Numerical Analysis of Turbulent Flow around Hydrofoils using the Finite-Volume Method

By

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Submitted to the
Department of Naval Architecture and Marine Engineering
In partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE
in
Naval Architecture and Marine Engineering

Bangladesh University of Engineering and Technology (BUET)
December 2017
The thesis titled “Numerical Analysis of Turbulent Flow around Hydrofoils using the Finite-Volume Method”, submitted by Miad Al Mursaline, Roll No. 1015122018, Session October 2015, to the Department of Naval Architecture and Marine Engineering, Bangladesh University of Engineering and Technology, has been accepted as satisfactory in partial fulfillment of the requirements for the degree of Master of Science in Naval Architecture and Marine Engineering on December 20, 2017.

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To

My Parents
ACKNOWLEDGEMENT

I wish to express deepest gratitude to my supervisor Dr. Md. Shahjada Tarefder, whose continuous interest, encouragement and support during this study I greatly appreciate.

I want to take this opportunity to thank Dr. Milovan Peric for providing permission to use and modify his code Computer Aided Fluid Flow Analysis (CAFFA).

I am thankful to Michail Kiričkov of Kaunas University of Technology for his assistance with Tecplot, and his constructive criticism. Moreover, I am indebted to Edward Lim of Imperial College, London for the computational fluid dynamics resources he kindly provided.

I would also want to thank the members of my thesis committee for their invaluable suggestions. I thank Dr. Goutam Kumar Shaha, Dr. Md. Shahidul Islam, and the external member Dr. Md. Abdus Salam Akanda.
ABSTRACT

The purpose of this research is to carry out numerical simulation of turbulent flow around two-dimensional hydrofoils by finite volume method with non-orthogonal body-fitted grid. The governing equations are expressed in Cartesian velocity components and solution is carried out using SIMPLE algorithm for collocated arrangement of scalar and vector variables. Turbulence is modeled by the k-ε turbulence model and wall functions are used to bridge the solution variables at the near wall cells and the corresponding quantities on the wall. The numerical procedure is employed to compute the pressure distribution on the surface of NACA 0012 and NACA 4412 hydrofoils for different angles of attack and the results are validated by comparing with experimental data. The grid dependency of the solution is studied by varying the number of cells of the C-type structured mesh. The computed lift coefficients of NACA 4412 hydrofoil at different angles of attack are also compared with experimental results to further substantiate the validity of the numerical methodology.
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LIST OF SYMBOLS

Latin letters

\( a_p, a_E, a_M, a_N, a_S \)  Coefficients in the algebraic Eq. 5.2.10 and Eq. 6.2.24

\( B \)  Constant in the law of the wall

\( c \)  Chord length of hydrofoil

\( C_\mu, C_{el}, C_{e2} \)  Constants in the \( k-\epsilon \) turbulence model

\( C_p \)  Pressure coefficient

\( C_L \)  Lift coefficient

\( D \)  Diffusion coefficient in the algebraic Eq. 5.2.10 and Eq. 6.2.24

\( F \)  Mass flux

\( G \)  Production of turbulent kinetic energy

\( J \)  Jacobian of transformation

\( K \)  Total number of computational points

\( k \)  Turbulent kinetic energy

\( N \)  Cross diffusion coefficient

\( n \)  Normal distance from a boundary

\( n^+ \)  Dimensionless normal distance in the law of the wall

\( p \)  Instantaneous pressure

\( P \)  Effective pressure

\( q \)  Velocity scale

\( S \)  Integrated source term in algebraic Eq. 5.2.10 and Eq. 6.2.24

\( S'_{\phi} \)  Source term in Eq. 5.1.1 and Eq. 6.1.1

\( S_{\phi} \)  Source term in Eq. 5.2.1 and Eq. 6.2.1

\( S_{\xi,\eta}^{\phi} \)  Source term in Eq. 6.1.5

\( S_{\text{max}} \)  Convergence criteria for outer iteration

\( t \)  Time

\( \Delta t \)  Time step

\( U \)  Contravariant velocity component

\( u \)  \( x \) component of instantaneous velocity

\( u^+ \)  Dimensionless velocity in the law of the wall

\( u_\tau \)  Shear velocity

\( V \)  Contravariant velocity component

\( \delta V \)  Volume of cell
\( v \) \hspace{1cm} y \text{ component of instantaneous velocity}

\( \mathbf{\bar{v}}_r \) \hspace{1cm} Resultant nodal velocity vector at the near wall cell

\( v_t \) \hspace{1cm} Velocity tangential to the wall

\( v_n \) \hspace{1cm} Velocity normal to the wall

\( x \) \hspace{1cm} Cartesian coordinate direction

\( y \) \hspace{1cm} Cartesian coordinate direction

**Greek letters**

\( \alpha \) \hspace{1cm} Angle of attack

\( \alpha_{\phi} \) \hspace{1cm} Under-relaxation factor

\( \alpha', \beta', \gamma' \) \hspace{1cm} Transformation parameters

\( \Gamma' \) \hspace{1cm} Diffusion coefficient in Eq. 5.1.1

\( \Gamma' \) \hspace{1cm} Diffusion coefficient in Eq. 5.2.1

\( \varepsilon \) \hspace{1cm} Turbulent kinetic energy dissipation rate

\( \eta \) \hspace{1cm} Body-fitted coordinate direction

\( \theta \) \hspace{1cm} Weighting parameter for time

\( \kappa \) \hspace{1cm} Constant in the law of the wall

\( \lambda \) \hspace{1cm} Bulk viscosity coefficient

\( \mu \) \hspace{1cm} Dynamic viscosity

\( \mu_t \) \hspace{1cm} Turbulent viscosity coefficient

\( \nu \) \hspace{1cm} Kinematic viscosity

\( \nu_t \) \hspace{1cm} Kinematic turbulent viscosity

\( \xi \) \hspace{1cm} Body-fitted coordinate direction

\( \rho \) \hspace{1cm} Density

\( \sigma_\varepsilon \) \hspace{1cm} Constant in the k-\( \varepsilon \) turbulence model

\( \tau_w \) \hspace{1cm} Wall shear stress

\( \tau_{xx} \) \hspace{1cm} Components of viscous stress

\( \tau_{xy} \) \hspace{1cm} Components of viscous stress

\( \tau_{yy} \) \hspace{1cm} Components of viscous stress

\( \phi \) \hspace{1cm} Generic transported variable

\( \psi \) \hspace{1cm} Blending factor for convective scheme
Subscripts

- $e$: East boundary of a cell
- $E$: East grid point
- $n$: North boundary of a cell
- $N$: North grid point
- $o$: Quantities at inlet
- $s$: South boundary of a cell
- $S$: South grid point
- $w$: West boundary of a cell
- $W$: West grid point

Superscripts

- $k$: Inner iteration level
- $m$: Outer iteration level
- $o$: Converged variable value at previous time step
- $\bar{}$: Time-averaged quantity
- $\sim{}$: Mass-averaged quantity
- $'$: Fluctuating quantity
- $\ast{}$: Tentative velocity
CHAPTER 1
INTRODUCTION

1.1 Literature Review

The outstanding feature of a turbulent flow, in opposite to a laminar flow, is that the molecules move in a chaotic fashion along complex irregular paths. The strong chaotic motion causes the various layers of the fluid to mix together intensely. Because of the increased momentum and energy exchange between the molecules and solid walls, turbulent flows lead to higher skin friction and heat transfer as compared to laminar flows under the same conditions.

Although the chaotic fluctuations of the flow variables are of deterministic nature, the simulation of turbulent flows still continues to present a significant problem. Despite the performance of modern supercomputers, a direct simulation of turbulence by the time-dependent Navier-Stokes equations known as the Direct Numerical Simulation (DNS) is applicable only to relatively simple flow problems at low Reynolds numbers (Re) of about 20000. A more widespread utilization of the DNS is prevented by the fact that the number of grid points needed for sufficient spatial resolution scales as $Re^{9/4}$ and the CPU-time as $Re^{3}$. Therefore, scientists are forced to account for the effects of turbulence in an approximate manner. For this purpose, a large variety of turbulence models was developed and the research still goes on.

One should be aware of the fact that there is no single turbulence model, which can predict reliably all kinds of turbulent flows. Each of the models has its strengths and weaknesses. For example, if a particular model works perfectly in the case of attached boundary layers, it may fail completely for separated flows. Thus, it is important always to ask whether the model includes all the significant features of the flow being investigated. Another point which should be taken into consideration is the computational effort versus the accuracy required by the particular application. This means that in many cases a numerically inexpensive turbulence model can predict some global measures with the same accuracy as a more complex model.

Pope (1978) developed a finite-difference procedure using k-epsilon turbulence model to calculate the mean properties of turbulent recirculating flows in general orthogonal coordinates. A novel method of transforming quantities into general orthogonal coordinates facilitated the incorporation of a variety of transport equations into the scheme. The grid-
generation procedure was based on the solution of Laplace’s equation by finite-difference methodology.

Shamroth and Gibeling (1979) used a transition-turbulence model and obtained a compressible time-dependent solution of the Navier-Stokes equations for the isolated airfoil flow field problem. The equations were solved by a consistently split linearized block implicit scheme. A non-orthogonal body fitted coordinate system was used which had maximum resolution near the airfoil surface and in the region of the airfoil leading edge.

Rhie (1981) used finite volume method for the solution of two-dimensional incompressible, steady turbulent flows over airfoils with and without trailing edge separation. To describe the turbulent flow process \( k-\varepsilon \) model was utilized and semi-empirical formulas called "wall functions" were used to bridge the viscosity-affected region between the wall and the fully-turbulent region. Instead of staggered grids, the body fitted grid utilized a collocated arrangement of vector and scalar variables. To circumvent the false pressure field special momentum interpolation was used.

Demirdzic et al. (1987) provided a complete exposition of a finite volume approach to the calculation of turbulent flows in geometrically complex domains based on a non-orthogonal coordinate formulation of the governing equations with contravariant physical velocity components. An implicit finite volume method of solution was used together with a simple transfinite mapping procedure for mesh generation. Finally the overall method was applied, by way of an example, to the prediction of flow and heat transfer in a staggered tube bank.

Karki and Patankar (1988) presented a general calculation procedure for computing fluid flow and related phenomenon in arbitrary-shaped domains. The scheme was developed for a generalized non-orthogonal coordinate system and was based on control volume approach with a staggered arrangement. The physical covariant velocity components were selected as the dependent variables in momentum equations. The coupling between the continuity and momentum equations was affected using the SIMPLER algorithm.

Masuko and Ogiwara (1990) carried out numerical simulation of viscous flow around ships having practical hull forms under propeller operating conditions. The governing equations were discretized by finite difference approximation and solved with SIMPLE algorithm adopting \( k - \varepsilon \) turbulence model and standard wall functions. The propeller was simulated by giving a pressure jump at its position. In order to avoid the skew grid around the practical hull
form, adjustment of grid angle was made to the grid generated by solving the elliptic differential equations.

Majumdar (1988) reported that solutions of steady-state problems from Rhie and Chow momentum interpolation were dependent on the underrelaxation factor. To eliminate this underrelaxation factor dependency, an iteration algorithm was proposed by him to calculate the cell-face velocity for steady-state problems.

Peric (1990) analyzed the full and simplified pressure correction equations when the grid nonorthogonality becomes appreciable. Peric (1990) demonstrated that the efficiency of the SIMPLE coupling algorithm is not affected by the grid nonorthogonality, provided that no additional simplifications are introduced in the pressure correction equation. However, the algorithm with the simplified equation was inefficient when the angle between grid lines approach 45 degrees and fails to converge for angles below 30 degrees.

Choi (1999) reported that, the solution using the original Rhie and Chow scheme was time step size dependent. He proposed a modified Rhie and Chow scheme for an unsteady problem which is quite similar to scheme for a steady problem used by Majumdar (1988).

Wang and Komori (1999) compared the use of Cartesian and Covariant velocity components on non-orthogonal collocated grids. The momentum equations were solved using SIMPLE like technique for lid driven cavity problem. The accuracy and convergence performance for the Cartesian and Covariant velocity case was compared which showed that both had the same numerical accuracy. The convergence rate of the covariant velocity method was found to be faster than that of the Cartesian velocity method if the relaxation factor for pressure was small enough.

Yu et al. (2002) discussed different momentum interpolation practices for collocated grid systems. Two new momentum interpolation methods, called MMIM1 and MMIM2, were proposed. Analysis showed that the two new methods achieved numerical solutions that were independent of both the under-relaxation factor and the time step size. Taking lid-driven cavity flow as an example, numerical computations were conducted for several Reynolds numbers and different mesh sizes using the SIMPLE algorithm, and the results were compared with benchmark solutions.

Li (2003) performed simulations of turbulent free-surface flows around ships in a numerical water tank, based on the FINFLO-RANS SHIP solver developed at Helsinki University of
Technology. The Reynolds-averaged Navier–Stokes (RANS) equations with the artificial
compressibility and the nonlinear free-surface boundary conditions were discretized by
means of a cell-centered finite-volume scheme utilizing two turbulence models, namely,
Chien’s low Reynolds number k–epsilon model and Baldwin–Lomax’s model. The
convergence performance was improved with the multigrid method.

Mulvany et al. (2004) carried out assessment of two-equation turbulence models for high
Reynolds number hydrofoil flows using finite volume method and SIMPLE solution
technique. Four widely applied two-equation RANS turbulence models were assessed
through comparison with experimental data. They were the standard $k - \varepsilon$ turbulence model,
the realizible $k - \varepsilon$ turbulence model, the standard $k - \omega$ turbulence model and the shear-
stress-transport (SST) $k - \omega$ model. It was found that the realizible $k - \varepsilon$ turbulence model
used with enhanced wall functions and near-wall modelling techniques consistently provided
superior performance in predicting the flow characteristics around the hydrofoil.

Chao-bang and Wen-cai (2012) presented a method to calculate the resistance of a high-speed
displacement ship taking the effect of sinkage and trim and viscosity of fluid into account. A
free surface flow field was evaluated by solving RANS equations with volume of fluid (VoF)
method. The sinkage and trim were computed by equating the vertical force and pitching
moment to the hydrostatic restoring force and moment. The software Fluent, Maxsurf and
MATLAB were used to implement the method.

Demirdzic (2015) discussed the discretization of diffusion term in finite volume continuum
Mechanics. A Review of the cell-centered finite-volume approximations of diffusive flux was
presented, starting with early methods that used Cartesian meshes and finishing with
contemporary methods that employ arbitrary polyhedral control volumes.

1.2 Objective of the Research
The aim of this research is to analyze turbulent flow past hydrofoils located far away from the
free surface using finite volume based FORTRAN source codes on collocated body-fitted
grid. The coupling between continuity and momentum equations is accomplished using
SIMPLE algorithm, and turbulence is modeled by k-epsilon model with wall functions. The
numerical methodology is employed to obtain the flow field around NACA 0012 and NACA
4412 hydrofoils. Pressure and lift coefficients are computed and validated against results
from literature. Moreover, to explain the flow physics TECPLOT is used to plot contour
diagrams, streamlines and velocity vectors.
CHAPTER 2
CONSERVATION OF MASS AND MOMENTUM

2.1 Instantaneous Navier-Stokes Equations

The well-known Navier–Stokes equations of motion for a two-dimensional, compressible and
viscous fluid may be written in the following form:

**Continuity equation**

\[
\frac{\partial p}{\partial t} + \frac{\partial (p u)}{\partial x} + \frac{\partial (p v)}{\partial y} = 0 \tag{2.1.1}
\]

**Momentum equations**

\[
\begin{align*}
\frac{\partial (p u)}{\partial t} + \frac{\partial (p u u)}{\partial x} + \frac{\partial (p u v)}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{\partial (\tau_{xx})}{\partial x} + \frac{\partial (\tau_{yx})}{\partial y} \tag{2.1.2} \\
\frac{\partial (p v)}{\partial t} + \frac{\partial (p v u)}{\partial x} + \frac{\partial (p v v)}{\partial y} &= -\frac{\partial p}{\partial y} + \frac{\partial (\tau_{xy})}{\partial x} + \frac{\partial (\tau_{yy})}{\partial y} \tag{2.1.3}
\end{align*}
\]

where the body force terms have been neglected.

In the late seventeenth century Isaac Newton stated that shear stress in a fluid is proportional
to the time-rate-of-strain, i.e. velocity gradients. Such fluids are called Newtonian fluids. In
virtually all practical problems, the fluid can be assumed to be Newtonian. For such fluids,
Stokes, in 1845, obtained:

\[
\begin{align*}
\tau_{xx} &= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu \frac{\partial u}{\partial x} \tag{2.1.4} \\
\tau_{yy} &= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu \frac{\partial v}{\partial y} \tag{2.1.5} \\
\tau_{xy} &= \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{2.1.6}
\end{align*}
\]

where \(\mu\) is the molecular viscosity coefficient and \(\lambda\) is the bulk viscosity coefficient. Stokes
made the hypothesis that

\[
\lambda = -\frac{2}{3}\mu \tag{2.1.7}
\]

Putting Eq. 2.1.4-2.1.6 in right side of Eq. 2.1.2-2.1.3 the momentum equations may be
written in the following form:

\[
\begin{align*}
\frac{\partial (p u)}{\partial t} + \frac{\partial (p u u)}{\partial x} + \frac{\partial (p u v)}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{2}{3}\mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \tag{2.1.8}
\end{align*}
\]
\[
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho vu)}{\partial x} + \frac{\partial (\rho vv)}{\partial y} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[ \frac{2}{3} \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial v}{\partial y} \right]
\] (2.1.9)

Eq. 2.1.1-2.1.9 are applicable to laminar as well as to turbulent flows. For the latter, however, the values of the dependent variables are to be replaced by their instantaneous values. A direct approach to the turbulence problem, namely the solution of the full time-dependent Navier–Stokes equations, then consists in solving the equations for a given set of boundary and initial values and computing mean values over the ensemble for solutions. Even for the most restricted problem, turbulence of an incompressible fluid appears to be a hopeless undertaking, because of the nonlinear terms in the equations. Thus, the standard procedure is to average over the equations rather than over the solutions. The averaging can be done either by the conventional time-averaging procedure or by the mass-weighted averaging procedure.

For incompressible flows Eq. 2.1.1 and 2.1.8-2.1.9 may be written in an alternative form as follows:

For incompressible flows, the divergence of velocity vector is zero, Hence the continuity Eq. 2.1.1 becomes:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\] (2.1.10)

The momentum equation in the \(x\) direction for incompressible flows may be written as:

\[
\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[ \frac{2}{3} \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial v}{\partial y} \right]
\]

\[
\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \mu \left( \frac{\partial u}{\partial y} \frac{\partial}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial}{\partial y} \right) + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

Using the rules of partial differentiation

\[
\frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial y} \right)
\]

We get

\[
\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \mu \left( \frac{\partial u}{\partial y} \frac{\partial}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial}{\partial y} \right) + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

By invoking continuity equation for incompressible flow the second term becomes zero and using the definition of kinematic viscosity \(\frac{\mu}{\rho} = \nu\) we finally get

\[
\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\] (2.1.11)
Similarly this form of momentum equation in the y direction is given as:

$$\frac{\partial v}{\partial t} + \frac{\partial (vu)}{\partial x} + \frac{\partial (vv)}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

(2.1.12)

### 2.2 Conventional Time-Averaging

In order to obtain the governing conservation equations for turbulent flows, it is convenient to replace the instantaneous quantities by their mean and their fluctuating quantities. In the conventional time-averaging procedure, the velocity, pressure, and density are usually written in the following forms:

$$u(x,y,t) = \overline{u(x,y)} + u'(x,y,t) \quad (2.2.1a)$$

$$v(x,y,t) = \overline{v(x,y)} + v'(x,y,t) \quad (2.2.1b)$$

$$p(x,y,t) = \overline{p(x,y)} + p'(x,y,t) \quad (2.2.2)$$

$$\rho(x,y,t) = \overline{\rho(x,y)} + \rho'(x,y,t) \quad (2.2.3)$$

where \( \overline{u(x,y)} \), \( \overline{p(x,y)} \) and \( \overline{\rho(x,y)} \) are the time averages of the bulk velocity, pressure and density respectively, and \( u'(x,y,t) \), \( v'(x,y,t) \), \( p'(x,y,t) \) and \( \rho'(x,y,t) \) are the superimposed velocity, pressure, and density fluctuations respectively. The time average or "mean" of any quantity \( f(t) \) is defined by:

$$\bar{f} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{t_0}^{t_0 + \Delta t} f(t) \, dt \quad (2.2.4)$$

In practice, \( \Delta t \) is taken to mean a time that is long compared to the reciprocal of the predominant frequencies in the spectrum of \( f \). Having understood this definition the limits are now changed with 0 replacing \( t_0 \) and \( T \) replacing \( t + \Delta t \). Eq. 2.2.4 now becomes:

$$\bar{f} = \frac{1}{T} \int_0^T f(t) \, dt \quad (2.2.5)$$

The rules of conventional time averaging may be summarized as follows:

$$\bar{f} = \frac{1}{T} \int_0^T \bar{f} \, dt = \bar{f} \quad (2.2.6)$$

$$\bar{f} = \frac{1}{T} \int_0^T f' \, dt = 0 \quad (2.2.7)$$

$$\bar{f} g' = \frac{1}{T} \int_0^T \bar{f} g' \, dt = \frac{1}{T} \bar{f} \int_0^T g' \, dt = \bar{f} \bar{g'} = \bar{f} \times 0 = 0 \quad (2.2.8)$$

$$\bar{f} + g = \frac{1}{T} \int_0^T (f + g) \, dt = \frac{1}{T} \int_0^T f \, dt + \frac{1}{T} \int_0^T g \, dt = \bar{f} + \bar{g} \quad (2.2.9)$$

$$\bar{f}' f' = \frac{1}{T} \int_0^T f' \, dt = \frac{1}{T} \int_0^T (f')^2 \, dt = (\bar{f}')^2 \neq 0 \quad (2.2.10)$$
\[ \overline{a'b'} = \frac{1}{T} \int_{0}^{T} a'b' \, dt \quad (2.2.11) \]

The average of \( a'b' \) depends on whether \( a' \) and \( b' \) are correlated. As illustrated in Fig. 2.1 the fluctuating variable \( a' \) has the same sign as the variable \( b' \) for most of the time; this makes \( a'b' > 0 \) and thus they are correlated. So for two correlated fluctuating components we have:

\[ \frac{1}{T} \int_{0}^{T} a'b' \, dt = \overline{a'b'} \neq 0 \quad (2.2.12) \]

The variable \( c' \), on the other hand, is uncorrelated with \( a' \) and \( b' \). For two uncorrelated variables we have:

\[ \frac{1}{T} \int_{0}^{T} b'c' \, dt = \overline{b'c'} = 0 \quad (2.2.13) \]

\[ \frac{1}{T} \int_{0}^{T} a'c' \, dt = \overline{a'c'} = 0 \quad (2.2.14) \]

(note that \( \overline{a'b'} \neq 0 \), \( \overline{a'c'} \neq 0 \) does not imply \( \overline{b'c'} \neq 0 \)).

\[ \overline{(f + f')}^2 = \frac{1}{T} \int_{0}^{T} \left( \overline{f^2} + 2\overline{ff'} + \overline{f'^2} \right) \, dt = \overline{f^2} + 2\overline{ff'} + \overline{f'^2} = \overline{f^2} + \overline{f'^2} \quad (2.2.15) \]

\[ \overline{\frac{\partial (u'^2)}{\partial x}} = \frac{\partial}{\partial x} \left( \frac{1}{T} \int_{0}^{T} (u'^2) \, dt \right) = \frac{\partial u'^2}{\partial x} = 2u' \frac{\partial u'}{\partial x} \neq 0 \quad (2.2.16) \]

**Mean momentum equations in conventional time averaged variables**

Since the real value of the inertia forces is always equal to the sum of the real values of the applied forces in any kind of motion (laminar or turbulent), the mean value of the inertia forces with respect to time is equal to the mean value of the applied forces with respect to time.

![Fig. 2.1: Fluctuating component of variable a, b, c](image-url)
Using this idea the time average of each term in Eq. 2.1.8 is taken to obtain the mean momentum equation in the $x$ direction as follows:

$$
\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[ \frac{2}{3} \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right]
$$

Using the formula given by Eq. 2.2.5 in the above equation we get

$$
\frac{1}{T} \int_0^T \left\{ \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} \right\} dt
$$

$$
= \frac{1}{T} \int_0^T \left\{ -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[ \frac{2}{3} \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \right\} dt
$$

Using the formula given by Eq. 2.2.1-2.2.3 in the above equation

$$
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial t} (\tilde{\rho} + \rho')(\bar{u} + u') + \frac{\partial}{\partial x} (\tilde{\rho} + \rho')(\bar{u} + u')(\bar{v} + v') \right\} dt
$$

$$
= \frac{1}{T} \int_0^T \left\{ -\frac{\partial}{\partial x} (\tilde{\rho} + \rho') \right\} + \frac{\partial}{\partial x} \left[ \frac{2}{3} \mu \left( \frac{\partial}{\partial x}(\bar{u} + u') + \frac{\partial}{\partial y}(\bar{v} + v') \right) + 2\mu \frac{\partial}{\partial x}(\bar{u} + u') \right]
$$

$$
+ \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial}{\partial y} (\bar{u} + u') + \frac{\partial}{\partial x}(\bar{v} + v') \right) \right] \right\} dt
$$

Linear Forces

Local inertia force term:

$$
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial t} (\tilde{\rho} + \rho')(\bar{u} + u') \right\} dt
$$

$$
= \frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial t} (\tilde{\rho} \bar{u} + \rho' u' + \rho' \bar{u} + \rho' u') \right\} dt
$$

$$
= \frac{\partial}{\partial t} \left( \frac{1}{T} \int_0^T (\tilde{\rho} \bar{u} + \rho' u' + \rho' \bar{u} + \rho' u') dt \right)
$$

But from the rules of averaging given by Eq. 2.2.6-2.2.16

$$
\frac{1}{T} \int_0^T \tilde{\rho} \bar{u} dt = \tilde{\rho} \bar{u}
$$

$$
\frac{1}{T} \int_0^T \tilde{\rho} u' dt = \tilde{\rho} \times 0 = 0
$$
\[
\frac{1}{T} \int_0^T \rho' \ddot{u} \, dt = 0
\]
\[
\frac{1}{T} \int_0^T \rho' u' \, dt = \overline{\rho' u'}
\]
we get
\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial t} (\bar{\rho} + \rho')(\bar{u} + u') \right\} \, dt
\]
\[
= \frac{\partial}{\partial t} (\bar{\rho} \bar{u} + \rho' u')
\]  \hspace{1cm} (2.2.19)

**Pressure force term:**
\[
\frac{1}{T} \int_0^T \frac{\partial \bar{p}}{\partial x} \, dt
\]
\[
= \frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T (\bar{p} + p') \right\} \, dt
\]
But from the rules of averaging given by Eq. 2.2.6-2.2.16
\[
\frac{1}{T} \int_0^T \bar{p} \, dt = \bar{p}
\]
\[
\frac{1}{T} \int_0^T p' \, dt = 0
\]
We get
\[
\frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T (\bar{p} + p') \, dt \right\} = \frac{\partial \bar{p}}{\partial x}
\]  \hspace{1cm} (2.2.20)

**Viscous force terms:**
\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial x} \left[ \frac{2}{3} \mu \left( \frac{\partial}{\partial x} (\bar{u} + u') + \frac{\partial}{\partial y} (\bar{v} + v') \right) + 2\mu \frac{\partial}{\partial x} (\bar{u} + u') \right] \right. \\
\left. + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial}{\partial y} (\bar{u} + u') + \frac{\partial}{\partial x} (\bar{v} + v') \right) \right] \right\} \, dt
\]
\[
= \frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T \left[ \frac{2}{3} \mu \left( \frac{\partial}{\partial x} (\bar{u} + u') + \frac{\partial}{\partial y} (\bar{v} + v') \right) + 2\mu \frac{\partial}{\partial x} (\bar{u} + u') \right] \right. \\
\left. + \frac{\partial}{\partial y} \left[ \frac{1}{T} \int_0^T \left[ \mu \left( \frac{\partial}{\partial y} (\bar{u} + u') + \frac{\partial}{\partial x} (\bar{v} + v') \right) \right] \right\} \, dt
\]
But from the rules of averaging given by Eq. 2.2.6-2.2.16
Similarly,
\[
\frac{1}{T} \int_0^T \frac{\partial}{\partial y} (\bar{u} + u') \, dt = \frac{\partial \bar{v}}{\partial y}
\]
\[
\frac{1}{T} \int_0^T \frac{\partial}{\partial x} (\bar{v} + v') \, dt = \frac{\partial \bar{v}}{\partial x}
\]
\[
\frac{1}{T} \int_0^T \frac{\partial}{\partial y} (\bar{v} + v') \, dt = \frac{\partial \bar{v}}{\partial y}
\]

Using the above results we get
\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial x} \left[ \frac{2}{3} \mu \left( \frac{\partial}{\partial x} (\bar{u} + u') + \frac{\partial}{\partial y} (\bar{v} + v') \right) + 2\mu \frac{\partial}{\partial x} (\bar{u} + u') \right]
\]
\[
+ \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial}{\partial y} (\bar{u} + u') + \frac{\partial}{\partial x} (\bar{v} + v') \right) \right] \right\} \, dt
\]
\[
= \frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial (\bar{\tau}_{yx})}{\partial y}
\]
\[
= \frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial (\bar{\tau}_{yx})}{\partial y}
\]

where,
\[
\bar{\tau}_{xx} = \frac{2}{3} \mu \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) + 2\mu \frac{\partial \bar{u}}{\partial x}
\]
\[
\bar{\tau}_{yy} = \frac{2}{3} \mu \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) + 2\mu \frac{\partial \bar{v}}{\partial y}
\]
\[
\bar{\tau}_{xy} = \tau_{yx} = \mu \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right)
\]

Convective inertia force terms (nonlinear terms)
\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial x} \left( \bar{\rho}'(\bar{u} + u')(\bar{u} + u') + \frac{\partial}{\partial y} \left( \bar{\rho}'(\bar{u} + u')(\bar{v} + v') \right) \right) \right\} \, dt
\]
\[
= \frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T \left[ (\bar{\rho}'(\bar{u} + u')(\bar{u} + u')) \right] \, dt \right\} + \frac{\partial}{\partial y} \left\{ \frac{1}{T} \int_0^T \left[ (\bar{\rho}'(\bar{u} + u')(\bar{v} + v')) \right] \, dt \right\}
\]
\[
\frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T (\bar{\rho} \bar{u} \bar{u} + 2 \bar{\rho} \bar{u} \bar{u}' + \rho' \bar{u} \bar{u} + 2 \rho' \bar{u}' \bar{u} + \bar{\rho} \bar{u}' \bar{u}' + \rho' \bar{u}' \bar{u}') dt \right\} + \frac{\partial}{\partial y} \left\{ \frac{1}{T} \int_0^T (\bar{\rho} \bar{u} \bar{v} + \bar{\rho} \bar{u}' \bar{v}' + \rho' \bar{u} \bar{v}' + \rho' \bar{u}' \bar{v}' + \rho' \bar{u} \bar{v} + \rho' \bar{u}' \bar{v}') dt \right\} 
\]

But from the rules of averaging given by Eq. 2.2.6-2.2.16

\[
\frac{1}{T} \int_0^T (\bar{\rho} \bar{u} \bar{u}) dt = \bar{\rho} \bar{u} \bar{u}, \quad \frac{1}{T} \int_0^T (2\bar{\rho} \bar{u} \bar{u}') dt = 0
\]

\[
\frac{1}{T} \int_0^T (\rho' \bar{u} \bar{u}) dt = 0, \quad \frac{1}{T} \int_0^T (2\rho' \bar{u}' \bar{u}) dt = 2\bar{u} \bar{u}'
\]

\[
\frac{1}{T} \int_0^T (\bar{\rho} \bar{u}' \bar{u}') dt = \bar{\rho} \bar{u}' \bar{u}', \quad \frac{1}{T} \int_0^T (\rho' \bar{u}' \bar{u}') dt = \rho' \bar{u}' \bar{u}'
\]

\[
\frac{1}{T} \int_0^T (\bar{u} \bar{u}') dt = \bar{u} \bar{u}', \quad \frac{1}{T} \int_0^T (\rho' \bar{u}' \bar{u}') dt = \rho' \bar{u}' \bar{u}'
\]

Using the above results we get

\[
\frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T (\bar{\rho} \bar{u} \bar{u} + 2 \bar{\rho} \bar{u} \bar{u}' + \rho' \bar{u} \bar{u} + 2 \rho' \bar{u}' \bar{u} + \bar{\rho} \bar{u}' \bar{u}' + \rho' \bar{u}' \bar{u}') dt \right\} + \frac{\partial}{\partial y} \left\{ \frac{1}{T} \int_0^T (\bar{\rho} \bar{u} \bar{v} + \bar{\rho} \bar{u}' \bar{v}' + \rho' \bar{u} \bar{v}' + \rho' \bar{u}' \bar{v}' + \rho' \bar{u} \bar{v} + \rho' \bar{u}' \bar{v} ) dt \right\} 
= \frac{\partial}{\partial x} (\bar{\rho} \bar{u} \bar{u} + 2\bar{u} \bar{u}' + \rho' \bar{u}' \bar{u}') + \frac{\partial}{\partial y} (\bar{\rho} \bar{u} \bar{v} + \bar{u} \bar{v}')
\]

Now putting results Eq. 2.2.19, 2.2.20, 2.2.22 and 2.2.26 in Eq. 2.2.18 we get the following form of momentum equation in the x direction
On rearranging we get
\[
\frac{\partial}{\partial t} (\bar{\rho} \bar{u} + \bar{\rho} \breve{u}') + \frac{\partial}{\partial x} (\bar{\rho} \bar{u} \bar{u} + \bar{\rho} \breve{u} \breve{u}') + \frac{\partial}{\partial y} (\bar{\rho} \bar{v} \breve{v} + \bar{\rho} \breve{v} \breve{v}') = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y}
\]

This is the Reynolds averaged momentum equation for unsteady, compressible flow in the \(x\) - direction in conventional time averaged variables. The time averaged momentum equations in the \(y\) direction may be obtained using a similar methodology. So finally the conventional time averaged momentum equations in \(x\) and \(y\) directions for unsteady, compressible flow can be written as follows:

\(x - \text{momentum equation:}\)
\[
\frac{\partial}{\partial t} (\bar{\rho} \bar{u} + \bar{\rho} \breve{u}') + \frac{\partial}{\partial x} (\bar{\rho} \bar{u} \bar{u} + \bar{\rho} \breve{u} \breve{u}') + \frac{\partial}{\partial y} (\bar{\rho} \bar{v} \breve{v} + \bar{\rho} \breve{v} \breve{v}')
- \frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial x} (\bar{\tau}_{xx} - \bar{u} \breve{u}' \breve{u}' - \bar{\rho} \breve{u}' \breve{u}' - \bar{\rho} \breve{u}' \breve{u}') + \frac{\partial}{\partial y} (\bar{\tau}_{yx} - \bar{v} \breve{v}' \breve{v}' - \bar{\rho} \breve{v}' \breve{v}' - \bar{\rho} \breve{v}' \breve{v}')
\]  \hspace{1cm} (2.2.27)

\(y - \text{momentum equation:}\)
\[
\frac{\partial}{\partial t} (\bar{\rho} \bar{v} + \bar{\rho} \breve{v}') + \frac{\partial}{\partial x} (\bar{\rho} \bar{u} \breve{v}' + \bar{\rho} \breve{u}' \breve{v}') + \frac{\partial}{\partial y} (\bar{\rho} \bar{v} \breve{v}' + \bar{\rho} \breve{v} \breve{v}')
= -\frac{\partial \bar{p}}{\partial y} + \frac{\partial}{\partial x} (\bar{\tau}_{yx} - \bar{u} \breve{v}' \breve{v}' - \bar{\rho} \breve{v}' \breve{v}' - \bar{\rho} \breve{v}' \breve{v}')
+ \frac{\partial}{\partial y} (\bar{\tau}_{yy} - \bar{v} \breve{v}' \breve{v}' - \bar{\rho} \breve{v}' \breve{v}' - \bar{\rho} \breve{v}' \breve{v}')
\]  \hspace{1cm} (2.2.28)

The averaging procedure for three dimensional case is given in appendix A.

