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# Evolutionary Algorithm Using Adaptive Fuzzy Dominance and Reference Point for Many-Objective Optimization

by

Siddhartha Shankar Das

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*Dedicated to my loving parents*

## Author's Contact

Siddhartha Shankar Das

Lecturer

Department of Computer Science & Engineering

Bangladesh University of Engineering & Technology (BUET).

Email: siddhartha047@cse.buet.ac.bd

The thesis titled "Evolutionary Algorithm Using Adaptive Fuzzy Dominance and Reference Point for Many-Objective Optimization", submitted by Siddhartha Shankar Das, Roll No. **1014052012P**, Session October 2014, to the Department of Computer Science and Engineering, Bangladesh University of Engineering and Technology, has been accepted as satisfactory in partial fulfillment of the requirements for the degree of Master of Science in Computer Science and Engineering and approved as to its style and contents. Examination held on 18 September, 2017.

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Professor
Department of Computer Science and Engineering
Bangladesh University of Engineering and Technology, Dhaka.

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Department of Computer Science and Engineering
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This is hereby declared that the work titled "Evolutionary Algorithm Using Adaptive Fuzzy Dominance and Reference Point for Many-Objective Optimization" is the outcome of research carried out by me under the supervision of Dr. Md. Monirul Islam, in the Department of Computer Science and Engineering, Bangladesh University of Engineering and Technology, Dhaka 1000. It is also declared that this thesis or any part of it has not been submitted elsewhere for the award of any degree or diploma.

Siddhartha Shankar Das

Candidate

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# Abstract

Many-objective optimization is very important for numerous practical applications. It, however, poses a great challenge to the Pareto dominance based evolutionary algorithms. In this thesis, a fuzzy dominance based evolutionary algorithm is proposed for many-objective optimization. The essence of the proposed algorithm is that it adaptively determines the fuzzy membership function for each objective of a given many-objective optimization problem. Furthermore, it emphasizes both convergence and diversity of all the evolved solutions in the same way by using one selection criterion. This is why our algorithm employs the reference points for clustering the evolved solutions and selects the best ones from different clusters in a round-robin fashion. The proposed algorithm has been tested extensively on a number of benchmark problems in evolutionary computing, including eight Waking-Fish-Group (WFG), three Deb-Thiele-Laumanns-Zitzler (DTLZ) problems having 2 to 25 objectives and three instances of Rectangle problem. The experimental results show that the proposed algorithm is able to solve many-objective optimization problems efficiently, and it is compared favorably with the other evolutionary algorithms devised for such problems. A parametric study is also provided to understand the influence of a key parameter of the proposed algorithm.

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# Chapter 1

# Introduction

## 1.1 Many Objective Optimization Problem

Multi-objective Optimization problems (MOPs) usually have more than one conflicting objectives and these objectives are needed to be optimized simultaneously under several constraints [4, 5]. MOPs can be formally defined as following,

$$
\begin{aligned}
\mathit{minimize},\ f(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}), ..., f_m(\mathbf{x})] \\
\mathit{constrainted\ by},\ p_j(\mathbf{x}) \geq 0, \qquad j = 1, ..., P \\
q_k(\mathbf{x}) = 0, \qquad k = 1, ..., Q \\
x_i^L \leq x_i \leq x_i^U, \qquad i = 1, ..., n
\end{aligned}
$$

Here a solution $\mathbf{x} = [x_1, x_2, x_3, ..., x_n]$ contains $n$ decision variables bounded by $x_i = [x_i^L, x_i^U]$, $m$ objective functions, and defined by $P, Q$ constraints.

For a MOPs having conflicting objective functions, no single solution exists that simultaneously optimizes each objective. In such cases there exists potentially infinite number of Pareto optimal solutions. A solution is called Pareto optimal, if none of the objective functions can be improved in value without degrading some of the other objective values. In absence of additional subjective preference information, all Pareto optimal solutions are considered equally good as they cannot be ordered properly. Researchers study MOPs from different viewpoints and there exist different solution philosophies and goals when setting and solving them. The goal may be to find a well distributed representative set of Pareto optimal solutions (Pareto Front) or finding a single solution that satisfies the subjective preferences of a human decision maker (DM).

These sets of problems and their solution strategies have been applied in many fields of science, including engineering, economics and logistics where optimal decisions need to be taken in the presence of trade-offs between multiple conflicting objectives. Minimizing cost and maximizing comfort while buying a car, maximizing performance while simultaneously minimizing fuel consumption and emission of pollutants of a vehicle are examples of MOPs involving two and three objectives, respectively. Among the solution approaches Evolutionary Algorithms (EAs), as a class of population based search heuristics, are able to obtain a set of solutions in a single run. Thanks to this attractive property Multi-Objective Evolutionary Algorithms (MOEAs) have shown significant progress in the past two decades. This property is very popular among DMs as it requires little to no domain knowledge beforehand.

Although approaches, proven to be effective for solving MOPs, have been developed throughout the years but they face substantial challenges when number of objective increases. Additionally, in practical problems, there can be many number of objectives [6–8]). Currently MOPs having more than three objective functions to be optimized are commonly referred as Many-Objective Optimization Problem (MaOPs). Due to their challenges MaOPs have attracted increasing attention in the Evolutionary Multi-objective (EMO) research community [9]. One major reason behind the failure of most conventional MOEAs for solving MaOPs is the reduction of selection pressure i.e., the pressure for the population to converge toward the Pareto Front. This happens when a selection scheme fails to discriminate solutions due to the increase non-dominated solutions. Another prominent problem is the conflict between convergence and diversity, which also aggravates with the increase of objectives [10]. Normally, Pareto Dominance relation is used as a primary criterion to provide selection pressure and diversity maintenance procedure as secondary criterion [1]. But since for MaOPs Pareto relation is ineffective, only diversity promotion mechanism remains active. To handle this issue many modified dominance relation based MOEAs have haven devised. Since these dominance relation offers comparability on non-dominated solutions, bias toward different objective regions seemed evident in MaOPs and lacks diversity among solutions. Some MOEAs depends on data-structure which rely on exponential growth of size in relation with objective space. These algorithms often falls into the curse of dimensionality and compromise it with lack of diversity or increasing solution numbers [10]. Furthermore, visualization of the objective space become difficult for MaOPs. This is very important to a decision maker since in any real-world application, a final solution must be chosen from the obtained non-dominated solution set. Although several multidimensional visualization techniques have been studied in the literature [11], of which Parallel Coordinate Plot [12] has been used extensively to visualize and interpret Pareto Front interactively. Many Objective Visualization is currently a promising research area for application to MOPs.

Thus designing effective algorithms for solving MaOPs has been one of the major research topics in recent years.

## 1.2 Solution Approaches for Multi-Objective Optimization Problems

There are many methods that convert original problem with multiple objectives into a single-objective optimization problem using some form of aggregation procedure. This is called scalarized problem. If scalarization is done properly Pareto optimal solution can be achieved. The major problem of these approaches is that they require several parameters to tune by decision maker (DM) which is very difficult.

The target of solving multi-objective optimization problem is to support decision maker in finding the most preferred Pareto optimal solution. The underlying assumption is that one solution to the problem must be identified to be implemented in practice. Hence to support decision maker finding the most preferred results, many algorithmic frameworks have been developed following different philosophies. Typically these Multi-objective optimization methods can be divided into four classes. In so-called no preference methods, no decision makers is expected to be available, but a neutral compromised solution is identified without preference information. The other classes are so-called a priori, a posteriori and interactive methods and they all involve preference information from the DM in different ways.

- *No Preference Methods* : Here decision maker (DM) does not provide any preference information and the multi-objective optimization problem are typically converted into a scalarized problem. An well known example is the method of global criterion [13]. This method is sensitive to the scaling of the objective functions, and it is recommended that the objectives are normalized into a uniform, dimensionless scale.

- *Priori Methods* : In a priori methods, preference information is first asked from the DM and then a solution best satisfying these preferences is found. Some well known priori methods are utility function method, lexicographic method and goal programming. In utility function method, custom utility function is defined by decision maker where in lexicographic method objective functions are ranked by their order or importance.

- *Posteriori Methods* : In a posteriori methods, a representative set of Pareto optimal solutions is first found and then the DM must choose one of them. Most a posteriori methods fall into either one of the following two classes: mathematical programming-based a posteriori methods, where an algorithm is repeated and each run of the algorithm produces one Pareto optimal solution, and evolutionary algorithms where one run of the algorithm produces a set of Pareto optimal solutions.

Well-known examples of mathematical programming-based a posteriori methods are the Normal Boundary Intersection (NBI) [14], Normal Constraint (NC) [15] and Successive Pareto Optimization (SPO) [16] methods that solve the multi-objective optimization problem by constructing several scalarizations. The solution to each scalarization yields a single Pareto optimal solution, whether locally or globally. The scalarizations of the NBI, NC and SPO methods are constructed with the target of obtaining evenly distributed Pareto points that give a good evenly distributed approximation of the real set of Pareto points.

Evolutionary algorithms are popular approaches to generating Pareto optimal solutions to a multi-objective optimization problem. Currently, most evolutionary multi-objective optimization (EMO) algorithms apply Pareto-based ranking schemes. Evolutionary algorithms such as the Non-dominated Sorting Genetic Algorithm-II (NSGA-II) [1] and Strength Pareto Evolutionary Algorithm 2 (SPEA-2) [17] have become standard approaches, although some schemes based on particle swarm optimization and simulated annealing are significant. The main advantage of evolutionary algorithms, when applied to solve multi-objective optimization problems, is the fact that they typically generate sets of solutions, allowing computation of an approximation of the entire Pareto front. The main disadvantage of evolutionary algorithms is their lower speed and the Pareto optimality of the solutions cannot be guaranteed.

- *Interactive Methods* : In interactive methods, the decision maker is allowed to iteratively search for the most preferred solution. In each iteration of the interactive method, the DM is shown Pareto optimal solutions and describes how the solutions could be improved. The information given by the decision maker is then taken into account while generating new Pareto optimal solutions for the DM to study in the next iteration. In this way, the DM learns about the feasibility of his/her wishes and can concentrate on solutions that are interesting to him/her. The DM may stop the search whenever he/she wants to.

Among these approaches Evolutionary algorithm based MOPs are particularly popular among decision makers primarily because they can generate several Pareto Optimal solutions in a single run. Also the posteriori approaches trivially doesn't require any preference or domain specific knowledge beforehand which helps researcher to develop a framework that can work on broad range of problems. Keeping this in mind, in this thesis we have proposed a Evolutionary Algorithm based framework for solving Many Objective optimization problem.

## 1.3 Evolutionary Algorithms for Solving MaOPs

Since early nineties, Multi-objective Evolutionary Algorithms (MOEAs) have gained popularity in solving such complex MOPs. The popularity of such algorithms is due to their ability of providing several candidate solutions in a single run. As it is impractical to find all Pareto optimal solutions, the evolutionary algorithm tries to approximate the Pareto optimal front constituted by the best trade-off solutions.

MOEAs primarily focus on genetic variations and selection mechanism phase. Algorithms could incorporate primary and secondary selection scheme or a single selection scheme. Some of the primary selection schemes are Pareto Dominance, weighted aggregation and secondary schemes are crowding distance, density Estimation, reference point based clustering etc. Based on the selection schemes MOEAs can be classified into four groups: modified Pareto dominance, active diversity promotion, decomposition and indicator based approaches.

Pareto based EMO algorithms usually employ Pareto dominance relation as a primary selection criterion and the solutions' density in the objective space as a secondary selection criterion [1]. The former criterion favors non-dominated solutions over dominated ones but the later one encourages diversity among solutions. Thus the selection pressure of these algorithms can be improved by modifying dominance concept and/or diversity maintenance mechanism.

A good number of modified dominance concepts and different ranking schemes have been introduced to improve selection pressure. Some of these concepts are subspace dominance comparison [18], dominance area control [19], path control strategy [20], grid dominance [10],

$\theta$-dominance [21], fuzzy dominance [22].

There are several studies that deal with improving diversity maintenance. These mechanisms are called active diversity promotion [23]. For example, Adra and Fleming [24] employ a diversity management operator to adjust diversity requirement in the mating and environmental selection of an EMO algorithm. In [25], Li *et al.* develop a general modification of density estimation, termed shift-based density estimation (SDE), to make Pareto-based algorithms suitable for MaOPs. Unlike traditional density estimation, SDE considers both distribution and convergence information of solutions. A knee point driven approach [26], on the other hand, prefers knee point among non-dominated solutions for maintaining diversity. The growing interests stemmed from the issues that some approaches (e.g. crowding distance [1]) prefer dominance resistant solutions (solutions having higher improvement in at least one objective but poor in others) and some approaches can maintain diversity well in objective space but with poor proximity to global Pareto front. Deb and Jain proposed NSGA-III [27], an improved version of established NSGA-II [1], where they replaced the Crowding Distance operator with clustering approach using a set of well-distributed Reference Points to guide individuals searching for different directions. In [28], Mario *et al.* use hierarchical clustering in decision space for diversity management.

As an alternative, non-Pareto based approaches are also used in solving MaOPs. Two types of such approaches have been found to be promising in the EMO literature. They are decomposition based approaches and indicator based approaches. The former approach first decomposes an MOP into a number of single objective optimization sub-problems and then solves them simultaneously by evolving a population of solutions. The objective of each sub-problem can be a linear or nonlinear weighted aggregation of all the individual objectives of the MOP. Multi-objective evolutionary algorithm based on decomposition (MOEA/D) [2] is the most typical implementation of this class. This algorithm and its variants [29,30] have been quite successful in solving various MaOPs. Another decomposition based algorithm RVEA (reference vector guided evolutionary algorithm) [31] uses reference vectors to decompose solutions into different sub-problems. The salient feature of RVEA is that it dynamically adjusts the distribution of the reference vectors according to the scales of the objective functions. In [3], a preference inspired co-evolutionary algorithms, PICEAg, is proposed based on a concept of co-evolving a population of candidate solutions with a set of goals.

The main idea of indicator based approaches is to use a single performance indicator to optimize a desired property of an evolving population. Among the different indicators, hypervolume [32] or S-metric is the only quality measure known to be Pareto-compliant and is ever used in multi-objective search. It has been known that maximizing the hypervolume indicator is equivalent to finding the Pareto front. However, one prominent problem of hypervolume based algorithms is their extreme computational overhead. Bader and Zitzler [33] proposed a hypervolume estimation algorithm for multi-objective optimization (HypE) to reduce computational burden. Recently, a two-archive algorithm (Two_Arch) for many-objective optimization [34] has been proposed which combines the indicator and Pareto based approaches in one algorithm.

## 1.4 Objective of the Thesis

Despite the recent advancements in solving MaOPs, more effective EMOs are still needed to tackle these challenges. Among the non-Pareto based approaches, the notion of fuzzy concept has proven to be effective for MaOPs. Its strength comes from the fact that fuzzy concept can continuously differentiate solutions into different degrees of optimality beyond the classification of the original Pareto dominance which is beneficial for MaOPs because solutions which were previously incomparable can be now compared and complete or partial order of solutions can be found. Moreover given the membership functions, the fuzzy procedure does not usually require any extra computational overhead even for a very large number of objectives. The simplicity inherent within the fuzzy dominance computation makes it an appealing candidate for solving MaOPs. But fuzzy concepts have the issues of diversity loss and selection of appropriate parameters in different fuzzy steps. There are a few studies [22,35–38] that utilize this concept in solving MaOPs using evolutionary algorithms and the field is still under-explored.

The main objective of our study are follows.

1. To propose a fuzzy dominance based evolutionary algorithm ($F$-DEA) for efficiently solving complex MaOPs on par with the state of the art approaches. The algorithm will be able to obtain Pareto Optimal solutions in the targeted regions given decision makers preference beforehand while in absence of any preference, will achieve uniform diverse solutions throughout the entire Pareto front.

2. To propose a reference points based clustering technique for active diversity promotion. The mechanism will be robust in high dimensional objective spaces, compatible with fuzzy dominance procedure ensuring convergence and be effective for degenerate, deceptive, disconnected problems. The diversity of solutions can be controlled by changing default parameter with the trade off for more converged solutions within the algorithms' termination criteria.

3. To propose adaptive fuzzy parameter settings that requires no domain knowledge beforehand and can effectively handle differently scaled objective spaces. The proposed method will be robust to handle impact of isolated solutions.

4. To evaluate and compare the performance of $F$-DEA with other state-of-the-art algorithms using benchmark problem suites. We will also investigate the contribution of different components of $F$-DEA.

## 1.5 Thesis Organization

The reminder of the thesis is organized as follows.

In Chapter 2 we briefly discuss the concepts that are necessary to understand the idea of the thesis. Here the basic concepts of fuzzy in relation with EMO algorithms, issues arises therein and related works are also introduced.

Chapter 3 presents the major contribution of this thesis. We introduce our proposed $F$-DEA at length. We present the different components of the framework and discuss why each component is necessary and important in handling the problem.

In Chapter 4, the experimental analysis, comparisons, and discussions regarding the performance of the proposed $F$-DEA is presented.

Finally, Chapter 5 concludes the thesis with a brief summary and a few remarks.

# Chapter 2

# Background

## 2.1 Introduction

This chapter begins by explaining general Multi-objective Evolutionary Algorithm (MOEA) framework and definition of different terminology. Then, in the following sections we broadly discussed about fuzzy notions, preliminaries and related fuzzy based works.

## 2.2 Evolutionary Algorithm framework

Mechanisms of Evolutionary Algorithm are inspired by biological evolution. For solving Multi or Many objective optimization problem, the algorithms starts with N candidate solutions, then various genetic variations (eg. crossover, mutation etc.) are applied for generating offspring solutions. Solutions are evaluated based on the objective functions definition and among the parent and offspring solutions best $N$ solutions are selected. The whole process is repeated until the stopping criteria meets. As the best solutions are selected throughout each iteration, candidate solutions are ensured to be more optimized than previous iteration. The general flow of the evolutionary algorithms are shown in Fig. 2.1.

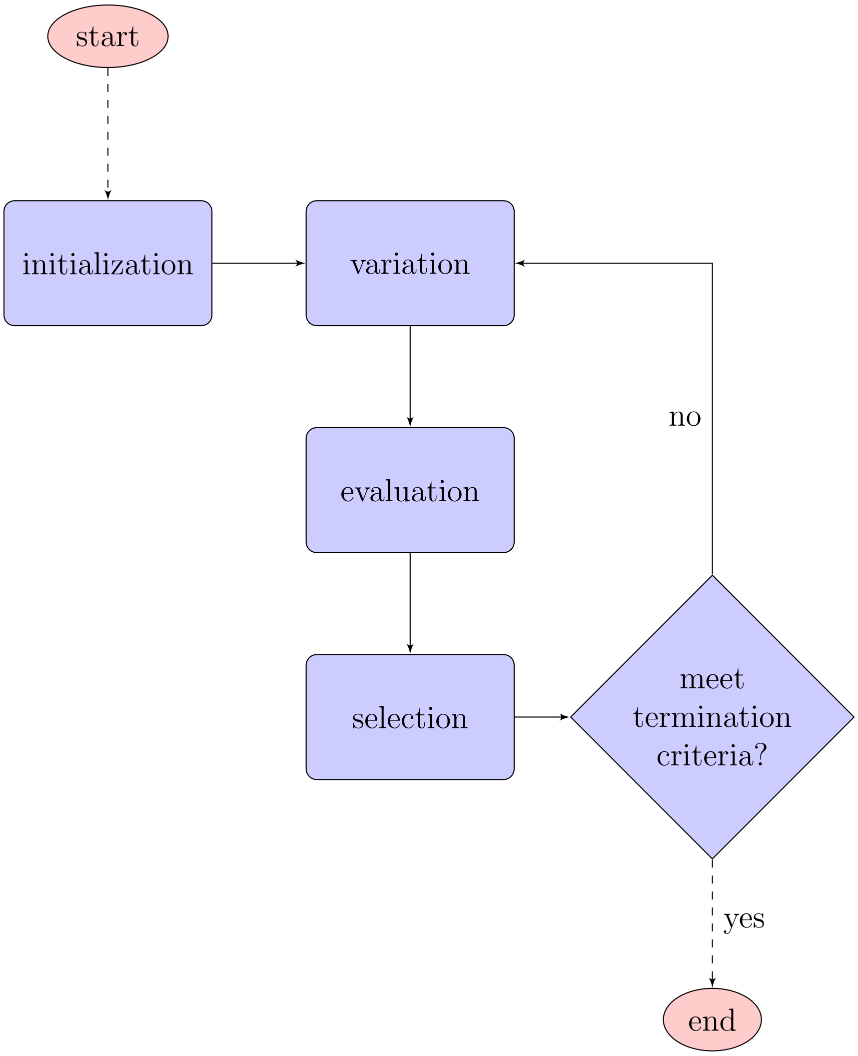



Figure 2.1: Flow chart of Evolutionary Algorithm for solving Multi-objective Optimization Problem

While solving Many-objective optimization problems (MaOPs), different algorithmic frameworks primarily focus on two things namely genetic variations and selection mechanism. Based on these two factors researchers have proposed myriads of techniques in the last few decades [1, 17, 32, 39] have gained popularity among the researchers.

## 2.3 Common definitions

### 2.3.1 Targets of MOEA

In the absence of any preference information, all Pareto optimal solutions are equally important. Hence it is important to find as many as Pareto optimal solutions. The two goals of multi-objective optimization can be summarized as follows

1. *Convergence :* To find a set of solutions as close as possible to the Pareto-optimal front.
2. *Diversity :* To find a set of solutions as diverse as possible.

Fig. 2.2 shows the goals of multi-objective optimization problems in case of minimization problem.

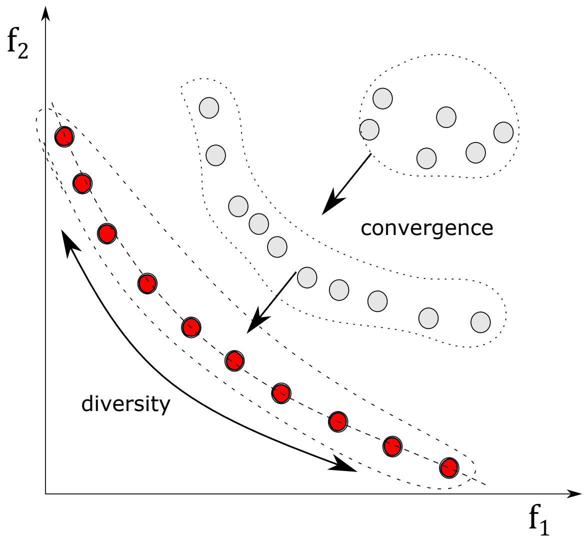



Figure 2.2: Convergence and diversity goal

Ensuring convergence is mandatory in any optimization task. Converging to a set of solutions which are not close to the true optimal set is not desirable. On the other hand, only with a diverse set of solutions we can get a good trade-off solutions. MOP deals with decision space and objective space. Two solutions said to be diverse if their Euclidean distance is large.

It is the decision makers choice to select which space for maintaining diversity. Following the similar existing state of the art approaches [4, 5, 10, 21, 27], in this study we have worked on maintaining diversity on objective space.

### 2.3.2 Special Solutions

#### Ideal Objective Vector

For each objective there can be individual optimal value and together they constituted an objective array called the ideal objective vector. In general ideal vector is a non-existent solution and can be defined formally as follows:

**Definition 2.3.1** *Ideal objective vector: For $m$ objective problem, the ideal objective vector will have $m$ elements.*

$$z^* = f^* = (f_1^*, f_2^*, ..., f_M^*) \tag{2.1}$$

*Here the $i^{th}$ element is subjected to minimum objective value of the constrained minimum solution of following problem for $i^{th}$ objective.*

$$\begin{aligned} &Minimize,\ \ f_i(\boldsymbol{x}) \\ &subject\ to\ , \boldsymbol{x} \in N \end{aligned} \tag{2.2}$$

#### Nadir Objective vector

Unlike the ideal objective vector which represents the lower bound of each objective in the entire feasible search space, the nadir objective vector, $z^{nad}$, represents the upper bound of each objective in the entire Pareto-optimal set and not the entire search space. This is different and not to be confused with worst feasible objective function vectors.

Fig. 2.3 shows the ideal vector $z^*$ and nadir vector $z^{nad}$ for a minimization problem.

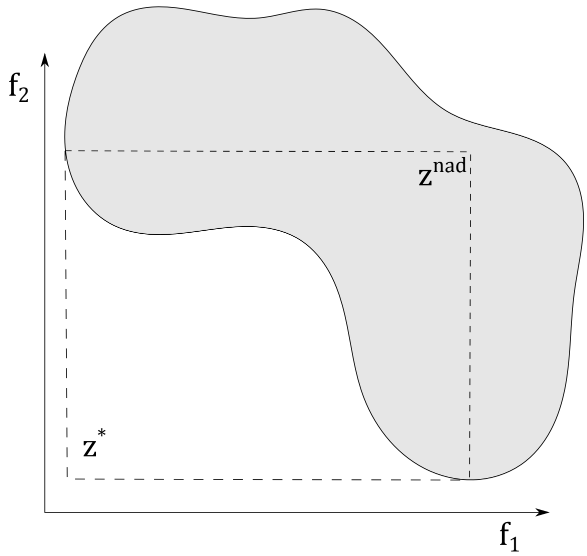



Figure 2.3: Ideal and Nadir vector in the feasible objective space

### 2.3.3 Concepts of domination

#### Pareto Dominance

Most multi-objective algorithms use the concept of domination. In these algorithms, two solutions are compared on the basis of whether one dominates the other solution or not.

