_o)_i$$  \hspace{1cm} (A-23)
(3) Elliptic Element.

If a segment $S_i$ of the meridian is a portion of an ellipse then there are no explicit expressions for $\Phi_0$ and $\Phi_0$ and we have to resort to the numerical integration of the differential equation (A-16). It should be noted that an additional information, the ratio of the minor to major axis of the ellipse of which the segment $S_i$ is a part, is necessary for the integration of (A-16).

(iii) TREATMENT OF BOUNDARY CONDITIONS

Equations (18a) written in terms of the normalised fundamental variables and in accordance with the statement of Eqs. (10c) appear as

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\bar{u} \\
\bar{\beta} \\
\bar{w} \\
\bar{V} \\
\bar{H} \\
\bar{M}_E \\
\end{bmatrix}
=
\begin{bmatrix}
\bar{u} \\
\bar{\beta} \\
\bar{w} \\
\bar{V} \\
\bar{H} \\
\bar{M}_E \\
\end{bmatrix}
\]  

(A-24)

In the matrix Eqs. (A-24) the elements of the column matrix on the left hand side remain in the same order, whereas, those
on the right hand side should be arranged in such a manner that the three prescribed elements at the boundary become the first three elements of this column matrix. According to Eqs. (10c), if \( \bar{u} \) is specified at the boundary, the first and the fifth rows of the unit-matrix of (A-24) remain the same, while specification of \( \bar{H} \) at the boundary will require the interchange of these two rows which will interchange \( \bar{u} \) and \( \bar{H} \) in the column matrix on the right hand side. Similarly, if \( \bar{\sigma} \) is specified at the boundary, the second and the last rows remain as they are, and interchanged when \( \bar{M}_S \) is specified. Lastly, the third and the fourth rows of the unit-matrix are kept the same or interchanged depending on whether \( \bar{w} \) or \( \bar{V} \) is specified at the boundary. The same operation is carried out for both the boundary points. The transformed unit-matrices of (A-24) are then designated by \( T_1 \) at the starting boundary and by \( T_{M+1} \) at the finishing boundary.

(iv) ON THE USE OF THE PROGRAM

In order to use the program for obtaining solutions of different problems the knowledge of the definition of in-put and out-put variables is essential. Therefore, these variables with their definitions are given in the table at the end of Appendix A.
Necessary informations to be read in are:

Card No. 20: This card reads in the amount of loading step \( EMl \) and the number of loading steps \( SOBl \). The loading step is necessary to obtain nonlinear solution at higher loading intensities. For a given load the linear solution is obtained first, and then the nonlinear solution is obtained by iteration starting from the linear solution as the initial guess. From there on nonlinear solutions are obtained for higher loadings with an increment of \( EMl \) at a time - the initial guess at each new loading being the nonlinear solution of the previous loading step. If at any loading the solution fails to converge, the loading step \( EMl \) is automatically halved by the program and the solution for the new loading is attempted. If the solution is desired for a particular loading the quantities \( EMl \) and \( SOBl \) should be read in as zero.

Card No. 21: \( M \), the number of segments of the shell meridian, and \( IZ \), indicator of the type of problem, are read in by this card. There is no exact formula for the calculation of the appropriate length of the segment. The normalized segment length \( \Delta \overline{S} \) is roughly given by

\[
\Delta \overline{S} \leq 3 \sqrt{R_m/R} \sqrt{h/R}/[R(3(1 - \nu^2))^{1/4}] \quad (A-25)
\]
where $R_m$ is the minimum principal radius of curvature of the shell generated by this segment,

$R$ is the radius of cylindrical part in case of pressure vessels, or the radius at the base of the shell in case of general shells of revolution,

and $h$ is the thickness of the shell for this segment.

Eq. (A-25) reduces to

$$\Delta s \leq 3/[R \sqrt{T} (3(1 - \nu^2))^{1/4}]$$

for a hemispherical head pressure vessel. It should be noted at this point that, while the shell meridian is divided into segments, the points of discontinuity of the shell meridian as well as thickness must be considered as nodal points. As the formula (A-25) gives only a rough estimation of segment length, it is quite possible that sometimes the segment length is over estimated. Under this circumstance, the program will print out a message saying that the particular segment is too long.

The indicator $IZ$ will have different values depending upon the type of problem to be solved. The appropriate values of $IZ$ in accordance with the types of problem are given below in tabular form.
Card No. 23: This card is used only for the general case of composite shells and will be skipped over in case of pressure vessel problems. It reads in the values of IG(I) which indicate the type of the segment $S_i$. The quantity IG(I) may have any one of the values given below in tabular form depending upon the type of the segment $S_i$.

<table>
<thead>
<tr>
<th>Type of Segment $S_i$</th>
<th>Value of IG(I)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line element</td>
<td>1</td>
</tr>
<tr>
<td>Circular element</td>
<td>2</td>
</tr>
<tr>
<td>Elliptic element</td>
<td>3</td>
</tr>
</tbody>
</table>
Card No. 24: This card also is used only for the general case of composite shells and skipped otherwise. It reads in the starting value of the meridional angle \( \phi_0 \) for the segment \( S_i \).

Card No. 25: Like Cards No. 23 and 24 this card is ignored for pressure vessel problems and is used only for composite shells. The value of \( RC \), the ratio of the total length of the shell meridian to the radius at the base of the shell, is read in by this card. In case of a shell which is open at the top the length of the meridian should be measured from the centre of the open top so that the value of \( \bar{\xi} \) at the edge of the open top is different from zero. This is necessary because \( \bar{\xi} = 0 \) is associated with the specialised equations valid only at the apex.

Card No. 26: This card reads in the values of Poisson's ratio \( \alpha_N \), normalized load \( \alpha_M \), meridional angle of the spherical cap \( \phi_I \) at the juncture, the semi-angle \( \alpha_{LP} \) of the conical head, the ratio \( \alpha_R \) of the minor to major axes of the ellipsoidal head, and the ratio \( \alpha_{XL} \) of the radius at the juncture of spherical tipping of conical head to the radius of the cylindrical part.

The last four quantities of this card, namely \( \phi_I \), \( \alpha_{LP} \), \( \alpha_R \), and \( \alpha_{XL} \) are not needed for general case of composite shells, and thus can be assigned arbitrary values. In case of pressure vessel problems only the relevant quantity should be assigned its appropriate value and others are ignored.
Card No. 27: This card reads in the thickness ratios TK(I) for the segments S_i.