**Mean continuity equation in conventional time averaged variables**

The time average of each term in Eq. 2.1.1 is taken to obtain the mean continuity equation as follows:
\[
\frac{\partial \bar{p}}{\partial t} + \frac{\partial (\bar{\rho} \bar{u})}{\partial x} + \frac{\partial (\bar{\rho} \bar{v})}{\partial y} = 0
\]

Using the formula given by Eq. 2.2.5 in the above equation we get
Using the formula given by Eq. 2.2.1-2.2.3 in the above equation we get
\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial t} (\bar{\rho} + \rho') + \frac{\partial}{\partial x} (\bar{\rho} + \rho') (\bar{u} + u') + \frac{\partial}{\partial y} (\bar{\rho} + \rho') (\bar{v} + v') \right\} dt = 0
\]

or,
\[
\frac{\partial}{\partial t} \left\{ \frac{1}{T} \int_0^T (\bar{\rho} + \rho') \right\} + \frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T (\bar{\rho} \bar{u} + \bar{\rho} u' + \rho' \bar{u} + \rho' u') dt \right\}
\]
\[
+ \frac{\partial}{\partial y} \left\{ \frac{1}{T} \int_0^T (\bar{\rho} \bar{v} + \bar{\rho} v' + \rho' \bar{v} + \rho' v') dt \right\} = 0
\text{(2.2.29)}
\]

But from the rules of averaging given by the Eq. 2.2.6-2.2.16
\[
\frac{1}{T} \int_0^T \bar{\rho} \bar{u} dt = \bar{\rho} \bar{u}, \quad \frac{1}{T} \int_0^T \bar{\rho} u' dt = 0, \quad \frac{1}{T} \int_0^T \rho' \bar{u} dt = 0
\]
\[
\frac{1}{T} \int_0^T \bar{\rho} \bar{v} dt = \bar{\rho} \bar{v}, \quad \frac{1}{T} \int_0^T \bar{\rho} v' dt = 0, \quad \frac{1}{T} \int_0^T \rho' \bar{v} dt = 0
\]
\[
\frac{1}{T} \int_0^T \bar{\rho} dt = \bar{\rho}, \quad \frac{1}{T} \int_0^T \rho' dt = 0, \quad \frac{1}{T} \int_0^T \rho' u' dt = \bar{\rho}' \bar{u}'
\]
\[
\frac{1}{T} \int_0^T \rho' v' dt = \bar{\rho}' \bar{v}'
\]

Using the above results we obtain mean continuity equation for unsteady compressible flow as follows
\[
\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x} (\bar{\rho} \bar{u} + \rho' \bar{u}') + \frac{\partial}{\partial y} (\bar{\rho} \bar{v} + \rho' \bar{v}') = 0
\text{(2.2.30)}
\]

So the (mean) continuity and momentum equations for unsteady, compressible flow in time averaged variables are:

**Continuity**
\[
\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x} (\bar{\rho} \bar{u} + \rho' \bar{u}') + \frac{\partial}{\partial y} (\bar{\rho} \bar{v} + \rho' \bar{v}') = 0
\text{(2.2.31)}
\]

**Momentum**
\[
\frac{\partial}{\partial t} (\bar{\rho} \bar{u} + \rho' \bar{u}') + \frac{\partial}{\partial x} (\bar{\rho} \bar{u} \bar{u} + \bar{u} \rho' \bar{u}') + \frac{\partial}{\partial y} (\bar{\rho} \bar{u} \bar{v} + \bar{u} \rho' \bar{v}')
\]
\[
= -\frac{\partial \bar{\rho}}{\partial x} + \frac{\partial}{\partial x} (\bar{r}_{xx} - \bar{u} \rho' \bar{u}' - \bar{u} \rho' \bar{u}')
\]
\[
+ \frac{\partial}{\partial y} (\bar{r}_{xy} - \bar{u} \rho' \bar{v}' - \bar{u} \rho' \bar{v}')
\text{(2.2.32)}
\]

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For incompressible flows, \( \rho = 0 \). As a result, Eq. 2.2.31–2.2.33 can be simplified considerably. For unsteady, incompressible flow the time averaged continuity and momentum equations take the following form:

**Continuity**

\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{2.2.34}
\]

**Momentum**

\[
\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u} \bar{u}}{\partial x} + \frac{\partial \bar{v} \bar{v}}{\partial y} = \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + \frac{1}{\rho} \frac{\partial (\bar{\tau}_{xx} - \rho \bar{u}' \bar{u}')}{\partial x} + \frac{1}{\rho} \frac{\partial (\bar{\tau}_{xy} - \rho \bar{u}' \bar{v}')}{\partial y} \tag{2.2.35}
\]

\[
\frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{u} \bar{u}}{\partial x} + \frac{\partial \bar{v} \bar{v}}{\partial y} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial y} + \frac{\partial (\bar{\tau}_{xx} - \rho \bar{v}' \bar{v}')}{\partial x} + \frac{\partial (\bar{\tau}_{xy} - \rho \bar{v}' \bar{v}')}{\partial y} \tag{2.2.36}
\]

where the mean viscous stresses are obtained from Eq. 2.2.23-2.2.25 and modified using continuity Eq. 2.2.34 to give

\[
\bar{\tau}_{xx} = 2\mu \frac{\partial \bar{u}}{\partial x} \tag{2.2.37a}
\]

\[
\bar{\tau}_{yy} = 2\mu \frac{\partial \bar{v}}{\partial y} \tag{2.2.37b}
\]

\[
\bar{\tau}_{xy} = \tau_{yx} = \mu \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \tag{2.2.37c}
\]

We see from Eq. 2.2.34–2.2.36 that the continuity and the momentum equations contain mean terms that have the same form as the corresponding terms in the instantaneous equations. However, they also have terms representing the mean effects of turbulence, which are additional unknown quantities. For that reason, the resulting conservation equations are undetermined. Consequently, the governing equations in this case, continuity and momentum, do not form a closed set. They require additional relations, which have to come from statistical or similarity considerations. The additional terms enter the governing equations of the fluid as turbulent transport terms such as \( \bar{\rho} \bar{u}' \bar{u}' \), \( \bar{\rho} \bar{v}' \bar{v}' \), \( \bar{\rho} \bar{u}' \bar{v}' \) and as density generated terms such as \( \bar{\rho}' \bar{u}' \), \( \bar{\rho}' \bar{v}' \), \( \bar{\rho}' \bar{u}' \bar{u}' \), \( \bar{\rho}' \bar{u}' \bar{v}' \), \( \bar{\rho}' \bar{v}' \bar{v}' \) for compressible flows. In incompressible
flows the density generated terms disappear and the additional unknowns are $\rho u'u', \rho v'v', \rho u'v'$.

The use of conventional time averaging for a compressible flow is cumbersome. Alternatively, more convenient mass-weighted-averaging procedure is employed. Mass-weighted averaging eliminates the mean-mass term as $\bar{\rho}u'\bar{u}'$, $\bar{\rho}v'\bar{v}'$ and some of the momentum transport terms such as $\bar{u}\bar{p}'\bar{u}'$ and $\bar{p}'\bar{u}'\bar{u}'$ across mean streamlines. This technique is discussed in the next section.

### 2.3 Mass-Weighted Averaging

For treatment of compressible flows and mixtures of gases in particular, mass-weighted averaging is convenient.

A mass-averaged quantity denoted by a tilde over an arbitrary variable $f$ is defined as

$$
\bar{f} = \frac{1}{\bar{\rho}} \frac{1}{T} \int_0^T \rho f \, dt = \frac{\rho f}{\bar{\rho}}
$$

(2.3.1)

Where a bar over a variable indicates a Reynolds’s averaged quantity. In this approach only the velocity components are mass averaged and the other fluid properties such as density and pressure are treated as before.

$$
\bar{u} = \frac{\bar{\rho}u}{\bar{\rho}}, \quad \bar{v} = \frac{\bar{\rho}v}{\bar{\rho}}
$$

(2.3.2)

$$
\rho = \bar{\rho} + \rho', \quad p = \bar{p} + p'
$$

(2.3.3)

To substitute into the conservations equations, we define new fluctuating quantities by

$$
u = \bar{u} + u'' \quad v = \bar{v} + v''
$$

(2.3.4)

Now the time average of mass-averaged variable $f$ is

$$
\bar{\bar{f}} = \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} \frac{\rho f}{\bar{\rho}} \, dt = \bar{\bar{f}}
$$

(2.3.5)

Any quantity $f$ may be decomposed as

$$
f = \bar{f} + f''
$$

Multiplying both sides by $\rho$ and then taking time average over the equation we obtain,

$$
\bar{\rho f} = \bar{\rho f} + \rho f''
$$

$$
\bar{\rho f} = \bar{\bar{f}} + \bar{f} f''
$$

From the definition of mass-weighted variables, $\bar{\rho} \bar{f} = \bar{f}$

$$
\bar{\rho f''} = 0
$$

(2.3.6)
Also,
\[ \bar{f} = \frac{\rho \hat{f}}{\bar{\rho}} \]
\[ \bar{\rho} \hat{f} = \bar{\rho} \hat{f} \]
\[ \bar{\rho} \hat{f} = (\bar{\rho} + \rho')(\bar{f} + \bar{f}') \]
\[ \bar{\rho} \hat{f} = \bar{\rho} \hat{f} + \bar{\rho} \hat{f}' + \rho' \bar{f} + \rho' \bar{f}' \]
\[ \bar{\rho} \hat{f} = \bar{\rho} \hat{f} + \rho' \bar{f}' \]
\[ \bar{f} = \frac{\bar{\rho} \hat{f} + \rho' \bar{f}'}{\bar{\rho}} \]  \hspace{1cm} (2.3.7)

Now,
\[ f'' = f - \bar{f} = f - \frac{\bar{\rho} \hat{f} + \rho' \bar{f}'}{\bar{\rho}} = f - \bar{f} - \frac{\rho' \bar{f}'}{\bar{\rho}} \]

Substituting \( f = \bar{f} + f \) we obtain
\[ f'' = \hat{f} - \frac{\rho' \bar{f}'}{\bar{\rho}} \]

Taking time averaging over the entire equation we obtain
\[ \bar{f}'' = -\frac{\rho' \bar{f}'}{\bar{\rho}} \]  \hspace{1cm} (2.3.8)

It is to be noted that the time average of \( f'' \) is not zero unless \( \rho' = 0 \)
\[ \bar{\rho} \hat{f} \hat{g} = \bar{\rho} (\hat{f} + f'')(\hat{g} + g'') = \bar{\rho} (\hat{f} \hat{g} + \hat{f} \hat{g}'' + f'' \hat{g} + f'' g'') \]
\[ = \bar{\rho} \hat{f} \hat{g} + \bar{\rho} \hat{f} \hat{g}'' + \rho' \bar{f} \hat{g} + \rho' \bar{f}'' \hat{g}'' \]
\[ = \bar{\rho} \hat{f} \hat{g} + \rho' \bar{f}'' \hat{g}'' \]  \hspace{1cm} (2.3.9)

We may thus write:
\[ \bar{f} = \hat{f} \]
\[ \bar{\rho} \bar{f}'' = 0 \]
\[ \bar{f}'' = -\frac{\rho' \bar{f}'}{\bar{\rho}} \]
\[ \bar{\rho} \hat{f} \hat{g} = \bar{\rho} \hat{f} \hat{g} + \rho' \bar{f}'' \hat{g}'' \]

2.3.1 Mean momentum equation in mass weighted variables

The mass weighted variables are substituted in Eq. 2.1.2 with rearrangement and taking time average of each term, the mean momentum equation in the \( x \) direction is obtained as follows:
Using Eq. 2.2.5 the above equation may be written as

\[
\frac{\partial}{\partial t} \left( \rho \ddot{u} + \rho u'' \right) + \frac{\partial}{\partial x} \left( \rho \ddot{u}u'' + 2\rho \dddot{u}u'' + \rho u''u'' + \ddot{p} + p' - \tau_{xx} \right) \\
+ \frac{\partial}{\partial y} \left( \rho \ddot{v} + \rho \dddot{v}u'' + \rho u''\ddot{v} + \rho u''v'' - \tau_{xy} \right) = 0
\]

Using the above results in Eq. 2.3.10 we get

\[
\frac{1}{T} \int_0^T \rho u'' dt = \frac{1}{T} \int_0^T \rho \ddot{u}u'' dt = \frac{1}{T} \int_0^T p' dt = 0 \\
\frac{1}{T} \int_0^T \rho \ddot{v}v'' dt = \frac{1}{T} \int_0^T \rho u''\ddot{v} dt = 0
\]

Using the above results in Eq. 2.3.10 we get

\[
\frac{\partial}{\partial t} \left[ \frac{1}{T} \int_0^T (\rho \ddot{u}) dt \right] + \frac{\partial}{\partial x} \left[ \frac{1}{T} \int_0^T (\rho \ddot{u}u'' + \ddot{p} - \tau_{xx}) dt \right] \\
+ \frac{\partial}{\partial y} \left[ \frac{1}{T} \int_0^T (\rho \ddot{v} + \rho u''v'' - \tau_{xy}) dt \right] = 0
\]

Using Eq. 2.2.40 in Eq. 2.2.44 we get

\[
\frac{\partial}{\partial t} \left[ \frac{1}{T} \int_0^T (\ddot{\rho} + \rho') \ddot{u} dt \right] + \frac{\partial}{\partial x} \left[ \frac{1}{T} \int_0^T ((\ddot{\rho} + \rho')\ddot{u}u'' + \rho u''u'') dt \right] \\
+ \frac{\partial}{\partial y} \left[ \frac{1}{T} \int_0^T ((\ddot{\rho} + \rho')\ddot{v} + \rho u''v'') dt \right] \\
= -\frac{\partial}{\partial x} \left[ \frac{1}{T} \int_0^T \ddot{p} dt \right] + \frac{\partial}{\partial x} \left[ \frac{1}{T} \int_0^T \tau_{xx} dt \right] + \frac{\partial}{\partial y} \left[ \frac{1}{T} \int_0^T \tau_{xy} dt \right]
\]
Linear Forces

*Local inertia force term:*

\[
\frac{\partial}{\partial t} \left[ \frac{1}{T} \int_0^T (\bar{\rho} + \rho') \ddot{u} \, dt \right]
= \frac{\partial}{\partial t} \left\{ \frac{1}{T} \int_0^T (\bar{\rho} \ddot{u} + \rho' \ddot{u}) \, dt \right\}
\]

Since

\[
\frac{1}{T} \int_0^T \rho' \ddot{u} \, dt = 0
\]

\[
\frac{1}{T} \int_0^T \bar{\rho} \ddot{u} \, dt = \bar{\rho} \ddot{u}
\]

We get

\[
\frac{\partial}{\partial t} \left\{ \frac{1}{T} \int_0^T (\bar{\rho} \ddot{u} + \rho' \ddot{u}) \, dt \right\}
= \frac{\partial}{\partial t} \left( \bar{\rho} \ddot{u} \right)
\]  \hspace{1cm} (2.3.13)

*Pressure force term:*

\[
\frac{1}{T} \int_0^T \frac{\partial p}{\partial x} \, dt
= \frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T (\bar{p} + p') \right\} \, dt
\]

Since

\[
\frac{1}{T} \int_0^T \bar{p} \, dt = \bar{p}
\]

\[
\frac{1}{T} \int_0^T p' \, dt = 0
\]

we get

\[
\frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T (\bar{p} + p') \right\} \, dt
= \frac{\partial \bar{p}}{\partial x}
\]  \hspace{1cm} (2.3.14)

*Viscous force terms:*

\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right\} \, dt
\]
Using the expression of stresses from Eq. 2.1.4, 2.1.6, and expression of fluctuating quantities from Eq. 2.3.4, the viscous terms may be written as:

\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial x} \left[ \frac{2}{3} \mu \left( \frac{\partial}{\partial x} (\ddot{u} + u'') + \frac{\partial}{\partial y} (\ddot{v} + v'') \right) + 2\mu \frac{\partial}{\partial x} (\ddot{u} + u'') \right] \\
+ \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial}{\partial y} (\ddot{u} + u'') + \frac{\partial}{\partial x} (\ddot{v} + v'') \right) \right] \right\} dt \\
= \frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T \left[ \frac{2}{3} \mu \left( \frac{\partial}{\partial x} (\ddot{u} + u'') + \frac{\partial}{\partial y} (\ddot{v} + v'') \right) + 2\mu \frac{\partial}{\partial x} (\ddot{u} + u'') \right] dt \right\} \\
+ \frac{\partial}{\partial y} \left\{ \frac{1}{T} \int_0^T \left[ \mu \left( \frac{\partial}{\partial y} (\ddot{u} + u'') + \frac{\partial}{\partial x} (\ddot{v} + v'') \right) \right] dt \right\}
\]

But

\[
\frac{1}{T} \int_0^T \frac{\partial \ddot{u}}{\partial x} \, dt = \frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T \ddot{u} \, dt \right\} = \frac{\partial \ddot{u}}{\partial x}
\]

Similarly,

\[
\frac{1}{T} \int_0^T \frac{\partial \ddot{v}}{\partial y} \, dt = \frac{\partial \ddot{v}}{\partial y}
\]

\[
\frac{1}{T} \int_0^T \frac{\partial \ddot{v}}{\partial x} \, dt = \frac{\partial \ddot{v}}{\partial x} \cdot \frac{1}{T} \int_0^T \frac{\partial \ddot{v}}{\partial y} \, dt = \frac{\partial \ddot{v}}{\partial y}
\]

Also

\[
\frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T \ddot{v}'' \, dt \right\} = \frac{\partial \dddot{v}''}{\partial x}
\]

\[
\frac{\partial}{\partial y} \left\{ \frac{1}{T} \int_0^T \ddot{v}'' \, dt \right\} = \frac{\partial \ddot{v}'''}{\partial y}
\]

Using the above results we get

\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial x} \left[ \frac{2}{3} \mu \left( \frac{\partial}{\partial x} (\ddot{u} + u'') + \frac{\partial}{\partial y} (\ddot{v} + v'') \right) + 2\mu \frac{\partial}{\partial x} (\ddot{u} + u'') \right] \\
+ \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial}{\partial y} (\ddot{u} + u'') + \frac{\partial}{\partial x} (\ddot{v} + v'') \right) \right] \right\} dt \\
= \frac{\partial}{\partial x} \left\{ \left\{ \frac{2}{3} \mu \left( \frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y} + \frac{\partial \ddot{w}}{\partial z} \right) + 2\mu \frac{\partial \ddot{u}}{\partial x} \right\} + \left\{ \frac{2}{3} \mu \left( \frac{\partial \ddot{u}''}{\partial x} + \frac{\partial \ddot{v}''}{\partial y} + \frac{\partial \ddot{w}''}{\partial z} \right) + 2\mu \frac{\partial \ddot{u}''}{\partial x} \right\} \right\} \\
+ \frac{\partial}{\partial y} \left\{ \left\{ \mu \left( \frac{\partial \ddot{u}}{\partial y} + \frac{\partial \ddot{v}}{\partial x} \right) \right\} + \left\{ \mu \left( \frac{\partial \ddot{u}''}{\partial y} + \frac{\partial \ddot{v}''}{\partial x} \right) \right\} \right\}
\]
\[ \frac{\partial \bar{r}_{xx}}{\partial x} + \frac{\partial \bar{r}_{yx}}{\partial y} \quad (2.3.15) \]

Where,
\[ \bar{r}_{xx} = \left\{ \frac{2}{3} \mu \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) + 2 \mu \frac{\partial \bar{u}}{\partial x} + \left\{ \frac{2}{3} \mu \left( \frac{\partial u''}{\partial x} + \frac{\partial v''}{\partial y} \right) + 2 \mu \frac{\partial u''}{\partial x} \right\} \quad (2.3.16) \]
\[ \bar{r}_{yy} = \left\{ \frac{2}{3} \mu \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) + 2 \mu \frac{\partial \bar{v}}{\partial y} + \left\{ \frac{2}{3} \mu \left( \frac{\partial u''}{\partial x} + \frac{\partial v''}{\partial y} \right) + 2 \mu \frac{\partial v''}{\partial y} \right\} \quad (2.3.17) \]
\[ \bar{r}_{xy} = \tau_{yx} = \left\{ \mu \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) + \left\{ \mu \left( \frac{\partial u''}{\partial y} + \frac{\partial v''}{\partial x} \right) \right\} \quad (2.3.18) \]

Convective inertia force terms (nonlinear terms)
\[ \frac{\partial}{\partial x} \left[ \frac{1}{T} \int_0^T \{ (\tilde{p} + \rho') \bar{u} \bar{u} + \rho u'' u'' \} \, dt \right] + \frac{\partial}{\partial y} \left[ \frac{1}{T} \int_0^T \{ (\tilde{p} + \rho') \bar{v} \bar{v} + \rho u'' v'' \} \, dt \right] + \]
\[ = \frac{\partial}{\partial x} \left[ \frac{1}{T} \int_0^T (\bar{p} \bar{u} \bar{u} + \rho' \bar{u} \bar{u} + \rho u'' u'') \, dt \right] + \frac{\partial}{\partial y} \left[ \frac{1}{T} \int_0^T (\bar{p} \bar{v} \bar{v} + \rho' \bar{v} \bar{v} + \rho u'' v'') \, dt \right] \]

Since
\[ \frac{1}{T} \int_0^T \rho' \bar{u} \bar{u} \, dt = \frac{1}{T} \int_0^T \rho' \bar{v} \bar{v} \, dt = 0 \]
\[ \frac{1}{T} \int_0^T \bar{p} \bar{u} \bar{u} \, dt = \bar{p} \bar{u} \bar{u} , \quad \frac{1}{T} \int_0^T \bar{p} \bar{v} \bar{v} \, dt = \bar{p} \bar{v} \bar{v} \]

Using the results, the convective terms become
\[ \frac{\partial}{\partial x} (\bar{p} \bar{u} \bar{u} + \rho u'' u'') + \frac{\partial}{\partial y} (\bar{p} \bar{v} \bar{v} + \rho u'' v'') \quad (2.3.19) \]

Using these results into Eq. 2.3.12 and little rearrangement the following form of mean momentum equation in the x-direction in mass weighted variables is obtained.
\[ \frac{\partial}{\partial t} (\bar{p} \bar{u}) + \frac{\partial}{\partial x} (\bar{p} \bar{u} \bar{u}) + \frac{\partial}{\partial y} (\bar{p} \bar{v} \bar{v}) \]
\[ = - \frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial x} (\bar{r}_{xx} - \bar{p} u'' u'') + \frac{\partial}{\partial y} (\bar{r}_{xy} - \bar{p} u'' v'') \quad (2.3.20) \]

Similarly the mean momentum equations in the y direction in mass weighted variables are obtained as follows
\[ \frac{\partial}{\partial t} (\bar{p} \bar{v}) + \frac{\partial}{\partial x} (\bar{p} \bar{v} \bar{u}) + \frac{\partial}{\partial y} (\bar{p} \bar{v} \bar{v}) \]
\[ = - \frac{\partial \bar{p}}{\partial y} + \frac{\partial}{\partial x} (\bar{r}_{yx} - \bar{p} v'' u'') + \frac{\partial}{\partial y} (\bar{r}_{yy} - \bar{p} v'' v'') \quad (2.3.21) \]
2.3.2 Mean continuity equation in mass weighted variables

Using Eq. 2.3.3 and 2.3.4 in the instantaneous continuity equation given by Eq. 2.1.1 we get

\[
\frac{\partial}{\partial t} (\bar{\rho} + \rho') + \frac{\partial}{\partial x} (\bar{\rho} + \rho')(\bar{u} + u'') + \frac{\partial}{\partial y} (\bar{\rho} + \rho')(\bar{v} + v'') = 0
\]

Taking time average over the entire equation, we have

\[
\frac{\partial}{\partial t} \left(\frac{1}{T} \int_0^T \left(\bar{\rho} + \rho'\right) dt\right) + \frac{\partial}{\partial x} \left(\frac{1}{T} \int_0^T (\bar{\rho} + \rho') \bar{u} dt\right) + \frac{\partial}{\partial y} \left(\frac{1}{T} \int_0^T (\bar{\rho} + \rho') \bar{v} dt\right) = 0
\]

Using Eq. 2.2.5 and expanding the above equation

\[
\frac{1}{T} \int_0^T \left\{\frac{\partial}{\partial t} (\bar{\rho} + \rho') + \frac{\partial}{\partial x} (\bar{\rho} + \rho') + \frac{\partial}{\partial y} (\bar{\rho} + \rho')\right\} dt = 0
\]

But

\[
\frac{1}{T} \int_0^T \bar{\rho} \bar{u} dt = \bar{\rho} \bar{u}, \quad \frac{1}{T} \int_0^T \rho' \bar{u} dt = 0, \quad \frac{1}{T} \int_0^T \bar{\rho} u'' dt = 0
\]

\[
\frac{1}{T} \int_0^T \bar{\rho} \bar{v} dt = \bar{\rho} \bar{v}, \quad \frac{1}{T} \int_0^T \rho' \bar{v} dt = 0, \quad \frac{1}{T} \int_0^T \bar{\rho} v'' dt = 0
\]

\[
\frac{1}{T} \int_0^T \bar{\rho} dt = \bar{\rho}, \quad \frac{1}{T} \int_0^T \rho' dt = 0, \quad \frac{1}{T} \int_0^T \rho' u'' dt = 0
\]

\[
\frac{1}{T} \int_0^T \rho' v'' dt = 0
\]

Using the above results the mean continuity equation in mass-weighted variables can be written as

\[
\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial (\bar{\rho} \bar{u})}{\partial x} + \frac{\partial (\bar{\rho} \bar{v})}{\partial y} = 0 \quad (2.3.22)
\]

For incompressible flows, \(\rho' = 0\) and then the differences between the conventional and mass-weighted variables vanish, so that the continuity equation can be written as

\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \quad (2.3.23)
\]

So it may be noted that in connection with continuity equation, there is no difference between the mass-weighted and conventional variables for incompressible flow. Finally the (mean)
continuity and momentum equations for unsteady, compressible flow in mass weighted variables are:

**Continuity**
\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 \tag{2.3.24}
\]

**Momentum**
\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u u)}{\partial x} + \frac{\partial (\rho u v)}{\partial y} = -\frac{\partial \rho}{\partial x} + \frac{\partial (\bar{e}_{xx} - \rho u'' u'')}{\partial x} + \frac{\partial (\bar{e}_{xy} - \rho u'' v'')}{\partial y} \tag{2.3.25}
\]
\[
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v u)}{\partial x} + \frac{\partial (\rho v v)}{\partial y} = -\frac{\partial \rho}{\partial y} + \frac{\partial (\bar{e}_{yx} - \rho v'' u'')}{\partial x} + \frac{\partial (\bar{e}_{yy} - \rho v'' v'')}{\partial y} \tag{2.3.26}
\]

**2.4 Closure Problem for Incompressible Flows**

For incompressible flow, the instantaneous continuity and momentum Eq. 2.1.1-2.1.3 form a closed set of three equations with three unknowns \(u, v, p\). However, in performing the time-averaging operation on the momentum equations we throw away all details concerning the state of the flow contained in the instantaneous fluctuations. As a result we obtain three additional unknowns \(\rho u'u'', \rho v'v'', \rho u'v''\), the Reynolds stresses, in the time averaged momentum equations for incompressible flow. These terms, the Reynolds stresses, represents the rate of momentum transfer due to turbulent velocity fluctuations. The complexity of turbulence usually precludes simple formulae for the extra stresses.
CHAPTER 3
TURBULENCE MODELING

3.1 Introduction

A turbulence model is a semi-empirical equation relating the fluctuating co-relation to mean flow variables with various constants provided from experimental investigations. When this equation is expressed as an algebraic equation, it is referred to as the zero-equation model. When partial differential equations (PDEs) are used, they are referred to as one equation or two-equation models, depending on the number of PDEs utilized. Some models employ ordinary differential equations, in which case they reduce to half-equation models. Finally, it is possible to write a partial differential equation directly for each of the turbulence correlations such as $u'u'$, $u'u'$, $v)v'$ in which case they compose a system of PDEs known as Reynolds stress equations. In the present work the two-equation $k-\varepsilon$ model is used to deal with turbulence.