**Definition 2.3.2** *Pareto Dominance: For a minimization problem, a solution* $\mathbf{x^a}$ *with objective vector* $f(\mathbf{x^a}) = [f_1(\mathbf{x^a}), f_2(\mathbf{x^a}), f_3(\mathbf{x^a}), ..., f_m(\mathbf{x^a})]$ *is said to dominate another solution* $\mathbf{x^b}$ *with objective vector* $f(\mathbf{x^b}) = [f_1(\mathbf{x^b}), f_2(\mathbf{x^b}), f_3(\mathbf{x^b}), ..., f_m(\mathbf{x^b})]$*, iff* $\forall i, f_i(\mathbf{x^a}) \leq f_i(\mathbf{x^b})$ *and* $\exists i, f_i(\mathbf{x^a}) < f_i(\mathbf{x^b})$.

Based on dominance relation there can be three possible outcomes,

1. $\mathbf{x^a}$ dominates $\mathbf{x^b}$

2. $\mathbf{x^b}$ dominates $\mathbf{x^a}$

3. $\mathbf{x^a}$ and $\mathbf{x^b}$ are non-dominated with each other.

The Pareto dominance is not *reflexive*, since any solution $\mathbf{x}^a$ doesn't dominant itself. The dominance relation is also not *symmetric*, because $\mathbf{x}^a$ dominates $\mathbf{x}^b$ does not imply that $\mathbf{x}^b$ dominates $\mathbf{x}^a$. Since dominance relation is not *symmetric*, it cannot be *antisymmetric* as well. The relation is *transitive* because if $\mathbf{x}^a$ dominates $\mathbf{x}^b$ and $\mathbf{x}^b$ dominates $\mathbf{x}^c$, then $\mathbf{x}^a$ dominates $\mathbf{x}^c$.

Fig. 2.4 shows the Pareto dominance relation of a 2- objective minimization problem.

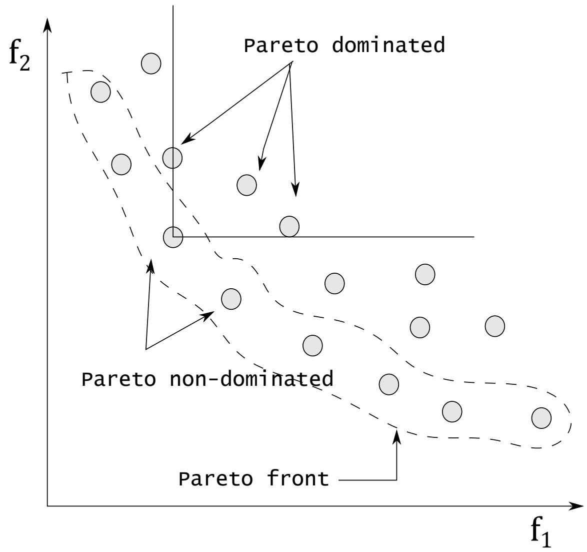



Figure 2.4: Pareto dominance relation and Pareto front of a simple 2- objective minimization problem

#### Pareto Optimality

Pareto optimality states that $\mathbf{x}$ is Pareto optimal if no feasible vector exists which would improve some criterion without causing a simultaneous worsening in at least another one. Contrary to single-objective optimization problems, the solution to a MOP is not a single solution, but a set of nondominated solutions called the Pareto optimal set. A solution that belongs to this set is said to be a Pareto optimum and when the solutions of this set are plotted in the objective space, they are collectively known as the Pareto front. Obtaining an accurate approximation of the Pareto front is the main goal in multi-objective optimization.

Fig. 2.5 shows the Pareto Optimal front of a 2 objective minimization optimization problem in the feasible objective space.

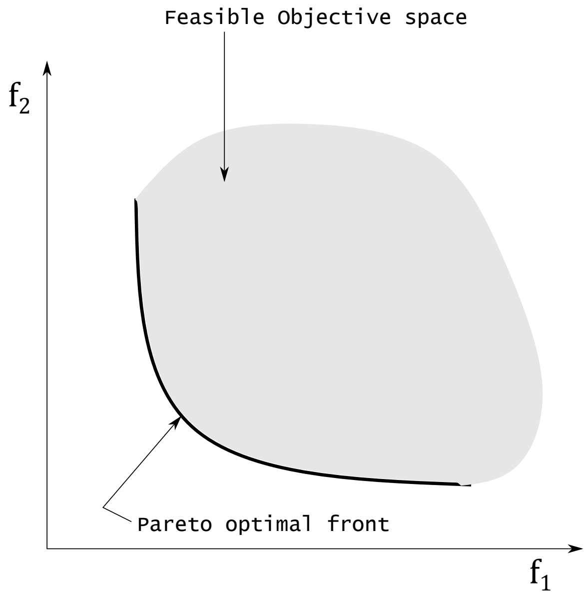



Figure 2.5: Pareto Front in the feasible objective space

## 2.4 Fuzzy Preliminaries and Related Work

The loss of selection pressure occurs when a selection scheme fails to discriminate solutions based on their objective values. To overcome this problem, a myriad of techniques have been proposed which are highlighted in the previous section. One such technique is the incorporation fuzzy concepts in fitness evaluation, which is also employed in the proposed $F$-DEA. In this section, some related fuzzy based works, basic ideas of fuzzy logic, the issues of using such concepts in the existing methodologies and other related works of $F$-DEA are discussed.

### 2.4.1 Fuzzy Preliminaries

The term fuzzy logic was first introduced by Lotfi Zadeh (1965) in his seminal work "Fuzzy sets. It refers to a form of many-valued logic in which the output of an input data may be any real value between 0 and 1, beyond crisp value (say, true and false). The application of fuzzy logic to solve a given problem usually requires three steps: fuzzification, fuzzy inference and defuzzification. Fuzzification converts input data into fuzzy data through membership functions. The second step combines the membership values to derive the fuzzy outputs. Finally, the fuzzy outputs are converted back to crisp outputs, if necessary. Let $F$ is a fuzzy set and $v$ is an element of any set $V$. We call $\gamma_F(v)$ as the membership degree of $v$ in $F$, which quantifies the membership grade of $v$ to $F$. If $\gamma_F(v)$ is 0 (or 1), it indicates $v$ is not a member (or fully a member) of $F$. The values between 0 and 1 characterize fuzzy members, representing $v$ belongs to $F$ only partially.

A variety of membership functions has been used in the literature [22, 35–38]. These include triangular shaped, trapezoidal-shaped, Gaussian, bell-shaped, and Sigmoidal membership functions. The choice of membership function, however, is dependent on the nature of applications. For example, the triangular-shaped or trapezoidal-shaped membership function is used for those applications that require significant variation within a short period of time, while the Gaussian or Sigmoid one is used for those applications that require high precision.

Pareto dominance based EMO algorithms compare two solutions $\mathbf{x}^a = x_1^a, x_2^a, \ldots, x_n^a$ and $\mathbf{x}^b = x_1^b, x_2^b, \ldots, x_n^b$ based on their objective vectors $\mathbf{f}(\mathbf{x}^a) = f_1(\mathbf{x}^a), f_2(\mathbf{x}^a), \ldots, f_m(\mathbf{x}^a)$ and $\mathbf{f}(\mathbf{x}^b) = f_1(\mathbf{x}^b), f_2(\mathbf{x}^b), \ldots, f_m(\mathbf{x}^b)$, respectively. In line with this, fuzzy based EMO algorithms can utilize the difference of the objective vectors as the argument of the membership function i.e., $v = f_i(\mathbf{x}^a) - f_i(\mathbf{x}^b)$ or $v = f_i(\mathbf{x}^b) - f_i(\mathbf{x}^a)$. As a membership function, the left Gaussian function, as depicted in Fig. 2.6, is usually employed in the EMO literature (e.g. [22], [37]). An interesting feature of this function is its monotonic decreasing nature. The analytic form of this function is

$$\gamma_g(v) = \mathrm{e}^{-\frac{1}{2}(\frac{v-\mu}{\sigma})^2} \tag{2.3}$$

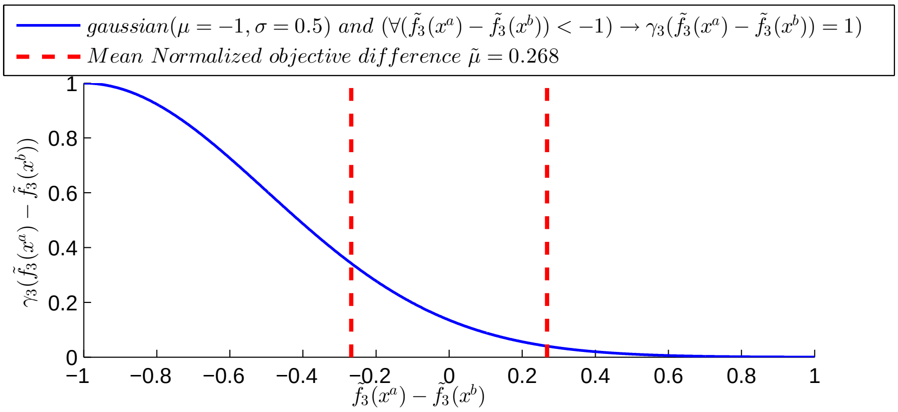



Figure 2.6: Left Gaussian Membership Function. This particular case shows the position of mean nomalized objective difference value ($\bar{\mu_3}$) for the third objective of WFG2 problem obtained from the $250^{th}$ generation of a particular seed.

It is seen from Eq. (2.3) that the Gaussian function is defined by two parameters, $\mu$ and $\sigma$. While $\mu$ represents mean and it is set to $-1$, $\sigma$ defines the spread of the Gaussian function and it is set to 0.5 as a compromise. It has been indicated that setting $\sigma$ too large or too small leads to the inability to discriminate $v \in [-1, 0]$ or $v \in [0, 1]$ [22]. Even setting $\sigma$ in this way may create uneven discrimination between the entire domain of $v \in [-1, 1]$. This can be attributed from Fig. 2.6 that the discrimination ability of $\gamma_g(v)$ for $v \in [-1, 0]$ and for $v \in [0, 1]$ is not similar because the curve's nature in the said ranges are not identical.

An MaOP may or may not have an identical range of values for its each objective, $f_i$. To handle this issue, we may use a different membership function for each of different objectives i.e., a separate $\sigma$ for each objective. A similar concept is used in [37] where $\sigma$ is chosen manually. This will, however, require rich domain knowledge for each problem we like to solve. Alternatively, we may normalize $v$ by the maximum difference among all pairs of solutions in that objective and can use the same membership function for all objectives. This alternative approach is used in some previous studies (e.g. [22]). The downside of normalization is the loss of information and influence of very large/small isolated value(s).

In the inference step, to compute dominance of one solution with respect to other (e.g. $\mathbf{x}^a$ over $\mathbf{x}^b$), the fuzzy membership values of $m$-objective differences $(f_i(\mathbf{x}^a) - f_i(\mathbf{x}^b), i = 1, 2, \cdots, m)$ are combined. Irrespective of the type of membership function, any suitable fuzzy set theoretic operation can be used for combination [40]. Table 2.1 shows some typical inference operations. The fuzzy intersection set operation is generally used for a minimization problem. The most popular intersection set operation used in MaOPs is the product operation, also known as $t$-norm operation.

Table 2.1: Some common fuzzy set operations to combine membership values in the inference step of fuzzy logic

| OR (Union) | AND (Intersection) |
|---|---|
| $Max(\{\gamma_1(v_1), \gamma_2(v_2)\})$ | $Min(\{\gamma_1(v_1), \gamma_2(v_2)\})$ |
| $\gamma_1(v_1) + \gamma_2(v_2) - \gamma_1(v_1) \times \gamma_2(v_2)$ | $\gamma_1(v_1) \times \gamma_2(v_2)$ |
| $Min(\{1, \gamma_1(v_1) + \gamma_2(v_2)\})$ | $Max(\{0, \gamma_1(v_1) + \gamma_2(v_2) - 1\})$ |

The following equations show the fuzzy inference and relative dominance computation procedure of two $m$-objective solutions, $\mathbf{x}^a$ and $\mathbf{x}^b$.

$$dom(\mathbf{x}^a, \mathbf{x}^b) = \prod_{i=1}^{m} \gamma_i(f_i(\mathbf{x}^a) - f_i(\mathbf{x}^b)) \tag{2.4}$$

$$dom(\mathbf{x}^b, \mathbf{x}^a) = \prod_{i=1}^{m} \gamma_i(f_i(\mathbf{x}^b) - f_i(\mathbf{x}^a)) \tag{2.5}$$

$$\phi(\mathbf{x}^a, \mathbf{x}^b) = \frac{dom(\mathbf{x}^a, \mathbf{x}^b)}{dom(\mathbf{x}^a, \mathbf{x}^b) + dom(\mathbf{x}^b, \mathbf{x}^a)} \tag{2.6}$$

$$\phi(\mathbf{x}^b, \mathbf{x}^a) = \frac{dom(\mathbf{x}^b, \mathbf{x}^a)}{dom(\mathbf{x}^a, \mathbf{x}^b) + dom(\mathbf{x}^b, \mathbf{x}^a)} \tag{2.7}$$

where $dom(\mathbf{x}^i, \mathbf{x}^j)$ is a scalar value and represents how much $\mathbf{x}^a$ dominates $\mathbf{x}^b$. The term $\phi(\mathbf{x}^a, \mathbf{x}^b)$ indicates the relative dominance of $\mathbf{x}^a$ to $\mathbf{x}^b$. We consider $\mathbf{x}^a$ dominates $\mathbf{x}^b$ iff

$dom(\mathbf{x}^a, \mathbf{x}^b) > dom(\mathbf{x}^b, \mathbf{x}^a)$ and non-dominated iff $dom(\mathbf{x}^a, \mathbf{x}^b) = dom(\mathbf{x}^b, \mathbf{x}^a)$. It is evident from the above relations that if one solution is Pareto dominated then it will also be fuzzy dominated and if two solutions are fuzzy non-dominated they are Pareto non-dominated. However, it is very unlikely that two different solutions will have a same fuzzy dominance value (fuzzy non-dominated) even though they are Pareto non-dominated. Because of the different objective differences, fuzzy dominance is able to discriminate two solutions even when they are Pareto non-dominated. Therefore Pareto dominance can be considered as a special case of fuzzy dominance.

In defuzzification step, the dominance impact of one solution with respect to other solutions can be combined, ranked or used instead of Pareto dominance [22, 35–38]. An important observation is that irrespective of a membership function and a set theoretic operation, we end up with only a scalar fuzzy value which does not tell anything about diversity. If the solutions are selected solely based on the fuzzy dominance values, the chosen solutions will be less diverse and will not able to cover the entire Pareto front of a given problem. This can be seen from the experimental evidences provided in a recent study [25]. Some existing approaches (e.g. [22], [38]) use a threshold parameter to divide solutions into different fronts but still fails to provide enough diversity in high-dimensional and non-regular Pareto [41].

### 2.4.2 Related Fuzzy Based Work

There are a very few fuzzy based EMO algorithms in the literature.

In [37], Farina and Amato proposed definitions of fuzzy-based optimality for multi-objective optimization. The main idea behind the given definitions is to introduce different degree of optimality. For each objective, the authors used a combination of three Gaussian functions as a non-adaptive membership function. The purpose of such a combined mechanism is to distinguish amongst better, equal and worse objective differences between two solutions. The proposed definitions have been applied on two simple multi-criteria decision making problems and two MOPs.

The work described in [36] studied the fuzzyfication of Pareto dominance relation and its application to the design of an EMO algorithm. In fuzzyfication, the authors used a non-symmetric membership function. Solutions those have high performance in one objective but poor in others will be preferred in the given fuzzy dominance definition. As no additional diversity measure was used in [36], the evolving population will lose diversity. To verify the usefulness of proposed EMO algorithm, an analytic study of the Pareto-Box problem was provided.

Nasir *et al.* [38] introduced a decomposition based fuzzy dominance algorithm (MOEA/DFD). For all objectives, the algorithm uses a general membership function of $Ae^{-x}$ with uneven discrimination power. As the same membership function is used for all objectives, scaling issues are not addressed here. In comparing solutions, MOEA/DFD employed fuzzy dominance when a solution's dominance level is found greater than a particular threshold value. Otherwise, it used decomposition based weighted approach, which is able to handle diversity. The performance of MOEA/DFD was evaluated on twelve benchmark problems having two-objective to five-objective.

In [22, 35], the authors utilized fuzzy dominance concept to continuously differentiate individuals of a population into different degrees of optimality. They used the same left Gaussian function as a membership function for all objectives of a given MaOP. To handle the scaling issue, the objective differences are normalized by corresponding maximum objective difference value. The fuzzy concept was incorporated into NSGA-II and SPEA2 as a case study and termed them FD-NSGA-II and FD-SPEA2. In FD-NSGAII, a threshold value is used to divide solutions into different fronts based on the fuzzy dominance value. However, the solutions in the last front are chosen based on crowding distance. As fuzzy dominance is capable of discriminating solutions, crowding distance will rarely be used. Moreover, crowding distance has already found to be ineffective on MaOPs [21, 23, 27]. In FD-SPEA2, the best $N$ solutions for next generation are taken based on fuzzy dominance values. The performance of the proposed algorithm was evaluated on seven DTLZ problems and two WFG problems having 5-, 10- and 20-objective.

In summary, with respect to existing fuzzy works, three issues need to be addressed while using fuzzy dominance in an EMO algorithm. These include uneven discrimination ability of the membership function, normalization of the objective difference and loss of diversity during environmental selection.

The proposed framework described in the following Chapter uses a preferred reference point based clustering along with adaptive Sigmoid membership function to deal with the issues commonly faced by other fuzzy based approaches. While the preferred reference point based clustering is used to deal with diversity promotion (Section 3.2.3), the adaptive membership function (3.2.4) is employed to handle the objective scaling issue.

# Chapter 3

# Proposed Framework

## 3.1 Introduction

From the previous chapter we can realize that the fuzzy concept is particularly suitable for many objective problem as it can effectively offer comparability among non-dominated solutions for large number of objectives. The proposed $F$-DEA utilizes this notion and handles the issues faced by existing approaches. In order to avoid the detrimental effect of uneven discrimination and objective normalization, $F$-DEA employs a separate membership function for each objective and determines its parameters adaptively. It also employs preferred reference points based clustering for ensuring diversity in its environmental selection. These features make the proposed algorithm different from others. In the following sections our proposed framework is described and the theoretical aspects of the components explained which are further supported through experiments in the Chapter 4. Finally in terms of existing works we mentioned the novel aspects of our proposed $F$-DEA.

## 3.2 Proposed Algorithm

The algorithm starts with a randomly generated parent population $P_t$ of $N$ solutions and a set of generated/supplied reference points $R^g/R^s$. It then creates an offspring population $Q_t$ of size $N$ by applying crossover and mutation. Based on the positional information in the objective space, $F$-DEA finds $p$ preferred points ($p \leq N$) and constructs clusters using the solutions as members and the preferred points as centers. It then adaptively constructs $m$ fuzzy membership functions i.e., one function for each objective and utilizes them to compute dominance degrees of the solutions. Finally, $F$-DEA assigns fitness to the solutions and selects the best ones from different clusters in a round-robin fashion to from a new population $P_{t+1}$ for the next generation. We summarize the steps of our method in Algorithm 3.1, which are explained further as follows.

**Algorithm 3.1** Generation $t$ of $F$-DEA

**Input:** $R^g/R^s$ (generated reference points or supplied points), $P_t$ (parent population of $t$-th generation), $N$ (population size)

**Output:** $P_{t+1}$

1: $Q_t \leftarrow$ Recombination and Mutation on $P_t$
2: $C_t \leftarrow P_t \cup Q_t$
3: Select $p \leq N$ preferred reference points:
$R^p \leftarrow PreferredReferencePoint(C_t, R^g/R^s, p)$
4: Construct clusters using $R^p$:
$R^p(\{\mathbf{r}^j, X(\mathbf{r}^j) | 1 \leq j \leq p\})$ % $\mathbf{r}^j$: $j$-th reference point in $R^p$, $X(\mathbf{r}^j)$: associated solution with $\mathbf{r}^j$
5: Construct $m$ membership functions:
$(\gamma_1, \gamma_2 \cdots \gamma_m) \leftarrow AdaptiveMembershipFunction(C_t)$
6: Assign fitness to solutions within individual cluster:
$FitnessAssignment(R^p, C_t, \gamma)$
7: Sample solution from each cluster:
$P_{t+1} \leftarrow SamplingSelection(R^p, N)$

**Algorithm 3.2** $PreferredReferencePoint(C_t, R^g/R^s, p)$

**Input:** $R^g/R^s$, $C_t$ (combined population), $p$ (maximum size of preferred reference points)
**Output:** $R^p(\{\mathbf{r}^j, X(\mathbf{r}^j)|1 \leq j \leq p\})$ (set of preferred reference points $\mathbf{r}^j$s with associated cluster of solutions $X(\mathbf{r}^j)s$)

**for** each solution $\mathbf{x} \in C_t$ **do**
  Normalize $f(\mathbf{x})$: $\tilde{f}(\mathbf{x}) = \tilde{f}_1(\mathbf{x}), \tilde{f}_2(\mathbf{x}), \ldots \tilde{f}_m(\mathbf{x})$
**end for**
$R^a = \{\emptyset\}$
**for** each solution $\mathbf{x} \in C_t$ **do**
  $\mathbf{r} = \mathbf{r^g} : \text{argmax}_{\mathbf{r^g} \in R^g/R^s}(S(\mathbf{r^g}, \tilde{\mathbf{f}}(\mathbf{x})))$
  **if** $\mathbf{r} \notin R^a$ **then**
    $X(\mathbf{r}) = \{\mathbf{x}\}$
    $R^a = R^a \bigcup \{\mathbf{r}\}$
  **else**
    $X(\mathbf{r}) = X(\mathbf{r}) \bigcup \{\mathbf{x}\}$
  **end if**
**end for**
**if** $|R^a| > p$ **then**
  Sort $R^a$: $MinMax(R^a)$
  $R^p =$ First $p$ points of $R^a$
  $R^r = (R^a - R^p)$
  **for** each $\mathbf{r}^r \in R^r$ **do**
    **for** each solution $\mathbf{x} \in X(\mathbf{r}^r)$ **do**
      $\mathbf{r} = \mathbf{r}^p : \text{argmax}_{\mathbf{r}^p \in R^p}(S(\mathbf{r}^p, \tilde{\mathbf{f}}(\mathbf{x})))$
      $X(\mathbf{r}) = X(\mathbf{r}) \bigcup \{\mathbf{x}\}$
    **end for**
  **end for**
**else**
  $R^p = R^a$
**end if**

### 3.2.1 Reference Point Generation

A number of existing studies [3, 20, 21, 27, 31, 42] employ reference points for assigning fitness values. However, $F$-DEA uses such points for clustering the solutions with a hope of maintaining both convergence and diversity. There are two important considerations for generating the reference points. Firstly, the reference points are to be generated in such a way so that they are uniformly distributed over the $m$-dimensional objective space. Secondly, the generation procedure has to be scaleable and computationally efficient. Keeping these considerations in

**Algorithm 3.3** $MinMax(R)$

**Input:** $R$ (reference points)
**Output:** $R^{st}$ (reference points sorted according to $MinMax$ distance)

```
R^st = ∅
R^m = A set of m extreme points (one for each objective) chosen from R
R^st = R^st ⋃ R^m
R' = R − R^st
for each r in R' do
    Distance measure: dist(r) = min_{∀r^st ∈ R^st} d(r, r^st)
end for
while R' ≠ ∅ do
    r^b = r : argmax_{r∈R'} dist(r)
    R^s = R^s ⋃ {r^b}
    R' = R' − {r^b}
    for each r in R' do
        Update: dist(r) = min(dist(r), d(r, r^b))
    end for
end while
```

mind, $F$-DEA uses the Das and Dennis's procedure [14] like others [2,21,27] for generating the reference points, which are distributed uniformly along the $m$-dimensional hyper-plane. The procedure generates a set of reference points, $R^g$, spanning the whole plane, each at $\delta = \frac{1}{\lambda}$ distance apart from the others. Analytically, $|R^g| = \binom{m+\lambda-1}{\lambda}$ reference points are generated where $\lambda$ denotes the number of divisions in each objective-coordinate.

