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A Study on Graceful Labeling of Trees

by

Jannatul Maowa

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Board of Examiners

1. ______________________________

Dr. M. Kaykobad — Chairman

Professor — (Supervisor)

Department of Computer Science and Engineering, BUET, Dhaka.

2. ______________________________

Dr. Md. Sohel Rahman — Member

Professor and Head — (Ex-Officio)

Department of Computer Science and Engineering, BUET, Dhaka

3. ______________________________

Dr. Md. Saidur Rahman — Member

Professor

Department of Computer Science and Engineering, BUET, Dhaka

4. ______________________________

Dr. Rifat Shahriyar — Member

Assistant Professor

Department of Computer Science and Engineering, BUET, Dhaka

5. ______________________________

Dr. Mohammad Nurul Huda — Member

Professor — (External)

Department of Computer Science and Engineering, United International University, Dhaka, Bangladesh.

DEDICATED TO PROPHET HAZRAT MUHAMMAD (SWM.)

Candidate's Declaration

This is hereby declared that the work titled "A Study on Graceful Labeling of Trees" is the outcome of research carried out by me under the supervision of Dr. M. Kaykobad, in the Department of Computer Science and Engineering, Bangladesh University of Engineering and Technology, Dhaka 1000. It is also declared that this thesis or any part of it has not been submitted elsewhere for the award of any degree or diploma.

Jannatul Maowa

Candidate

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Abstract

A tree is a connected acyclic graph on n vertices and $n-1$ edges. Graceful labeling of a tree is a labeling of its vertices with the numbers from 0 to $n-1$, so that no two vertices share a label, labels of edges, being absolute difference of the labels of its end points, are also distinct. There is a famous conjecture named Graceful tree conjecture or Ringel-Kotzig Conjecture that says “All trees are graceful”. Almost 50-year old conjecture is yet to be proved. However, researchers have been able to prove that many classes of trees are graceful. In this thesis, we have proved that the classes of Superstar and Extended Superstarare graceful. A tree with one internal node and k leaves is said to be a star $S_{1,k}$ or a complete bipartite graph $K_{1,k}$. Superstar is a tree that consists of several stars all connected to a single star by sharing their leaves. If we remove all the leaves of a Superstar then we will get a Spider tree which has already been proved to be graceful. Extended Superstar is a tree that consists of several Superstars all connected to a single star by sharing their leaves. We have also proved that extended superstars are graceful.

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Chapter 1

Introduction

Graph labeling is the assignment of labels, where the vertices or edges or both are assigned real values subject to certain conditions, have often been motivated by their use in various applied fields and their intrinsic mathematical interest. Most graph labeling methods trace their origin to one introduced by Rosa [76] in 1967 or given by Graham and Sloane [33] in 1980. Graph labeling was first introduced in the mid 1960s. In the intervening 50 years, nearly 200 graph labeling techniques have been studied in over 2000 papers and is still getting embellished due to increasing number of application driven concepts.

Graph labeling, where the vertices are assigned values subject to certain conditions, have often been motivated by practical problems. Labeled graphs serve as useful mathematical models for a broad range of applications such as Coding theory, including the design of good radar type codes, sync-set codes, missile guidance codes and convolution codes with optimal auto correlation properties. They facilitate the optimal nonstandard encoding of integers.

In graph theory, a tree is an undirected graph in which any two vertices are connected by exactly one path. In other words, any acyclic connected graph is a tree. Given a tree $T(V,E)$ with $|V|=n$ and $|E|=n-1$, if we can label any vertex $u \in V$ by $f(u)$ using integers $\{0,1,\cdots,n-1\}$ so that an edge (u,v) is labelled by $|f(u)-f(v)|$ in such a way that every edge is labelled by a distinct integer from the set $\{1,2,\ldots,n-1\}$, then the tree is said to be gracefully labelled. The Ringel-Kotzig conjecture that all trees are graceful has been the focus of many papers [2, 3, 4, 14, 17, 19, 22, 30, 41, 43, 50, 66, 74, 78, 96]. Many classes of trees have been shown to be graceful [74]. However, it has not yet been possible to prove the conjecture for all trees. A lot of work have been done by many researchers towards proving this conjecture. So far some special classes of trees have been shown to be graceful. For example, paths, caterpillars, symmetrical trees, spider trees, lobster trees, star trees, firecrackers, banana trees etc. Trees of diameter at most five and trees with up to 35 vertices have also been shown graceful. Our aim is to discover new classes of graceful trees.

1.1 Background Study

There are many graph labeling techniques like Graceful Labeling, Harmonious Labeling, Magic-type Labeling, Antimagic-type Labeling, Prime and Vertex Prime Labelings, Edge-graceful Labelings, Radio Labelings, Line-graceful Labelings, k-sequential Labelings, Product and Divisor Cordial, Edge Product Cordial, Difference Cordial Labelings, Prime Cordial labelings, Geometric labelings, Mean Labelings, Irregular Total Labelings, Square Sum Labelings and Square Difference Labeling and so on. However, we shall concentrate on graceful labeling that has received perhaps attention of a wider scientific community. The name "Graceful Labeling" has come up thanks to Solomon W. Golomb. In 1967 paper on graph labeling [76], Alexander Rosa originally gave the name β-labelings to this class of labeling.

A graceful labeling of a graph with m edges is a labeling of its vertices with the numbers from 0 to m, so that no two vertices share a label, and so that each edge is uniquely identified by the positive, or absolute difference between its endpoints. A graph which admits a graceful labeling is called a graceful graph. A tree with n vertices and m edges is called graceful if there exists a labeling of its vertices with the numbers from 0 to m such that the set of absolute values of the differences of the numbers assigned to (vertices) the ends of each edge is the set $\{1, 2, \ldots, m\}$.

In graph theory, a major unproven conjecture is the Graceful Tree conjecture (GTC) or Ringel–Kotzig conjecture, named after Gerhard Ringel and Anton Kotzig, which hypothesizes that "all trees are graceful". The Ringel-Kotzig conjecture is also known as the "Graceful Labeling Conjecture". To prove the conjecture Kotzig once called the effort a "disease" [43].

Here we describe how the graceful tree problem first came up. We follow the description of [26].

A decomposition of a graph G is a collection $\{H_i\}$ of nonempty subgraphs such that $H_i = \langle E_i \rangle$ for some nonempty subset E_i of $E(G)$, where $\{E_i\}$ is a partition of $E(G)$. If $\{H_i\}$ is a decomposition of a graph G such that, for each i, $H_i = H$ for some graph H, then G is said to be H-decomposable. If G is an H-decomposable graph, then we write $H|G$ and say that H decomposes G.

A cyclic decomposition is a decomposition of a graph G into k copies of a subgraph H that can be obtained in the following manner:

- draw G appropriately
- select a subgraph H1 of G that is isomorphic to H
- rotate the vertices and edges of H_1 through an appropriate angle $k-1$ times to produce k copies of H in the decomposition.

If $H\ |G$ then the size of H necessarily divides the size of G, and H is necessarily a subgraph of G. However, the fact that the size of H divides the size of G is not a sufficient condition for $H\ |G$. For example, $4\ |12$ but $H - G$ where $G = K_{2,2,2}$ and $H = K_{1,4}$. It is easy to see that every nonempty graph is K_2-decomposable.

The following theorem of Kirkman characterizes when such a decomposition is attainable:

Theorem 1. *K_n is k_3-decomposable if and only if n is odd and* $3 \mid \binom{n}{2}$

We have an example of a balanced incomplete block design whenever K_n is K_k decomposable for natural numbers $k \geq 3$. Graph decompositions may be viewed as generalized block design.

In 1963, when Ringel posed the following problem, since then this problem is known as Ringel's Conjecture [76].

Conjecture 2. Every tree with $m+1$ vertices decomposes K_{2m+1}.

Till date this problem is still unsolved. In [76], according to Rosa Kotzig conjectured a stronger statement than Ringel's, Kotzig's Conjecture:

Conjecture 3. Every tree with $m+1$ vertices cyclically decomposes K_{2m+1}.

Intention of Rosa behind publishing the paper [76] was for providing insight into Ringel's Conjecture for efficiently addressing the issue. The idea was to use a labeling of the vertices of a graph H of order m to show that it can cyclically decompose K_{2m+1}. Rosa referred to a labeling as a valuation of the graph. Consider the following conditions where O_G be a labeling of the vertices of G, V_{O_G} be the numbers assigned to the vertices of G and E_{O_G} be the set of numbers assigned to the edges of G.

1. $V_{O_G} \subseteq \{1,2,\ldots,n\}$,
2. $V_{O_G} \subseteq \{1,2,\ldots,2n\}$,
3. $E_{O_G} \subseteq \{1,2,\ldots,n-1\}$,
4. $E_{O_G} \subseteq \{x_1,x_2,\ldots,x_n\}$, where $x_i = i$ or $x_i = 2n+1-i$,
5. There exists $x \in \{1,2,\ldots,n\}$, such that for an arbitrary edge v_iv_j of the graph either $a_i \leq x < a_j$ or $a_j \leq x < a_i$, where a_k is the label assigned to v_k for each k.

From above conditions Rosa defines four types of labelings which are given below :

- A ρ-valuation satisfies conditions (2) and (4).
- A σ-valuation satisfies conditions (2) and (3).
- A β-valuation satisfies conditions (1) and (3).
- A α-valuation satisfies conditions (1), (3) , and (5).

Graceful labeling is the other name of Rosa's β-valuation. A lot of work have been done by many researchers to prove the Graceful Tree conjecture (GTC). But The problem still remains open. There are two main approaches in the existing literature to proving the graceful tree conjecture. Showing that all trees having a particular structure are graceful is the more mathematical approach and showing that all trees with up to a certain number of vertices are graceful is more computational approach. The most recent result using the second approach is that all trees with up to 35 vertices are graceful which was claimed by Wenjie Fang [29] and trees of diameter at most five have also been shown graceful. All path graphs, caterpillar graphs, lobster graphs with a perfect matching are graceful which is the result of first approach.

So far research on settling the conjecture has expanded in three directions. The first is to discover new classes of trees that are graceful. This is how caterpillars, fire crackers, banana trees, symmetrical trees have been proved to be graceful. The second direction is establish that all classes of trees of up to certain sizes have been shown graceful by enumerating all non-isomorphic trees and then finding graceful labeling by extensive computation [17]. Yet the third direction is to apply integer programming formulation of the problem and then solve it. We are following the first direction towards proving the Graceful tree conjecture.

1.2 Aims and Objectives of the Thesis

After going through a number of research paper related to graceful labeling of trees we have set our objective to identifying yet another class of trees that are amenable to graceful labeling. With this objective we have divided the whole research into

(i) Study the results so far obtained in this area

(ii) Utilize the knowledge to expand the classes of graceful trees

(iii) Develop algorithms for graceful labeling

(iv) Discover more general class of gracefully labeled trees

(v) Develop a generalized algorithm for graceful labeling of new class of tree

1.3 Organization of the Thesis

In Chapter 1 we introduce the problem of graceful labeling. Chapter 2 contains a list of graph labeling and their applications. Chapter 3 discusses different classes of trees that have been proved graceful. Chapter 4 introduces the new classes of trees that we have proved graceful. Experimental results and discussions about parameter are presented in Chapter 5.Chapter 6 discusses our findings and possibility of extending the results. The thesis ends with a bibliography on the topic.

Chapter 2

Graph Labeling and its Application

A graph labeling is the assignment of labels, traditionally represented by non negative integers to the vertices or edges, or both, subject to certain conditions. If the domain is the set of vertices we speak about the vertex labeling. If the domain is the set of edges, then the labeling is called the edge labeling. If the labels are assigned to the vertices and also to the edges of a graph, such a labeling is called total. Moreover, if we consider the plane graphs, it is also possible to label the faces of these graphs. Graph labellings were first introduced by Alex Rosa in the mid 1960s. In the last 50 years nearly 200 graph labelling techniques have been studied in over 2000 papers. Keeping up with new discoveries and finding out what has been done for any particular kind of labeling is difficult because of the numerous number of papers and many of the papers have appeared in journals that are not widely available. Therefore, in this chapter we have tried to give a brief idea on recent results on graph labeling. Here we have avoided the technical details of any labeling of graphs rather we included definitions and examples and listed a detailed table of contents and index(as in [30]).

2.1 Graceful Labeling

In this section we follow the definition as in [5]. A vertex labeling f of a graph G is called graceful if f is an injective mapping from the set of vertices to the set of integers

$\{0,1,\dots|E(G)|\}$ such that the induced mapping

$$f(xy) \quad = \quad |f(x)-f(y)|, \quad \text{for every } xy \in E(G),$$

assigns different labels to different edges of G. The difference $|f(x)-f(y)|$ is called the weight of the edge xy. A graph G is called graceful, if G admits a graceful labeling.

Graceful labeling was first introduced by Rosa in 1967 [76]. However, Rosa called this labeling β-valuation. After Several years Golomb [32] studied the same type of labeling and called this labeling graceful labeling. Graceful labelings were introduced to attack Ringel's conjecture [36], i.e. that the complete graph K_{2n+1} is decomposable into $2n+1$ sub-graphs that are all isomorphic to a given tree of size n.

An example of graceful labeling of K4 is illustrated in the figure 2.1.

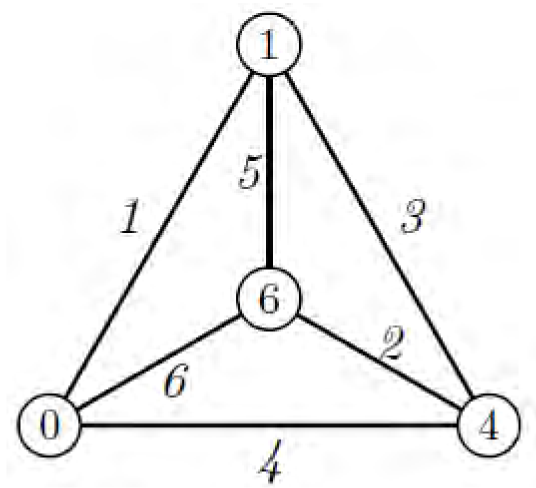

Figure 2.1: Graceful labeling of K_4.

The Ringel-Kotzig conjecture that all trees are graceful is a very popular open problem. Some methods for constructing the graceful labelings and α-labelings for certain families of trees can be found in [4], [22] and [26]. An example of graceful labeling of a Caterpillar is illustrated in the figure 3.2 in the page 25.

2.2 Harmonious Labeling

Harmonious labeling is a another kind of vertex labeling. It was first introduced by Graham and Sloane in 1980 [33] to study additive bases. A vertex labeling f of a graph G is called

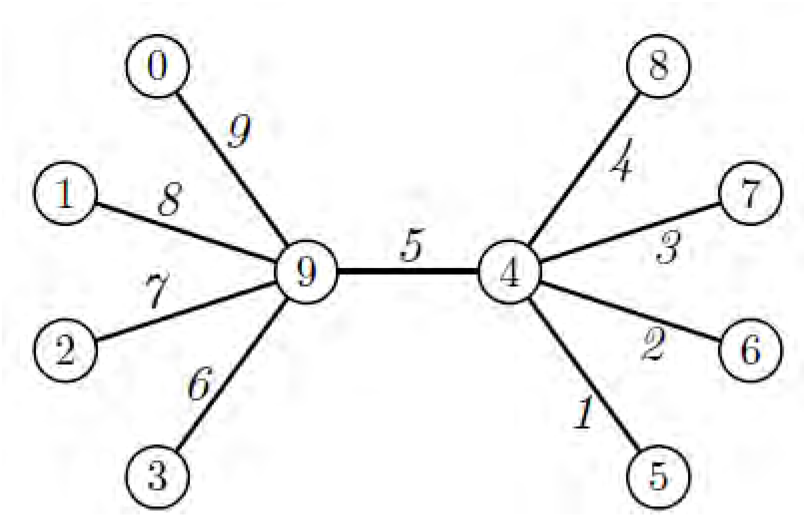

Figure 2.2: Graceful labeling of Caterpillar.

harmonious, if f is an injective mapping from the vertex set $V(G)$ to the additive group $(Z,+)$, such that the mapping f' from the edge set $E(G)$ to $(Z,+)$ defined by

$$f'(uv) = f(u) + f(v), \quad \text{for every } uv \in E(G),$$

assigns different labels to the edges of G. If the graph G admits a harmonious labeling, then it is said to be harmonious.

In other words, a connected labeled graph with n number of edges in which all vertices can be labeled with distinct integers(mod n) so that the sums of the pairs of numbers at the ends of each edge are also distinct (mod n). The ladder graph, fan, wheel graph,Petersen graph, tetrahedral graph, dodecahedral graph, and icosahedral graph are all harmonious (Graham and Sloane 1980) [33].

There are some properties of harmonious labeling [92] which are summarised below,

1. Harmonious labeling is not unique.
2. If f is a Harmonious labeling of any graph G with q edges, then $af(x)+b$ is also harmonious labeling of G, where a is invertible element of Z and b is any arbitrary element of q (Set of integers modulo q).
3. Any vertex in a harmonious graph can be assigned the label 0.
4. In the case of trees exactly two vertices are assigned the same vertex label which have been observed by Graham and Sloane [33].

5. Graham and Sloane [33] conjectured that every tree is harmonious.

6. Aldred and Mckay [3] provided an algorithm and used computer to show that all trees with at most 26 vertices are harmonious.

7. Golomb [32] proved that complete graph is harmonious if and only if $n \leq 4$.

8. Graham and Sloane [33] proved that $K_{m,n}$ is harmonious if and only if m or $n = 1$.

9. Every graph with less than or equal to 5 vertices is harmonious excepting the six graphs shown in the figure 2.3.

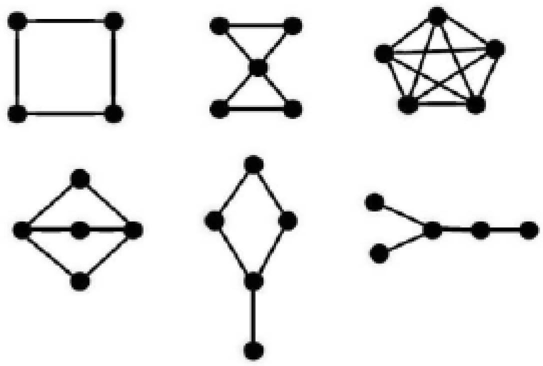

Figure 2.3: Example of graphs ($n \leq 5$) which are not harmonious.

10. The Peterson graph is Harmonious.

11. Graham and Sloane [33], proved that wheel $W_n = C_n + K_1$ is harmonious.

12. $K_n^{(2)}$ is harmonious if $n = 4$ but not harmonious if n is odd or $n = 6$.

The example of harmonious labeling of cycle C_5 and fan graph F_8 are shown in figures 2.4 and 2.5 in pages 10 and 10 respectively.

Although it is proved that almost no graphs are either graceful (Erdos' unpublished result) nor harmonious (Graham and Sloane's original paper) many authors are still dealing with these labelings [30]. Motivated by these two types of labelings, they have defined a large number of different vertex labelings that Gallian [30] divides into two main groups named variations of graceful labelings and variations of harmonious labelings.

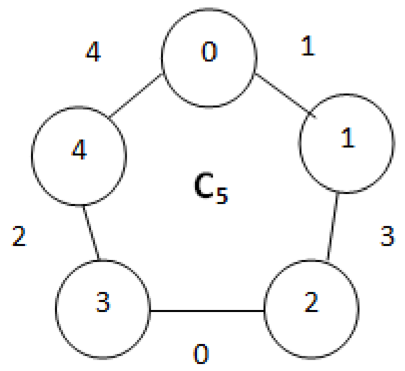

Figure 2.4: Harmonious labeling of C_5.

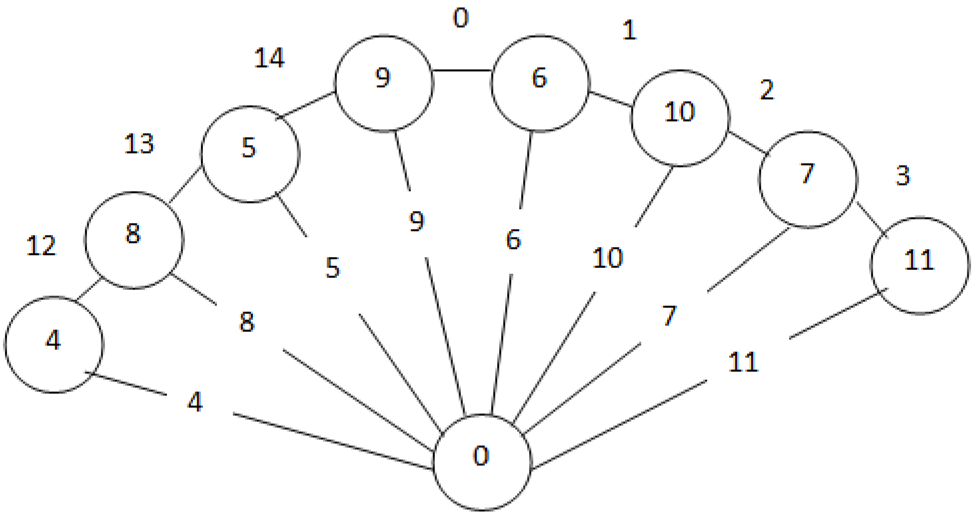

Figure 2.5: Harmonious labeling of F_8.

2.3 Magic Type Labeling

In 1963, motivated by the notion of magic squares in number theory, magic labelings were introduced by Sedl*áč*ek [79] . Stewart [89] and [90] studied various ways to label the edges of a graph in the mid 1960s to solve the problems raised by Sedl*áč*ek. Therefore, Stewart calls a connected graph semi-magic if there is a labeling of the edges with integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. A semi-magic labeling where the edges are labeled with distinct positive integers is called a magic labeling.

An example of magic labeling of K_5 is shown in the figure 2.6 in page 11.

Stewart calls a magic labeling supermagic if the set of edge labels consists of consecutive

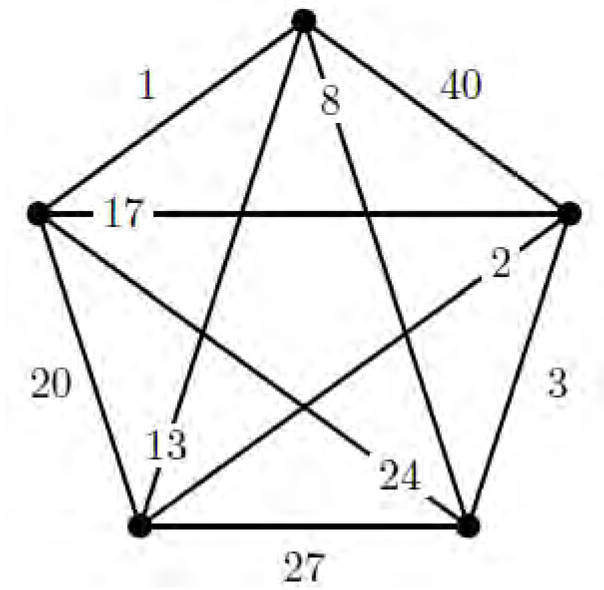

Figure 2.6: Magic labeling of K_5.

positive integers. A graph G is called a supermagic graph if it admits a supermagic labeling. An example of supermagic labeling of $K_{3,3}$ is shown in figure 2.7 in page 11.

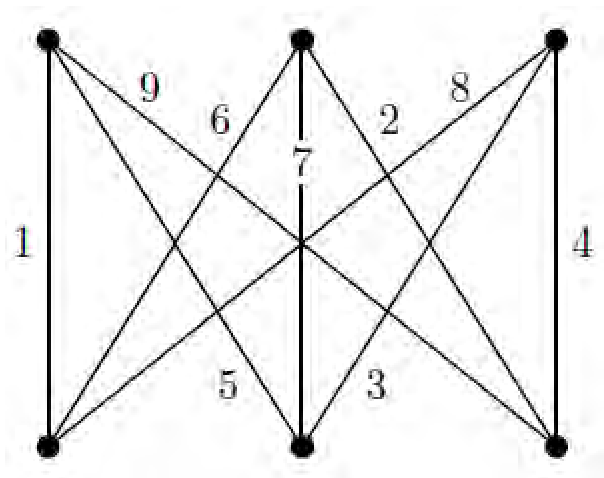

Figure 2.7: Supermagic labeling of $K_{3,3}$.

A bijection $f : V(G) \cup E(G) \to \{1,2,\ldots|V(G)|+|E(G)|\}$ is called a vertexmagic total labeling if there is a constant h, such that vertex-weight of every vertex in G is equal to this constant, i.e.

$$wt(x) = f(x) + \sum_{y \in N(x)} f(xy) = h, \quad \text{for every } x \in V(G)$$

The constant h is called the magic constant for the labeling f. A graph G is called vertex-magic total, if it admits a vertex-magic total labeling often abbreviated VMT. The concept of vertex-magic total labeling was given by MacDougall, Miller, Slamin and Wallis [45]. The Figure 2.8 illustrates vertex-magic total graphs with their vertex magic total labelings and Figure 2.9 is an example of super vertex-magic total labeling of K_5.

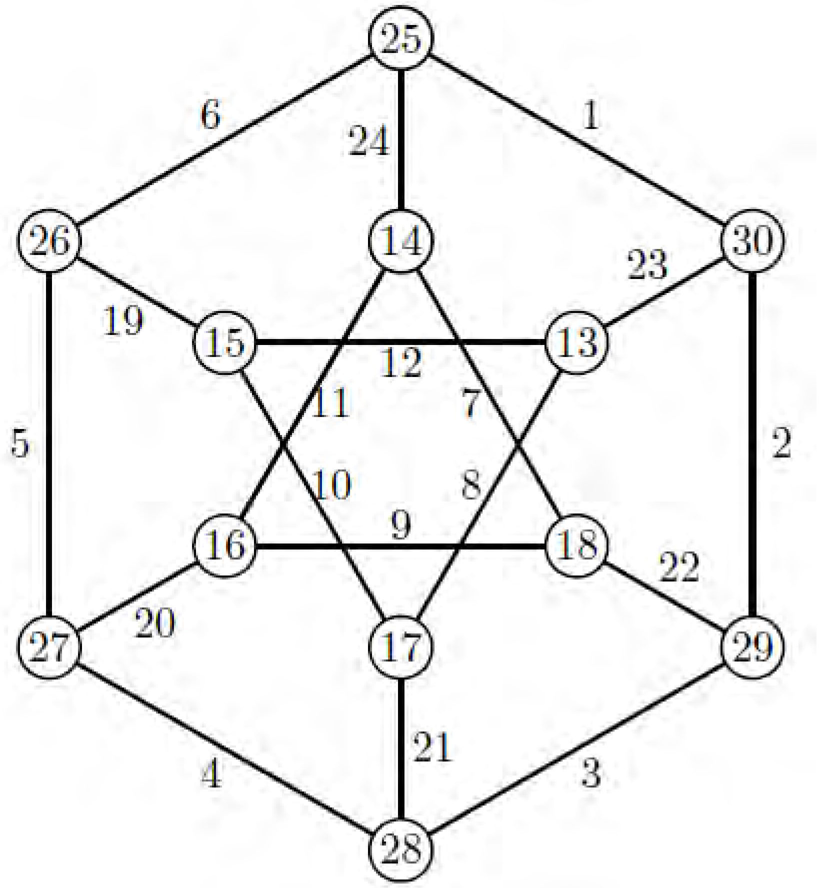

Figure 2.8: Vertex-magic total labeling of the generalised Petersen graph $P(6,2)$.

Stewart proved the following [89] and [90]:

- K_n is magic for $n = 2$ and all $n \geq 5$.
- K_n, n is magic for all $n \geq 3$.
- fans F_n are magic if and only if n is odd and $n \geq 3$.
- wheels W_n are magic for $n \geq 4$ and W_n with one spoke deleted is magic for $n = 4$ and for $n \geq 6$.
- $K_{m,n}$ is semi-magic if and only if $m = n$.
- K_n is supermagic for $n \geq 5$ if and only if $n > 5$ and $n \not\equiv 0 \pmod 4$.