3.2 Additional unknowns in Turbulent Incompressible Flow

According to Eq. 2.2.34-2.2.36, for unsteady, incompressible flows the continuity equation and the momentum equations can be summarized as

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \quad (3.2.1)$$

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial (\bar{u} \bar{u})}{\partial x} + \frac{\partial (\bar{u} \bar{v})}{\partial y} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + \frac{1}{\rho} \frac{\partial}{\partial x} (\bar{t}_{xx} - \rho \bar{u}'u') + \frac{1}{\rho} \frac{\partial}{\partial y} (\bar{t}_{xy} - \rho \bar{u}'v') \quad (3.2.2)$$

$$\frac{\partial \bar{v}}{\partial t} + \frac{\partial (\bar{u} \bar{u})}{\partial x} + \frac{\partial (\bar{v} \bar{v})}{\partial y} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial x} (\bar{t}_{yx} - \rho \bar{v}'u') + \frac{1}{\rho} \frac{\partial}{\partial y} (\bar{t}_{yy} - \rho \bar{v}'v') \quad (3.2.3)$$

Where,

$$\bar{t}_{xx} = 2\mu \frac{\partial \bar{u}}{\partial x}$$

$$\bar{t}_{yy} = 2\mu \frac{\partial \bar{v}}{\partial y}$$

$$\bar{t}_{xy} = \bar{t}_{yx} = \mu \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right)$$

The equations are formally identical to the Navier-Stokes equation with the exception of the additional terms $\rho u'u'$, $\rho v'v'$, $\rho u'v'$, $\rho v'u'$ which constitutes the so-called Reynolds stress tensor. It represents the transfer of momentum due to turbulent fluctuations. The Reynolds-stress tensor consists of four components in 2D:
But $\bar{\rho \overline{u u'}} = \rho \overline{v' u'}$. So, the Reynolds-stress tensor contains only three independent components and the first task at hand is to model the Reynolds stress terms. As we can see the fundamental problem of turbulence modeling based on the two-dimensional Reynolds-averaged Navier-Stokes equations is to find three additional relations in order to close the RANS equations. The system is not closed because the components of the Reynolds stress tensor are unknown. There are more variables than equations. To be exact, there are 6 unknowns and only 3 equations in two dimensional cases. The system can be solved numerically after we find a way to approximate the Reynolds stresses in terms of the mean flow quantities.

### 3.3 Eddy Viscosity Hypothesis

The Boussinesq hypothesis assumes that the turbulent shear stress is related linearly to mean rate of strain, as in a laminar flow. The proportionality factor is the eddy viscosity. The Boussinesq hypothesis for Reynolds averaged incompressible flow can be written as

1. $-\rho \overline{u''} = 2\mu_T \frac{\partial \overline{u'}}{\partial x} - \frac{2}{3} \rho \overline{k}$  
2. $-\rho \overline{v''} = 2\mu_T \frac{\partial \overline{v'}}{\partial y} - \frac{2}{3} \rho \overline{k}$  
3. $-\rho \overline{u'v'} = \mu_T \left( \frac{\partial \overline{u'}}{\partial y} + \frac{\partial \overline{v'}}{\partial x} \right)$

where $k = \frac{1}{2} \left( \overline{u'^2} + \overline{v'^2} \right)$ is the turbulent kinetic energy and $\mu_T$ stands for the eddy viscosity.

Unlike the molecular viscosity $\mu$, the eddy viscosity $\mu_T$ represents no physical characteristic of the fluid, but it is a function of the local flow conditions. Additionally, $\mu_T$ is also strongly affected by flow history effects.

In section 2.1, for incompressible flow, the momentum equations are written in a convenient form given by Eq. 2.1.11 and 2.1.12. The mean momentum equations may be similarly modified using Boussinesq’s hypothesis to approximate the Reynolds stresses.

From Eq. 3.2.2, the momentum equation in $x$ direction is given by

$$\frac{\partial \overline{u'}}{\partial t} + \frac{\partial (\overline{u} \overline{u'})}{\partial x} + \frac{\partial (\overline{v} \overline{v'})}{\partial y} = -\frac{1}{\rho} \frac{\partial \overline{p}}{\partial x} + \frac{1}{\rho} \frac{\partial \overline{u' u'}}{\partial x} + \frac{1}{\rho} \frac{\partial \overline{v' v'}}{\partial y}$$

Using Eq. 2.2.23, 2.2.25, 3.3.1, 3.3.3, the right side of Eq. 3.3.4 may be modified as follows:
\[
\frac{1}{\rho} \frac{\partial}{\partial x} (\bar{\tau}_{xx} - \rho \bar{u} \bar{u}') + \frac{1}{\rho} \frac{\partial}{\partial y} (\bar{\tau}_{xy} - \rho \bar{u} \bar{v}') \\
= \frac{1}{\rho} \frac{\partial}{\partial x} \left( 2\mu \frac{\partial \bar{u}}{\partial x} + (2\mu_T \frac{\partial \bar{u}}{\partial x} - \frac{2}{3} \rho k) \right) + \frac{1}{\rho} \frac{\partial}{\partial y} \left( \mu \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) + \mu_T \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \right) \\
= \frac{1}{\rho} \frac{\partial}{\partial x} \left[ (\mu + \mu_T) \frac{\partial \bar{u}}{\partial x} \right] + \frac{1}{\rho} \frac{\partial}{\partial y} \left[ (\mu + \mu_T) \frac{\partial \bar{u}}{\partial y} \right] + \frac{1}{\rho} \frac{\partial}{\partial x} \left( \mu \frac{\partial \bar{u}}{\partial x} + \mu_T \frac{\partial \bar{u}}{\partial x} \right) \\
+ \frac{1}{\rho} \frac{\partial}{\partial y} \left( \mu \frac{\partial \bar{v}}{\partial x} + \mu_T \frac{\partial \bar{v}}{\partial x} \right) - \frac{1}{\rho} \frac{\partial}{\partial x} \left( \frac{2}{3} \rho k \right) \\
= \frac{1}{\rho} \frac{\partial}{\partial x} \left[ (\mu + \mu_T) \frac{\partial \bar{u}}{\partial x} \right] + \frac{1}{\rho} \frac{\partial}{\partial y} \left[ (\mu + \mu_T) \frac{\partial \bar{u}}{\partial y} \right] - \frac{1}{\rho} \frac{\partial}{\partial x} \left( \frac{2}{3} \rho k \right) + \frac{1}{\rho} \frac{\partial}{\partial x} \left( (\mu + \mu_T) \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) \right)
\]

The last term is zero from continuity principle for incompressible flow.

Also $\frac{\mu}{\rho} = v, \frac{\mu_T}{\rho} = v_T$.

Using the above result in Eq. 3.3.4

\[
\frac{\partial \bar{u}}{\partial t} + \frac{\partial (\bar{u} \bar{u})}{\partial x} + \frac{\partial (\bar{u} \bar{v})}{\partial y} = - \frac{1}{\rho} \frac{\partial}{\partial x} \left( \bar{p} + \frac{2}{3} \rho k \right) + \frac{\partial}{\partial x} \left[ (\nu + \nu_T) \frac{\partial \bar{u}}{\partial x} \right] + \frac{\partial}{\partial y} \left[ (\nu + \nu_T) \frac{\partial \bar{u}}{\partial y} \right]
\]

\[
\frac{\partial \bar{u}}{\partial t} + \frac{\partial (\bar{v} \bar{u})}{\partial x} + \frac{\partial (\bar{v} \bar{v})}{\partial y} = - \frac{1}{\rho} \frac{\partial}{\partial y} \left( \bar{p} \right) + \frac{\partial}{\partial x} \left[ (\nu + \nu_T) \frac{\partial \bar{u}}{\partial x} \right] + \frac{\partial}{\partial y} \left[ (\nu + \nu_T) \frac{\partial \bar{u}}{\partial y} \right] \tag{3.3.5}
\]

Where

$\bar{p} + \frac{2}{3} \rho k = \bar{P}$ is the modified mean pressure.

Similarly for $y$ direction the modified form of momentum equation is

\[
\frac{\partial \bar{v}}{\partial t} + \frac{\partial (\bar{v} \bar{u})}{\partial x} + \frac{\partial (\bar{v} \bar{v})}{\partial y} = - \frac{1}{\rho} \frac{\partial}{\partial y} \left( \bar{P} \right) + \frac{\partial}{\partial x} \left[ (\nu + \nu_T) \frac{\partial \bar{v}}{\partial x} \right] + \frac{\partial}{\partial y} \left[ (\nu + \nu_T) \frac{\partial \bar{v}}{\partial y} \right] \tag{3.3.6}
\]

By applying the eddy-viscosity approach to the Reynolds-averaged form of the governing equations, the only change in the final form of the momentum equations is that, the viscosity coefficient $\nu$ is simply replaced by the sum of a laminar and a turbulent component, $\nu + \nu_T$.

The unknown Reynolds stress tensor has now been replaced with a function of dependent variables from the Navier-Stokes equations as well as two additional unknown scalars, the viscosity $\nu_T$ and the turbulent kinetic energy $k$. Once we know the viscosity $\nu_T$, we can easily extend the Navier-Stokes equations to simulate turbulent flow by introducing averaged flow variables and by adding $\nu_T$ to the laminar viscosity.

### 3.4 K-epsilon (k-ε) Turbulence Model

#### 3.4.1 Motivation behind the k-ε Turbulence Model

On the simplest level of description, turbulence is fully characterized by kinetic energy of fluctuations $k$ or their root-mean-square velocity and by a certain length scale $l$. 

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Dimensionality analysis shows that
\[ \mu_T = \rho C_\mu q l \]  
(3.4.1)
The velocity scale is defined as
\[ q = k^{1/2} \]  
(3.4.2)
To determine the length scale we employ properties of turbulent energy cascade. The kinetic energy of fluctuations with characteristic length scale \( l \) is related to this scale and to the total rate of dissipation as
\[ \varepsilon \approx \frac{k^{2/3}}{l} \]  
(3.4.3)
Assuming that \( l \) gives the length scale of the eddy viscosity formula, we obtain
\[ \mu_T = C_\mu \frac{k^2}{\varepsilon} \]  
(3.4.4)
where \( C_\mu \) is a dimensionless proportionality constant. We can modify Eq. 3.4.4 as follows
\[ \nu_T = C_\mu \frac{k^2}{\varepsilon} \]  
(3.4.5)
where \( \nu_T = \frac{\mu_T}{\rho} \)
Assuming that the constant \( C_\mu \) can be determined empirically, knowledge of \( k \) and \( \varepsilon \) would result in knowledge of the Reynolds stresses.

### 3.4.2 Turbulent Kinetic Energy Transport Equation

In two-equation models, the velocity and length or time scales of turbulence are determined as solution of two additional partial differential equations. In the k-\( \varepsilon \) model (most commonly used two-equation model) partial differential equations for turbulent kinetic energy and dissipation serve to provide value of required velocity and length scale and hence the eddy viscosity.

The \( x \)-component of momentum equation for **unsteady, incompressible** flow is given by Eq. 2.1.11 as:
\[
\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]
But
\[
u \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} - u \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\]
But the last term is zero according to continuity equation. Hence we get
\[
u \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y}
\]
Using this result, the momentum equation may be represented in **non-conservative** form in the following manner:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{3.4.6}
\]

Substituting Eq. 2.2.1-2.2.3 in above equation and multiplying by \(u\) yields:

\[
u' \frac{\partial}{\partial t} (\bar{u} + u') + u' (\bar{u} + u') \frac{\partial}{\partial x} (\bar{u} + u') + u' (\bar{v} + v') \frac{\partial}{\partial y} (\bar{u} + u')
\]

\[= -\frac{1}{\rho} \frac{\partial (\bar{p} + p')}{\partial x} + nu' \left\{ \frac{\partial^2}{\partial x^2} (\bar{u} + u') + \frac{\partial^2}{\partial y^2} (\bar{u} + u') \right\}
\]

Which can be time averaged to get

\[
u' \frac{\partial}{\partial t} (\bar{u} + u') + u' (\bar{u} + u') \frac{\partial}{\partial x} (\bar{u} + u') + u' (\bar{v} + v') \frac{\partial}{\partial y} (\bar{u} + u')
\]

\[= -\frac{1}{\rho} \frac{\partial (\bar{p} + p')}{\partial x} + \left\{ nu' \frac{\partial^2}{\partial x^2} (\bar{u} + u') + nu' \frac{\partial^2}{\partial y^2} (\bar{u} + u') \right\} \tag{3.4.7}
\]

But

\[
u' \frac{\partial}{\partial t} (\bar{u} + u') = \frac{1}{T} \int_0^T \left[ u' \frac{\partial \bar{u}}{\partial t} + u' \frac{\partial u'}{\partial t} \right] dt
\]

But from the rules of averaging given by Eq. 2.2.6-2.2.16

\[1 \int_0^T \left[ u' \frac{\partial \bar{u}}{\partial t} \right] dt = \frac{\partial \bar{u}}{\partial t} \int_0^T u' dt = \frac{\partial}{\partial t} \bar{u} \times 0 = 0
\]

So we get

\[
u' \frac{\partial}{\partial t} (\bar{u} + u') = \frac{1}{T} \int_0^T \left[ u' \frac{\partial u'}{\partial t} \right] dt = \bar{u}' \frac{\partial u'}{\partial t} \tag{3.4.8}
\]

Now

\[u' (\bar{u} + u') \frac{\partial}{\partial x} (\bar{u} + u') = \frac{1}{T} \int_0^T \left[ \bar{u} u' \frac{\partial \bar{u}}{\partial x} + u' u \frac{\partial \bar{u}}{\partial x} + \bar{u} u' \frac{\partial u'}{\partial x} + u' u' \frac{\partial u'}{\partial x} \right] dt
\]

But from the rules of averaging given by the Eq. 2.2.6-2.2.16

\[1 \int_0^T \bar{u} u' \frac{\partial \bar{u}}{\partial x} dt = 0
\]

So we get,

\[u' (\bar{u} + u') \frac{\partial}{\partial x} (\bar{u} + u') = \frac{1}{T} \int_0^T \left[ u' u' \frac{\partial \bar{u}}{\partial x} + \bar{u} u' \frac{\partial u'}{\partial x} + u' u' \frac{\partial u'}{\partial x} \right]
\]

\[= u' \frac{\partial \bar{u}}{\partial x} + \bar{u} u' \frac{\partial u'}{\partial x} + u'^2 \frac{\partial u'}{\partial x}
\]
Similarly

\[
\frac{u'(\bar{v} + v')}{\frac{\partial}{\partial y}(\bar{u} + u')} = \frac{\bar{v}'u'}{\frac{\partial}{\partial y}} + \frac{\bar{v}'u'}{\frac{\partial}{\partial y}} + \frac{v'u'}{\frac{\partial}{\partial y}} \tag{3.4.10}
\]

Now

\[
-u' \frac{1}{\rho} \frac{\partial (\bar{p} + p')}{\frac{\partial}{\partial x}} = \frac{1}{T} \int_0^T \left\{ -u' \frac{1}{\rho} \frac{\partial \bar{p}}{\frac{\partial}{\partial x}} - u' \frac{1}{\rho} \frac{\partial p'}{\frac{\partial}{\partial x}} \right\} dt
\]

But from the rules of averaging given by the Eq. 2.2.6-2.2.16

\[
\frac{1}{T} \int_0^T \left\{ -u' \frac{1}{\rho} \frac{\partial \bar{p}}{\frac{\partial}{\partial x}} \right\} dt = 0
\]

So we get

\[
-u' \frac{1}{\rho} \frac{\partial (\bar{p} + p')}{\frac{\partial}{\partial x}} = \frac{1}{T} \int_0^T \left\{ -u' \frac{1}{\rho} \frac{\partial p'}{\frac{\partial}{\partial x}} \right\} dt = -\frac{1}{\rho} u' \frac{\partial p'}{\frac{\partial}{\partial x}} \tag{3.4.11}
\]

Now

\[
u u' \frac{\partial^2}{\partial x^2} (\bar{u} + u') = \frac{1}{T} \int_0^T \left\{ \nu u' \frac{\partial^2}{\partial x^2} \bar{u}' + \nu u' \frac{\partial^2}{\partial x^2} u' \right\} dt
\]

But from the rules of averaging given by the Eq. 2.2.6-2.2.16

\[
\frac{1}{T} \int_0^T \left\{ \nu u' \frac{\partial^2}{\partial x^2} \bar{u}' \right\} dt = 0
\]

So we get

\[
u u' \frac{\partial^2}{\partial x^2} (\bar{u} + u') = \frac{1}{T} \int_0^T \left\{ \nu u' \frac{\partial^2}{\partial x^2} u' \right\} dt = \nu u' \frac{\partial^2}{\partial x^2} u' \tag{3.4.12}
\]

Similarly,

\[
u u' \frac{\partial^2}{\partial y^2} (\bar{u} + u') = \nu u' \frac{\partial^2}{\partial y^2} u' \tag{3.4.13}
\]

Using Eq. 3.4.8 - 3.4.13 in Eq. 3.4.7 we get

\[
\frac{u'}{\frac{\partial}{\partial t}} + u'^2 \frac{\partial}{\partial x} + u' \frac{\partial}{\partial x} + u' \frac{\partial}{\partial x} + v' \frac{\partial}{\partial y} + v' \frac{\partial}{\partial y} + v' \frac{\partial}{\partial y} = -\frac{1}{\rho} u' \frac{\partial p'}{\frac{\partial}{\partial x}} + \nu \left( u' \frac{\partial^2}{\partial x^2} + u' \frac{\partial^2}{\partial y^2} \right)
\]
The above equation can be rearranged to yield:

\[
\frac{1}{2} \frac{\partial}{\partial t} (\overline{u'^2}) \quad + \quad \frac{1}{2} \overline{u'} \frac{\partial}{\partial x} (\overline{u'^2}) \quad + \quad \frac{1}{2} \overline{v'} \frac{\partial}{\partial y} (\overline{u'^2}) \quad + \quad \frac{\partial^2 u'}{\partial x^2} \quad + \quad \frac{\partial^2 u'}{\partial y^2}
\]

(3.4.15)

To further simplify the above equation the last term on left of the equation may be written as

\[
\overline{u'^2} \frac{\partial u'}{\partial x} \quad + \quad \overline{v'u'} \frac{\partial u'}{\partial y} = \frac{1}{2} \left( \frac{\partial \overline{u'^3}}{\partial x} \quad + \quad \frac{\partial \overline{u'^2} v'}{\partial y} \right) \quad - \quad \frac{1}{2} \overline{u'^2} \left( \frac{\partial u'}{\partial x} \quad + \quad \frac{\partial v'}{\partial y} \right)
\]

But the second term in right side of the above equation is zero from continuity principle.

So,

\[
\overline{u'^2} \frac{\partial u'}{\partial x} \quad + \quad \overline{v'u'} \frac{\partial u'}{\partial y} = \frac{1}{2} \left( \frac{\partial \overline{u'^3}}{\partial x} \quad + \quad \frac{\partial \overline{u'^2} v'}{\partial y} \right)
\]

Thus, Eq. 3.4.15 becomes

\[
\frac{1}{2} \frac{\partial}{\partial t} (\overline{u'^2}) \quad + \quad \frac{1}{2} \overline{u'} \frac{\partial}{\partial x} (\overline{u'^2}) \quad + \quad \frac{1}{2} \overline{v'} \frac{\partial}{\partial y} (\overline{u'^2}) \quad = \quad - \left( \overline{u'^2} \frac{\partial \overline{u'} \overline{u'}}{\partial x} \quad + \quad \overline{v'u'} \frac{\partial \overline{u'}}{\partial y} \right) \quad - \quad \frac{1}{\rho} \overline{u'} \frac{\partial p'}{\partial x} \quad - \quad \frac{1}{2} \overline{u'^2} \left( \frac{\partial u'}{\partial x} \quad + \quad \frac{\partial v'}{\partial y} \right)
\]

(3.4.16a)

Similarly for the momentum equations in y direction we obtain

\[
\frac{1}{2} \frac{\partial}{\partial t} (\overline{v'^2}) \quad + \quad \frac{1}{2} \overline{u'} \frac{\partial}{\partial x} (\overline{v'^2}) \quad + \quad \frac{1}{2} \overline{v'} \frac{\partial}{\partial y} (\overline{v'^2}) \quad = \quad - \left( \overline{v'^2} \frac{\partial \overline{v'} \overline{v'}}{\partial x} \quad + \quad \overline{v'u'} \frac{\partial \overline{v'}}{\partial y} \right) \quad - \quad \frac{1}{\rho} \overline{v'} \frac{\partial p'}{\partial y} \quad - \quad \frac{1}{2} \overline{v'^2} \left( \frac{\partial u'}{\partial x} \quad + \quad \frac{\partial v'}{\partial y} \right)
\]

(3.4.16b)

The mathematical definition of turbulent kinetic energy for two-dimensional flow is:

\[
k = \frac{1}{2} (\overline{u'^2} \quad + \quad \overline{v'^2})
\]

Adding the Eq. 3.4.16a-3.4.16b and using the definition above, the transport equation of of turbulent kinetic energy is obtained in the following form:
\[
\frac{\partial k}{\partial t} + \bar{u} \frac{\partial k}{\partial x} + \bar{v} \frac{\partial k}{\partial y} = - \left( u'' \frac{\partial \bar{u}}{\partial x} + v'' \frac{\partial \bar{v}}{\partial y} + \bar{u} v'' \frac{\partial \bar{v}}{\partial y} + v'' \frac{\partial \bar{v}}{\partial y} \right) - \left( \frac{1}{\rho} u'' \frac{\partial p'}{\partial x} + \frac{1}{\rho} v'' \frac{\partial p'}{\partial y} \right) \\
- \frac{1}{2} \left( \frac{\partial u''^2}{\partial x} + \frac{\partial u'' v''}{\partial x} + \frac{\partial v''^2}{\partial x} + \frac{\partial v''^3}{\partial x} \right) \\
+ \nu \left\{ u'' \left( \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} \right) + v'' \left( \frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} \right) \right\}
\]

(3.4.17)

The above equation may be further modified by considering the term containing pressure and the last term in bracket.

\[
\frac{1}{\rho} u'' \frac{\partial p'}{\partial x} + \frac{1}{\rho} v'' \frac{\partial p'}{\partial y} = \frac{1}{\rho} \left[ \left( \frac{\partial p'}{\partial x} + \frac{\partial p'}{\partial y} \right) - p' \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) \right]
\]

The second term on the right is zero. So we get

\[
\frac{1}{\rho} u'' \frac{\partial p'}{\partial x} + \frac{1}{\rho} v'' \frac{\partial p'}{\partial y} = \frac{1}{\rho} \left[ \left( \frac{\partial p'}{\partial x} + \frac{\partial p'}{\partial y} \right) \right]
\]

Also

\[
\nu \left\{ u'' \left( \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} \right) + v'' \left( \frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} \right) + w'' \left( \frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} \right) \right\}
\]

\[
= \nu \left\{ \frac{\partial^2}{\partial x^2} \left( \frac{1}{2} u'' \right) - \frac{\partial u'}{\partial x} \frac{\partial u'}{\partial x} + \frac{\partial^2}{\partial y^2} \left( \frac{1}{2} u'' \right) - \frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y} \right\}
\]

\[
+ \nu \left\{ \frac{\partial^2}{\partial x^2} \left( \frac{1}{2} v'' \right) - \frac{\partial v'}{\partial x} \frac{\partial v'}{\partial x} + \frac{\partial^2}{\partial y^2} \left( \frac{1}{2} v'' \right) - \frac{\partial v'}{\partial y} \frac{\partial v'}{\partial y} \right\}
\]

\[
+ \nu \left\{ \frac{\partial^2}{\partial x^2} \left( \frac{1}{2} w'' \right) - \frac{\partial w'}{\partial x} \frac{\partial w'}{\partial x} + \frac{\partial^2}{\partial y^2} \left( \frac{1}{2} w'' \right) - \frac{\partial w'}{\partial y} \frac{\partial w'}{\partial y} \right\}
\]

Rearranging and using the definition of turbulent kinetic energy

\[
\nu \left[ \left( \frac{\partial^2 k}{\partial x^2} + \frac{\partial^2 k}{\partial y^2} \right) - \left\{ \left( \frac{\partial u'}{\partial x} \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \frac{\partial v'}{\partial y} + \frac{\partial u'}{\partial y} \frac{\partial v'}{\partial y} \right) \right\} \right]
\]

Also

\[
\bar{u} \frac{\partial k}{\partial x} + \bar{v} \frac{\partial k}{\partial y} = \frac{\partial k \bar{u}}{\partial x} + \frac{\partial k \bar{v}}{\partial y} - k \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right)
\]

But from mean continuity equation for incompressible flow

\[
\left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) = 0
\]

So we get
Using the above results in Eq. 3.4.17 and rearranging we get

\[
\frac{\partial k}{\partial t} + \frac{\partial (k \bar{u})}{\partial x} + \frac{\partial (k \bar{v})}{\partial y} = \frac{\partial}{\partial x} \left[ -\frac{1}{2} \left( u'^{3} + u'v'^{2} \right) - \frac{1}{\rho} p' \bar{u}' - \nu \frac{\partial k}{\partial x} \right] \\
+ \frac{\partial}{\partial y} \left[ -\frac{1}{2} (v'^{3} + v'u'^{2}) - \frac{1}{\rho} p' v' + \nu \frac{\partial k}{\partial y} \right] \\
- \left\{ u'^{2} \frac{\partial \bar{u}}{\partial x} + u'v' \frac{\partial \bar{v}}{\partial x} + v'^{2} \frac{\partial \bar{v}}{\partial y} + v'u' \frac{\partial \bar{u}}{\partial y} \right\} \\
- \nu \left\{ \left( \frac{\partial u' \partial u'}{\partial x} \frac{\partial \bar{u}}{\partial x} + \frac{\partial v' \partial v'}{\partial x} \frac{\partial \bar{v}}{\partial x} \right) + \left( \frac{\partial u' \partial u'}{\partial y} \frac{\partial \bar{u}}{\partial y} + \frac{\partial v' \partial v'}{\partial y} \frac{\partial \bar{v}}{\partial y} \right) \right\} \tag{3.4.18}
\]

The first two terms in square brackets on the right-hand side of the above equation contains three parts. The first part is known as the \textit{turbulent transport} and is regarded as the rate at which turbulence energy is transported through the fluid by turbulent fluctuations. The second part is called \textit{pressure diffusion} and is another form of turbulent transport resulting from correlation of pressure and velocity fluctuations. The third one is the diffusion of turbulence energy caused by the fluid's natural molecular transport process. The third term on the right-hand side which is enclosed in curly brackets is known as the \textit{production term} and represents the rate at which kinetic energy is transferred from the mean flow to the turbulence. Finally, the last term is the dissipation rate of the turbulent kinetic energy.

\textit{Closure approximations}

The left-hand side of Eq. 3.4.18 and the term representing the molecular diffusion are exact while production, dissipation, turbulent transport and pressure diffusion involve unknown correlations. To close the equation these terms have to be approximated. The standard way to approximate turbulent transport of scalar quantities is to use the \textit{gradient-diffusion} hypothesis.

In analogy with molecular transport processes, we say that

\[
-\bar{u}' \phi' \approx \mu_{T} \frac{\partial \phi}{\partial x}, \quad -\bar{v}' \phi' \approx \mu_{T} \frac{\partial \phi}{\partial y}
\]

where \( \phi \) is some conserved scalar, \( \rho \) or \( k \) for example. Thus

\[
\frac{\partial}{\partial x} \left[ -\frac{1}{2} (u'^{3} + u'v'^{2}) - \frac{1}{\rho} p' \bar{u}' \right] + \frac{\partial}{\partial y} \left[ -\frac{1}{2} (v'^{3} + v'u'^{2}) - \frac{1}{\rho} p' v' \right]
\]
where $\sigma_k$ is the turbulent Prandtl number for kinetic energy and is generally taken to be equal to unity. For the production term, denoted by $G$, the eddy viscosity hypothesis given by 3.3.1-3.3.3 is used as follows:

$$G = \left\{ \frac{u'^2}{\partial x} + \frac{u'v'}{\partial x} + \frac{v'^2}{\partial y} + \frac{v'u'}{\partial y} \right\}$$

Putting the Reynolds stresses from Eq. 3.3.1-3.3.3 in the above equation

$$G = \left\{ \left( 2\nu_T \frac{\partial \bar{u}}{\partial x} - \frac{2}{3} k \right) \frac{\partial \bar{u}}{\partial x} + \nu_T \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \frac{\partial \bar{v}}{\partial y} + \left( 2\nu_T \frac{\partial \bar{v}}{\partial y} - \frac{2}{3} k \right) \frac{\partial \bar{v}}{\partial y} + \nu_T \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \frac{\partial \bar{u}}{\partial y} \right\}$$

$$G = \left\{ \left( 2\nu_T \frac{\partial \bar{u}}{\partial x} - \frac{2}{3} k \right) \frac{\partial \bar{u}}{\partial x} + \nu_T \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \frac{\partial \bar{v}}{\partial y} + \left( 2\nu_T \frac{\partial \bar{v}}{\partial y} - \frac{2}{3} k \right) \frac{\partial \bar{v}}{\partial y} \right\}$$

The last term is zero from mean continuity equation. Hence we get:

$$G = 2\nu_T \left[ \left( \frac{\partial \bar{u}}{\partial x} \right)^2 + \left( \frac{\partial \bar{v}}{\partial y} \right)^2 \right] + \nu_T \left[ \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \right] \frac{\partial \bar{v}}{\partial y}$$

The last term,

$$\nu \left\{ \left( \frac{\partial u'}{\partial x} \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \frac{\partial v'}{\partial y} \right) + \left( \frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \frac{\partial v'}{\partial x} \right) \right\} = \varepsilon$$

where $\varepsilon$ is referred to as dissipation of turbulent kinetic energy. Using the above results in Eq. 3.4.18 we obtain

$$\frac{\partial k}{\partial t} + \frac{\partial (k\bar{u})}{\partial x} + \frac{\partial (k\bar{v})}{\partial y} = \frac{\partial}{\partial x} \left[ \left( \nu + \frac{\nu_T}{\sigma_k} \right) k \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \left( \nu + \frac{\nu_T}{\sigma_k} \right) k \frac{\partial u}{\partial y} \right] + G - \varepsilon \quad (3.4.19)$$

Where $G$ denotes the production of turbulent kinetic energy given by

$$G = 2\nu_T \left[ \left( \frac{\partial \bar{u}}{\partial x} \right)^2 + \left( \frac{\partial \bar{v}}{\partial y} \right)^2 \right] + \nu_T \left[ \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \right]$$

And $\varepsilon$ denotes the dissipation of turbulent kinetic energy given by

$$\varepsilon = \nu \left\{ \left( \frac{\partial u'}{\partial x} \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \frac{\partial v'}{\partial y} \right) + \left( \frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \frac{\partial v'}{\partial x} \right) \right\}$$

3.4.3 The $\varepsilon$ Equation

To eliminate the need for specifying the turbulent length scale, in addition to the $k$-equation, a transport equation for one more turbulence quantity can be used. In $k - \varepsilon$ model, transport
equations for dissipation of turbulent kinetic energy, $\epsilon$, is also solved. The exact equation for $\epsilon$ can be derived in a similar manner as the $k$-equation, but it is not a useful starting point for a model equation. The exact equation for $\epsilon$ belongs to processes in the dissipative range, in the end of the cascade. So the standard model equation for $\epsilon$ is best viewed as being entirely empirical. It is much easier to look at the $k$ equation, Eq. 3.4.19, and to setup a similar equation for $\epsilon$. The transport equation should include a local inertia term, $L^{\epsilon}$, convective term, $C^{\epsilon}$, a diffusion term, $D^{\epsilon}$, a production term, $G^{\epsilon}$, and a destruction term, $\Psi^{\epsilon}$, i.e.

$$L^{\epsilon} + C^{\epsilon} = D^{\epsilon} + G^{\epsilon} - \Psi^{\epsilon} \quad (3.4.20)$$