Card No. 29: This card reads in the values of the independent variable X(J, 1) and the initial values of the six fundamental variables X(J, I), (I = 2, 7), for nodal points J, (J = 1, M+1). For the general case of composite shells the nodal point (J = 1) coincides with the base of the shell where X(1, 1) = 1.0, while the other edge of the shell corresponds to J = M + 1 where X(M+1, 1) should be either zero or some other value depending on whether the shell is open or closed at the crown. In case of pressure vessel problems (J = 1) will be some point in the cylindrical part where the solution of momentless theory has not been disturbed by the 'edge-effect'. This point is usually a function of the type of head used, its thickness ratio, and the loading. In any case X(1, 1) should not be greater than 1.5, i.e., half of the length of the head from the juncture. The initial values of the fundamental variables should be read in as zeros or any arbitrary values.

Card No. 31: The boundary values for any three of the six fundamental variables at the starting boundary are accepted through this card. In case of pressure vessels these three
prescribed boundary conditions are:

\[
\begin{align*}
XX(1, 1) &= \bar{H} = 0.0 \\
XX(2, 1) &= \bar{\phi} = 0.0 \\
XX(3, 1) &= \bar{w} = 0.0 \\
\end{align*}
\] (A-26)

Card No. 32: This card reads in the three prescribed boundary conditions at the final boundary. For pressure vessel problems these boundary conditions refer to apex and they are

\[
\begin{align*}
XY(1, 1) &= \bar{u} = 0 \\
XY(2, 1) &= \bar{\psi} = 0 \\
XY(3, 1) &= \bar{V} = 0 \\
\end{align*}
\] (A-27)

Card No. 33: The values of the boundary conditions indicators at the starting boundary are read in by this card. The appropriate values of the indicators 'IS1', 'IS2', and 'IS3' are given in the following table.

<table>
<thead>
<tr>
<th>Specified quantity</th>
<th>Indicator and its value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{u} )</td>
<td>IS1 = 0</td>
</tr>
<tr>
<td>( \bar{\psi} )</td>
<td>IS2 = 0</td>
</tr>
<tr>
<td>( \bar{w} )</td>
<td>IS3 = 0</td>
</tr>
<tr>
<td>( \bar{V} )</td>
<td>IS3 = 1</td>
</tr>
<tr>
<td>( \bar{H} )</td>
<td>IS1 = 1</td>
</tr>
<tr>
<td>( \bar{M}_\xi )</td>
<td>IS2 = 1</td>
</tr>
</tbody>
</table>
Card No. 34: Here the values of the boundary conditions indicators at the final boundary are read in. Their appropriate values are given in the above table where the quantities 'IS1', 'IS2', and 'IS3' should be replaced by 'IF1', 'IF2', and 'IF3', respectively.

(v) OUT-PUT OF THE PROGRAM

The first out-put will be the given initial nodal values of the independent variable $\bar{S}$ and the six fundamental variables $\bar{u}, \bar{v}, \bar{w}, \bar{V}, \bar{H}$, and $\bar{M}_{\bar{S}}$, in their written order columnwise and in tabular form. The second out-put gives the value of number of pass, residue - the sum of the differences of the absolute values of the fundamental variables at the nodal points of the two recent consecutive passes, and the present value of the normalised load.

The first out-put is then repeated for solution based on linear theory. The next out-put presents the details of the solution based on the linear theory. Here the following quantities are printed out in tabular form and in the order of $\bar{S}$, $\bar{u}$, $\bar{w}$, $\bar{M}_\theta$, $\bar{M}_{\bar{S}}$, $\bar{N}_\theta$, $\bar{N}_{\bar{S}}$, $\bar{C}_{ci}$, $\bar{C}_{co}$, $\bar{C}_{ai}$, $\bar{C}_{ao}$ columnwise. For each segment these quantities are printed out at eleven equispaced points. This can be changed to either twenty-one
or six points by changing the last digit on card No. 294 to either one or four. After the print-out of the linear solution there will be repetition of the second and the first output (now based on nonlinear theory) for a number of times until the solution converges. When convergence is attained the details of the nonlinear solution will be printed out. The solutions at the nodal points are printed out twice, first - based on the initial value integration and second - based on the solution of matrix equations, to check on the accuracy of the results.

From this point onward the nonlinear solutions will be repeatedly printed out for increasing loadings up to the desired number of loading steps.

(vi) FURTHER BRIEFING

In order to analyse composite shells constituted of different materials Card No. 26 should be replaced by the following two cards.

516 Read 110, (ANK(I), I = 1, M) [for reading in the poisson's ratio of different segments.]

Read 110, (EMK(I), I = 1, M) [for reading in the normalised loadings for the segments which will be different for different segment as the value of Young's modulus is different.]
In addition, two more cards containing

\[ AN = ANK(J1) \]

\[ EM = EMK(J1) \]

should be added immediately before card No. 113. It should be noted that a card containing dimensions of ANK and EMK is also necessary. For pressure vessel problems another card should be added to read in the values of the appropriate quantity or quantities from the group 'PHI', 'ALP', 'ER', and 'XL'. The quantity 'ER' is necessary also for general composite shells if a segment of the shell is made-up of elliptic element.

(vii) COMPUTATIONAL TIME AND COMPUTER STORAGE REQUIREMENT

The computational time depends on the number of segments into which the shell meridian has to be divided. The length of a segment may be approximately estimated by Eq. (A-25). For an average problem which consists of ten segments the computational time including both the linear and nonlinear solutions is approximately 4 minutes on the G. E. 415. If nonlinear solutions are required for a number of loadings, then the time for each additional loading will be approximately 2½ minutes.
As evident from the program listing, no overlay is used in the program and no special tape is required to run the program. The program, as listed in Appendix B, has a dimension for 32 segments of the shell meridian which require a core storage of 11315 words on the G. E. 415 (available storage is 30,000 words). Each additional segment will require a core storage of 100 words. Experience suggests that most of the practical problems are well within the 32 segments range. Unless the shell is too thin or exceptionally long additional core storage will not be necessary. In the case of long thin shells it may be expected that there will be points along the shell meridian where the membrane solution is applicable and, thus, can be treated as a new boundary. For example, the membrane solution is valid in the cylindrical parts of pressure vessels at points away from the junction.
### TABLE OF INPUT-OUTPUT VARIABLES OF THE PROGRAM