The main problem of this generation procedure or others [2] is the exponential increase of reference points with the increase of objectives. A simple technique to handle this issue is to increase the population size with respect to $|R^g|$. This in turn will increase evolution time. As an alternative, some studies (e.g. [21, 27]) generate a smaller number of reference points and impose this number as a constraint on the population size. $F$-DEA, on the other hand, first generates a large number reference points and then selects a set of preferred points based on the population size. This alleviates the problem of imposing constraint on the population size.

### 3.2.2 Preferred Reference Point

All the generated reference points may not be equally important with respect to the existing solutions of an evolving population. The proposed $F$-DEA thus selects a set of preferred reference points, $R^p$. It first finds the active reference points from $R^g/R^s$ and then applies the *MinMax* procedure on them to choose $p$ diverse points. A reference point is called an active reference point if it maintains some associations with one (or more) solution(s). The upper bound of $p$ i.e., $|R^p|$ is $N$, the population size. The solutions of a population may reside in some (not all) regions of the $m$-dimensional objective space. In that case, $p$ will be less than $N$. Our algorithm constructs $p$ clusters using the preferred points as their centers and the solutions of $C_t$ as their members. The procedure for selecting the preferred reference points and clusters is given in Algorithm 3.2.

**Active Reference Point**

As the values of the generated reference points lie in the range between 0 and 1, we adaptively normalize the objective values of the solutions to measure how close a solution is with respect to a particular reference point. Following the adaptive normalization procedure of [27], the $i$-th objective value, $f_i(\mathbf{x}^a)$, of any solution $\mathbf{x}^a$ can be normalized as

$$\tilde{f}_i(\mathbf{x}^a) = \frac{f_i(\mathbf{x}^a) - f_i^{min}(\mathbf{x}^u)}{\mathbf{z}^i - f_i^{min}(\mathbf{x}^u)} \quad \forall i \in 1, 2, \cdots, m \tag{3.1}$$

where $\tilde{f}_i(\mathbf{x}^a)$ is the $i$-th normalized objective value of the solution $\mathbf{x}^a$. The symbol $f_i^{min}(\mathbf{x}^u)$ refers to the minimum value of the $i$-th objective with respect to all solutions in $C_t$ and the solution $\mathbf{x}^u$ has the minimum $i$-th objective value. The symbol $\mathbf{z}_i$ refers to the intercept computed from the $i$-th objective axis and a $m$-dimensional linear hyper-plane. The hyper-plane is constituted from $m$ solutions, where a solution $\mathbf{x}^h \in C_t$ makes the following achievement scalarizing function (ASF) minimum for an objective direction $\mathbf{w}_i$.

$$ASF(\mathbf{x}^h, \mathbf{w}_i) = \max_{i=1}^{m} \frac{f(\mathbf{x}^h) - f^{min}(\mathbf{x}^u)}{\mathbf{w}_i} \quad \mathbf{x}^h \in C_t \tag{3.2}$$

Normalization using intercepts helps solutions to expand it's objective space.

The closeness of solutions indicates their associations with the reference points. We utilize cosine similarity measure for this purpose. The cosine similarity measure, $S(\mathbf{r^j}, \tilde{\mathbf{f}}(\mathbf{x}^a))$, between the normalized fitness vector $\tilde{\mathbf{f}}(\mathbf{x}^a) = \tilde{f}_1(\mathbf{x}^a), \tilde{f}_2(\mathbf{x}^a), \ldots \tilde{f}_m(\mathbf{x}^a)$ of any solution $\mathbf{x}^a$ and the $m$-dimensional $j$-th reference point $\mathbf{r}^j$ can be computed as

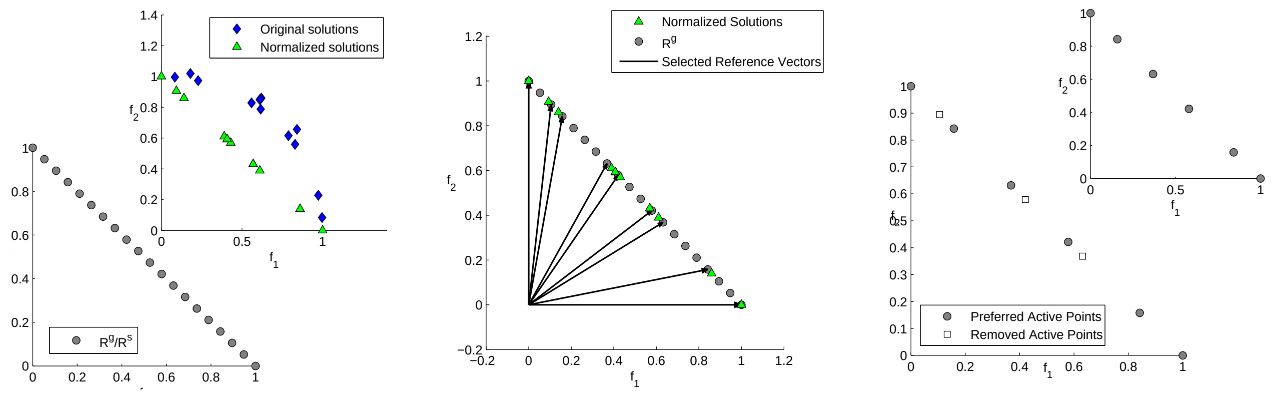
$$S(\mathbf{r^j}, \tilde{\mathbf{f}}(\mathbf{x}^a)) = \frac{\tilde{\mathbf{f}}(\mathbf{x}^a).\mathbf{r}^j}{|\tilde{\mathbf{f}}(\mathbf{x}^a)||\mathbf{r}^j|} \quad \forall j \in 1, 2, \cdots, |R^g| \tag{3.3}$$

As we generate a set of reference points, $R^g$, there will be $|R^g|$ similarity measures for $\mathbf{x}^a$. The largest similarity measure obtained for any of the reference points is considered as the active reference point that maintains the highest association with $\mathbf{x}^a$. While associating the solutions with the reference points, it is possible that some points may associate more than one solutions, some may associate one solution or some may associate no solution at all.

### MinMax Measure

The objective space size increases exponentially with an increasing number of objectives. It would be beneficial if we can cover the whole objective space by a reasonable number of active reference points in the least crowded manner. We devise a procedure based on the *MinMax* measure to select $p$ diverse active reference points, which $F$-DEA employs as the clusters' centers, from the crowded ones (Algorithm 3.3).

The procedure first selects $m$ number of most extreme active reference points i.e., one for each objective and then incrementally selects the other points. To select the other points, it calculates the minimum Euclidean distance between the already selected points and the remaining ones. The point with the maximum distance is selected as the next point and the minimum distance of the remaining points are updated based on the selected point. This procedure continues until all the points are selected. In this way, we finally obtain a sorted list of active reference points that are far apart from each other i.e., diverse. This procedure will be effective for disconnected, degenerated and other irregular shaped geometry because we are using the reference points only where solutions exist. Thus we are effectively maintaining cluster uniformity using the same number of reference points as other approaches but we are using them where it is needed. It may possible to devise other techniques as well, which can be a future research topic. Fig. 3.1 demonstrates the process of selecting the preferred reference points from a set of generated reference points.




(a) $|R^g| = 20$, Original Solutions $|C_t| = 12$

(b) Active reference points $|R^a| = 9$ (marked with vector line)

(c) Selection of a number of preferred reference points ($p = 6$)

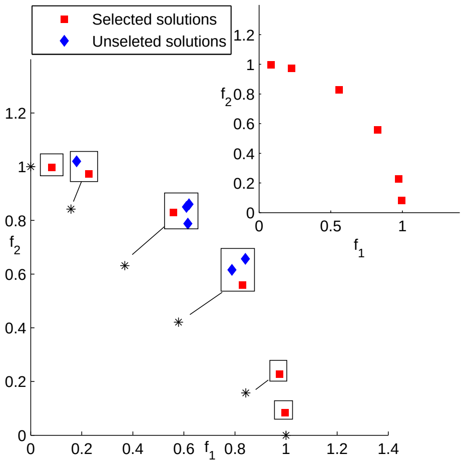
Figure 3.1: Process of selecting the preferred reference points from a set of generated reference points

### 3.2.3 Clustering

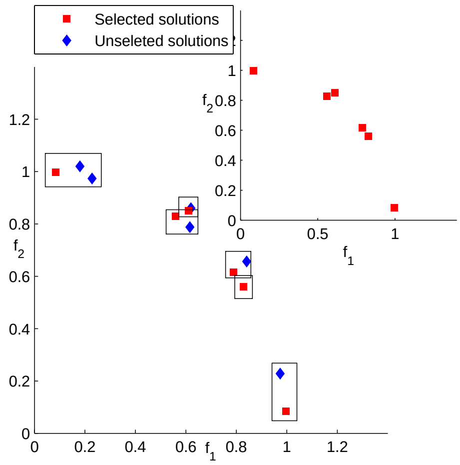
The proposed algorithm constructs clusters where the preferred reference points and solutions (data points) are used as the clusters' centers and members, respectively. An important question may arise why we do not use a classical approach for clustering. The classical approach usually partition a set of data points into a set of meaningful sub-classes with an aim of understanding the natural grouping/structure among them. This is why the clusters' centers are determined from the existing data points. On the other hand, the aim of our clustering is to facilitate diversity in selecting solutions for the next generation. We thus use a set of preferred points, which are diverse and selected from a set of uniformly generated reference points, as the clusters' centers. Although the generation of uniform reference points may suffer in case of an irregular Pareto front, the use of the preferred reference points as the clusters' centers alleviates this problem to some extent.

Fig. 3.2 shows the effect of preferred reference points based clustering and traditional $k$-means clustering. It is clear from the figure that $k$-means clustering with Euclidean distance measure won't work as it considers locally crowded regions to form clusters. Thus, the solutions that are Pareto dominated might be grouped together in the objective space. Although $k$-means clustering with cosine measure handles the aforementioned issue, it does not consider and maintain uniformity in clusters. In contrast, the reference points based clustering we employ in $F$-DEA maintains clusters uniformity by employing a uniformly distributed reference points as the clusters' centers.

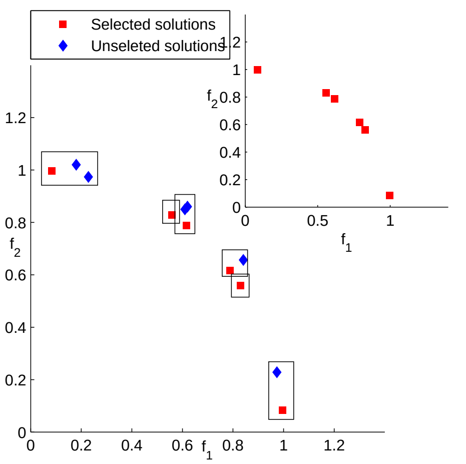
The reference points and evolved solutions are considered in $F$-DEA as the clusters' centers and members, respectively. Existing works (e.g. [3, 20, 21, 27, 31, 42] generally utilize reference points in some way for assigning fitness to the solutions. Some previous non-fuzzy EMO algorithms [21,27,28] also employ clustering in solving MaOPs. The clustering procedure of $F$-DEA differs from the one used in [27] with two notable exceptions. Firstly, $F$-DEA employs preferred reference points based clustering to provide better cluster uniformity, remove dependency on population size and handle irregular shaped Pareto fronts. Secondly, it uses cosine similarity instead of Euclidean distance to find the solutions' association with the generated reference points. It is suitable because angle between a reference point and a candidate solution remains constant irrespective of exact distance from the ideal point, which handles the scaling issue of the generated reference points. The $\theta$-dominance algorithm [21] also uses reference points based clustering similar to [27] with an exception in normalization procedure. CEGA [28], a clustering based elitist genetic algorithm, utilizes a bottom up hierarchical approach in the decision variable space not in the objective space. The basic notion of CEGA is that convergence with a poorly spread set of Pareto-optimal solutions is preferred than a well-spread set of solutions which are far from the Pareto-optimal surface. The procedure and notion of this clustering is different from the one we employ in $F$-DEA. In RVEA [31], reference vectors are used to decompose solutions into different sub-problems. Cosine similarity measure is used for association but unlike $F$-DEA vectors are dynamically adjusted on each generation according to the scales of the objective functions.



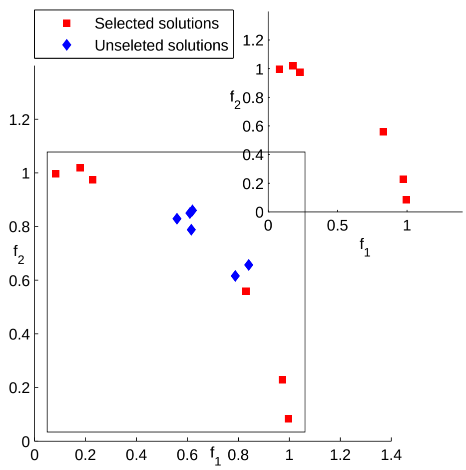

(a) Preferred Reference points based clustering




(b) $k$-means clustering with Euclidean distance measure




(c) $k$-means clustering with cosine similarity measure




(d) No clustering i.e., all solutions in one cluster

Figure 3.2: Solutions are grouped (rectangle box) by different clustering mechanisms. The red squared solutions are the selected solutions obtained by applying fuzzy fitness based environmental selection procedure within each cluster.

### 3.2.4 Adaptive Membership Function

In this work, we for the first time use the Sigmoid membership function (Fig. 3.3) into a fuzzy based EMO algorithm. This function is not only monotonically decreasing but also anti-symmetric at mean. We set its growth parameter i.e., $\alpha$ in such a way so that it handles the extreme values to some extent and work in support of our clustering approach. The Sigmoid membership function for the $i$-th objective can be defined as

$$\gamma_i(f_i(\mathbf{x}^a) - f_i(\mathbf{x}^b)) = \frac{1}{1 + e^{-\alpha_i((f_i(\mathbf{x}^a) - f_i(\mathbf{x}^b)))}} \tag{3.4}$$

where mean is set to zero for making the membership function anti-symmetric at that value.

An MOP may or may not have an identical range of values for each objective. To capture this notion, it is better to determine $\alpha_i$ adaptively. We calculate it for each objective at every generation of evolution. The mean ($\mu_i$) and variance ($\sigma_i^2$) of objective differences are used to compute $\alpha_i$. We obtain $\mu_i$ and $\sigma_i^2$ using absolute objective differences of all pairs of solutions. The growth parameter $\alpha_i$ has been defined in such a manner so that the membership value obtained from Eq. (3.4) is 0.99 at the point $(-\mu_i - \sigma_i)$. We compute $\alpha_i$ as

$$\alpha_i = \frac{\ln p - \ln(1-p)}{q_i} \tag{3.5}$$

where $p = 0.99$ and $q_i = -\mu_i - \sigma_i$. The way we define $\alpha_i$ is advantageous in three aspects.

- Firstly, as $\alpha_i$ is defined based on mean and variance instead of maximum or minimum objective difference, inappropriate normalization effect of a significantly large/small objective difference mentioned in Section 2.4 will be minimized to some degree.

- Secondly, if the objectives are in different scales, the corresponding mean and variance of the objective differences will be different which in turn will produce different $\alpha_i$s i.e., different membership functions.

- Thirdly, as we cluster the solutions, all objective differences in the same cluster will be small. It ensures that most of the objective differences will lie in the range between $-\mu_i - \sigma_i$ and $\mu_i + \sigma_i$. The shape of membership function indicates that the solutions are evenly discriminable within this range.

In short, the anti-symmetric property and $\alpha_i$'s definition resolve the issues of uneven discrimination ability, objective difference normalization and objective scaling.

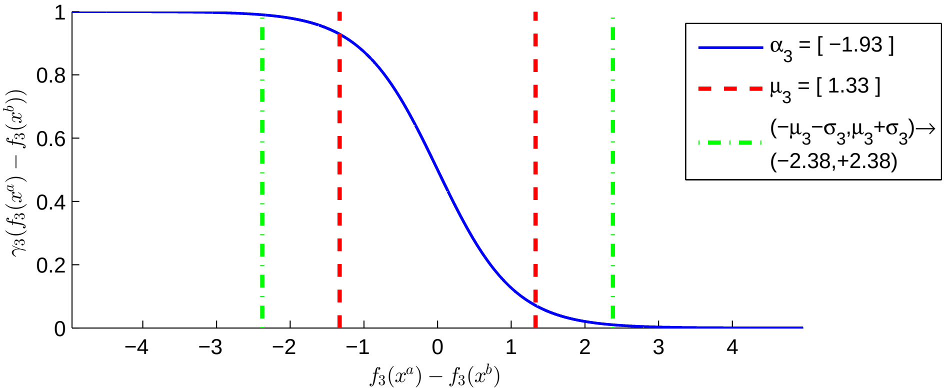



Figure 3.3: Sigmoid Membership Function. This particular case shows the position of mean objective difference value ($\mu_3$) for the third objective of WFG2 problem obtained from the $250^{th}$ generation of a particular seed.

### 3.2.5 Fuzzy Dominance and Fitness Assignment

The essence of our fuzzy dominance computation and fitness assignment is that they are local. We utilize the solutions in the same cluster for computing their dominance degrees and assigning their fitness scores. An advantage of this approach is that there is an opportunity to employ parallelism for such computation and assignment. We employ the membership function represented by Eq. (3.4) to compute membership values which in turn are used for obtaining fuzzy dominance among all pairs of solutions in each cluster. We use Eqs. (2.4) and (2.5) for obtaining fuzzy dominance.

For assigning fitness to a solution of any cluster, we first compute the relative dominance degrees, $\phi$s, using Eqs. (2.6) and (2.7). Note that the relative dominance degree of any solution $\mathbf{x}^a$ in a particular cluster is considered only with respect to all other solutions in that cluster. We then add all these degrees and assign it as the solution's fitness as represented by Eq. (3.6).

$$fit(\mathbf{x}^a) = \sum_{b=1, b\neq a}^{b=n} \phi(\mathbf{x}^b, \mathbf{x}^a) \tag{3.6}$$

Here $n$ is the number of solutions in any cluster. If a solution is least dominated by other solutions of the same cluster, then its fitness value will be smallest and it will be selected for the next generation.

### 3.2.6 Environmental Selection

As mentioned before, $F$-DEA constructs clusters using the evolved solutions as their members and the preferred reference points as their centers. The aim of our environmental selection procedure is to choose solutions for the next generation by considering not only their convergence but also their diversity. To achieve these goals, $F$-DEA first sorts the clusters by applying the *MinMax* procedure based on the clusters' centers. This is done to give priority on the distant clusters. The algorithm then selects the solutions from the sorted clusters in a round robin fashion. For a minimization problem, a solution having a minimum dominance score compared to the remaining ones of the same cluster is considered for selection. It means the selected solution is least dominated by the remaining ones of the same cluster. Giving preference to the distant clusters and selecting the least dominated solutions from them indicates $F$-DEA's emphasis on both diversity and convergence in its environmental selection.

To visualize the essence of our reference points based clustering, Fig. 3.2 shows the effect of reference points based clustering, no clustering, $k$-means clustering with Euclidean distance and $k$-means clustering with cosine similarity measure. It is evident from this figure that reference points based clustering is able to maintain well both convergence and diversity in selecting solutions for the next generation. In contrast, when no clustering is employed, fuzzy dominance ranks all solutions based on their scalar values and the corner solutions are selected in concave surface due to bias introduced by fuzzy dominance (Fig. 3.2(d)). Diversity maintenance of $k$-means clustering with Euclidean distance (Fig. 3.2(b)) is better than no clustering (Fig. 3.2(d)) but worse than $k$-means clustering with cosine similarity measure (Fig. 3.2(c)).

### 3.2.7 Contribution of Fuzzy Ranking and Reference Points

To get a further insight of how much contribution we get from reference points based clustering and fuzzy dominance over Pareto dominance, the theoretical aspects are discussed in this section and detail experimentation has been conducted and presented in Experimental Studies Section 4.10.

**Impact of Fuzzy Fitness Assingment**

In many-objective problems almost all solutions become non-dominated during evolutionary progress that is why Pareto based approaches fail to provide enough selection pressure. Fuzzy domination principle can offer comparability among Pareto incomparable solutions. Depending on fuzzy domination principle, there exists bias to which direction solutions are preferred. To handle this bias and make it work in favor of selection pressure we have modified fuzzy membership functions and incorporated reference points based cluster mechanism to provide more comparability and maintain diversity throughout generation.

From the fitness assignment procedure of $F$-DEA we get a complete order of solutions even if all of them are non-dominated to one another. Therefore if the solutions are selected based on this fitness value a bias will be noticed. The following examples shows that the fuzzy definition handles scaling issue, appropriately assigns better fitness and prefers those solutions that dominate others and are least dominated.

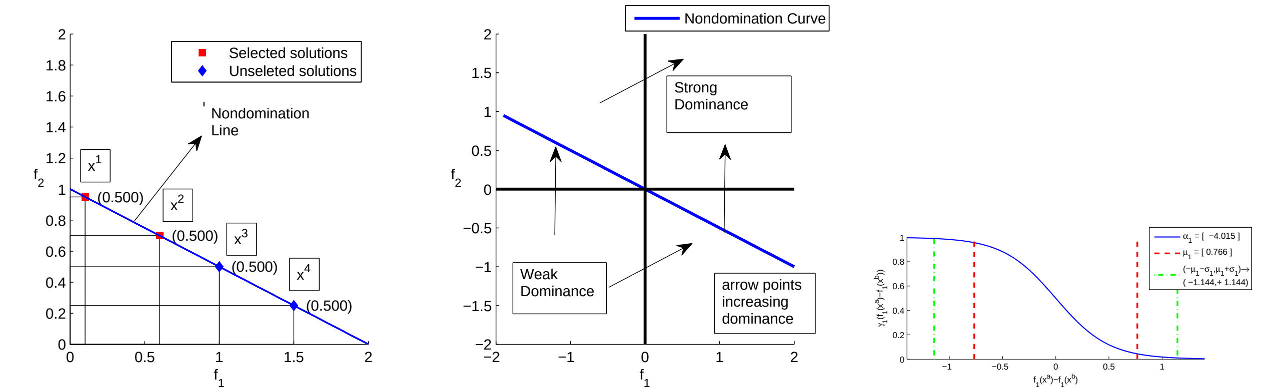
- *Scaling issue:* Let solutions are distributed along a straight line where scale of the two objective may be different. Fig 3.4(a) shows such a case where we have 4 solutions in a straight line where scale of two objectives are different. The membership functions of the two objectives in Fig. 3.4(c) and Fig. 3.4(d) are derived based on the solution objective values.

  Here we can see the span of $f_1$ objectives' membership function is longer than span of $f_2$ objectives' membership function. Therefore in case of dominance relation between solution $\mathbf{x}^1$ and $\mathbf{x}^2$, we see that improvement of solution $\mathbf{x}^1$ to $\mathbf{x}^2$ in objective $f_1$ is relatively same with the improvement of solution $\mathbf{x}^2$ to $\mathbf{x}^1$ in objective $f_2$ although their objective difference is different.

  The span of the membership functions handles the scaling factor of solutions. To understand more precisely in Fig. 3.4(b), we have shown the domination curve derived from the two membership functions. This plot shows the domination impact of a solution to other solution respective to it. The blue line represents the non-domination curve, any solution along that curve will be non-dominated with respective to centered solution.

  We have calculated non-domination curve from the following relation, $\gamma_1(dx_1) * \gamma_2(dx_2) = \gamma_1(-dx_1) * \gamma_2(-dx_2)$
  Here $dx_1$ represents the 1st objective difference of two solutions and $dx_2$ represents 2nd objective difference of two solutions. Two solutions will return same dominance value if the above relation satisfies. Therefore solving for $dx_2$ for given different values of $dx_1$ we can obtain the non-domination curve.

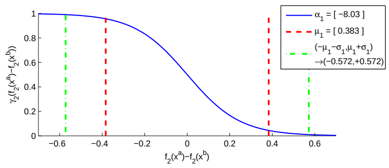
  From the dominance relation plot we can see how the scale issue is handled. As we move to upper quadrant the domination impact increases non-linearly based on the objective difference value because of the multiplication of Sigmoid functions. In Fig. 3.4(a) we have plotted non-domination curve with respect to individual solution. In this case all solutions resides on same line, so they are non-dominated to each other means domination impact to each other is same. Assigned normalized fitness value is same and 0.5 for all solutions. The two selected solutions in figure are selected based on randomly according to the solutions order of appearance.




(a) Solutions with two differently scaled objective values.

(b) Domination Curve adjusts itselfs according to the scale of objectives.

(c) Membership function of $f_1$ objective



(d) Membership function of $f_2$ objective

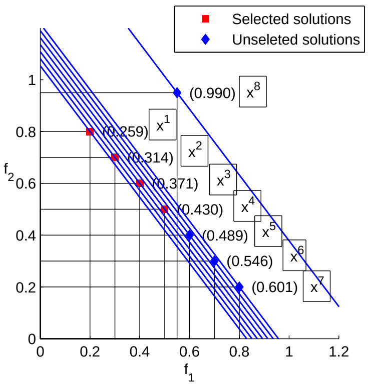
Figure 3.4: This figures explain how fuzzy fitness assignment procedure handles different scale of objectives. a) The red solutions are those which will be selected for next generation (here fitness is same for all thus any two solutions can be selected)

- *Domination Impact of solutions :* In Fig.3.5(a), let scale of the two objectives are same. Solutions, $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^7$ lies in straight line and solution $\mathbf{x}^8$ resides outside the line dominated by $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4$. Due to solution $\mathbf{x}^8$, the membership function of the two objective will be different as pairwise objective differences will be now different. Fig.3.5 shows the membership functions and domination curve.