The production and destruction terms, $G$ and $\epsilon$ in the $k$ equation are used to formulate the corresponding terms in the $\epsilon$ equation. The terms in the $k$ equation have the dimension $[m^2/s^3]$ (look at the unsteady term, $\frac{\partial k}{\partial t}$) whereas the terms in the $\epsilon$ equation have the dimension $[m^2/s^4]$. Hence, we must multiply $G$ and $\epsilon$ by a quantity which has the dimension $[1/s]$. One quantity with this dimension is the mean velocity gradient which might be relevant for the production term, but not for the destruction. A better choice should be $\frac{\epsilon}{k} [1/s]$. Hence

$$G^{\epsilon} - \Psi^{\epsilon} = \frac{\epsilon}{k} (C_{\epsilon 1} G - C_{\epsilon 2} \epsilon)$$

where $C_{\epsilon 1}$ and $C_{\epsilon 2}$ are constants. The turbulent diffusion term is expressed in the same way as that in the $k$ equation but with its own turbulent Prandtl number, $\frac{\epsilon}{\sigma_{\epsilon}}$, i.e.

$$D^{\epsilon} = \frac{\partial}{\partial x} \left[ \left( v + \frac{\nu_T}{\epsilon} \right) \frac{\partial \epsilon}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \left( v + \frac{\nu_T}{\epsilon} \right) \frac{\partial \epsilon}{\partial y} \right]$$

The final form of the $\epsilon$ equation reads:

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial (\bar{u} \epsilon)}{\partial x} + \frac{\partial (\bar{v} \epsilon)}{\partial y} = \frac{\partial}{\partial x} \left[ \left( v + \frac{\nu_T}{\sigma_{\epsilon}} \right) \frac{\partial \epsilon}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \left( v + \frac{\nu_T}{\sigma_{\epsilon}} \right) \frac{\partial \epsilon}{\partial y} \right] + \frac{\epsilon}{k} (C_{\epsilon 1} G - C_{\epsilon 2} \epsilon) \quad (3.4.21)$$

The $k$-epsilon ($k-\epsilon$) turbulence model equations can finally be summarized as follows:

$$\frac{\partial k}{\partial t} + \frac{\partial (\bar{u} k)}{\partial x} + \frac{\partial (\bar{v} k)}{\partial y} = \frac{\partial}{\partial x} \left[ \left( v + \frac{\nu_T}{\sigma_k} \right) \frac{\partial k}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \left( v + \frac{\nu_T}{\sigma_k} \right) \frac{\partial k}{\partial y} \right] + G - \epsilon \quad (3.4.22)$$

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial (\bar{u} \epsilon)}{\partial x} + \frac{\partial (\bar{v} \epsilon)}{\partial y} = \frac{\partial}{\partial x} \left[ \left( v + \frac{\nu_T}{\sigma_{\epsilon}} \right) \frac{\partial \epsilon}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \left( v + \frac{\nu_T}{\sigma_{\epsilon}} \right) \frac{\partial \epsilon}{\partial y} \right] + \frac{\epsilon}{k} (C_{\epsilon 1} G - C_{\epsilon 2} \epsilon) \quad (3.4.23)$$

where

$$C_{\mu} = 0.09, C_{\beta 1} = 1.44, C_{\beta 2} = 1.92, \sigma_k = 1.0 \text{ and } \sigma_{\epsilon} = 1.3$$

$$\nu_T = C_{\mu} \frac{k^2}{\epsilon}, \quad \epsilon = v \left\{ \left( \frac{\partial u' \partial u'}{\partial x} + \frac{\partial v' \partial v'}{\partial x} \right) \right\} + \left( \frac{\partial u' \partial u'}{\partial y} \right) + \left( \frac{\partial v' \partial v'}{\partial y} \right)$$

$$G = 2\nu_T \left[ \left( \frac{\partial \bar{u}}{\partial x} \right)^2 + \left( \frac{\partial \bar{v}}{\partial y} \right)^2 \right] + v_T \left[ \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \right]$$
CHAPTER 4
THE BOUNDARY VALUE PROBLEM

4.1 Governing Equations

In the equations throughout the remainder of the present work, all tildes and overbars are dropped except for those indicating correlations between fluctuating quantities for convenience. So quantities such as $\tilde{u}, \tilde{v}$ will be represented simply as $u, v$. Using such notations, the governing equations to be solved become:

**Continuity equation**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

**Momentum equations**

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} = \frac{\partial}{\partial x} \left[ (v + \nu_T) \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[ (v + \nu_T) \frac{\partial u}{\partial y} \right] - \frac{1}{\rho} \frac{\partial P}{\partial x}$$

$$\frac{\partial v}{\partial t} + \frac{\partial (vu)}{\partial x} + \frac{\partial (vv)}{\partial y} = \frac{\partial}{\partial x} \left[ (v + \nu_T) \frac{\partial v}{\partial x} \right] + \frac{\partial}{\partial y} \left[ (v + \nu_T) \frac{\partial v}{\partial y} \right] - \frac{1}{\rho} \frac{\partial P}{\partial y}$$

**Turbulent kinetic energy transport equation**

$$\frac{\partial k}{\partial t} + \frac{\partial (uk)}{\partial x} + \frac{\partial (vk)}{\partial y} = \frac{\partial}{\partial x} \left[ (v + \nu_T) \frac{\partial k}{\partial x} \right] + \frac{\partial}{\partial y} \left[ (v + \nu_T) \frac{\partial k}{\partial y} \right] + G - \varepsilon$$

**Turbulent kinetic energy dissipation equation**

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial (u\varepsilon)}{\partial x} + \frac{\partial (v\varepsilon)}{\partial y} = \frac{\partial}{\partial x} \left[ (v + \nu_T) \frac{\partial \varepsilon}{\partial x} \right] + \frac{\partial}{\partial y} \left[ (v + \nu_T) \frac{\partial \varepsilon}{\partial y} \right] + \frac{\varepsilon}{k} \left( C_{\varepsilon 1} G - C_{\varepsilon 2} \varepsilon \right)$$

4.2 Boundary Conditions

The equations given in section 4.1 are to be solved by applying the following boundary conditions

**Inlet:** The inlet values of $u$-velocity, $v$-velocity, turbulent kinetic energy and dissipation are to be provided. (Shown in Fig. 4.1)

For the specification of the turbulent kinetic energy $k$, appropriate values can be specified through turbulence intensity $I$ that is defined by the ratio of the fluctuating component of the velocity to the mean velocity. Approximate values for $k$ may be obtained from the following:

$$k_o = \frac{3}{2} (u_{inlet} I)^2$$

For the present case, the turbulence intensity is taken as 1.15%

Dissipation can be approximated by the following assumed form:
where \( L \) is the characteristic length scale.

The following conditions exists at the inlet

\[ u = U, \quad v = 0, \quad k = k_0, \quad \varepsilon = \varepsilon_0 \]

**Outlet:** If the location of the outlet is selected far away from geometric disturbances the flow often reaches a fully developed state where no change occurs in the flow direction. So at outlet the Neumann boundary conditions are applied for all quantities except pressure.

\[
\frac{\partial u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0, \quad \frac{\partial k}{\partial n} = 0, \quad \frac{\partial \varepsilon}{\partial n} = 0
\]

**Top and bottom boundary:** Symmetry boundary conditions as written below are applied at the top and bottom boundaries

\[
\frac{\partial u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0, \quad \frac{\partial k}{\partial n} = 0, \quad \frac{\partial \varepsilon}{\partial n} = 0
\]

**On the hydrofoil:** The no-slip condition persists on the hydrofoil as given below:

\[ u = 0, \quad v = 0, \quad k = 0, \quad \varepsilon = 0 \]

The detailed implementation of the boundary conditions will be discussed in chapter 8.

The boundary value problem is illustrated below for a hydrofoil of chord length \( c \), and at an angle of attack \( \alpha \) of zero degree.

**Fig. 4.1:** Boundary value problem
CHAPTER 5
THE FINITE VOLUME METHOD IN CARTESIAN COORDINATES

5.1 Generic form of Governing Equations

From the governing equations derived for either the laminar or turbulent conditions, there are significant commonalities between these various equations. Now the two-dimensional form of the governing equations for the conservations of mass, momentum, and turbulent quantities for incompressible flow are given below.

**Continuity equation**
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

**Momentum equations**
\[
\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} = \frac{\partial}{\partial x}\left[\left(v + v_T\right)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[\left(v + v_T\right)\frac{\partial u}{\partial y}\right] - \frac{1}{\rho} \frac{\partial P}{\partial x}
\]
\[
\frac{\partial v}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (vv)}{\partial y} = \frac{\partial}{\partial x}\left[\left(v + v_T\right)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left[\left(v + v_T\right)\frac{\partial v}{\partial y}\right] - \frac{1}{\rho} \frac{\partial P}{\partial y}
\]

**Turbulent kinetic energy transport equation**
\[
\frac{\partial k}{\partial t} + \frac{\partial (uk)}{\partial x} + \frac{\partial (vk)}{\partial y} = \frac{\partial}{\partial x}\left[\left(v + v_T\right)\frac{\partial k}{\partial x}\right] + \frac{\partial}{\partial y}\left[\left(v + v_T\right)\frac{\partial k}{\partial y}\right] + G - \varepsilon
\]

**Turbulent kinetic energy dissipation equation**
\[
\frac{\partial \varepsilon}{\partial t} + \frac{\partial (u\varepsilon)}{\partial x} + \frac{\partial (v\varepsilon)}{\partial y} = \frac{\partial}{\partial x}\left[\left(v + v_T\right)\frac{\partial \varepsilon}{\partial x}\right] + \frac{\partial}{\partial y}\left[\left(v + v_T\right)\frac{\partial \varepsilon}{\partial y}\right] + \frac{\varepsilon}{k}\left(C_{\varepsilon 1}G - C_{\varepsilon 2}\varepsilon\right)
\]

If we introduce a general variable \( \Phi \), then all the above equations can be expressed in the following generic form:
\[
\frac{\partial \Phi}{\partial t} + \frac{\partial (u\Phi)}{\partial x} + \frac{\partial (v\Phi)}{\partial y} = \frac{\partial}{\partial x}\left[\Gamma' \frac{\partial \Phi}{\partial x}\right] + \frac{\partial}{\partial y}\left[\Gamma' \frac{\partial \Phi}{\partial y}\right] + S_{\Phi}
\] (5.1.1)

Eq. 5.1.1 is the so-called transport equation for the property \( \Phi \). It illustrates the various physical transport processes occurring in the fluid flow: the local acceleration and advection terms on the left-hand side are equivalent to the diffusion term (\( \Gamma' = \) diffusion coefficient) and the source term (\( S_{\Phi} \)) on the right-hand side, respectively. The definition of \( \Phi \) and other terms in Eq. 5.1.1 are given in table 5.1.
Table 5.1: Definition of $\Phi$

<table>
<thead>
<tr>
<th>$\Phi$</th>
<th>$\Gamma'$</th>
<th>$S_\Phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$u$</td>
<td>$v + v_T$</td>
<td>$\frac{-1}{\rho} \frac{\partial p}{\partial x}$</td>
</tr>
<tr>
<td>$v$</td>
<td>$v + v_T$</td>
<td>$\frac{-1}{\rho} \frac{\partial p}{\partial y}$</td>
</tr>
<tr>
<td>$k$</td>
<td>$v + \frac{v_T}{\sigma_k}$</td>
<td>$G-\varepsilon$</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>$v + \frac{v_T}{\sigma_\varepsilon}$</td>
<td>$\frac{\varepsilon}{\kappa} (C_{e1}G - C_{e2}\varepsilon)$</td>
</tr>
</tbody>
</table>

5.2 Finite Volume Discretization in Cartesian Coordinates

For solving fluid flow problems using finite volume method the first step is to subdivide the flow domain into discrete control volumes with a grid arrangements which may be staggered grids or non-staggered. For the nonstaggered grids, vector variables and scalar variables are stored at the same locations, while for the staggered grids, vector components and scalar variables are stored at different locations, shifted half control volume in each coordinate direction. Staggered grids are popular because of their ability to prevent checkerboard pressure in the flow solution. However, programming of the staggered grid method is tedious since $u$ and $v$-momentum equations are discretized at different control volumes shifted in different directions from the main control volumes. The programming difficulties increase when one deals with curvilinear or unstructured grids. As a result, nearly all codes written on curvilinear or unstructured grids use non-staggered grid arrangement for the solution of fluid flow problems. On the other hand non-staggered grids are prone to produce a false pressure field-checkerboard pressure if special precautions are not taken. Rhie (1981) proposed a momentum interpolation method to eliminate the checkerboard pressure problem on collocated grid shown in Fig 5.1. Multiplying Eq. 5.1.1 by the constant density $\rho$, we obtain the following form of generic transport equation for two-dimensional flow:

$$\frac{\partial (\rho \Phi)}{\partial t} + \frac{\partial (\rho u \Phi)}{\partial x} + \frac{\partial (\rho v \Phi)}{\partial y} = \frac{\partial}{\partial x} \left[ \Gamma \frac{\partial \Phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \Gamma \frac{\partial \Phi}{\partial y} \right] + S_\Phi \quad (5.2.1)$$
where
\[ \Gamma = \rho \Gamma, \quad S_\Phi = \rho S_\Phi \] (5.2.2)

Integrating Eq. 5.2.1 with respect to space and time we get:
\[ \int_t^{t+\Delta t} \left[ \int_{\Delta V} \left\{ \frac{\partial (\rho \Phi)}{\partial t} + \frac{\partial (\rho u \Phi)}{\partial x} + \frac{\partial (\rho v \Phi)}{\partial y} \right\} dV \right] dt = \int_t^{t+\Delta t} \left[ \int_{\Delta V} \left\{ \frac{\partial \Phi}{\partial x} \left[ \Gamma \frac{\partial \Phi}{\partial x} \right] + \frac{\partial \Phi}{\partial y} \left[ \Gamma \frac{\partial \Phi}{\partial y} \right] + S_\Phi \right\} dV \right] dt \] (5.2.3)

Performing the space integral term by term we get:
\[ \int_{\Delta V} \frac{\partial (\rho \Phi)}{\partial t} dV = \frac{\partial (\rho \Phi)}{\partial t} \Delta V = \frac{\partial (\rho \Phi)}{\partial t} \Delta x \Delta y \Delta z = \frac{\partial (\rho \Phi)}{\partial t} \Delta x \cdot \Delta y \cdot 1 = \frac{\partial (\rho \Phi)}{\partial t} \Delta x \Delta y \]

Where \( \frac{\partial (\rho \Phi)}{\partial t} \) is assumed to be constant over the whole control volume
\[ \int_{\Delta V} \left\{ \frac{\partial (\rho u \Phi)}{\partial x} \right\} dV = \int_{\Delta A} (\rho u \Phi) dA_x = \sum_{i=1}^{4} \rho u \Phi_i A_i^x = ((\rho u \Phi)_e - (\rho u \Phi)_w) \Delta y \]

Where,
\( \Delta A \) is the surface bounding the control volume \( \Delta V \)
\( dA_x \) is the projected area in the \( x \) direction

Similarly
\[ \int_{\Delta V} \left\{ \frac{\partial (\rho v \Phi)}{\partial y} \right\} dV = ((\rho v \Phi)_n - (\rho v \Phi)_s) \Delta x \]
\[ \int_{\Delta V} \left\{ \Gamma \frac{\partial \Phi}{\partial x} \right\} dV = \int_{\Delta A} \left[ \Gamma \frac{\partial \Phi}{\partial x} \right] dA_x = \sum_{i=1}^{4} \left\{ \left[ \Gamma \frac{\partial \Phi}{\partial x} \right] A_i^x \right\} = \left\{ \left[ \Gamma \frac{\partial \Phi}{\partial x} \right]_e - \left[ \Gamma \frac{\partial \Phi}{\partial x} \right]_w \right\} \Delta y \]

Using central differencing scheme for the above diffusion term we get:
\[ \int_{\Delta V} \left\{ \frac{\partial \Phi}{\partial x} \right\} dV = \left\{ \frac{\Gamma}{\delta x_e} (\Phi_e - \Phi_p) - \frac{\Gamma}{\delta x_w} (\Phi_p - \Phi_w) \right\} \Delta y \]

Similarly,
In practical situations the source term may be a function of the dependent variable. In such cases the finite volume method approximates the source term by means of a linear form:

$$\bar{S} \Delta x \Delta y = (S_c + S_p \Phi_p) \Delta x \Delta y$$

Using the results of the space integrals in Eq. 5.2.3 we get

$$\int_{t}^{t+\Delta t} \left[ \frac{\partial (\rho \Phi)}{\partial t} \Delta x \Delta y + \{(\rho v \Phi)_e - (\rho v \Phi)_w\} \Delta y + \{(\rho v \Phi)_n - (\rho v \Phi)_s\} \Delta x \right] dt$$

$$= \int_{t}^{t+\Delta t} \left[ \left\{ \frac{\Gamma}{\delta x_e} (\Phi_e - \Phi_p) - \frac{\Gamma}{\delta x_w} (\Phi_p - \Phi_w) \right\} \Delta y + \left\{ \frac{\Gamma}{\delta x_e} (\Phi_N - \Phi_p) - \frac{\Gamma}{\delta x_w} (\Phi_p - \Phi_S) \right\} \Delta x + (S_c + S_p \Phi_p) \Delta x \Delta y \right] dt$$

(5.2.4)

Now,

$$\int_{t}^{t+\Delta t} \frac{\partial (\rho \Phi)}{\partial t} \Delta x \Delta y \ dt = \rho \Delta x \Delta y (\Phi - \Phi^o)$$

(5.2.5)

Where the superscript ‘o’ refers to quantities at time $t$ and the quantity at $t + \Delta t$ is not superscripted. For carrying out the time integration of convective and diffusion terms, it is necessary to assume how $\Phi$ vary with time. We may achieve this by means of a weighting parameter $\theta$ between 0 and 1 and write the integral of $\Phi$ as

$$\int_{t}^{t+\Delta t} \Phi \ dt = [\theta \Phi + (1 - \theta) \Phi^o] \Delta t$$

Using this idea and Eq. 5.2.5 in Eq. 5.2.4 we get

$$\rho \Delta x \Delta y (\Phi - \Phi^o) + \theta \{(\rho u \Phi )_e - (\rho u \Phi )_w\} \Delta y + \{(\rho v \Phi )_n - (\rho v \Phi )_s\} \Delta x \Delta t +$$

$$(1 - \theta) \{(\rho u \Phi^o) _e - (\rho u \Phi^o) _w\} \Delta y + \{(\rho v \Phi^o) _n - (\rho v \Phi^o) _s\} \Delta x \Delta t$$

$$= \theta \left[ \left\{ \frac{\Gamma}{\delta x_e} (\Phi_e - \Phi_p) - \frac{\Gamma}{\delta x_w} (\Phi_p - \Phi_w) \right\} \Delta y + \left\{ \frac{\Gamma}{\delta x_e} (\Phi_N - \Phi_p) - \frac{\Gamma}{\delta x_w} (\Phi_p - \Phi_S) \right\} \Delta x \right] \Delta t +$$

$$(1 - \theta) \left[ \left\{ \frac{\Gamma}{\delta x_e} (\Phi^o_e - \Phi^o_p) - \frac{\Gamma}{\delta x_w} (\Phi^o_p - \Phi^o_w) \right\} \Delta y + \left\{ \frac{\Gamma}{\delta x_e} (\Phi^o_N - \Phi^o_p) - \frac{\Gamma}{\delta x_w} (\Phi^o_p - \Phi^o_S) \right\} \Delta x \right] \Delta t$$

$$+ \left[ \theta \{(S_c + S_p \Phi_p) \Delta x \Delta y\} + (1 - \theta) \{(S_c + S_p \Phi^o_p) \Delta x \Delta y\} \right] \Delta t$$

(5.2.6)
where
\[ 0 \leq \theta \leq 1 \] (5.2.7)
Depending on the value of \( \theta \) the resulting schemes may be forward Euler (explicit), backward Euler (implicit) or Crank Nicolson scheme. For the present case implicit scheme is utilized.

Using \( \theta = 1 \) (which corresponds to implicit scheme) in Eq. 5.2.6 and dividing by \( \Delta t \) we get
\[
\frac{\rho \Delta x \Delta y}{\Delta t} (\Phi - \Phi^{0}) + \{ (\rho u \Phi)_e - (\rho u \Phi)_w \} \Delta y + \{ (\rho v \Phi)_n - (\rho v \Phi)_s \} \Delta x
\]

\[ = \left\{ \frac{\Gamma}{\delta x_e} (\Phi_{e} - \Phi_{p}) - \frac{\Gamma}{\delta x_w} (\Phi_{p} - \Phi_{w}) \right\} \Delta y \]

\[ + \left\{ \frac{\Gamma}{\delta x_e} (\Phi_{N} - \Phi_{p}) - \frac{\Gamma}{\delta x_w} (\Phi_{p} - \Phi_{S}) \right\} \Delta x + (S_c + S_p \Phi_p) \Delta x \Delta y \] (5.2.8)

The superscript ‘o’ refers to quantities at time \( t \) and the quantities at \( t + \Delta t \) are not superscripted. For convective terms, CDS results in negative coefficients for the downstream neighbor nodes if convection dominates strongly over diffusion. The negative coefficients may cause unphysical oscillations in the solution. Sometimes the solution does not converge.

The UDS, on the other hand, is unconditionally stable. However UDS introduces numerical errors known as “artificial” or “false” diffusion. One can choose the option between using CDS, UDS or a combination of the two. The combination can be achieved via the so called “deferred correction” approach, and also blending the two schemes by a blending factor of \( \psi \) (value of \( \psi \) is between 0 and 1) as follows:

\[
[\rho u_e \Delta y \Phi_e - \rho u_w \Delta y \Phi_w + \rho v_n \Delta x \Phi_n - \rho v_s \Delta x \Phi_s]_{\text{UDS}}
\]

\[ + \psi \left\{ [\rho u_e \Delta y \Phi_e - \rho u_w \Delta y \Phi_w + \rho v_n \Delta x \Phi_n - \rho v_s \Delta x \Phi_s]_{\text{CDS}}
\]

\[ - [\rho u_e \Delta y \Phi_e - \rho u_w \Delta y \Phi_w + \rho v_n \Delta x \Phi_n - \rho v_s \Delta x \Phi_s]_{\text{UDS}} \}

Using such a technique in Eq. 5.2.8 we get:

\[
\frac{\rho \Delta x \Delta y}{\Delta t} (\Phi - \Phi^{0}) + \left\{ \rho u_e \Delta y \Phi_e - \rho u_w \Delta y \Phi_w + \rho v_n \Delta x \Phi_n - \rho v_s \Delta x \Phi_s \right\}_{\text{UDS}}
\]

\[ = \left\{ \frac{\Gamma}{\delta x_e} (\Phi_{e} - \Phi_{p}) - \frac{\Gamma}{\delta x_w} (\Phi_{p} - \Phi_{w}) \right\} \Delta y \]

\[ + \left\{ \frac{\Gamma}{\delta x_e} (\Phi_{N} - \Phi_{p}) - \frac{\Gamma}{\delta x_w} (\Phi_{p} - \Phi_{S}) \right\} \Delta x + (S_c + S_p \Phi_p) \Delta x \Delta y
\]

\[- \psi \left\{ [\rho u_e \Delta y \Phi_e - \rho u_w \Delta y \Phi_w + \rho v_n \Delta x \Phi_n - \rho v_s \Delta x \Phi_s]_{\text{CDS}}
\]

\[ - [\rho u_e \Delta y \Phi_e - \rho u_w \Delta y \Phi_w + \rho v_n \Delta x \Phi_n - \rho v_s \Delta x \Phi_s]_{\text{UDS}} \} \] (5.2.9)

The first part of the convective term is known as the implicit term and the terms multiplied by \( \psi \) are called explicit terms. The terms multiplied by \( \psi \) become part of the source term and
are determined explicitly from previous iterations. As the solution converges the values of field variables for consecutive iterations tend to be equal, so that both UDS terms cancel and only the convective term interpolated with CDS which of higher accuracy than UDS scheme remains.

The upwind (UDS) and central difference (CDS) schemes are explained below:

**UDS**

When the flow is in the positive direction, \( u_w > 0, u_e > 0, u_n > 0, u_s > 0 (F_w > 0, F_e > 0, F_n > 0, F_s > 0) \)

\[
\phi_w = \phi_w, \phi_e = \phi_p, \phi_s = \phi_s, \phi_n = \phi_p
\]

When the flow is in the negative direction, \( u_w < 0, u_e < 0, u_n < 0, u_s < 0 (F_w < 0, F_e < 0, F_n < 0, F_s < 0) \)

\[
\phi_w = \phi_p, \phi_e = \phi_E, \phi_s = \phi_p, \phi_n = \phi_N
\]

**CDS**

The central differencing approximations are:

\[
\phi_w = \frac{\phi_w + \phi_p}{2}, \phi_e = \frac{\phi_E + \phi_p}{2}, \phi_n = \frac{\phi_N + \phi_p}{2}, \phi_s = \frac{\phi_S + \phi_p}{2}
\]

Using UDS for the implicit convective term followed by rearrangement, Eq. 5.2.9 may be written in the following algebraic form:

\[
a_p \phi_p = a_w \phi_w + a_E \phi_E + a_N \phi_N + a_s \phi_s + a_p^b \phi_p^b + b^\phi + p \text{ term}
\]

\[
a_p \phi_p = \sum_{nb} a_{nb} \phi_{nb} + a_p^b \phi_p^b + S \tag{5.2.10}
\]

where

\[
\sum_{nb} a_{nb} \phi_{nb} = a_w \phi_w + a_E \phi_E + a_N \phi_N + a_s \phi_s
\]

\[
a_p = a_w + a_E + a_N + a_s + a_p^b - S_p \Delta x \Delta y + (F_e - F_w + F_n - F_s)
\]

The term into brackets corresponds to the continuity equation. After each outer iteration steps, the mass fluxes are corrected so that the last term of \( a_p \) vanishes and, therefore, are not considered.

\[
a_p^b = \frac{\rho \Delta x \Delta y}{\Delta t}
\]

\[
b^\phi = S_c \Delta x \Delta y + \psi \{[\rho u_e \Delta y \phi_e - \rho u_w \Delta y \phi_w + \rho v_n \Delta x \phi_n - \rho v_s \Delta x \phi_s]_{CDS}
\]

\[
- [\rho u_e \Delta y \phi_e - \rho u_w \Delta y \phi_w + \rho v_n \Delta x \phi_n - \rho v_s \Delta x \phi_s]_{UDS}\}
\]
\[
\begin{array}{|c|c|c|c|}
\hline
a_W & a_E & a_N & a_S \\
\hline
D_w + \text{ max } (F_w, 0) & D_e + \text{ max } (0, -F_e) & D_n + \text{ max } (0, -F_n) & D_s + \text{ max } (F_s, 0) \\
\hline
\end{array}
\]

\[F_e = \rho u_e \Delta y, F_w = \rho u_w \Delta y, F_n = \rho v_n \Delta x, F_s = \rho v_s \Delta x,\]

\[D_e = \frac{\Gamma}{\delta x_e} \Delta y, D_w = \frac{\Gamma}{\delta x_w} \Delta y, D_n = \frac{\Gamma}{\delta y_n} \Delta x, D_s = \frac{\Gamma}{\delta y_s} \Delta x,\]

\[\begin{align*}
p_{\text{term}} &= -(p_e - p_w) \Delta y \text{ for } x - \text{momentum} \\
p_{\text{term}} &= -(p_n - p_s) \Delta x \text{ for } y - \text{momentum} \\
p_{\text{term}} &= 0 \text{ for all other equations}
\end{align*}\]

\[S = b^\Phi + p_{\text{term}}\]
CHAPTER 6

THE FINITE VOLUME METHOD IN BODY-FITTED COORDINATES

6.1 Transformation of Governing Equation

The generic form of the governing equation for a dependent variable $\Phi$ is:

$$\frac{\partial \Phi}{\partial t} + \frac{\partial (u\Phi)}{\partial x} + \frac{\partial (v\Phi)}{\partial y} = \frac{\partial}{\partial x} \left[ \Gamma' \frac{\partial \Phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \Gamma' \frac{\partial \Phi}{\partial y} \right] + S_\Phi$$  \hspace{1cm} (6.1.1)

If new independent coordinates $\xi$ and $\eta$ are introduced, the above generic transport equation changes according to the general transformation

$$\xi = \xi(x,y), \quad \eta = \eta(x,y)$$  \hspace{1cm} (6.1.2)

Partial derivatives of any function $f$ are transformed according to

$$\frac{\partial f}{\partial x} = \frac{1}{J} \frac{\partial f}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial f}{\partial \eta}, \quad \frac{\partial f}{\partial y} = \frac{1}{J} \frac{\partial f}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial f}{\partial \xi}$$  \hspace{1cm} (6.1.3)

Where $J$ is the Jacobian of the transformation given by

$$J = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} = \frac{\partial (x,y)}{\partial (\xi,\eta)}$$  \hspace{1cm} (6.1.4)

The transformation is schematically illustrated in Fig. 6.1 to 6.2

Fig. 6.1: Physical and transformed planes showing boundaries
Fig. 6.2: Physical and transformed planes showing constant $\xi$ and $\eta$ lines

Transformation of Convective term

The following notations are used for partial derivatives

\[
\frac{\partial \Phi}{\partial x} = \Phi_x, \quad \frac{\partial \Phi}{\partial y} = \Phi_y, \quad \frac{\partial \Phi}{\partial \xi} = \Phi_{\xi}, \quad \frac{\partial \Phi}{\partial \eta} = \Phi_{\eta}
\]

So we get

\[
\frac{\partial (u\Phi)}{\partial x} + \frac{\partial (v\Phi)}{\partial y} = (u\Phi)_x + (v\Phi)_y
\]

Using Eq. 6.1.3

\[
(u\Phi)_x + (v\Phi)_y = \frac{y_\eta (u\Phi)_\xi - y_\xi (u\Phi)_\eta}{J} + \frac{x_\xi (v\Phi)_\eta - x_\eta (v\Phi)_\xi}{J}
\]

where \( J = x_\xi y_\eta - x_\eta y_\xi \)

Using differentiation rule for products

\[
(u\Phi)_x + (u\Phi)_y
\]

\[
= \frac{1}{J} \left\{ (u\Phi y_\eta)_\xi - u\Phi (y_\eta)_\xi - (u\Phi y_\xi)_\eta + u\Phi (y_\xi)_\eta + (v\Phi x_\xi)_\eta - v\Phi (x_\xi)_\eta - (v\Phi x_\eta)_\xi + v\Phi (x_\eta)_\xi \right\}
\]

But,
\[ u \Phi(y_\eta)_\xi = u \Phi(y_\xi)_\eta, \quad v \Phi(x_\xi)_\eta = v \Phi(x_\eta)_\xi \]

So we get

\[
(u \Phi)_x + (v \Phi)_y = \frac{1}{j} \{ (u \Phi y_\eta)_\xi - (u \Phi y_\xi)_\eta + x_\xi (v \Phi x_\xi)_\eta - (v \Phi x_\eta)_\xi \}
\]

\[
= \frac{1}{j} \{ (u \Phi y_\eta - v \Phi x_\eta)_\xi + (v \Phi x_\xi - u \Phi y_\xi)_\eta \}
\]

Using

\[
U = uy_\eta - vx_\eta, \quad V = vx_\xi - uy_\xi
\]

we get

\[
(u \Phi)_x + (v \Phi)_y = \frac{1}{j} \{ (U \Phi)_\xi + (V \Phi)_\eta \}
\]

Using the original notation for partial derivative we can write the convective term as:

\[
\frac{\partial (u \Phi)}{\partial x} + \frac{\partial (v \Phi)}{\partial y} = \frac{1}{j} \left\{ \frac{\partial (U \Phi)}{\partial \xi} + \frac{\partial (V \Phi)}{\partial \eta} \right\}
\]