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>EM</td>
<td>EM = P/E, Normalised load</td>
</tr>
<tr>
<td>EML</td>
<td>Increasing step of EM</td>
</tr>
<tr>
<td>SOBL</td>
<td>Number of desired loading step</td>
</tr>
<tr>
<td>M</td>
<td>Number of segments</td>
</tr>
<tr>
<td>IZ</td>
<td>Indicator of the type of problem</td>
</tr>
<tr>
<td>APH(I)</td>
<td>Meridional angle at the nodal point I</td>
</tr>
<tr>
<td>IG(I)</td>
<td>Indicator of the type of segment $S_i$</td>
</tr>
<tr>
<td>RC</td>
<td>Constant $R = \xi_e / R$</td>
</tr>
<tr>
<td>AN</td>
<td>Poisson's ratio $\nu$</td>
</tr>
<tr>
<td>PHI</td>
<td>$\phi_e$, meridional angle of the spherical cap at the juncture</td>
</tr>
<tr>
<td>ALP</td>
<td>$\alpha$, semi-angle of the conical head</td>
</tr>
<tr>
<td>ER</td>
<td>The ratio of minor to major axis of ellipsoidal head</td>
</tr>
<tr>
<td>XL</td>
<td>$\eta$, Eq. (A-9)</td>
</tr>
<tr>
<td>TK(I)</td>
<td>$R/h$, thickness ratio for the segment $S_i$</td>
</tr>
<tr>
<td>X(1, I)</td>
<td>$\tilde{\xi}$ at the nodal point I</td>
</tr>
<tr>
<td>X(2, I)</td>
<td>$\tilde{u}$ at the nodal point I</td>
</tr>
<tr>
<td>X(3, I)</td>
<td>$\tilde{\phi}$ at the nodal point I</td>
</tr>
<tr>
<td>X(4, I)</td>
<td>$\tilde{w}$ at the nodal point I</td>
</tr>
<tr>
<td>X(5, I)</td>
<td>$\tilde{V}$ at the nodal point I</td>
</tr>
<tr>
<td>Variable</td>
<td>Definition</td>
</tr>
<tr>
<td>-------------</td>
<td>-----------------------------------</td>
</tr>
<tr>
<td>X(6, I)</td>
<td>$\bar{H}$ at the nodal point I</td>
</tr>
<tr>
<td>X(7, I)</td>
<td>$\bar{M}_x$ at the nodal point I</td>
</tr>
<tr>
<td>XX(1, 1)</td>
<td>value of $\bar{u}$ or $\bar{H}$ at the starting boundary</td>
</tr>
<tr>
<td>XX(2, 1)</td>
<td>value of $\bar{p}$ or $\bar{M}_x$ at the starting boundary</td>
</tr>
<tr>
<td>XX(3, 1)</td>
<td>value of $\bar{w}$ or $\bar{V}$ at the starting boundary</td>
</tr>
<tr>
<td>XY(1, 1)</td>
<td>value of $\bar{u}$ or $\bar{H}$ at the finishing boundary</td>
</tr>
<tr>
<td>XY(2, 1)</td>
<td>value of $\bar{p}$ or $\bar{M}_x$ at the finishing boundary</td>
</tr>
<tr>
<td>XY(3, 1)</td>
<td>value of $\bar{w}$ or $\bar{V}$ at the finishing boundary</td>
</tr>
<tr>
<td>IS1, IS2, IS3</td>
<td>indicators of boundary conditions at the starting boundary</td>
</tr>
<tr>
<td>IF1, IF2, IF3</td>
<td>indicators of boundary conditions at the finishing boundary</td>
</tr>
<tr>
<td>NP</td>
<td>Number of Pass; NP = 1 indicates linear solution</td>
</tr>
<tr>
<td>T22(N)</td>
<td>$\bar{N}_x = N_x/(R.P)$</td>
</tr>
<tr>
<td>T7(N)</td>
<td>$\bar{N}_z = N_z/(P.R)$</td>
</tr>
<tr>
<td>T9(N)</td>
<td>$\bar{M}<em>\theta = M</em>\theta/(P.R.h)$</td>
</tr>
<tr>
<td>Y(1, N)</td>
<td>$\bar{\varepsilon} = \varepsilon/\varepsilon_e$</td>
</tr>
<tr>
<td>Y(2, N)</td>
<td>$\bar{u} = uEh/(P.R^2)$</td>
</tr>
<tr>
<td>Y(3, N)</td>
<td>$\bar{p} = \beta$</td>
</tr>
<tr>
<td>Y(4, N)</td>
<td>$\bar{w} = wEh/(P.R^2)$</td>
</tr>
<tr>
<td>Y(5, N)</td>
<td>$\bar{V} = V/(P.R)$</td>
</tr>
<tr>
<td>Variable</td>
<td>Definition</td>
</tr>
<tr>
<td>----------</td>
<td>------------</td>
</tr>
<tr>
<td>Y(6, N)</td>
<td>$\bar{H} = H/(P.R)$</td>
</tr>
<tr>
<td>Y(7, N)</td>
<td>$\bar{M}_N = M_c/(P.R.h)$</td>
</tr>
<tr>
<td>ST1</td>
<td>$\bar{\sigma}<em>{ci} = \sigma</em>{ci}/E$, normalised circumferential stress at the inner surface of the shell</td>
</tr>
<tr>
<td>ST2</td>
<td>$\bar{\sigma}<em>{co} = \sigma</em>{co}/E$, normalised circumferential stress at the outer surface</td>
</tr>
<tr>
<td>ST3</td>
<td>$\bar{\sigma}<em>{ai} = \sigma</em>{ai}/E$, normalised axial stress at the inner surface</td>
</tr>
<tr>
<td>ST4</td>
<td>$\bar{\sigma}<em>{ao} = \sigma</em>{ao}/E$, normalised axial stress at the outer surface</td>
</tr>
</tbody>
</table>