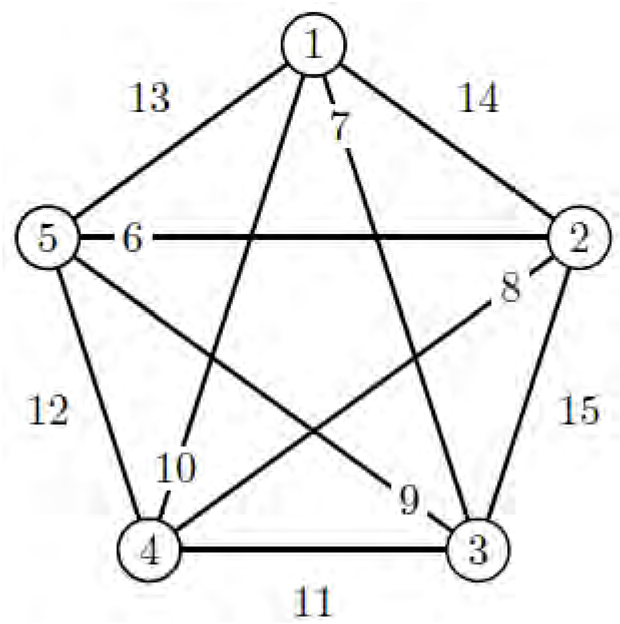

Figure 2.9: Super vertex-magic total labeling of K_5.

Sedl*áč*ek [80] showed that M*ö*bius ladders M_n are supermagic when $n \geq 3$ and n is odd and that $C_n \times P_2$ is magic, but not supermagic, when $n \geq 4$ and n is even. The M*ö*bius ladder M_n is the graph obtained from the ladder $P_n \times P_2$ by joining the opposite end points of the two copies of P_n.

Shiu, Lam, and Lee have proved the following [87]:

- the composition of C_m and $\overline{K_n}$ is supermagic when $m \geq 3$ and $n \geq 2$.
- the complete m-partite graph $K_{n,n,\ldots,n}$ is supermagic when $n \geq 3$, $m > 5$ and $m \not\equiv 0 \pmod 4$.
- if G is an r-regular supermagic graph, then so is the composition of G and $\overline{K_n}$ for $n \geq 3$.

That the composition of K_m and $\overline{K_n}$ is supermagic for $m = 3$ or 5 and $n = 2$ or n odd have been shown by Ho and Lee [39]. Ba*č*a, Holl*ä*ander, and Lih [9] have found two families of 4-regular supermagic graphs. Shiu, Lam, and Cheng [86] proved that for $n \geq 2$, $mK_{n,n}$ is supermagic if and only if n is even or both m and n are odd. Ivan*č*o [44] proved that Q_n is supermagic if and only if $n = 1$ or n is even and greater than 2 and that $C_n \times C_n$ and $C_{2m} \times C_{2n}$ are supermagic. He conjectures that $C_m \times C_n$ is supermagic for all m and n. Sun, Guan, and Lee give an efficient algorithm for finding a magic labeling of a graph. Trenkl*é*r [93] has proved that a connected magic graph with p vertices and q edges other than P_2 exists if and only if $\frac{5p}{4} < q \leq \frac{p(p-1)}{2}$. One can find more results on magic labeling of graphs in [30].

2.4 Anti-magic Type Labeling

In this section we follow the definitions and notation of paper [5]. A graph G is called antimagic if the n edges of G can be distinctly labeled 1 through n in such a way that when taking the sum of the edge labels incident to each vertex, the sums will all be different. An example of an antimagic labeling of graph $K4$ is illustrated in the figure 2.10. Note that the vertex labels are the sums of the labels for the edges incident to the closest vertex.

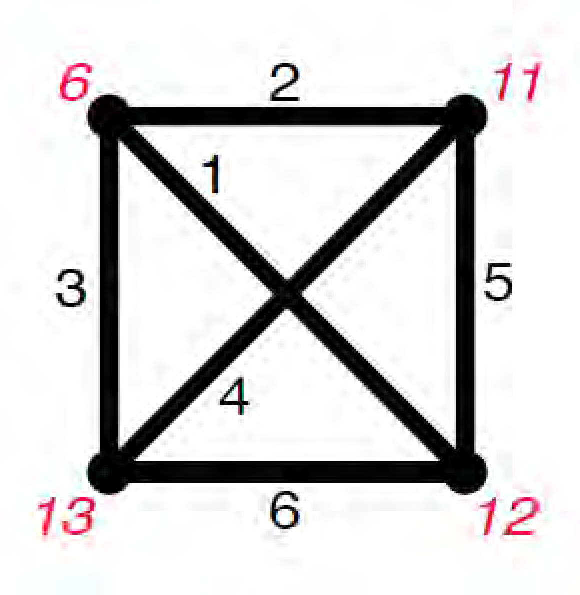

Figure 2.10: Antimagic labeling of K_4.

This type of labeling is sometimes referred to as a strong antimagic labeling due to the fact that there is also a weak antimagic labeling. A graph is said to have a weak antimagic labeling if you can label the edges in an antimagic way, still allowing the edges to be integers less than or equal to the number of edges, without the edge labels necessarily being distinct. An example of an weak antimagic labeling of graph K_4 is illustrated in Figure 2.11. Any strong antimagic labeling is also a weak antimagic labeling.

In 1989, the concept of an antimagic graph was introduced by Hartsfield and Ringel (see in the papers [34] and [35]). A bijection $f, f : E(G) \to \{1,2,\ldots,|E(G)|\}$ is called an

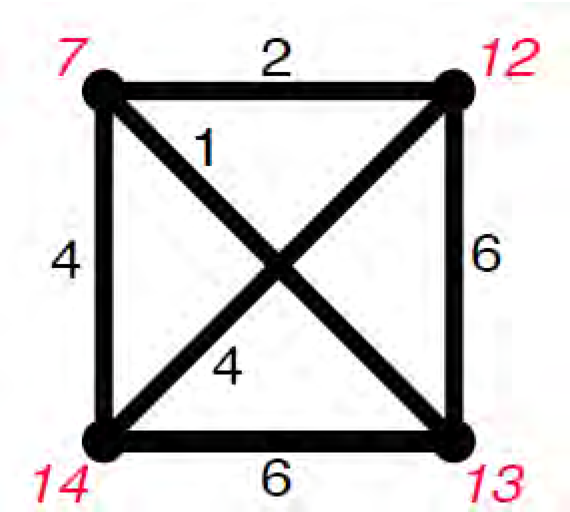

Figure 2.11: Weak Antimagic labeling of K_4.

antimagic labeling if all vertex-weights are distinct. More precisely,

$$w(x) \neq w(y), \quad \text{for all } x \neq y \in V(G)$$

where

$$w(x) = \sum_{y \in N(x)} f(xy).$$

This labelings is also known as vertex-antimagic. Hartsfield and Ringel [34] conjecture that every tree except P_2 is antimagic and, moreover, every connected graph except P_2 is antimagic. Alonet al [6] proved that this conjecture is true for all graphs having minimum degree $\Omega(log|V(G)|)$.

As it is easy to label almost all graphs using antimagic labelings researchers started putting some sort of restrictions. Bodendiek and Walther [20] introduced the concept of an (a,d)-antimagic labeling in 1996. An antimagic labeling f is called an (a,d)-antimagic if the set of vertex-weights

$$\begin{aligned} W &= \{w(x) : w(x) = \sum_{y \in N(x)} f(xy), \quad x \in V(G)\} \\ &= \{a, a+d, a+2d, \ldots, a+(|V(G)-1|)d\} \end{aligned}$$

where $a > 0$ and $d \geq 0$ are two fixed integers. Lin et al [59] called this labeling (a,d)-vertex-antimagic edge labeling, for short an (a,d)-VAE labeling. A graph that admits an (a,d)-VAE labeling is called an (a,d)-VAE graph.

An example of $(20,1)$-VAE labeling of K_5 is illustrated in Figure 2.12.

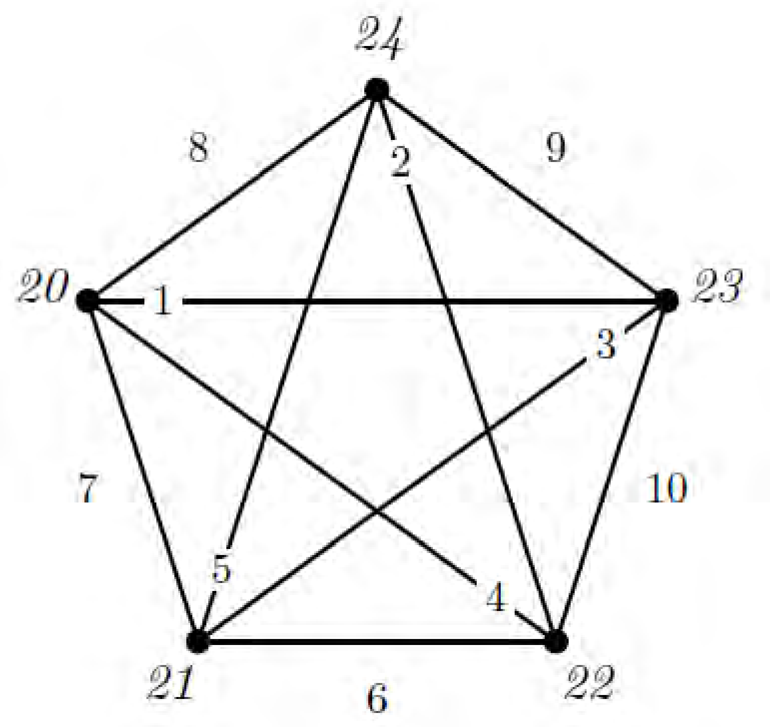

Figure 2.12: A $(20,1)$-VAE labeling of K_5.

A bijection $f, f : V(G) \cup E(G) \to \{1,2,\ldots,|V(G)|+|E(G)|\}$ is called an vertex antimagic total labeling if the vertex-weights are different, thus

$$wt(x) \neq wt(y), \quad \text{for all } x \neq y \in V(G)$$

where

$$wt(x) = \quad f(x) + \sum_{y \in N(x)} f(xy).$$

Bać́a, Bertault et al [8] introduced the notion of an (a,d)-vertex-antimagic total labeling in 2000. A vertex-antimagic total labeling f is called an (a,d)-vertex-antimagic total, (a,d)-VAT, if the set of vertex-weights is

$$\begin{aligned} Wt &= \{wt(x) : wt(x) = \quad f(x) + \sum_{y \in N(x)} f(xy)\} \\ &= \{a, a+d, a+2d, \ldots, a+(|V(G)-1|)d\} \end{aligned}$$

where $a > 0$ and $d \geq 0$ are two fixed integers. An (a,d)-VAT labeling f is called super (a,d)-VAT if the vertices are labeled with the smallest possible numbers, i.e. with the numbers $1,2,\ldots,|V(G)|$. An example of the super $(11,1)$-VAT labeling of Cycle graph C_3 is depicted

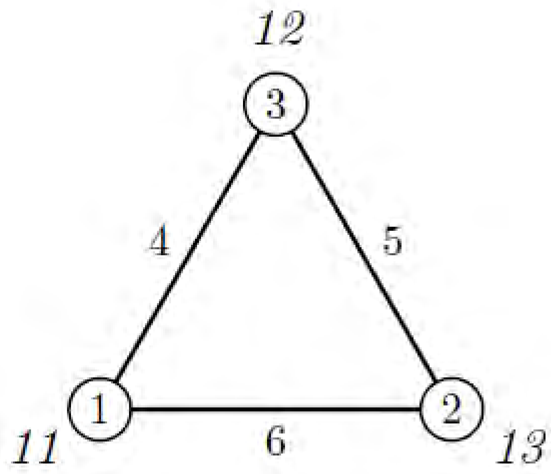

Figure 2.13: A $(11,1)$-VAT labeling of C_3.

in Figure 2.13 in the page 17. Baća et al [8] investigated and introduced basic properties of (a,d)-VAT labeling and constructed such labeling for some families of graphs. They established the relationship between (a,d)-VAT labelings and SVMT labelings, and described the labeling schemes for P_n, cycle C_n and complete bipartite graphs $K_{m,n}$. They also presented VAT labeling for the families of generalized Petersen graphs, prisms, antiprisms and convex polytope graphs in [7]. Detailed results related to antimagic labelings of graphs can be found at [30].

2.5 Miscellaneous Labeling

As there are so much of practical applications of graph labeling, researcher got interested in graph labeling techniques. Therefore, a huge number graph labeling techniques have been discovered. In this thesis paper we have tried to highlight the most famous graph labeling technique which have been already discussed above. Moreover, there are more graph labeling techniques which also have application in practical fields. These are Sum Graph Labeling, Prime and Vertex Prime Labeling, Edge-graceful Labelings, Radio Labelings, Line-graceful Labelings, k-sequential Labelings, Product and Divisor Cordial Labelings, Edge Product Cordial Labelings, Difference Cordial Labelings, Prime Cordial Labelings, Geometric Labelings, Mean Labelings, Irregular Total Labelings etc. Some of them are briefly discussed in rest of this chapter. One can find the details results of those labelings in [30].

2.5.1 Edge-Graceful Labelings

Lo [60] introduced the notion of edge-graceful graphs in 1985. A graph $G(V,E)$ is said to be edge-graceful if there exists a bijection f from E to $\{1,2,\ldots,|E|\}$ such that the induced mapping f^+ from V to $0,1,\ldots,|V|-1\}$ given by $f^+(x) = (\sum f(xy)) \pmod{|V|}$ taken over all edges xy is a bijection. Though an edge-graceful graph is antimagic also. A necessary condition for a graph with p vertices and q edges to be edge-graceful is that $q(q+1) \equiv p(p+1)/2 \pmod{p}$. Lee [57] conjectured that any connected simple (p,q)-graph with $q(q+1) \equiv p(p-1)/2 \pmod{p}$ vertices is edge-graceful. Lee, Kitagaki, Young, and Kocay [58] prove that the conjecture is true for maximal outerplanar graphs in the year of 2006.

2.5.2 Line-Graceful Labelings

In paper [31] Gnanajothi has defined a concept similar to edge-graceful. She defined a graph with n vertices is line-graceful if it is possible to label its edges with $0,1,2,\ldots,n$ such that when each vertex is assigned the sum modulo n of all the edge labels incident with that vertex the resulting vertex labels are $0,1,2,\ldots,n-1$. A necessary condition for the line-gracefulness of a graph is that its order is not congruent to $2 \pmod{4}$. Among all, some line-graceful graphs are P_n if and only if $n \not\equiv \pmod{4}$, C_n if and only if $n \not\equiv 2 \pmod{4}$, $K_{1,n}$ if and only if $n \not\equiv 1 \pmod{4}$, $P_n \odot K_1$ (combs) if and only if n is even, $(Pn \odot K1) \odot K1$ if and only if $n \not\equiv 2 \pmod{4}$, mC_n when mn is odd, $Cn \odot K1$ (crowns) if and only if n is even, mC_4 for all m, complete n-ary trees when n is even, $K_{1,n} \cup K_{1,n}$ if and only if n is odd, odd cycles with a chord, even cycles with a tail, even cycles with a tail of length 1 and a chord etc. She conjectures that all trees with $p \not\equiv 2 \pmod{4}$ vertices are line-graceful and proved this conjecture for $p \leq 9$.

2.5.3 k-Sequential Labelings

Bange, Barkauskas, and Slater [12] defined a k-sequential labeling f of a graph $G(V,E)$ as one for which f is a bijection from $V \cup E$ to $\{k, k+1, \ldots, |V \cup E| + k - 1\}$ such that for each edge xy in E, $f(xy) = |f(x) - f(y)|$. Bange, Barkauskas, and Slater showed that cycles are 1-sequential and if G is 1-sequential, then $G + K_1$ is graceful. Hegde and Shetty [37] have shown that every T_p-tree is 1-sequential. Slater proved: K_n is 1-sequential if and only if $n \leq 3$, for $n \geq 2$, K_n is not k-sequential for any $k \geq 2$, and $K_{1,n}$ is k-sequential if and only if k divides n. There are some more result on k-sequential Labelings which can be found in [30].

2.6 Applications of Graph Labeling

Graph labeling were first introduced in the mid sixties. In the middle of the time dozens of graph labeling technique have been introduced in over 1000 papers. An enormous amount of literature has grown around the subject and is still getting flourished due to increasing number of application oriented concepts. Labeled graphs are becoming an increasingly useful family of mathematical models for a broad range of applications. In diverse fields of human enquiry a qualitative labeling of graph elements have inspired research such as Conflict resolution in social psychology, electrical circuit theory and energy crisis etc. and quantitative labeling of graphs have led to quite intricate fields of application such as coding theory problems, including the design of good Radar location codes, Synch-set codes; missile guidance codes and convolution codes with optimal autocorrelation properties. Moreover, labeled graphs have also been applied, in determining ambiguities in X-Ray Crystallographic analysis, to Design Communication Network addressing Systems, in determining Optimal Circuit Layouts and Radio-Astronomy etc.

Graceful labeling technique is a most popular graph labeling technique and traces its origin to one introduced by Rosa [76]. However, Graceful labeling is not just an appealing research problem that a non-researcher or non-specialist can understand but a problem with extraordinary versatile applications. Graceful labeling of trees have been used in Multi Protocol Label Switching (MPLS) routing platform in IP networks. Graceful labeling of directed graphs have been used to describe some algebraic structures such as cyclic difference sets, sequenceable groups, generalized complete mappings and neofeilds. The odd graceful labeling is one of the most famous and widely used labeling methods of graphs [97]. The labeling of graphs serves as models in a wide range of applications as listed below.

2.6.1 The Coding Theory

The design of good non periodic codes for pulse radar and missile guidance is equivalent to labeling the complete graph in such a way that all the edge labels are distinct. Therefore, node labels can determine the time positions at which pulses are transmitted.

2.6.2 The X-ray Crystallography

In modern days, X-ray diffraction is one of the most powerful techniques for characterizing the structural properties of crystalline solids, in which a beam of X-rays strikes a crystal and diffracts into many specific directions. Position of atom in a crystal structures are made by X – ray diffraction patterns. Measurements indicate the set of inter atomic distances in crystal lattices. Mathematically, one can find the finite set of integers to one atom position , so that diffraction is equivalent to the distinct edge lengths between these two integers. In some cases more than one structure has the same diffraction information. This problem is mathematically equivalent to determining all labeling of the appropriate graphs which produce a pre-specified set of edge labels.

2.6.3 The Communications Network Addressing

A communication network is composed of nodes, each of which has computing power and can transmit and receive messages over communication links, wireless or cabled. The basic network topologies are include fully connected, mesh, star, ring, tree, bus. A single network may consist of several interconnected subnets of different topologies. If one had a communication network with a fixed number $n+1$ of communication centers (i.e. vertex) and they were numbered $0,1,\ldots,n$ then the lines between any two centers could be labeled with the difference between two center labels (i.e. vertex labels) If the communication center grid was laid out in a graceful graph, we would then be able to label the connections between each center such that each connection would have a distinct label. One good advantage of such a labeling is that if a link goes out, a simple algorithm could detect which two centers are no longer linked.

Chapter 3

Existing Classes of Graceful Trees

Ringel conjecture stating that, the complete graph K_{2m+1} can be decomposed into trees isomorphic to a given tree with m edges was published by Rosa [76] in 1967 . Rosa showed that if a tree has a graceful labeling, then the conjecture of Ringel will be followed. Right away Attempts to prove Ringel's conjecture have therefore focused on obtaining the stronger result that *every tree is graceful*, a term given by Golomb [32]. This is the primary motivation that inspired us to analyse the results that will be covered in this section.

Conjecture 4. (Ringel-Kotzig Conjecture [1964]) All trees are graceful.

In the attempt to prove that all trees are graceful, many classes of trees have been proven graceful. Initially, the gracefulness of several classes of trees was established by Rosa in [76]. Since then, other classes have been shown to admit graceful labeling. This referred paper [30] is contain a great source for finding a list of graceful classes of trees. Yet, knowing all of them is still not enough to conclude that all trees are graceful. In this section, we are going to exhibit a lot of those classes.

The list of trees which are known to be graceful are: caterpillars [76], trees with at most 4 end-vertices [43], [98] and [49], trees with diameter at most 5 [98] and [42], symmetrical trees [16], [72], rooted trees where the roots have odd degree and the lengths of the paths from the root to the leaves differ by at most one and all the internal vertices have the same parity [23], rooted trees with diameter D where every vertex has even degree except for one root and the leaves in level $\lfloor D/2 \rfloor$ [11], rooted trees with diameter D where every vertex has even degree except for one root and the leaves, which are in level $\lfloor D/2 \rfloor$ [11], rooted trees with diameter D where every vertex has even degree except for one root, the vertices in level $\lfloor D/2 \rfloor - 1$, and the leaves which are in level $\lfloor D/2 \rfloor$ [11], the graph obtained by identifying the endpoints any number of paths of a fixed length except for the case that the length has the form $4r+1$, $r > 1$ and the number of paths is of the form $4m$ with $m > r$ [81], regular bamboo trees [81] and olive trees [71], [1]. In 2010 Fang [29] Motivated by Horton's work [40] used a deterministic back-tracking algorithm to prove that all trees with at most 35 vertices are graceful. Bahls, Lake, and Wertheim [10] proved that spider trees are graceful.

In [25] Chen, Lu, and Yeh proved that firecrackers are graceful and conjectured that banana trees are graceful. Sethuraman and Jesintha [84] and [85] and [46] proved that all banana trees and extended banana trees are graceful, various kinds of bananas trees had been shown to be graceful by Bhat-Nayak and Deshmukh [17], by Murugan and Arumugam [67], [68] and by Vilfred [94].

Forhad et al [41] proved that two new classes of trees named Super Caterpillars and Extended Super Caterpillars are graceful. Sourabh et al provide an idea in [78] to construct larger classes of graceful trees which consist of paths and caterpillars. Afsana, Maowa, Tania and Kaykobad also introduced a new class of graceful trees [66] called Super stars.

The graphs obtained by starting with any number of identical stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex have been shown graceful by Zhenbin [99]. Another result that graphs obtained by starting with any two stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are graceful have also been shown graceful. To generate graceful trees from a graceful star with n edges, in [47] Jesintha and Sethuraman use a method of Hrnciar and Havier [42].

Eshghi and Azimi in [27] and [28] discuss a programming model for finding graceful labelings of large graphs. This method is used to verify that all trees with 30, 35, or 40 vertices are graceful. Koh, Rogers, and Tan [52], [53], [55] and Stanton and Zarnke [88] gave methods for combining graceful trees to yield larger graceful trees. In [75] Rogers and in [54] Koh, Tan, and Rogers provide recursive constructions to create graceful trees.

The graph obtained from any graceful tree by subdividing every edge has been shown graceful by Burzio and Ferrarese [22]. In year of 1979 Bermond [14] conjectured that lobsters are graceful. Morgan [65] has shown that all lobsters with perfect matchings are graceful. Mishra and Panigrahi [62] and [70] found classes of graceful lobsters of diameter at least five. They also showed other classes of lobsters graceful [63] and [64]. In [83] Sethuraman and Jesintha explores how one can generate graceful lobsters from a graceful caterpillar while in [82] and [85] also in [46] they show how to generate graceful trees from a graceful star. More special cases of Bermond's conjecture have been done by Ng [69], by Wang, Jin, Lu, and Zhang [95], Abhyanker [2], and by Mishra and Panigrahi [63].

Barrientos [13] introduced a new tree named y-tree as a graph obtained from a path by appending an edge to a vertex of a path adjacent to an end point. He proves that graphs obtained from a y-tree T by replacing every edge e_i of T by a copy of K_{2,n_i} in such a way that the ends of e_i are merged with the two independent vertices of K_{2,n_i} after removing the edge e_i from T are graceful.

Despite the efforts of many researchers, still the graceful tree conjecture remains open even for trees with maximum degree 3. Many more results about graceful trees are contained in [14], [18], [51], [61], [24], [48], and [77]. In [26] Edwards and Howard provide a detailed and elaborate survey paper on graceful trees. Now some of the trees which are proved to be graceful in an attempt to prove Graceful Tree Conjecture are discussed below in detail.

3.1 Paths

In graph theory, a path is a tree whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the edges are $\{v_i, v_{i+1}\}$ where $i = 1, 2, \ldots, n-1$. Equivalently, a path with at least two vertices is connected and has two leaves (vertices's that have degree 1), while all others (if any) have degree 2.

Let P_n be the path with n edges and $n+1$ vertices's.

Theorem 5. *All paths are graceful*

Proof. Label P_n by starting at one end of the path and alternating between the least and greatest remaining label along the path, so that the labels are

$$0, n, 1, n-1, 2, \ldots$$

□

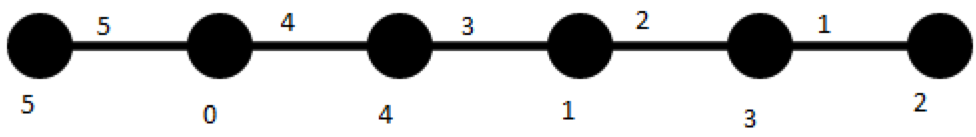

Figure 3.1: An example of a gracefully labeled Path tree on 6 vertices

3.2 Caterpillars

A caterpillar is a tree such that if one removes all of its leaves, the remaining graph is a path. This path can be termed as backbone of the caterpillar.

In [76] Rosa proved that all caterpillars are graceful. Now We are going to exhibit a proof of that fact. Since paths are also caterpillars, it will follow Theorem 5 that paths are also graceful.

Theorem 6. *All caterpillars are graceful.*

Proof. Let C be a caterpillar on 14 vertices's as depicted in Figure 3.2. A caterpillar is labeled from one end of its backbone with 0 and its adjacent vertices are labeled using so far unused largest labels ending in labeling the next vertex on the path with the smallest of the largest labels used. Its adjacent vertices are labeled using the smallest so far unused labels alternately. In the mean time while we label vertices largest unused edge labels are generated. It can be noted that for a tree with m edges if $f(i)$ is a graceful labeling then so is $m - f(i)$. That is why we can label any end of a caterpillar by 0 or by m. □

3.3 Super Caterpillars

In this section we follow the definition of paper [41].

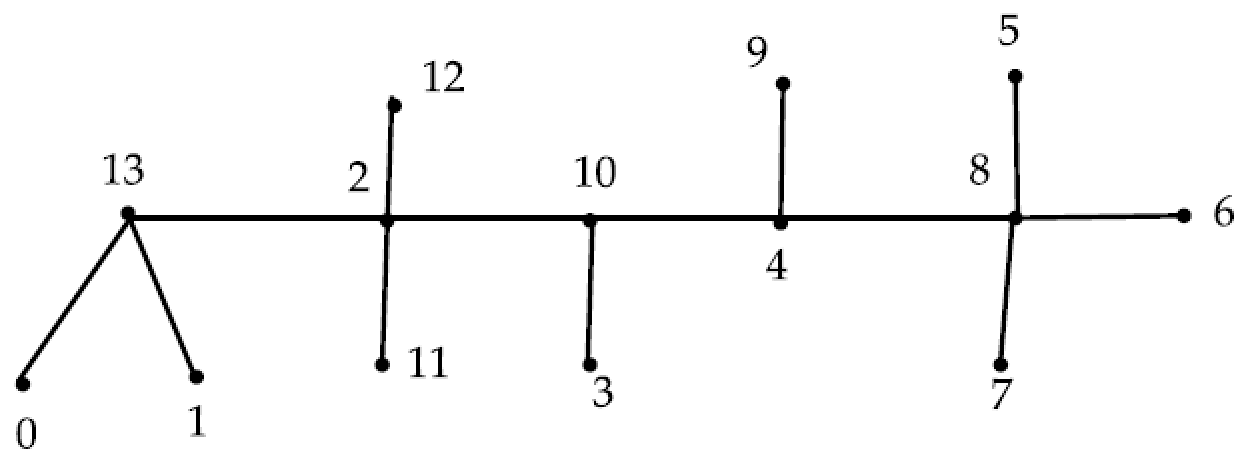

Figure 3.2: An example of gracefully labeled 14-vertex Caterpillar

Let T_0 be any arbitrary caterpillar and $T_i, i = 1, \ldots, k$ be caterpillars with $|T_i| = m$ number of vertices and sum total of vertices's is the same in odd levels of all pairs T_{2i+1} and T_{2i+2}. In case k being an odd number, one caterpillar will be without a pair. Let one end of each backbone be joined to the vertex v by an edge. Then the resulting tree is called a super-caterpillar.

Example 7. Two super-caterpillars are illustrated in Figure 3.3 with odd and even k.