**Transformation of Diffusion term**

\[
\frac{\partial}{\partial x} \left[ \Gamma' \frac{\partial \Phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \Gamma' \frac{\partial \Phi}{\partial y} \right] = (\Gamma'' \Phi_x)_x + (\Gamma'' \Phi_y)_y
\]

\[
(\Gamma'' \Phi_x)_x + (\Gamma'' \Phi_y)_y = \frac{1}{j} \{ y_\eta (\Gamma' \Phi_x)_\xi - y_\xi (\Gamma' \Phi_x)_\eta + x_\xi (\Gamma' \Phi_y)_\eta - x_\eta (\Gamma' \Phi_y)_\xi \}
\]

\[
= \frac{\Gamma''}{j^2} \{ y_\eta (\Phi_x)_\xi - y_\xi (\Phi_x)_\eta + x_\xi (\Phi_y)_\eta - x_\eta (\Phi_y)_\xi \}
\]

\[
= \frac{\Gamma''}{j^2} \{ y_\eta [y_\xi \Phi_\xi - y_\xi \Phi_\eta]_\xi - y_\xi [y_\eta \Phi_\xi - y_\xi \Phi_\eta]_\eta + x_\xi [x_\xi \Phi_\eta - x_\eta \Phi_\xi]_\eta
\]

\[
- x_\eta [x_\xi \Phi_\eta - x_\eta \Phi_\xi]_\xi \}
\]

\[
= \frac{\Gamma''}{j^2} \{ y_\eta [y_\xi \Phi_\xi - y_\xi \Phi_\eta]_\xi - y_\eta [x_\xi \Phi_\eta - x_\eta \Phi_\xi]_\xi + x_\xi [x_\xi \Phi_\eta - x_\eta \Phi_\xi]_\eta
\]

\[
- y_\xi [y_\eta \Phi_\xi - y_\xi \Phi_\eta]_\eta \}
\]

\[
= \frac{\Gamma''}{j^2} \{ ([y_\eta \Phi_\xi - y_\xi \Phi_\eta] y_\eta)_\xi - (y_\eta \Phi_\xi - y_\xi \Phi_\eta) [y_\eta]_\xi - ([x_\xi \Phi_\eta - x_\eta \Phi_\xi] x_\eta)_\xi
\]

\[
+ (x_\xi \Phi_\eta - x_\eta \Phi_\xi) [x_\eta]_\xi + ([x_\xi \Phi_\eta - x_\eta \Phi_\xi] x_\xi)_\eta - (x_\xi \Phi_\eta - x_\eta \Phi_\xi) [x_\xi]_\eta
\]

\[
- ([y_\eta \Phi_\xi - y_\xi \Phi_\eta] y_\xi)_\eta + (y_\eta \Phi_\xi - y_\xi \Phi_\eta) [y_\xi]_\eta \}
\]

Using
Using the original notation for partial derivative we can write the diffusion term as:

\[
(y_\eta)_{\xi} = (y_\xi)_{\eta} - (x_\xi)_{\eta}
\]

\[
= \frac{\Gamma''}{f^2} \left\{ \left[ (y_\eta \Phi_\xi - y_\xi \Phi_\eta) y_\eta \right]_{\xi} - \left[ (x_\xi \Phi_\eta - x_\eta \Phi_\xi) x_\eta \right]_{\xi} + \left[ (x_\xi \Phi_\eta - x_\eta \Phi_\xi) x_\xi \right]_{\eta}
\]

\[
- \left[ (y_\eta \Phi_\xi - y_\xi \Phi_\eta) y_\xi \right]_{\eta} \right\}
\]

\[
= \frac{\Gamma''}{f^2} \left\{ \left[ (y_\eta \Phi_\xi - y_\xi \Phi_\eta) y_\eta - (x_\xi \Phi_\eta - x_\eta \Phi_\xi) x_\eta \right]_{\xi}
\]

\[
+ \left[ (x_\xi \Phi_\eta - x_\eta \Phi_\xi) x_\xi - (y_\eta \Phi_\xi - y_\xi \Phi_\eta) y_\xi \right]_{\eta} \right\}
\]

Using the original notation for partial derivative we can write the diffusion term as:

\[
\frac{\partial}{\partial x} \left[ \frac{\Gamma''}{f} \frac{\partial \Phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \frac{\Gamma''}{f} \frac{\partial \Phi}{\partial y} \right]
\]

\[
= \frac{1}{f} \frac{\partial}{\partial \xi} \left[ \frac{\Gamma''}{f} \left( y_\eta \Phi_\xi - y_\xi \Phi_\eta \right) y_\eta - \frac{\Gamma''}{f} \left( x_\xi \Phi_\eta - x_\eta \Phi_\xi \right) x_\eta \right]
\]

\[
+ \frac{1}{f} \frac{\partial}{\partial \eta} \left[ \frac{\Gamma''}{f} \left( x_\xi \Phi_\eta - x_\eta \Phi_\xi \right) x_\xi - \frac{\Gamma''}{f} \left( y_\eta \Phi_\xi - y_\xi \Phi_\eta \right) y_\xi \right]
\]

\[
= \frac{1}{f} \frac{\partial}{\partial \xi} \left[ \frac{\Gamma''}{f} \Phi_\xi (x_\eta^2 + y_\eta^2) - \frac{\Gamma''}{f} \Phi_\eta (x_\xi x_\eta + y_\xi y_\eta) \right]
\]

\[
+ \frac{1}{f} \frac{\partial}{\partial \eta} \left[ - \frac{\Gamma''}{f} \Phi_\xi (x_\xi x_\eta + y_\xi y_\eta) + \frac{\Gamma''}{f} \Phi_\eta (x_\xi^2 + y_\xi^2) \right]
\]

\[
= \frac{1}{f} \frac{\partial}{\partial \xi} \left[ \frac{\Gamma''}{f} (\alpha' \Phi_\xi - \beta' \Phi_\eta) \right] + \frac{1}{f} \frac{\partial}{\partial \eta} \left[ \frac{\Gamma''}{f} (\gamma' \Phi_\eta - \beta' \Phi_\xi) \right]
\]

Where

\[
\alpha' = x_\eta^2 + y_\eta^2, \quad \beta' = x_\xi x_\eta + y_\xi y_\eta, \quad \gamma' = x_\xi^2 + y_\xi^2
\]

Using these results in Eq. 6.1.1 we finally get the following transformed form of the generic governing equation

\[
\frac{\partial \Phi}{\partial t} + \frac{1}{f} \frac{\partial (U \Phi)}{\partial \xi} + \frac{1}{f} \frac{\partial (V \Phi)}{\partial \eta}
\]

\[
= \frac{1}{f} \frac{\partial}{\partial \xi} \left[ \frac{\Gamma''}{f} (\alpha' \Phi_\xi - \beta' \Phi_\eta) \right] + \frac{1}{f} \frac{\partial}{\partial \eta} \left[ \frac{\Gamma''}{f} (\gamma' \Phi_\eta - \beta' \Phi_\xi) \right] + S^\Phi_{\xi, \eta} \quad (6.1.5)
\]

where \( S^\Phi_{\xi, \eta} \) is the lumped source term

### 6.2 Finite Volume Discretization in Body-Fitted Grids

Multiplying Eq. 6.1.5 by the constant density \( \rho \), and jacobian \( f \) we obtain the following form of generic transport equation for two dimensional flow
In the present treatment, governing equations are discretized by finite volume method on a nonstaggered grid system in which all variables are stored at the center of the control volume (Fig. 6.3). The essence of finite volume method is to express the governing equations in integral form and apply Gauss’ divergence theorem to the volume integrals.

$$\int \frac{\partial (\rho \Phi)}{\partial t} + \frac{\partial (\rho U \Phi)}{\partial \xi} + \frac{\partial (\rho V \Phi)}{\partial \eta} = \frac{\partial}{\partial \xi} \left[ \Gamma \left( \alpha' \Phi_\xi - \beta' \Phi_\eta \right) \right] + \frac{\partial}{\partial \eta} \left[ \Gamma \left( \gamma' \Phi_\eta - \beta' \Phi_\xi \right) \right] + JS^\Phi(\xi, \eta) \quad (6.2.1)$$

where

$$\Gamma = \rho \Gamma', \quad S^\Phi = \rho S^\Phi$$

Eq. 6.2.1 contains both space and time coordinates and is therefore integrated with respect to both space and time to yield:

$$\int_t^{t+\Delta t} \left[ \int_{\Delta \nu'} \left\{ \frac{\partial \rho \Phi}{\partial t} + \frac{\partial (\rho U \Phi)}{\partial \xi} + \frac{\partial (\rho V \Phi)}{\partial \eta} \right\} d\nu' \right] dt$$

$$= \int_t^{t+\Delta t} \left[ \int_{\Delta \nu'} \left\{ \frac{\partial}{\partial \xi} \left[ \Gamma \left( \alpha' \Phi_\xi - \beta' \Phi_\eta \right) \right] + \frac{\partial}{\partial \eta} \left[ \Gamma \left( \gamma' \Phi_\eta - \beta' \Phi_\xi \right) \right] 

+ JS^\Phi(\xi, \eta) \right\} d\nu' \right] dt \quad (6.2.2)$$

Performing the space integral term by term we get:

$$\int_{\Delta \nu'} \frac{\partial (\rho \Phi)}{\partial t} d\nu' = \int \frac{\partial (\rho \Phi)}{\partial t} \Delta V = \frac{\partial (\rho \Phi)}{\partial t} J\Delta \xi \Delta \eta$$

where \(\frac{\partial (\rho \Phi)}{\partial t}\) is assumed to be constant over the whole control volume

$$\int_{\Delta \nu'} \left\{ \frac{\partial (\rho U \Phi)}{\partial \xi} \right\} d\nu' = \int_{\Delta \nu'} \left( \rho U \Phi \right) d\xi = \sum_{i=1}^{4} \rho U \Phi_i A_i^{\xi} = \left( \rho U \Phi \right)_e - \left( \rho U \Phi \right)_w \Delta \eta = \left( \rho U \Phi \right)_w^{\xi} \Delta \eta$$
where,
\[ \Delta A' \] is the surface bounding control volume \( \Delta V' \)
\[ dA^\xi \] is the projected area in the \( \xi \) direction

Similarly
\[
\int_{\Delta V'} \left\{ \frac{\partial (\rho V \Phi)}{\partial \eta} \right\} dv' = \{(\rho V \Phi)_n - (\rho V \Phi)_s\} \Delta \xi = \{(\rho V \Phi)_n\} \Delta \xi
\]
\[
\int_{\Delta V'} \frac{\partial}{\partial \xi} \left[ \int \left( \alpha' \Phi_\xi - \beta' \Phi_\eta \right) dv' \right]
\]
\[
= \int_{\Delta A'} \left[ \int \left( \frac{\Gamma}{\int} \alpha' \Phi_\xi - \beta' \Phi_\eta \right) dv' \right] dA^\xi
= \sum_{i=1}^{4} \left\{ \left[ \frac{\Gamma}{\int} \left( \alpha' \Phi_\xi - \beta' \Phi_\eta \right) \right] \right\} A_i^\xi
\]
\[
= \left\{ \left( \frac{\Gamma}{\int} \alpha' \Phi_\xi \right)_e - \left( \frac{\Gamma}{\int} \alpha' \Phi_\xi \right)_w \right\} \Delta \eta - \left\{ \left( \frac{\Gamma}{\int} \beta' \Phi_\eta \right)_e - \left( \frac{\Gamma}{\int} \beta' \Phi_\eta \right)_w \right\} \Delta \eta
\]
\[
= \left\{ \left( \frac{\Gamma}{\int} \alpha' \frac{\partial \Phi}{\partial \xi} \right)_e - \left( \frac{\Gamma}{\int} \alpha' \frac{\partial \Phi}{\partial \xi} \right)_w \right\} \Delta \eta - \left\{ \left( \frac{\Gamma}{\int} \beta' \frac{\partial \Phi}{\partial \eta} \right)_e - \left( \frac{\Gamma}{\int} \beta' \frac{\partial \Phi}{\partial \eta} \right)_w \right\} \Delta \eta
\]

Using central differencing scheme the above diffusion term becomes:
\[
\int_{\Delta V'} \frac{\partial}{\partial \xi} \left[ \int \left( \alpha' \Phi_\xi - \beta' \Phi_\eta \right) dv' \right]
\]
\[
= \left\{ \frac{\Gamma}{\int} \alpha' \left( \Phi_E - \Phi_p \right) - \frac{\Gamma}{\int} \alpha' \left( \Phi_p - \Phi_w \right) \right\} \Delta \eta - \left\{ \left( \frac{\Gamma}{\int} \beta' \frac{\partial \Phi}{\partial \eta} \right)_e - \left( \frac{\Gamma}{\int} \beta' \frac{\partial \Phi}{\partial \eta} \right)_w \right\} \Delta \eta
\]
\[
= \left\{ \frac{\Gamma}{\int} \alpha' \left( \Phi_E - \Phi_p \right) - \frac{\Gamma}{\int} \alpha' \left( \Phi_p - \Phi_w \right) \right\} \Delta \eta - \left\{ \left( \frac{\Gamma}{\int} \beta' \frac{\partial \Phi}{\partial \eta} \right)_e \Delta \eta - \left( \frac{\Gamma}{\int} \beta' \frac{\partial \Phi}{\partial \eta} \right)_w \right\}
\]

Similarly,
\[
\int_{\Delta V'} \frac{\partial}{\partial \eta} \left[ \int \left( \frac{\Gamma}{\int} \gamma' \Phi_\eta - \beta' \Phi_\xi \right) dv' \right]
\]
\[
= \left\{ \frac{\Gamma}{\int} \gamma' \left( \Phi_N - \Phi_p \right) - \frac{\Gamma}{\int} \gamma' \left( \Phi_p - \Phi_s \right) \right\} \Delta \xi - \left\{ \frac{\Gamma}{\int} \beta' \frac{\partial \Phi}{\partial \xi} \Delta \xi \right\}
\]
\[
\int_{\Delta V'} JS^\phi \, dv' = \bar{S} J \Delta \xi \Delta \eta
\]

In practical situations the source term may be a function of the dependent variable. In such cases the finite volume method approximates the source term by means of a linear form:
\[
\bar{S} J \Delta \xi \Delta \eta = (S_c + S_p \Phi_p) J \Delta \xi \Delta \eta
\]

Using the results of the space integrals in Eq. 6.2.2 we get
\[
\int_{t}^{t+\Delta t} \left[ \frac{\partial \rho \Phi}{\partial t} \Delta \xi \Delta \eta + \{(\rho U \Phi)_e - (\rho U \Phi)_w\} \Delta \eta + \{(\rho V \Phi)_n - (\rho V \Phi)_s\} \Delta \xi \right] dt \\
= \int_{t}^{t+\Delta t} \left\{ \left[ \frac{\Gamma \alpha'}{\Delta \xi} (\Phi_E - \Phi_p) - \frac{\Gamma \alpha'}{\Delta \xi} (\Phi_p - \Phi_w) \right] \Delta \eta \\
+ \left\{ \frac{\Gamma \gamma'}{\Delta \eta} (\Phi_N - \Phi_p) - \frac{\Gamma \gamma'}{\Delta \eta} (\Phi_p - \Phi_s) \right\} \Delta \xi \\
+ \left\{ - \left( \frac{\Gamma}{\Delta \eta} \frac{\partial \Phi}{\partial \eta} \right)_w^e - \left( \frac{\Gamma}{\Delta \xi} \frac{\partial \Phi}{\partial \xi} \right)_s^n \right\} + J(S_c + S_p \Phi_p) \Delta \xi \Delta \eta \right\} dt \quad (6.2.3)
\]

Now,
\[
\int_{t}^{t+\Delta t} J \frac{\partial (\rho \Phi)}{\partial t} \Delta \xi \Delta \eta \, dt = J \rho \Delta \xi \Delta \eta (\Phi - \Phi^o) \quad (6.2.4)
\]

Where the superscript ‘o’ refers to quantities at time \( t \) and the quantities at \( t + \Delta t \) is not superscripted. For carrying out the time integration of convective and diffusion terms, it is necessary to assume how \( \Phi \) vary with time. We may achieve this by means of a weighting parameter \( \theta \) between 0 and 1 and write the integral of \( \Phi \) as
\[
\int_{t}^{t+\Delta t} \Phi \, dt = [\theta \Phi + (1 - \theta) \Phi^o] \Delta t
\]

Using this idea and Eq. 6.2.4 in Eq. 6.2.3 we get
\[
J \rho \Delta \xi \Delta \eta (\Phi - \Phi^o) + \theta \left\{ \{(\rho U \Phi)_e - (\rho U \Phi)_w\} \Delta \eta + \{(\rho V \Phi)_n - (\rho V \Phi)_s\} \Delta \xi \right\} dt + \\
(1 - \theta) \left\{ \{(\rho U \Phi^o)_e - (\rho U \Phi^o)_w\} \Delta \eta + \{(\rho V \Phi^o)_n - (\rho V \Phi^o)_s\} \Delta \xi \right\} \Delta t \\
= \theta \left\{ \left[ \frac{\Gamma \alpha'}{\Delta \xi} (\Phi_E - \Phi_p) - \frac{\Gamma \alpha'}{\Delta \xi} (\Phi_p - \Phi_w) \right] \Delta \eta \\
+ \left\{ \frac{\Gamma \gamma'}{\Delta \eta} (\Phi_N - \Phi_p) - \frac{\Gamma \gamma'}{\Delta \eta} (\Phi_p - \Phi_s) \right\} \Delta \xi \right\} dt \\
+(1 - \theta) \left\{ \left[ \frac{\Gamma \alpha'}{\Delta \xi} (\Phi^o_E - \Phi^o_p) - \frac{\Gamma \alpha'}{\Delta \xi} (\Phi^o_p - \Phi^o_w) \right] \Delta \eta \\
+ \left\{ \frac{\Gamma \gamma'}{\Delta \eta} (\Phi^o_N - \Phi^o_p) - \frac{\Gamma \gamma'}{\Delta \eta} (\Phi^o_p - \Phi^o_s) \right\} \Delta \xi \right\} \Delta t \\
+ \left\{ \theta \left\{ - \left( \frac{\Gamma}{\Delta \eta} \frac{\partial \Phi}{\partial \eta} \right)_w^e - \left( \frac{\Gamma}{\Delta \xi} \frac{\partial \Phi}{\partial \xi} \right)_s^n \right\} + (S_c + S_p \Phi_p) \Delta \xi \Delta \eta \right\} + (1 - \theta) \left\{ - \left( \frac{\Gamma}{\Delta \eta} \frac{\partial \Phi}{\partial \eta} \right)_w^e - \left( \frac{\Gamma}{\Delta \xi} \frac{\partial \Phi}{\partial \xi} \right)_s^n \right\} \\
+ (S_c + S_p \Phi^o_p) \Delta \xi \Delta \eta \right\} \Delta t \quad (6.2.5)
\]
Depending on the value of $\theta$ the resulting schemes may be forward Euler (explicit), backward Euler (implicit) or crank Nicolson scheme. For the present case implicit scheme is utilized.

Using $\theta = 1$ in Eq. 6.2.5 and dividing by $\Delta t$ we get

$$\frac{\rho J_{\xi}}{\Delta t} \Delta \eta \left( \Phi - \Phi^{o} \right) + \{ \{ (\rho U \Phi)_{e} - (\rho U \Phi)_{w} \} \Delta \eta + \{ (\rho V \Phi)_{n} - (\rho V \Phi)_{s} \} \Delta \xi \}
$$

$$= \left\{ \left( \frac{\Gamma}{J} \alpha' \left( \Phi_{E} - \Phi_{P} \right) - \frac{\Gamma}{J} \alpha' \left( \Phi_{P} - \Phi_{W} \right) \right) \Delta \eta + \left( \frac{\Gamma}{J} \gamma' \left( \Phi_{N} - \Phi_{P} \right) - \frac{\Gamma}{J} \gamma' \left( \Phi_{P} - \Phi_{S} \right) \right) \Delta \xi \right\}$$

$$+ \left\{ - \left( \frac{\Gamma}{J} \beta' \frac{\partial \Phi}{\partial \eta} \Delta \eta \right)_{w}^{e} - \left( \frac{\Gamma}{J} \beta' \frac{\partial \Phi}{\partial \xi} \Delta \xi \right)_{s}^{n} + J (S_{c} + S_{p} \Phi_{P}) \Delta \xi \Delta \eta \right\}$$

$$J \frac{\rho J_{\xi} \Delta \eta}{\Delta t} \left( \Phi - \Phi^{o} \right) + \{ \{ (\rho U \Phi)_{e} - (\rho U \Phi)_{w} \} \Delta \eta + \{ (\rho V \Phi)_{n} - (\rho V \Phi)_{s} \} \Delta \xi \}
$$

$$= \left\{ \left( \frac{\Gamma}{J} \alpha' \left( \Phi_{E} - \Phi_{P} \right) - \frac{\Gamma}{J} \alpha' \left( \Phi_{P} - \Phi_{W} \right) \right) \Delta \eta \right\}$$

$$+ \left\{ \left( \frac{\Gamma}{J} \gamma' \left( \Phi_{N} - \Phi_{P} \right) - \frac{\Gamma}{J} \gamma' \left( \Phi_{P} - \Phi_{S} \right) \right) \Delta \xi + S^{\xi, \eta} \right\} \quad (6.2.6)$$

Where

$$S^{\xi, \eta} = \left\{ - \left( \frac{\Gamma}{J} \beta' \frac{\partial \Phi}{\partial \eta} \Delta \eta \right)_{w}^{e} - \left( \frac{\Gamma}{J} \beta' \frac{\partial \Phi}{\partial \xi} \Delta \xi \right)_{s}^{n} + J (S_{c} + S_{p} \Phi_{P}) \Delta \xi \Delta \eta \right\}$$

is the new source term.
Now we may write using Fig. 6.4 and 6.5:

\[
\left( \frac{\partial x}{\partial \xi} \right)_e \approx \frac{x_E - x_p}{\xi_E - \xi_p}, \quad \left( \frac{\partial y}{\partial \eta} \right)_e \approx \frac{y_E - y_p}{\eta_E - \eta_p}
\]

\[
\left( \frac{\partial x}{\partial \eta} \right)_e \approx \frac{x_n - x_s}{(\eta_n - \eta_s)_e}, \quad \left( \frac{\partial y}{\partial \xi} \right)_e \approx \frac{y_n - y_s}{(\eta_n - \eta_s)_e}
\]

Using the results the Jacobian at face e becomes:

\[
J_e = \frac{\partial x \partial y}{\partial \xi \partial \eta} - \frac{\partial x \partial y}{\partial \eta \partial \xi}
\]

\[
\approx \frac{(x_E - x_p)(y_n - y_s)_e}{(\xi_E - \xi_p)(\eta_n - \eta_s)_e} - \frac{(x_n - x_s)(y_E - y_p)}{(\eta_n - \eta_s)_e(\xi_E - \xi_p)}
\]

\[
\approx \frac{(x_E - x_p)(y_n - y_s)_e - (x_n - x_s)(y_E - y_p)}{(\xi_E - \xi_p)(\eta_n - \eta_s)_e} \quad \text{(6.2.7)}
\]

**Diffusion terms:**

Using Eq. 6.2.7 in the diffusion terms of Eq. 6.2.6 we get

\[
\left( \frac{\Gamma \alpha'}{\int \Delta \xi \Delta \eta} \right)_e (\Phi_E - \Phi_p)
\]

\[
\approx \frac{\Gamma(\xi_E - \xi_p)(\eta_n - \eta_s)_e(\eta_n - \eta_s)_e}{\{x_E - x_p\}y_n - y_s)_e - \{x_n - x_s\}(y_E - y_p)}\left( \frac{y_n^2 + x_n^2}{(\xi_E - \xi_p)} \right)(\Phi_E - \Phi_p)
\]

\[
= \frac{\Gamma(\xi_E - \xi_p)(\eta_n - \eta_s)_e^2}{\{x_E - x_p\}y_n - y_s)_e - \{x_n - x_s\}(y_E - y_p)}\left( \frac{y_n^2}{(\eta_n - \eta_s)_e} + \frac{x_n^2}{(\eta_n - \eta_s)_e} \right)(\Phi_E - \Phi_p)
\]

\[
= \frac{\Gamma(y_n - y_s)_e^2 + (x_n - x_s)_e^2}{\{x_E - x_p\}y_n - y_s)_e - \{x_n - x_s\}(y_E - y_p)}(\Phi_E - \Phi_p)
\]
\[ D_e = (\Phi_E - \Phi_p) \] 

Where

\[ D_e = \frac{\Gamma \{(y_n - y_s)^2_e + (x_n - x_s)^2_e\}}{(x_E - x_p)(y_n - y_s)_e - (x_n - x_s)_e(y_E - y_p)} \]

Similarly,

\[ \left( \frac{\Gamma \alpha'}{\Delta \xi \Delta \eta} \right)_w (\Phi_p - \Phi_W) \approx \frac{\Gamma \{(y_n - y_s)^2_w + (x_n - x_s)^2_w\}(\Phi_p - \Phi_W)}{(x_p - x_w)(y_n - y_s)_w - (x_n - x_s)_w(y_p - y_w)} = D_w(\Phi_p - \Phi_W) \] 

Where

\[ D_w = \frac{\Gamma \{(y_n - y_s)^2_w + (x_n - x_s)^2_w\}}{(x_p - x_w)(y_n - y_s)_w - (x_n - x_s)_w(y_p - y_w)} \]

\[ \left( \frac{\Gamma \gamma'}{\Delta \eta \Delta \xi} \right)_n (\Phi_N - \Phi_p) \approx \frac{\Gamma \{(x_e - x_w)^2_n + (y_e - y_w)^2_n\}(\Phi_N - \Phi_p)}{-(x_N - x_p)(y_e - y_w)_n + (x_e - x_w)_n(y_N - y_p)} = D_n(\Phi_N - \Phi_p) \]

Where

\[ D_n = \frac{\Gamma \{(x_e - x_w)^2_n + (y_e - y_w)^2_n\}}{-(x_N - x_p)(y_e - y_w)_n + (x_e - x_w)_n(y_N - y_p)} \]

\[ \left( \frac{\Gamma \gamma'}{\Delta \eta \Delta \xi} \right)_s (\Phi_p - \Phi_s) \approx \frac{\Gamma \{(x_e - x_w)^2_s + (y_e - y_w)^2_s\}(\Phi_p - \Phi_s)}{-(x_p - x_s)(y_e - y_w)_s + (x_e - x_w)_s(y_p - y_s)} = D_s(\Phi_p - \Phi_s) \]

Where

\[ D_s = \frac{\Gamma \{(x_e - x_w)^2_s + (y_e - y_w)^2_s\}}{-(x_p - x_s)(y_e - y_w)_s + (x_e - x_w)_s(y_p - y_s)} \]

**Convective terms:**

\[ (\rho U \Phi)_e(\Delta \eta)_e = \rho \left( u \eta - v \eta \right)_e (\Delta \eta)_e \Phi_e \]

\[ \approx \rho \left( u_e \frac{(y_n - y_s)_e}{(\eta_n - \eta_s)_e} - v_e \frac{(x_n - x_s)_e}{(\eta_n - \eta_s)_e} \right) (\eta_n - \eta_s)_e \Phi_e \]

\[ = \rho (u_e(y_n - y_s)_e - v_e(x_n - x_s)_e) \Phi_e \]
\[ = F_e \Phi_e \quad \text{(6.2.12)} \]

Where

\[ F_e = \rho \{u_e(y_n - y_s)_e - v_e(x_n - x_s)_e\} \]

Similarly,

\[ (\rho U \Phi \Delta \eta)_w = F_w \Phi_w, (\rho V \Phi \Delta \xi)_n = F_n \Phi_n, (\rho V \Phi \Delta \xi)_s = F_s \Phi_s \quad \text{(6.2.13)} \]

where

\[ F_w = \rho \{u_w(y_n - y_s)_w - v_w(x_n - x_s)_w\}, F_n = \rho \{-u_n(y_e - y_w)_n + v_n(x_e - x_w)_n\} \]

\[ F_s = \rho \{-u_s(y_e - y_w)_s + v_s(x_e - x_w)_s\} \]

**Source terms:**

**Cross diffusion terms**

\[ \left( \Gamma \frac{\beta'}{\beta} \frac{\partial \Phi}{\partial \eta} \Delta \eta \right)_e \]

\[ \approx \frac{\Gamma(\xi_E - \xi_P)(\eta_n - \eta_s)_e}{(x_E - x_P)(y_n - y_s)_e - (x_n - x_s)_e(y_E - y_P)} \left\{ \frac{(x_E - x_P)(x_n - x_s)_e}{\xi_E - \xi_P} (\eta_n - \eta_s)_e \right\} \]

\[ + \left( \frac{y_E - y_P}{\xi_E - \xi_P} \right) \frac{(y_n - y_s)_e}{(\eta_n - \eta_s)_e} \frac{(\Phi_n - \Phi_s)_e}{(\eta_n - \eta_s)_e} \]

\[ = \frac{\Gamma[(x_E - x_P)(x_n - x_s)_e + (y_E - y_P)(y_n - y_s)_e]}{(x_E - x_P)(y_n - y_s)_e - (x_n - x_s)_e(y_E - y_P)} (\Phi_n - \Phi_s)_e \quad \text{(6.2.14)} \]

Where

\[ N_e = \frac{\Gamma[(x_E - x_P)(x_n - x_s)_e + (y_E - y_P)(y_n - y_s)_e]}{(x_n - x_s)_e(y_E - y_P) - (x_E - x_P)(y_n - y_s)_e} \]

Similarly,

\[ \left( \Gamma \frac{\beta'}{\beta} \frac{\partial \Phi}{\partial \eta} \Delta \eta \right)_w \]

\[ = N_w(\Phi_n - \Phi_s)_w \quad \text{(6.2.15)} \]

where

\[ N_w = \frac{\Gamma[(x_P - x_w)(x_n - x_s)_w + (y_P - y_w)(y_n - y_s)_w]}{(x_n - x_s)_w(y_P - y_w) - (x_P - x_w)(y_n - y_s)_w} \]

\[ \left( \Gamma \frac{\beta'}{\beta} \frac{\partial \Phi}{\partial \xi} \Delta \xi \right)_e = N_e(\Phi_n - \Phi_s)_e - N_w(\Phi_n - \Phi_s)_w \quad \text{(6.2.16)} \]

Similarly,

\[ \left( \Gamma \frac{\beta'}{\beta} \frac{\partial \Phi}{\partial \xi} \Delta \xi \right)_s \]
Where,

\[ N_n = \frac{\Gamma[(x_N - x_p)(y_N - y_p)(y_N - y_w)n]}{(x_N - x_p)(y_N - y_w)n - (x_e - x_w)n(y_N - y_p)} \]

\[ N_s = \frac{\Gamma[(x_p - x_s)(x_e - x_w)s + (y_p - y_s)(y_e - y_w)s]}{(x_p - x_s)(y_e - y_w)s - (x_e - x_w)s(y_p - y_s)} \]

\[(S_c + S_p \Phi_p)\Delta \xi \Delta \eta \]

\[ = (S_c + S_p \Phi_p) \left[ \frac{(x_e - x_w)e(y_n - y_s) - (x_n - x_s)(y_e - y_w)}{(\xi_e - \xi_w)(\eta_n - \eta_s)} \right] \]

\[(S_c + S_p \Phi_p)[(x_e - x_w)(v_n - v_s) - (x_n - x_s)(v_e - v_w)] \]

\[ = (S_c + S_p \Phi_p) \delta V \quad (6.2.18) \]

Where

\[ \delta V = [(x_e - x_w)(y_n - y_s) - (x_n - x_s)(y_e - y_w)] \]

is the volume(area in 2D) of the control volume around the node P.

\[ x_e \]

is the x coordinate of midpoint joining the two corner points at the east face of the cell.

That is

\[ x_e = \frac{x_{ne} + x_{se}}{2} \]

Similarly the other quantities in \( \delta V \) may be defined.