* N denotes points in a segment at which the variables are evaluated.
APPENDIX B

PROGRAM LISTING

1     DIMENSION IG(32), APH(32), TK(32)
2     DIMENSION X7(33,7)
3     DIMENSION T22(21)
4     DIMENSION AY(3,1), BY(3,1), A18(3,3), AK(4)
5     DIMENSION TS1(3,3), TS2(3,3), TS3(3,3), TS4(3,3),
       TSF1(3,3), TSF2(3,3)
6     DIMENSION TS3(3,3), TS4(3,3), A15(3,1), A16(3,1),
       A17(3,1), A14(3,1)
7     DIMENSION C(33,3,3), E(33,3,3), A(33,3), B(33,3), X1(3,1)
8     DIMENSION C1(21), C2(21), T9(21), T10(21), T7(21),
       PH(21), RO(21), R(21)
9     DIMENSION Y3(33,3), Y2(33,3), Z1(3,1), Z2(3,1)
10    DIMENSION H(32), A1(3,3), A2(3,3), A3(3,3), A4(3,3),
       A6(3,3), A7(3,3)
11    DIMENSION Z(7,6), XX(3,1), XY(3,1), X(33,7), Y(7,21),
       Y1(7,21), F(7,21)
12    DIMENSION A8(3,3), A9(3,1), A10(3,1), A11(3,1), A12(3,1),
       U(6,6)
13    NP=0
14    IN=1
15    SOB2=0.
16    SS=1.
17    N2=6
18    N3=3
19    PB2=1.5707963268
20    C PART A. (READING IN INFORMATION)
21    READ 110, EM1, SOB1
22    25 READ 59, M, IZ
23    IF(IZ-5)515, 516, 516
24    516 READ 59, (IG(I), I=1,M)
25    READ 110, (APH(I), I=1,M)
26    READ 110, RC
27    515 READ 110, AN, EM, PHI, ALP, ER, XL
28    READ 110, (TK(I), I=1,M)
29    MO=M+1
30    READ 41, ((X(J, I), I=1,7), J=1, MO)
31    PRINT 41, ((X(J, I), I=1,7), J=1, MO)
32    READ 41, (XY(I, 1), I=1, 3)
33    READ 41, (XY(I, 1), I=1, 3)
34    READ 59, IS1, IS2, IS3
35    READ 59, IF1, IF2, IF3
36    C PART B. (TREATMENT OF BOUNDARY CONDITIONS)
37    DO 21 I=1,N3
38    DO 21 J=1,N3
39    TS1(I,J)=0.
38 TS2(I,J)=0.0
39 TS3(I,J)=0.0
40 TS4(I,J)=0.0
41 TF4(I,J)=0.0
42 TF3(I,J)=0.0
43 TF2(I,J)=0.0
44 21 TF1(I,J)=0.0
45 IF(IS1)23,23,24
46 23 TS1(1,1)=1.0
47 TS4(2,2)=1.0
48 GO TO 27
49 24 TS2(1,2)=1.0
50 TS3(2,1)=1.0
51 27 IF(IS2)28,28,29
52 28 TS1(2,2)=1.0
53 TS4(3,3)=1.0
54 GO TO 30
55 29 TS2(2,3)=1.0
56 TS3(3,2)=1.0
57 30 IF(IS3)33,33,34
58 33 TS1(3,3)=1.0
59 TS4(1,1)=1.0
60 GO TO 35
61 34 TS2(3,1)=1.0
62 TS3(1,3)=1.0
63 35 IF(IF1)36,36,37
64 36 TF2(1,2)=1.0
65 TF3(2,1)=1.0
66 GO TO 38
67 37 TF1(1,1)=1.0
68 TF4(2,2)=1.0
69 38 IF(IF2)39,39,40
70 39 TF2(2,3)=1.0
71 TF3(3,2)=1.0
72 GO TO 819
73 40 TF1(2,2)=1.0
74 TF4(3,3)=1.0
75 819 IF(IF3)84,84,87
76 84 TF2(3,1)=1.0
77 TF3(1,3)=1.0
78 GO TO 88
79 87 TF1(3,3)=1.0
80 TF4(1,1)=1.0
81 88 CONTINUE
82 DO 31 J=1,M
83 31 H(J)=(X(J+1,1)-X(J,1))*.05
C PART C. (CALCULATION OF RC. FOR PRESSURE VESSELS ONLY)
GO TO (401,402,403,404,405),IZ
401 RC=PHI/SIN(PHI)
GO TO 405
402 RC=1.
GO TO 405
403 RC=(1.-XL)/SIN(ALP)+(PB2-ALP)*XL/COS(ALP)
X(M,1)=(PB2-ALP)*XL/COS(ALP)/RC
GO TO 405
404 I=1
AL=1.
BL=2.
AKL=1.-ER**2
EL=1.
CL=1.
406 EL=EL*(AL/BL)**2
FL=EL*AKL**I/AL
CL=CL-FL
AL=AL+2.
BL=BL+2.
I=I+1
IF(ABS(FL)-.1E-08)407,401,406
407 RC=PB2*CL
405 CONTINUE
IF(IZ=5)521,522,522
521 DP=PB2
GO TO 523
522 DP=APH(1)
523 DR=1./RC
26 DO 1 J1=1,M
C PART D. CALCULATION OF CONSTANTS
113 T2=1.+AN
114 T1=RC*(1.-AN*AN)
115 T=TK(J1)
116 T21=EM*T
117 T0=1./(T21*EM*EM)
118 TL=RC/T/EM
119 TM=EM*EM*T
120 PR=EM*T
121 N=1
122 DO 32 I=1,7
123 32 Y(I,N)=X(J1,I)
124 DO 300 I=1,21
125 IF(I-21)312,313,313
126 312 Y(1,I+1)=Y(1,I)+H(J1)
127 313 IF(Y(1,I)-1.)306,308,305
128 308 IF(I-5)306,306,305
305 PH(I)=PB2
306 RO(I)=1./RC
309 GO TO 300
306 GO TO (301,302,303,304,509),IZ
301 PH(I)=Y(1,I)*PHI
304 RO(I)=SIN(PH(I))/PHI
305 GO TO 300
302 PH(I)=0.
307 RO(I)=Y(1,I)
306 GO TO 300
303 IF(Y(1,I)-X(M,I))307,309,309
309 PH(I)=PB2-ALP
302 RO(I)=Y(1,I)
306 GO TO 300
303 IF(Y(I,I)-X(M,I))307,309,309
307 PH(I)=Y(1,I)*RC/XL*COS(ALP)
304 RO(I)=SIN(PH(I))/RC/COS(ALP)
306 GO TO 300
304 PH(I)=DP
308 RO(I)=DR
309 ZZ=PH(I)
310 DO 310 J=1,4
311 FF=RC/ER**2*(ER**2+AKL*SIN(ZZ)*SIN(ZZ)**1.5
312 AK(J)=H(J1)*FF
314 V=1.
313 V=.5
316 GO TO 310,311,314,313,316
310 CONTINUE
315 DR=RO(I)+H(J1)*COS(APH(J1))
316 GO TO 300
309 IJK=IG(J1)
310 GO TO (510,511,304),IJK
510 PH(I)=APH(J1)
511 RM=I-1
314 RO(I)=DR
315 DR=RO(I)+H(J1)*COS(APH(J1))
316 GO TO 300
311 PH(I)=APH(J1)+RM*H(J1)*SIN(APH(J1))/DR
312 RO(I)=DR*SIN(PH(I))/SIN(APH(J1))
313 300 CONTINUE
314 DR=RO(21)
315 IF(IZ-5)512,513,513
316 512 DP=PH(21)
317 512 GO TO 514
513 DP=APH(J1+1)
514 N1=1