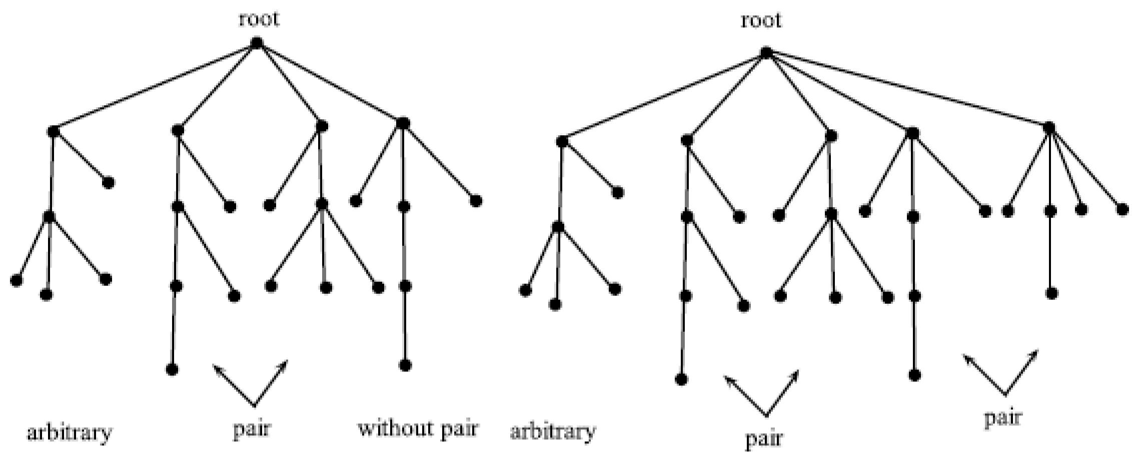

Figure 3.3: Example of Super-caterpillar with (a) odd k, (b)even k, with an arbitrary caterpillar joined with a root.

Theorem 8. *All super-caterpillars are graceful.*

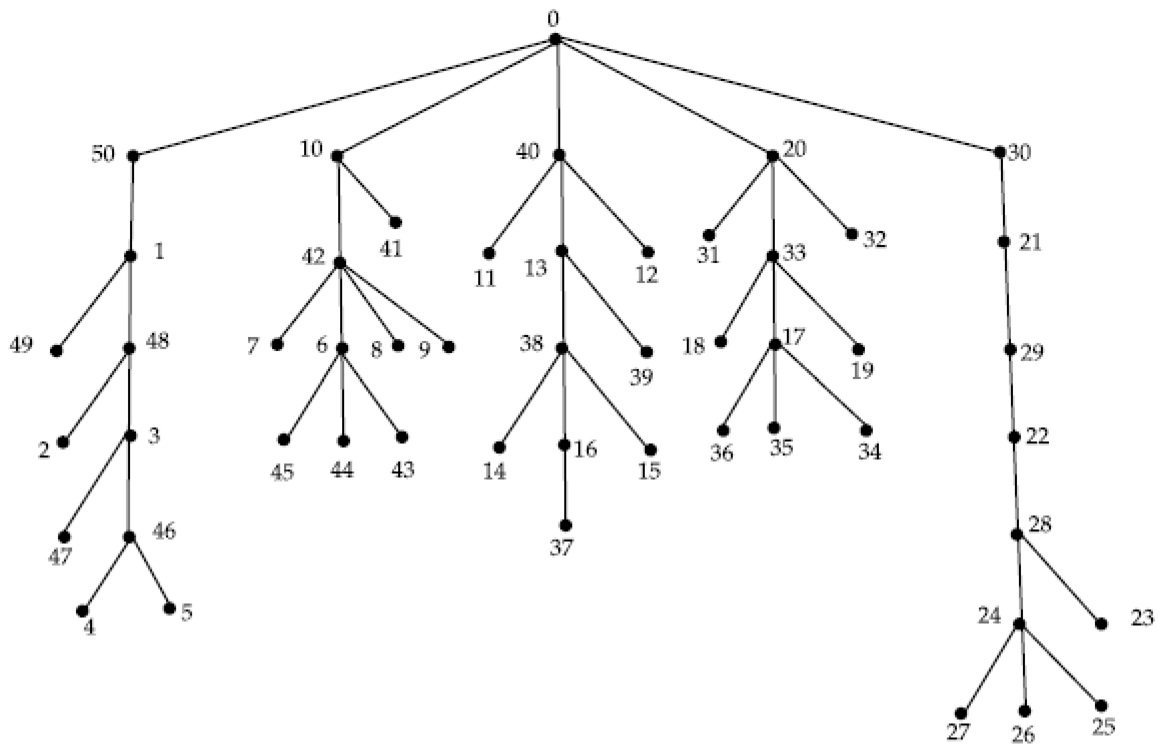

Figure 3.4: Example of gracefully labeled Super-caterpillar

3.4 Extended Super Caterpillars

In this section we follow the definition of paper [41]. Let there be an even number kp caterpillars, each having m vertices's and sum total number of vertices's in odd (or even) levels of those caterpillars are the same. These caterpillars are grouped in k groups each having p caterpillars. Let backbones of the group i of caterpillars be connected to a vertex v_i that is connected to vertex v. Then the resulting tree is called a extended super-caterpillar.

An extended super-caterpillar is illustrated in Figure 3.6 in the page 28 with six caterpillars grouped in two.

Theorem 9. *All Extended Super Caterpillars are graceful.*

The following examples show the verification of the proof as there are six caterpillars in figure 3.7 which can be grouped in three (figure 3.8) or two(figure 3.10) ($k = 2$ or $k = 3$). Here each caterpillar has 9 vertices's ($m = 9$). Then both these groups are connected with v and gracefully labeled in figure 3.9 and figure 3.11.

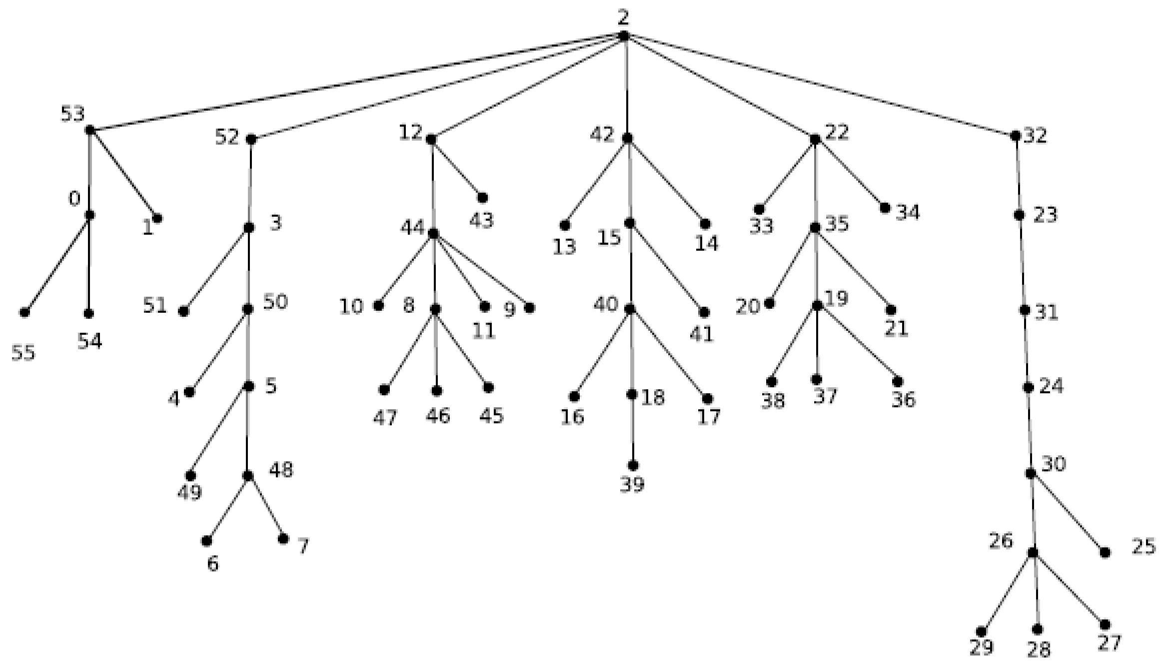

Figure 3.5: Another Example of gracefully labeled super-caterpillar with an arbitrary Caterpillar where $s_0 = 2$.

3.5 Symmetrical Trees

In this section we follow the definition and proof of paper [41] and [74].

A rooted tree in which every level contains vertices of the same degree is called symmetrical trees. J.C. Bermond and J. Schonheim [15] proved that all symmetrical trees are graceful.

A gracefully labeled symmetrical tree on 15 vertices is illustrated in Figure 3.12 in the page 31.

Theorem 10. *All symmetrical trees are graceful*

The proof has been shown by induction on the number of layers that all symmetrical trees are graceful and there exists a graceful labeling which assigns the number 1 to the root.

If T is a symmetrical tree with 0 layers, then, it consists of 0 edges and just one vertex, and clearly there is a graceful labeling which assigns 1 to that vertex. Suppose we have proved that for some $l > 0$ all symmetrical trees with $\leq l-1$ layers are graceful and each of them has a graceful labeling which assigns the number 1 to the root.

The idea of the induction step is to consider a rooted symmetrical tree for which we know

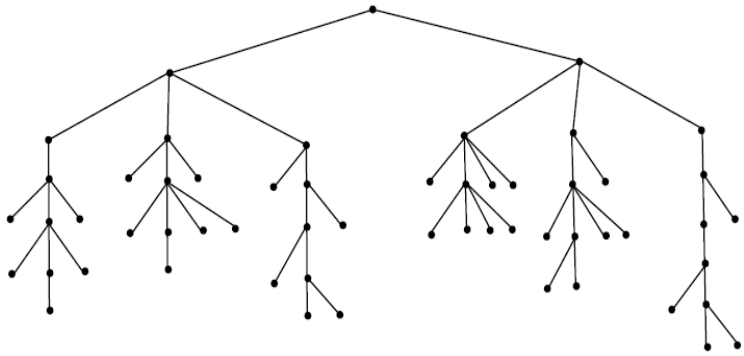

Figure 3.6: Example of extended super-caterpillar with two groups consisting three Caterpillars.

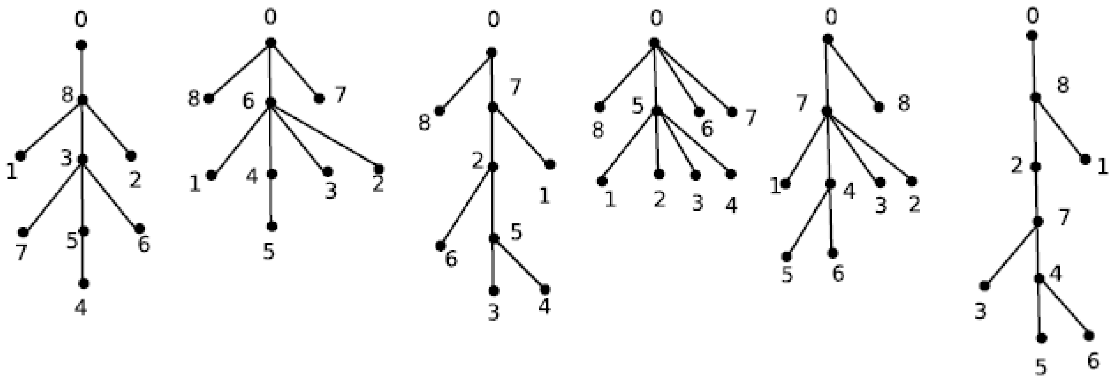

Figure 3.7: Six graceful Caterpillars.

that its k children $T_1, T_2, \ldots, T_k$ are graceful (and isomorphic to each other). We label the children with their (identical) graceful labeling and then add certain numbers to each of the vertices's. The way we do this is illustrated in the figure 3.13 in the page 32.

We order the children from left to right. Then, if n is the number of vertices's in each child, we start from the $0th$ layer of the children and add $(k-1)n$ to the root of T_1, $(k-2)n$ to the root of $T_2, \ldots$, and 0 to the root of the k-th one. Then, for the first layer, we start from right to left this time and add $(k-1)n$ to each of the vertices's in the 1st layer of T_k, then, we add $(k-2)n$ to each of the vertices's in the 1st layer of $T_{k-1}, \ldots$, and 0 to each of the vertices's in the first layer of T_1. So, then we go on with the second layer and we start from left to right, and so on until we finish with the last layer. Then, we write $nk+1$ on the root of the new tree. Then, we do the transformation $x \mapsto nk+2-x$ to each of the vertices's, so that we can have 1 at the root and the resulting labeling, as we show in the sequel, is graceful.

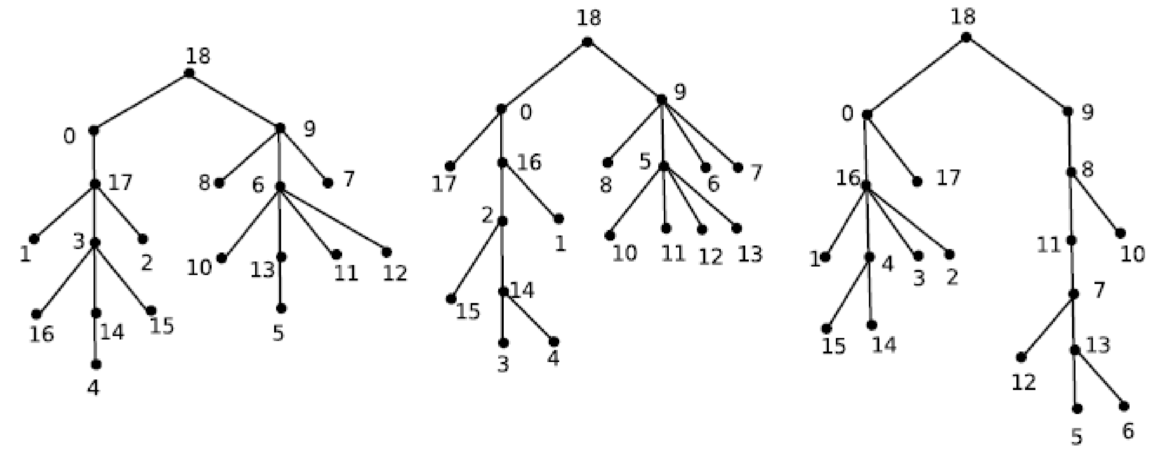

Figure 3.8: Three Grouped Caterpillars are joined with a single vertex v_i which are graceful.

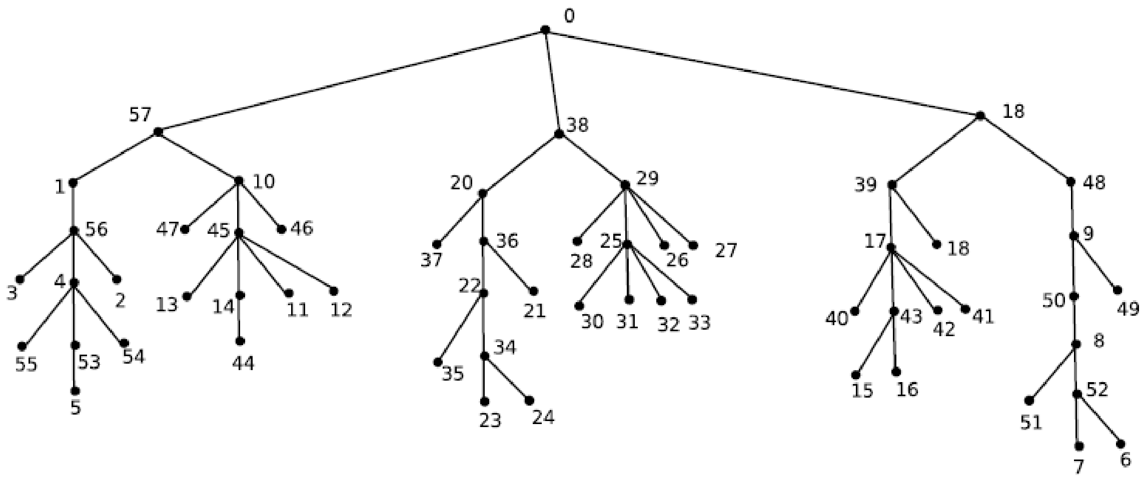

Figure 3.9: All groups joined with a root vertex v and gracefully labeled.

3.6 Star Trees

A tree with one internal node and k leaves is said to be a star $S_{1,k}$ that happen to be a complete bipartite graph $K_{1,k}$.

Stars can be gracefully labelled by labeling its centre by 0 and other leaves by $j, j = 1, ..., k$. An example of gracefully labelled star is shown in the figure 4.1.

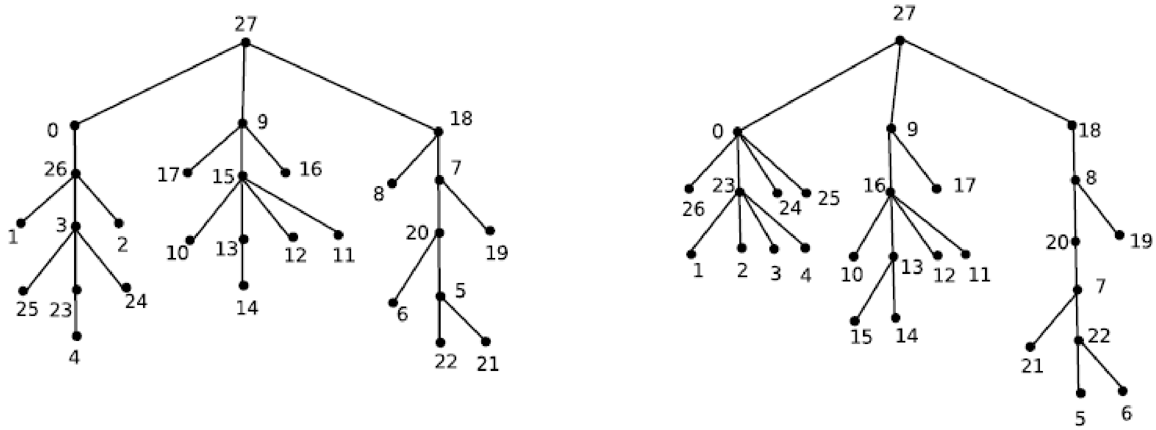

Figure 3.10: Six graceful caterpillars grouped into two each containing three Caterpillars.

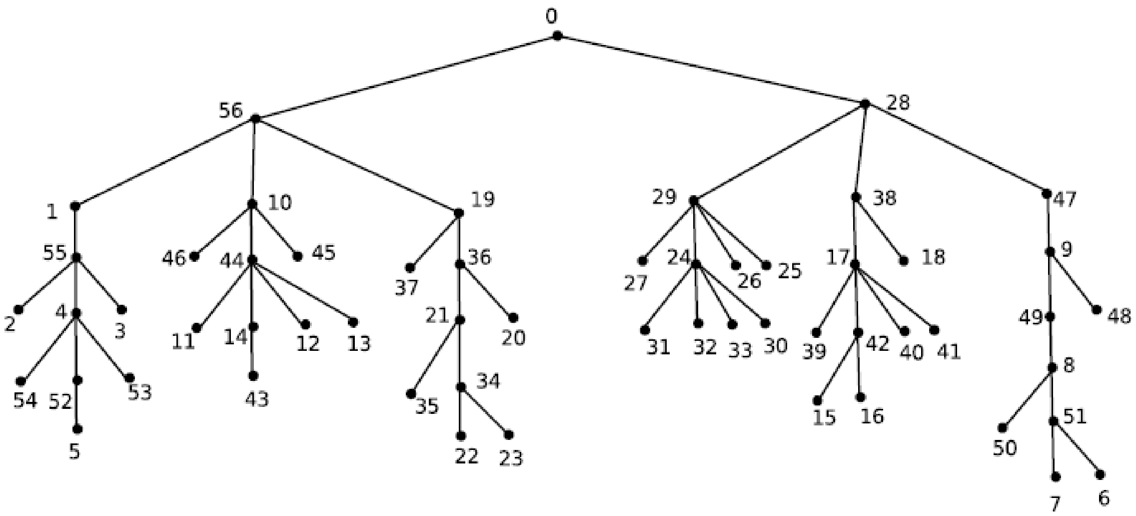

Figure 3.11: Both groups joined with a single root vertex.

3.7 *m*-Stars

A *m*-Star has a single root node with any number of paths of length *m* attached to it. An example of gracefully labelled *m*-star is shown in the figure 3.15.

3.8 Spider Trees

A spider tree is a tree with at most one vertex of degree greater than 2. If such a vertex exists, it is called the branch point of the tree. A leg of a spider tree is any one of the paths from the branch points to a leaf of the tree.

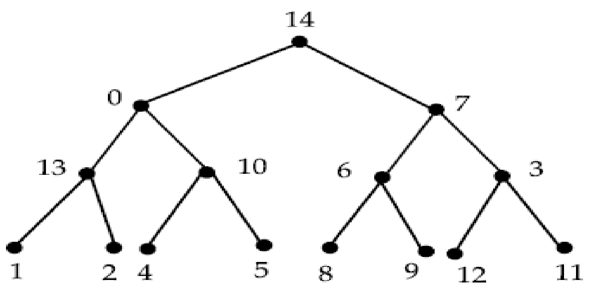

Figure 3.12: An example of gracefully labeled 15-vertex Symmetrical Tree.

Let S be a spider tree on 31 vertices's as shown in Figure 3.16.

Bahls, Lake and Wertheim has proved in [10] that all spider trees are graceful.

Theorem 11. *Let T be a spider tree with l leg, each of which has length in $\{m, m+1\}$ for some $m > 1$. Then, T is gracefil.*

Proof. (as in [10]) We assume that $l \geq 3$ since otherwise T is a path and we already showed that paths are graceful. We look at two cases.

Case 1. l is odd. Let $l = l_0 + l_1$, where l_i is the number of legs of length $m+i$ for $i \in \{0, 1\}$. Then, T has $n = lm + l_1 + 1$ vertices's. We call the legs $L_1, L_2, \ldots, L_l$ where $L_1, L_2, \ldots, L_{l_1}$ are the legs of length $m+1$ and $L_{l_1+1}, L_{l_2+1}, \ldots, L_l$ are the legs of length m. Let v^* be the branch point of T and let $v_{i,j}$ be the vertex in L_i of distance j from v^*.

We exhibit the following labeling f:

1. f$(v^*) = 1$,
2. If i and j are both odd, then $f(v_{i,j}) = n - \frac{i-1}{2} - \frac{(j-1)l}{2}$,
3. If i and j are both even, then $f(v_{i,j}) = n - \frac{l-1}{2} - \frac{i}{2} - \frac{(j-2)l}{2}$,
4. If i is even and j is odd, then $f(v_{i,j}) = \frac{i}{2} + \frac{(j-1)l}{2} + 1$, and,
5. If i is odd and j is even, then $f(v_{i,j}) = \frac{l-1}{2} + \frac{i+1}{2} + \frac{(j-2)l}{2} + 1$,

This labeling assigns all numbers from 1 to n to the vertices's of T since it starts by assigning 1 to v^* and then traverses the longer legs first, alternating between highest and lowest remaining unused labels, spirally away from the center. This is illustrated in Figure 3.17 in the page 36, in which $l_0 = 2$, $l_1 = 3$, and $m = 4$.

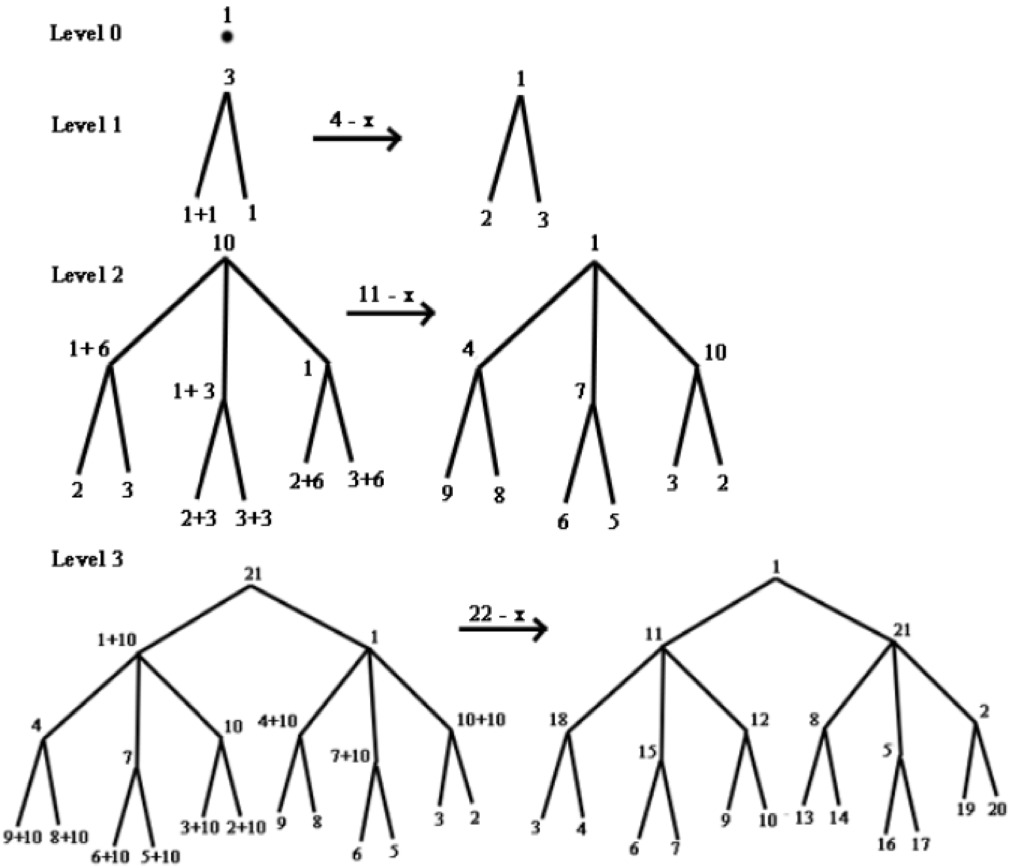

Figure 3.13: All the steps in a graceful labeling of a Symmetrical Tree.

Then, we have that for $i \equiv j (\text{mod } 2)$, $f(v_{i,j}) - f(v_{i,j+1}) = n - 1 - \frac{l-1}{2} - i + (1-j)l > 0$ and $f(v_{i,j}) - f(v_{i,j-1}) = n - 1 - \frac{l-1}{2} - i + (2-j)l > 0$. Suppose that there exist $(i,j) \neq (i',j')$ and $i' \equiv j'$ (mod 2) and $f(v_{i,j}) - f(v_{i,j+1}) = f(v_{i',j'}) - f(v_{i',j'+1})$. So we get that $i - i' + (j - j')l = 0$, so, $l = \frac{i-i'}{j-j'}$ (note that if $j = j'$, then also $i = i'$, so, $(i,j) = (i',j')$, so, $j \neq j'$. Thus, $|i - i'| < l$ and $|j - j'| > 1$, and $l = |\frac{i-i'}{j-j'}| < \frac{l}{1} = l$, which gives us a contradiction. Thus, $f(v_{i,j}) - f(v_{i,j+1}) \neq f(v_{i',j'}) - f(v_{i',j'+1})$. Similarly, we get that $f(v_{i,j}) - f(v_{i,j+1}) \neq f(v_{i',j'}) - f(v_{i',j'-1})$ and $f(v_{i,j}) - f(v_{i,j-1}) \neq f(v_{i',j'}) - f(v_{i',j'-1})$.

Case 2. l is even. Without loss of generality L_l is a leg of length m (otherwise the tree is symmetric which we already proved is graceful). Remove the leg L_l to get a tree T_0 with an odd number of legs $l-1$. From above we get a graceful labeling f_0 of T_0 with $f_0(v^*) = 1$. Let $V(T_0) = n'$. Define a new graceful labeling f_0' of T_0 by $f_0'(v) = n' + 1 - f_0(v)$ for each $v \in V(T_0)$.

Now, construct a new tree T_1 by appending a new vertex, w_1, to T_0's center. Extend f_1 on

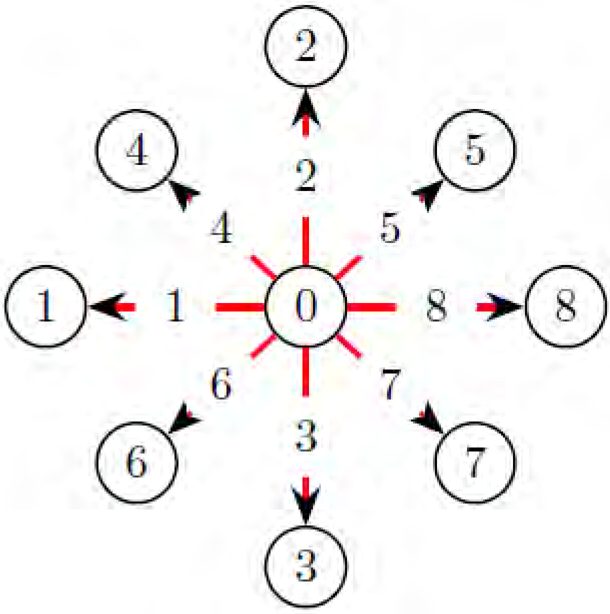

Figure 3.14: An example of gracefully labelled star.