In terms of coordinates of corner points of the cell, by taking the magnitude of the cross product of diagonal vectors of a control volume it is easy to show that the volume may be also written as

\[ \delta V = \frac{1}{2}[(x_{ne} - x_{sw})(y_{nw} - y_{se}) - (y_{ne} - y_{sw})(x_{nw} - x_{se})] \]

The above expression of volume may also be derived by putting the definition of terms such as

\[ x_e = \frac{x_{ne} + x_{se}}{2} \]

in to \( \delta V = [(x_e - x_w)(y_n - y_s) - (x_n - x_s)(y_e - y_w)] \)

So, \( \delta V = [(x_e - x_w)(y_n - y_s) - (x_n - x_s)(y_e - y_w)] \)

\[ = \frac{1}{2}[(x_{ne} - x_{sw})(y_{nw} - y_{se}) - (y_{ne} - y_{sw})(x_{nw} - x_{se})] \]

The source term in equation for \( k \) and \( \varepsilon \) require special attention. From section 3.4, it is evident that the turbulent kinetic energy transport equation has additional source terms \( G - \varepsilon \). This may be integrated to yield:
\[ \int_{\Delta V'} (G - \varepsilon) J dV' \approx (G_p - \varepsilon_p) J \Delta \xi \Delta \eta = (G_p - \varepsilon_p) \delta V = G_p \delta V - \frac{\varepsilon_p^*}{k_p} \delta V \]

Where \textit{asterisk} is used on quantities computed from previous iteration

Similarly for turbulent dissipation, the source term is written as

\[ \int_{\Delta V'} \frac{\varepsilon}{k} (C_{\varepsilon 1} G - C_{\varepsilon 2} \varepsilon) J dV' \approx C_{\varepsilon 1} \frac{\varepsilon_p^*}{k_p} G_p \delta V - \frac{\varepsilon_p^*}{k_p} \varepsilon_p \]

**Unsteady terms**

\[ \int \frac{\rho \Delta \xi \Delta \eta}{\Delta t} (\Phi_p - \Phi_p^o) = \frac{\rho \delta V}{\Delta t} (\Phi - \Phi^o) \]  

(6.2.19)

Using Eq. 6.2.8-6.2.19 in Eq. 6.2.6 we get

\[ \frac{\rho \delta V}{\Delta t} (\Phi_p - \Phi_p^o) + [F_e \Phi_e - F_w \Phi_w + F_n \Phi_n - F_s \Phi_s] \]

\[ = D_e (\Phi_e - \Phi_p) - D_w (\Phi_p - \Phi_w) + D_n (\Phi_n - \Phi_p) - D_s (\Phi_p - \Phi_s) + S^{\Phi'} \]  

(6.2.20)

Where

\[ S^{\Phi'} = N_e (\Phi_e - \Phi_s) + N_w (\Phi_w - \Phi_p) + N_n (\Phi_n - \Phi_s) + (S_c + S_p \Phi_p) \delta V \]

In the convective terms the values of \( \Phi \) are at face ("e", "w", "n", "s") of the control volume. These need to be expressed in terms of nodal values by means of suitable interpolation schemes.

For convective terms, CDS results in negative coefficients for the downstream neighbor nodes if convection dominates strongly over diffusion. The negative coefficients may cause unphysical oscillations in the solution. Sometimes the solution does not converge. The UDS, on the other hand, is unconditionally stable. However UDS introduces numerical errors known as "artificial" or "false" diffusion. One can chose the option between using CDS, UDS or a combination of the two. The combination can be achieved via the so called \textit{deferred correction} approach as discussed in chapter 5, and also blending the two schemes by a blending factor of \( \beta'' \) (value of \( \beta'' \) is between 0 and 1) as follows

\[ [F_e \Phi_e - F_w \Phi_w + F_n \Phi_n - F_s \Phi_s]_{UDS} \]

\[ + \psi \{ [F_e \Phi_e - F_w \Phi_w + F_n \Phi_n - F_s \Phi_s]_{CDS} \]

\[ - [F_e \Phi_e - F_w \Phi_w + F_n \Phi_n - F_s \Phi_s]_{UDS} \]  

(6.2.21)

Using Eq. 6.2.21 in Eq. 6.2.20 we get
\[
\frac{\rho \delta V}{\Delta t} (\Phi_p - \Phi_p^0) + [F_e \Phi_e - F_w \Phi_w + F_n \Phi_n - F_s \Phi_s]_{\text{UDS}} \\
= D_e (\Phi_e - \Phi_p) - D_w (\Phi_p - \Phi_w) + D_n (\Phi_N - \Phi_p) - D_s (\Phi_p - \Phi_s) + S^\phi' \\
+ \psi \{(F_e \Phi_e - F_w \Phi_w + F_n \Phi_n - F_s \Phi_s)_{\text{UDS}} - (F_e \Phi_e - F_w \Phi_w + F_n \Phi_n - F_s \Phi_s)_{\text{CDS}}\}
\]  
(6.2.22)

\[
\frac{\rho \delta V}{\Delta t} (\Phi_p - \Phi_p^0) + [F_e \Phi_e - F_w \Phi_w + F_n \Phi_n - F_s \Phi_s]_{\text{UDS}} \\
= D_e (\Phi_e - \Phi_p) - D_w (\Phi_p - \Phi_w) + D_n (\Phi_N - \Phi_p) - D_s (\Phi_p - \Phi_s) + S^\phi
\]  
(6.2.23)

Where

\[S^\phi = S^\phi' + \psi \{(F_e \Phi_e - F_w \Phi_w + F_n \Phi_n - F_s \Phi_s)_{\text{UDS}} - (F_e \Phi_e - F_w \Phi_w + F_n \Phi_n - F_s \Phi_s)_{\text{CDS}}\}\]

The first term is known as the implicit term and the terms multiplied by \(\psi\) are called explicit terms. The terms multiplied by \(\psi\) become part of the source term and are determined explicitly from previous iterations as discussed in chapter 5. The same is done with the cross diffusion terms: these are also evaluated on each CV face, summed up and added to the source term.

The upwind (UDS) and central difference (CDS) schemes are described below:

**UDS**

When the flow is in the positive direction, \(u_w > 0, u_e > 0, u_n > 0, u_s > 0, F_w > 0, F_e > 0, F_n > 0, F_s > 0\)

\[\Phi_w = \Phi_w, \quad \Phi_e = \Phi_p, \quad \Phi_s = \Phi_s, \quad \Phi_n = \Phi_p\]

When the flow is in the negative direction, \(u_w < 0, u_e < 0, u_n < 0, u_s < 0, F_w < 0, F_e < 0, F_n < 0, F_s < 0\)

\[\Phi_w = \Phi_p, \quad \Phi_e = \Phi_E, \quad \Phi_s = \Phi_p, \quad \Phi_n = \Phi_N\]

**CDS**

The central differencing approximations are:

\[\Phi_w = \frac{\Phi_w + \Phi_p}{2}, \quad \Phi_e = \frac{\Phi_e + \Phi_p}{2}, \quad \Phi_n = \frac{\Phi_N + \Phi_p}{2}, \quad \Phi_s = \frac{\Phi_s + \Phi_p}{2}\]

Using UDS for the implicit convective term Eq. 6.2.23 may be written in the following algebraic form:

\[a_p \Phi_p = a_w \Phi_w + a_e \Phi_E + a_N \Phi_N + a_s \Phi_s + S\]  
(6.2.24)

\[a_p \Phi_p = \sum_{nb} a_{nb} \Phi_{nb} + S\]  
(6.2.25)
where
\[
\sum_{n} a_{nb} \Phi_{nb} = a_W \Phi_W + a_E \Phi_E + a_N \Phi_N + a_S \Phi_S
\]
\[
a_p = a_W + a_E + a_N + a_S + a_p^0 - (S_p) \delta V + [F_e - F_w + F_n - F_s]
\]
\[
= a_E + a_N + a_S + a_p^0 - (S_p) \delta V
\]
The term into brackets corresponds to the continuity equation. After each outer iteration steps, the mass fluxes are corrected so that the bracketed term vanishes identically and, therefore, are not considered.
\[
a_p^0 = \frac{\rho \delta V}{\Delta t}
\]
\[
S = N_e(\Phi_e - \Phi_s)_e - N_w(\Phi_e - \Phi_s)_w + N_n(\Phi_e - \Phi_s)_n - N_s(\Phi_e - \Phi_s)_s
\]
\[
+ \psi \{[F_e \Phi_e - F_w \Phi_w + F_n \Phi_n - F_s \Phi_s]_{UDS} - [F_e \Phi_e - F_w \Phi_w + F_n \Phi_n - F_s \Phi_s]_{CDS}\}
\]
\[
+ (S_e) \delta V + \text{pterm} + a_p^0 \Phi_p^0
\]
\[
\text{pterm} \begin{cases} 
-(P_e - P_w)(y_e - y_s) + (P_n - P_s)(y_e - y_w) & \text{for } x - \text{momentum} \\
-(P_n - P_s)(x_e - x_w) + (P_e - P_w)(x_e - x_s) & \text{for } y - \text{momentum} \\
0 & \text{for all other equations}
\end{cases}
\]

<table>
<thead>
<tr>
<th>(a_W)</th>
<th>(a_E)</th>
<th>(a_N)</th>
<th>(a_S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_w + \max (F_w, 0))</td>
<td>(D_E + \max (0, -F_e))</td>
<td>(D_n + \max (0, -F_n))</td>
<td>(D_s + \max (F_s, 0))</td>
</tr>
</tbody>
</table>

\[
D_e = \frac{\Gamma\{(x_e - x_s)^2_e + (y_e - y_s)^2_s\}}{(x_E - x_p)(y_e - y_s) + (x_n - x_s)(y_e - y_p)}
\]
\[
D_w = \frac{\Gamma\{(x_n - x_s)^2_w + (y_n - y_s)^2_s\}}{(x_p - x_w)(y_n - y_s) + (x_n - x_s)(y_p - y_w)}
\]
\[
D_n = \frac{\Gamma\{(x_e - x_w)^2_e + (y_e - y_w)^2_w\}}{-(x_n - x_p)(y_e - y_w) + (x_e - x_w)(y_n - y_p)}
\]
\[
D_s = \frac{\Gamma\{(x_e - x_s)^2_e + (y_e - y_s)^2_e\}}{-(x_p - x_s)(y_n - y_s) + (x_e - x_s)(y_p - y_s)}
\]
\[
F_e = \rho\{u_e(y_n - y_s)_e - v_e(x_n - x_s)_e\}, \quad F_w = \rho\{u_w(y_n - y_s)_w - v_w(x_n - x_s)_w\}
\]
\[
F_n = \rho\{-u_n(y_e - y_w)_n + v_n(x_e - x_w)_n\}, \quad F_s = \rho\{-u_s(y_e - y_w)_s + v_s(x_e - x_w)_s\}
\]
\[
N_e = \frac{\Gamma\{(x_E - x_p)(x_n - x_s)_e + (y_E - y_p)(y_n - y_s)_e\}}{(x_n - x_s)_e(y_E - y_p) + (x_E - x_p)(y_n - y_s)_e}
\]
\[ N_w = \Gamma \left\{ \frac{(x_p - x_w)(x_n - x_s)w + (y_p - y_w)(y_n - y_s)w}{(x_n - x_s)_w(y_p - y_w) - (x_p - x_w)(y_n - y_s)_w} \right\} \]

\[ N_n = \Gamma \left\{ \frac{(x_N - x_p)(x_e - x_w)_n + (y_N - y_p)(y_e - y_w)_n}{(x_N - x_p)(y_e - y_w)_n - (x_e - x_w)_n(y_N - y_p)} \right\} \]

\[ N_s = \Gamma \left\{ \frac{(x_p - x_s)(x_e - x_w)_s + (y_p - y_s)(y_e - y_w)_s}{(x_p - x_s)(y_e - y_w)_s - (x_e - x_w)_s(y_p - y_s)} \right\} \]

\[ \delta V = [(x_e - x_w)(y_n - y_s) - (x_n - x_s)(y_e - y_w)] \]

\[ = \frac{1}{2} \left[ (x_{ne} - x_{sw})(y_{nw} - y_{se}) - (y_{ne} - y_{sw})(x_{nw} - x_{se}) \right] \]
CHAPTER 7
SOLUTION STRATEGY

7.1 Main Difficulty
In previous chapters the procedure for discretizing the transport equation for the general variable \( \Phi \) was formulated. However examination of the coefficients of the discretized equations reveal that, they contain velocity terms which itself must be generated as a solution of the problem. That is, the velocity field is not known and has to be computed by solving the set of Navier-Stokes equations. This can be dealt with by a picard type iterative procedure. For incompressible flows the task is also complicated by the strong coupling that exist between pressure and velocity and by the fact that pressure does not appear as a primary variable in either the momentum or continuity equations. The density does appear in the continuity equation, but for incompressible flows, the density is unrelated to the pressure and cannot be used instead. Thus, if we want to use sequential, iterative methods, it is necessary to find a way to introduce the pressure into the continuity equation. Methods which use pressure as the solution variable are called \textit{pressure-based} methods. They are very popular in the incompressible flow community. In the present work, we will use the SIMPLE algorithm on collocated grids. The idea behind the algorithm is similar to that used by Patankar and Spalding (1992) but the arrangement of the vector and scalar variables is different. Unlike the method of Patankar and Spalding (1992), in the collocated arrangement, the scalar and vectors are both stored at cell centres. The problem of a spurious pressure field arising from such an arrangement is solved by using the momentum interpolation method suggested by Rhie and Chow (1983). At the beginning of the chapter however, method used to solve linear system of equations by incomplete LU decomposition of Stone (1968) will be discussed first.

7.2 Incomplete LU Decomposition: Stone’s Method
The generic transport equation given by Eq. 6.2.4 may be written as:
\[
a_p \Phi_p - (a_W \Phi_W + a_E \Phi_E + a_N \Phi_N + a_S \Phi_S) = S
\]
Which after picard type linearization becomes amenable to solution techniques applied to other linear systems.
Eq. 7.2.1 may be written in the following matrix form
\[
[A][\Phi] = [S]
\]
Where \( [\Phi] \) denotes the field of the unknown variables arranged in vector form which in this case may be \( u, v, p', k, \epsilon \), \( [S] \) is a similar vector containing source terms and \( [A] \) is the coefficient matrix. The latter is a square matrix with dimensions \( K \times K \) where \( K \) is the total
number of computational points. In two-dimensional flows, each row contains of $A$ five (for orthogonal) nonzero coefficients or nine nonzero coefficients (for nonorthogonal) coefficients. However, by treating the cross derivative terms explicitly, the coefficient matrix gets reduced to five diagonal form. The arrangement of the nonzero coefficients in $[A]$ depend on the way the vector $\{\Phi\}$ is formed. If the vector $\{\Phi\}$ is formed such that the points follow each other along one line, line after line in given order (for example $j = 1$ to $N_j$ for $i = 1$ and in the same way for $i = 2$ to $N_i$) matrix $[A]$ will have the form shown in Fig. 7.1. $N_i$ and $N_j$ denotes the number of nodes in $i$ and $j$ direction respectively.

![Fig. 7.1: Schematic representation of Eq. 7.2.2](image)

For a particular arrangement of $\Phi$,

$[A] \{\Phi\} = \{S\}$

non-zero diagonals

From Fig. 7.1 it is evident that $A$ is a sparse matrix which can be decomposed in to lower triangular matrix $L$ and upper triangular matrix $U$ in such a way that both $L$ and $U$ have nonzero on diagonals on which $A$ has nonzero diagonals. If this is done, unlike $LU$ decomposition, the product of $L$ and $U$ does not give back $A$ exactly. The product of $L$ and $U$, whose elements are constructed in the manner indicated above results in two additional diagonals corresponding nodes NW and SE or NE and SW, depending on the ordering of the vector $\Phi$. For the ordering used in the present case, the extra diagonals correspond to the nodes NW and SE. Also the variables are to be stored in one-dimensional arrays. The conversion between the grid locations, compass notation, and storage locations is indicated in Table 7.1 and illustrated by Fig. 7.2.
From the discussion above, it is evident that in the ILU decomposition

\[ LU = M \]

But

\[ LU = M \approx A \]

\[ LU = M = A + N \]  \hspace{1cm} (7.2.3)

Where the structure of \(L, U, M\) are shown in Fig. 7.3

To make these matrices unique, every element on the main diagonal of \(U\) is set to unity. Thus five sets of elements (three in \(L\) and two in \(U\)) need to be determined. For the matrices of the form shown in Fig. 7.3, the rules of matrix multiplication gives the elements of \(L\) and \(U\) for the \(lth\) row:

\[ M_W^l = L_W^l \]  \hspace{1cm} (7.2.4a)

\[ M_{NW}^l = L_W^l U_W^{l-N_j} \]  \hspace{1cm} (7.2.4b)

\[ M_S^l = L_S^l \]  \hspace{1cm} (7.2.4c)

\[ M_P^l = L_W^l U_E^{l-N_j} + L_S^l U_N^{l-1} + L_P^l \]  \hspace{1cm} (7.2.4d)

\[ M_N^l = U_N^l L_P^l \]  \hspace{1cm} (7.2.4e)

\[ M_{SE}^l = L_S^l U_E^{l-1} \]  \hspace{1cm} (7.2.4f)
\[ M_E^1 = U_E^1 L_P^1 \]  

(7.2.4g)

\( L \) and \( U \) have to be selected in such a way that \( M \) is as good an approximation to \( A \) as possible. At minimum, \( N \) must contain the two additional diagonals of \( M \) that correspond to zero diagonals of \( A \). An obvious choice is to let \( N \) have nonzero elements, on these two diagonals and force the other diagonals of \( M \) to equal the corresponding diagonals of \( A \) like the standard ILU method. Unfortunately, this method converges slowly. This was realized by Stone (1968) who asserted that convergence could be improved by allowing \( N \) to have nonzero elements on the diagonals corresponding to all seven nonzero diagonals of \( LU \). This method is most easily understood by considering the following vector \( M\Phi \):

\[(M\Phi)_p = M_p\phi_p + M_E\phi_E + M_W\phi_W + M_S\phi_S + M_N\phi_N + M_{NW}\phi_{NW} + M_{SE}\phi_{SE} \]  

(7.2.5)

The last two terms are the ‘extra ones’ arising after \( L \) and \( U \) are multiplied and each term in this equation correspond to a diagonal of \( LU = M \). The matrix \( N \) must contain these two ‘extra’ diagonals of \( M \), and we want to choose the elements on the remaining diagonals of \( N \) so that \( N\Phi \approx 0 \) and thereby fulfilling the requirement of making \( M \) as good an approximation to \( A \) as possible. That is,

\[ N_p\phi_p + N_E\phi_E + N_W\phi_W + N_S\phi_S + N_N\phi_N + M_{NW}\phi_{NW} + M_{SE}\phi_{SE} \approx 0 \]  

(7.2.6)

This requires that the contribution of the two ‘extra terms’ are nearly canceled by the contribution of the other diagonals. In other words, Eq. 7.2.6 reduces to the following:

\[ M_{NW}(\phi_{NW} - \phi_{NW}^*) + M_{SE}(\phi_{SE} - \phi_{SE}^*) \approx 0 \]  

(7.2.7)

Since the equations of the present case approximate elliptic partial differential equations, the solution is expected to be smooth. This being so, \( \phi_{NW}^* \) and \( \phi_{SE}^* \) can be approximated in terms of values of \( \Phi \) at nodes corresponding to diagonals of \( A \) by truncated Taylor series expansion which yields after little manipulation:

\[ \phi_{NW}^* \approx \alpha(\phi_W + \phi_N - \phi_p) \]  

(7.2.8)

\[ \phi_{SE}^* \approx \alpha(\phi_S + \phi_E - \phi_p) \]  

(7.2.9)

It must be noted that, the error in the Taylor series implies that \( M \) is still an approximation of \( A \) and thereby iterations are necessary to diminish the error. If \( \alpha = 1 \), one gets the exact result obtained from truncated Taylor series approximation which are second order accurate interpolations but Stone found that stability requires \( \alpha < 1 \). If Eq. 7.2.8 and Eq. 7.2.9 are inserted to 7.2.7 and the result is equated with 7.2.6, we obtain all elements of \( N \) as a linear combination of \( M_{NW} \) and \( M_{SE} \). The elements of \( M \) given by Eq. 7.2.4a-7.2.4g can now be set equal to the sum of elements of \( A \) and \( N \) to yield the elements of the upper and lower triangular matrices \( L \) and \( U \) for the \( l \)th row as follows:
\( L^l_W = \frac{A^l_W}{1 + \alpha U^{l-Nj}_N} \)  
(7.2.10)

\( L^l_S = \frac{A^l_S}{1 + \alpha U^{l-1}_E} \)  
(7.2.11)

\[ L^l_p = A^l_p + \alpha \left( L^l_W U^{l-Nj}_N + L^l_S U^{l-1}_E \right) - L^l_W U^{l-Nj}_E - L^l_S U^{l-1}_N \]  
(7.2.12)

**Fig. 7.3: \( U, L \) and \( M \) matrices**

---

- non-zero diagonals
- additional non-zero diagonals in \( M \)
The iteration begins by guessing the value of $\Phi$ for all nodes, which we call $\Phi_k$ at the $k^{th}$ inner iteration. The elements of matrix $A$ needs to be calculated next and hence the elements of $L$ and $U$ are obtained from Eq. 7.2.10-7.2.14. Assuming the value $\Phi^k$ is not correct, there exists some residual $R^k$ at the $k^{th}$ inner iteration such that:

$$R^k = \{S\} - [A]\{\Phi^k\}$$  \hspace{1cm} (7.2.15)

Using Eq. 7.2.15 the residual needs to be computed next followed by computation of $L^1\text{Norm}$ given by

$$L^1\text{Norm} = \sum_{i=1}^{K} |R^i|$$

computation of sum of absolute values of residual aids to monitor convergence.

Now the residual may also be written as

$$[A]\{\Phi^{k+1}\} - [A]\{\Phi^k\} = R^k$$

Where $\delta^k = \Phi^{k+1} - \Phi^k$ is the correction

Eq. 7.2.15 may be written as

$$LU\{\delta^k\} = R^k$$  \hspace{1cm} (7.2.16)

$$U\{\delta^k\} = L^{-1}R^k$$  \hspace{1cm} (7.2.17)

With residual vector and elements of $L$ known from Eq. 7.2.15 and 7.2.10-7.2.12, one can solve Eq. 7.2.18 for $U\{\delta^k\}$ by forward substitution. Thus we have

$$U\{\delta^k\} = Y^k$$  \hspace{1cm} (7.2.18)

Where $Y^k = L^{-1}R^k$

Since $L$ is a convenient lower triangular matrix, $Y^k$ is easily obtained by forward substitution, marching in the order of increasing $l$ for the $l^{th}$ row as:

$$Y^l = \frac{R^l - L^l_S R^{l-1} - L^l_W R^{l-N_j}}{L^l_p}$$

Now, $U$ being an upper triangular matrix, whose main diagonal elements are 1 and other diagonal elements are known from Eq. 7.2.13-7.2.14, Eq. 7.2.18 can be solved by backward substitution, to yield the correction $\delta^k$ as follows:
\[
\{\delta^k\} = U^{-1}Y^k
\]  

(7.2.19)

Since backward substitution is used, this equation is to be solved by marching in the order of decreasing \(l\) as:

\[
\delta^l = Y^l - U^l_N \delta^{l+1} - U^l_P \delta^{l+N_j}
\]

Once \(\delta^k\) is known, the updated or corrected value of \(\Phi\) is obtained as:

\[
\Phi^{k+1} = \Phi^k + \delta^k
\]

If convergence does not take place, the above steps are to be repeated using \(\Phi^{k+1}\) as the ‘guessed value’ for the next inner iteration. In case of nonlinear equations such as momentum equations, the elements of matrix \(A\) are functions of velocity field. Thus an outer iteration is also needed. During inner iteration solutions are obtained for each variable for \textbf{given coefficients}, and during outer iterations, the coefficients are updated. The whole process is repeated until there is no significant variation in coefficients or dependent variables.

The summary of the algorithm is given below:

\textit{Evaluate incomplete LU decomposition of matrix} \(A\)

\[
A\Phi = (M - N)\Phi = (LU - N)\Phi = S
\]

\[
M\Phi^{k+1} = M\Phi^k - (A\Phi^k - S) = M\Phi^k + R^k, \text{ such that } ||M|| \gg ||N||
\]

\[
M(\Phi^{k+1} - \Phi^k) = M\delta^k = LU\delta^k = R^k
\]

\[
LU\delta^k = L(U\delta^k) = LU\delta^k = LY^k = R^k
\]

\text{Set a guess}

\[
k = 0, \Phi^k
\]

\[
R^k = S - A\Phi^k
\]

\text{While } ||R^k||_1 \geq \varepsilon

\[
R^k = S - A\Phi^k
\]

\text{Solve } LY^k = R^k \text{ by forward substitution}

\[
Y^k = L^{-1}R^k
\]

\text{Solve } U\{\delta^k\} = Y^k \text{ by backward substitution}

\[
\{\delta^k\} = U^{-1}Y^k
\]

\text{Update Solution}

\[
\Phi^{k+1} = \Phi^k + \delta^k
\]

\text{End While}
7.3 Momentum Interpolation

The concept of momentum interpolation in body fitted grids is similar to that of Cartesian grids. However, due to the transformation of coordinates, equations contain more terms compared to their Cartesian counterparts. So the algebra involved is relatively more tedious.

From Eq. 6.2.25, using $\Phi = u$ the discretized $u$ momentum equation about the node P may be written as:

$$a_p u_p = \sum a_{nb} u_{nb} + a_p^0 u_p^0 + S_p^u - (P_e - P_w) (y_n - y_s) + (P_n - P_s) (y_e - y_w) \quad (7.3.1)$$

In order to slow down the change of dependent variables in consecutive solutions an under relaxation factor is introduced in the above equations to get:

$$\frac{a_p}{\alpha u} u_p = \sum a_{nb} u_{nb} + a_p^0 u_p^0 + S_p^u - (P_e - P_w) (y_n - y_s) + (P_n - P_s) (y_e - y_w) \quad (7.3.2)$$

where

$$S_p^u = S_p^u + \frac{1 - \alpha u}{\alpha u} a_p u_p^{m-1}$$

$$u_p = \frac{\alpha u}{(a_p)_p} \left[ \sum a_{nb} u_{nb} + \left\{ a_p^0 u_p^0 + S_p^u + \frac{1 - \alpha u}{\alpha u} a_p u_p^{m-1} + (P_n - P_s) (y_e - y_w) \right\} \right]_p$$

$$- \frac{\alpha u}{(a_p)_p} [(P_e - P_w) (y_n - y_s)]_p$$

$$u_p = \frac{\alpha u}{(a_p)_p} \left[ \sum a_{nb} u_{nb} + B_p \right]_p - \frac{\alpha u}{(a_p)_p} [(P_e - P_w) (y_n - y_s)]_p \quad (7.3.2)$$

Where

$$B_p = a_p^0 u_p^0 + S_p^u + \frac{1 - \alpha u}{\alpha u} a_p u_p^{m-1} + (P_n - P_s) (y_e - y_w)$$

Similarly for node E we may write

$$u_E = \frac{\alpha u}{(a_p)_E} \left[ \sum a_{nb} u_{nb} + B_p \right]_E - \frac{\alpha u}{(a_p)_E} [(P_e - P_w) (y_n - y_s)]_E \quad (7.3.3)$$

Mimicking the formulation of $u_p$ and $u_E$ we can obtain the following expression for the interface velocity at the cell face $e$

$$u_e = \frac{\alpha u}{(a_p)_e} \left[ \sum a_{nb} u_{nb} + B_p \right]_e - \frac{\alpha u}{(a_p)_e} (y_n - y_s) (P_e - P_P)$$

$$u_e = \alpha u \left[ \frac{\sum a_{nb} u_{nb} + B_p}{a_p} \right]_e - \alpha u \frac{(y_n - y_s) e (P_e - P_P)}{(a_p)_e} \quad (7.3.4)$$

The overbar in Eq. 7.3.4 denotes linear interpolation from values at nodes on either side of the cell face using the following CDS expressions:
Where $f_e^+$ is a linear interpolation factor defined as

$$f_e^+ = \frac{|\vec{P}_e|}{|\vec{P}_e| + |\vec{e}E|}$$

In order to have a better understanding of Eq. 7.3.4 substituting $\left(\frac{\sum a_{nb}u_{nb} + B_P}{a_p}\right)_E$ from Eq. 7.3.5 and $\left(\frac{\sum a_{nb}u_{nb} + B_P}{a_p}\right)_p$ and $\left(\frac{\sum a_{nb}u_{nb} + B_P}{a_p}\right)_E$ from Eq. 7.3.2 and Eq. 7.3.3 into Eq. 7.3.4 we get:

$$u_e = [f_e^+ u_E + (1 - f_e^+) u_P]$$

$$+ \left[ -\alpha_u \frac{(y_n - y_s)_E}{(a_p)_E} (P_E - P_P) + f_e^+ \left\{ \frac{\alpha_u}{(a_p)_E} [(P_E - P_w) (y_n - y_s)]_E \right\} + (1$$

$$- f_e^+) \left\{ \frac{\alpha_u}{(a_p)_p} [(P_E - P_w) (y_n - y_s)]_P \right\} \right]$$

Eq. 7.3.4 and 7.3.7 are essentially the same. However, Eq. 7.3.7 separates the interfacial velocity into two parts: a linear interpolation part and the additional one. The term in first set of brackets of Eq. 7.3.7 is the arithmetic averaged values(linear interpolation) of the two nodal velocities. The second term in brackets can be regarded as a correction term, which has the function of smoothing the pressure field, and remove the unrealistic pressure field.

### 7.4 The SIMPLE Algorithm on Body-Fitted Collocated Grids

The SIMPLE (Semi-Implicit Method for Pressure Linked Equations) algorithm and its variants are a set of pressure-based methods widely used in the incompressible flow community. Here, we use the discretized continuity and momentum equations using a non-staggered or co-located mesh (Fig. 7.4). The present formulation employs Cartesian velocity components, storing both pressure and velocity at the cell centroid. Specialized interpolation schemes are used to prevent checker boarding as discussed in section 7.3. The change to a co-located or non-staggered storage scheme is a change in the discretization practice. The iterative methods used to solve the resulting discrete set are the same as those for staggered grids, albeit with a few minor changes to account for the change in storage scheme.
The $u$ and $v$ momentum equations about the node $P$ may be written as:

\[
\frac{(\alpha_p)}{(\alpha_v)} u_p = \sum a_{nb} u_{nb} + a_{pb} u_p + S_u - (P_e - P_w)(y_n - y_s) + (P_n - P_s)(y_e - y_w) \tag{7.4.1}
\]

\[
\frac{(\alpha_p)}{(\alpha_v)} v_p = \sum a_{nb} u_{nb} + a_{pb} v_p + S_v + (P_e - P_w)(x_n - x_s) - (P_n - P_s)(x_e - x_w) \tag{7.4.2}
\]

Writing Eq. 7.4.1 and 7.4.2 in matrix form similar to Eq. 7.2.2 we get:

\[
[A_u]\{u\} = \{S_u\} \tag{7.4.3}
\]

\[
[A_v]\{v\} = \{S_v\} \tag{7.4.4}
\]

The first stage of the solution process is the solution of the discretized versions of the momentum equations using the current estimate of the pressure field, and using a cell face mass flux that is interpolated in the manner discussed in section 7.3 from the current estimate of the velocity field. So the $(m+1)$th iteration begins by guessing velocities $u^m$, $v^m$, turbulent quantities $k^m$, $\varepsilon^m$ and pressure $P^m$ for all nodes ($m$ denotes the outer iteration level). The guessed velocity field, grid data and eddy viscosity calculated from $k^m$, $\varepsilon^m$ are used to calculate the coefficients of momentum equation and therefore linearize it in a way similar to picard linearization. Once the coefficients have been obtained, they form the elements of the matrix $A_u$. The pressure at the faces $e,w,n,s$ are not known since the pressure is stored at the cell centroids. So interpolation about the cell centroids is required to obtain pressure at the cell faces. The interpolated guessed pressures, the terms treated explicitly in a deferred correction manner and the contribution from previous time steps form part of the source column vector $S_u$. One can now solve the following linear system (inner iteration) at the $(m+1)$th outer iteration by the method described in section 7.2:

\[
[A_u]^m\{u^\ast\} = [S_u]^m \tag{7.4.5}
\]
Solving Eq. 7.4.5 (inner iterations) yields the nodal velocity field $u_p^*$. The asterisk is used to indicate that the computed velocity satisfies momentum but necessarily continuity equation. Also the superscript $m$ indicates that the coefficient matrix and source column vector is based on previous outer iteration data. Similarly solving Eq. 7.4.6 yields the $v^*$ velocity field.

$$[A_p]^m \{v^*\} = [S_p]^m \quad (7.4.6)$$

The face velocities and hence mass fluxes at the $(m+1)$th outer iteration may be obtained from these nodal velocity field $u_p^*, v_p^*$ corresponding to the guessed pressure field by momentum interpolation discussed in section 7.3.