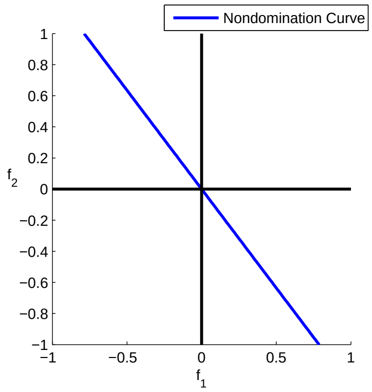
  When in comparison $\mathbf{x}^1$ and $\mathbf{x}^2$ although the objective wise difference is same but relative to the membership function, improvement in objective $f_1$ is significant than improvement in objective $f^2$, so $\mathbf{x}^1$ will dominate $\mathbf{x}^2$.

  Blue lines in Fig.3.5(a) shows the approximation non-domination curve respective to individual solutions. The fact that $\mathbf{x}^1$ dominates $\mathbf{x}^2$ can be also justified from the domination curve.

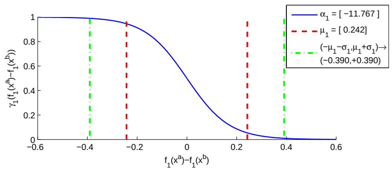
In this case we can see $\mathbf{x}^8$ is Pareto dominated by $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ and $\mathbf{x}^4$ so domination impact of $\mathbf{x}^8$ to those solutions will be less and solutions $\mathbf{x}^5, \mathbf{x}^6, \mathbf{x}^7$ are non-dominated so domination impact of $\mathbf{x}^8$ to these will be more. Thus after considering overall domination impact we will find $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ and $\mathbf{x}^4$ with minimum domination impact in that order.



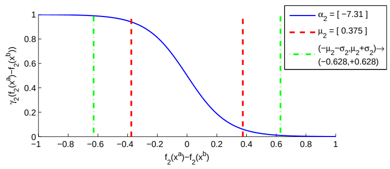

(a) Scale of two objectives are same. Solution $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ and $\mathbf{x}^4$ have least dominantion impact as these Pareto dominates $\mathbf{x}^8$




(b) Approximation domination curve



(c) Membership function for $f_1$ objective

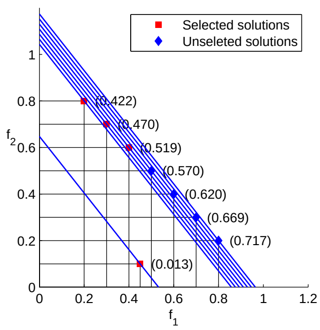


(d) Membership function for $f_2$ objective

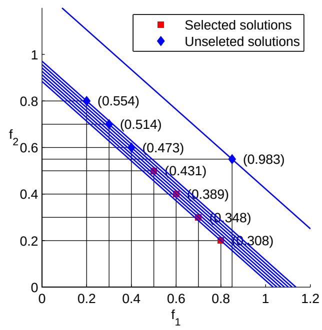
Figure 3.5: Membership function of 3.5(a) and domination curve

- *Other cases :* Fig. 3.6 shows some other cases. From these figures we can conclude that solutions dominate others solutions and solutions that are least dominated by others are usually preferred.

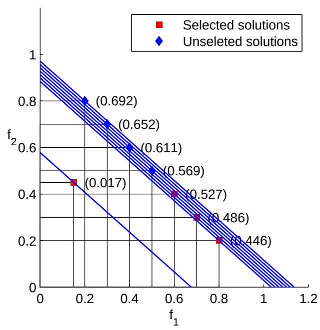
From these figures we can see that convergence of solutions is ensured but no diversity mechanism is inherent with this. Therefore if we continue next generation selection in this approach solutions will be converged to some points.



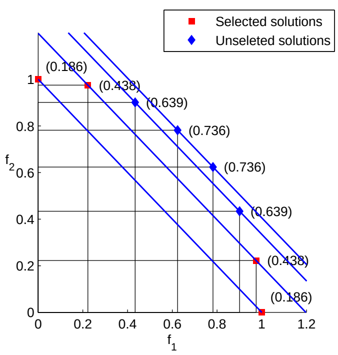

(a) Solutions that are less dominated will be preferred



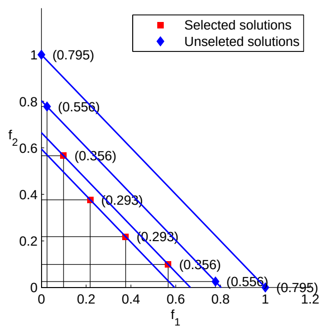

(b) Solutions that dominate others prefered




(c) Solutions that are less dominated will be preferred




(d) Concave




(e) Convex

Figure 3.6: Different scenarios showing how fitness is assigned to solution.

### Impact of Reference Points based Clustering

To support diversity throughout next generation fuzzy dominance alone is not enough because of the bias. Some existing approaches (eg. FD-NSGAII) use $\alpha$ parameter to divide solutions into fronts but still fails to provide enough diversity in high dimensional and non-regular Pareto surfaces( [22, 35]). To promote diversity we have incorporated the concept of reference points to guide solutions in certain direction. We have used reference points uniformly throughout the generations to maintain diversity. Also we know that uniform reference points may suffer from irregular Pareto front. To alleviate this problem and have a control over diversity and convergence we have introduced computation of preferred reference points. Together they will promote diversity as well as convergence in many objective problems.

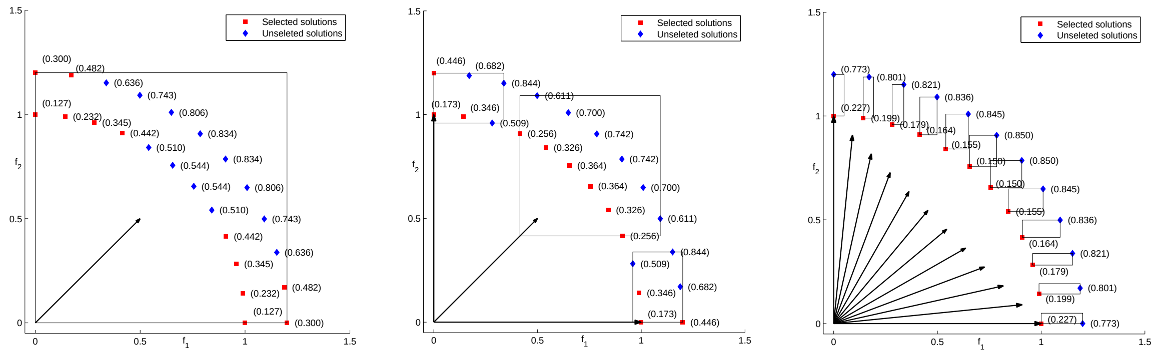
Here in the following two examples we have shown how reference points based clustering approach handles the problem of biases created by dominance operation and how using preferred reference points we can solve the irregular Pareto problem.

- Solutions in Fig. 3.7 are in two fronts so we can consider them in a single big cluster and the bias to the solutions will be to the corners as expected. The membership functions for both objective will be same in this case due to same objective values in two axis.

  Fig. 3.7(c) shows case for 3 uniform reference points and formed three clusters. As membership functions are computed from entire solution set so same domination principle will be applicable for any pair of solutions within clusters. From the figure we can see that from the middle clusters we have selected converged solutions so overall it promotes more diversity than before.

  In case of 12 uniform reference points, we get clusters of 2 solutions each and the selected solutions within the clusters guarantee uniformity as well as convergence.

- The second scenario in Fig. 3.8 explain how clustering with preferred reference points in effect works for irregular shaped Pareto problem of WFG2.




(a) Concave surface points in a single cluster

(b) Concave surface with 3 clusters

(c) Concave surface with 12 clusters

Figure 3.7: Concave surface

The motivation behind computing preferred points comes from the fact that, we use sampling from each cluster to obtain best solution for next generation. So for selecting $N$ next generation solutions there cannot be more than $N$ clusters and thus we are limited with $N$ reference points. During evaluation many reference points may have no solution in association. Our ultimate goal is to find desired $p$ reference points such that they uniformly represents the objective surface where solution exists.

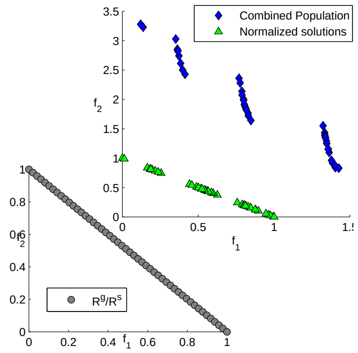
To do this first we generate lots of uniform reference points and associate solutions with the reference points. Solutions are normalized by following NSGAIII normalization procedure which will keep diversity tension to outwards ( [27]).

Using Cosine similarity measures we get first level of reference points which represents objective surface, we call these points active reference points. Fig. 3.8(a) and 3.8(b) shows steps.

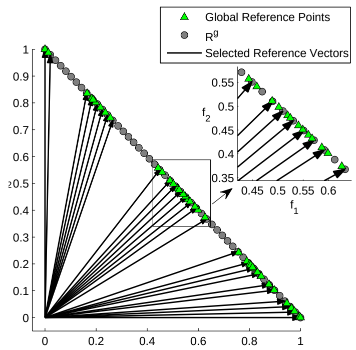
Now from the active reference points we need to select $p$ reference points. So here we applied MinMax procedure to remove crowded reference points, shown in Fig. 3.8(c). After reassigning solutions that was assigned to the removed reference points we will find desired clusters of solutions shown in Fig. 3.8(d).

Fig. 3.8(e) and Fig. 3.8(e) shows how with and without cluster next generation solutions is selected from that generation.

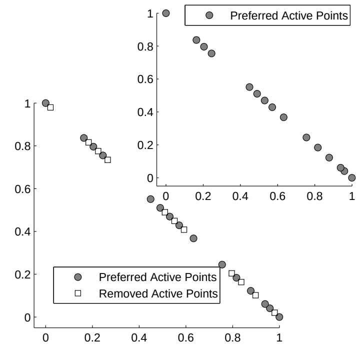
(The preferred reference points can be used in another algorithms, which we intend to show in future work.)



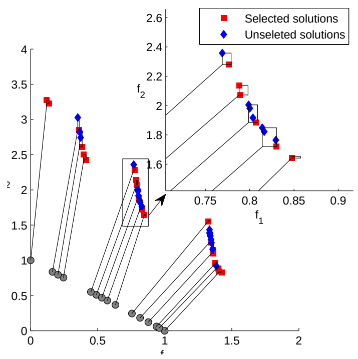

(a) Generated reference points $R^g$, the original solutions in combined population $C_t$ and their normalized ones.



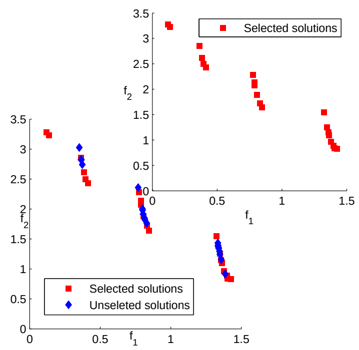

(b) Global reference points to active reference points (marked with vector line)



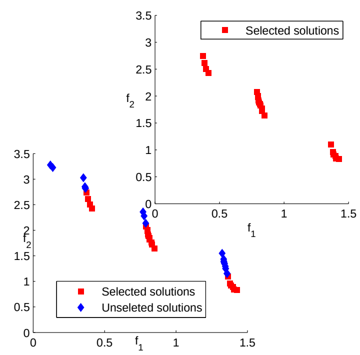

(c) Reduction of active reference points to preferred size $p = 15$




(d) Cluster of solutions associated with preferred points




(e) Selected solutions based after clustering




(f) Selected solutions without clustering. The bias is toward objective $f_1$

Figure 3.8: Construction of clusters using the solutions in the combined population and generated reference points for the WFG2 problem.

### 3.2.8 Computational Complexity

The basic $F$-DEA algorithm contains seven lines (Algorithm 3.1). As the first two lines are common in an evolutionary algorithm, the remaining five lines that call five procedures actually determine $F$-DEA's complexity. Algorithm 3.2 finds $p$ preferred reference points from $|R^g/R^s|$ generated/supplied points and constructs clusters using such points. The overall computation including adaptive normalization (O($mN$)) can be performed with $max(O(mN\,|R^g|\,/\,|R^s|\,, O(mN^2))$ operations. The construction of adaptive membership function using mean $\mu$ and variance $\sigma^2$ requires O($mN^2$) operations.To calculate $\mu$ and $\sigma^2$ in a single pass, we use the Knuth's running-mean-variance approximation process [43]. Let the maximum number of solutions in a cluster is $\chi$. As there can be $p$ different clusters, the time complexity of fitness assignment is at most $O(mp\chi^2)$, including the selection of best solutions that requires $max(O(mp^2), O(p\chi))$ or $O(mN^2)$ in the worst case. Thus the overall time complexity of $F$-DEA is $max(O(mN\,|R^g/R^s|), O(mN^2))$.

## 3.3 Novel features of the Proposed Algorithm

In this thesis we propose a fuzzy dominance based evolutionary algorithm ($F$-DEA) for efficiently solving MaOPs. The proposed algorithm exploits the strength of fuzzy dominance in conjunction with clustering based active diversity promotion mechanism in order to efficiently solve MaOPs. The novelty of $F$-DEA comes from the following new features,

- Diversity maintenance is the primary issue faced by any fuzzy based approaches. To maintain diversity among solutions, $F$-DEA employs reference points based clustering to select solutions for the next generation of an evolutionary process. To the best of our knowledge, $F$-DEA is the first algorithm that employs fuzzy dominance and reference point synergistically. Unlike existing reference point based approaches [3,20,21,27,31,42].
- The preferred reference points based clustering provides better cluster uniformity, removes dependency on population size and effectively handles irregular-shaped Pareto fronts. Also the preferred reference points size parameter gives control over convergence and diversity.
- $F$-DEA introduces the anti-symmetric Sigmoid membership function with an aim of ensuring proper discrimination ability of the membership function. Existing approaches either use domain knowledge [37] or approximation procedure [22].

- To handle the scaling issue of objectives, $F$-DEA uses a separate membership function for each objective and estimates its parameter(s) adaptively. The estimation procedure handles the bias induced by isolated solutions.

- The proposed algorithm emphasizes both convergence and diversity in the same way from beginning to end of an evolutionary process. These aspects relate to the consideration of both diversity and convergence in the same selection process. Existing fuzzy based algorithms use fuzzy dominance as a primary selection criterion and diversity measure as a secondary one. This is problematic in the sense that algorithms would rarely employ the secondary criterion as fuzzy dominance is capable of discriminating solutions.

# Chapter 4

# Experimental Studies

## 4.1 Introduction

We perform a series of experiments to investigate and compare the optimization capability of our algorithm. Two well known benchmark test suites WFG [44] and DTLZ [45] are utilized for this purpose. The WFG problems are truly non-linear, non-separable and multimodal, and they do not have an identical range of values for each objective [44]. These characteristics make the WFG problems more challenging than DTLZ ones. To investigate the performance on degenerate problems, we also include the Rectangle problem [41] for experimentation.

## 4.2 Benchmark Problems

The evaluation and comparison of our method is based on several benchmark problems taken from the WFG and DTLZ test suites. The WFG test suite contains nine problems and we choose eight of them for experimentation. We exclude the WFG3 problem because it has a non-degenerate part [46] which might create erroneous result during performance evaluation. We choose five (DTLZ1, DTLZ2, DTLZ3, DTLZ4 and DTLZ7) out of seven problems the DTLZ problem suite. We omit the DTLZ5, DTLZ6 problems due to their ambiguity in Pareto fronts beyond 3-objective [44]. Table 4.1 shows characteristics of the WFG and DTLZ problems. We also choose three instances of Rectangle problem.

The problems in WFG and DTLZ test suites can be scaled to any number of objectives and decision variables. We consider the number of objectives $m \in \{2, 3, 5, 7, 10, 12, 15, 20, 25\}$. As per recommendation from the WFG Toolkit[1], we set the distance related parameter $l = 20$, the position related parameter $k = 4$ for $m = 2$, $k = 2 \times (m - 1)$ for $3 \leq m \leq 10$, and $k = (m - 1)$ for $m > 10$. The number of decision variables, $n$, is set equal to $l + k$. We also follow the suggestions of [27, 45] in setting $n$ and $k$ of DTLZ problems. We set $n$ equal to $m + k - 1$ for all DTLZ problems we consider in this work. We set $k = 5$ for DTLZ1, $k = 10$ for DTLZ3 and $k = 20$ for DTLZ7. In this study, WFGX-Y refers to the problem WFGX with Y objective. The similar notation is used for the DTLZ problems.

## 4.3 Performance metrics

The performance of any evolutionary algorithm for an MOP is usually measured from two aspects: convergence and diversity. Inverse generalization distance (IGD) [47] and hypervolume (HV) [32] are two performance metrics that capture in one scalar both convergence and diversity. To calculate IGD, a well uniform sample set of true Pareto front is required. It is, however, challenging to get such a set for an increasing number of objectives [34]. Deb and Jain [27] recently introduced a direct procedure for calculating IGD based on the reference points, where for each reference direction we can exactly locate the intersecting point of a known true Pareto front.

The other performance metric HV has nicer mathematical properties and is the only quality measure known to be strictly Pareto-compliant [47]. These good features make HV a fair

[1] http://www.wfg.csse.uwa.edu.au/toolkit/README.txt

Table 4.1: CHARACTERISTICS OF DIFFERENT WFG AND DTLZ TEST PROBLEMS.

| Problem | Features |
|---|---|
| WFG1 | Separable, Uni-modal, Biased, Mixed, Scaled |
| WFG2 | Nonseparable, Multi-modal<br>Convex, Disconnected, Scaled |
| WFG4 | Separable, Multi-modal, Concave, Scaled |
| WFG5 | Separable, Deceptive, Concave, Scaled |
| WFG6 | Nonseparable, Uni-modal, Concave, Scaled |
| WFG7 | Separable, Uni-modal, Biased, Concave, Scaled |
| WFG8 | Nonseparable, Uni-modal, Biased Concave, Scaled |
| WFG9 | Nonseparable, Multi-modal, Deceptive<br>Biased, Concave, Scaled |
| DTLZ1 | Separable, Multi-modal, Linear |
| DTLZ2 | Separable, Uni-modal, Concave |
| DTLZ3 | Separable, Multi-modal, Concave |
| DTLZ4 | Separable, Uni-modal, Biased, Concave |
| DTLZ7 | Multi-modal, Disconnected |

indicator for comparing different algorithms. For a non-dominated solution set $A$ obtained in final generation by an algorithm, HV is calculated with respect to a reference point $\mathbf{r}$. HV of $A$ with respect to $\mathbf{r}$ is the volume of region dominated by $A$ and bounded by $\mathbf{r}$.

$$HV(A, \mathbf{r}) = volume(\bigcup_{f \in A} [f_1, r_1] \times \cdots \times [f_m, r_m]) \quad (4.1)$$

The choice of $\mathbf{r}$ is crucial in computing HV and it has been known that choosing $\mathbf{r}$ slightly larger than the nadir point, $\mathbf{z}^{nad}$, is suitable [48]. In our experiments, we set $\mathbf{r}$ to $1.1\mathbf{z}^{nad}$, which can be analytically obtained or approximated [44]. The Pareto front of WFG4-WFG9 is part of a hyper-ellipse with radii $R_i = 2 \times i$ ( $i = 1, 2, \cdots, m$) and has a regular geometry [3, 44]. It of DTLZ1-DTLZ4 also has a regular shape. As the Pareto fronts of WFG1 (mixed), WFG2 (disconnected) and DTLZ7 (disconnected) do not have a regular geometry, we obtain $\mathbf{z}^{nad}$ for these problems by an using an approximation procedure. Following [49], the points which do not dominate $\mathbf{r}$ are discarded in computing HV. For the problems having Pareto fronts with differently scaled objective values, we first normalize the objective values of the points in $A$ and the reference point $\mathbf{r}$ using $\mathbf{z}^{nad}$ and $\mathbf{z}^*$ before computing HV. Here $\mathbf{z}^*$ indicates the optimal objective value, which is 0 for all adopted problems we consider in this study. Thus the computed HV for an $m$-objective problem would be between 0 and $1.1^m - V_m$, where $V_m$ is hypervolume of the region enclosed by the normalized Pareto front and coordinate axes. We use exact HV calculation for problems with objective less than 5 and the Monte Carlo based fast HV approximation algorithm [33] with $10, 00, 000$ sampling points for others. The large number of sampling points is considered here to ensure accuracy in computing HV.

In experimental studies, we have used HV, IGD and visualization figures to evaluate performances of the algorithms.

## 4.4 Other Algorithms in Comparison

There exists a few fuzzy dominance based EMO algorithms in the literature. FD-NSGAII [22] is one such algorithm, which has been found better than other similar algorithms. We thus select FD-NSGAII for comparison. We also choose NSGAIII [27] for comparison because it exhibits superior performance compared to several well-known algorithms. Decomposition based algorithm MOEA/D [2] with weighted Tchebycheff approach has been chosen as a representative of decomposition, aggregation and reference weights/points based approach. Although a new version of MOEA/D (e.g. MOEA/D-DE [29]) with differential evolution has been proposed to deal with complicated Pareto surfaces, it has been found to be shown poor performance in MaOPs [27]. We choose a variant of SDE [25], SPEA2+SDE, for comparison. This particular variant shows the best overall performance among its other variants (NSGA-II+SDE, SPEA2+SDE and PESA-II+SDE) [25]. Preference based co-evolutionary algorithm (PICEAg) [3] and HypE [33], an indicator based algorithm, are also chosen for comparison. HypE adopts Monte Carlo simulation to approximate exact hypervolume. Its core idea is that only the rankings of the solutions induced by the hypervolume indicator are important, while the actual indicator values are not. All the aforementioned algorithms cover standard approaches for solving MaOPs.

## 4.5 Parameter Setting

The population size $N$ of NSGAIII cannot be arbitrarily specified, rather it has to be set equal to the number of reference points. The procedure employed for generating such points uses a division parameter $\lambda$ that determines this number. Although our algorithm uses reference points, it does not put any constraint in choosing $N$. Table 4.2 shows $\lambda$ of NSGAIII and $F$-DEA, the weights $Z$ of MOEA/D and the number of goals $G$ for PICEAg. To make a fair comparison, the population size is set same for all competing algorithms.

All competing algorithms employ simulated binary crossover and polynomial mutation for generating offspring. The crossover and mutation probabilities are set to 1 and $1/n$, respectively. We also use the same mutation distribution index i.e., 20 for these algorithms. The crossover distribution index is set to 30 for $F$-DEA and NSGAIII, 20 for SDE, and 15 for HypE, MOEA/D, FD-NSGAII, and PICEAg. Beside the general and common parameters, there are some specific parameters for competing algorithms.

1. $F$-DEA: The division number, $\lambda$, of the Das and Dennis's procedure [14] used in $F$-DEA is shown in Table 4.2 and $p$ is set equal to $N$.
2. HypE [33]: The bound of reference point and the number of sampling points have been set to 200 and $10,000$, respectively.
3. MOEA/D [2, 29]: The neighborhood size, $T$, is chosen 5% of the population and the maximum number of population slots, $\eta_r$, has been chosen 1% of $T$.
4. FD-NSGAII [22]: The fuzzy ranking threshold parameter $\beta$ has been set to 0.50.
5. NSGAIII [27]: The division number, $\lambda$, of the Das and Dennis's procedure [14] used in NSGAIII is shown in Table 4.2.
6. SDE [25]: the archive is set equal to population size, $N$.
7. PICEAg [3]: The number of goal $G$ is set to $m \times 100$.