$V(T_1)$ by $f_1(w_1) = 1$ and $f_1(v) = f_0'(v) + 1$ for all $v \in V(T_0)$. Define f_1' on T_1 by $f_1(v) = n' + 2 - f_1(v)$ for all v. Note that $f_1'(w_1) = n' + 1$. We can consecutively append vertices's $w_2, w_3, \ldots, w_m$ to our l-th leg to obtain a graceful labeling of T. Note that we can append as many vertices's as we want, not just m.

□

3.9 Lobster

A lobster tree is a tree such that if you remove all of its leaves, it becomes a caterpillar. An exmaple of graceful lobster tree has been illustrated in the figure 3.18 in the page 37.

In [14] Bermond conjectured that all lobsters are graceful in the year of 1979. Many researchers have attempted to resolve this conjecture, although no one has been able to do it yet. In the year of 2002 Morgan [65] proved that all lobster trees with perfect matching are graceful. Mishra and Panigrahi [62, 70] found classes of graceful lobsters of diameter at least five. They also showed [63, 64] that some other classes of lobsters are graceful. They observed that a lobster having diameter at least five has a unique path $H = x_0x_1 \ldots x_m$ satisfying the property that, besides the adjacency's in H, both x_0 and x_m are adjacent to the centers of at least one $K_{1,s}$ (which is a star with s leaves), where $s > 0$, and each x_i, for

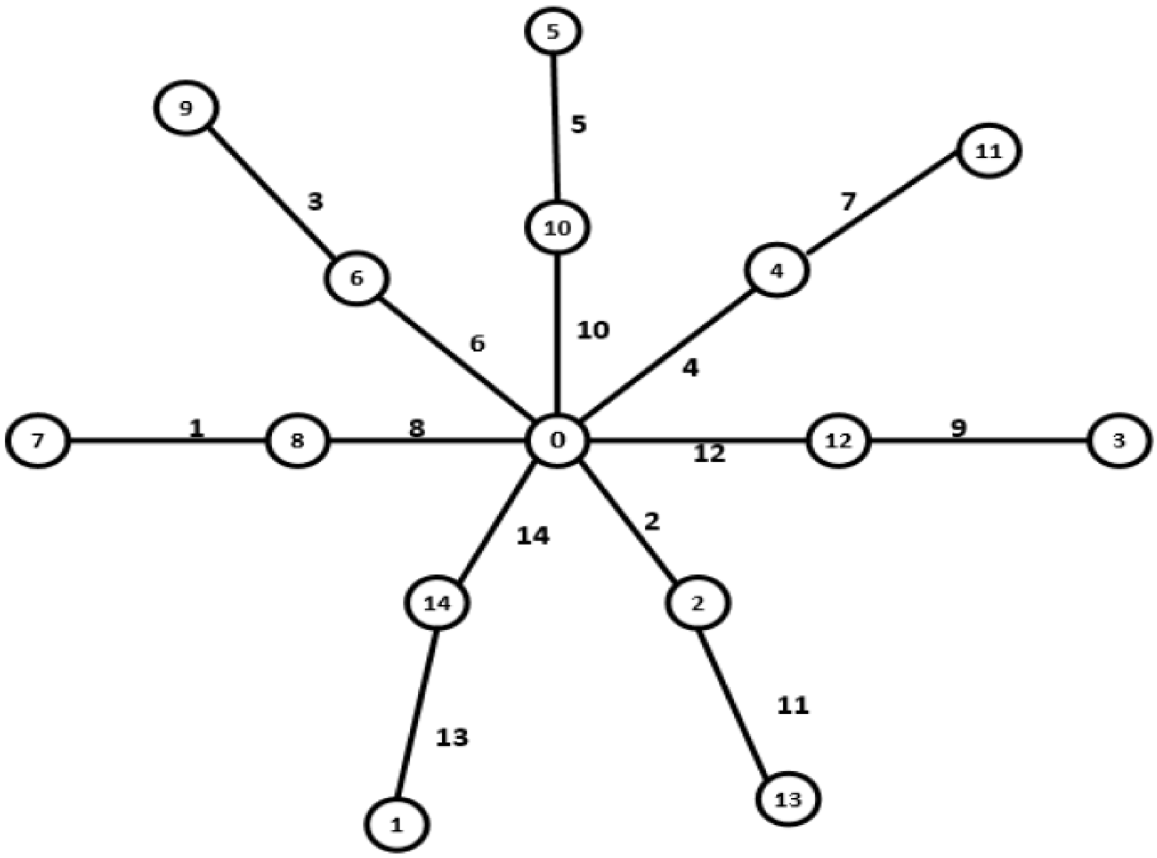

Figure 3.15: An example of gracefully labelled m-star where $m = 2$.

$1 \leq i \leq m-1$, is at most adjacent to the centers of some $K_{1,s}$, where $s \geq 0$. The unique path H is called the central path of the lobster. There are three types of branches that the vertices's of H can be adjacent to: even, odd, and pendant. $K_{1,s}$ is an even branch, if x_i is adjacent to the center of a $K_{1,s}$ where $s \geq 2$ is even. Again $K_{1,s}$ is an odd branch, if x_i is adjacent to the center of a $K_{1,s}$ where s is odd. $K_{1,s}$ is called pendant, if x_i is adjacent to the center of a $K_{1,s}$ where $s = 0$, i.e. x_i is adjacent to a leaf. Mishra and Panigrahi in the paper [63], they give graceful labelings to the lobsters having some special features.

More special cases of classes of graceful lobsters have been found by Sethuraman and Jesintha in [82], by Ng in [69], by Wang, Jin, Lu, and Zhang in [95], by Abhyanker in [2].

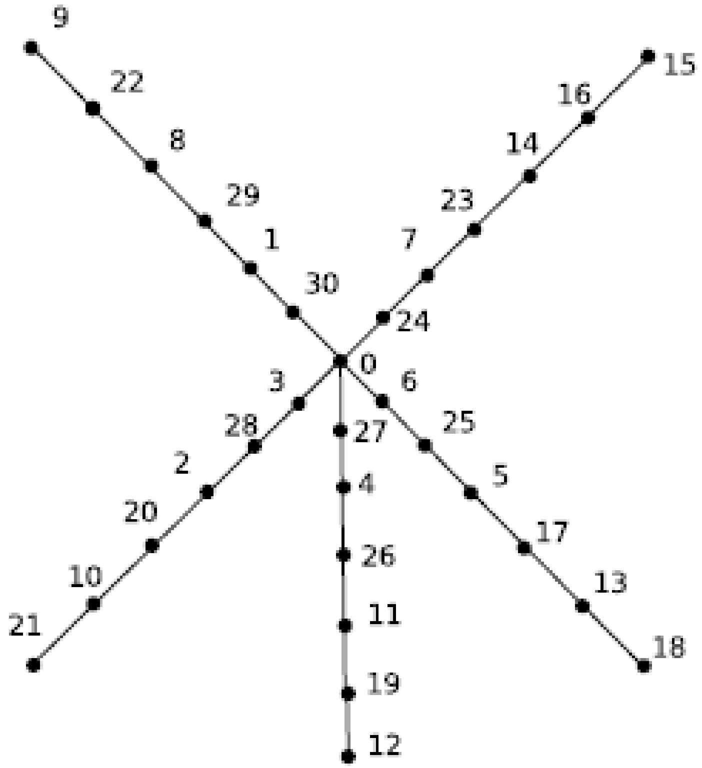

Figure 3.16: An example of gracefully labeled 31-vertex Spider Tree.

3.10 Firecrackers

A firecracker F consists of a set of stars, centres of which are connected to a single vertex. Chen, Lu and Yeh in [25] proved that firecrackers are graceful.

Theorem 12. *All firecrackers are graceful.*

Proof. (as in [74]) Let F be a firecracker tree, let $P(F)$ have k vertices's and let each of the stars attached to them have $m-1$ vertices's (excluding the vertex in $P(F)$ each of them is attached to, so, m vertices's with it). So, the total number of vertices's of F is km.

An example of graceful labeling of firecracker has been illustrated in the figure 3.19.

We number the verticess on the central path with the numbers $1, 1+(k-1)m, 1+m, 1+(k-2)m, 1+2m, \ldots$ from left to right. Then, we number the centers of the stars, from left to right with the numbers $km, m, (k-1)m, 2m, (k-2)m, \ldots$. Then, the remaining $m-2$ vertices's of each star we number with the following: We start from left to right and

1. if the star we are looking at is the $2i+1-st$ from left to right, then we have numbered

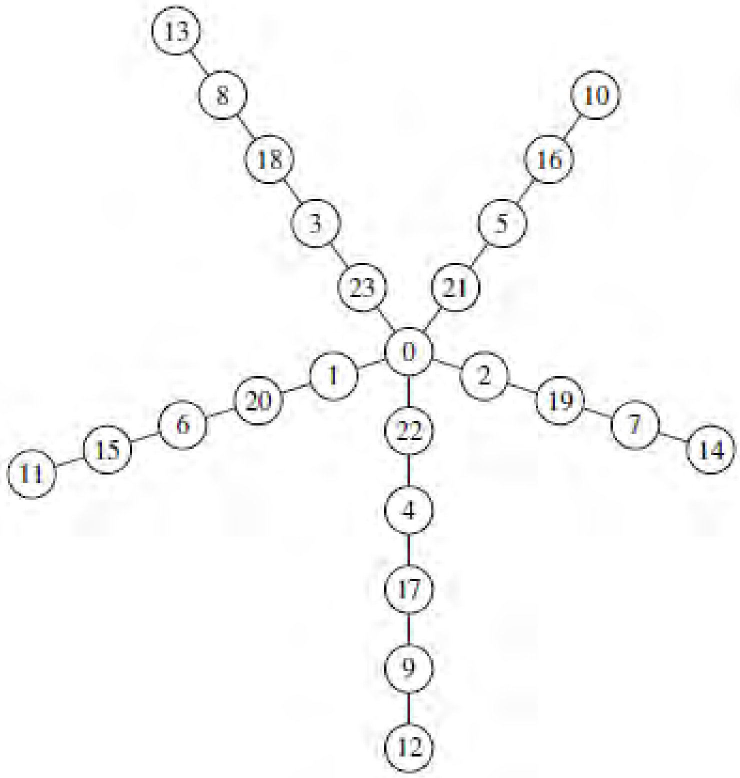

Figure 3.17: The labeling f for $l_0 = 2$, $l_1 = 3$, and $m = 4$.

its center vertex with $(k-i)m$ and the top vertex with $1+im$. Then, we number the remaining $m-2$ vertices's of this star with $2+im, 3+im, \ldots, m-1+im$.

Thus, we get induced edge valuations (since $2i < k$): $(k-2i)m-1, (k-2i)m-2, \ldots, (k-2i)m-(m-1)$.

2. if the star we are looking at is the $2i$th from left to right, then we have numbered its center vertex with im and its top vertex with $1+(k-i)m$. Then, we number the remaining $m-2$ vertices's of this star with $2+(k-i)m, 3+(k-i)m, \ldots, m-1+(k-i)m$.

   Thus, we get induced edge valuations (since $2i < k$): $(k-2i)m+1, (k-2i)m+2, \ldots, (k-2i)m+m-1$.

   Thus, all the edge valuations that we get are different, hence we have exhibited a graceful labeling.

□

A generalized firecracker tree is one in which the stars can have different numbers of ver-

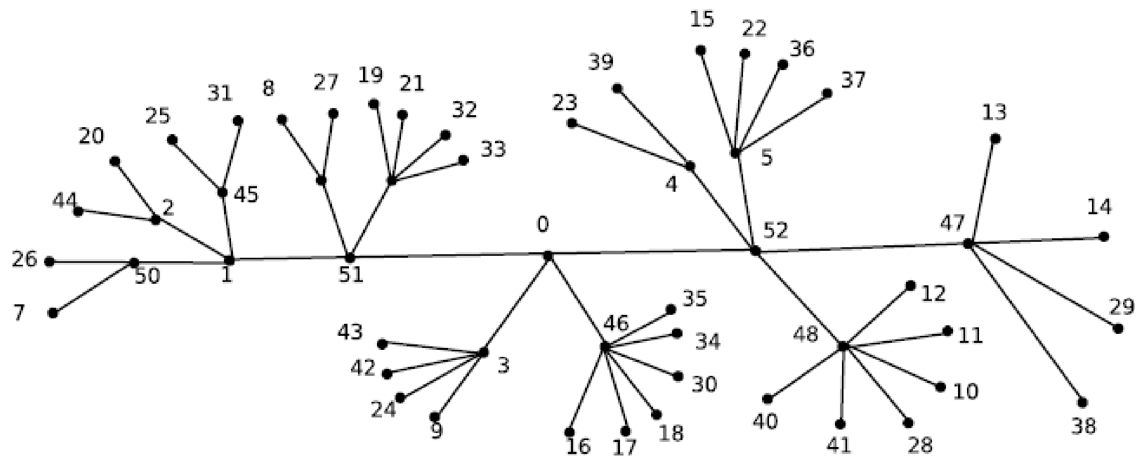

Figure 3.18: An example of gracefully labeled 53-vertex *Lobster*

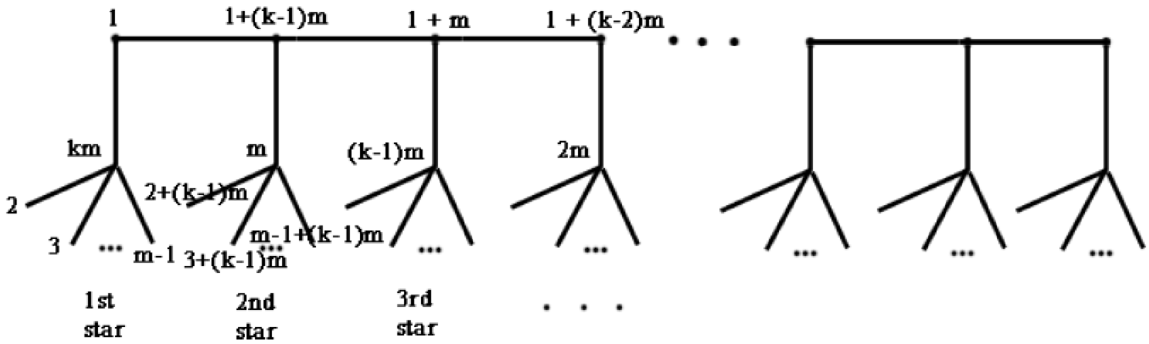

Figure 3.19: A graceful labeling of a firecracker

tices's. According to [26], Chen, Lu and Yeh proved in [25] that all generalized firecracker trees are also graceful.

3.11 Banana Trees

A banana tree consists of a vertex v joined to one leaf of any number of stars. An example of graceful labeling of banana tree has been illustrated in the figure 3.20.

Let $(2K_{1,1},\ldots,2K_{1,n})$ be the tree obtained by adding a vertex to the union of two copies of each of $K_{1,1},\ldots,K_{1,n}$ and joining it to a leaf of each star. The banana tree obtained in this way is interlaced and therefore graceful. Chen, Lu, and Yeh conjectured in [25] that all banana trees are graceful.

Bhat-Nayak and Deshmukh [17] have constructed three new families of graceful banana

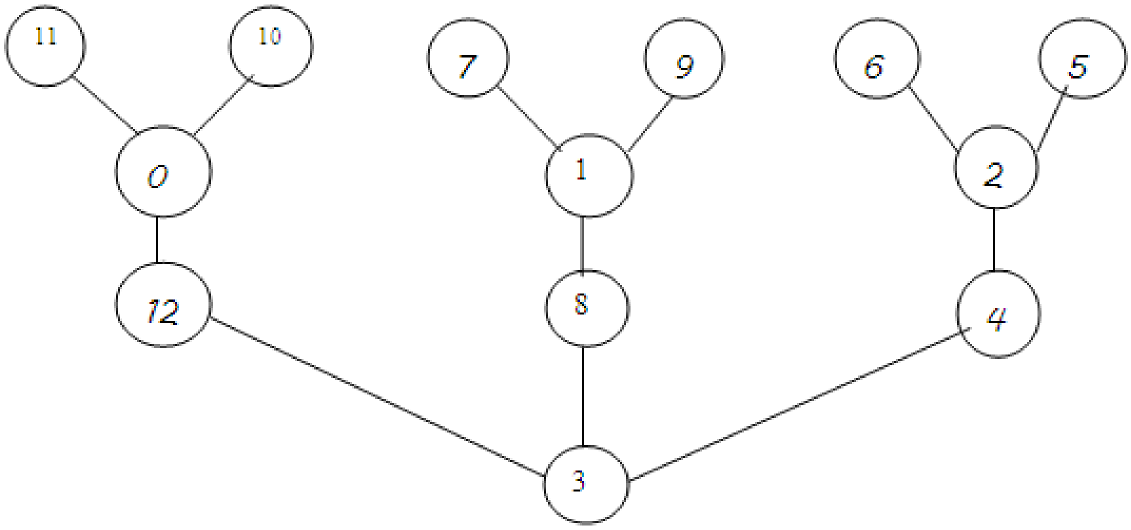

Figure 3.20: An example of graceful labeling of a banana tree

trees using an algorithmic labeling proof. Extending the results of Chen, Lu and Yeh [25], they have shown that the following are graceful:

1. $(K_{1,1},\ldots,K_{1,t-1},(\alpha+1)K_{1,t},K_{1,t+1},\ldots,K_{1,n})$, where $0 \le \alpha < t$;
2. $(2K_{1,1},\ldots,2K_{1,t-1},(\alpha+2)K_{1,t},2K_{1,t+1},\ldots,2K_{1,n})$ where $0 \le \alpha < t$;
3. $(3K_{1,1},3K_{1,2},\ldots,3K_{1,n})$

Moreover, Murugan and Arumugam [67] showed that any banana tree where all the stars have the same size is graceful by constructing a graceful labeling of these banana trees. Note that a banana tree, in which all the stars have the same size is also a symmetrical tree, so, it is also graceful by what we have shown before.

3.12 Regular Bamboo Trees

A regular bamboo tree is a rooted tree consisting of one central vertex, and several legs of equal length attached to it, the leaves of which are identified with leaves of stars of equal size.

As referenced in [26], Regular bamboo trees were shown to be graceful by C. Sekar in [81] in the year of 2002.

V. Ramachandran and C. Sekar in [73] in the year of 2014 has proved Every regular bamboo tree is one modulo N graceful for every positive integer $N > 1$. An example of One modulo 5 graceful labeling of regular bamboo tree, where $k = 5, n = 5, m = 3$ is depicted in the figure 3.21 in the page 39.

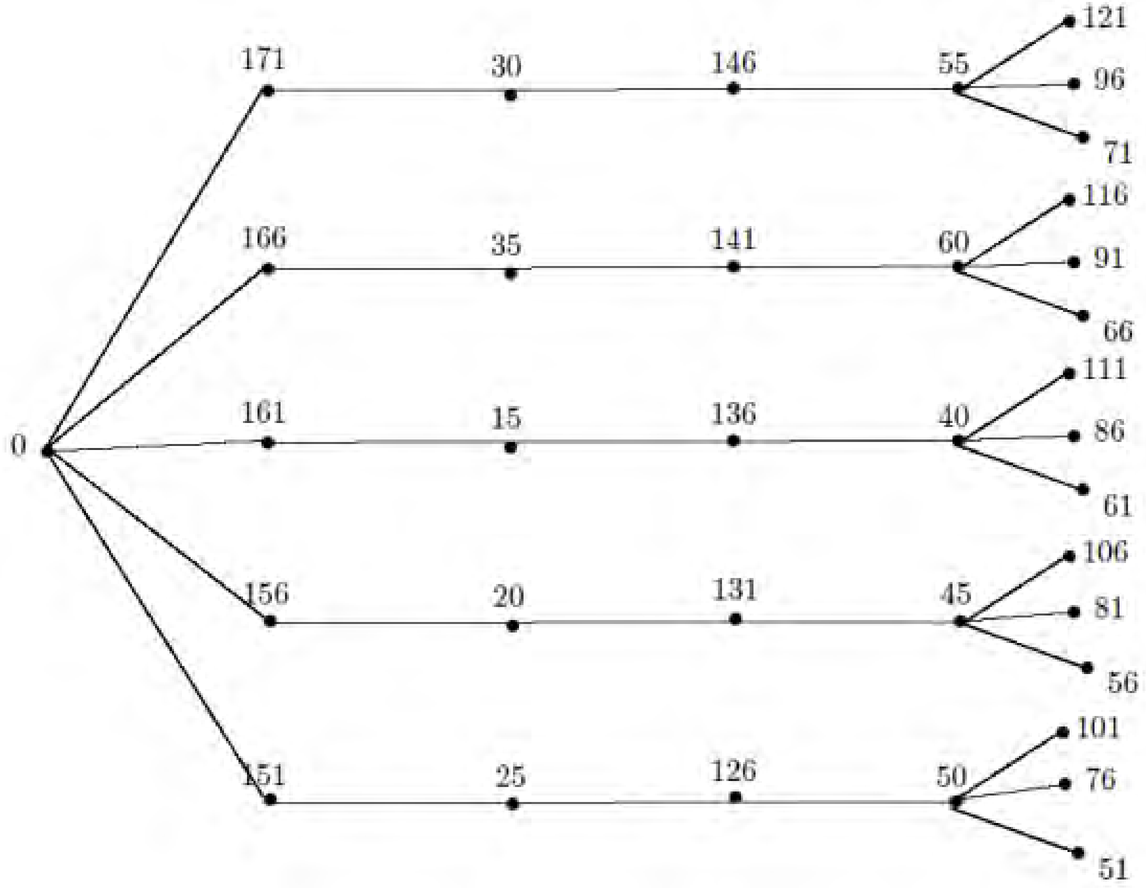

Figure 3.21: An example of One modulo 5 graceful labeling of a *Regular Bamboo tree*

3.13 Coconut Trees

A coconut Tree $CT(m,n)$ is the graph obtained from the path P_n by appending m new pendent edges at an end vertex of P_n.

V. Ramachandran and C. Sekar in [73] in the year of 2014 has proved Every coconut tree is one modulo N graceful for every positive integer N. An example of One modulo 3 graceful labeling of coconut tree is depicted in the figure 3.22 in the page 40.

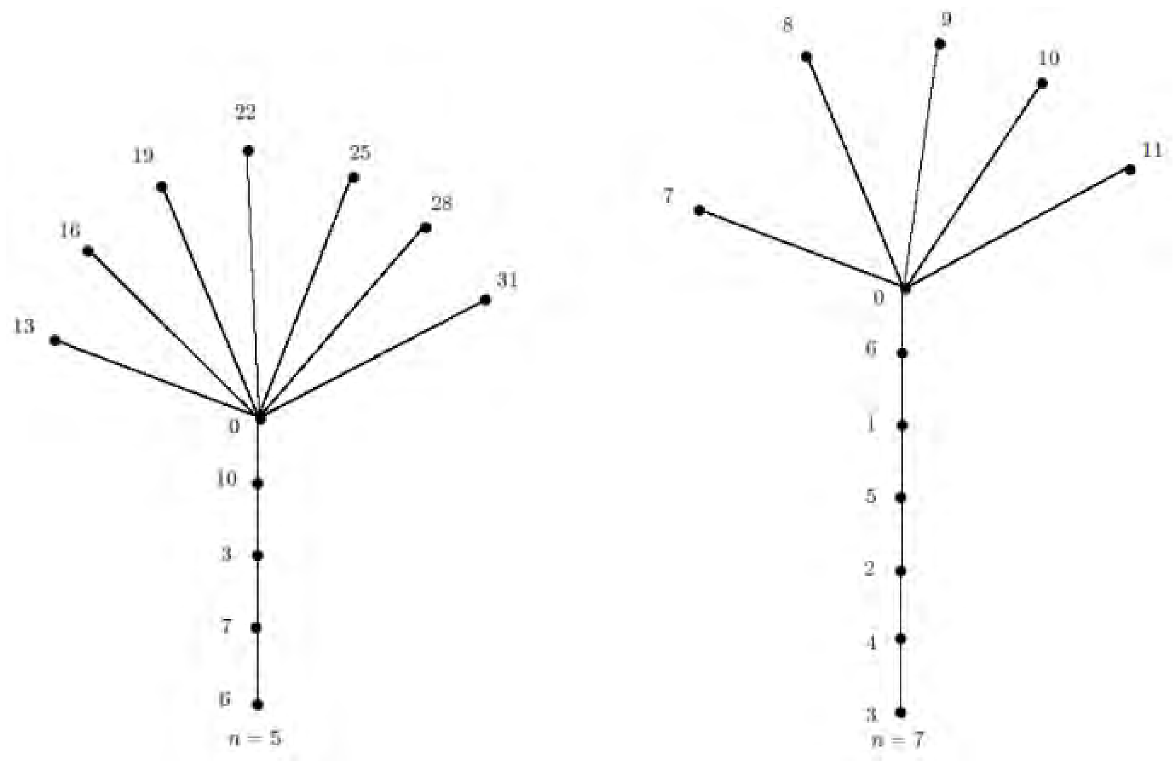

Figure 3.22: An example of One modulo 3 graceful labeling of a *coconut tree*

3.14 Olive Trees

An olive tree T_k is a spider tree with k legs with lengths $1, 2, \ldots, k$ respectively.

As referenced in [26], Abhyankar and Bhat-Nayak in [1] gave direct graceful labeling methods for T_{2n+1} and T_{2n}. Both of these methods involve assigning labels $q = (n+1)(2n+1)$ or n to the roots of the trees T_{2n+1} and T_{2n} respectively and then assigning labels to the vertices's on the k paths adjacent to the root depending on the parity of the path label and the tree in question. Finally, the labels are assigned to the remaining vertices's of the tree so that the sum of any two adjacent vertices's is either $q-1$ or q in the case of T_{2n+1}, or q or $q+1$ in the case of T_{2n}.

Here two example of Olive trees T_5 and T_4 are depicted in the figures in 3.23 and 3.24 in the pages 41 and 42 respectively.

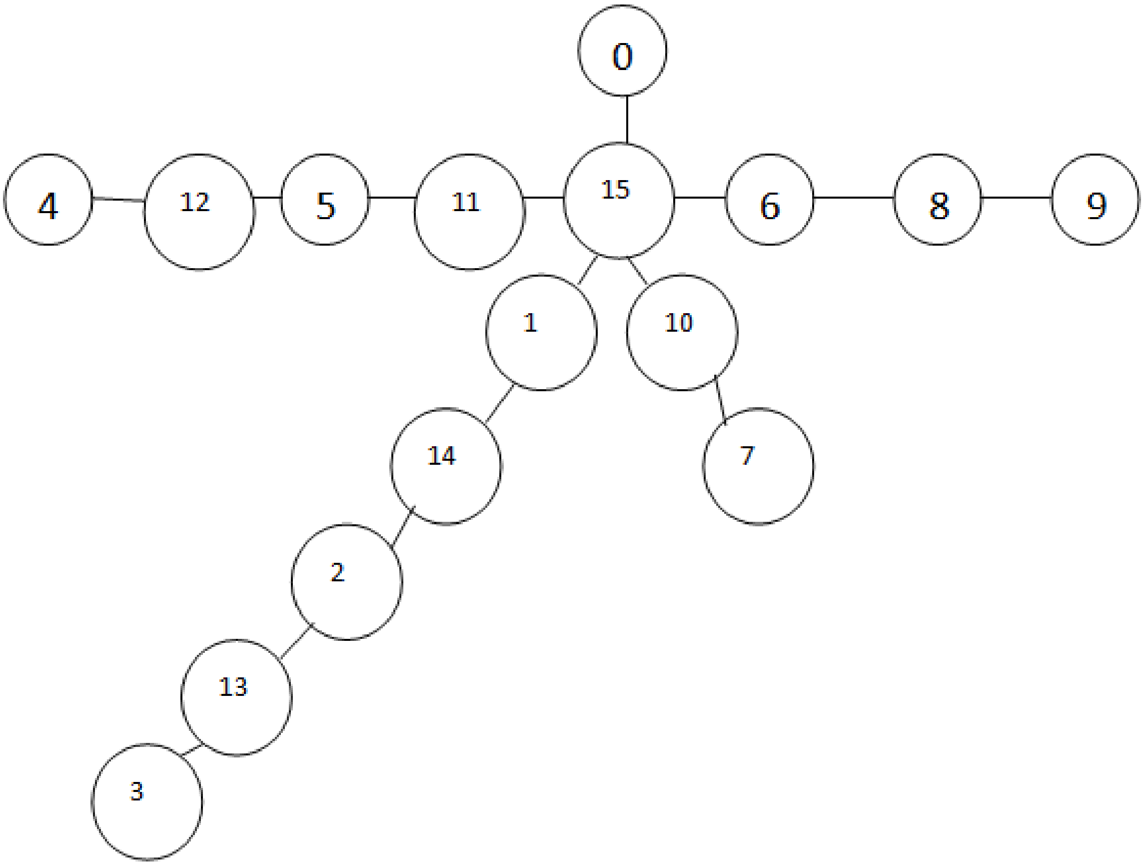

Figure 3.23: An example of *Olive trees* T_5

3.15 Spraying Pipes

A spraying pipe tree is a path $v_1, v2, \ldots, v_n$ such that each vertex v_i is joined to m_i paths at a leaf of each path, and all paths have the same length. Cheng, Lu and Yeh [25] (as referenced in [26]) proved that a spraying pipe tree is interlaced if n is even and $m_{2i-1} = m_{2i}$ for each $1 \leq i \leq \frac{n}{2}$.