At east face, face velocity $u_e^*$ is obtained as follows:

$$u_e^* = \alpha_u \left( \sum a_{nb} u_{nb}^* + B_p \right) - \alpha_u \left( \frac{(y_n - y_e)}{(a_p)\,e} \right) \left( p_{E}^m - p_{p}^m \right) \quad (7.4.7)$$

Other face values of velocity may be similarly obtained. The discretized continuity equation for incompressible flow is:

$$\text{or, } F_e - F_w + F_n - F_s = 0 \quad (7.4.8)$$

The face velocities $u_e^*, u_w^*, v_n^*, v_s^*$ found by interpolating $u^*, v^*$ using momentum interpolation are not guaranteed to satisfy the discrete continuity equation. Mathematically:

$$\text{or, } F_e^* - F_w^* + F_n^* - F_s^* \neq 0$$

$$\text{or, } F_e^* - F_w^* + F_n^* - F_s^* = b$$

Thus, we propose face velocity corrections

$$u_e = u_e^* + u_e \quad (7.4.9)$$

$$u_w = u_w^* + u_w \quad (7.4.10)$$

$$v_s = v_s^* + v_s \quad (7.4.11)$$

$$v_n = v_n^* + v_n \quad (7.4.12)$$

The correct face velocity at face e is given by:

$$u_{e}^{m+1} = \alpha_u \left( \sum a_{nb} u_{nb}^{m+1} + B_p \right) - \alpha_u \left( \frac{(y_n - y_e)}{(a_p)\,e} \right) \left( p_{E}^{m+1} - p_{p}^{m+1} \right) \quad (7.4.13)$$

Subtracting Eq. 7.4.7 from Eq. 7.4.13 we get

$$u_{e}' = \alpha_u \left( \sum a_{nb} u_{nb}^{'} + B_p \right) - \alpha_u \left( \frac{(y_n - y_e)}{(a_p)\,e} \right) \left( p_{E}' - p_{p}^{'} \right) \quad (7.4.14)$$

$$u_{e}' = \alpha_u \left( \sum a_{nb} u_{nb}^{'} + B_p \right) + \alpha_u \left( \frac{(p_n^' - p_e^')}{(a_p)\,e} \right) \left( y_e - y_w \right) - \alpha_u \left( \frac{(y_n - y_e)}{(a_p)\,e} \right) \left( p_{E}' - p_{p}^{'} \right)$$

Where
As an approximation in SIMPLE algorithm the first term in the above equation is neglected giving

\[ u_e = \alpha_u \frac{(p_n - p_s) e (y_e - y_w) e}{(a_p) e} - \alpha_u \frac{(y_n - y_s) e}{(a_p) e} (p_E' - p_p') \]  

(7.4.15)

For \( v_e \) we may analogously write

\[ v_e = -\alpha_v \frac{(p_n - p_s) e (x_e - x_w) e}{(a_p) e} + \alpha_v \frac{(x_n - x_s) e}{(a_p) e} (p_E' - p_p') \]  

(7.4.16)

The corrected velocity \( u_e \) and \( v_e \) may be written as:

\[ u_e = u_e^* + \alpha_u \frac{(p_n - p_s) e (y_e - y_w) e}{(a_p) e} - \alpha_u \frac{(y_n - y_s) e}{(a_p) e} (p_E' - p_p') \]  

(7.4.16)

\[ v_e = v_e^* - \alpha_v \frac{(p_n - p_s) e (x_e - x_w) e}{(a_p) e} + \alpha_v \frac{(x_n - x_s) e}{(a_p) e} (p_E' - p_p') \]  

(7.4.17)

Corrections for other face velocities may be obtained similarly. The flux \( F_e \) is given by:

\[ F_e = \rho \{ u_e (y_n - y_s) e - v_e (x_n - x_s) e \} \]

Using Eq. 7.4.16 and 7.4.17 in above equation we obtain the corrected flux as:

\[ F_e = \rho \left\{ u_e^* + \alpha_u \frac{(p_n - p_s) e (y_e - y_w) e}{(a_p) e} - \alpha_u \frac{(y_n - y_s) e}{(a_p) e} (p_E' - p_p') \right\} (y_n - y_s) e \]

\[ - \left\{ v_e^* - \alpha_v \frac{(p_n - p_s) e (x_e - x_w) e}{(a_p) e} + \alpha_v \frac{(x_n - x_s) e}{(a_p) e} (p_E' - p_p') \right\} (x_n - x_s) e \]

\[ = \rho \left\{-\alpha_u \frac{(y_n - y_s) e}{(a_p) e} (p_E' - p_p') \right\} (y_n - y_s) e - \left\{ \alpha_v \frac{(x_n - x_s) e}{(a_p) e} (p_E' - p_p') \right\} (x_n - x_s) e \]

\[ + \rho \{ u_e^* (y_n - y_s) e - v_e^* (x_n - x_s) e \} \]

\[ + \rho \left\{ \alpha_u \frac{(p_n - p_s) e (y_e - y_w) e}{(a_p) e} (y_n - y_s) e + \alpha_v \frac{(x_n - x_s) e}{(a_p) e} (x_n - x_s) e \right\} \]

The last term containing cross diffusion terms are neglected in deriving the pressure correction equation. The pressure-correction equation is only a means of ensuring that the velocities satisfy the continuity constraint after each iteration; any simplification that does not violate this requirement is justifiable as long as the solution process converges.

\[ F_e = -\rho \left\{ \alpha_u \frac{(y_n - y_s) e}{(a_p) e} (y_n - y_s) e + \alpha_v \frac{(x_n - x_s) e}{(a_p) e} (x_n - x_s) e \right\} (p_E' - p_p') \]

\[ + \{ u_e^* (y_n - y_s) e - v_e^* (x_n - x_s) e \} \]
Corrected fluxes $F_w, F_n, F_s$ at the other faces may be similarly written as:

$$F_w = -\rho \left\{ \alpha_u \frac{(y_n - y_s)}{(a_p)_w} (y_n - y_s)w + \alpha_v \frac{(x_n - x_s)}{(a_p)_w} (x_n - x_s)w \right\} (p'_w - p_W)$$

$$+ \left[ u'_w (y_n - y_s)_w - v'_w (x_n - x_s)_w \right]$$

$$F_n = -\rho \left\{ \alpha_u \frac{(y_w - y_e)}{(a_p)_n} (y_w - y_e)n + \alpha_v \frac{(x_w - x_e)}{(a_p)_n} (x_w - x_e)n \right\} (p'_n - p_p)$$

$$+ \left[ -u'_n (y_w - y_e)_n + v'_n (x_w - x_e)_n \right]$$

$$F_s = -\rho \left\{ \alpha_u \frac{(y_w - y_e)}{(a_p)_s} (y_w - y_e)s + \alpha_v \frac{(x_w - x_e)}{(a_p)_s} (x_w - x_e)s \right\} (p'_s - p_s)$$

$$+ \left[ -u'_s (y_w - y_e)_s + v'_s (x_w - x_e)_s \right]$$

Substituting the corrected fluxes, $F_w, F_n, F_s$ into discretized continuity Eq. 7.4.8 followed by rearrangement gives

$$a_p p'_p = a_w p'_w + a_E p'_E + a_N p'_N + a_s p'_s + b$$ \hspace{1cm} (7.4.18)

The source term $b$ in the pressure correction equation is the mass imbalance resulting from the solution of the momentum equations.

where

$$a_E = \rho \left\{ \alpha_u \left( \frac{1}{(a_p)_e} \right) (y_n - y_s)^2_e + \alpha_v \left( \frac{1}{(a_p)_e} \right) (x_n - x_s)^2_e \right\}$$

$$a_w = \rho \left\{ \alpha_u \left( \frac{1}{(a_p)_w} \right) (y_n - y_s)^2_w + \alpha_v \left( \frac{1}{(a_p)_w} \right) (x_n - x_s)^2_w \right\}$$

$$a_N = \rho \left\{ \alpha_u \left( \frac{1}{(a_p)_n} \right) (y_e - y_w)^2_n + \alpha_v \left( \frac{1}{(a_p)_n} \right) (x_e - x_w)^2_n \right\}$$

$$a_s = \rho \left\{ \alpha_u \left( \frac{1}{(a_p)_s} \right) (y_e - y_w)^2_s + \alpha_v \left( \frac{1}{(a_p)_s} \right) (x_e - x_w)^2_s \right\}$$

$$b = \left[ u'_w (y_n - y_s)_w - v'_w (x_n - x_s)_w \right] - \left[ u'_e (y_n - y_s)_e - v'_e (x_n - x_s)_e \right]$$

$$+ \left[ -u'_s (y_e - y_w)_s + v'_s (x_e - x_w)_s \right] - \left[ -u'_n (y_e - y_w)_n + v'_n (x_e - x_w)_n \right]$$

Solving Eq. 7.4.18 using strongly implicit method of Stone (1968) in a manner similar to momentum equations, yield the nodal pressure correction $p'$. Using the values of pressure correction, the face velocities are corrected using expressions like Eq. 7.4.16 and 7.4.17. The nodal pressure field is corrected using

$$p^{m+1} = p^m + a_p p'$$ \hspace{1cm} (7.4.19)

and the nodal velocity is corrected as
Where

\[ \alpha_p = \text{the pressure under-relaxation factor chosen as 0.1} \]
\[ \alpha_u = \text{the u momentum under-relaxation factor chosen as 0.7} \]
\[ \alpha_v = \text{the v momentum under-relaxation factor chosen as 0.7} \]

The scalar transport equation for turbulent kinetic energy and dissipation are then solved to obtain an updated value of eddy viscosity to be used in the momentum equations for the next iteration.

\[ u_{m+1}^p = u_p + \alpha_u \frac{(p_n - p_s')(y_e - y_w)}{(a_p)_p} - \alpha_u \frac{(y_n - y_s)(p_e' - p_w')}{(a_p)_p} \] (7.4.20)

\[ v_{m+1}^p = v_p - \alpha_v \frac{(p_n - p_s')(x_e - x_w)}{(a_p)_p} + \alpha_v \frac{(x_n - x_s)}{(a_p)_p} (p_e' - p_w') \] (7.4.21)

Eq. 7.4.22 and 7.4.23 can be solved using strongly implicit method of Stone (1968) just like momentum and pressure correction equation. In the solution of the turbulent quantities according to Eq. 7.4.22 and 7.4.23, the coefficient matrices and source vectors are superscripted as \(m+1\) to point out the availability of updated velocities (Eq. 7.4.20 and 7.4.21). However, any quantity depending on \(k\) and \(\varepsilon\) (which may arise from the production and dissipation terms existing as additional sources in these transport equation) are obtained from previous iteration (\(m\)th outer iteration). Under relaxation of turbulent quantities is also necessary. In the present case we have \(\alpha_k = 0.7, \alpha_\varepsilon = 0.7\). This completes the (\(m+1\))th outer iteration yielding \(u^{m+1}, v^{m+1}, k^{m+1}, \varepsilon^{m+1}, p^{m+1}\) to act as initial ‘guesses’ for the next outer iteration. This process is continued until convergence. A summary of the steps is illustrated in Fig. 7.5

7.5 Convergence Criteria

Starting from initial guess for all field values the process of solving the equations is repeated until convergence. Due to coupling of variables and the nonlinearity of the equations, it is not necessary to solve exactly the discretized equations for a given set of coefficients (inner iteration); these are only approximate and need to be updated. So inner iterations of momentum equations and equations of turbulent quantities are terminated by limiting the number of iteration to 1. Convergence of the pressure correction equation is monitored by comparing the sum of the absolute residuals after each sweep to its initial value.
For outer iterations (solution with updated coefficients), the sum of the absolute values of the residuals over all control volumes is calculated, and normalized by inlet flux of the relevant quantity, $f_{\text{inlet}}^\phi$. That is:

$$\overline{R_\phi} = \frac{\sum_{i=1}^{K} R_\phi^i}{f_{\text{inlet}}^\phi}$$

The residual is obtained in the manner discussed in section 7.2. For convergence of outer iteration to take place the following must be satisfied:

$$\max(\overline{R_x}, \overline{R_1}, \overline{R_k}, \overline{R_e}) \leq S_{\text{max}}$$

The above criterion ensures that the relative changes in the variables from one iteration to the next are of the order of $S_{\text{max}}$ or less.
Fig. 7.5: The SIMPLE algorithm

Initialize $u^m, v^m, P^m, k^m, \varepsilon^m$

Calculate coefficients and source terms of momentum equations

Solve linear momentum equations to obtain nodal velocity field

$u^*, v^*$

Calculate face velocities using Rhie-Chow interpolation and mass imbalance

Assemble and solve Pressure correction equation

$p^*$

Correct velocity and pressure field

$u^{m+1}, v^{m+1}, p^{m+1}$

Calculate coefficients and source terms of turbulent kinetic energy and dissipation equation

Solve linear equations for turbulent quantities $k^{m+1}, \varepsilon^{m+1}$

Converged?

NO

Outter iteration

$u^{m+1}, v^{m+1}, p^{m+1}$

$u^{m+1}, v^{m+1}, p^{m+1}$

Converged solution of $\varepsilon$ at $t$ for $t+T \Delta t$

Next time step $T+=1$
CHAPTER 8
BOUNDARY CONDITIONS AND THEIR IMPLEMENTATION

8.1 Type of Boundary Conditions
All CFD problems are defined in terms of initial and boundary condition. Unless the boundary constraints are compatible with the physics of the problem, spurious results may be generated. So careful implementation of boundary constraints is vital for a successful simulation. The fluxes at the boundary must be either known, or be expressed as a combination of interior values and boundary values. Since boundary nodes do not give additional equations, they should not introduce additional unknowns. Typically the following types of boundary conditions are used in FVM:

- Inlet
- Outlet
- Wall
- Prescribed pressure
- Symmetry
- Periodicity

In the present work only inlet, outlet, wall and symmetry conditions are relevant and additional boundary conditions will arise due to the pressure correction equation which will be discussed in section 8.2.

8.2 Implementation of Boundary Conditions
The algebraic form of the generic transport equation is given as:

\[ a_p \Phi_p = a_w \Phi_w + a_e \Phi_e + a_n \Phi_n + a_s \Phi_s + S \]  (8.2.1)

Now equation 8.2.1 is to be solved applying the following boundary conditions:

**Inlet boundary**
At inlet, values of flow variable \( \Phi \) are provided. This is known in literature as **Dirichlet** type boundary condition. This in turn leads to a **Neumann** type condition for pressure correction equation which will be discussed subsequently.
Momentum equations

For u-momentum Eq. 8.2.1 becomes:

\[ a_p u_p = a_W u_W + a_E u_E + a_N u_N + a_S u_S + S^u \]  

(8.2.2)

Where \( S^u \) also includes the \( pterm \).

At the inlet velocities \( u \) and \( v \) are prescribed. In Fig. 8.1, inlet boundary is the west face of the cell with center P. When a Dirichlet type boundary condition is applied, for momentum equations this means, the mass flux \( F_w \) is known from known velocities at inlet.

That is

\[ F_w = \rho (u_{inlet} (y_n - y_s)_{w} - v_{inlet} (x_n - x_s)_{w}) \]

The convective flux at the boundary thus can be directly computed and the diffusive flux estimated by one sided finite difference approximation.

This implies that the terms \( a_W \) and \( u_W \) in Eq. 8.2.2 are replaced by known boundary quantities. This may be implemented by setting

\( a_W = 0 \)

\[ a_p = \{D_w + \max(F_w, 0)\} + a_E + a_N + a_S + a_p^0 + [F_e - F_w + F_n - F_s] \]

\[ S^u = \{N_e (u_n - u_s)_e - N_w (u_n - u_s)_w + N_n (u_e - u_w)_n - N_s (u_e - u_w)_s\} \]

\[ + \beta \{[F_e u_e - F_w u_w + F_n u_n - F_s u_s]_{UDS} - [F_e u_e - F_w u_w + F_n u_n - F_s u_s]_{CDS}\} \]

\[ + pterm + \{D_w + \max(F_w, 0)\} u_{inlet} + a_p^0 u_p^0 + pterm \]

Where,

\[ D_w = \frac{\Gamma}{(x_p - x_w)(y_n - y_s)_{w} - (x_n - x_s)_{w}(y_p - y_w)} \{ (y_n - y_s)^2_{w} + (x_n - x_s)^2_{w} \} \]

\[ N_w = \frac{\Gamma((x_p - x_w)(x_n - x_s)_{w} + (y_p - y_w)(y_n - y_s)_{w})}{(x_n - x_s)_{w}(y_p - y_w) - (x_p - x_w)(y_n - y_s)_{w}} \]
Other quantities remain unchanged.

**Turbulent kinetic energy and dissipation equations**

\[
\begin{align*}
\alpha_p k_p &= a_W k_W + a_E k_E + a_N k_N + a_S k_S + S^k \quad (8.2.3) \\
\alpha_p \varepsilon_p &= a_W \varepsilon_W + a_E \varepsilon_E + a_N \varepsilon_N + a_S \varepsilon_S + S^\varepsilon \quad (8.2.4)
\end{align*}
\]

where for turbulent kinetic energy we have extra terms embedded in \( \alpha_p \) and \( S^k \) arising from production and dissipation as follows:

\[
\begin{align*}
\alpha_p &= a_W + a_E + a_N + a_S + a_p^o + \frac{\varepsilon_p^o}{k_p} \delta V + [F_e - F_w + F_n - F_s] \\
S^k &= \{N_e (k_n - k_e) e - N_w (k_n - k_w) e + N_n (k_e - k_w) n - N_s (k_e - k_w) s \\
&\quad + \beta \{[F_e k_e - F_w k_w + F_n k_n - F_s k_s]_{UDS} - [F_e k_e - F_w k_w + F_n k_n - F_s k_s]_{CDS} \} \\
&\quad + (G_p) \delta V + a_p^o k_p^o 
\end{align*}
\]

and similarly for turbulent dissipation equation we have extra terms embedded in \( \alpha_p \) and \( S^\varepsilon \) as follows:

\[
\begin{align*}
\alpha_p &= a_W + a_E + a_N + a_S + a_p^o + \frac{\varepsilon_p^o}{k_p} \varepsilon_p + [F_e - F_w + F_n - F_s] \\
S^\varepsilon &= \{N_e (\varepsilon_n - \varepsilon_e) e - N_w (\varepsilon_n - \varepsilon_w) e + N_n (\varepsilon_e - \varepsilon_w) n - N_s (\varepsilon_e - \varepsilon_w) s \\
&\quad + \beta \{[F_e \varepsilon_e - F_w \varepsilon_w + F_n \varepsilon_n - F_s \varepsilon_s]_{UDS} - [F_e \varepsilon_e - F_w \varepsilon_w + F_n \varepsilon_n - F_s \varepsilon_s]_{CDS} \} \\
&\quad + C_{\varepsilon 1} \frac{\varepsilon_p^o}{k_p} \delta V + a_p^o \varepsilon_p^o 
\end{align*}
\]

With mass flux given by prescribed velocities, one needs to provide known values at inlet for turbulent kinetic energy and dissipation. That is, \( \Phi_W = k_w = k_{inlet} \) and \( \Phi_W = \varepsilon_w = \varepsilon_{inlet} \).

For Turbulent kinetic energy this means

\[
a_W = 0
\]

\[
a_p = \{D_w + \max (F_w, 0)\} + a_E + a_N + a_S + a_p^o + \frac{\varepsilon_p^o}{k_p} k_p + [F_e - F_w + F_n - F_s]
\]

\[
S^k = \{N_e (k_n - k_e) e - N_w (k_n - k_w) e + N_n (k_e - k_w) n - N_s (k_e - k_w) s \\
\quad + \beta \{[F_e k_e - F_w k_w + F_n k_n - F_s k_s]_{UDS} - [F_e k_e - F_w k_w + F_n k_n - F_s k_s]_{CDS} \} \\
\quad + \{D_w + \max (F_w, 0)\} k_{inlet} + G_p \delta V + a_p^o k_p^o
\]

For Turbulent dissipation equation this means

\[
a_W = 0
\]

\[
a_p = \{D_w + \max (F_w, 0)\} + a_E + a_N + a_S + a_p^o + \frac{\varepsilon_p^o}{k_p} \varepsilon_p + [F_e - F_w + F_n - F_s]
\]
Pressure correction equations

When the velocities at the inlet boundaries are known, there is no need to correct the velocities at the boundaries in the derivation of the pressure correction equation. In this case, the velocity known at inlet is at the west side of the control volume with centre P as shown in Fig. 8.1. So, with known velocities at the west boundary we can write:

\[ u_w = u_{inlet}, \quad v_w = v_{inlet} \]

The other face velocities remain the same as before. Substituting the results into the discretized continuity equation we obtain the following pressure correction equation for a control volume near the west boundary

\[ a_p p'_p = a_w p'_w + a_E p'_e + a_N p'_n + a_s p'_s + b \]

Where

\[ a_w = a_{inlet} + v_{inlet} \]

\[ a_{inlet} = 0 \]

\[ a_E = -\rho \left( \frac{\bar{y}_n - y_s}{(a_p)_e} (y_n - y_s) e + \frac{\bar{x}_n - x_s}{(a_p)_e} (x_n - x_s) e \right) \]

\[ a_N = -\rho \left( \frac{\bar{y}_w - y_e}{(a_p)_n} (y_w - y_e) n + \frac{\bar{x}_w - x_e}{(a_p)_n} (x_w - x_e) n \right) \]

\[ a_s = -\rho \left( \frac{\bar{y}_w - y_e}{(a_p)_s} (y_w - y_e) s + \frac{\bar{x}_w - x_e}{(a_p)_s} (x_w - x_e) s \right) \]

\[ b = \left[ u_{inlet} (y_n - y_s) w + v_{inlet} (x_n - x_s) w \right] - u'_e (y_n - y_s) e + v'_e (x_n - x_s) e \]

\[ + \left[ -u'_n (y_w - y_e) n + v'_n (x_w - x_e) n \right] - \left[ -u'_s (y_w - y_e) s + v'_s (x_w - x_e) s \right] \]

The bar is used on quantities that are interpolated.

For a near boundary control volume the same definition of the coefficients as used for the interior points can be used for a near boundary control volume by setting the corresponding coefficient (\(a_w\) in this case) to zero and using \(u_{inlet}, v_{inlet}\) in the \(b\) term. The above formulation corresponds to a Neumann boundary condition for pressure correction. As a result no value of pressure correction at the boundary \(p'_w\) is involved in this formulation. However, the value of the pressure correction is needed for correcting the nodal velocities near boundaries. For example, for correcting the velocity at a nodal point P near a west boundary, \(p'_w\) at the west boundary is needed. This value can be obtained by using \(p'_w = p'_p\).
A similar treatment of pressure correction equation also applies for outlet, symmetry and wall boundaries.

**Outlet boundary**

If the outlet boundary (Fig. 8.2) is sufficiently away from regions of significant change, then a Neumann type boundary condition can be applied.

**Fig. 8.2: Cell adjacent to outlet boundary**

That is gradient of variables in taken to be zero in the direction normal to the boundary. Mathematically this implies:

\[
\frac{\partial \phi}{\partial n} = 0
\]

**Momentum equations**

Using Neumann type boundary condition for momentum equation we can write

\[
\frac{\partial u}{\partial n}, \quad \frac{\partial v}{\partial n} = 0
\]

For the outlet boundary shown in Fig. 2 this can be implemented simply by setting

\[
u_p = u_E, \quad v_p = v_E
\]  

(8.2.5)

The modification of the algebraic form the equations may be carried out in a similar fashion as shown earlier.

However, during iteration cycles of the SIMPLE algorithm there is no guarantee that these velocities will conserve mass over the whole computational domain as a whole. That is, the sum of the outlet mass fluxes \( F^o \) must equal sum of inlet mass fluxes \( F^I \). This can be done by calculating outlet mass fluxes using newly estimated values of velocity components and correcting these components by multiplying them with the ratio \( \frac{F^o}{F^I} \).

**Turbulent kinetic energy and dissipation equations**

Using Neumann type boundary condition we can write
\[ \frac{\partial k}{\partial n}, \quad \frac{\partial \varepsilon}{\partial n} = 0 \]

Hence,
\[ k_p = k_e, \quad \varepsilon_p = \varepsilon_e \]

The modification of the algebraic form the equations may be carried out in a similar fashion as shown earlier.

**Pressure correction equations**

With velocities known at the outlet boundary from the Neumann type condition, the treatment of pressure correction equation is analogous to that at inlet where a Neumann type condition for pressure correction arises.

**Symmetry boundary**

At a symmetry boundary there is no flow across the boundary (zero convective flux) and diffusion of dependent variable is zero. For a boundary aligned with the x-axis as shown in Fig. 8.3 zero convection is implemented by setting the normal velocity at the boundary to be zero and the zero diffusion condition is implemented by setting all quantities at the boundary equal to their nodal values of the boundary adjacent cell. For the case given in Fig. 8.3, this means:

\[ v = 0, \quad \Phi_n = \Phi_p \]

where,
\[ \Phi = u, k, \varepsilon \]

![Symmetry boundary diagram](image)

**Wall boundary**

**Implementation of Wall functions**

The equations describing the velocity profile in the inner region of the turbulent boundary layer are collectively called the "law of the wall" and are given by Eq. 8.2.6 in viscous sublayer and Eq. 8.2.7 in the log region.
\[ u^+ = \frac{v_t}{u_\tau} = n^+ \]  
\[ u^+ = \frac{v_t}{u_\tau} = \frac{1}{\kappa} \ln(n^+) + B = \frac{1}{\kappa} \ln(En^+) \]  

where,

\[ B = \frac{1}{\kappa} \ln(E), \kappa = 0.41 \text{ and } B = 5.5 \text{ for a boundary layer over a smooth flat plate (Ferziger and Peric, 2002).} \]

\[ u_\tau = \sqrt{\frac{|\tau_w|}{\rho}} \]  
\[ n^+ = \frac{u_\tau n}{\nu} \]

The wall parallel component of resultant velocity \( \overline{v}_r \) is \( v_t \), \( \tau_w \) is the wall shear stress, \( n \) is the distance in the direction normal to the wall and \( u_\tau \) is the shear velocity. A typical turbulent boundary layer velocity profile, similar to that shown in Anderson, Tannehill, and Fletcher (1985), is shown in Fig. 8.4 and a control volume next to the wall is shown in figure 8.5.

Fig. 8.4: The turbulent boundary layer: velocity profile as function of distance normal to the wall

In the implementation of wall functions, the first task is to ensure that the first computational node (P in Fig. 8.5) is at a distance from the wall such that \( n^+ > 30 \).
Momentum equation:

The algebraic form of the $u$-momentum equation is

$$a_p u_p = a_W u_W + a_E u_E + a_N u_N + a_S u_S + S^u$$  \hspace{1cm} (8.2.10)

Since there can be no flow through an impermeable wall, the convective flux for momentum is zero there. This can easily be taken into account by setting the convective flux through the corresponding control volume face, in this case the control volume or cell shown in Fig. 8.5, to zero. However, the treatment of the diffusive fluxes in the momentum equations deserves special attention where the normal stress is zero and the exchange of momentum is transmitted only by the shear stress, i.e the wall shear stress. This implies that the wall shear force obtained from the wall shear stress is to be inserted in the momentum equation as a boundary condition at the wall. In the present approach this wall shear stress is not computed from standard discretization but the algebraic wall functions.

The shear stress at the wall is defined as:

$$\tau_w = \mu \frac{\partial v_t}{\partial n}$$  \hspace{1cm} (8.2.11)

Now,

$$\tau_w = \mu \frac{\partial v_t}{\partial n} \neq \mu \frac{v_t}{n_p}$$  \hspace{1cm} (8.2.12)

Eq. 8.2.12 is true because the velocity gradient at the wall is very large. However this issue can be circumvented by using an effective viscosity. That is,

$$\tau_w = \mu \frac{\partial v_t}{\partial n} \approx \mu_{\text{effective}} \frac{v_t}{n_p} \approx \mu_{\text{effective}} \frac{v_t}{n_p}$$  \hspace{1cm} (8.2.13)

Where the error incurred in the velocity gradient is offset by the use of an effective viscosity instead of the original viscosity.
Using Eq. 8.2.7 and Eq. 8.2.8 we get for the near wall node

\[ \tau_w = \frac{\rho v_t u_t}{\frac{1}{k} \ln(En^+)} \]  

(8.2.14)

In the local equilibrium boundary layers, the eddy viscosity formula gives

\[ -\overline{\rho u v} = \mu_t \frac{\partial v_t}{\partial n} \]

And thus the production of turbulent kinetic energy may be written as

\[ G \approx -\overline{u v} \frac{\partial v_t}{\partial n} = v_t \frac{\partial v_t}{\partial n} \left( v_t \frac{\partial v_t}{\partial n} \right) \]

(8.2.15)

The gradients of velocity in the other directions in \( G \) are neglected. Since \( v_t = C_{\mu} \frac{k^2}{\varepsilon} \) and \( G = \varepsilon \) in local equilibrium

\[ \left( \frac{-\overline{u v}}{k} \right)^2 = \left( \frac{v_t}{k^2} \right)^2 = \frac{v_t G}{k^2} = \frac{C_{\mu} G}{\varepsilon} = C_{\mu} \]

(8.2.16)

But

\[ u_t = \sqrt{\frac{\tau_w}{\rho}} \]

Also

\[ \tau_w \approx -\overline{\rho u v} \]

Since turbulent stresses dominate in the log region, which is evident from Fig. 8.6. So we get

\[ u_t \approx \sqrt{-u v} \]

(8.2.17)

Using Eq. 8.2.16 and Eq. 8.2.17 we can write

\[ u_t \approx C_{\mu} \frac{1}{k^2} \]

(8.2.18)

---

![Fig. 8.6: Total shear stress near wall](image_url)
Using Eq. 8.2.18 in 8.2.9 yields

\[ n^+ = \frac{C_{\mu}^{1/2} \frac{1}{k} n}{\nu} = \frac{\rho C_{\mu}^{1/2} \frac{1}{k} n}{\mu} \]  \hspace{1cm} (8.2.19)

Using Eq. 8.2.18 in Eq. 8.2.14 gives

\[ \tau_w^{m+1} = \{v_t\} \left( \frac{\rho C_{\mu}^{1/2} \frac{1}{k} n}{\nu} \right)^m \left( \frac{1}{\kappa \ln(En^+_p)} \right) \]  \hspace{1cm} (8.2.20)

Where the superscript \( m \) is used to emphasize that the right side is calculated from previous iteration.

Using Eq. 8.2.19, and Eq. 8.2.20 may be rewritten as

\[ \tau_w = \frac{\kappa \mu n^+_p}{\ln(En^+_p)} \frac{v_t}{n_p} \]  \hspace{1cm} (8.2.21)

Comparing Eq. 8.2.21 with 8.2.13 we get

\[ \mu_{\text{effective}} = \frac{\kappa \mu n^+_p}{\ln(En^+_p)} \]

By calculating \( \mu_{\text{effective}} \) and the wall parallel velocity, the shear stress can be computed from Eq. 8.2.13, and the sheaf force that would serve as a boundary condition for momentum equation is calculated from the product of the wall shear stress and the area as follows:

\[ T_w = -\tau_w * A_w. \]

So that one may write

\[ T_w = -\frac{\kappa \mu n^+_p A_w v_t}{\ln(En^+_p)} n_p \]  \hspace{1cm} (8.2.22)

Since the \( u \) and \( v \) momentum equations are resolved in the Cartesian coordinate directions, namely \( x \) and \( y \) directions, the resultant shear force also needs to be expressed in Cartesian coordinates. Its components in the \( x \) and \( y \) directions serving as boundary conditions for \( u \) and \( v \) momentum equations respectively are given by:

\[ T_{wx} = -\frac{\kappa \mu n^+_p A_w \{u_p(1 - n_1^2) - v_p n_1 n_2\}}{\ln(En^+_p)} n_p = -\frac{\kappa \mu n^+_p A_w \{u_p(n_2^2) - v_p n_1 n_2\}}{\ln(En^+_p)} n_p \]  \hspace{1cm} (8.2.23)

\[ T_{wy} = -\frac{\kappa \mu n^+_p A_w \{v_p(1 - n_2^2) - u_p n_1 n_2\}}{\ln(En^+_p)} n_p = -\frac{\kappa \mu n^+_p A_w \{v_p(n_1^2) - u_p n_1 n_2\}}{\ln(En^+_p)} n_p \]  \hspace{1cm} (8.2.24)

Where the Cartesian components of \( \mathbf{n} \) (unit normal to the wall boundary) are given by \( n_1 \) and \( n_2 \). Since the wall is at the south face (Fig. 8.5), to implement this force as a boundary...
condition for $u$ momentum equation, one has to set the coefficient $a_S = 0$ at the right side of Eq. 8.2.10 and add $\frac{\mu n^+ A_w}{\ln(En^+)} (1 - n_1^2)$ or $\frac{\mu n^+ A_w}{\ln(En^+)} n_2^2$ to central coefficient $a_p$ and $-\frac{\mu n^+ A_w}{\ln(En^+)} v_p n_1 n_2$ to the source term, $S^u$.