C INTEGRATION OF FUNDAMENTAL SET STARTS

60 NO=0
46 CONTINUE
111 IF(NP-1)111,111,112
112 IF(Y(1,N)-.1E-07)198,198,199
198 F(2,N)=T1*Y(6,N)/TZ
199 F(3,N)=Y(7,N)/TO/TZ
198 TO=TL+F(2,N)
198 F(5,N)=TQ*PR/2.
185 F(4,N)=0.
186 F(6,N)=0.
187 F(7,N)=0.
188 GO TO 200
199 T2=Y(2,N)/RO(N)
190 T3=PH(N)-Y(3,N)
191 C1(N)=COS(T3)
192 C2(N)=SIN(T3)
193 T4=(SIN(PH(N))-SIN(T3))/RO(N)
194 T5=Y(6,N)*C1(N)+Y(5,N)*C2(N)
195 T22(N)=T5
196 T8=T1*T5-AN*T2
197 T6=(Y(7,N)-AN*TO*T4)/T0
198 T7(N)=(T2+AN*T8)/T1
199 T9(N)=TO*(T4+AN*T6)
200 T10(N)=TL+T8
201 R(N)=TL*RO(N)+Y(2,N)
202 F(2,N)=T10(N)*C1(N)-COS(PH(N))*TL
203 F(3,N)=T6
204 F(4,N)=T10(N)*C2(N)-SIN(PH(N))*TL
205 F(5,N)=T10(N)*C2(N)+Y(5,N)*C1(N)/R(N)+PR*C1(N)
206 F(6,N)=T10(N)*((Y(6,N)*C1(N)-T7(N))/R(N)+PR*C2(N))
207 F(7,N)=(T10(N)*C1(N)/R(N))*T9(N)-Y(7,N)-T10(N)
1 *(Y(6,N)*C2(N)-Y(5,N)*C1(N))*TM
208 GO TO 200
111 C1(N)=COS(PH(N))
210 C2(N)=SIN(PH(N))
211 IF(Y(1,N)-.1E-07)598,598,599
212 598 F(2,N)=T1*Y(6,N)/TZ
213 F(3,N)=Y(7,N)/TO/TZ
214 F(4,N)=0.
215 F(5,N)=RC/2.
216 F(6,N)=0.
217 F(7,N)=0.
218 GO TO 200
219 599 T2=Y(2,N)/RO(N)
220 T4=Y(3,N)*C1(N)/RO(N)
221 \[ T5 = Y(6,N) \times C1(N) + Y(5,N) \times C2(N) \]
222 \[ T22(N) = T5 \]
223 \[ T8 = T1 \times T5 - A N \times T2 \]
224 \[ T6 = Y(7,N) / T0 - A N \times T4 \]
225 \[ T7(N) = (T4 + A N \times T6) \times T0 \]
226 \[ F(2,N) = T8 \times C1(N) + Y(3,N) \times C2(N) \times T0 \]
227 \[ F(3,N) = T6 \]
228 \[ F(4,N) = T8 \times C2(N) - Y(3,N) \times C1(N) \times T0 \]
229 \[ F(5,N) = -(Y(5,N) / R0(N) - R C) \times C1(N) \]
230 \[ F(6,N) = -(Y(6,N) \times C1(N) - T7(N)) / R0(N) - R C \times C2(N) \]
231 \[ T1 = -Y(7,N) / T0 \]
232 \[ T9(N) = (T4 + A N \times T6) \times T0 \]
233 \[ F(7,N) = T8 \times C1(N) - R C \times T8 \times C2(N) - Y(5,N) \times C1(N) \]
234 \[ 200 \text{ IF}(N-2)42,43,43 \]
235 \[ 43 \text{ IF}(N-6)44,47,45 \]
236 \[ 44 \ N=N+1 \]
237 \[ \text{ GO TO 46} \]
238 \[ 42 \text{ DO 81} \ J=2,6 \]
239 \[ \ P2=J-1 \]
240 \[ \ P3=P2 \times H(J1) \]
241 \[ Y(I,J) = Y(I,1) + P3 \]
242 \[ \text{ DO 81} \ I=2,7 \]
243 \[ 81 \ Y(I,J) = Y(I,1) + P3 \times F(I,1) \]
244 \[ N=2 \]
245 \[ \text{ IP}=1 \]
246 \[ \text{ GO TO 46} \]
247 \[ 47 \text{ DO 48} \ I=2,7 \]
248 \[ Z(1,2) = Y(I,1) + (H(J1) / 1440.) \times (493 \times F(I,1) + 1337.) \]
249 \[ 1 \times F(I,2) - 618 \times F(I,3) + 302 \times F(I,4) - 83 \times F(I,5) + 9 \times F(I,6) \]
250 \[ Z(1,3) = Y(I,1) + (H(J1) / 90.) \times (28 \times F(I,1) + 129 \times F(I,2) \]
251 \[ 1 \times 14 \times F(I,3) + 14 \times F(I,4) - 6 \times F(I,5) + F(I,6) \]
252 \[ Z(1,4) = Y(I,1) + (3 \times H(J1) / 160.) \times (17 \times F(I,1) + 73 \times F(I,2) \]
253 \[ 1 \times 38 \times F(I,3) + F(I,4) - 7 \times F(I,5) + F(I,6) \]
254 \[ Z(1,5) = Y(I,1) + (4 \times H(J1) / 90.) \times (7 \times F(I,1) + F(I,5) \)
255 \[ 1 \times 32 \times F(I,2) + F(I,4) + 12 \times F(I,3) \]
256 \[ Z(1,6) = Y(I,1) + (5 \times H(J1) / 288.) \times (19 \times F(I,1) + F(I,6) \)
257 \[ 1 \times 75 \times F(I,2) + F(I,5) + 50 \times F(I,4) + F(I,3) \]
258 \[ R1=0. \]
259 \[ \text{ IP}=\text{IP}+1 \]
260 \[ \text{ IF}(\text{IP}-15)141,45,45 \]
261 \[ 141 \text{ IF}(R1=.1E-07)45,45,50 \]
50 N=2
GO TO 46

45 IF(NO-1)53,53,55
53 N=N+1
IF(N-21)61,61,62
61 Y(I,N)=Y(I,N-1)+H(J1)
DO 51 I=2,7
51 Y(I,N)=Y(I,N-6)+(.3*H(J1))*(11.*F(I,N-5)+F(I,N-1))
1-14.*(F(I,N-4)+F(I,N-2))+26.*F(I,N-3))