3.16 Other Results

3.16.1 Two Theorems on Graceful Labelings of Trees

In this section we follow the definition of the paper [78].

Theorem 13. *A tree* $T = (V, E)$ *has a graceful labeling if* V *has an ordered partition* $\{V_1, V_2, \ldots, V_k\}$ *such that the following two conditions hold.*

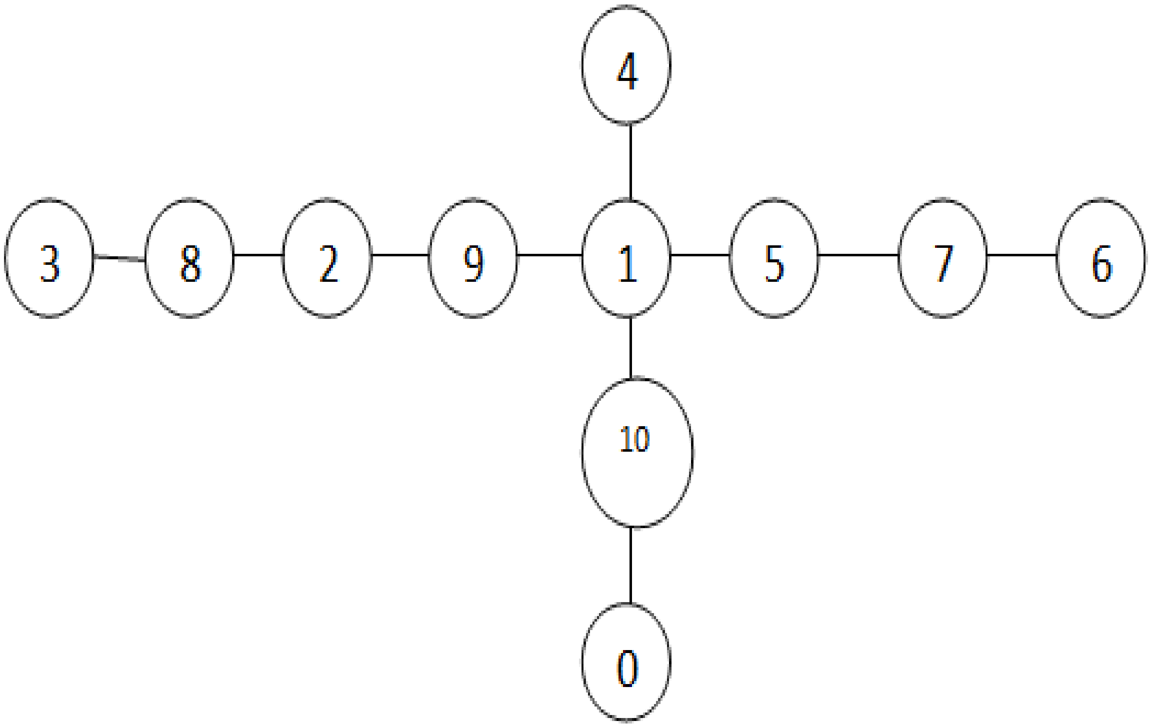

Figure 3.24: An example of *Olive trees* T_4

1. $|V_1| = |V_2| = |V_3| = ... = |V_k|$ *and each* $|V_i|, 1 \leq i \leq k$*, induce a path.*

2. *Exactly one vertex* x *in* $V_i, 1 \leq i \leq k-1$*, has a neighbor* y *in* V_{i+1} *, and for each connected component* I *of* $G(V_i - \{x\})$ *has a corresponding connected component* J *of* $G(V_i - \{y\})$ *such that* I *and* J *have the same number of vertices.*

Theorem 14. *A tree* $T = (V,E)$ *has a graceful labeling if* V *has an ordered partition* $\{V_1, V_2, ..., V_k\}$ *such that the following three conditions hold.*

1. $|V_1| = |V_2| = |V_3| = ... = |V_k|$ *and each* $|V_i|, 1 \leq i \leq k$ *induce a caterpillar.*

2. *For* $1 \leq i, j \leq k$ *and* $i \neq j$*, caterpillars induced by* V_i *and* V_j *are pairwise isomorphic.*

3. *Exactly one vertex* x *in* $V_i, 1 \leq i \leq k-1$*, has a neighbor* y *in* V_{i+1} *, and for each connected component* I *of* $G(V_i - \{x\})$ *has a corresponding connected component* J *of* $G(V_i - \{y\})$ *such that* I *and* J *have the same number of vertices.*

The examples of graceful labeling of a tree satisfying the conditions of Theorem 13 and Theorem 14 are illustrated in figure **??**.

3.16.2 Transformed Trees (T_P-Trees)

A class of tree called T_P-trees (transformed trees) are created by taking a gracefully labeled chain and shifting some of the edges.

$A(p,q)$-graph $G=(V,E)$ is said to be (k,d)-graceful, where k and d are positive integers, if its p vertices's admits an assignment of a labeling of numbers $0,1,2,\ldots,k+(q-1)d$ such that the values on the edges defined as the absolute difference of the labels of their end vertices's form the set $\{k,k+d,\ldots,k+(q-1)d\}$.

Suresh Manjanath Hegde and Sudhakar Shetty in [38] proved that a class of trees called T_P trees and subdivision of T_P trees are (k,d)-graceful for all positive integers k and d.

Theorem 15. *Every T_P tree is (k,d)-graceful for all positive integers k and d.*

Proof. (as in [38]) Let T be a T_P tree with $n+1$ vertices's. By the definition of a T_P tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T))=V(T)$ and (ii) $E(P(T))=(E(T)-E_d)\cup E_P$, where E_d is the set of edges deleted from T and E_P is the set of edges newly added through the sequence $P=(P1,P2,\ldots,P_k)$ of the ept's P used to arrive at the path $P(T)$. Clearly E_d and E_P have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1,v_2,v_3,\ldots,v_{n+1}$ starting from one pendant vertex of $P(T)$ right up to other. The labeling f defined by

$$f(v_i)=\begin{cases} k+(q-1)d-i[(i-1)/2]d \quad \text{for odd i}, & 1\leq i\leq n+1. \\ [(i/2)-1]d \quad \text{for even i}, & 2\leq i\leq n+1. \end{cases}$$

where k and d are positive integers and q is the number of edges of T, $f(v_i)$ is a (k,d)-graceful labeling of the path $P(T)$.

Let v_iv_j be an edge in T for some indices i and j, $1\leq i<j\leq n+1$ and let P_1 be the ept that deletes this edge and adds the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} as also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent epts. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$ it follows that $i+t+1=j-t$ which implies $j=i+2t+1$. Therefore i and j are of opposite parity, i.e., i is odd and j is

even or vice-versa. The value of the edge $v_i v_j$ is given by,

$$g_f(v_i v_j) = g_f(v_i v_{i+2t+1}) = |f(v_i) - f(v_{i+2t+1})|. \tag{3.1}$$

If i is odd and $1 \leq i \leq n$, then

$$\begin{aligned} f(v_i) - f(v_{i+2t+1}) &= k + (q-1)d - [(i-1)/2]d - [((i+2t+1)/2) - 1]d \\ &= k + (q-1)d - (i+t-1)d \end{aligned} \tag{3.2}$$

If i is even and $2 \leq i \leq n$, then

$$\begin{aligned} f(v_i) - f(v_{i+2t+1}) &= [(i/2) - 1]d - [k + (q-1)d] + [(i+2t+1-1)/2]d \\ &= (i+t-1)d - [k + (q-1)d]. \end{aligned} \tag{3.3}$$

Therefore from 3.1, 3.2 and 3.3,

$$g_f(v_i v_j) = |k + (q-1)d - (i+t-1)d|, \quad 1 \leq i \leq n. \tag{3.4}$$

Now

$$\begin{aligned} g_f(v_{i+t} v_{j-t}) &= g_f(v_{i+t} v_{i+t+1}) \\ &= |f(v_{i+t}) - f(v_{i+t+1})| \\ &= |k + (q-1)d - (i+t-1)d|, 1 \leq i \leq n. \end{aligned} \tag{3.5}$$

Therefore from 3.4 and 3.5,

$$g_f(v_i v_j) = g_f(v_{i+t} v_{j-t}). \tag{3.6}$$

Hence f is a (k,d)-graceful labeling of T_P -tree T. The proof is complete.

□

For example, a $(1,1)$-graceful labeling of a T_P -tree T using Theorem 15, is shown in figure 3.25 in the page 45.

Theorem 16. *If T is a T_P -tree with q edges then the subdivision tree $S(T)$ is (k,d)-graceful for all positive integers k and d.*

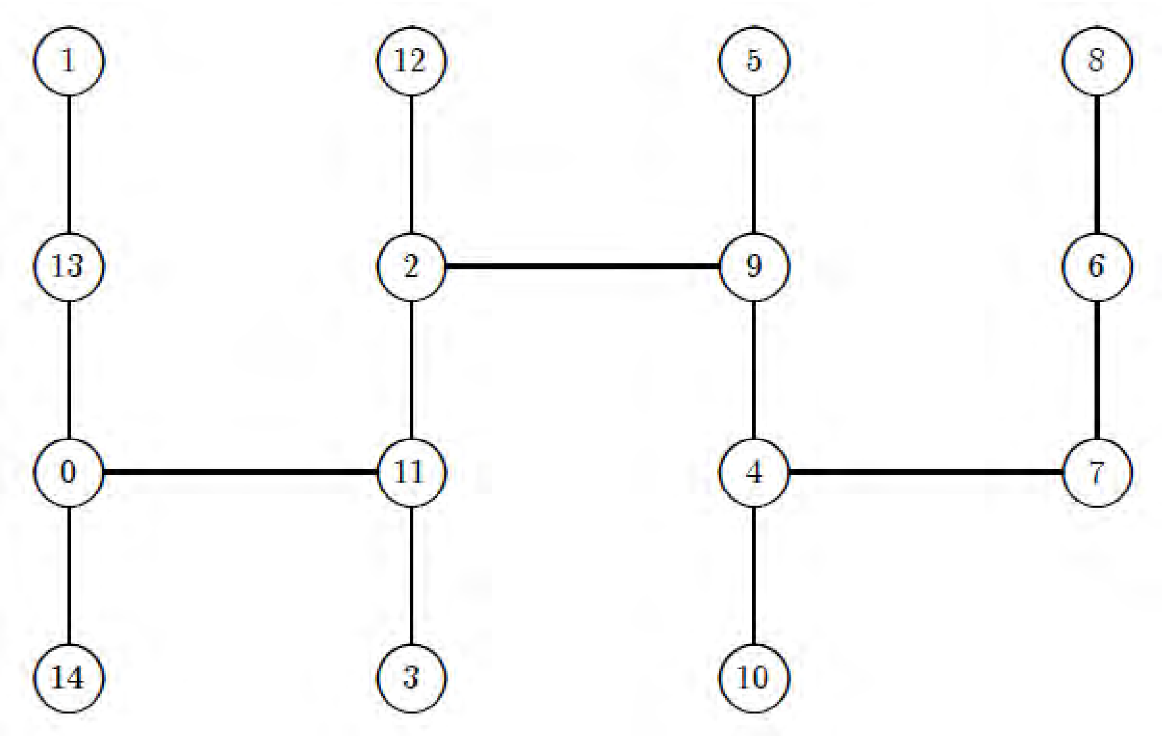

Figure 3.25: A graceful labeling of a T_P -tree using theorem 15

Proof. If anyone interested to know the proof of this theorem they can find it in [38]. We intentionally omit it here since it is very long and technical. □

For example, a $(1,1)$-graceful labeling of subdivision of a T_P -tree using theorem 16, is shown in Figure 3.26 in the page 46.

3.16.3 Trees of Diameter at Most Five

The example of Trees with diameter 2 are star trees and they are instances of caterpillars, hence they are graceful. Rosa proved that trees of diameter at most three are graceful. In 1989 Zhao [98] showed that all trees of diameter four are graceful. In 2001, Hrnciar and Haviar [42] showed that all trees of diameter five are graceful.

Let T be a tree and let $uv \in E(T)$. Then, $T_{u,v}$ is the sub tree of T induced by the set $V(T_{u,v}) = \{w \in (T) : w = u$ or v is in a $u-w$ path$\}$.

Hrnciar and Haviar use ransformations in their paper [42]. They use two types of transfers of end-edges. A $u \to v$ transfer is a transfer of the first type if the end vertices's of the transferred end edges have labels $k, k+1, \ldots, k+m$ for some k and m. A $u \to v$ transfer is

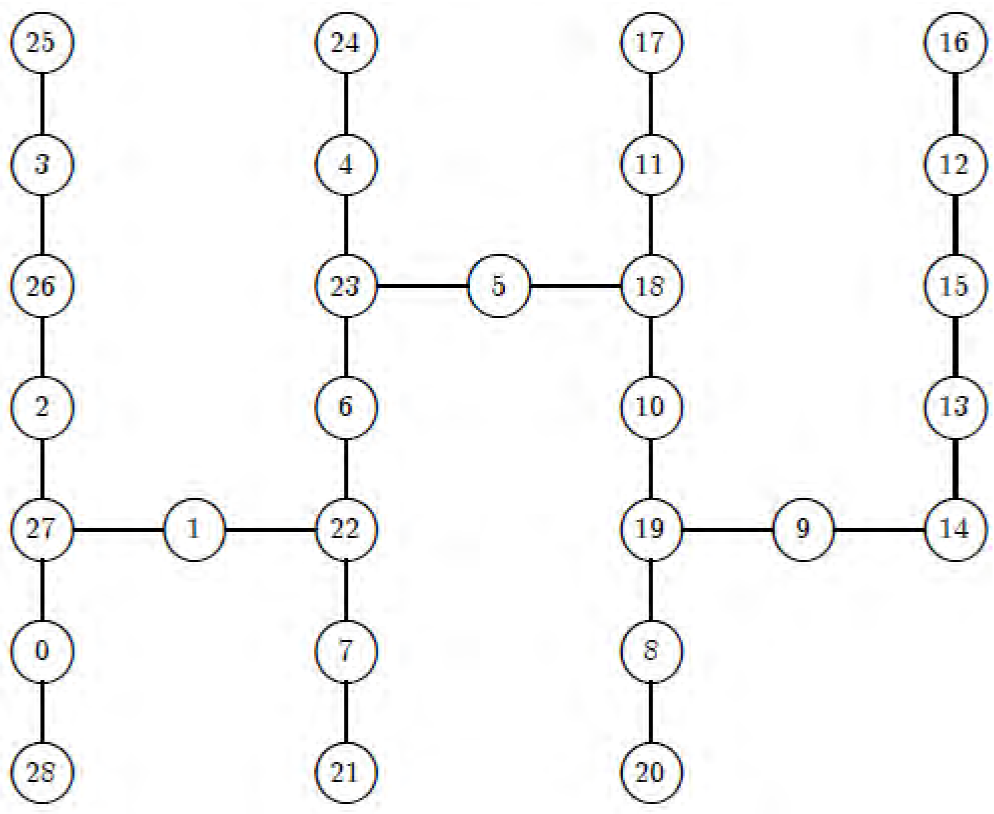

Figure 3.26: A graceful labeling of subdivision of a T_P -tree using theorem 16

a transfer of the second type if the labels of the end vertices's of the transferred end edges form two sections $k, k+1, \ldots, k+m$ and $l, l+1, \ldots, l+m$ for some k, l, m.

Transfers of the first type work if $f(u) + f(v) = k + (k+m)(= k+1+(k+m-1) = k+2+(k+m-2) = \ldots)$. Transfers of the second type work if $f(u) + f(v) = k+l+m(= k+1+(l+m-1) = k+2+(l+m-2) = \ldots)$.

Theorem 17. *Every tree of diameter 4 is graceful.*

Proof. (as appears in [42]) It is sufficient, to prove that every tree T of diameter 4 having the central vertex of an odd degree has a graceful labeling such that the label of the central vertex is maximal.

Let x be the number of vertices's of an even degree that are adjacent to the central vertex of T. Let y be the number of vertices's of odd degree greater than 1 that are adjacent to the central vertex of T. Let the degree of the central vertex of T be $2k+1$ and let T have n edges. We can obtain a graceful labeling of T starting with the tree in the figure on the right in the figure above by carrying out the following transfers:

$$0 \to n-1 \to 1 \to n-2 \to 2 \to n-3 \to \dots$$

where the first x transfers are of the first type and the next $y-1$ (if $y > 1$) transfers are of the second type (to get the desirable sets of end edges of even cardinality).

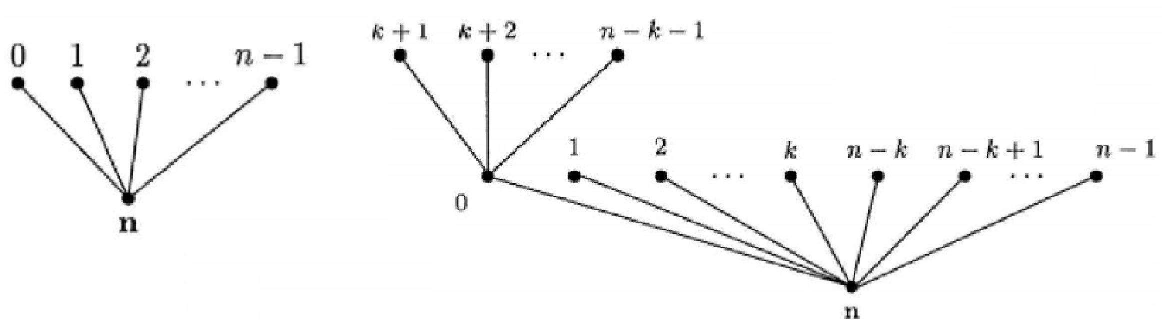

Figure 3.27: An example of Transformation of trees

□

Now we are going to look at trees of diameter 5. Hrnciar and Haviar in [42] first show using the previously explained methods that every tree with diameter 5 is "nearly" graceful and after that they prove the main result.

Therefore, they assume T be a tree of diameter 5 and it has two central vertices's which they denote by a and b. Let x be a vertex adjacent to the central vertex a such that $x \neq b$. The sub-tree $T_{a,x}$ is a branch (at the vertex a) if $T_{a,x}$ is a sub-tree of diameter 2. A branch $T_{a,x}$ is an odd branch if the degree of the vertex x is even, otherwise, $T_{a,x}$ is an even branch. Similarly, they define even and odd branches $T_{b,y}$ adjacent to b.

Now, let $p =$ No of odd branches at a, $r =$ No of even branches at a, and i = No of endedges at a. Similarly, let q = No of odd branches at b, s = No of even branches at b, and j = No of endedges at b. The graceful labelings defined in the sequel depend on those cardinality, mostly on their parties. In fact, Hrnciar and Haviar in [42] introduced the following notation: for example $(p,r,i;q,s,j) \equiv (e,o,o;e,e,e)$ if p,q,s,j are even and $r,i$are odd.

Theorem 18. *Every tree T of diameter 5 is graceful or nearly graceful, i.e. if the cardinality of its edge set is n, then, there exists a vertex labeling with the numbers from 1 to n such that the cardinality of the induced edge labeling is either $n-1$ or $n-2$, i.e. at most 2 edges have the same label.*

Theorem 19. *Every tree of diameter* 5 *is graceful.*

The proof of this theorem looks at a number of cases and exhibits specific transfers which give graceful or nearly graceful labelings of the given tree. They produce graceful labelings via the transfers defined in the beginning of this section. If anyone interested to know the proof of this theorem they can find it in [42]. We intentionally omit it here since it is very long and technical.

3.16.4 A Class of Graceful Diameter-6 Trees

Matthew C. Superdock in [91] surveyed the current state of progress on the Graceful Tree Conjecture, and then they present several new results toward the conjecture, driven by three new ideas:

1. It has been proven that generalized banana trees are graceful by rearranging the branches at the root.
   - Consider rearranging branches at all internal vertices's.
2. The method of transfers has typically involved type-1 transfers and type-2 transfers.
   - All type-2 transfers are type-1 transfers in disguise, and hence can be removed from the discussion.
3. The method of transfers has typically used the sequence of transfers

$$0 \to n \to 1 \to n-1 \to 2 \to n-2 \to \dots$$

   - Transfer backwards to manipulate the resulting labels.

Using these ideas, author proved that several classes of diameter-6 trees are graceful, and therefore generalize some of these classes to larger trees.

Theorem 20. *Let T be a rooted diameter-6 tree, with central vertex and root v, with the following properties:*

- *The vertex v, and all vertices's of distance* 1 *from v, have an odd number of children.*

- *All leaves have distance* 3 *from v.*

Then T has a graceful labeling f with $f(v) = 0$.

Proof. The proof of this theorem are very long and lengthy. Therefore we have omit it here. One can find the proof of this theorem in [91] for better understanding. □

Theorem 21. *Let T be a rooted diameter-6 tree, with central vertex and root v, with the following properties:*

- *The vertex v, and all vertices's of distance* 1 *from v, have an odd number of children.*
- *All leaves have distance* 2 *or* 3 *from v, such that*
    - *No two leaves of distance* 2 *from v have the same parent.*
    - *Each leaf of distance* 2 *from v has a sibling with an even number of children.*

Then T has a graceful labeling f with $f(v) = 0$.

Proof. As above, with the following adjustments:

- Remove only the leaves of T with distance 3 from v to get S.

The rest of the proof of this theorem is same as above theorems proof. Therefore we have omit it here. One can find the proof of this theorem in [91] for better understanding. □

Theorem 22. *Let T be a rooted diameter-2r tree, with central vertex and root v, with the following properties:*

- *The vertex v, and all vertices's of distance at most $r-2$ from v, have an odd number of children.*
- *The number of vertices of distance $r-1$ from v, with an even number of children, is not* 3 *(mod* 4*).*
- *All leaves have distance r from v.*

Then T has a graceful labeling f with $f(v) = 0$.

Chapter 4

New Classes of Graceful Trees

In the last 50 years, a lot of effort have been given by researchers to prove Graceful Tree Conjecture. Although the conjecture has not been proved for all trees researchers have been continuing their quest for discovering new classes of graceful trees. We have also joined the quest and proved the conjecture for Superstars and Extended Superstars.

4.1 Superstar

Definition 23. Let S_k be a tree with k leaves joined to a central vertex. Then S_k is said to be a star.

We know stars can be gracefully labelled by labeling its centre by 0 and other leaves by r where $r = 1,...,k$.

Definition 24. Let a tree T consist of stars $S(i,k_i)$, $i\{i = 0,1,\ldots,I\}$ with k_i leaves. Each $Si,k_i,i = 1,...,I$ shares exactly one leaf with $S0,k_0$. This $S0,k_0$ is called the root star whereas $Si,k_i,i = 1,...,I$ are called leaf stars. Then T is said to be a superstar denoted SS.

Example 25. An example of Superstar is illustrated in Figure 4.2 where $m = 34$, $I = 6$ and $l_0 = MAX_k$ and $MAX_k = 5$.

Example 26. Another example of Superstar is illustrated in Figure 4.3 in page 53 where $m = 32$, $I = 5$ and $l_0 = I$.

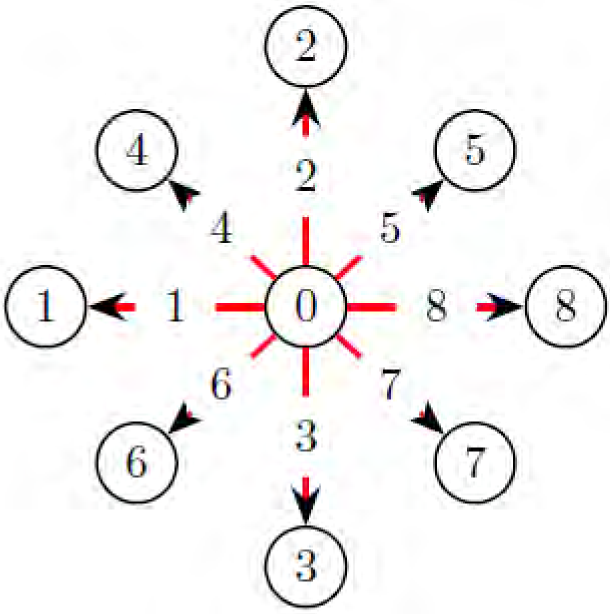

Figure 4.1: An example of gracefully labelled star.

4.2 Labeling of Superstars

First of all let us discuss how to label a superstar gracefully. The idea is to start labeling leaf stars consecutively generating edge labels from m sequentially downwards. We must ensure that in shifting from one leaf star to the second edge connecting vertex common with the root leaf star must have the immediate next edge label. For this we must label centres of stars by smallest possible labels and leafs should be labelled with the largest possible labels. Let centre of $S_(i,k_i)$ be labelled by l_i so that $l_0 - l_i \geq k_i$. This necessitates that the center vertex of root star is labelled by $l_0 = \min\{I, MAX_k\}$ where $MAX_k = \max_{i \in I}\{k_i\}$. Then centres of other stars(leaf stars $S(i,k_i)$) will be labelled by $l_i = 0,...,I$ bypassing l_0.