Each algorithm is run independently with 20 different seeds for each problem instance. We set the termination criterion to 250 generations for each run. The Wilcoxon rank-sum test [50], which is equivalent to the MannWhitney $U$ test [51] (MATLAB implementation[2]) with a 5%-significance level is used while comparing two algorithms on any problem instance over 20 runs. For reducing Type-I error in pairwise testing, Šidák corrections [52] are also employed. $F$-DEA has been implemented by the authors in *JMetal*[3] framework and it's source

[2]http://www.mathworks.com/help/stats/ranksum.html
[3]www.jmetal.org

Table 4.2: Population Size $N$, number of divisions $\lambda$ used in NSGAII [1] and $F$-DEA, number of weight vectors $Z$ used in MOEA/D [2] and goals in PICEAg [3] for different objectives

| Obj. No. ($m$) | $N$ used in all algorithms | $\lambda$ used in NSGAIII | $\lambda$ used in F-DEA | $Z$ used in MOEA/D | $G$ used in PICEAg |
|---|---|---|---|---|---|
| 2 | 204 | 200 | 2000 | 20400 | 200 |
| 3 | 204 | 18 | 100 | 20400 | 300 |
| 5 | 212 | 6 | 30 | 21200 | 500 |
| 7 | 212 | 4 | 13 | 21200 | 700 |
| 10 | 220 | 3 | 9 | 22000 | 1000 |
| 12 | 160 | 2, 2 | 7 | 16000 | 1200 |
| 15 | 240 | 2, 2 | 6 | 24000 | 1500 |
| 20 | 212 | 2 | 5 | 21200 | 2000 |
| 25 | 328 | 2 | 4 | 32800 | 2500 |

code is available online[4]. HypE and MOEA/D from MOEA framework[5]. We use a C++ implementation[6] for NSGAIII. FD-NSGAII has been implemented by the authors from [22] in *JMetal* framework. The source codes of SDE and PICEAg algorithms are received from the authors of the respective algorithms. The true Pareto front of the WFG problems is generated using the *WFG-Toolkit* [7]. The uniform Pareto front of DTLZ1, DTLZ3 is generated from NSGAIII implementation tools[6]. For the DTLZ7 problems, the Pareto front is generated using *MOEAFramework* [5]. All algorithms were run on Intel 2.40GHz core i5 processor with 4GB RAM. The performance evaluation and visualization codes are shared in *Github* [8].

[4] https://github.com/siddhartha047/FDEA
[5] www.moeaframework.org
[6] http://web.ntnu.edu.tw/ tcchiang/publications/nsga3cpp/nsga3cpp.htm
[7] http://www.wfg.csse.uwa.edu.au/toolkit/
[8] https://github.com/siddhartha047/MOEAevaluation-plot

## 4.6 Experiment on WFG Problems

This section presents evaluation and comparison of our $F$-DEA and six other algorithms on eight WFG problems with 2-, 3-, 5-, 7-, 10-, 12-, 15-, 20- and 25-objective. Table 4.3 and Table 4.4 shows the average HV with standard deviation and ranks of different algorithms obtained by the Wilcoxon rank-sum test based on HVs of 20 independent runs. The lower a rank is, the better an algorithm is.

For a particular problem, the performance score is a count that indicates the number of times the algorithm is significantly beaten by the competing algorithms based the Wilcoxon rank-sum test. If there are $h$ algorithms, the lowest score could be 0 (none found better) and the highest one could be $h-1$ (all the competing algorithms are better). The lower the score is, the better the algorithm is. The detail results of these problems are given in appendix for brevity.

### 4.6.1 WFG1 Problem

The WFG1 problem has most transformation functions among the other problems of the same suite. Hence, it is difficult to maintain diversity with sufficient convergence for this problem and algorithms that stress over diversity will fail to achieve convergence. This problem unimodal, biased and the Pareto front is mixed (concave and convex). $F$-DEA exhibited the best performance among all competing algorithms from 2-objective to 25-objective. PICEAg, NSGAIII, SDE were the close competitors for a lower number of objectives. As the number of objective increases, FD-NSGAII secured the second position and exhibited better results compared to others.

### 4.6.2 WFG2 Problem

This is the only disconnected problem in the WFG test suite. To achieve good performance, it is necessary for an algorithm to distribute the obtained solutions in all the disconnected regions. FD-NSGAII exhibited the worst performance on this problem with nine different objectives we considered in this work. NSGAIII was the top performer in 2-objective, while PICEAg and SDE were the top two performers from 3-objective to 25-objective. $F$-DEA secured overall the third position and showed very competitive performance with respect to average HVs achieved by the top performers.

### 4.6.3 WFG4 Problem

From WFG4 to WFG9, the Pareto front is part of a hyper-ellipse with radii $R_i = 2 \times i$ ( $i = 1, 2, \cdots, m$) and has a regular geometry [3, 44]. Although the Pareto fronts are same but the problem nature and transformations are different from problem to problem. WFG4 is a concave multi-modal problem. For a smaller number of objectives, SDE and PICEAg were the two top performing algorithms while $F$-DEA showed competitive performance. $F$-DEA, however, tied with SDE in 7-objective and outperformed all other competing algorithms from 10-objective to 25-objective. These results indicate that the performance of $F$-DEA improves as the number of objectives increases.

### 4.6.4 WFG5 Problem

An important aspect of this problem is its deceptive nature, which challenges the ability of an algorithm to find good quality solutions. $F$-DEA handled the challenge successfully and outperformed all competing algorithms from 15-objective to 25-objective. From 7-objective to 12-objective, $F$-DEA, SDE and PICEAg were top performers. Our algorithm also showed competitive performance for 2-, 3- and 5-objective.

### 4.6.5 WFG6 Problem

$F$-DEA was the top performer from 12-objective to 25-objective of the non-separable reduced problem, WFG6. The proposed algorithm shared the top position with SDE and PICEAg for 2-objective and with PICEAg for 10-objective. For other objectives, $F$-DEA showed competitive performance with the top performers.

### 4.6.6 WFG7 Problem

This is a separable unimodal problem. Table shows $F$-DEA achieved better performance with an increasing number of objectives. For example, it outperformed all other algorithms from 15-objective to 25-objective and showed competitive performance with SDE, PICEAg and NSGAIII for a smaller number of objectives.

### 4.6.7 WFG8 Problem

SDE, PICEAg and $F$-DEA were the top performing algorithms with close average HV values for this non-separable problem. SDE was the top performer for 2-, 3- and 20-objective, PICEAg for 5- and 10-objective, and $F$-DEA for 15- and 25-objective. $F$-DEA together with SDE became the top performer for the 12-objective and it secured the second position for the 2-, 7-, 10 and 20-objective with relatively close average HV values.

### 4.6.8 WFG9 Problem

This is a non-separable deceptive problem for which NSGAIII and SDE were two top performers for 2-, 3- and 5-objective. As the number objective increases, the performance of $F$-DEA enhanced and shared the top position with SDE, PICEAg for the 7- and 10-objective. However, $F$-DEA outperformed all other algorithms from 12-objective to 25-objective.

Table 4.3: Part I: Average HV of different algorithms on WFG1, WFG2, WFG4 and WFG5 problems over 20 independent runs. The best result based on the Wilcoxon rank sum test with a significance level of 0.05 is marked in bold-face. The rank of a particular algorithm is shown in bracket.

| Prob. | $m$ | HYPE | MOEA/D | FD-NSGAII | SDE | NSGAIII | PICEA-g | F-DEA |
|---|---|---|---|---|---|---|---|---|
| wfg1 | 2 | 0.0970±0.0257(6) | 0.1467±0.0264(5) | 0.0495±0.0214(7) | 0.2324±0.0434(4) | **0.3333±0.0689(1.5)** | 0.3233±0.0037(3) | **0.3438±0.0371(1.5)** |
| | 3 | 0.2968±0.0248(7) | 0.4042±0.0485(5) | 0.3959±0.0226(5) | 0.5614±0.0497(2.5) | 0.4094±0.0396(5) | 0.5689±0.0052(2.5) | **0.6120±0.0413(1)** |
| | 5 | 0.3515±0.0347(7) | 0.5187±0.0246(5) | 0.5979±0.0474(3) | 0.5622±0.0232(4) | 0.4065±0.0732(6) | 0.7117±0.0114(2) | **0.7516±0.0337(1)** |
| | 7 | 0.4164±0.0279(7) | 0.5985±0.0402(5) | 0.8524±0.0531(2.5) | 0.7058±0.0256(4) | 0.4824±0.0275(6) | 0.8429±0.0297(2.5) | **0.9058±0.0560(1)** |
| | 10 | 0.6005±0.0225(6) | 0.8432±0.0692(5) | 1.2047±0.0754(2) | 0.9937±0.0418(4) | 0.3671±0.0924(7) | 1.0310±0.0307(3) | **1.4206±0.0684(1)** |
| | 12 | 0.6543±0.0292(6) | 0.8989±0.0669(5) | 1.3867±0.1012(2) | 1.1189±0.0517(4) | 0.2839±0.0633(7) | 1.2233±0.0310(3) | **1.6017±0.1296(1)** |
| | 15 | 0.8732±0.0501(6) | 1.2230±0.0545(5) | 2.1247±0.1903(2) | 1.6596±0.0671(3) | 0.4128±0.1145(7) | 1.5930±0.0473(4) | **2.3314±0.1024(1)** |
| | 20 | 1.2481±0.0439(6) | 1.6443±0.1007(5) | **3.4625±0.3626(1.5)** | 2.7029±0.1280(3) | 0.3940±0.1587(7) | 2.5527±0.0635(4) | **3.5738±0.2874(1.5)** |
| | 25 | 2.1095±0.1037(6) | 2.4165±0.1568(5) | 6.6621±0.3718(2) | 2.8201±0.0539(4) | 1.8261±0.2503(7) | 4.0829±0.0980(3) | **7.1479±0.4528(1)** |
| wfg2 | 2 | 0.6806±0.0149(5.5) | 0.6820±0.0183(5.5) | 0.3459±0.0572(7) | 0.7407±0.0054(2) | **0.7452±0.0087(1)** | 0.7406±0.0091(3) | 0.7363±0.0064(4) |
| | 3 | 0.9702±0.0767(5.5) | 1.0575±0.0972(5.5) | 0.3862±0.0656(7) | 1.2207±0.0581(2) | 1.1573±0.0929(3.5) | **1.2450±0.0022(1)** | 1.2119±0.0418(3.5) |
| | 5 | 1.3316±0.1315(4.5) | 1.3283±0.1175(6) | 0.4409±0.0009(7) | 1.5401±0.0840(2) | 1.3872±0.1229(4.5) | **1.5546±0.1049(1)** | 1.5277±0.0838(3) |
| | 7 | 1.6032±0.1593(5.5) | 1.4962±0.1489(5.5) | 0.5498±0.0678(7) | 1.8686±0.1022(2) | 1.6951±0.1574(4) | **1.8875±0.1285(1)** | 1.8347±0.1241(3) |
| | 10 | 2.2659±0.1715(4.5) | 2.1991±0.1021(6) | 0.5655±0.2030(7) | 2.5531±0.0064(2) | 2.3060±0.0507(4.5) | **2.5889±0.0018(1)** | 2.5280±0.0097(3) |
| | 12 | 2.6969±0.0885(4.5) | 2.5541±0.1337(6) | 0.6473±0.2821(7) | 3.0745±0.0166(2) | 2.6757±0.0975(4.5) | **3.1266±0.0034(1)** | 3.0375±0.0228(3) |
| | 15 | 3.7255±0.1172(4) | 3.5520±0.1362(5.5) | 0.8039±0.3350(7) | 4.1263±0.0084(2) | 3.5388±0.0700(5.5) | **4.1692±0.0022(1)** | 4.0798±0.0196(3) |
| | 20 | 5.7851±0.4480(4.5) | 5.6843±0.2581(4.5) | 1.2816±0.6254(7) | 6.6467±0.0169(2) | 4.0451±0.7806(6) | **6.7064±0.0094(1)** | 6.5645±0.0299(3) |
| | 25 | 9.6551±0.3095(4.5) | 9.1922±0.3394(6) | 2.2394±0.9957(7) | 10.6314±0.0470(2.5) | 8.9972±1.8322(4.5) | **10.8188±0.0074(1)** | 10.6293±0.0517(2.5) |
| wfg4 | 2 | 0.3946±0.0041(6) | 0.4058±0.0044(5) | 0.1278±0.0377(7) | **0.4184±0.0010(1.5)** | **0.4182±0.0010(1.5)** | 0.4143±0.0014(4) | 0.4172±0.0006(3) |
| | 3 | 0.6447±0.0203(6) | 0.6697±0.0115(5) | 0.1220±0.0053(7) | **0.7501±0.0028(1)** | 0.7096±0.0038(4) | 0.7432±0.0021(2) | 0.7321±0.0024(3) |
| | 5 | 0.9151±0.0969(6) | 1.0343±0.0251(5) | 0.2472±0.0915(7) | 1.2164±0.0083(2.5) | 1.0503±0.0134(4) | **1.2475±0.0262(1)** | 1.2220±0.0083(2.5) |
| | 7 | 1.1010±0.1397(6) | 1.3328±0.0423(4) | 0.3230±0.1273(7) | **1.6016±0.0149(1.5)** | 1.3015±0.0559(4) | 1.3295±0.0982(4) | **1.6127±0.0221(1.5)** |
| | 10 | 1.5518±0.1823(6) | 2.0844±0.0399(4.5) | 0.4548±0.1907(7) | 2.3121±0.0292(2) | 2.0664±0.0923(4.5) | 2.1635±0.0753(3) | **2.3892±0.0111(1)** |
| | 12 | 1.6225±0.2620(6) | 2.4331±0.1179(3) | 0.4192±0.1924(7) | 2.7870±0.0416(2) | 1.9808±0.0874(5) | 2.3082±0.1245(4) | **2.8607±0.0273(1)** |
| | 15 | 2.3364±0.3730(6) | 3.4223±0.0955(3.5) | 0.6341±0.2520(7) | 3.7551±0.0348(2) | 2.7353±0.1271(5) | 3.3657±0.1431(3.5) | **3.9442±0.0311(1)** |
| | 20 | 3.6517±0.3742(6) | 5.2892±0.1486(3.5) | 0.7661±0.2789(7) | 5.9939±0.2018(2) | 4.0528±0.2925(5) | 5.1166±0.3164(3.5) | **6.3855±0.0460(1)** |
| | 25 | 7.4075±0.6866(6) | 8.8991±0.1289(3.5) | 1.1546±0.1966(7) | 9.4890±0.4354(2) | 8.2805±0.4261(5) | 8.7945±0.5704(3.5) | **10.5396±0.0492(1)** |
| wfg5 | 2 | 0.3243±0.0122(6) | 0.3603±0.0068(5) | 0.0861±0.0202(7) | **0.3743±0.0005(1.5)** | 0.3740±0.0006(3.5) | **0.3741±0.0009(1.5)** | 0.3737±0.0008(3.5) |
| | 3 | 0.5445±0.0262(6) | 0.6056±0.0145(5) | 0.1142±0.0305(7) | **0.7010±0.0035(1)** | 0.6648±0.0056(4) | 0.6966±0.0033(2) | 0.6882±0.0022(3) |
| | 5 | 0.8947±0.0602(6) | 0.9683±0.0248(5) | 0.1278±0.0006(7) | 1.1715±0.0074(3) | 0.9993±0.0119(4) | **1.1854±0.0038(1)** | 1.1800±0.0073(2) |
| | 7 | 1.0418±0.1275(6) | 1.2624±0.0405(4.5) | 0.1554±0.0003(7) | 1.5562±0.0097(3) | 1.2482±0.0517(4.5) | **1.5814±0.0173(1.5)** | **1.5809±0.0085(1.5)** |
| | 10 | 1.4758±0.1146(6) | 1.9581±0.0466(4) | 0.2064±0.0005(7) | 2.2843±0.0079(2) | 1.8706±0.0613(5) | 2.1577±0.0755(3) | **2.2964±0.0062(1)** |
| | 12 | 1.6040±0.1641(6) | 2.2598±0.0914(4) | 0.2487±0.0005(7) | **2.7525±0.0127(1.5)** | 2.0394±0.1186(5) | 2.3688±0.0823(3) | **2.7670±0.0236(1.5)** |
| | 15 | 2.4435±0.1978(6) | 3.2303±0.0904(4) | 0.3316±0.0016(7) | 3.6807±0.0185(2) | 2.7944±0.1447(5) | 3.3723±0.0774(3) | **3.7649±0.0091(1)** |
| | 20 | 3.6012±0.4559(6) | 5.0711±0.2051(3.5) | 0.5312±0.0011(7) | 5.9066±0.0244(2) | 4.5393±0.3738(5) | 5.1737±0.1697(3.5) | **6.0356±0.0172(1)** |
| | 25 | 6.5479±0.7306(6) | 8.3672±0.3236(4.5) | 0.8543±0.0013(7) | 8.6133±0.2848(4.5) | 8.9789±0.2287(2.5) | 8.8711±0.1808(2.5) | **9.7850±0.0161(1)** |

Table 4.4: Part II: Average HV of different algorithms on WFG6, WFG7, WFG8 and WFG9 problems over 20 independent runs. The best result based on the Wilcoxon rank sum test with a significance level of 0.05 is marked in bold-face. The rank of a particular algorithm is shown in bracket.

| Prob. | $m$ | HYPE | MOEA/D | FD-NSGAII | SDE | NSGAIII | PICEA-g | F-DEA |
|---|---|---|---|---|---|---|---|---|
| wfg6 | 2 | 0.2660±0.0187(6) | 0.3587±0.0115(5) | 0.0864±0.0027(7) | **0.3845±0.0040(2)** | 0.3735±0.0127(4) | **0.3836±0.0041(2)** | **0.3817±0.0045(2)** |
| | 3 | 0.3448±0.0424(6) | 0.6295±0.0123(5) | 0.1070±0.0052(7) | **0.7094±0.0039(1)** | 0.6736±0.0066(4) | 0.7039±0.0069(2) | 0.6874±0.0048(3) |
| | 5 | 0.5599±0.0876(6) | 0.9626±0.0238(5) | 0.1341±0.0026(7) | 1.1874±0.0072(2) | 1.0276±0.0183(4) | **1.2105±0.0074(1)** | 1.1804±0.0129(3) |
| | 7 | 0.6954±0.1325(6) | 1.2821±0.0561(4.5) | 0.1632±0.0021(7) | 1.6131±0.0125(2) | 1.2492±0.0339(4.5) | **1.6319±0.0137(1)** | 1.5948±0.0155(3) |
| | 10 | 1.2015±0.1278(6) | 1.9249±0.0507(4) | 0.2175±0.0022(7) | 2.2412±0.0276(3) | 1.8219±0.0262(5) | **2.2302±0.0846(1.5)** | **2.2819±0.0258(1.5)** |
| | 12 | 1.3825±0.1510(6) | 2.1664±0.1233(4) | 0.2638±0.0023(7) | 2.6691±0.0246(2) | 2.0114±0.1159(5) | 2.5231±0.1406(3) | **2.7368±0.0532(1)** |
| | 15 | 2.1409±0.2052(6) | 3.1680±0.1164(4) | 0.3519±0.0029(7) | 3.5523±0.0467(2.5) | 2.7916±0.1132(5) | 3.5061±0.1279(2.5) | **3.7578±0.0354(1)** |
| | 20 | 3.0136±0.4601(6) | 4.7739±0.2878(4) | 0.5655±0.0055(7) | 5.5752±0.1867(2) | 3.9772±0.5521(5) | 5.3687±0.3096(3) | **6.0223±0.0860(1)** |
| | 25 | 5.6551±0.6803(6) | 8.2901±0.2623(4) | 0.9077±0.0050(7) | 8.4299±0.4208(4) | 8.5485±0.5225(4) | 9.1704±0.3702(2) | **9.8641±0.0942(1)** |
| wfg7 | 2 | 0.3460±0.0130(6) | 0.3829±0.0105(5) | 0.1092±0.0007(7) | **0.4204±0.0004(1)** | 0.4196±0.0007(2) | 0.4188±0.0002(3.5) | 0.4188±0.0005(3.5) |
| | 3 | 0.5615±0.0440(6) | 0.6549±0.0211(5) | 0.1209±0.0000(7) | **0.7583±0.0010(1)** | 0.7360±0.0017(4) | 0.7548±0.0010(2) | 0.7449±0.0014(3) |
| | 5 | 0.8518±0.0860(6) | 1.0421±0.0267(5) | 0.1463±0.0000(7) | 1.2677±0.0038(2) | 1.0804±0.0163(4) | **1.2840±0.0031(1)** | 1.2584±0.0045(3) |
| | 7 | 1.1269±0.0937(6) | 1.2974±0.0650(5) | 0.1771±0.0004(7) | **1.7049±0.0086(1.5)** | 1.3397±0.0280(4) | 1.6038±0.0907(3) | **1.7000±0.0106(1.5)** |
| | 10 | 1.3918±0.1735(6) | 2.0242±0.0660(4.5) | 0.2357±0.0000(7) | **2.4482±0.0060(2)** | 2.0276±0.0773(4.5) | **2.4141±0.0638(2)** | **2.4440±0.0099(2)** |
| | 12 | 1.4725±0.2122(6) | 2.1892±0.2464(4.5) | 0.2851±0.0004(7) | **2.9726±0.0110(1)** | 2.3038±0.1306(4.5) | 2.6645±0.1339(3) | 2.9359±0.0194(2) |
| | 15 | 2.2205±0.2748(6) | 2.7870±0.3252(5) | 0.3800±0.0008(7) | 3.9866±0.0159(2) | 3.2219±0.1321(4) | 3.7738±0.1628(3) | **4.0371±0.0115(1)** |
| | 20 | 3.6111±0.3759(6) | 4.2830±0.3362(5) | 0.6168±0.0177(7) | 6.4104±0.0543(2) | 4.8939±0.4791(4) | 5.8699±0.1624(3) | **6.5277±0.0246(1)** |
| | 25 | 6.4233±1.0674(6) | 7.5748±0.4440(5) | 0.9969±0.0541(7) | 10.2048±0.2123(2.5) | 9.4596±0.4048(4) | 10.1284±0.3726(2.5) | **10.6904±0.0235(1)** |
| wfg8 | 2 | 0.2809±0.0075(6) | 0.3027±0.0064(4.5) | 0.0076±0.0230(7) | **0.3299±0.0017(1)** | 0.3009±0.0015(4.5) | 0.3221±0.0024(3) | 0.3243±0.0016(2) |
| | 3 | 0.4898±0.0258(6) | 0.5556±0.0176(5) | 0.1204±0.0004(7) | **0.6538±0.0024(1)** | 0.6176±0.0046(4) | 0.6422±0.0030(2) | 0.6304±0.0031(3) |
| | 5 | 0.7401±0.0472(6) | 0.7954±0.0399(5) | 0.1461±0.0001(7) | 1.0653±0.0072(2) | 0.9567±0.0099(4) | **1.0781±0.0072(1)** | 1.0411±0.0074(3) |
| | 7 | 0.9667±0.0709(5.5) | 0.9092±0.1073(5.5) | 0.1769±0.0000(7) | **1.4248±0.0154(2)** | 1.2062±0.0381(4) | **1.4170±0.0538(2)** | **1.4186±0.0123(2)** |
| | 10 | 1.1224±0.0912(6) | 1.5561±0.1768(5) | 0.2354±0.0005(7) | 2.1759±0.0216(2.5) | 1.8744±0.0598(4) | **2.1823±0.0500(1)** | 2.1766±0.0124(2.5) |
| | 12 | 1.3276±0.1107(6) | 1.9617±0.2471(4.5) | 0.2849±0.0004(7) | **2.6802±0.0418(1.5)** | 2.0247±0.1341(4.5) | 2.5040±0.1345(3) | **2.6891±0.0191(1.5)** |
| | 15 | 1.9463±0.1602(6) | 2.9531±0.3379(4) | 0.3793±0.0007(7) | 3.6871±0.0869(2) | 2.8278±0.1333(5) | 3.5647±0.0864(3) | **3.7733±0.0353(1)** |
| | 20 | 2.9824±0.3292(6) | 5.1240±0.3526(4) | 0.6108±0.0013(7) | **6.2214±0.0311(1)** | 4.1653±0.2093(5) | 5.7492±0.2372(3) | 6.1634±0.0429(2) |
| | 25 | 5.5585±0.6066(6) | 8.9032±0.2442(4.5) | 1.0571±0.3308(7) | 10.2317±0.0798(2) | 9.1665±0.4875(4.5) | 9.7334±0.1773(3) | **10.3434±0.0374(1)** |
| wfg9 | 2 | 0.3037±0.0237(6) | 0.3564±0.0169(3.5) | 0.1030±0.0271(7) | **0.3835±0.0330(1.5)** | **0.4055±0.0039(1.5)** | 0.3493±0.0231(5) | 0.3758±0.0335(3.5) |
| | 3 | 0.4596±0.0595(6) | 0.6116±0.0240(5) | 0.0919±0.0000(7) | **0.6848±0.0383(1.5)** | **0.6743±0.0251(1.5)** | 0.6370±0.0023(4) | 0.6669±0.0339(3) |
| | 5 | 0.7079±0.0664(6) | 0.9108±0.0357(5) | 0.1165±0.0000(7) | **1.0942±0.0492(2)** | **1.0267±0.0549(2)** | **1.0618±0.0028(2)** | 1.0627±0.0219(4) |
| | 7 | 0.9327±0.0564(6) | 1.0914±0.0761(5) | 0.1431±0.0001(7) | **1.4376±0.0731(2)** | 1.2560±0.1225(4) | **1.3869±0.0063(2)** | **1.3916±0.0126(2)** |
| | 10 | 1.4499±0.1661(6) | 1.7477±0.1010(4.5) | 0.1892±0.0001(7) | 1.9608±0.1248(3) | 1.7120±0.1009(4.5) | **2.1912±0.0919(1.5)** | **2.2331±0.0414(1.5)** |
| | 12 | 1.5584±0.1890(6) | 1.8958±0.1707(4.5) | 0.2287±0.0000(7) | 2.2009±0.1204(3) | 2.0178±0.1406(4.5) | 2.3813±0.1246(2) | **2.6000±0.0902(1)** |
| | 15 | 2.5248±0.1976(5.5) | 2.6371±0.2223(5.5) | 0.3067±0.0123(7) | 2.9727±0.1766(3.5) | 2.9336±0.1684(3.5) | 3.4368±0.1596(2) | **3.6800±0.1132(1)** |
| | 20 | 3.9541±0.3817(5) | 3.9678±0.4635(6) | 0.5114±0.0422(7) | 4.2424±0.2701(3.5) | 4.4592±0.4513(3.5) | 5.1012±0.3868(2) | **5.8761±0.2987(1)** |
| | 25 | 6.9389±0.4429(5) | 6.8244±0.4366(5) | 0.8860±0.0942(7) | 6.7878±0.3133(5) | 8.9492±0.7306(2.5) | 8.9584±0.5207(2.5) | **9.8070±0.4047(1)** |

Table 4.5 summarizes the obtained results of different algorithms with respect to the number of objectives. We count the number of times $F$-DEA is better, worse or equal than any competing algorithm based on the Wilcoxon rank-sum test. For better understanding, we present the results into three groups: 2-objective to 3-objective, 5-objective to 7-objective and 10-objective to 25-objective. For 2-objective and 3-objective, $F$-DEA was outperformed by SDE and PICEAg, but it was found better than MOEA/D, HypE, FD-NSGAII and NSGAIII. SDE and PICEAg were also the top performing algorithms for 5-objective to 7-objective, while $F$-DEA secured the overall third position and outperformed others. This scenario is totally different for a higher number of objectives i.e., 10-objective to 25-objective for which $F$-DEA was found better than all competing algorithms.