99 NO=2
IP=1
GO TO 46

55 R1=0.
IP=IP+1
DO 56 I=2,7
56 Z(I,1)=Y(I,N-6)+(.3*H(J1))*(F(I,N-6)+5.*F(I,N-5)+F(I,N-4)+6.*
1 F(I,N-3)+F(I,N-2)+5.*F(I,N-1)+F(I,N))
R1=R1+ABS ( Y(I,N)-Z(I,1))
IF(IP-10)142,60,60
142 IF(R1-.1E-07)60,46,46
62 IF(NP-1)662,762,912
912 IF(AA-.10)911,911,914
914 IF(NP-10)662,911,911
911 IN=2
GO TO 764

762 RRR=0.
DO 763 I=2,7
763 RRR=RRR+ABS(Y(I,21)-X(J1+1,I))
IF(RRR-.1)764,764,766
766 PRINT 767
767 FORMAT(20H SEGMENT IS TOO LONG)
CONTINUE

PART E. OUTPUT OF RESULTS

PRINT 508
PRINT 507
DO 793 N=1,21,2
793 PRINT105,Y(1,N),Y(2,N),Y(4,N),T9(N),Y(7,N),T22(N),T7(N),
1ST1,ST2,ST3,ST4
GO TO 1
INTEGRATION OF DERIVED SET STARTS

662 N1=N1+1
N=1
Y1(1,N)=X(J1,1)
DO 63 I=2,7
63 Y1(I,N)=.0
Y1(N1,N)=1.
90 N0=0
76 CONTINUE
IF(NP-1)113,113,114
114 IF(Y1(1,N)-.1E-07)201,201,202
201 F(2,N)=T1*Y1(6,N)/TZ
F(3,N)=Y1(7,N)/TO/TZ
F(5,N)=F(2,N)*PR/2.
F(4,N)=0.
F(6,N)=0.
F(7,N)=0.
GO TO 203
202 T2=Y1(2,N)/RO(N)
T3=Y1(3,N)*C1(N)/RO(N)
T4=Y1(6,N)*C1(N)+Y1(5,N)*C2(N)-Y1(3,N)*(Y(5,N)*C(N)
1-Y(6,N)*C2(N))
T5=T1*T4-AN*T2
T6=Y1(7,N)/TO-AN*T3
Q1=(T2+AN*T5)/T1
T8=TO*(T3+AN*T6)
F(2,N)=T5*C1(N)+T10(N)*Y1(3,N)*C2(N)
F(4,N)=T5*C2(N)-T10(N)*Y1(3,N)*C1(N)
F(3,N)=T6
TA=(Y1(6,N)*C1(N)-T7(N))/R(N)
F(6,N)=T5*(TA+PR*C2(N))-T10(N)*((Y1(6,N)*C1(N)
1+Y1(3,N))*Y1(6,N)*C2(N)-Q1-TA*Y1(2,N))/R(N)-PR*Y1
(3,N)*C1(N))
F(5,N)=-F(2,N)*(Y(5,N)/R(N)-PR)-T10(N)*C1(N)*(Y1(5,N)
1-Y(5,N)*Y1(2,N)/R(N))/R(N)
TX=(T9(N)-Y(7,N))/R(N)
F(7,N)=F(2,N)*(TX+TM*Y(5,N))+T10(N)*(C1(N)*(TM*Y1(5,N)
1+Y1(7,N)+T8-TX*Y1(2,N))/R(N))-TM*C2(N)*Y1(6,N))
1-TM*F(4,N)*Y(6,N)
GO TO 203
113 IF(Y1(1,N)-.1E-07)501,501,502
501 F(2,N)=T1*Y1(6,N)/TZ
F(3,N)=Y1(7,N)/TO/TZ
F(4,N)=0.
F(5,N)=0.
339 \[ F(6,N) = 0. \]
340 \[ F(7,N) = 0. \]
341 \[ \text{GO TO 203} \]
342 \[ 502 \]
343 \[ T2 = Y1(2,N) / RO(N) \]
344 \[ T4 = Y1(3,N) * C1(N) / RO(N) \]
345 \[ T5 = Y1(6,N) * C1(N) + Y1(5,N) * C2(N) \]
346 \[ T8 = T1 * T5 - AN * T2 \]
347 \[ T6 = Y1(7,N) / T0 - AN * T4 \]
348 \[ T7(N) = (T2 + AN * T8) / T1 \]
349 \[ T9(N) = (T4 + AN * T6) * T0 \]
350 \[ F(2,N) = T8 * C1(N) + Y1(3,N) * C2(N) * T1 \]
351 \[ F(3,N) = T6 \]
352 \[ F(5,N) = -Y1(6,N) / RO(N) * C1(N) \]
353 \[ F(6,N) = -Y1(6,N) * C1(N) \]
354 \[ T8 = (T1(7,N) - T9(N)) / RO(N) \]
355 \[ F(7,N) = T8 * C1(N) - RC * T*(Y1(6,N) * C2(N) - Y1(5,N) * C1(N)) \]
356 \[ \text{GO TO 76} \]
357 \[ \text{GO TO 76} \]
358 \[ N = N + 1 \]
359 \[ \text{GO TO 76} \]
360 \[ \text{DO 82 J = 2, 6} \]
361 \[ P2 = J - 1 \]
362 \[ P3 = P2 * H(J1) \]
363 \[ Y1(1,J) = Y1(1,1) + P3 \]
364 \[ \text{DO 82 I = 2, 7} \]
365 \[ Y1(I, J) = Y1(I, 1) + P3 * F(I, 1) \]
366 \[ N = 2 \]
367 \[ IP = 1 \]
368 \[ \text{GO TO 76} \]
369 \[ \text{GO TO 76} \]
370 \[ \text{DO 82 I = 2, 7} \]
371 \[ Z(I, 2) = Y1(I, 1) + (H(J1) / 1440.) * (493 * F(I, 1) + 1337 * F(I, 2) \]
372 \[ 1 + 618 * F(I, 3) + 302 * F(I, 4) - 83 * F(I, 5) + 9. * F(I, 6)) \]
373 \[ Z(I, 3) = Y1(I, 1) + (H(J1) / 90.) * (28 * F(I, 1) + 129 * F(I, 2) + 14. \]
374 \[ 1 * F(I, 3) + 14. * F(I, 4) - 6. * F(I, 5) + F(I, 6)) \]
375 \[ Z(I, 4) = Y1(I, 1) + (3. * H(J1) / 160.) * (17. * F(I, 1) + 73. * F(I, 2) \]
376 \[ 1 + 38. * (F(I, 3) + F(I, 4)) - 7. * F(I, 5) + F(I, 6)) \]
377 \[ Z(I, 5) = Y1(I, 1) + (4. * H(J1) / 90.) * (7. * F(I, 1) + F(I, 5)) + 32. \]
378 \[ 1 * (F(I, 2) + F(I, 4)) + 12. * F(I, 3)) \]
379 \[ R1 = 0. \]
380 \[ IP = IP + 1 \]
381 \[ \text{DO 82 I = 2, 7} \]
382 \[ \text{DO 79 J = 2, 7} \]
C