If $l_0 = I$ then we have to find an arbitrary leaf star i who satisfies the condition, $s_i \geq l_0 - i$ and label the center vertex of the leaf star by i. In this way, we have to label all the center vertex of the leaf stars. Now we have to label the leaves of all leaf star starting from the leaf star whose center vertex got the label 0 then from $1,...,I-1$. During the process of labeling leaves of leaf star we have to first label the shared leaf of root star and leaf star. Let $j = m$ and s_i is the number of leaves of the star whose centre has been labelled l_i. The shared leaf will be labelled by $j - i - s_i + l_0$ and other leaves will be labelled from j down to $j - s_i + 1$. Therefore, the label of edges will be produced from $j - i, ..., j - s_i + 1$ for each leaf star and the edge which incident to the center vertex of the root star will got label $j - s_i - i$. However,

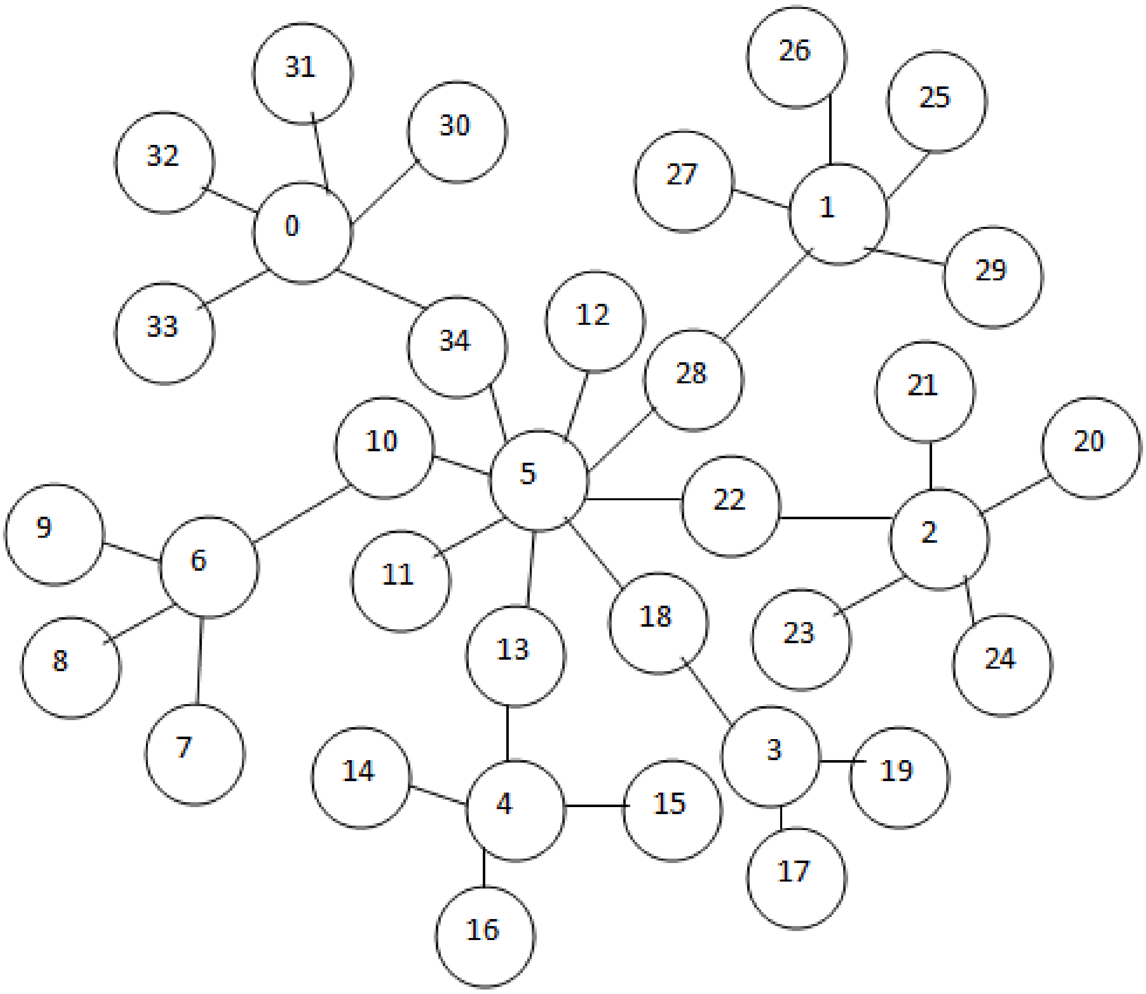

Figure 4.2: An example of gracefully labelled Superstar where $m = 34$, $I = 6$ and $l_0 = MAX_k$.

we can say that we are producing edge label in descending order that is $j, \ldots, 1$. As we have done labeling of all leaf star, now we have to label the unshared leaf of root star from j down to $I+1$.

If $l_0 = MAX_k$ then for $i = 0, \ldots, MAX_k - 1$, we have to find an arbitrary leaf star i who satisfies the condition, $s_i \geq l_0 - i$ and label the center vertex of the leaf star by i and for $i = MAX_k + 1, \ldots, I$, find an arbitrary leaf star i who satisfies the condition, $k_i \geq l_0 - i$ and label the center vertex of the leaf star by i. Now we have to label the leaves of all leaf star starting from the leaf star whose center vertex got the label 0 then from $1, \ldots, MAX_k - 1$. During the process of labeling leaves of leaf star we have to first label the shared leaf of root star and leaf star. The shared leaf will be labelled by $j - i - s_i + l_0$ and other leaves will be labelled from j down to $j - s_i + 1$. Therefore, the label of edges will be produced from $j - i, \ldots, j - s_i + 1$ for each leaf star and the edge which incident to the center vertex

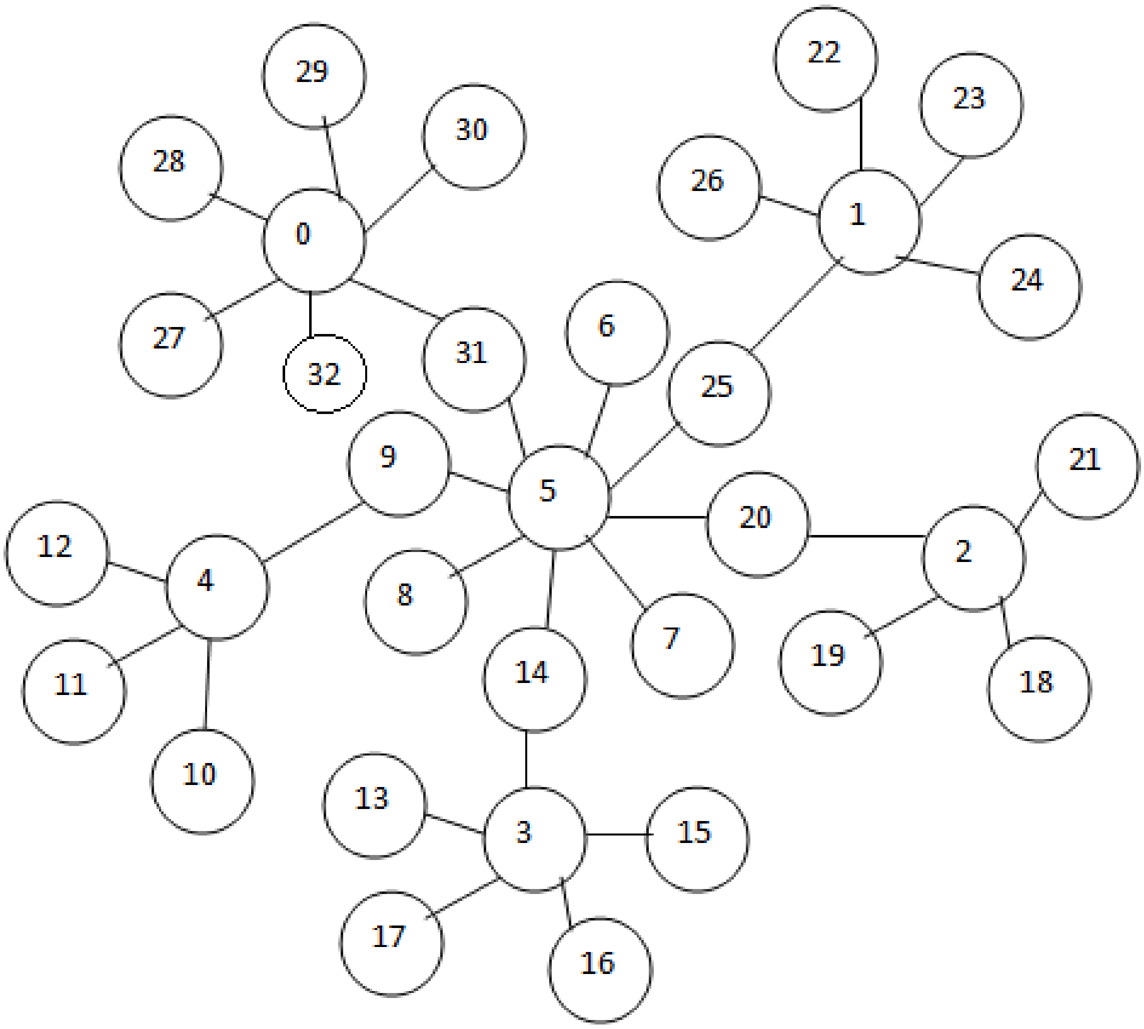

Figure 4.3: An example of gracefully labelled Superstar where $m = 32$, $I = 5$ and $l_0 = I$.

of the root star will got label $j - s_i - i$. Now we have to label the unshared leaf of root star by j down to $j-$total number of unshared leaf of root star. After that, we have to label the leaves of all leaf star starting from the leaf star whose center vertex got the label from $i = MAX_k + 1, ..., I$. We have to first label the shared leaf of root star and leaf star. The shared leaf will be labelled by $j - C$ where $C = 0, 1, ..., I - MAX_k$ and other leaves will be labelled from j down to $j - s_i + 1$. Therefore, the label of edges will be produced from $j - i, ..., j - s_i - i + 1$ for each leaf star and the edge which incident to the center vertex of the root star will got label $j - l_0$. However, we can say that we are producing edge label in descending order that is from j down to 1.

Example 1

The steps of graceful labeling of superstar for the example 4.4 which are illustrated in the figures 4.5,4.6,4.7 and 4.8 where $m = 18$ and $I = 4$.

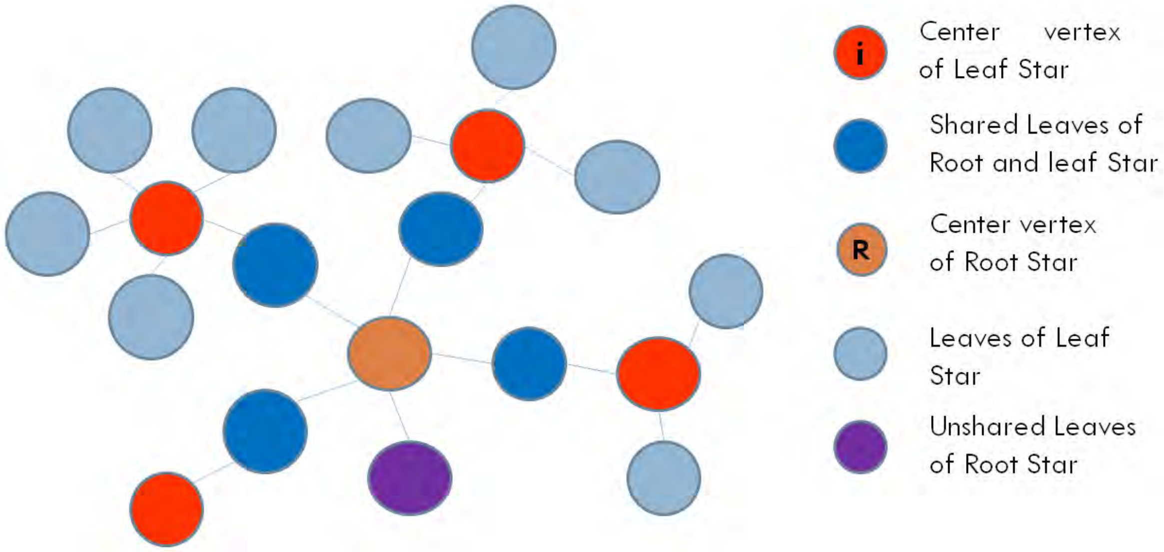

Figure 4.4: An example of Superstar.

Step 1: First of all we have to find out the value of I and MAX_k and the center vertex of root star, l_0 will be labeled by $\min\{I, MAX_k\}$. As for the above example $I = 4$ and $MAX_k = 5$, therefore, $l_0 = 4$ which is illustrated in the figure 4.5.

Step 2: Next we have to find an arbitrary star, i which satisfies the condition $k_i \geq l_0 - i$ where k_i is the number of leaves of ith star and have to label the center vertex of all the leaf stars by l_i where $l_i = 0, 1, ..., I-1$. Therefore, we have labelled all the center vertex of leaf star by $0, 1, 2, 3$ which is illustrated in the figure 4.6.

Step 3: Now we have to label the leafs of all star starting from the star whose center vertex is labelled by 0 and then $1, 2, 3, 4$. Therefore, First we have to label the shared vertex by $j - l_i - k_i + l_0$ and then label rest of the leafs starting from $j = m$ that means $j = 18$ down to $j - k_i + 1 = 14$ of all the leafs of star i for which $l_i = 0$ which is illustrated in the figure 4.7. After labeling ith star the value of j will be $j - k_i$ and i will be incremented by 1.

Step 4: Now we have labelled the leafs of star i for which $l_i = 1, 2, 3$ which is illustrated in the figure 4.8. If we follow the procedure to label all the vertices then we got the edge label in descending order starting from $m, ..., 1$ consecutively. In this way we have labelled

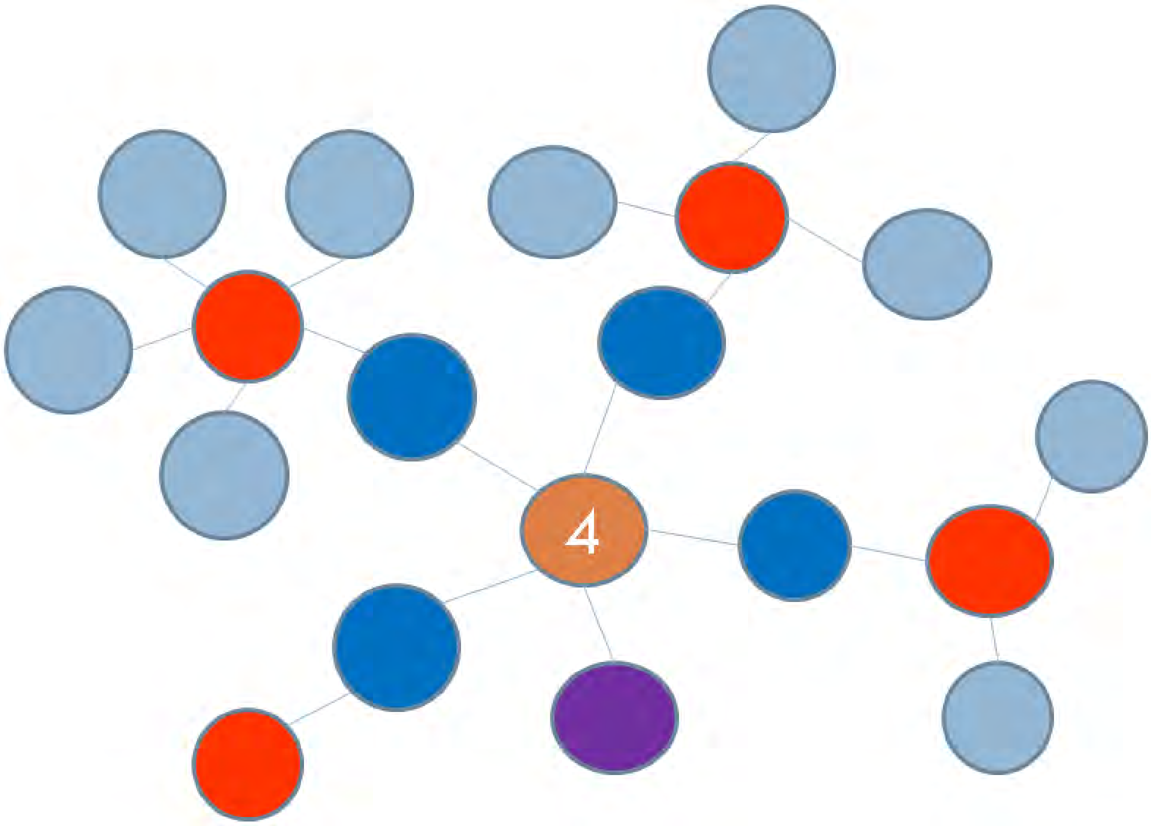

Figure 4.5: An example of Superstar to be labelled gracefully where $l_0 = 4$.

the superstar gracefully.

Example 2

The steps of graceful labeling of superstar for the example 4.9 which are illustrated in the figures 4.10,4.11,4.12,4.13 and 4.14 where $m = 23$ and $I = 5$.

Step 1: In the same way explained in the example 1 in the section 4.2 we have to label $l_0 = \min\{I, MAX_k\}$. Therefore, we have labelled $l_0 = 4$ which is illustrated in the figure 4.10 where $I = 5$ and $MAX_k = 4$.

Step 2: Now we have to label the center vertex of the leaf star in the same way explained in the example 4.2 for $0,\ldots,3$ and then bypasses 4 the next stars center vertex will be labelled by 5 as 4 has already been used to label center vertex of root star. The entire situation is illustrated in the figure 4.11.

Step 3: In this step, we have to label the leafs of star i for which $l_i = 0,\ldots,3$ using previously explained technique.Therefore, First we have to label the shared vertex by $j - l_i - k_i + l_0$ and

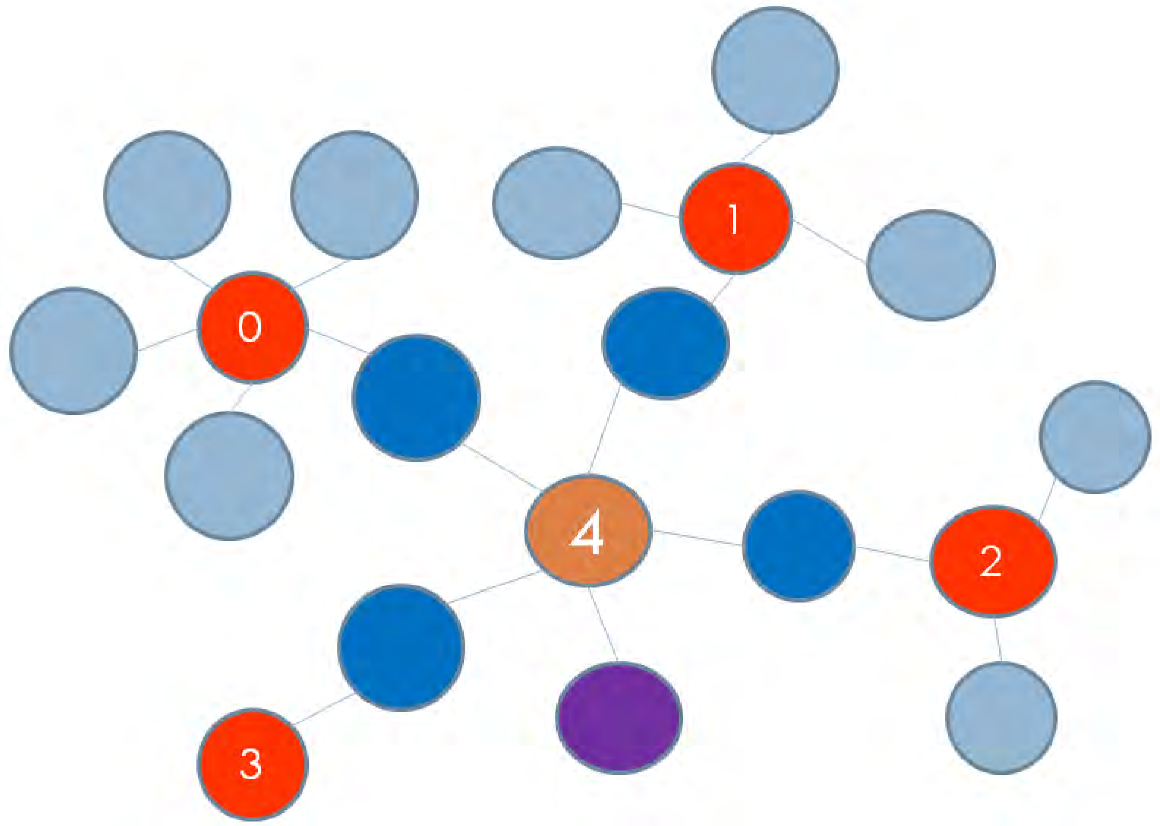

Figure 4.6: An example of Superstar to be labelled gracefully where $l_i = 0, 1, 2, 3$.

then label rest of the leafs starting from $j = m$ that means $j = 23$ down to $j - k_i + 1 = 20$ of all the leafs of star i for which $l_i = 0$ which is illustrated in the figure 4.12. After labeling ith star the value of j will be $j - k_i = 19$ and i will be incremented by 1. The next stars leaf will be labelled starting from $j = 19$. This process will be continued up to $l_i = 3$ which is illustrated in the figure 4.13.

Step 4: In this step first we have labelled the unshared leaf of root star by $j = 9$.

Step 5: For the ith star for which $l_i = 5$, the shared vertex will be labelled by $j - C = 8$ where $C = 0, \ldots, I - MAX_k - 1 = 0$ and rest of the leafs will be labelled by $j = 7$ down to 6. The entire situation is illustrated in the figure 4.15.

If we follow the above explained procedure to label all the vertices then we got the edge label in descending order starting from $m, \ldots, 1$ consecutively. In this way we have labelled the superstar gracefully.

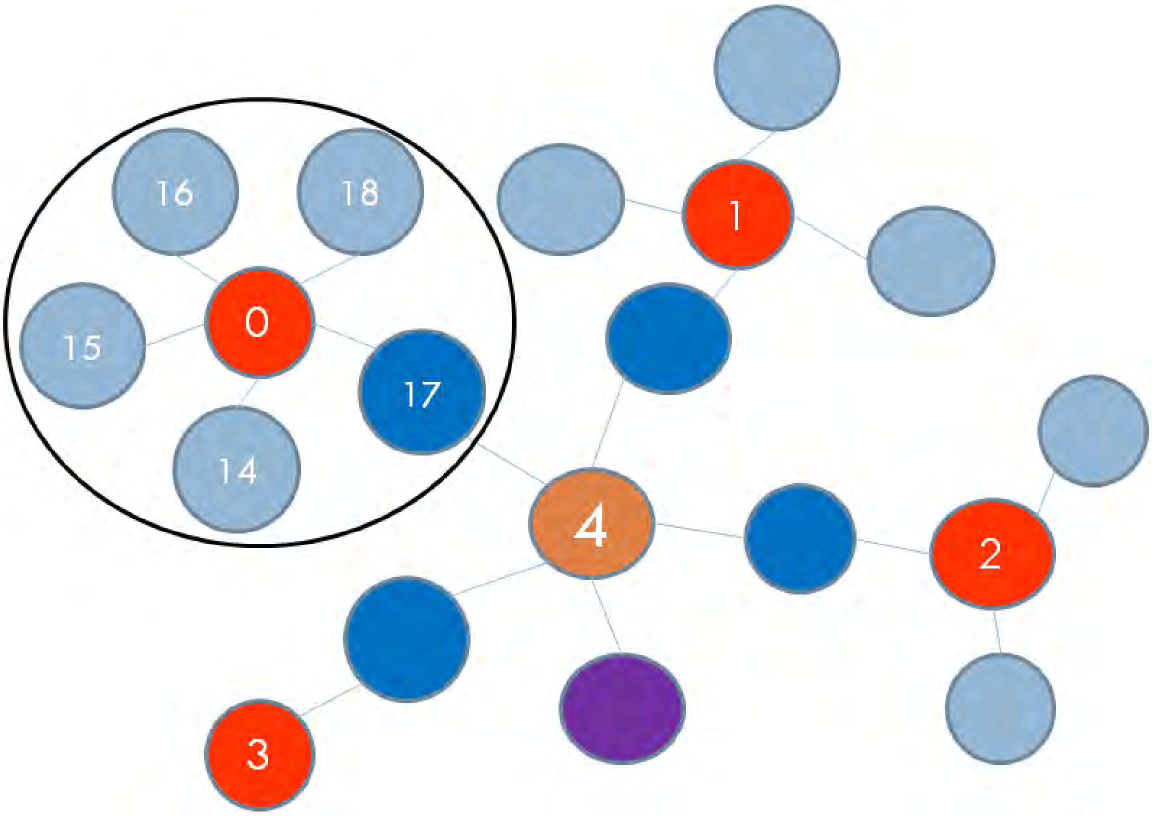

Figure 4.7: An example of Superstar to be labelled gracefully where the leafs of the star for which $l_i = 0$ are labelled.

4.2.1 Algorithm

Algorithm 1 Superstar

$I+1 \leftarrow$ total number of star of a Superstar, where $i = 0, 1, \ldots, I$

$m \leftarrow$ total number of edge of a Superstar

$k \leftarrow$ number of leaf of each star, $S_{i,k}$

$S_{i,k} \leftarrow$ leaf star of a Superstar

$SS_i \leftarrow$ root star

$SS_{ik} \leftarrow$ number of leaf of root star

$Max_k \leftarrow 0$

$l_0 \leftarrow$ root of SS_i

$C = 0$, where $C = 0, 1, ..., I - MAX_k$

for $i = 0$ to I **do**

  Count k_i for each $S_{i,k}$

  if $Max_k < k_i$ **then**

    $Max_k = k_i$

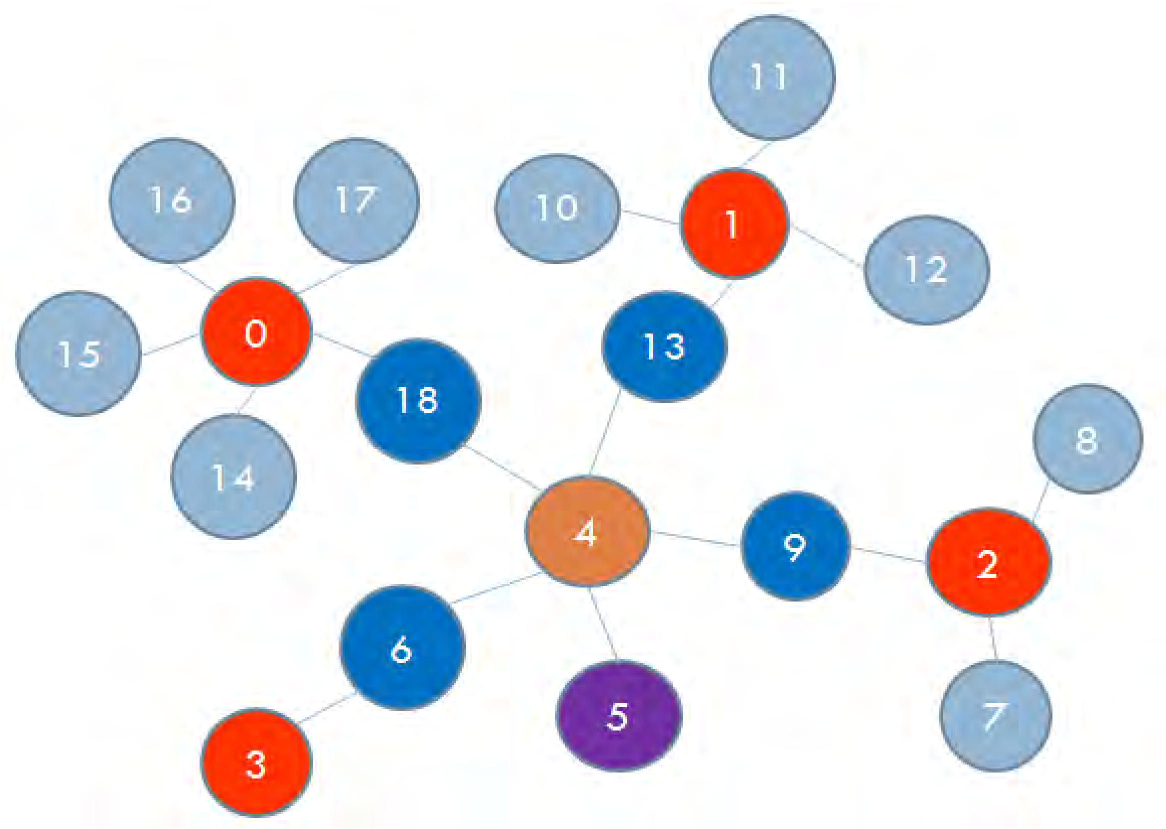

Figure 4.8: An example of gracefully labelled Superstar.