Table 4.5: Summary of HV performance of competing algorithms for eight WFG problems. Here B, E AND W indicate the number of times $F$-DEA was found better, equal and worse compared to a particular algorithm.

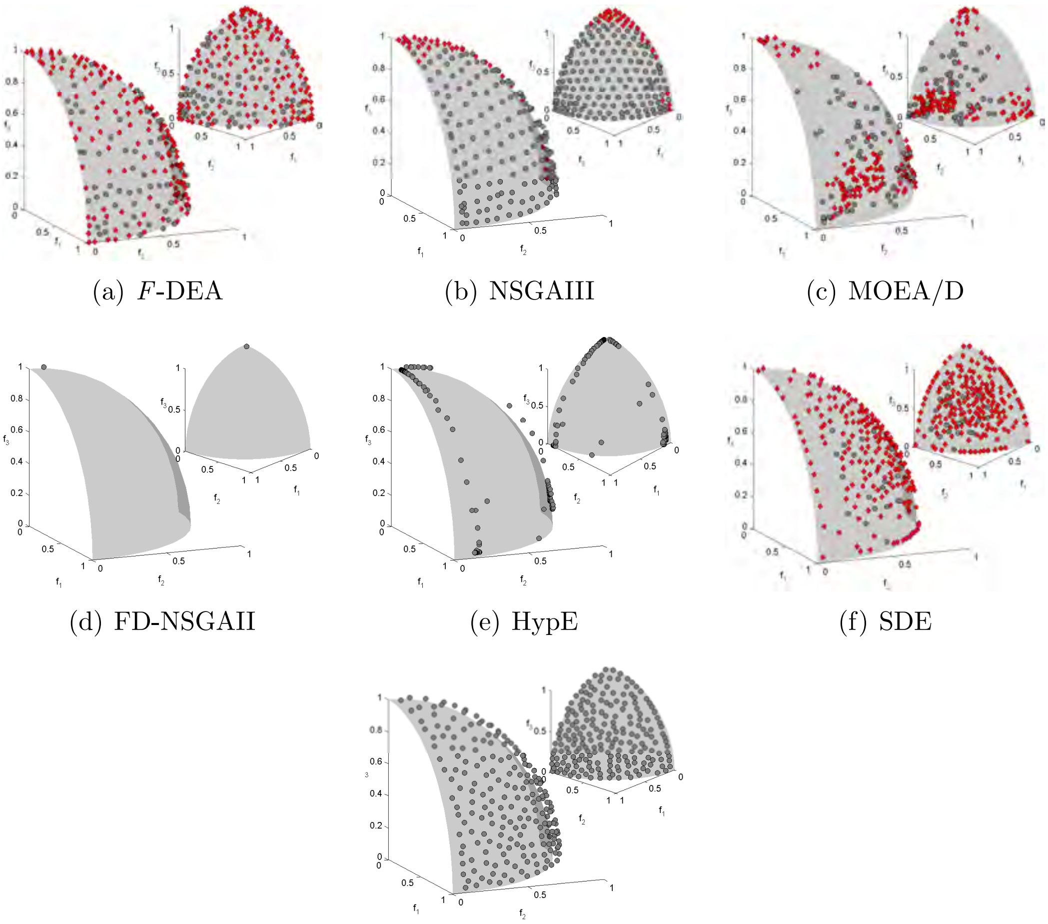
| WFG | HV Performance on 2- and 3- objective | | | | | |
|---|---|---|---|---|---|---|
| F-DEA vs | HypE | MOEA/D | FD-NSGAII | SDE | NSGAIII | PICEA-g |
| B | 16 | 15 | 16 | 2 | 8 | 5 |
| E | 0 | 1 | 0 | 1 | 3 | 4 |
| W | 0 | 0 | 0 | 13 | 5 | 7 |
| | HV Performance on 5- and 7- objective | | | | | |
| B | 16 | 16 | 15 | 4 | 15 | 4 |
| E | 0 | 0 | 0 | 5 | 0 | 3 |
| W | 0 | 0 | 1 | 7 | 1 | 9 |
| | HV Performance on 10-, 12-, 15-, 20- and 25-objective | | | | | |
| B | 40 | 40 | 35 | 29 | 40 | 31 |
| E | 0 | 0 | 2 | 4 | 0 | 3 |
| W | 0 | 0 | 3 | 7 | 0 | 6 |

We use the non-dominated solutions obtained in the final generation for obtaining plots. In terms of HV, the solutions were taken from the 10th run of the sorted 20 independent runs. For brevity, we consider the WFG9 problem with 3- and 15-objective for attainment surface plots and parallel coordinate plots, respectively. Note that the deceptive, biased, non-separable, and multi-modal nature make WFG9 a very difficult problem.

To visualize the achieved convergence and diversity more clearly, the obtained non-dominated solutions of a particular algorithm are divided into two categories: converged and non-converged. The solutions that are at most $d$-distance apart from the normalized surface are considered as converged and remaining are considered as non-converged.

Figure 4.1 shows the attainment surfaces of five different algorithms on the WFG9 problem with 3-objective. We here call an obtained non-dominated solution as the converged one if it is at most $d = 0.03$ distance apart from the points of normalized Pareto surface. Otherwise, it is called as the non-converged solution. It can be seen from Fig. 4.1(a) that both the converged solutions (red colored diamond points) and the non-converged ones (gray colored circle points) of $F$-DEA's are almost uniformly distributed in the entire Pareto surface. Although the obtained solutions of NSGAIII are evenly distributed, but a few of them converges (Fig. 4.1(b)). One reason is that while $F$-DEA employ reference point for constructing diverse clusters, NSGAIII uses the reference point directly for maintaining diversity among the solutions. However, some experimental observations [23, 49, 53] have indicated that favoring too much diversity has the potential detrimental effect on the convergence of EMO algorithms for MaOPs. For SDE, most of the obtained solutions were converged (Fig. 4.1(f)), but they were not evenly diverse like those of NSGAIII and $F$-DEA. PICEAg maintained good diversity without convergence (Fig. 4.1(g)). It is evident from Fig. 4.1(d) that the solutions of FD-NSGAII, whether they converged or non-converged, cover only a very small portion of the normalized Pareto surface.

The situation, however, changes as the number of objectives increases. For example, Fig. 4.2 shows the parallel coordinate plots of the five algorithms for the WFG9 problem with 15-objective. In terms of the number of converged solutions, FD-NSGAII 4.2(d), $F$-DEA 4.2(a) and PICEAg 4.2(g) secured the first, second and third positions, respectively. However, only $F$-DEA was able to maintain good diversity which could be attributed to the variations in the objective values of the solutions. While FD-NSGAII failed miserably to maintain diversity, PICEAg was able to maintain diversity moderately. Although most of the solutions from SDE (Fig. 4.2(f)), NSGAIII (Fig. 4.2(b)) were not converged but they maintained better diversity than FD-NSGAII and PICEAg. The bottom figures of five competing algorithms show the zoomed version of all the obtained non-dominated solutions. It is clear from these figures that $F$-DEA is better in terms of simultaneous minimization of all the objectives while maintaining diversity.

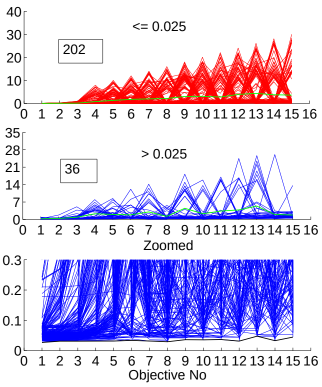


(a) $F$-DEA (b) NSGAIII (c) MOEA/D

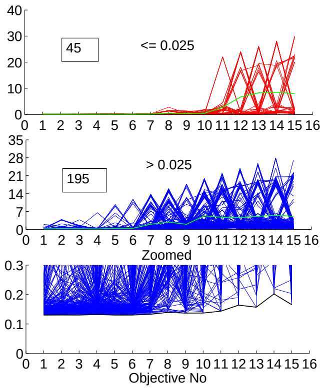
(d) FD-NSGAII (e) HypE (f) SDE

(g) PICEAg

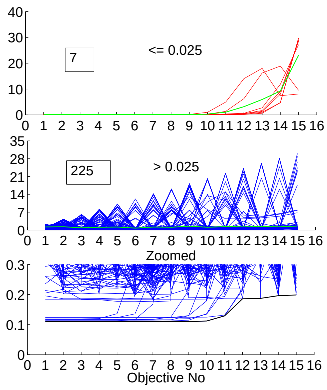
Figure 4.1: Attainment surface of different algorithms for the WFG9 problem with 3-objective. For better visualization, the obtained non-dominated solutions are categorized into converged (red diamond points) and non-converged (gray circle points) based on threshold distance 0.03 from the points in the normalized Pareto surface to the normalized solution value.



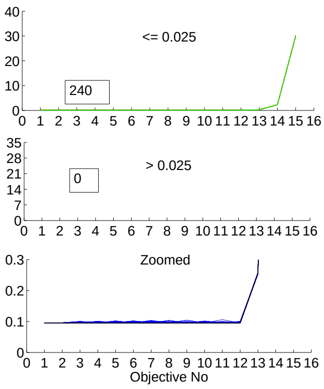

(a) *F*-DEA



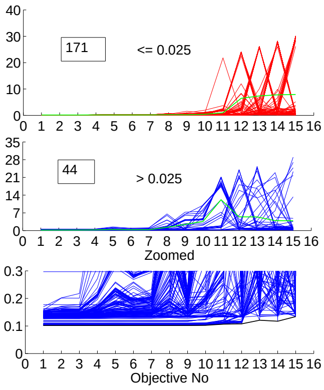

(b) NSGAIII



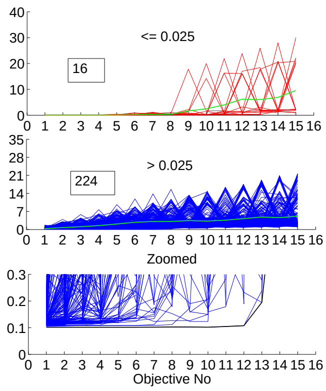

(c) MOEA/D



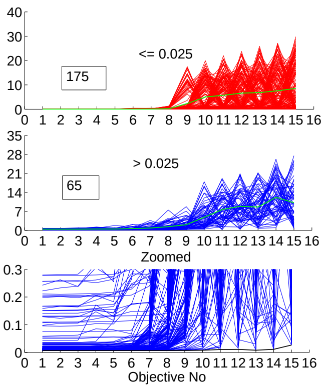

(d) FD-NSGAII




(e) HypE




(f) SDE




(g) PICEAg

Figure 4.2: Parallel coordinate plot of different algorithms for the WFG9 problem with 15-objective. Here the non-dominated solutions are separated into two categories based on a threshold distance value from the normalized Pareto front. The solutions with a distance less than or equal to 0.025 is regarded as converged solutions (top figure, red colored), while the other ones are regarded as non-converged solutions (middle figure, blue colored). Also, to observe the simultaneous minimization of different objectives, the bottom figure shows closer inspection of all the solutions.

## 4.7 Experiments on DTLZ Problems

We apply $F$-DEA and other competing algorithms on five DTLZ problems, DTLZ1, DTLZ2, DTLZ3, DTLZ4 and DTLZ7. We use HV and IGD for comparison. But as DTLZ7 is a disconnected problem, it is difficult to get the reliable estimation of IGD value for this case. Hence, we did not employ IGD for DTLZ7.

Tables 4.6 and 4.7 show the performances of different algorithms on DTLZ problems in terms of HV and IGD, respectively.

### 4.7.1 DTLZ1 Problem

Although DTLZ1 has a simple linear Pareto Front ($\sum_{i=1}^{m} f_i = 0.5$), a large number of local optima ($= 11^5 - 1$) makes it difficult for an algorithm to converge into the hyper-plane. In terms of average HV and IGD, NSGAIII was better than all other competing algorithms for a smaller number of objectives. $F$-DEA, MOEA/D, PICEAg, SDE were able to solve this problem with competitive performance. $F$-DEA, however, outperformed others as the number of objectives increased. For example, in terms of HV, $F$-DEA was found better than all competing algorithms from 10-objective to 25-objective. In terms of IGD, it beat all others in 7-, 10-, 15- and 20-objective and secured the second position after SDE in 12- and 25-objective.

### 4.7.2 DTLZ2 Problem

This is relatively an easy problem with concave geometrical shape ($\sum_{i=1}^{m} f_i^2 = 1$). All of the competing algorithms were able to solve it. In terms of IGD, NSGAIII was the top performer from 2-objective to 7-objective. SDE secured the top position based on HV for all but 2-objective. It also secured the top position in terms of IGD obtained for 10-objective to 25-objective. The performances of $F$-DEA were close to the top performers both with respect to

IGD and HV. In terms of IGD for the 10-objective to 25-objective, the proposed algorithm secured the second position after SDE.

### 4.7.3 DTLZ3 Problem

This problem has the same shape as DTLZ2 but it has a large number of local Pareto fronts parallel to the global one.

This problem has concave geometrical shape ($\sum_{i=1}^{m} f_i^2 = 1$) with a large number of local Pareto fronts parallel to the global one. This property makes it a very challenging problem. SDE and MOEA/D were top performers on this problem. Not all but $F$-DEA, SDE, MOEA/D, and PICEAg were able to solve this problem for a larger number of objectives, which could be seen by their non-zero HV values. In terms of HV and IGD, $F$-DEA was one of the best performers in 3-objective and secured the second position after SDE for the 25-objective. It shared the second position with MOEA/D in many cases while compared with respect to HV and for a larger number of objectives. MOEA/D, however, outperformed $F$-DEA in terms of IGD values, which caused $F$-DEA to achieve overall the third position.

### 4.7.4 DTLZ4 Problem

Although DTLZ4 has same geometrical shape as DTLZ2 and DTLZ3, it challenges the ability of an algorithm to maintain diversity in the objective space by introducing variable density of solutions along the Pareto front. The IGD values showed that $F$-DEA secured the first position in the 20-objective and the second position in rest of the objectives. NSGAIII showed the best performance for a smaller number of objectives and SDE for a larger ones. In terms of HV, $F$-DEA secured the third position after SDE and NSGAIII.

### 4.7.5 DTLZ7 Problem

This problem has disconnected regions which make it interesting and challenging. In terms of HV, $F$-DEA exhibited superior performance for a larger number of objectives while SDE showed superior performance for a smaller ones (Table 4.6). For example, SDE and MOEA/D jointly secured the first position for 2-objective and the former one independently secured the first position from the 3-objective to 7-objective. NSGAIII obtained the second position for 3-objective and $F$-DEA secured the second position for 5-objective and 7-objective. For a larger number of objectives (from 10-objective to 25-objective), $F$-DEA was the best and PICEAg was next to it. Most of the algorithms did not able to converge within the reference point bound which constituted their small HV values for a larger number of objectives.

Tables 4.8 and 4.9, respectively, show the number of times $F$-DEA is found better, worse or equal than any competing algorithm based on the Wilcoxon rank-sum test applied on the HV and IGD values of the DTLZ problems. In terms of HV and IGD, the three algorithms SDE, NSGAIII, PICEAg and $F$-DEA showed a very similar performance for a smaller number of objectives, 2-objective to 7-objective. However, for higher number of objectives $F$-DEA outperforms others in DTLZ1, DTLZ7 problems and shows competitive performance with SDE and MOEA/D at DTLZ3 problem.

Fig. 4.3 shows the parallel coordinate plot of the competing algorithms for the DTLZ7 problem with 10-objective. The upper bound of the last objective for this problem is $2 \times m$ or $f_{10} \leq 20$. It can be seen from the figure that $F$-DEA was able to maintain diversity and convergence together within the Pareto optimal front. SDE (Fig. 4.3(f)) maintained diversity and convergence well but $F$-DEA outperforms SDE by having more objective value variation in the first 9 objectives. PICEAg (Fig. 4.3(g)) converged in the first nine objective but few solutions converge on 10-th objective. Similarly, NSGAIII (Fig. 4.3(b)) also converges in the first 9 objectives but only few converges in the 10-th objective. FD-NSGAII (Fig. 4.3(d)) converged solutions into a region as expected.

Table 4.6: Average HV of different algorithms on DTLZ1, DTLZ3 and DTLZ7 problems over 20 independent runs. The best result based on the Wilcoxon rank sum test with a significance level of 0.05 is marked in bold-face. The rank of a particular algorithm is shown in bracket.

| Prob. | $m$ | HYPE | MOEA/D | FD-NSGAII | SDE | NSGAIII | PICEA-g | F-DEA |
|---|---|---|---|---|---|---|---|---|
| dtlz1 | 2 | 0.6596±0.0422(6) | **0.7069±0.0005(2)** | 0.0000±0.0000(7) | **0.7066±0.0002(2)** | **0.7063±0.0012(2)** | 0.7061±0.0005(4.5) | 0.7063±0.0002(4.5) |
| | 3 | 0.6849±0.5932(6) | 1.0884±0.0088(5) | 0.0000±0.0000(7) | 1.1261±0.0023(3.5) | **1.1246±0.0277(1.5)** | **1.1313±0.0045(1.5)** | 1.1249±0.0020(3.5) |
| | 5 | 0.0067±0.0116(6) | 1.5142±0.0098(5) | 0.0000±0.0000(7) | 1.5481±0.0046(4) | **1.5707±0.0076(1)** | 1.5297±0.0459(2.5) | 1.5582±0.0030(2.5) |
| | 7 | 0.6145±0.7117(6) | 1.8803±0.0094(4) | 0.0000±0.0000(7) | 1.9108±0.0031(3) | **1.9095±0.1091(1)** | 1.7799±0.0819(5) | 1.9272±0.0026(2) |
| | 10 | 0.6077±1.0526(6) | 2.4924±0.0229(3) | 0.0090±0.0331(7) | 2.5591±0.0044(2) | 1.7479±0.9172(4.5) | 2.2692±0.3215(4.5) | **2.5829±0.0024(1)** |
| | 12 | 0.0141±0.0231(6) | 2.9727±0.0335(3) | 0.0306±0.1175(7) | 3.0855±0.0093(2) | 1.8917±1.1930(4.5) | 2.5134±0.6594(4.5) | **3.1253±0.0034(1)** |
| | 15 | 0.7807±0.6792(6) | 4.0354±0.0273(3) | 0.0152±0.0680(7) | 4.1336±0.0064(2) | 3.8412±0.3188(4.5) | 3.4812±1.0382(4.5) | **4.1680±0.0021(1)** |
| | 20 | 0.0000±0.0000(6.5) | 6.4428±0.0967(3) | 0.1024±0.2214(6.5) | 6.6532±0.0101(2) | 3.2618±2.7249(4.5) | 5.1065±2.0933(4.5) | **6.7005±0.0072(1)** |
| | 25 | 0.3519±0.6122(5.5) | 0.0000±0.0000(7) | 0.2169±0.3563(5.5) | 10.7567±0.0128(2) | 6.9000±4.0868(4) | 10.3842±0.3817(3) | **10.8130±0.0069(1)** |
| dtlz2 | 2 | 0.3953±0.0042(6) | 0.4223±0.0000(1) | 0.1099±0.0000(7) | 0.4222±0.0000(2.5) | 0.4222±0.0001(2.5) | 0.4210±0.0002(5) | 0.4217±0.0001(4) |
| | 3 | 0.6160±0.0487(6) | 0.7172±0.0085(5) | 0.1209±0.0000(7) | **0.7646±0.0007(1)** | 0.7638±0.0006(2) | 0.7600±0.0007(3) | 0.7569±0.0012(4) |
| | 5 | 0.9862±0.0894(6) | 1.1694±0.0120(5) | 0.1463±0.0000(7) | **1.3040±0.0028(1)** | 1.2995±0.0007(2) | 1.2975±0.0021(3) | 1.2847±0.0026(4) |
| | 7 | 1.1859±0.0833(6) | 1.6068±0.0124(5) | 0.1770±0.0000(7) | **1.7729±0.0024(1)** | 1.7193±0.1551(3) | 1.7582±0.0069(2) | 1.7469±0.0045(4) |
| | 10 | 1.5584±0.0502(6) | 2.3023±0.0203(4) | 0.2357±0.0000(7) | **2.5098±0.0017(1)** | 1.8555±0.2055(5) | 2.4646±0.0380(2.5) | 2.4854±0.0059(2.5) |
| | 12 | 1.5474±0.1067(6) | 2.6923±0.0658(3.5) | 0.2852±0.0000(7) | **3.0618±0.0032(1)** | 2.1792±0.2725(5) | 2.7169±0.1602(3.5) | 3.0037±0.0180(2) |
| | 15 | 2.0251±0.2760(6) | 3.7780±0.0643(4) | 0.3796±0.0000(7) | **4.1362±0.0030(1)** | 3.1109±0.2047(5) | 3.8769±0.1392(3) | 4.1171±0.0083(2) |
| | 20 | 2.7181±0.9549(6) | 5.9789±0.1165(3.5) | 0.6114±0.0000(7) | **6.7068±0.0018(1)** | 5.2380±0.5474(5) | 5.9443±0.2673(3.5) | 6.6499±0.0300(2) |
| | 25 | 5.8982±0.5908(6) | 9.6726±0.0691(4) | 0.9846±0.0001(7) | **10.8225±0.0023(1)** | 8.9515±0.9008(5) | 10.4096±0.3050(3) | 10.8131±0.0059(2) |
| dtlz3 | 2 | 0.0830±0.1437(5.5) | 0.4139±0.0068(1.5) | 0.0000±0.0000(7) | **0.4104±0.0096(1.5)** | 0.3796±0.0911(4) | 0.1540±0.1810(5.5) | 0.4041±0.0120(3) |
| | 3 | 0.1338±0.2317(5.5) | 0.6931±0.0150(3) | 0.0000±0.0000(7) | 0.4101±0.2904(4) | 0.1595±0.2052(5.5) | **0.4626±0.3281(1.5)** | **0.7261±0.0130(1.5)** |
| | 5 | 0.0000±0.0000(6.5) | 1.1638±0.0221(2) | 0.0000±0.0000(6.5) | **1.2779±0.0197(1)** | 0.0673±0.1730(4.5) | 0.6202±0.4698(3) | 0.1174±0.2130(4.5) |
| | 7 | 0.0000±0.0000(6) | 1.4667±0.3553(2) | 0.0000±0.0000(6) | **1.7475±0.0093(1)** | 0.1311±0.3230(6) | 0.5283±0.6051(3.5) | 0.8720±0.7061(3.5) |
| | 10 | 0.0000±0.0000(6) | 1.9846±0.4769(2.5) | 0.0000±0.0000(6) | **2.4098±0.2289(1)** | 0.0000±0.0000(6) | 0.6794±0.7619(4) | 1.7546±0.9708(2.5) |
| | 12 | 0.0000±0.0000(5.5) | 1.6366±1.0472(2.5) | 0.0000±0.0000(5.5) | **2.5020±1.0833(1)** | 0.0000±0.0000(5.5) | 0.1847±0.5736(5.5) | 1.3304±1.3433(2.5) |
| | 15 | 0.0000±0.0000(6) | 3.4835±0.0759(2.5) | 0.0000±0.0000(6) | **4.0894±0.0245(1)** | 0.0000±0.0000(6) | 0.9834±1.3996(4) | 2.2773±1.8582(2.5) |
| | 20 | 0.0000±0.0000(6) | 4.6077±1.7082(2) | 0.0000±0.0000(6) | **6.5836±0.0571(1)** | 0.0000±0.0000(6) | 1.3989±2.1580(3.5) | 1.5242±2.5204(3.5) |
| | 25 | 0.0000±0.0000(5.5) | 0.0000±0.0000(5.5) | 0.0000±0.0000(5.5) | **10.7668±0.0279(1)** | 0.0000±0.0000(5.5) | 5.5089±3.6702(2.5) | 3.9419±4.2086(2.5) |
| dtlz4 | 2 | 0.2107±0.1745(6) | 0.3911±0.0961(1) | 0.1099±0.0000(7) | 0.4222±0.0000(3) | 0.4223±0.0000(2) | 0.4212±0.0001(5) | 0.4217±0.0001(4) |
| | 3 | 0.5398±0.1510(6) | 0.6848±0.0957(5) | 0.1209±0.0000(7) | 0.7501±0.0672(1) | 0.7219±0.1024(2) | 0.7591±0.0007(3) | 0.7573±0.0009(4) |
| | 5 | 0.9247±0.1222(6) | 1.1728±0.0561(5) | 0.1463±0.0000(7) | **1.3001±0.0329(1)** | 1.3031±0.0010(2) | 1.2928±0.0035(3) | 1.2889±0.0019(4) |
| | 7 | 1.0358±0.3297(6) | 1.6415±0.0852(5) | 0.1770±0.0000(7) | 1.7707±0.0036(1) | 1.7616±0.0051(2) | 1.7483±0.0167(3.5) | 1.7540±0.0025(3.5) |
| | 10 | 1.2239±0.0970(6) | 2.4014±0.0554(5) | 0.2356±0.0000(7) | **2.5058±0.0029(1)** | 2.4908±0.0366(2) | 2.4802±0.0111(4) | 2.4951±0.0023(3) |
| | 12 | 0.5542±0.5997(6) | 2.8696±0.0733(5) | 0.2850±0.0001(7) | **3.0586±0.0060(1)** | 2.9287±0.1026(4) | 2.9963±0.0209(3) | 3.0480±0.0044(2) |
| | 15 | 0.9156±0.3491(6) | 4.0652±0.0213(4.5) | 0.3793±0.0002(7) | **4.1406±0.0020(1.5)** | 4.0400±0.1085(4.5) | 4.1160±0.0078(3) | **4.1402±0.0019(1.5)** |
| | 20 | 0.5678±0.2077(7) | 6.6212±0.0255(5) | 0.6630±0.1608(6) | **6.7076±0.0020(1)** | 6.6739±0.0408(3.5) | 6.6902±0.0047(3.5) | 6.7038±0.0034(2) |
| | 25 | 1.1185±0.7458(6.5) | 10.7742±0.0087(5) | 1.1514±0.3481(6.5) | **10.8284±0.0006(1.5)** | 10.8250±0.0153(1.5) | 10.8247±0.0013(4) | 10.8277±0.0013(3) |
| dtlz7 | 2 | 0.3190±0.0101(6) | **0.3337±0.0000(1.5)** | 0.1133±0.0143(7) | **0.3337±0.0000(1.5)** | 0.3337±0.0000(3) | 0.3165±0.0372(5) | 0.3334±0.0001(4) |
| | 3 | 0.3753±0.0141(6) | 0.4103±0.0070(5) | 0.1207±0.0001(7) | **0.4351±0.0008(1)** | 0.4327±0.0006(2) | 0.4033±0.0560(3.5) | 0.4281±0.0015(3.5) |
| | 5 | 0.4074±0.0018(5) | 0.3423±0.0376(6) | 0.1436±0.0003(7) | **0.5248±0.0036(1)** | 0.4318±0.0241(3.5) | 0.4079±0.0494(3.5) | 0.4834±0.0038(2) |
| | 7 | 0.3774±0.0159(3) | 0.2361±0.0553(6) | 0.1535±0.0030(7) | **0.5093±0.0074(1)** | 0.3231±0.0322(5) | 0.3547±0.0204(4) | 0.4822±0.0061(2) |
| | 10 | 0.2845±0.0445(3.5) | 0.0345±0.0243(7) | 0.1649±0.0014(5) | 0.3119±0.0479(3.5) | 0.0848±0.0555(6) | 0.3405±0.0253(2) | **0.4563±0.0149(1)** |
| | 12 | 0.2441±0.0042(2.5) | 0.0052±0.0063(6) | 0.1735±0.0048(4.5) | 0.1591±0.0329(4.5) | 0.0010±0.0023(7) | 0.2522±0.0418(2.5) | **0.5079±0.0507(1)** |
| | 15 | 0.2748±0.0394(3) | 0.0002±0.0003(6) | 0.1636±0.0012(4) | 0.0283±0.0099(5) | 0.0001±0.0005(7) | 0.3346±0.0274(2) | **0.3831±0.0245(1)** |
| | 20 | 0.1714±0.0362(2) | 0.0000±0.0000(6.5) | 0.0834±0.0066(3.5) | 0.0086±0.0125(5) | 0.0000±0.0000(6.5) | 0.0889±0.0836(3.5) | **0.3275±0.0571(1)** |
| | 25 | 0.0000±0.0000(5.5) | 0.0000±0.0000(5.5) | 0.0000±0.0000(5.5) | 0.0005±0.0005(3) | 0.0000±0.0000(5.5) | 0.1203±0.0984(2) | **0.1932±0.0399(1)** |