SOLUTION OF MATRIX EQUATIONS STARTS

N1=J1

DO 4 I=1,N3

DO 4 J=1,N3
\[ A_1(J,I) = U(I,J) \]
\[ A_2(J,I) = U(I+3,J) \]
\[ A_3(J,I) = U(I,J+3) \]
\[ A_4(J,I) = U(I+3,J+3) \]
\[ X_1(I,1) = X(N_1,I+1) \]
\[ X_2(I,1) = X(N_1,I+4) \]
\[ Y_3(N_1+1,I) = Y(I+1,21) \]
\[ Y_2(N_1+1,I) = Y(I+4,21) \]
\[ \text{DO 20 I=1,N3} \]
\[ A_1(I,1) = Y_3(N_1+1,I) \]
\[ B_1(I,1) = Y_2(N_1+1,I) \]
\[ \text{CALL MATM}(A_1, X_1, A_9, N_3, N_3, 1) \]
\[ \text{CALL MATM}(A_2, X_2, Z_1, N_3, N_3, 1) \]
\[ \text{CALL MATS}(A_9, Z_1, N_3, 1) \]
\[ \text{CALL MATSB}(Z_1, N_3, 1) \]
\[ \text{CALL MATM}(A_3, X_1, A_9, N_3, N_3, 1) \]
\[ \text{CALL MATM}(A_4, X_2, Z_2, N_3, N_3, 1) \]
\[ \text{CALL MATS}(A_9, Z_2, N_3, 1) \]
\[ \text{CALL MATSB}(Z_2, N_3, 1) \]
\[ \text{CALL MATS}(B_1, Z_2, N_3, 1) \]
\[ \text{IF}(N_1-I)6,6,7 \]
\[ \text{CALL MATM}(A_1, TS_1, A_6, N_3, N_3, 1) \]
\[ \text{CALL MATM}(A_1, TS_2, A_7, N_3, N_3, 1) \]
\[ \text{CALL MATS}(A_6, A_1, N_3, 1) \]
\[ \text{CALL MATM}(A_2, TS_3, A_1, N_3, N_3) \]
\[ \text{CALL MATS}(A_6, A_1, N_3) \]
\[ \text{CALL MATM}(A_2, TS_4, A_6, N_3, N_3, 3) \]
\[ \text{CALL MATS}(A_6, A_7, N_3, 1) \]
\[ \text{CALL MATM}(A_3, TS_1, A_6, N_3, N_3, 1) \]
\[ \text{CALL MATM}(A_3, TS_2, A_8, N_3, N_3, 3) \]
\[ \text{CALL MATM}(A_3, TS_3, A_3, N_3, N_3, 3) \]
\[ \text{CALL MATS}(A_6, A_3, N_3, 3) \]
\[ \text{CALL MATM}(A_4, TS_4, A_6, N_3, N_3) \]
\[ \text{CALL MATS}(A_6, A_8, N_3, 3) \]
\[ \text{CALL MATM}(A_4, TS_4, A_6, N_3, N_3) \]
\[ \text{CALL MATS}(A_6, A_8, N_3, N_3) \]
\[ \text{DO 2 I=1,N3} \]
\[ \text{DO 2 J=1,N3} \]
\[ A_4(I,J) = A_8(I,J) \]
\[ A_1(I,J) = A_7(I,J) \]
\[ \text{CALL MATI}(A_2, A_6, N_3) \]
\[ \text{CALL MATM}(A_4, A_6, A_7, N_3, N_3, 1) \]
\[ \text{CALL MATI}(A_7, A_8, N_3) \]
\[ \text{CALL MATM}(A_1, XX, A_9, N_3, N_3, 1) \]
\[ \text{CALL MATS}(Z_1, A_9, N_3, 1) \]
\[ \text{CALL MATSB}(A_9, N_3, 1) \]
\[ \text{CALL MATM}(A_3, XX, A_{10}, N_3, N_3, 1) \]
CALL MATS(Z2,A10,N3,1)
CALL MATM(A4,A6,A7,N3,N3,N3)
CALL MATM(A7,A9,A11,N3,N3,1)
CALL MATS(A11,A10,N3,1)
CALL MATSB(A10,N3,1)
GO TO 8
7 IF(N1-M)3,5,5
5 CALL MATM(TF1,A1,A6,N3,N3,N3)
CALL MATM(TF4,A3,A6,N3,N3,N3)
CALL MATS(A6,A1,N3,N3)
CALL MATM(TF4,A3,A6,N3,N3,N3)
CALL MATS(A6,A7,N3,N3)
CALL MATM(TF1,A2,A6,N3,N3,1)
CALL MATM(TF3,A2,A18,N3,N3,1)
CALL MATM(TF2,A4,A2,N3,N3,N3)
CALL MATS(A6,A2,N3,N3)
CALL MATM(TF4,A4,A6,N3,N3,N3)
CALL MATS(A6,A18,N3,N3)
CALL MATM(TF1,Z1,A14,N3,N3,1)
CALL MATM(TF3,Z1,A15,N3,N3,1)
CALL MATM(TF2,Z2,Z1,N3,N3,1)
CALL MATS(A14,Z1,N3,1)
CALL MATM(TF4,Z2,A14,N3,N3,1)
CALL MATS(A14,A15,N3,1)
DO 19 I=1,N3
Z2(I,1)=A15(I,1)
DO 19 J=1,N3
A3(I,J)=A7(I,J)
3 CALL MATM(A1,A9,A7,N3,N3,N3)
CALL MATS(A2,A7,N3,N3)
CALL MATM(A7,A6,N3)
CALL MATM(A1,A8,A7,N3,N3,N3)
CALL MATM(A7,A10,A9,N3,N3,1)
CALL MATS(Z1,A9,N3,1)
CALL MATSB(A9,N3,1)
CALL MATM(A3,A8,A7,N3,N3,N3)
CALL MATM(A7,A10,A11,N3,N3,N3)
CALL MATS(A4,A7,N3,N3)
CALL MATM(A6,A9,A12,N3,N3,1)
CALL MATM(A7,A12,A10,N3,N3,1)
CALL MATS(A11,A10,N3,1)
CALL MATS(Z2,A10,N3,1)
CALL MATSB(A10,N3,1)
CALL MATM(A3,A8,A7,N3,N3,N3)
CALL MATS(A4,A7,N3,N3)
508 CALL MATM(A7,A6,A1,N3,N3,N3)
509 CALL MATM(A1,A8,N3)
510 IF(N1-M)8,9,9
511 9 CALL MATS(XY,A10,N3,1)
512 8 DO 1 I=1,N3
513 1 DO 1 J=1,N3
514 E(N1,I,J)=A6(I,J)
515 C(N1,I,J)=A8(I,J)
516 A(N1,I)=A9(I,1)
517 B(N1,I)=A10(I,1)
518 1 CONTINUE
519 IF(NP-1)117,115,117
520 117 GO TO (718,108),IN
521 718 AA=0.
522 15 DO 11 I1=1,M