Algorithm 1 Superstar continued...

  end if
end for
if $Max_k > I$ **then**
  $l_0 = I$
  for $i = 0$ to $I - 2$ **do**
    repeat
      Find $S_{i,k}$
    until $k_i \geq l_0 - i$
    Root of $S_{i,k} = i$
    Connect l_0 to the leaf of $S_{i,k}$ labeled with $m - i - k_i + l_0$
    for $m + 1$ down to $m - k_i + 1$ **do**
      Leaf of $S_{i,k} = m - 1$
      $m = m - 1$
    end for
  end for
  for m down to $m - I$ **do**

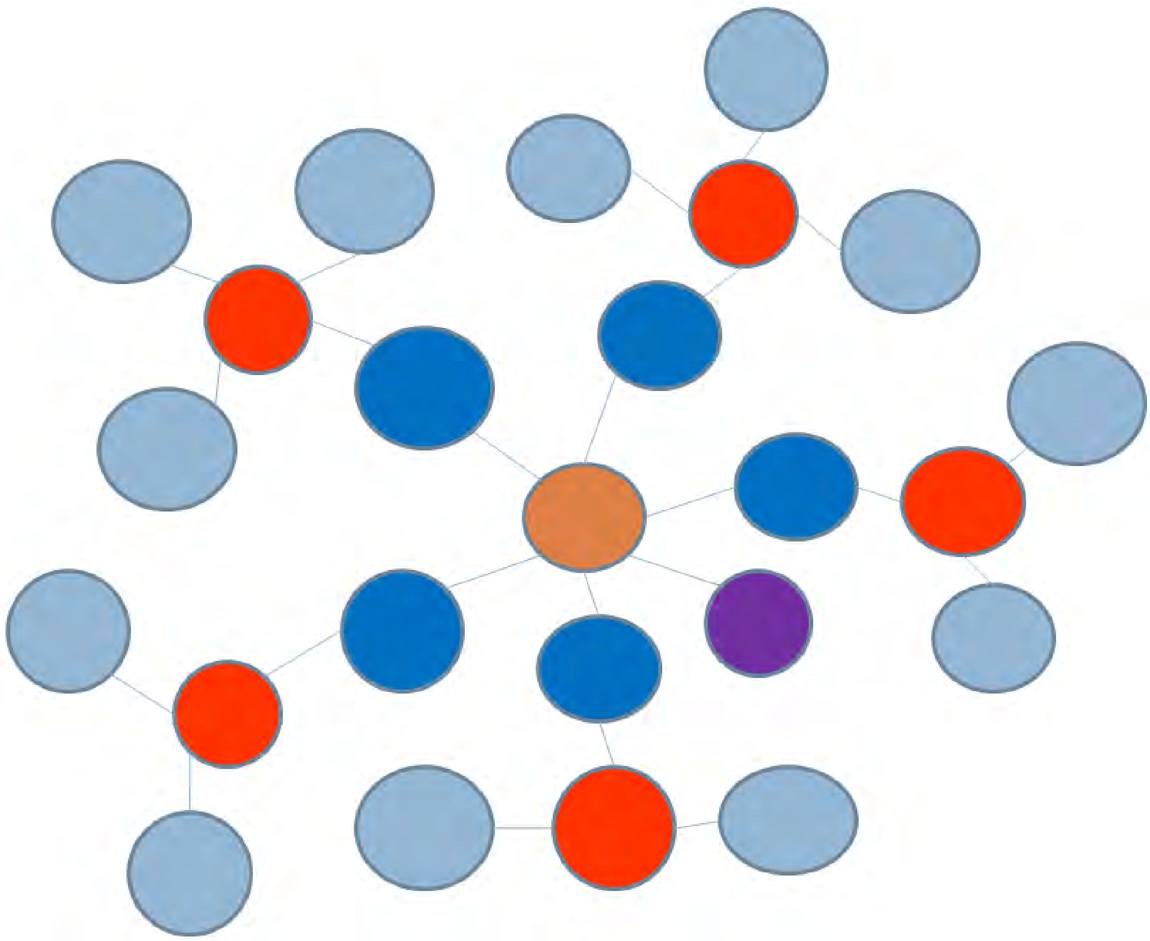

Figure 4.9: An example of Superstar.

Algorithm 1 Superstar continued...

Leaf of $SS_i = m - 1$

$m = m - 1$

end for

else

$l_0 = Max_k$

for $i = 0$ to $Max_k - 1$ **do**

repeat

Find $S_{i,k}$

until $k_i \geq l_0 - i$

Root of $S_{i,k} = i$

Connect l_0 to the leaf of $S_{i,k}$ labeled with $m - i - k_i + l_0$

for $m + 1$ down to $m - k_i + 1$ **do**

Leaf of $S_{i,k} = m - 1$

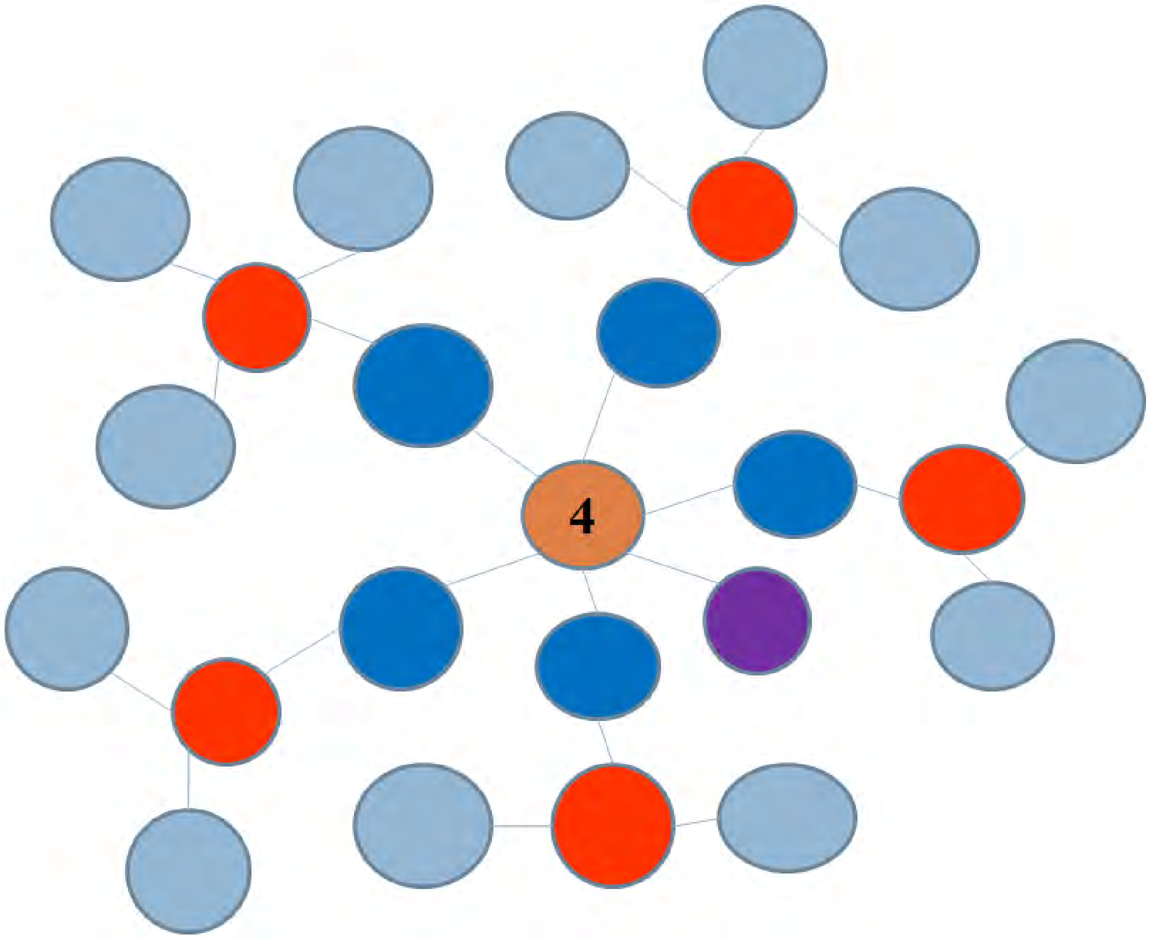

Figure 4.10: An example of Superstar to be labelled gracefully where $l_0 = 4$.

Algorithm 1 Superstar continued...

$m = m - 1$
end for
end for
for m down to $m - SS_{i,k}$ **do**
Unshared Leaf of $SS_i = m - 1$
$m = m - 1$
end for
for $i = Max_k + 1$ to I **do**
repeat
Find $S_{i,k}$
until $l_0 - i \geq k_i$
Root of $S_{i,k} = i$
Connect l_0 to the leaf of $S_{i,k}$ labeled with $m - C$

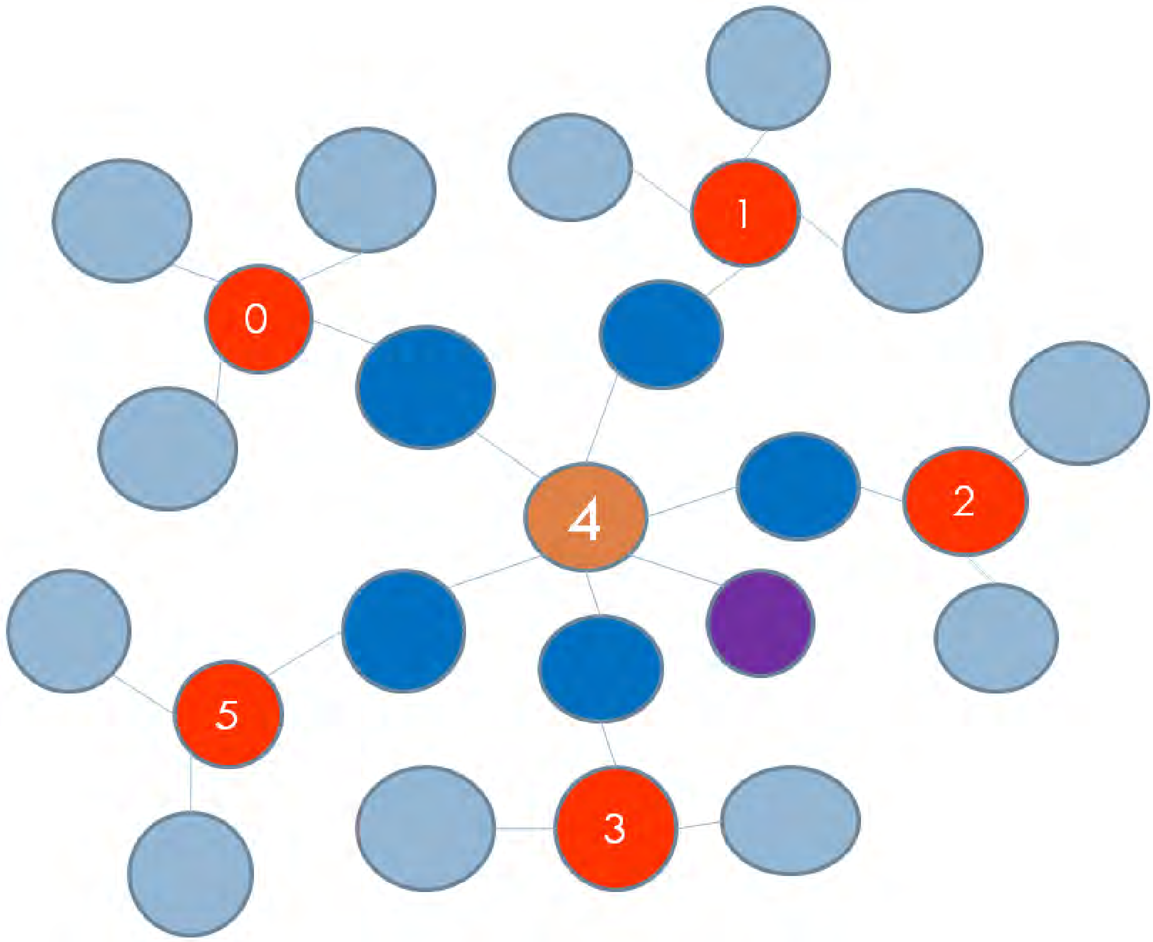

Figure 4.11: An example of Superstar to be labelled gracefully where we have labelled $l_i = 0, 1, 2, 3$ and $l_i = 5$ except l_0 as we have labelled it earlier.

Algorithm 1 Superstar continued...

        $C = C + 1$
        for $m+1$ down to $m - k_i + 1$ **do**
            Leaf of $S_{i,k} = m - 1$
            $m = m - 1$
        end for
    end for
end if

Lemma 27. *Algorithm 1 labels center vertices of stars by labels* $0, 1, \ldots, J$*, whereas leaves of stars with root labels* $0, 1, \ldots, i$ *labelled consecutively with labels from* m *to* $m - \sum_{j=1}^{i} k_i + 1$ *and edges get labels* m *down to* $m - \sum_{j=1}^{i} k_i - i$.

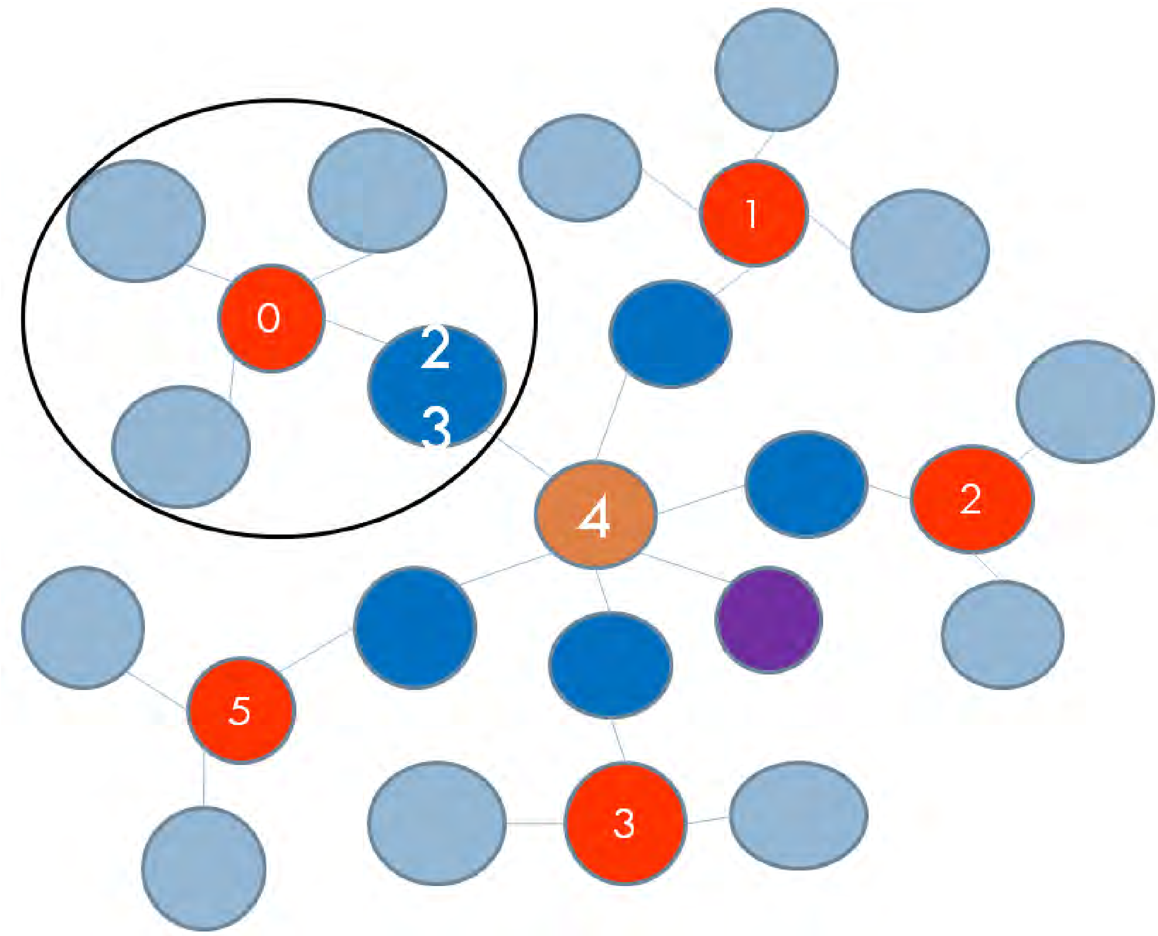

Figure 4.12: An example of Superstar to be labelled gracefully where shared vertex of ith star labeled by $j - l_i - k_i + l_0 = 23$ for which $l_i = 0$.

Proof. Let us label leaves of the star centre of which has been labelled $i = 1$. Since $l_0 - l_1 \leq k_1$, the leaf common to root star and the star being labelled can be labelled in a way that root star edge gets label $m - k_1$, leaves get labels from m down to $m - k_1$, edges are labelled consecutively from m down to $m - k_1 - 1 + 1$. Assume that we have labeled $i + 1$ stars with vertex labels of centres from 0 to i and leaf labels from m down to $m - \sum_{j=1}^{i} k_i + 1$ inducing edge labels from m down to $m - \sum_{j=1}^{i} k_i - i$. Now we are labelling leaves of star centre of which has been labelled $i + 1$. We label the leaf common to root star and star centre of which has been labelled by $i + 1$ in such a way that it induces edge label $m - \sum_{j=1}^{i} k_i - i$ (we can do it by virtue of the inequality satisfied by labels of centers and number of leaves of the star), then the other edge labels up to $m - \sum_{j=1}^{i+1} k_i - i$ can be generated by using vertex labels from $m - \sum_{j=1}^{i} k_i$ to $m - \sum_{j=1}^{i+1} k_i$. In case root star vertices are labelled not at the last then its yet unlabeled vertices should be labelled. $\square$

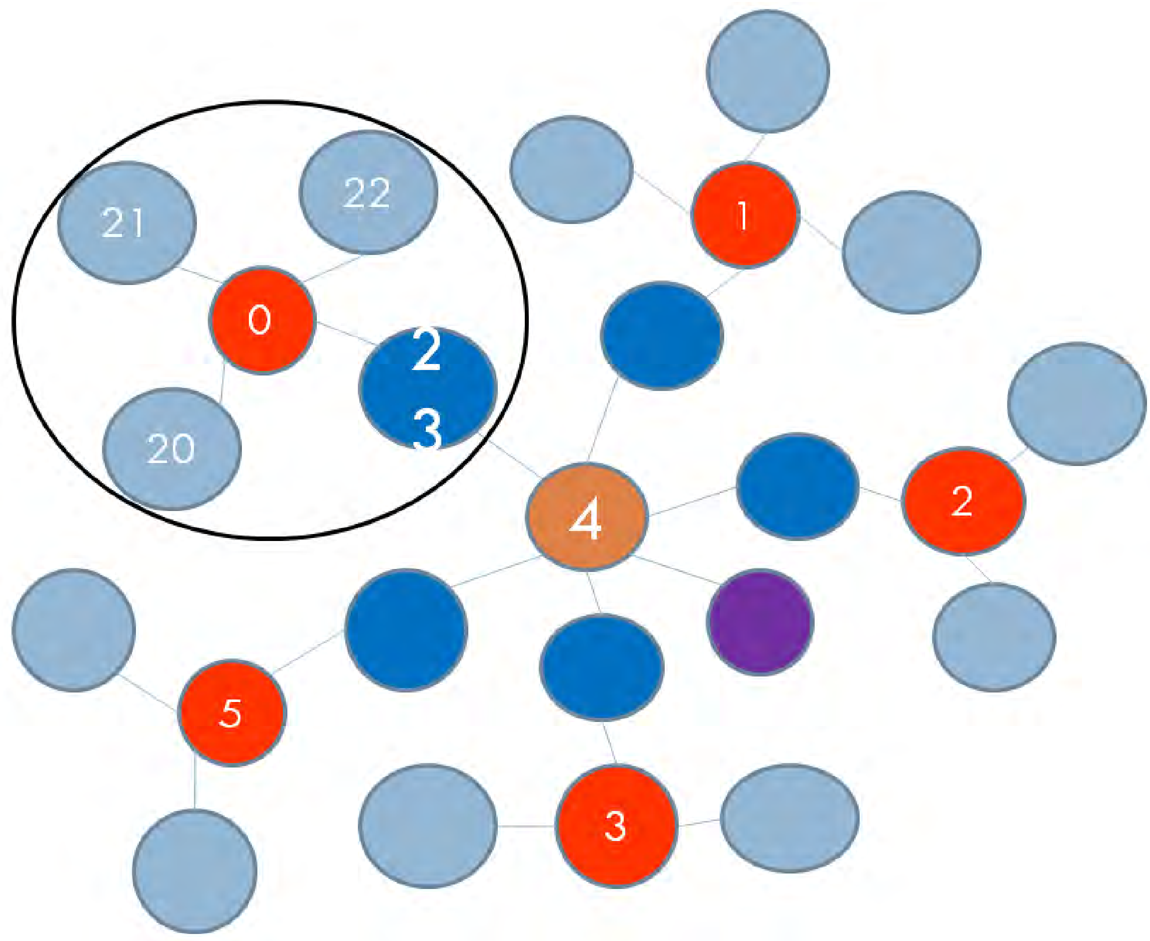

Figure 4.13: An example of Superstar to be labelled gracefully where we have labelled all the leaf of ith star for which $l_i = 0$.

Theorem 28. *All Superstars are graceful.*

Proof. By lemma 27, Algorithm 1 systematically labels leaves with labels from m down to $J+1$, centres of stars already labeled by 0 to J. This induces edge labels from m down to 1. Hence this is a graceful labelling of a Superstar.

4.3 Extended Superstar

If a tree T consist of stars $S(i,k_i)$, $i\{i = 0,1,\ldots,I\}$ with k_i leaves. Each $S(i,k_i), i = 1,...,I$ shares exactly one leaf with $S0,k_0$. This $S(0,k_0)$ is called the root star whereas $S(i,k_i), i = 1,...,I$ are called leaf stars. Then T is said to be a superstar denoted SS.

Definition 29. Let ESS be an Extended Superstar with m edges and Stars $S(i,k_i)$, $i \in I_j$

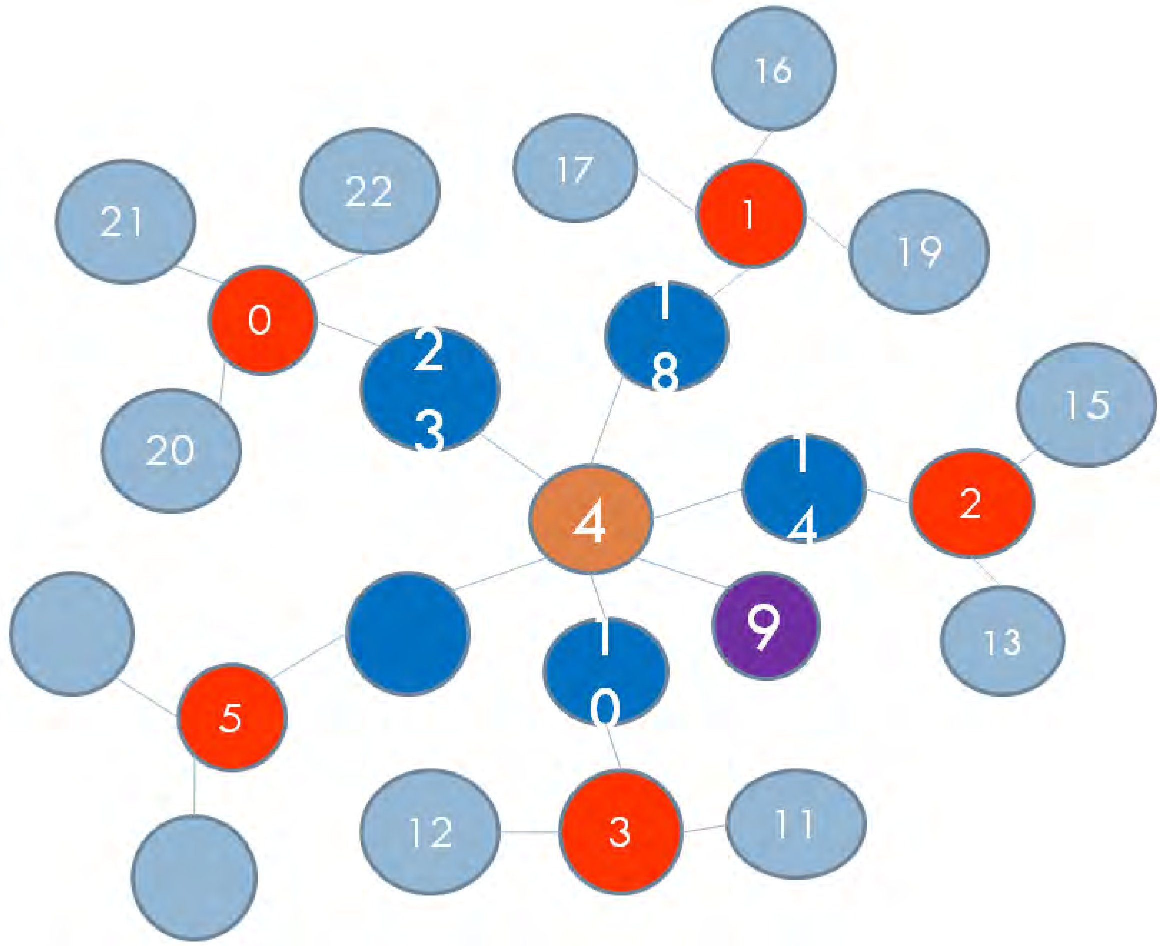

Figure 4.14: An example of Superstar to be labelled gracefully where we have labelled all the leaf of ith star for which $l_i = 0, \dots, 3$ and unshared leaf of root star by 9

contained in the Superstar SS_j where $j = 1, 2, \dots, J$. Among all the Stars $S(i, k_i)$ one star is a root star and rest of them are included in leaf Superstars. Therefore, total number of leaf Superstars is J. If all the leaf Superstars SS_j share exactly one leaf with the leaf of the root then the resulting tree is called an Extended Superstar ESS.

Example 30. An example of gracefully labelled Extended Superstar is illustrated in Figure 4.16.

Example 31. Another example of Gracefully labelled Extended Superstar is illustrated in Figure 4.17 where three Superstar are share their leaves with one root star.

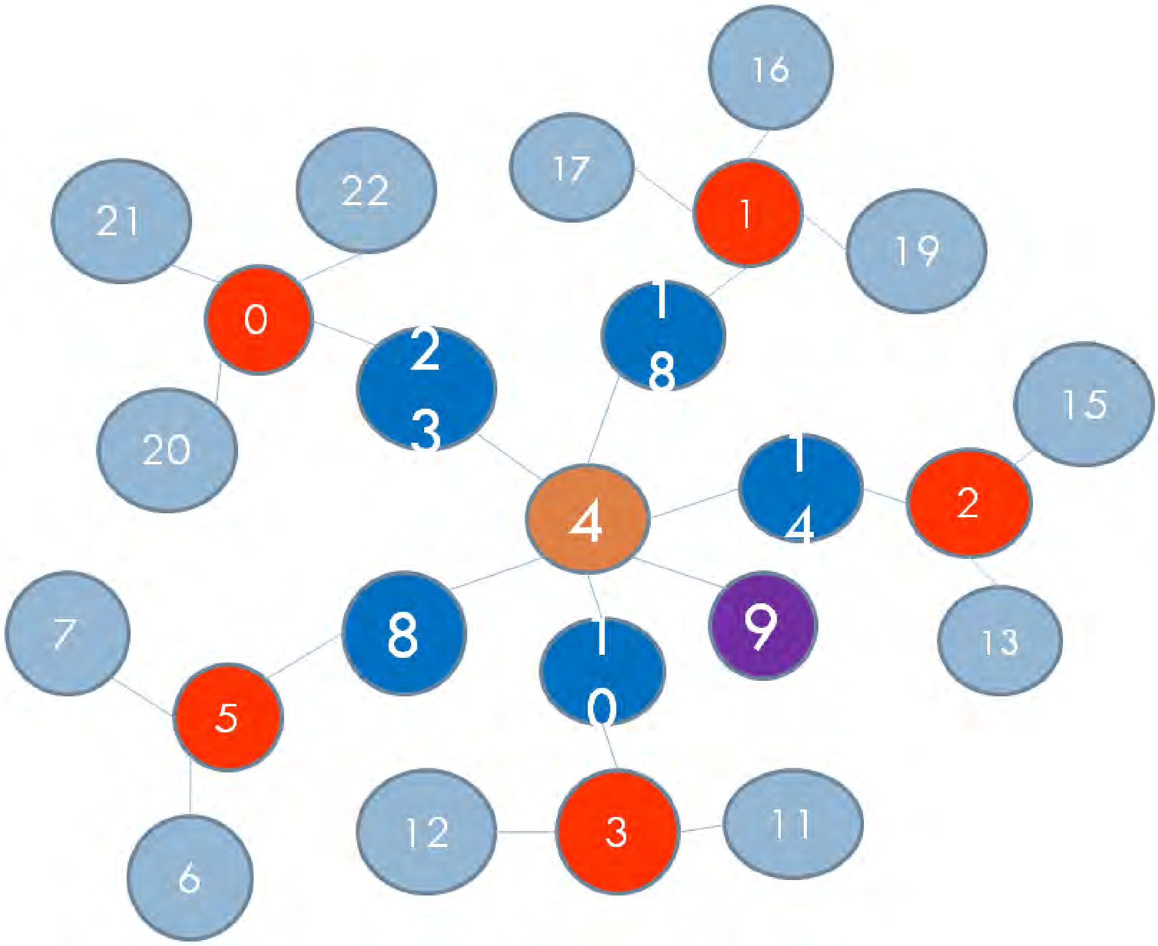

Figure 4.15: An example of gracefully labelled Superstar.

4.4 Labeling of Extended Superstars

Consider the example given in the figure 4.17. In this section we will discus how to label this tree gracefully which are illustrated in the figures 4.18,4.19,4.20 and 4.21.