Table 4.7: Average IGD of different algorithms on DTLZ1 and DTLZ3 problems over 20 independent runs. The best result based on the Wilcoxon rank sum test with a significance level of 0.05 is marked in bold-face. The rank of a particular algorithm is shown in bracket.

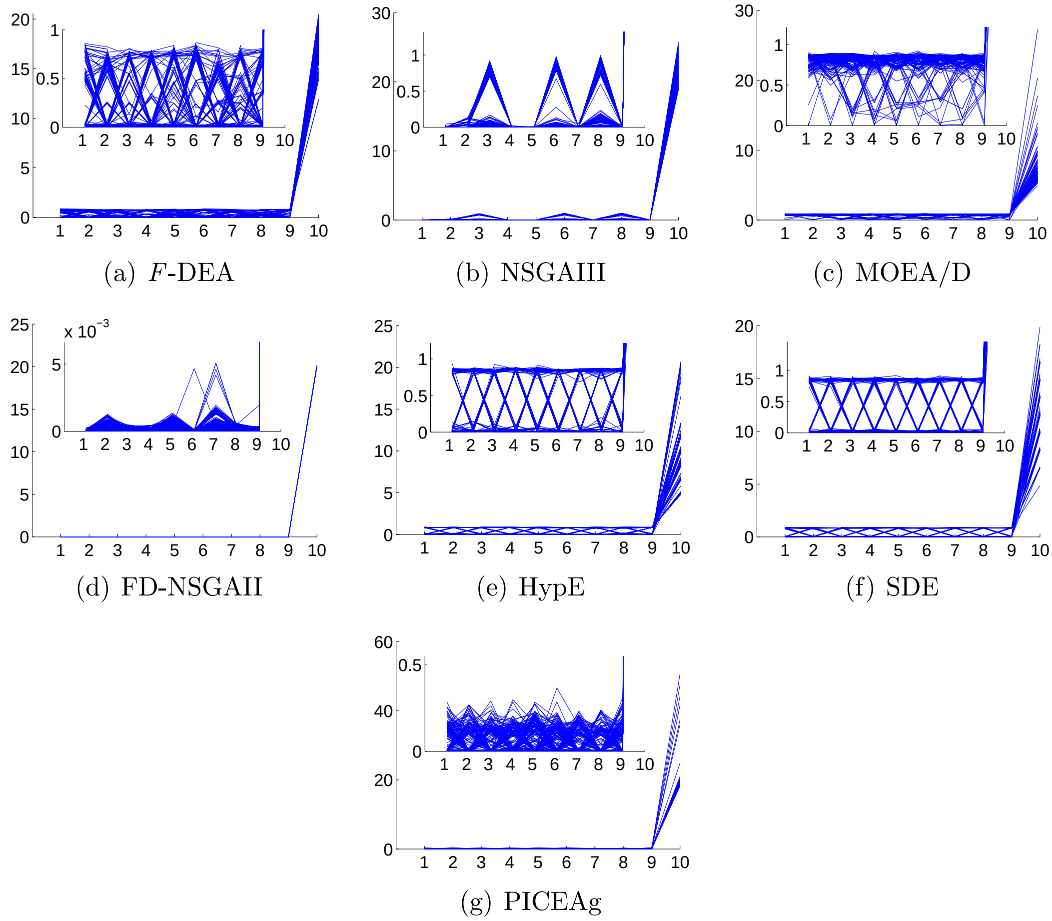
| Prob. | $m$ | HYPE | MOEA/D | FD-NSGAII | SDE | NSGAIII | PICEA-g | F-DEA |
|---|---|---|---|---|---|---|---|---|
| dtlz1 | 2 | 0.0016±0.0010(6) | 0.0000±0.0000(2) | 0.3873±0.1489(7) | 0.0000±0.0000(3) | **0.0000±0.0000(1)** | 0.0000±0.0000(4) | 0.0001±0.0000(5) |
| | 3 | 0.0117±0.0164(6) | 0.0021±0.0001(5) | 0.5881±0.2721(7) | **0.0011±0.0000(1)** | 0.0009±0.0011(4) | 0.0011±0.0002(2) | 0.0012±0.0001(3) |
| | 5 | 0.0512±0.0312(6) | 0.0052±0.0000(4) | 0.3606±0.1076(7) | 0.0041±0.0000(2) | **0.0020±0.0013(1)** | 0.0076±0.0033(5) | 0.0042±0.0001(3) |
| | 7 | 0.0592±0.0444(6) | 0.0077±0.0001(4) | 0.2067±0.0495(7) | 0.0067±0.0001(2.5) | 0.0040±0.0044(2.5) | 0.0182±0.0020(5) | **0.0063±0.0001(1)** |
| | 10 | 0.0631±0.0359(6) | 0.0116±0.0007(3) | 0.0821±0.0310(7) | 0.0092±0.0001(2) | 0.0254±0.0083(5) | 0.0226±0.0021(4) | **0.0083±0.0002(1)** |
| | 12 | 0.1058±0.0682(7) | 0.0152±0.0007(3) | 0.1077±0.0509(6) | **0.0113±0.0002(1)** | 0.0335±0.0125(4) | 0.0301±0.0106(5) | 0.0116±0.0004(2) |
| | 15 | 0.0929±0.0674(7) | 0.0132±0.0010(3) | 0.0775±0.0366(6) | 0.0095±0.0002(2) | 0.0202±0.0016(4) | 0.0235±0.0058(5) | **0.0094±0.0002(1)** |
| | 20 | 0.1201±0.0407(7) | 0.0204±0.0006(3) | 0.0728±0.0335(6) | 0.0155±0.0003(2) | 0.0370±0.0128(5) | 0.0320±0.0110(4) | **0.0141±0.0009(1)** |
| | 25 | 0.0165±0.0093(6) | 0.1045±0.0217(7) | 0.0150±0.0049(5) | **0.0033±0.0000(1)** | 0.0077±0.0029(4) | 0.0060±0.0003(3) | 0.0036±0.0002(2) |
| dtlz2 | 2 | 0.0032±0.0017(6) | 0.0001±0.0000(2) | 0.0611±0.0000(7) | 0.0006±0.0001(5) | **0.0000±0.0000(1)** | 0.0002±0.0000(4) | 0.0002±0.0000(3) |
| | 3 | 0.0153±0.0024(6) | 0.0054±0.0003(5) | 0.0744±0.0000(7) | 0.0046±0.0001(4) | **0.0002±0.0001(1)** | 0.0028±0.0000(2) | 0.0031±0.0000(3) |
| | 5 | 0.0250±0.0016(6) | 0.0152±0.0002(5) | 0.0808±0.0000(7) | 0.0127±0.0002(4) | **0.0008±0.0001(1)** | 0.0111±0.0002(2) | 0.0115±0.0002(3) |
| | 7 | 0.0350±0.0042(6) | 0.0230±0.0002(5) | 0.0858±0.0000(7) | 0.0165±0.0004(2) | 0.0046±0.0132(3.5) | **0.0153±0.0010(1)** | 0.0175±0.0005(3.5) |
| | 10 | 0.0501±0.0045(6) | 0.0298±0.0004(4) | 0.0876±0.0000(7) | **0.0197±0.0005(1)** | 0.0583±0.0136(5) | 0.0241±0.0072(3) | 0.0223±0.0006(2) |
| | 12 | 0.0688±0.0056(6) | 0.0399±0.0037(3) | 0.1044±0.0000(7) | **0.0242±0.0015(1)** | 0.0759±0.0166(5) | 0.0642±0.0088(4) | 0.0305±0.0016(2) |
| | 15 | 0.0622±0.0029(5) | 0.0359±0.0030(3) | 0.0854±0.0000(7) | **0.0198±0.0013(1)** | 0.0668±0.0030(6) | 0.0534±0.0061(4) | 0.0250±0.0009(2) |
| | 20 | 0.0755±0.0024(6) | 0.0528±0.0031(3) | 0.0941±0.0000(7) | **0.0202±0.0016(1)** | 0.0752±0.0074(5) | 0.0729±0.0037(4) | 0.0306±0.0035(2) |
| | 25 | 0.0197±0.0004(6) | 0.0151±0.0003(3) | 0.0239±0.0000(7) | **0.0094±0.0000(1)** | 0.0191±0.0015(5) | 0.0170±0.0013(4) | 0.0102±0.0002(2) |
| dtlz3 | 2 | 0.0435±0.0396(5) | **0.0004±0.0003(1)** | 1.6814±0.7393(7) | 0.0008±0.0003(2) | 0.0039±0.0122(4) | 0.0586±0.0755(6) | 0.0008±0.0005(3) |
| | 3 | 0.1586±0.1130(6) | 0.0056±0.0004(2) | 1.1440±0.4942(7) | 0.0271±0.0269(3) | 0.0493±0.0292(4) | 0.0317±0.0395(5) | **0.0037±0.0003(1)** |
| | 5 | 0.6058±0.2773(6) | 0.0150±0.0004(2) | 0.9245±0.2767(7) | **0.0130±0.0005(1)** | 0.1037±0.0649(5) | 0.0608±0.0397(3) | 0.0904±0.0624(4) |
| | 7 | 0.9081±0.3801(7) | 0.0272±0.0120(2) | 0.8025±0.2808(6) | **0.0173±0.0006(1)** | 0.1592±0.1150(5) | 0.1012±0.0586(4) | 0.0528±0.0418(3) |
| | 10 | 0.9161±0.2690(7) | 0.0404±0.0119(2) | 0.6995±0.2610(5) | **0.0245±0.0048(1)** | 0.6902±0.3276(6) | 0.1096±0.0593(4) | 0.0396±0.0246(3) |
| | 12 | 1.3816±0.4275(6) | 0.0786±0.0540(2) | 1.1271±0.4369(5) | **0.0494±0.0359(1)** | 1.0651±0.6395(7) | 0.2625±0.1429(4) | 0.1004±0.0884(3) |
| | 15 | 0.8515±0.3101(6) | 0.0469±0.0046(2) | 0.5108±0.1996(5) | **0.0289±0.0031(1)** | 0.9245±0.4516(7) | 0.1055±0.0299(4) | 0.0510±0.0317(3) |
| | 20 | 1.2758±0.4770(6) | 0.0761±0.0293(2) | 0.6600±0.2519(5) | **0.0374±0.0033(1)** | 1.4316±0.4919(7) | 0.1644±0.0839(4) | 0.1337±0.0769(3) |
| | 25 | 2.8646±0.7739(6) | 3.0273±0.3555(7) | 0.1204±0.0454(4) | **0.0107±0.0010(1)** | 0.3112±0.0964(5) | 0.0249±0.0084(3) | 0.0230±0.0098(2) |
| dtlz4 | 2 | 0.0162±0.0266(6) | 0.0062±0.0188(5) | 0.0612±0.0000(7) | 0.0006±0.0001(4) | **0.0000±0.0000(1)** | 0.0002±0.0000(3) | 0.0002±0.0000(2) |
| | 3 | 0.0383±0.0204(6) | 0.0118±0.0160(5) | 0.0746±0.0000(7) | 0.0068±0.0099(4) | 0.0065±0.0158(3) | **0.0029±0.0000(1)** | 0.0031±0.0000(2) |
| | 5 | 0.0475±0.0080(6) | 0.0193±0.0087(5) | 0.0809±0.0000(7) | 0.0140±0.0050(4) | **0.0010±0.0002(1)** | 0.0113±0.0002(3) | 0.0112±0.0002(2) |
| | 7 | 0.0561±0.0093(6) | 0.0251±0.0077(5) | 0.0858±0.0000(7) | 0.0180±0.0004(3) | **0.0019±0.0009(1)** | 0.0173±0.0033(4) | 0.0178±0.0003(2) |
| | 10 | 0.0579±0.0057(6) | 0.0310±0.0046(5) | 0.0876±0.0000(7) | 0.0225±0.0007(3) | **0.0036±0.0071(1)** | 0.0216±0.0028(4) | 0.0225±0.0005(2) |
| | 12 | 0.0808±0.0023(6) | 0.0457±0.0069(5) | 0.1044±0.0000(7) | **0.0248±0.0025(1)** | 0.0328±0.0156(4) | 0.0368±0.0042(3) | 0.0269±0.0008(2) |
| | 15 | 0.0652±0.0026(6) | 0.0327±0.0025(5) | 0.0854±0.0000(7) | **0.0203±0.0009(1)** | 0.0279±0.0109(4) | 0.0271±0.0021(3) | 0.0224±0.0005(2) |
| | 20 | 0.0804±0.0048(6) | 0.0420±0.0021(5) | 0.0938±0.0010(7) | 0.0281±0.0013(2) | 0.0299±0.0126(3) | 0.0327±0.0015(4) | **0.0247±0.0019(1)** |
| | 25 | 0.0203±0.0009(6) | 0.0117±0.0001(5) | 0.0237±0.0003(7) | **0.0097±0.0000(1)** | 0.0100±0.0005(4) | 0.0104±0.0001(3) | 0.0102±0.0001(2) |

Table 4.8: Summary of HV performance of competing algorithms for DTLZ1, DTLZ3 and DTLZ7 problems. Here B, E AND W indicate the number of times $F$-DEA was found better compared to a particular algorithm.

| DTLZ | HV Performance on 2- and 3- objective | | | | | |
|---|---|---|---|---|---|---|
| F-DEA vs | HypE | MOEA/D | FD-NSGAII | SDE | NSGAIII | PICEA-g |
| B | 10 | 5 | 10 | 1 | 2 | 4 |
| E | 0 | 0 | 0 | 1 | 0 | 3 |
| W | 0 | 5 | 0 | 8 | 8 | 3 |
| | HV Performance on 5- and 7- objective | | | | | |
| B | 10 | 8 | 10 | 2 | 3 | 3 |
| E | 0 | 0 | 0 | 0 | 1 | 3 |
| W | 0 | 2 | 0 | 8 | 6 | 4 |
| | HV Performance on 10-, 12-, 15-, 20- and 25-objective | | | | | |
| B | 25 | 21 | 25 | 10 | 23 | 22 |
| E | 0 | 3 | 0 | 1 | 0 | 3 |
| W | 0 | 1 | 0 | 14 | 2 | 0 |

Table 4.9: Summary of IGD performance of competing algorithms for four DTLZ problems. Here B, E AND W indicate the number of times $F$-DEA was found better compared to a particular algorithm.

| DTLZ | IGD Performance on 2- and 3- objective | | | | | |
|---|---|---|---|---|---|---|
| F-DEA vs | HypE | MOEA/D | FD-NSGAII | SDE | NSGAIII | PICEA-g |
| B | 8 | 5 | 8 | 5 | 4 | 4 |
| E | 0 | 0 | 0 | 0 | 0 | 0 |
| W | 0 | 3 | 0 | 3 | 4 | 4 |
| | IGD Performance on 5- and 7- objective | | | | | |
| B | 8 | 6 | 8 | 4 | 3 | 5 |
| E | 0 | 0 | 0 | 0 | 1 | 0 |
| W | 0 | 2 | 0 | 4 | 4 | 3 |
| | IGD Performance on 10-, 12-, 15-, 20- and 25-objective | | | | | |
| B | 20 | 16 | 20 | 5 | 19 | 20 |
| E | 0 | 0 | 0 | 0 | 0 | 0 |
| W | 0 | 4 | 0 | 15 | 1 | 0 |



(a) *F*-DEA (b) NSGAIII (c) MOEA/D

(d) FD-NSGAII (e) HypE (f) SDE

(g) PICEAg

Figure 4.3: Parallel coordinate plot of all competing algorithms in 10− objective DTLZ7 problem. The inset figure shows the closer inspection of the first 9 objectives.

## 4.8 Experiment on Rectangle Problem

The rectangle problem is a 4-objective test problem. The interesting properties of it are that the degenerate Pareto optimal solutions lie in a rectangle of the two variable decision space and have similar images in the objective space. We choose three different instances of the problem: Rectangle Problem I, Rectangle Problem II and Rectangle Problem III. Each instances differs from others with respect to the domain of two decision variables, $x_1$ and $x_2$. Note that the domain ranges of three different instances were chosen according to [41].

We apply all the competing algorithms on the three instances of Rectangle problem. For brevity we show results for $F$-DEA, NSGAIII, FD-NSGAII, SDE, PICEAg and results for MOEA/D, HypE are included in the supplementary materials. The division parameter, $\lambda$, of NSGAIII's and $F$-DEA's were set 10 and 60, respectively. The number of weights used in MOEA/D was chosen as 28800, while the number of goals for PICEAg was 400. The population size was set to 288 by considering the constraint of NSGAIII and the termination criterion was chosen to 250 generations. All the other parameters of the algorithms are in usual settings.

### 4.8.1 Rectangle Problem I

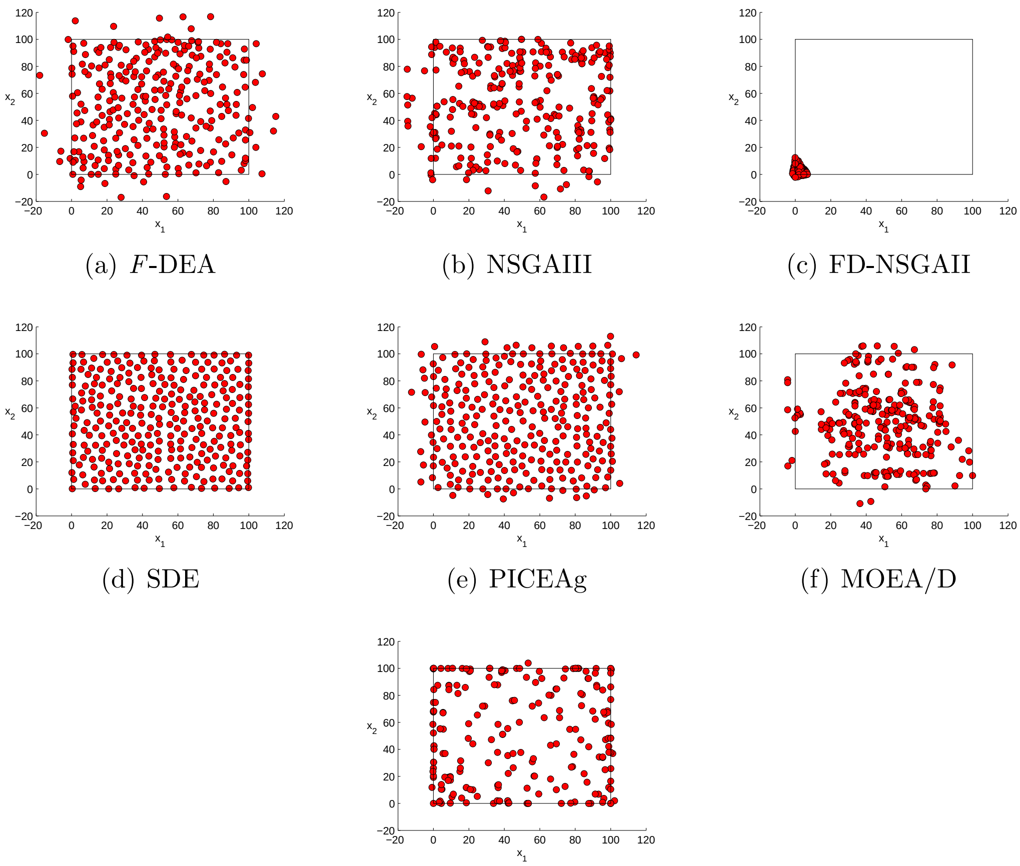
The domain of $x_1$ and $x_2$ for Rectangle Problem I is set to $[-20, 120]$. This problem has the smaller search region compared to its other two counter parts. Fig. 4.4 shows the non-dominated solutions of different algorithms obtained at the final generation of a particular run. It is evident that the solutions of FD-NSGAII (Fig. 4.4(c)) were concentrated into a particular location, indicating the algorithm's good convergence but poor diversity. Although the solutions from algorithm NSGAIII (Fig. 4.4(b)) converged but its distribution was not uniform. In contrast, most of the solutions of $F$-DEA (Fig 4.4(a)) and PICEAg (Fig 4.4(e)) were converged and maintained better uniformity. The solutions of SDE (Fig 4.4(d)) showed superior performance as all of them were converged and their distribution was uniform.