523 N1=M-I1+1.
524 10 DO 10 I=1,N3
525 9 DO 10 J=1,N3
526 A6(I,J)=E(N1,I,J)
527 A8(I,J)=C(N1,I,J)
528 A9(I,1)=A(N1,1)
529 10 A10(I,1)=B(N1,1)
530 IF(N1-M)11,12,12
531 12 CALL MATM(A8,A10,A11,N3,N3,1)
532 CALL MATS(A11,A9,N3,1)
533 CALL MATM(A6,A9,A12,N3,N3,1)
534 CALL MATM(TF1,A11,A14,N3,N3,1)
535 CALL MATM(TF2,XY,A15,N3,N3,1)
536 CALL MATM(TF3,A11,A16,N3,N3,1)
537 CALL MATM(TF4,XY,A17,N3,N3,1)
538 89 DO 89 I=1,N3
539 X(MO,I+1)=A15(I,1)+A14(I,1)
540 89 X(MO ,I+4)=A17(I,1)+A16(I,1)
541 GO TO 16
542 11 CALL MATS(A12,A10,N3,1)
543 CALL MATM(A8,A10,A11,N3,N3,1)
544 CALL MATS(A11,A9,N3,1)
545 CALL MATM(A6,A9,A12,N3,N3,1)
546 17 DO 17 I=1,N3
547 17 X(N1+I,1)=A11(I,1)
548 IF(N1-1)93,93,16
549 93 CALL MATM(TS1,XX,A14,N3,N3,1)
550 CALL MATM(TS2,A12,A15,N3,N3,1)
CALL MATM(TS3,XX,A16,N3,N3,1)
CALL MATM(TS4,A12,A17,N3,N3,1)
DO 98 I=1,N3
  X(1,I+1)=A15(I,1)+A14(I,1)
  X(1,I+4)=A17(I,1)+A16(I,1)
GO TO 18
16 DO 13 I=1,N3
  X(N1,I+4)=A12(I,1)
18 DO 13 I=1,N3
  AA=ABS(Y3(N1+1,I)-X(N1+1,I+1)+AA
  AA=ABS(Y2(N1+1,I)-X(N1+1,I+4))+AA
115 NP=NP+1
RES=AA/SS
SS=AA
PRINT 505,NP,AA,EM
IF(NP-5)151,152,152
152 IF(RES-1.)151,151,153
153 DO 154 I=2,7
154 X(J,I)=X7(J,I)
   EM=EM-EM1
   EM1=EM1/2.
   NP=3
151 PRINT 104,((X(J,I),I=1,7),J=1,MO)
GO TO 405
108 DO 155 I=2,7
157 DO 155 J=1,MO
155 X7(J,I)=X(J,I)
   IN=1
   NP=3
   AA=1.
   SOB2=SOB2+1.
   EM=EM+EM1
   IF(SOB2-SOB1)405,405,109
109 CALL EXIT
END
SUBROUTINE MATS (A5, B5, L, K)
DIMENSION A5(3,3), B5(3,3)
DO 99 L1 = 1, L
   DO 99 K1 = 1, K
   99 B5(L1, K1) = A5(L1, K1) + B5(L1, K1)
RETURN
END
SUBROUTINE MATSB (A5,L,K)
DIMENSION A5(3,3)
DO 98 L1=1,L
   DO 98 K1=1,K
      A5(L1,K1)=-A5(L1,K1)
98 RETURN
END
SUBROUTINE MATM (A5, B5, C5, L, K, K2)
DIMENSION A5(3,3), B5(3,3), C5(3,3)
DO 97 L1 = 1, L
  DO 97 K1 = 1, K2
    C5(L1, K1) = 0.
  DO 97 J1 = 1, K
    C5(L1, K1) = C5(L1, K1) + A5(L1, J1) * B5(J1, K1)
97 RETURN
END
SUBROUTINE MATI(A5,B5,K1)
DIMENSION A5(3,3),B5(3,3)
P=0.0
DO 9 L=1,3
  DO 9 K=1,3
    GO TO (2,3,4),L
  2 I1=L+1
  I2=L+2
  GO TO 5
  3 I1=L+1
  I2=1
  GO TO 5
  4 I1=1
  I2=2
  GO TO (6,7,8),K
  5 GO TO (6,7,8),K
  6 J1=K+1
  J2=K+2
  GO TO 9
  7 J1=K+1
  J2=1
  GO TO 9
  8 J1=1
  J2=2
  9 B5(K,L)=A5(I1,J1)*A5(I2,J2)-A5(I2,J1)*A5(I1,J2)
DO 11 L=1,3
  11 P=P+A5(1,L)*B5(L,1)
DO 12 L=1,3
  12 B5(L,K)=B5(L,K)/P
RETURN
END
CURRICULUM VITAE

Md. Wahhaj Uddin was born in Jessore, East Pakistan on April 1, 1939. He attended Jhenidah High English School, East Pakistan and passed the Matriculation Examination of the East Pakistan Board of Education in 1954. He then entered Rajshalin College and passed the Intermediate Science Examination of Rajshahi University in 1956. He received a Bachelor of Science degree with Honours in Mathematics from the University of Rajshahi in 1958.

He then entered East Pakistan University of Engineering and Technology and for four years was the recipient of a Merit Scholarship. He received a Bachelor of Science degree in Mechanical Engineering, first class, in November 1962.

In November, 1962, he joined East Pakistan University of Engineering and Technology as a Lecturer in Mechanical Engineering and served there until September 1964.

In September, 1964, he received a Colombo Plan Scholarship awarded by the Government of Canada and entered Carleton University, Ottawa, as a graduate student. He obtained his Master of Engineering degree in May, 1966.