Step 1: First we have to identify total number superstar in the given Extended Superstar which is $j = 3$ for the example in the figure 4.17 and total number of edges $m = 69$. Therefore, We have to take an arbitrary Superstar $j = 1$ and label the Superstar gracefully using 1 where $p = m$. After labeling jth Superstar the value of $p = 51$ and $i = 4$. Then we have to label the center vertex of root star, $l_{0,0}$ by $i = 5$ and shared vertex of root star by p which is illustrated in the figure 4.18.

Step 2: Now we have to label the unshared leaf of root star by p down to p- total number of Unshared leaf of root star that is 2. Therefore the unshared leaf of root star will be labelled by 50 and 49 which is shown in the figure 4.19.

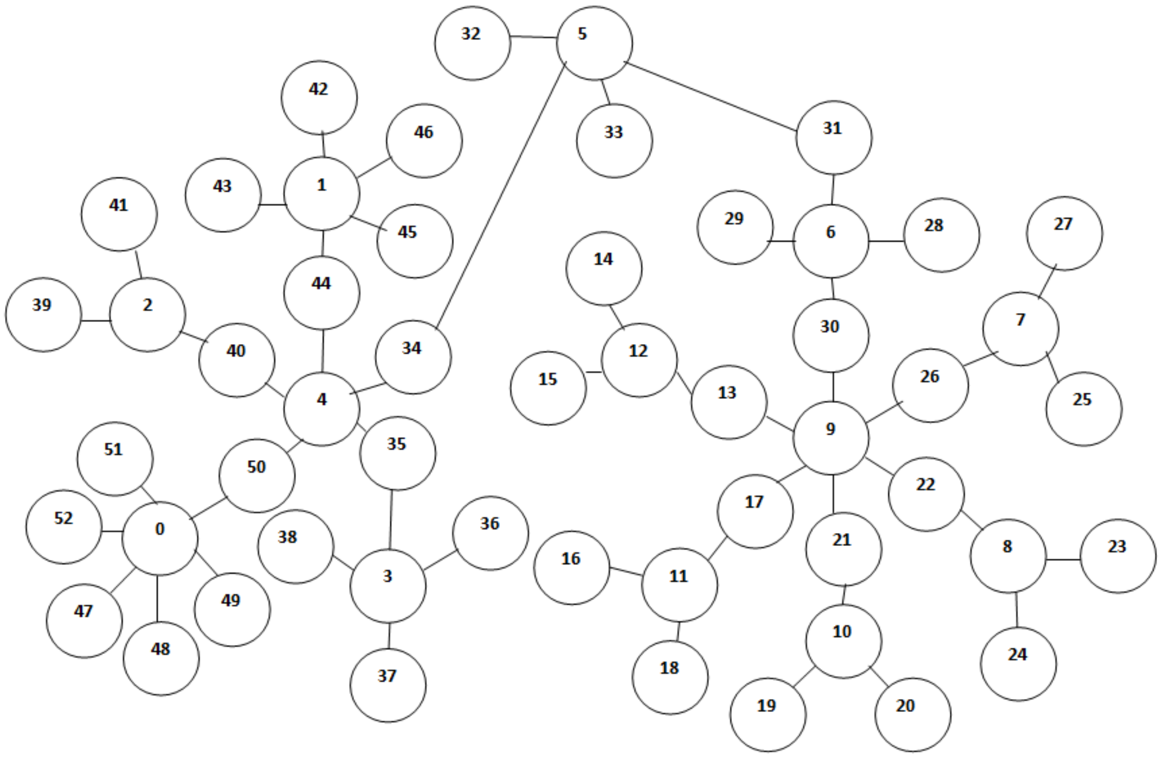

Figure 4.16: An example of gracefully labelled Extended Superstar.

Step 3: Now take another Superstar and first label the shared leaf of root star by $p = 48$. Now $p = 47$ and $i = 6$. Then label the Superstar in the same way which is illustrated in the figure 4.20.

Step 4: In this step we have to label the Superstar for $j = 3$ and we have $p = 31$ and $i = 12$. As $j > 2$, therefore, first we have to label the Superstar gracefully in the same way then have to label the shared vertex of root star by $p - i + 1 + l_{0,0} = 25$ which is shown in the figure 4.21 .

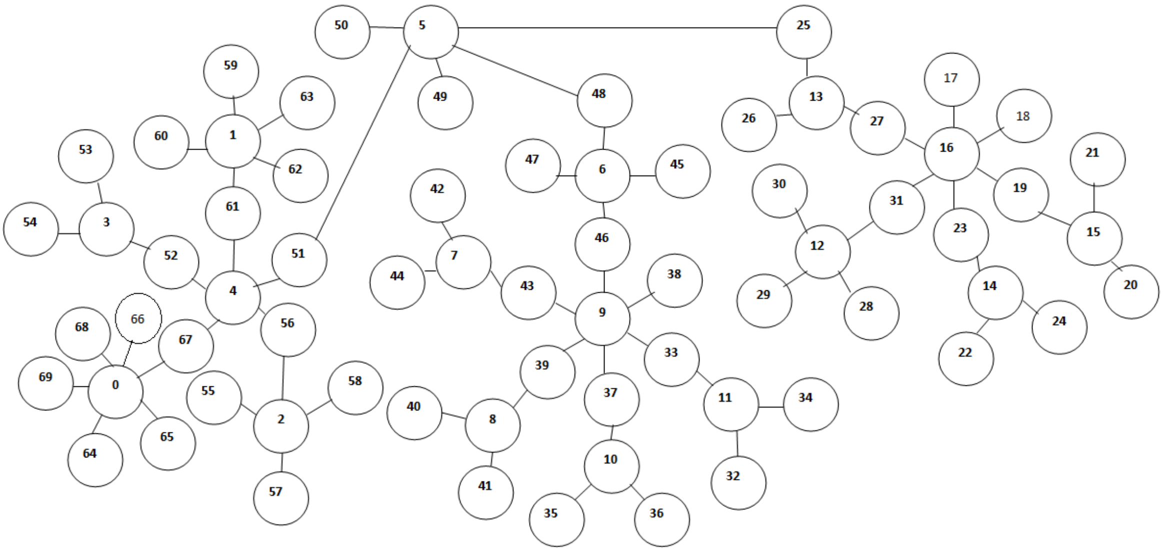

Figure 4.17: Another example of gracefully labelled Extended Superstar.

4.4.1 Algorithm

Algorithm 2 Extended Superstar

$i = \leftarrow$ least possible label
$p = m \leftarrow$ largest possible label
call Superstar(i, p)
label center of the root star $\leftarrow i$
label shared vertex of root star and leaf Superstar, $SS_1 \leftarrow p$
label shared vertex of root star and leaf Superstar, $SS_2 \leftarrow p - 1$
call Superstar(i, p)
for $j = 3$ to J **do**
    call Superstar(i, p)
    label shared vertex of root star and leaf Superstar, $SS_j \leftarrow p - 1 - i + l_{0,0}$
end for

Theorem 32. *All Extended Superstars are graceful.*

Proof. We have already discussed how to label Superstar gracefully therefore, we have omitted the labeling technique of Superstar here.

Let us assume for simplicity, SS_j is a Superstar and ESS is an Extended Superstar. Now let $l_{0,0}$ be the label of the center of root star. First take an arbitrary Superstar SS_j for $j =$

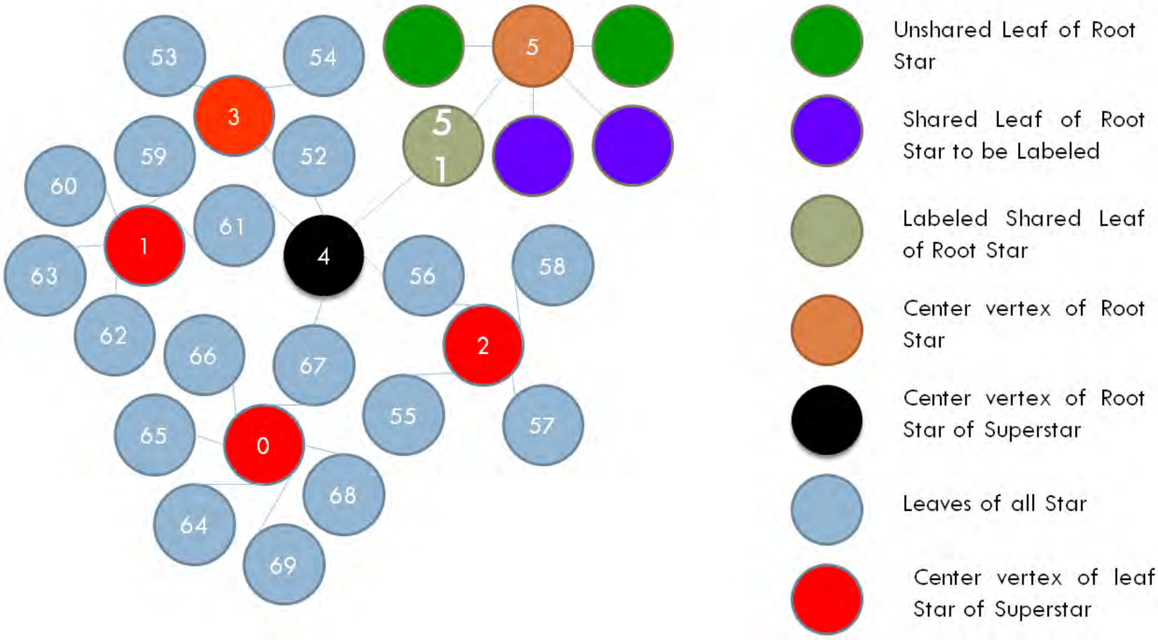

Figure 4.18: An example of Extended Superstar to be labelled gracefully where first Superstar have already been labelled.

$1,2,...,J$ and using previously described algorithm 1 label the Superstar gracefully. Let $p =$ be the largest possible label yet be used after labeling SS_1. Then label the center of root star, $l_{0,0} = i$ where i is the immediate next least possible label and label the shared leaf of root star and Superstar SS_1 by p. Now we have to label the unshared leaf of root star from p down to total number of unshared leaf of root star. After that, the shared leaf of root star and Superstar SS_2 will be labelled by $p-1$. Then for $j = 2$ take another arbitrary Superstar and label the Superstar in the same way. But for $j = 3,...,J$ the shared leaf of root star and Superstar SS_j will be labelled by $p-1-i+l_{0,0}$ and the remaining Superstars will be labelled by using the same algorithm1 which is used to label the Superstar. Therefore, all the label of vertices and edges of Extended Superstar will be distinct and from the set $\{1,2,...,m\}$ and $\{0,2,...,m\}$ respectively. This way we have labelled all the vertices and edges of the Extended Superstar gracefully. $\square$

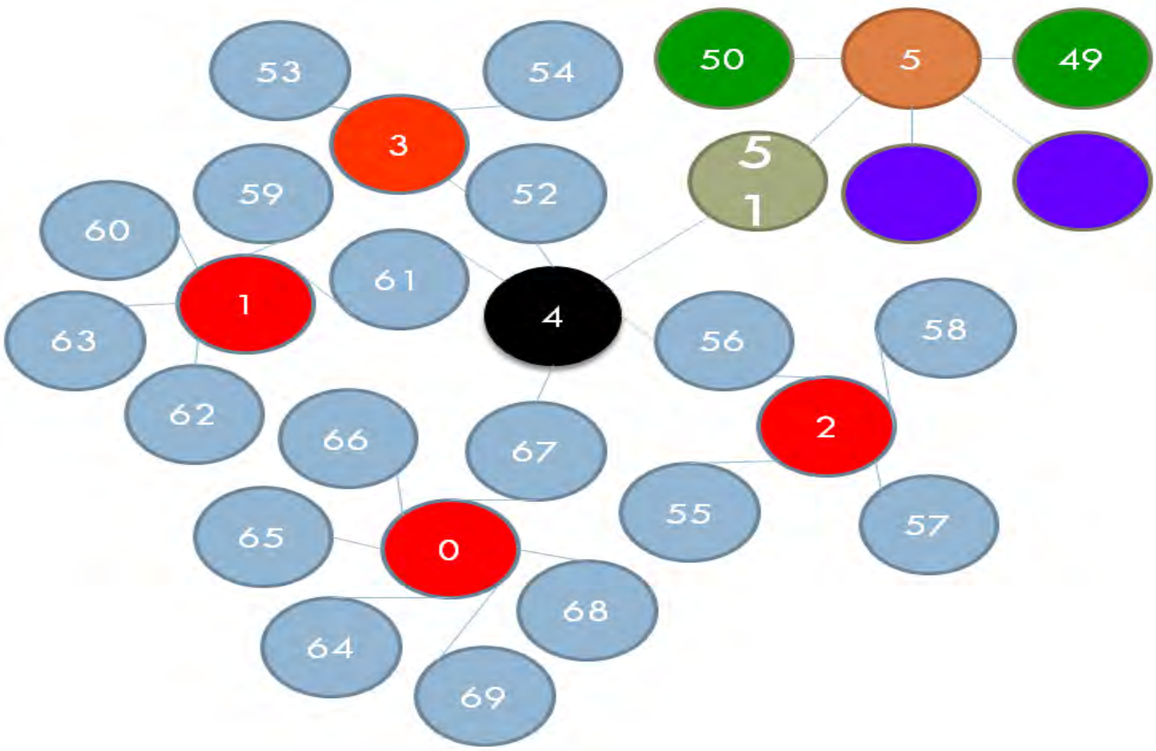

Figure 4.19: An example of Extended Superstar to be labelled gracefully where unshared leaf of root star have been labelled.

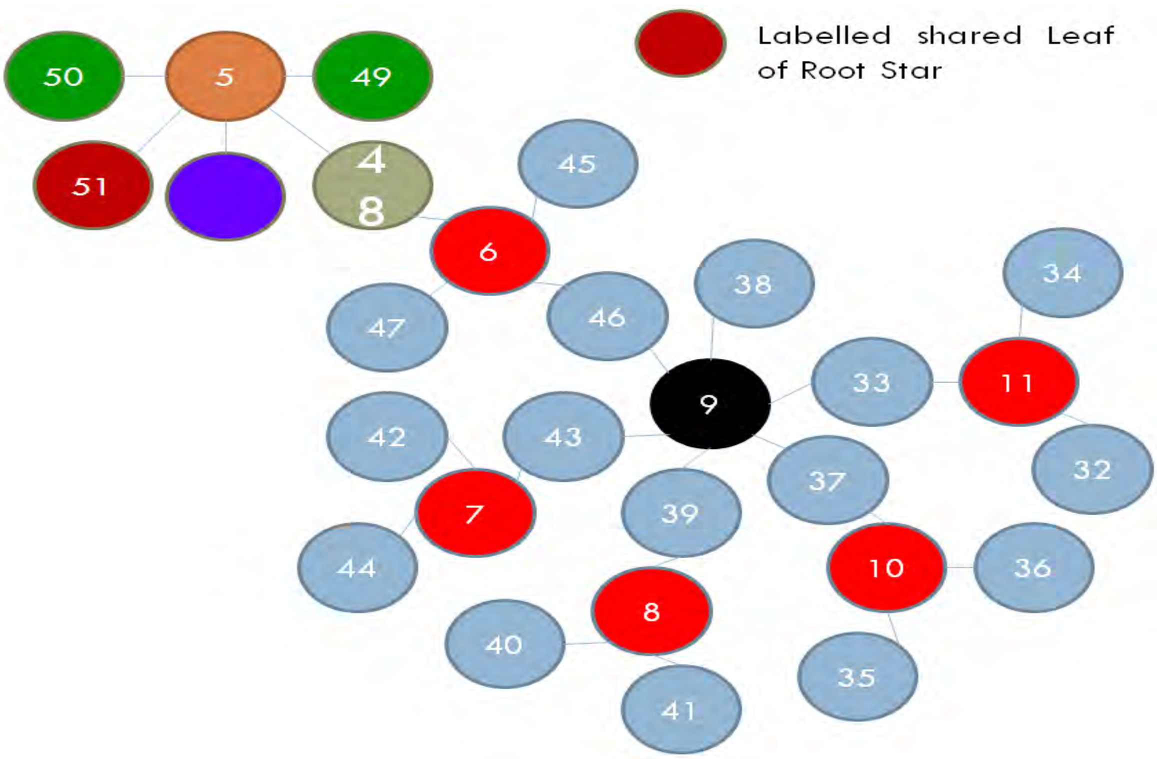

Figure 4.20: An example of Extended Superstar to be labelled gracefully where another Superstar have been gracefully labelled.

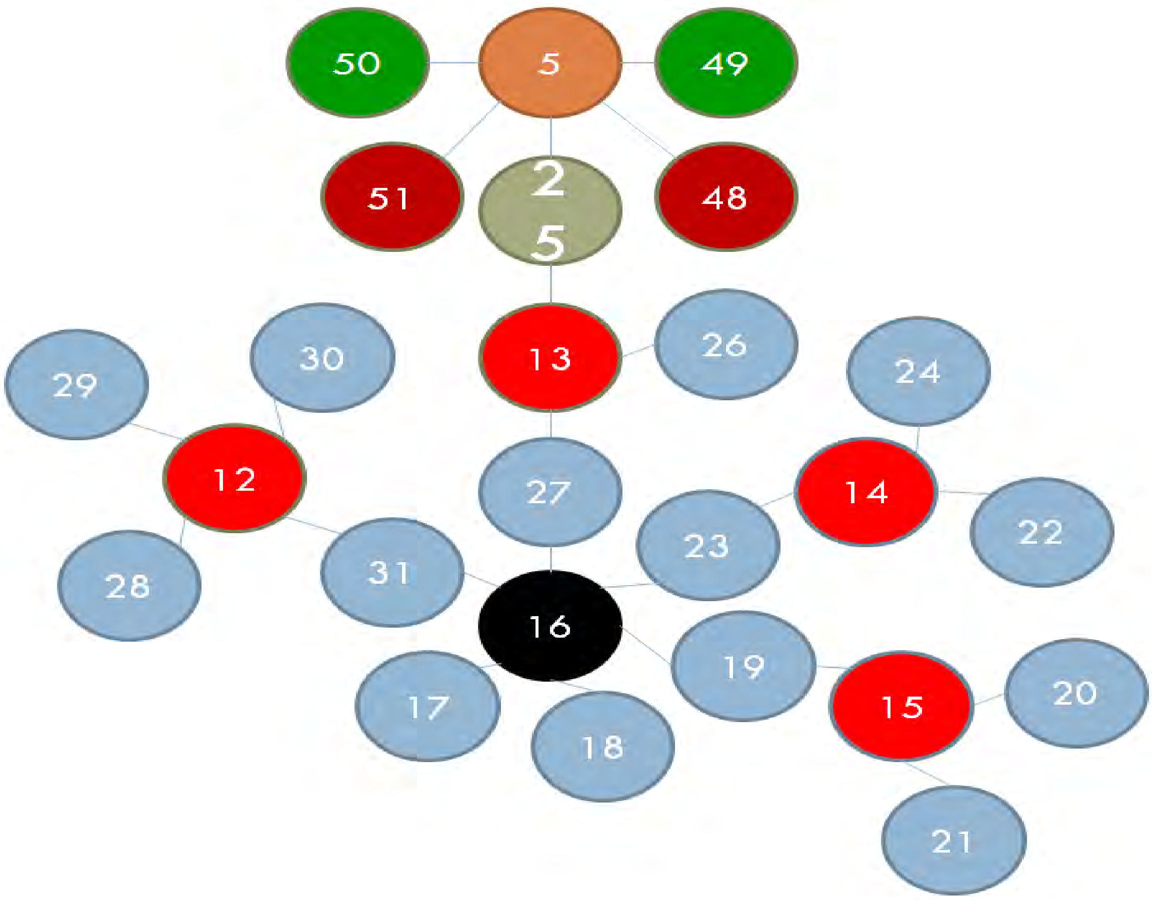

Figure 4.21: An example of gracefully labelled Extended Superstar.

Chapter 5

Experimental Analysis

5.1 Device

The codes implemented for our research were developed in a device running ofWindow OS. The minimum requirement for the example demonstration is the installation of Flash Player (minimum version 11). An update web browser will help in that case. The codes can be executed smoothly in Windows or Linux based devices.

5.2 Language

The language used for the development of the samples provided with this paper is ActionScript 3.0. It is an object-oriented programming language which was originally developed by Macromedia Inc., later dissolved by Adobe Systems. It is a superset of the syntax and semantics of the language JavaScript. The language is primarily used for the development of websites and software targeting the Adobe Flash Player platform.

5.3 IDE

The IDE used for the development of the algorithms and the example codes provided with this paper was Adobe Flash Professional CS6. It is a part of the Adobe CS6 Master Collection. Adobe Flash Professional is a multimedia authoring program used to create content for Adobe Engagement Platform. It is used to develop web applications, games, movies, content for mobile phones and other embedded devices. The platform supports the scripting language ActionScript 3.0 in case of user interaction and graphical manipulation. The version, Adobe Flash Professional CS6 as released in 2012. It was upgraded from the previous versions by integrating the support of HTML5 and the ability to generate spread sheets.

5.4 Code Specification for Superstar

In order to execute the algorithm 1 , two integer variables has to be provided to the function. The number of arms for each star will be provided as armsNumber1 and armsNumber2 respectively, where armsNumber1 is a positive integer and armsNumber2 is a non-negative integer. After the simulation, the vertexNumber array will contain the labels of nodes in graceful manner. The sequence will be as follows :

- Initial index (0) will contain label of the center node of first star
- Following armsNumber1 indices will contain the labels of the arms of the first star
- The next index will contain the label of the center node of the second star
- Following armsNumber2 indices will contain the labels of the arms of the second star
- The last index, i.e. the index indicating the number of vertices will contain the label of the joining node (nth vertex).

Apart from the array, two variables join1 and join2 will contain the edge numbers of the edges joining the nth vertex with the observed arms of the first star and second star respectively.

5.5 Input Format for Superstar

The input for the simulation of the Superstar are integer number. The first input textbox is for the number of arms of the first star, which must be a positive integer. The second textbox is for the number of arms of the second star. Pressing simulate will provide the required graphical illustration of the gracefully labeled Superstar. In the provided test case, the inputs were 5 and 7 in the figure 5.1.

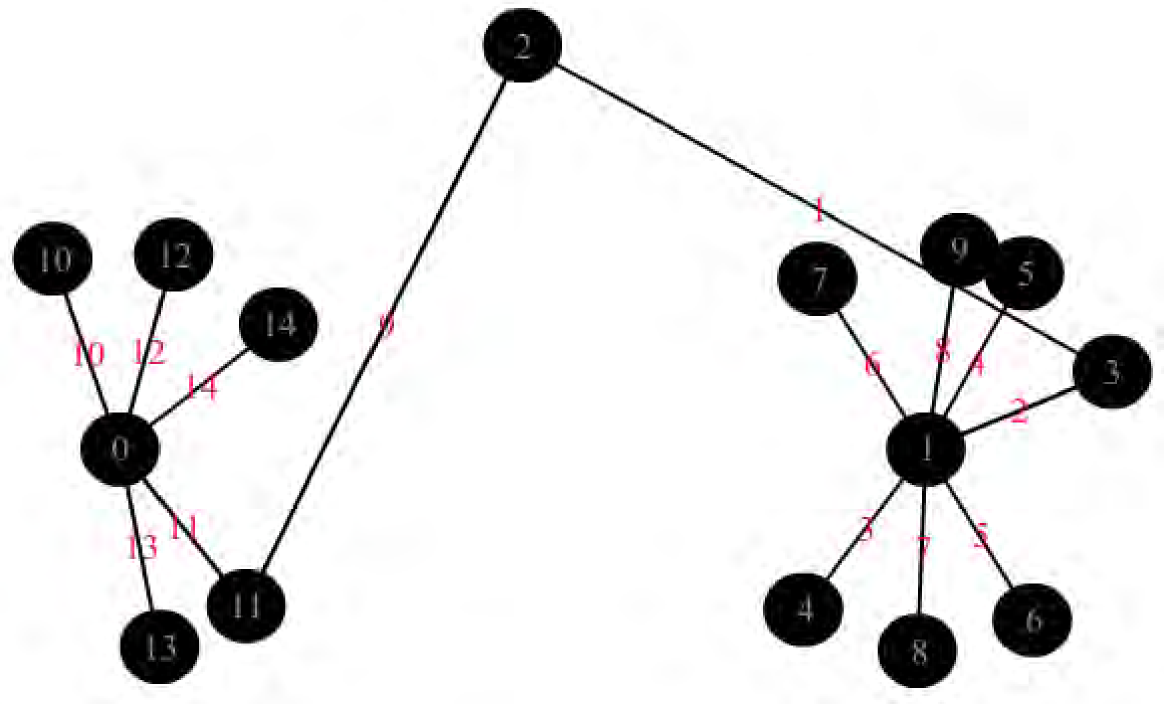

Figure 5.1: An example of gracefully labelled Superstar.

Chapter 6

Conclusion & Recommendation

In this chapter, we draw conclusion by highlighting the major contributions made in this thesis. We have also provided some directions for future research. In Section 6.1, the contribution of our thesis is elaborated. Then in Section 6.2 we shed light into possible future research directions. Finally, the Section 6.3 summarizes the whole thesis.

6.1 Contribution

In this thesis paper we described Graceful and Graceful-like various graph labelings techniques and their connections to a few well known combinatorial problems. Moreover, we also describe how the graceful tree problem first came up. Additionally, we have discussed some known results of Graceful labeling. As advances in the area of graceful labeling would have an impact on such areas as map colourings, graph decomposition and Latin squares the traditional approach to this problem has been to construct graceful and graceful-like labelings for particular classes of graphs or trees. We have particularly focused on the Graceful labeling of trees. It seems that there is a general consensus among the researchers working in the area that a different approach is needed to prove Graceful Tree Conjecture. So far research on settling the conjecture has expanded in three directions. The first is to discover new classes of trees that are graceful. This is how caterpillars, fire crackers, banana trees, symmetrical trees have been proved to be graceful. The second direction is to establish that all classes of trees of up to certain sizes have been shown graceful by enumerating all non-isomorphic trees and then finding graceful labeling by extensive computation. Yet the third direction is to apply integer programming formulation of the problem and then solve it. We have followed the first direction.

The search for a class of Gracefully labeled trees began in an effort to prove Graceful Tree Conjecture. Therefore, We have proved that two fairly general classes of trees named Superstar and Extended Superstar are graceful. In particular, if a tree containing number of Stars where one is a root Star and the remaining Stars are leaf Star and all the leaf Stars share exactly one leaf with the root star's leaf then the resulting tree is called a Superstar. In the same way if a tree containing number of Superstars and a root Star and the leaf of Superstars share exactly one leaf with the root star's leaf then the resulting tree is called an Extended Superstar. Moreover, we have provided a generalized algorithm to label all Superstars and Extended Superstars gracefully.

6.2 Suggestion for Future Works

During the research period a number of future research directions arise out of our work. For example, it would be interesting to see whether we can extend the level of Extended Superstars and prove them graceful.

In addition to these, we propose the following research directions for the general area of graceful labeling.

- Identify new classes of trees that are graceful. The ultimate goal within this research direction would be to give a complete characterization of graceful trees, but this is clearly a very challenging task. For example, it would imply a solution to the *Graceful Tree Conjecture*. We propose that making use of spectral graph theory, a powerful method with a potential to eliminate certain classes of trees as graceful candidates.

- Improve the bounds of known results for size of graceful trees. In particular, it would be interesting to focus on trees with bounded degree and a perfect matching. As trees with a perfect matching are particularly important as the existence of graceful labeling for them implies the existence of α-labeling for other related trees. Therefore, Kotzig [56] showed that if at least one connected subgraph S of a tree T containing the base of T has a graceful labeling then T has a α-labeling. Although, every tree T has a subgraph that contains the base of T and has a perfect matching [21] therefore if every tree with a perfect matching has a graceful labeling then every tree has a α-labeling. In turn, α-labeling for all trees implies Ringel's conjecture.

- For Graceful labelings of a path with n vertices investigate the allowed combinations for labels of end vertices of the path.

6.3 Summary

In Chapter 1 we introduce the problem of graceful labeling and the origin of this problem. Chapter 2 contains a list of graph labeling and their applications in the real world problem. Chapter 3 discusses different classes of trees that have been proved graceful to strengthen the Graceful Tree Conjecture that All trees are Graceful. Chapter 4 introduces the new classes of trees that we have proved graceful which implies a partial effort to prove Graceful Tree Conjecture. Experimental results and discussions about parameter are presented in Chapter 5.

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