### 4.8.2 Rectangle Problem II

In this instance, the domain of $x_1, x_2$ is increased from $[-20, 120]$ to $[-10000, 10000]$, which poses a challenge to an algorithm for maintaining both convergence and diversity. SDE (Fig. 4.5(d)) achieved both good convergence and diversity in this problem. The solutions of FD-NSGAII (Fig. 4.5(c)) failed to converge in this problem. NSGAIII (Fig. 4.5(b)) and PICEAg (Fig. 4.5(e)) had very few converged solutions and they formed a cross. In the rectangle problem, any solution inside the parallel objectives' line can be only dominated by a solution in the same line whereas the solutions in corner are dominated by those reside in the broader region [41]. This property creates difficulty for Pareto based algorithm to converge. As the primary selection criterion of NSGAIII and PICEAg was based on Pareto, it makes problem for them reaching to the Pareto optimal front and solutions are distributed in crisscross. Our $F$-DEA applied fuzzy-fitness criterion in it's selection mechanism which alleviates the problem faced by Pareto dominance. Fig. 4.5(a) shows that most of the solutions of $F$-DEA was able to converge in optimal and near-optimal regions with uniformity.

### 4.8.3 Rectangle Problem III

In this test instance, the decision variables' domain is increased further $x_1, x_2 \in [-10^{12}, 10^{12}]$ to evaluate the ability of an algorithm in a very large search space. Fig. 4.6 shows the obtained non-dominated solutions of different algorithms. Solutions from SDE (Fig. 4.6(d)), PICEAg (Fig. 4.6(e)), and FD-NSGAII (Fig. 4.6(c)) have failed to converge into the optimal region i.e. inside the rectangle. NSGAIII (Fig. 4.6(b)) had solutions distributed in the crisscross manner and few of them only converged, which can be seen from the closer inspection of the inset figure. With respect to the other competing algorithms, $F$-DEA (Fig. 4.6(a)) showed good convergence and diversity.

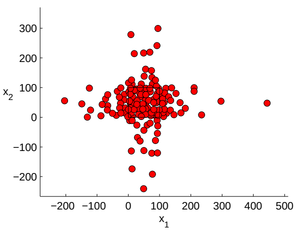



(a) $F$-DEA (b) NSGAIII (c) FD-NSGAII

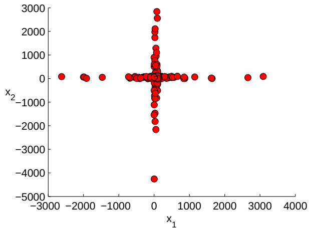
(d) SDE (e) PICEAg (f) MOEA/D

(g) HypE

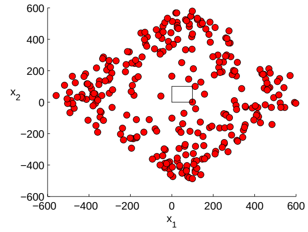
Figure 4.4: Final non-domination solution set of $F$-DEA and six others competing algorithms in decision space on Rectangle Problem Instance I where $x_1, x_2 \in [-20, 120]$



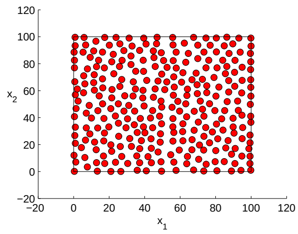

(a) *F*-DEA



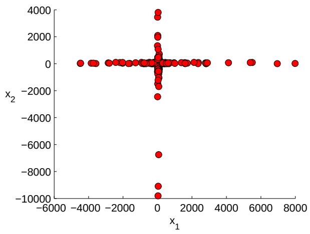

(b) NSGAIII



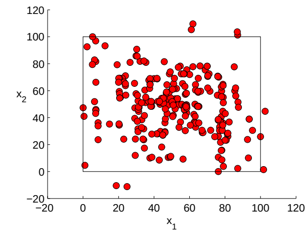

(c) FD-NSGAII



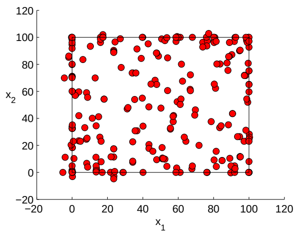

(d) SDE




(e) PICEAg

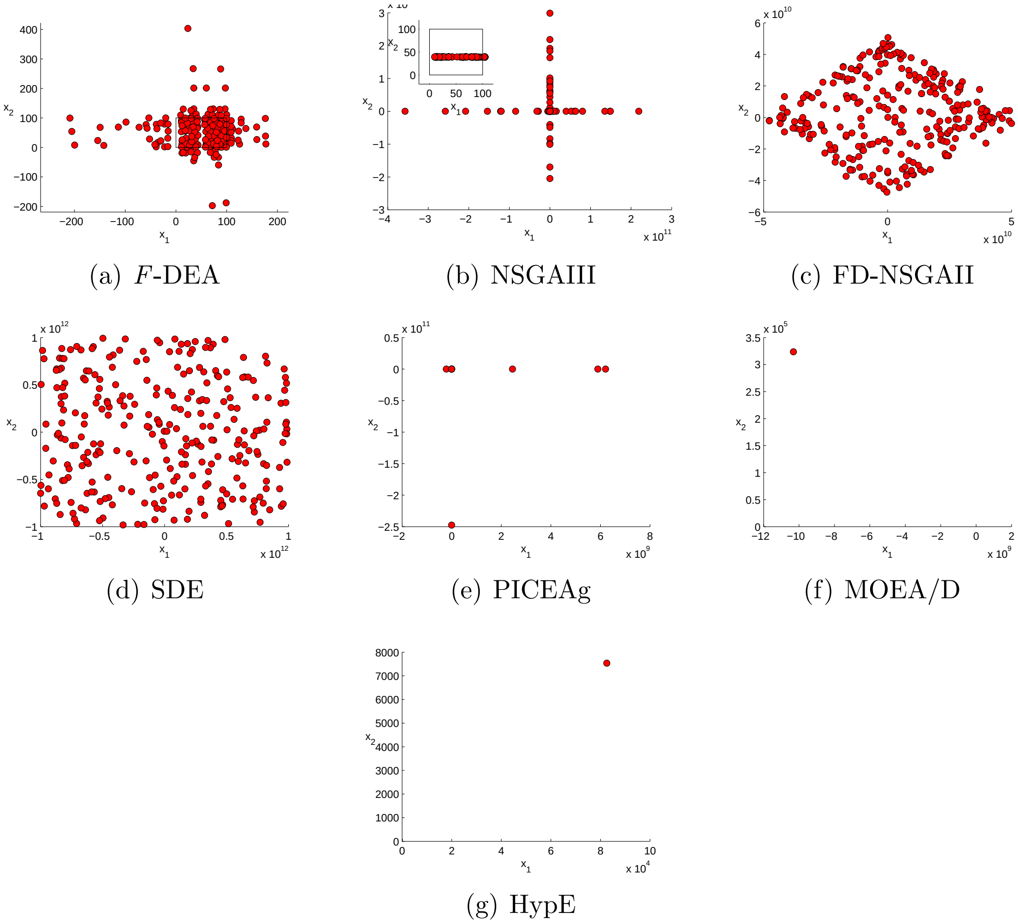



(f) MOEA/D




(g) HypE

Figure 4.5: Final non-domination solution set of *F*-DEA and six others competing algorithms in decision space on Rectangle Problem Instance II where $x_1, x_2 \in [-10000, 10000]$



(a) $F$-DEA (b) NSGAIII (c) FD-NSGAII

(d) SDE (e) PICEAg (f) MOEA/D

(g) HypE

Figure 4.6: Final non-domination solution set of $F$-DEA and six others competing algorithms in decision space on Rectangle Problem Instance III where $x_1, x_2 \in [-10^{12}, 10^{12}]$

## 4.9 Discussion

The results presented in the previous sections and supplementary material give an idea about the performances of $F$-DEA with respect to different competing algorithms. This section briefly explains the reasons behind such performances.

$F$-DEA maintains cluster uniformity based on preferred reference points. If a parent population lies in crowded regions and any offspring solution lies in a different region, then the solution will form a new cluster and $F$-DEA will select it for the next generation. The next generation clusters will thus have a broader span. This property helps $F$-DEA expanding its search region whenever possible. And this is beneficial for a deceptive problem containing large-size hill and disconnected problems having isolated regions. However, this property is not beneficial for a degenerate problem. As the isolated solution increase the search region, it will take some generations to converge. This can be seen for the Rectangle Problem where few solutions reside outside the optimal region. The fuzzy dominance with adaptive membership functions can optimize different objectives well in case of a large number of objectives (Fig. 4.2), and the bias induced for that was mitigated from the cluster uniformity. The fitness assignment procedure can maintain extreme point well in $F$-DEA which helps to retain cluster uniformity for the next generation.

The poor performance of FD-NSGAII [35] was due to lack of diversity among solutions (4.2(d) and 4.3(d)). NSGAIII [27] emphasizes on solutions that are non-dominated and close to reference line. When the number of objectives is large, the Pareto-dominance relied on by NSGAIII lacks enough selection pressure to push the population towards Pareto front. In a sense, NSGA-III stresses diversity more than convergence [21]. Our $F$-DEA uses fuzzy dominance that is able to maintain good selection pressure in high dimensional objective space. As the number of objectives increases fuzzy dominance becomes more effective to differentiating which solutions are better. Also the preferred reference points based clustering method promotes diversity using those reference points where solution exists rather than entire high dimensional objective space.

SDE [25] performs relatively well in the DTLZ problem suite than the WFG one. The reason most likely is the normalized nature of the former problem suite. The performance of $F$-DEA for the WFG problem signifies that the proposed algorithm was able to handle the scaling issue well due to the use of scale independent adaptive fuzzy membership function.

The co-evolutionary algorithm PICEAg is goal oriented and Pareto based algorithm. The maintenance of an increasing number of goal vectors by this algorithm enhances the comparability among solutions in a high dimensional objective space. The inherent tendency to maintain diversity sometimes responsible for reducing selection pressure. This can be understood by looking the results of the algorithm for the WFG1 problem for which it performed worse compared to its performance on other problems (Table 4.3).

The obtained solutions of MOEA/D in high dimension might achieve good aggregation values but far away from the corresponding weight vector. This property makes harder for MOEA/D to maintain diversity for problems with a large number of objectives. HypE failed to maintain diversity in some cases and pushed the solutions into the corner of the Pareto front.

The test problems in the WFG suite are far more difficult to solve and hard to maintain diversity. It is because the WFG problems have more transformation functions and they have different ranges for different objectives. The proposed algorithm $F$-DEA was able to outperform competing algorithms on these problems with a larger number of objectives. From the results of the DTLZ suite, we see that for a larger number of objectives, $F$-DEA was able to solve difficult to converge DTLZ1 and disconnected DTLZ7 problems better than others.

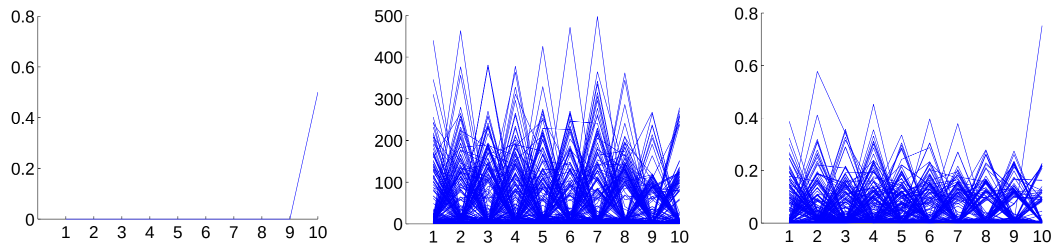
## 4.10 Effect of Reference Point Based Clustering and Fuzzy Dominance

To investigate how much benefit we get from reference point based clustering and fuzzy dominance over Pareto dominance, we evaluate two variants, $F$-DEA$^*$ and $F$-DEA$^\#$, of basic $F$-DEA on DTLZ1 and DTLZ3 problems with the usual settings. While $F$-DEA$^*$ does not employ reference points based clustering, $F$-DEA$^\#$ employs Pareto-dominance based fitness assignment instead of fuzzy dominance based fitness assignment. Tables 4.10 shows the comparative performances of $F$-DEA$^*$ and $F$-DEA$^\#$ against basic $F$-DEA in terms of HV and IGD. For brevity, we consider DTLZ1 and DTLZ3 problems with 7-, 10- and 15-objective.

It is clear from Tables 4.10 that the basic $F$-DEA outperforms both $F$-DEA$^*$ and $F$-DEA$^\#$ significantly in all problem instances. The large IGD values and small (in fact zero) HV values of $F$-DEA$^\#$ in DTLZ1 and DTLZ3 suggest that the Pareto based selection failed to create necessary selection pressure on hard to converge problems. $F$-DEA$^*$, on the other hand, convergences into a part of Pareto front due to the bias created from fuzzy dominance. The IGD and HV values of $F$-DEA$^*$ are better than $F$-DEA$^\#$ as no solution obtained by $F$-DEA$^\#$ converges into Pareto front.

Fig.4.7 shows the non-dominated solutions obtained by $F$-DEA$^*$, $F$-DEA$^\#$ and basic $F$-DEA for DTLZ1 problem with in 10-objective. Basic $F$-DEA achieved good convergence (overall objective range $[0 - 0.5]$) while maintaining good diversity (Fig. 4.7(c)). In contrast, $F$-DEA$^\#$ (Fig. 4.7(b)) maintained diversity due to reference points based clustering but failed to converge (overall objective range $[0 - 500]$) because of Pareto based selection. $F$-DEA$^*$ (Fig. 4.7(a)) converged to a part of Pareto front but severely lacked diversity.

The performances of two variants $F$-DEA$^*$ and $F$-DEA$^\#$ indicate the importance of using clustering and fuzzy dominance based selection in solving MaOPs. The fuzzy based selection alone cannot maintain diversity and clustering does not work with Pareto based approach due to its lack of comparability for a large number of objective. This is why $F$-DEA harvests the benefits from these two techniques, clustering and fuzzy dominance, and compliments each others' weakness.




(a) $F$-DEA* (b) $F$-DEA# (c) Basic $F$-DEA

Figure 4.7: Parallel coordinate plots of variants of $F$-DEA on 10− objective DTLZ1 problem.

## 4.11 Parameter Sensitivity

In the absence of supplied reference points, $F$-DEA employs the Das and Dennis [14] procedure for generating such points. This procedure requires a parameter $\lambda$, the number of division in an objective. It is worth mentioning that $\lambda$ is necessary if any evolutionary algorithm (see, for example, [27]) employs this procedure. The aim of this section is to show why we select $p$ ($=N$, the population size) preferred points from a large number of generated points.

Trivially, an evolutionary algorithm maintains diversity and convergence by its own way, where the decision maker has no control. In $F$-DEA, the decision maker can control them using $p$. Furthermore, $\lambda$ and $p$ together remove the constraint to population size for any number of objectives unlike NSGAIII [21, 27] in which the population size is bounded by $\lambda$ and not flexible for an arbitrary number of objectives.

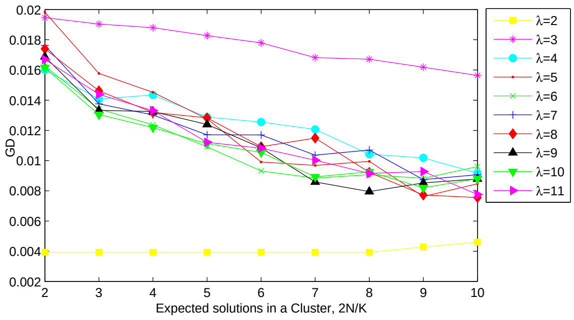
Fig. 4.8 shows impact of $\lambda$ and $p$ on the GD, IGD and HV performances of DTLZ2 problem with 10-objective. The experiments were conducted with the usual parameter settings but varying $\lambda$ and $p$. For convenience, $p$ was changed in such a way that the expected number of solutions in each cluster, $\chi$, becomes $2, 3, \cdots, 10$.

Table 4.10: Comparison among three different versions of the proposed algorithm, basic version ($F$-DEA), $F$-DEA without clustering ($F$-DEA$^*$), and $F$-DEA with Pareto dominance ($F$-DEA$^\#$), on DTLZ problems based on IGD and HV value.

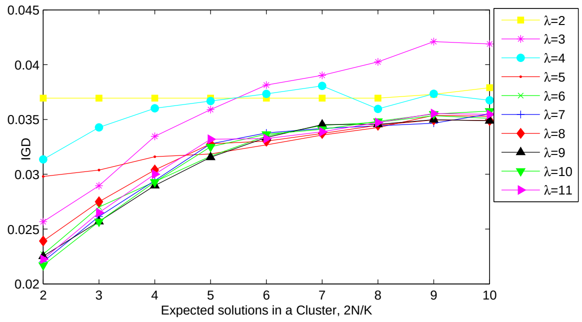
| Prob. | Obj. | $F$-DEA$^*$ | $F$-DEA$^\#$ | $F$-DEA |
|---|---|---|---|---|
| | | IGD | | |
| DTLZ1 | 7 | 0.0296±0.0152(2) | 9.0918±1.5703(3) | **0.0063±0.0001(1)** |
| | 10 | 0.0302±0.0155(2) | 7.6205±1.1507(3) | **0.0083±0.0002(1)** |
| | 15 | 0.0269±0.0159(2) | 5.8475±1.1274(3) | **0.0094±0.0002(1)** |
| DTLZ2 | 7 | 0.0858±0.0000(2) | 0.1602±0.0014(3) | **0.0175±0.0005(1)** |
| | 10 | 0.0877±0.0000(2) | 0.1357±0.0108(3) | **0.0223±0.0006(1)** |
| | 15 | 0.0855±0.0000(2) | 0.1015±0.0074(3) | **0.0250±0.0009(1)** |
| DTLZ3 | 7 | 0.0861±0.0001(2) | 83.6979±7.0178(3) | **0.0528±0.0418(1)** |
| | 10 | 0.0905±0.0120(2) | 70.5014±6.9913(3) | **0.0396±0.0246(1)** |
| | 15 | 0.0856±0.0001(2) | 56.6080±6.5905(3) | **0.0510±0.0317(1)** |
| DTLZ4 | 7 | 0.0858±0.0000(2) | 0.1517±0.0034(3) | **0.0178±0.0003(1)** |
| | 10 | 0.0877±0.0000(2) | 0.1056±0.0069(3) | **0.0225±0.0005(1)** |
| | 15 | 0.0855±0.0000(3) | 0.0729±0.0066(2) | **0.0224±0.0005(1)** |
| | | HV | | |
| DTLZ1 | 7 | 0.1410±0.0723(2) | 0.0000±0.0000(3) | **1.9272±0.0026(1)** |
| | 10 | 0.1877±0.0963(2) | 0.0000±0.0000(3) | **2.5829±0.0024(1)** |
| | 15 | 0.2839±0.1682(2) | 0.0000±0.0000(3) | **4.1680±0.0021(1)** |
| DTLZ2 | 7 | 0.1771±0.0000(2) | 0.0000±0.0000(3) | **1.7469±0.0045(1)** |
| | 10 | 0.2357±0.0000(2) | 0.0004±0.0019(3) | **2.4854±0.0059(1)** |
| | 15 | 0.3797±0.0000(2) | 0.0000±0.0000(3) | **4.1171±0.0083(1)** |
| DTLZ3 | 7 | 0.1684±0.0053(2) | 0.0000±0.0000(3) | **0.8720±0.7061(1)** |
| | 10 | 0.2156±0.0511(2) | 0.0000±0.0000(3) | **1.7546±0.9708(1)** |
| | 15 | 0.3683±0.0096(2) | 0.0000±0.0000(3) | **2.2773±1.8582(1)** |
| DTLZ4 | 7 | 0.1771±0.0000(2) | 0.0000±0.0000(3) | **1.7540±0.0025(1)** |
| | 10 | 0.2357±0.0000(2) | 0.0001±0.0004(3) | **2.4951±0.0023(1)** |
| | 15 | 0.3797±0.0000(2) | 0.0491±0.0596(3) | **4.1402±0.0019(1)** |

We can see as $\lambda$ increases so does overall HV (Fig. 4.8(c)) and reduces GD (Fig. 4.8(a)) and IGD (Fig. 4.8(b)) up to a certain level. Increasing $\lambda$ increases the number of generated reference points exponentially. A small number of reference points do not cover the entire objective space. Generating a large number of reference points maintains better cluster uniformity which in turn improves overall performance. But if we continue increasing reference points then after a certain $\lambda$, the clusters' representative points will be nearly same as before and performance will be clipped. It is also expensive to generate a huge number of reference points. Hence, a reasonable number of reference points should be used. By inspecting the performance on DTLZ2, it is clear that after $\lambda = 9$ the performance doesn't improve increasing $\lambda$ (Fig. 4.8). It is to be noted that HV for $\lambda = 3$ is greater than the HV value for $\lambda = 4$ and this behavior is problem dependent. Therefore it is sufficient to use $\lambda = 9$ for 10-objective problem.

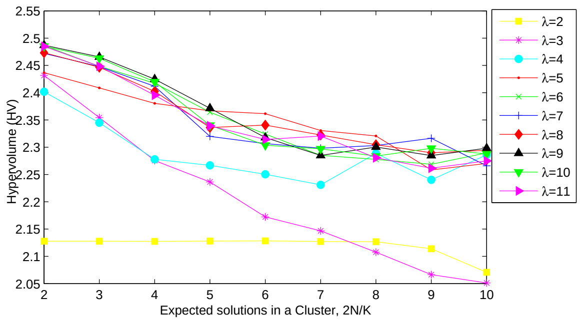
As $p$ decreases, the number of clusters decreases and the expected number of solutions, $\chi$, in a cluster increases for a fixed-size combined population, $|C_t| = 2N$ (e.g. for $p = N$, $\chi = 2N/N = 2$ and for $p = \frac{N}{2}$, $\chi = 2N/\frac{N}{2} = 4$). The fuzzy dominance relation within a cluster promotes faster convergence. By decreasing $p$, the solutions achieve faster convergence while losing some diversity. A closer inspection of Fig. 4.8(a) reveals that as $p$ decreases (i.e., increasing $\chi$), the overall GD value decreases i.e., convergence increases. The overall performance of HV decreases (Fig. 4.8(c)) and IGD value increases (Fig. 4.8(b)), which indicates reduction in diversity. Therefore $p = N$ or $\chi = 2$ is preferable as it works as the best compromise between convergence and diversity. We might require more converged optimal solutions compromising some diversity in some real-world applications. It is possible to achieve this goal by decreasing $p$. An opposite scenario i.e., obtaining more diverse solutions can also be achieved by increasing $p$.




(a) GD values of DTLZ2 with 10-objective




(b) IGD values of DTLZ2 with 10-objective




(c) HV values of DTLZ2 with 10-objective

Figure 4.8: Effect of $\lambda$ and $\chi$ on GD, IGD and HV performance of $F$-DEA on the DTLZ2 problem with 10-objective. The plots show expected number of solutions in a cluster($\chi$) for a fixed population size $N = 250$ vs HV performance for incremental values of $\lambda$ in horizontal line. The stable parameter for $\lambda$ selected as $9,\ 6$ for objective $10,\ 15$ respectively.

# Chapter 5

# Conclusion

## 5.1 Conclusion and Future Work

Evolutionary algorithms can provide several candidate solutions in a single run, which make them popular to solve many practical problems including MaOPs. However, the loss of selection pressure is a challenging issue for such algorithms while solving MaOPs. In this thesis, we have incorporated fuzzy-dominance and reference points in the environmental selection mechanism of the proposed $F$-DEA with an aim of improving selection pressure. The introduction of reference point in conjunction of fuzzy-dominance not only helps in maintaining diversity of the evolved solutions but also convergence.

$F$-DEA has been extensively evaluated and compared using eight WFG and three DTLZ problems having 2- to 25-objectives. The simulation results reveal that $F$-DEA in general performs better than other algorithms on complex problems with an increasing number of objectives. It has also been found that $F$-DEA can balance between the conflicting goal of convergence and diversity well in comparison with other algorithms, especially for complex problems.

In its current implementation, the reference point generation procedure [14] used in $F$-DEA has one user-specified parameter, which was set after some preliminary experiments. One of the future avenues would be to make it adaptive. It would be interesting in the future to analyze $F$-DEA further and identify its strength and weakness. It would also be interesting to apply $F$-DEA to real-world problems.

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