The wall boundary condition for $v$ momentum equation can be similarly applied.

**Turbulent kinetic energy and dissipation equation:**

\begin{align*}
\alpha_p k_p &= a_W k_W + a_E k_E + a_N k_N + a_S k_S + S_k \\
\alpha_p \varepsilon_p &= a_W \varepsilon_W + a_E \varepsilon_E + a_N \varepsilon_N + a_S \varepsilon_S + S^\varepsilon
\end{align*}

(8.2.25) (8.2.26)

The equation for $k$ and $\varepsilon$ need special treatment for the near wall cells. The diffusion of turbulent kinetic energy is zero at the wall (Launder and Spalding, 1974) which in this case is the $s$ face in Fig. 8.5. The further modification to the near wall transport equation is to the source terms, that is, production and dissipation terms, because the usual linear interpolation for the computation of the gradients of the mean velocity components would yield erroneous results. The modified production term is obtained as follows

\begin{equation}
G \approx -u v \frac{\partial v_t}{\partial n} = v_t \frac{\partial v_t}{\partial n} \left( \frac{\partial v_t}{\partial n} \right) \tag{8.2.27}
\end{equation}

Where only $\frac{\partial v_t}{\partial n}$ is an approximation to the dominant gradient in the original formula for $G$.

But, in the log region where turbulent stresses dominate (Fig. 8.6), we have

\begin{equation}
-uv = v_t \frac{\partial v_t}{\partial n} \approx \tau_{total} \approx \tau_w
\end{equation}

Where, $\tau_{total} = \tau_{turbulent} + \tau_{laminar}$

Using the result in Eq. 8.2.26 we get

\begin{equation}
G \approx \tau_w \frac{\partial v_t}{\partial n}
\end{equation}

The velocity derivative $\frac{\partial v_t}{\partial n}$ can be derived from the logarithmic velocity profile given by Eq. 8.2.7 (Ferziger and Peric, 2002). So we get:

\begin{equation}
\left( \frac{\partial v_t}{\partial n} \right)_p = \frac{u_t}{\kappa n_p}
\end{equation}

Using from $u_t$ from Eq. 8.2.18 we get

\begin{equation}
\left( \frac{\partial u_t}{\partial n} \right)_p = \frac{C_{\mu}^{1/3} k_p^{1/3}}{\kappa n_p}
\end{equation}

Hence,
For $\varepsilon$, instead of solving the transport equation, the value is set on the first computational node near wall from the following formula:

$$\varepsilon_p = \frac{C_\mu 3 k_p \frac{3}{2}}{\kappa n_p}$$

This is readily implemented by setting in Eq. 8.2.26

$$a_p = 1, \quad a_W = a_E = a_N = a_S = 0, \quad S^\varepsilon = \varepsilon_p = \frac{C_\mu 3 k_p \frac{3}{2}}{\kappa n_p}$$
CHAPTER 9

RESULTS AND DISCUSSION

The turbulent flow over NACA 0012 and NACA 4412 hydrofoil forms a rich blend of fluid mechanics phenomena and its analysis has been carried out in the present work. The major aim was to see the influence of number of grid points and angle of attack (\(\alpha\)) on the pressure coefficient of the hydrofoils. Moreover, effort was put to predict the flow past NACA 4412 hydrofoil at maximum lift condition with angle of attack 13.9 degrees. To achieve these objectives C type structured grid was utilized. A typical grid is shown in Fig. 9.1. However, before studying the effect of grid refinement and angle of attack on flow properties, the obtained results were compared with experimental and established theoretical data. The experimental results by Gregory and O’Reilly (1970) on NACA 0012 were available for comparison at various angles of attack. Also the work by Rhie (1981) on turbulent flow around airfoil provided results at 0 and 6 degrees of incidence for comparison. For NACA 4412 hydrofoil however, experimental results are not so readily available. For validation purpose, experimental data of Pinkerton (1936) was utilized. For the case of foil with separation, pressure coefficients were compared with Coles and Wadcock (1979).

Fig. 9.1: Overview of typical grid used
9.1 NACA 0012 Hydrofoil

The external flow past an NACA 0012 hydrofoil with 2.5 m chord length was simulated at $Re_c = 2.8 \times 10^6$ with grid sizes of $50 \times 14, 88 \times 20, 176 \times 40$ and 0, 6, 10 degrees of incidence (a grid size of $50 \times 14$ implies that, 50 control volumes existed in the $\xi$ direction and 14 in the $\eta$ direction). The obtained pressure coefficients ($C_p$) and experimental results for each angle and a grid size of $176 \times 40$ are shown in Fig. 9.2 to 9.4 respectively.

Fig. 9.2: Pressure coefficients for NACA 0012 at 0 degree angle of attack

Fig. 9.3: Pressure coefficients for NACA 0012 at 6 degree angle of attack
From Fig. 9.2 to 9.4, it is evident that obtained results agree very well with the experiment. In fact at 0 and 6 degree the result agrees better than that obtained by Rhie (1981). However, discrepancy can be noted at the leading and trailing edges which may be attributed partly to inadequate turbulence modeling. Moreover, the mesh may not be sufficiently fine to capture the very strong gradients existing in those regions.

In addition to the pressure coefficient curves, pressure line contour, the streamlines, pressure coefficient and turbulent quantity flood contours, and vector plots at 6 degree incidence are shown in Fig. 9.5 to 9.10 for the same grid at $Re_c = 2.8 \times 10^6$.

C type structured mesh with three different sizes were used to investigate the influence on surface pressure coefficients. The first mesh consisted of 50 control volumes in the $\xi$ direction and 14 in the $\eta$ direction ($50 \times 14$) with 520 cells in total. The subsequent finer meshes had sizes of $88 \times 20$ and $176 \times 40$ with 1760 and 7040 control volumes respectively. At a given angle of attack, pressure coefficients were computed at each of the three grids and plotted on the same axes. Results for 0, 6 and 10 degree of incidence are shown in Fig. 9.11. By observing Fig. 9.11 and comparing with experimental data in Fig. 9.2 to 9.4 it is evident that, refinement of grid gives increased numerical accuracy in the present case. Significant improvement is found on the suction side of the hydrofoil. However the effect on the pressure side (where the discrepancy is small) is less considerable.
Fig. 9.5: Isobars at 6 degree incidence with 176x40 grid size for NACA 0012 hydrofoil

Fig. 9.6: Streamlines at 6 degree incidence with 176x40 grid size for NACA 0012 hydrofoil
Fig. 9.7: Flooded pressure coefficient contours and streamlines at 6 degree incidence for 176x40 grid size

Fig. 9.8: Turbulent kinetic energy contours at 6 degree incidence for 176x40 grid size
Fig. 9.9: Turbulent kinetic energy dissipation contours at 6 degree incidence for 176x40 grid size

Fig. 9.10: Velocity vectors at 6 degree incidence for 176x40 grid size
Fig. 9.11: Grid dependency of pressure coefficients for NACA 0012 hydrofoil at 0, 6 and 10 degree incidence.
The improvement in result is particularly marked at the point of lowest pressure coefficient near the leading edge. The percentage errors compared to experimental findings at this point for $50 \times 14, 88 \times 20$ and $176 \times 40$ were found to be 50.0%, 33.0% and 16.6% respectively. There was a reduction in discrepancy because the steep gradients could be better resolved by the finer mesh used. For the $176 \times 40$ grid, effect of angle of attack was studied by plotting the pressure coefficients at 0, 6 and 10 degree on the same axes as shown in Fig. 9.12. Also the convergence history for NACA 0012 at 10 degree incidence is illustrated in Fig. 9.13 for the finest grid.

Fig. 9.12: Effect of angle of attack on pressure coefficients of NACA 0012 hydrofoil

Fig. 9.13: Convergence history for NACA 0012 and NACA 4412 at 10 degree incidence
9.2 NACA 4412 Hydrofoil

Turbulent flow past NACA 4412 hydrofoil was analyzed using a method similar to that of NACA 0012. In this case the first mesh consisted of 76 control volumes in the $\xi$ direction and 20 in the $\eta$ direction ($76 \times 20$) with 1520 cells in total. The subsequent finer meshes had sizes of ($134 \times 25$) and ($268 \times 50$) with 3350 and 13400 control volumes respectively. The most comprehensive experimental tests on the NACA 4412 section available for validation are those of Pinkerton (1936). Unhappily, Pinkerton's measurements are actually for three-dimensional flow over a rectangular foil with a 30-inch span and a 5-inch chord. The equivalent two-dimensional flow is found by subtracting the theoretically calculated induced angle of attack from the geometric angle of incidence. Nevertheless, these tests are often used in two-dimensional comparisons because of a general lack of experimental data, and for that reason, we do so also. Additionally, the lift coefficient, (Fig. 9.18) was computed and validated against experimental findings of Coles and Wadcock (1979) and Kermeen (1956). Like NACA 0012 foil, grid dependency and effect of angle of attack was also studied for NACA 4412 hydrofoil (See Fig. 9.19 to 9.22).

The convergence history for NACA 4412 at 10 degree incidence is illustrated in Fig. 9.13 for the finest grid which substantiate that the solutions satisfy the continuity principle. The pressure coefficients were computed at $Re_c = 3.1 \times 10^6$ and compared with Pinkerton (1936) for a $268 \times 50$ C type structured grid (see Fig. 9.14 to 9.17). Since Pinkerton’s experiment was for three dimensional flows, comparisons are made at effective angles of attack.

Fig. 9.14: Pressure coefficients for NACA 4412 at 1.2 degree angle of attack
Fig. 9.15: Pressure coefficients for NACA 4412 at 2.9 degree angle of attack

The effective angles of attack (1.2 degree, 2.9 degree, 6.4 degree, 10 degree) were obtained according to Pinkerton (1936) by subtracting the theoretically calculated induced angle of attack from the geometric angle of incidence. From the curves it can be seen that, at all the angles of attack, the agreement on the pressure side is excellent with slight discrepancy on the trailing edge. Compared to the pressure side, discrepancy is more notable on the suction side, particularly near the leading edge.

Fig. 9.16: Pressure coefficients for NACA 4412 at 6.4 degree angle of attack

For the 2.9 and 6.4 degree angle of incidence, little discrepancy also exists between $x/c$ of 0.1 to 0.3 which is believed to occur partly due to error incurred in calculating the effective angle of attack by Pinkerton (1936).
The determination of the effective angle of attack of the section entails certain assumptions that are subject to considerable uncertainty. First, the angle of attack of this section may be in error because of the assumption that the deviation of the air-stream axis from the tunnel axis is uniform along the span of the model; i.e. that the geometric angle of attack is the same for all sections along the span. Actually there is some variation of the air-stream direction across the tunnel. Because of the interference of the support strut, the deflection of the stream in this region might reasonably be expected to exceed the deflection at the midspan section; hence, the deflection at the midspan section is probably less than the effective mean value. The lift coefficients shown in Fig. 9.18 shows excellent agreement at lower angles of attack. As angles are increased however, the agreement seems to be less satisfactory. At very high incidences, flow separation takes place, and since the k-epsilon turbulence model is known to perform poorly in separated flows, discrepancy in the result
arises. The obtained value of lift coefficient at the highest lift condition of around 13.9 degree is found to be 1.5162 which differs from the experimental value of Coles and Wadcock (1979) by 9.2%. Rhie (1981) obtained a value of 1.51 by theoretical calculation which varies from the experimental value by 9.6%.

The pressure coefficients at a given angle of attack were plotted for three grid sizes on the same axes at $Re_c = 3.1 \times 10^6$. The plots for 1.2, 2.9, 6.4 and 10 degree of incidence are shown in Fig. 9.19, 9.20, 9.21 and 9.22.

Fig. 9.19: Grid dependency of pressure coefficients for NACA 4412 at 1.2 degree angle of attack

Fig. 9.20: Grid dependency of pressure coefficients for NACA 4412 at 2.9 degree angle of attack
Moreover, plots were also made for the pressure coefficients at 1.2, 2.9, 6.4 and 10 degree angles of attack on the same graph for a mesh size of $268 \times 50$. This is shown in Fig. 9.23 at $Re_c = 3.1 \times 10^6$
Using a $268 \times 50$ grid size, flow over NACA 4412 hydrofoil was simulated at $Re_c = 3.1 \times 10^6$ with 13.9 degree incidence. The pressure coefficients on both suction and pressure side was computed and compared with Coles and Wadcock (1979) as shown in Fig. 9.24. Streamlines and velocity contours showing the separation near trailing edge was drawn (see Fig. 9.25 and 9.26).
The applicability of two-equation ($k-\epsilon$) turbulent model is in serious doubt in separated regions. Despite this doubt, the pressure coefficients agreed quite well. However, discrepancy in pressure coefficient values exists near the leading edge and trailing edge. Also the error on the suction side of the hydrofoil is more considerable than the pressure side and the flow separation occurs earlier than found in the experiment. The pressure computed on the suction side is less negative compared to Coles and Wadcock (1979). This may overestimate the adverse pressure gradient resulting in early flow separation.
CONCLUSIONS

In this research the numerical simulation of turbulent flow is performed around two-dimensional hydrofoils by finite volume method with non-orthogonal body-fitted grid. The governing equations are expressed in Cartesian velocity components and solution is carried out using SIMPLE algorithm for collocated arrangement of scalar and vector variables. Turbulence is modeled by the k-ε turbulence model and wall functions are used to bridge the solution variables at the near wall cells and the corresponding quantities on the wall. A simplified pressure correction equation is derived, and proper under-relaxation factors are used so that computational cost is reduced without adversely affecting convergence rate. The numerical procedure is validated by comparing the computed grid independent pressure distribution on the surface of NACA 0012 and NACA 4412 hydrofoils for different angles of attack with experimental data. The following conclusions can be drawn from the numerical study:

- The numerical results are found to show excellent agreement with experimental data until flow begins to separate
- As the number of cells is increased the accuracy of the solution increases and for a certain grid density the solution becomes grid independent which bears out that the numerical methodology is ‘well-behaved’.
- The maximum lift for NACA 4412 hydrofoil occurs at 13.9 degree incidence which marks the onset of flow separation. Beyond this angle numerical accuracy is found to be less satisfactory
- The performance of k-ε turbulence model deteriorates at high angles of attack when flow separation takes place
REFERENCES


Pinkerton, R. M. (1936): Calculated and Measured Pressure Distributions over the Mid Span Section of the NACA 4412 Airfoil, NACA Report No. 563, National Advisory Committee for Aeronautics.


APPENDIX A

Mean three dimensional momentum equations in conventional time averaged variables for three dimensional flows

\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho uu)}{\partial x} + \frac{\partial (\rho vv)}{\partial y} + \frac{\partial (\rho ww)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[ \frac{2}{3} \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] + 2\mu \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \\
+ \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] \\
\]

\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho uu)}{\partial x} + \frac{\partial (\rho vv)}{\partial y} + \frac{\partial (\rho ww)}{\partial z} \right\} dt \\
= \frac{1}{T} \int_0^T \left\{ -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[ \frac{2}{3} \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] + 2\mu \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \\
+ \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] \right\} dt \\
\]

(A.1)

\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial t} (\bar{u} + \rho' \bar{u} + \bar{v} + v') + \frac{\partial}{\partial x} (\bar{u} + \rho' \bar{u} + \bar{v} + v') (\bar{u} + \bar{u}') \right\} dt \\
+ \frac{\partial}{\partial y} \left( \bar{v} + v' \right) + \frac{\partial}{\partial z} \left( \bar{w} + w' \right) \right\} dt \\
= \frac{1}{T} \int_0^T \left\{ -\frac{\partial p}{\partial x} \right\} dt \\
+ \frac{\partial}{\partial x} \left[ \frac{2}{3} \mu \left( \bar{u} + u' \right) + \frac{\partial}{\partial y} (\bar{v} + v') + \frac{\partial}{\partial z} (\bar{w} + w') \right) \\
+ 2\mu \frac{\partial}{\partial x} (\bar{u} + u') + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial}{\partial y} (\bar{u} + u') + \frac{\partial}{\partial x} (\bar{v} + v') \right) \right] \\
+ \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial}{\partial z} (\bar{u} + u') + \frac{\partial}{\partial x} (\bar{w} + w') \right) \right] \right\} dt \\
\]

(A.2)

Linear Forces

Local inertia force term:

\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial t} (\bar{u} + \rho' \bar{u} + \bar{u}') \right\} dt \\
= \frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial t} (\bar{u} + \rho' \bar{u} + \bar{u}') \right\} dt \\
= \frac{\partial}{\partial t} \left\{ \frac{1}{T} \int_0^T (\bar{u} + \rho' \bar{u} + \bar{u}') dt \right\} \\
\]

But from the rules of averaging
We get
\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial t} (\bar{\rho} + \rho') (\bar{u} + u') \right\} dt
\]
\[
= \frac{\partial}{\partial t} \left\{ \bar{\rho} \bar{u} + \lim_{T \to \infty} \frac{1}{T} \int_0^T (\rho' u') dt \right\}
\]
\[
= \frac{\partial}{\partial t} (\bar{\rho} \bar{u} + \bar{\rho}' u')
\]  \hspace{1cm} \text{(A.3)}

Pressure force term:
\[
\frac{1}{T} \int_0^T \frac{\partial p}{\partial x} dt
\]
\[
= \frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T (\bar{p} + \rho') dt \right\}
\]
But from the rules of averaging
\[
\frac{1}{T} \int_0^T \bar{p} dt = \bar{p}
\]
\[
\frac{1}{T} \int_0^T p' dt = 0
\]
We get
\[
\frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T (\bar{p} + \rho') \right\} dt = \frac{\partial \bar{p}}{\partial x}
\]  \hspace{1cm} \text{(A.4)}

Viscous force terms:
\[
\frac{1}{T} \int_0^T \left\{ \frac{\partial}{\partial x} \left( \frac{2}{3} \mu \left( \frac{\partial}{\partial x} (\bar{u} + u') + \frac{\partial}{\partial y} (\bar{v} + v') + \frac{\partial}{\partial z} (\bar{w} + w') \right) + 2 \mu \frac{\partial}{\partial x} (\bar{u} + u') \right) \right\} dt
\]
\[
= \frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T \left[ \frac{2}{3} \mu \left( \frac{\partial}{\partial y} (\bar{u} + u') + \frac{\partial}{\partial z} (\bar{v} + v') + \frac{\partial}{\partial z} (\bar{w} + w') \right) + 2 \mu \frac{\partial}{\partial x} (\bar{u} + u') \right] dt \right\}
\]
\[
+ \frac{\partial}{\partial y} \left\{ \frac{1}{T} \int_0^T \left[ \mu \left( \frac{\partial}{\partial y} (\bar{u} + u') + \frac{\partial}{\partial x} (\bar{v} + v') \right) \right] dt \right\}
\]
\[
+ \frac{\partial}{\partial z} \left\{ \frac{1}{T} \int_0^T \left[ \mu \left( \frac{\partial}{\partial z} (\bar{u} + u') + \frac{\partial}{\partial x} (\bar{w} + w') \right) \right] dt \right\}
\]
But from the rules of averaging
\[
\frac{1}{T} \int_0^T \frac{\partial \bar{u}}{\partial x} \, dt = \frac{\partial}{\partial x} \left( \frac{1}{T} \int_0^T \bar{u} \, dt \right) = \frac{\partial \bar{u}}{\partial x}
\]
Similarly,
\[
\frac{1}{T} \int_0^T \frac{\partial \bar{v}}{\partial y} \, dt = \frac{\partial \bar{v}}{\partial y}, \quad \frac{1}{T} \int_0^T \frac{\partial \bar{w}}{\partial z} \, dt = \frac{\partial \bar{w}}{\partial z}
\]
\[
\frac{1}{T} \int_0^T \frac{\partial \bar{v}}{\partial x} \, dt = \frac{\partial \bar{v}}{\partial x}, \quad \frac{1}{T} \int_0^T \frac{\partial \bar{w}}{\partial y} \, dt = \frac{\partial \bar{w}}{\partial y}
\]
\[
\frac{1}{T} \int_0^T \frac{\partial \bar{w}}{\partial x} \, dt = \frac{\partial \bar{w}}{\partial x}, \quad \frac{1}{T} \int_0^T \frac{\partial \bar{w}}{\partial z} \, dt = \frac{\partial \bar{w}}{\partial z}
\]
Also
\[
\frac{\partial}{\partial x} \left( \frac{1}{T} \int_0^T \bar{u}' \, dt \right) = \frac{\partial}{\partial y} \left( \frac{1}{T} \int_0^T \bar{v}' \, dt \right) = \frac{\partial}{\partial z} \left( \frac{1}{T} \int_0^T \bar{w}' \, dt \right) = 0
\]
Using the above results we get
\[
\frac{1}{T} \int_0^T \left\{ \frac{2}{3} \mu \left( \frac{\partial}{\partial x} (\bar{u} + \bar{u}') + \frac{\partial}{\partial y} (\bar{v} + \bar{v}') + \frac{\partial}{\partial z} (\bar{w} + \bar{w}') \right) + 2\mu \frac{\partial}{\partial x} (\bar{u} + \bar{u}') \right\} \, dt
\]
\[
= \frac{\partial}{\partial x} \left[ \frac{2}{3} \mu \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} \right) + 2\mu \frac{\partial \bar{u}}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \right] + \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right) \right] \quad (A.5)
\]
\[
= \frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z} \quad (A.6)
\]
Where,
\[
\bar{\tau}_{xx} = \frac{2}{3} \mu \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} \right) + 2\mu \frac{\partial \bar{u}}{\partial x}
\]
\[
\bar{\tau}_{yy} = \frac{2}{3} \mu \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} \right) + 2\mu \frac{\partial \bar{v}}{\partial y}
\]
\[
\bar{\tau}_{zz} = \frac{2}{3} \mu \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} \right) + 2\mu \frac{\partial \bar{w}}{\partial z}
\]
\[
\bar{\tau}_{xy} = \bar{\tau}_{yx} = \mu \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right)
\]
\[
\bar{\tau}_{xz} = \bar{\tau}_{zx} = \mu \left( \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right)
\]
\[ \tau_{xy} = \tau_{yx} = \mu \left( \frac{\partial \bar{w}}{\partial y} + \frac{\partial \bar{v}}{\partial z} \right) \quad (A.7) \]

Convective inertia force terms (nonlinear terms)

\[
\begin{align*}
\frac{1}{T} \int_{0}^{T} & \left\{ \frac{\partial}{\partial x} (\bar{\rho} + \rho')(\bar{u} + u')(\bar{u} + u') + \frac{\partial}{\partial y} (\bar{\rho} + \rho')(\bar{u} + u')(\bar{v} + v') \right. \\
& + \frac{\partial}{\partial z} (\bar{\rho} + \rho')(\bar{u} + u')(\bar{w} + w') \bigg\} \, dt \\
= & \frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_{0}^{T} (\bar{\rho} + \rho')(\bar{u} + u')(\bar{u} + u') \bigg\} \, dt + \frac{\partial}{\partial y} \left\{ \frac{1}{T} \int_{0}^{T} (\bar{\rho} + \rho')(\bar{u} + u')(\bar{v} + v') \bigg\} \, dt \right. \\
& + \frac{\partial}{\partial z} \left\{ \frac{1}{T} \int_{0}^{T} (\bar{\rho} + \rho')(\bar{u} + u')(\bar{w} + w') \bigg\} \, dt \bigg\} \\
& \frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_{0}^{T} (\bar{\rho} \bar{u} u' + 2\bar{\rho} \bar{u} u' + \rho' \bar{u} u' + 2\rho' \bar{u} u' + \rho' \bar{u} u' + \rho' \bar{u} u') \bigg\} \\
& + \frac{\partial}{\partial y} \left\{ \frac{1}{T} \int_{0}^{T} (\bar{\rho} \bar{u} v' + \bar{\rho} \bar{u} v' + \bar{\rho} \bar{u} v' + \rho' \bar{u} \bar{v} + \rho' \bar{u} v' + \rho' \bar{u} v' + \rho' \bar{u} v' + \rho' \bar{u} v') \bigg\} \\
& + \frac{\partial}{\partial z} \left\{ \frac{1}{T} \int_{0}^{T} (\bar{\rho} \bar{w} \bar{u} + \bar{\rho} \bar{w} \bar{u} + \bar{\rho} \bar{w} \bar{u} + \rho' \bar{w} \bar{u} + \rho' \bar{w} \bar{u} + \rho' \bar{w} \bar{u}) \bigg\} \\
& + \rho' \bar{u} \bar{w} \bigg\} \\
\end{align*}
\]

But from the rules of averaging

\[
\begin{align*}
\frac{1}{T} \int_{0}^{T} (\bar{\rho} \bar{u} \bar{u}) \, dt &= \bar{\rho} \bar{u} \bar{u}, \frac{1}{T} \int_{0}^{T} (2\bar{\rho} \bar{u} u') \, dt = 0 \\
\frac{1}{T} \int_{0}^{T} (\rho' \bar{u} \bar{u}) \, dt &= 0, \frac{1}{T} \int_{0}^{T} (2\bar{\rho} \bar{u} \bar{u}) \, dt = 2\bar{\rho} \bar{u} \bar{u} \\
\frac{1}{T} \int_{0}^{T} (\bar{\rho} \bar{u} u') \, dt &= \bar{\rho} \bar{u} u', \frac{1}{T} \int_{0}^{T} (\rho' \bar{u} u') \, dt = \bar{\rho} \bar{u} u' \\
\frac{1}{T} \int_{0}^{T} (\bar{\rho} \bar{v} \bar{v}) \, dt &= \bar{\rho} \bar{v} \bar{v}, \frac{1}{T} \int_{0}^{T} (\rho' \bar{v} \bar{v}) \, dt = \bar{\rho} \bar{v} \bar{v} \\
\frac{1}{T} \int_{0}^{T} (\bar{\rho} \bar{v} \bar{v}) \, dt &= \bar{\rho} \bar{v} \bar{v}, \frac{1}{T} \int_{0}^{T} (\rho' \bar{v} \bar{v}) \, dt = \bar{\rho} \bar{v} \bar{v} \\
\frac{1}{T} \int_{0}^{T} (\rho' \bar{v} \bar{v}) \, dt &= 0, \frac{1}{T} \int_{0}^{T} (\bar{v} \rho' \bar{v}) \, dt = \bar{v} \rho' \bar{v} \\
\frac{1}{T} \int_{0}^{T} (\bar{u} \rho' \bar{v}) \, dt &= \bar{u} \rho' \bar{v}, \frac{1}{T} \int_{0}^{T} (\rho' \bar{u} \bar{v}) \, dt = \bar{u} \rho' \bar{v} \\
\frac{1}{T} \int_{0}^{T} (\bar{u} \rho' \bar{v}) \, dt &= \bar{u} \rho' \bar{v}, \frac{1}{T} \int_{0}^{T} (\rho' \bar{u} \bar{v}) \, dt = \bar{u} \rho' \bar{v} \\
\frac{1}{T} \int_{0}^{T} (\bar{u} \rho' \bar{v}) \, dt &= 0, \frac{1}{T} \int_{0}^{T} (\rho' \bar{u} \bar{v}) \, dt = \bar{u} \rho' \bar{v} \\
\frac{1}{T} \int_{0}^{T} (\bar{u} \rho' \bar{v}) \, dt &= 0, \frac{1}{T} \int_{0}^{T} (\rho' \bar{u} \bar{v}) \, dt = \bar{u} \rho' \bar{v} \\
\end{align*}
\]
Using the above results we get

\[
\frac{1}{T} \int_0^T (\rho' \bar{u} \bar{w}) dt = 0, \quad \frac{1}{T} \int_0^T (\bar{w} \rho' \bar{u}') dt = \bar{w} \rho' \bar{u}'
\]

\[
\frac{1}{T} \int_0^T (\bar{u} \rho' \bar{w}') dt = \bar{u} \rho' \bar{w}' - \frac{1}{T} \int_0^T (\rho' \bar{u}' \bar{w}') dt = \bar{u}' \bar{w}'
\]

Using the above results we get

\[
\frac{\partial}{\partial x} \left\{ \frac{1}{T} \int_0^T (\rho \bar{u} \bar{u} + 2 \rho \bar{u} \bar{u}' + \rho \bar{u}' \bar{u}' + 2 \rho' \bar{u}' \bar{u}' + \rho' \bar{u}' \bar{u}' + \rho' \bar{u}' \bar{u}') dt \right\}
+ \frac{\partial}{\partial y} \left\{ \frac{1}{T} \int_0^T (\rho \bar{v} \bar{u}' + \rho \bar{v} \bar{u}' + \rho \bar{v} \bar{u}' + \rho \bar{v} \bar{u}' + \rho \bar{v} \bar{u}' + \rho \bar{v} \bar{u}') dt \right\}
+ \frac{\partial}{\partial z} \left\{ \frac{1}{T} \int_0^T (\rho \bar{w} \bar{w}' + \rho \bar{w} \bar{w}' + \rho \bar{w} \bar{w}' + \rho \bar{w} \bar{w}' + \rho \bar{w} \bar{w}') dt \right\}
\]

\[
= \frac{\partial}{\partial x} (\rho \bar{u} \bar{u} + 2 \bar{u} \rho \bar{u}' + \rho \bar{u}' \bar{u}') + \frac{\partial}{\partial y} (\rho \bar{v} \bar{u}' + \bar{v} \rho \bar{v}' + \bar{v} \rho \bar{v}' + \bar{v} \rho \bar{v}') + \frac{\partial}{\partial z} (\rho \bar{w} \bar{w}' + \bar{w} \rho \bar{w}' + \bar{w} \rho \bar{w}')
\]

\[
(A.8)
\]

Now putting results in (A.1) we get the following form of momentum equation in the x direction

\[
\frac{\partial}{\partial t} (\rho \bar{u} + \rho' \bar{u}') + \frac{\partial}{\partial x} (\rho \bar{u} \bar{u} + 2 \bar{u} \rho \bar{u}' + \rho \bar{u}' \bar{u}') + \frac{\partial}{\partial y} (\rho \bar{u} \bar{v}' + \bar{v} \rho \bar{v}' + \bar{v} \rho \bar{v}') + \frac{\partial}{\partial z} (\rho \bar{u} \bar{w}' + \bar{w} \rho \bar{w}' + \bar{w} \rho \bar{w}')
\]

\[
= - \frac{\partial \bar{p}}{\partial x} + \frac{\partial \bar{t}_{xx}}{\partial x} + \frac{\partial (\bar{t}_{yx} \bar{u})}{\partial y} + \frac{\partial (\bar{t}_{xz} \bar{u})}{\partial z}
\]

On rearranging we get,

\[
\frac{\partial}{\partial t} (\rho \bar{u} + \rho' \bar{u}') + \frac{\partial}{\partial x} (\rho \bar{u} \bar{u} + \bar{u} \rho \bar{u}') + \frac{\partial}{\partial y} (\rho \bar{v} \bar{u}' + \bar{u} \rho \bar{v}') + \frac{\partial}{\partial z} (\rho \bar{w} \bar{u}' + \bar{u} \rho \bar{w}')
\]

\[
= - \frac{\partial \bar{p}}{\partial x} + \frac{\partial \bar{t}_{xx} \bar{u}}{\partial x} + \frac{\partial (\bar{t}_{yx} \bar{u})}{\partial y} + \frac{\partial (\bar{t}_{xz} \bar{u})}{\partial z}
\]

This is the Reynolds averaged momentum equation for unsteady, compressible flow in the x direction in conventional time averaged variables. The time averaged momentum equations in the y and z directions may be obtained using a similar methodology.

So finally the conventional time averaged momentum equations in x, y and z directions for unsteady, compressible flow may be written as follows:
Mean continuity equation in conventional time averaged variables for three dimensional flows

The time averaged form of continuity equation is as follows:

$$\frac{\partial}{\partial t}(\bar{\rho}\bar{u}) + \frac{\partial}{\partial x}(\bar{\rho}\bar{u} + u'u') + \frac{\partial}{\partial y}(\bar{\rho}\bar{v} + v'v') + \frac{\partial}{\partial z}(\bar{\rho}\bar{w} + w'w') = 0$$

But from the rules of averaging

$$\frac{1}{T} \int_0^T \bar{\rho} \bar{u} dt = \bar{\rho} \bar{u} = \bar{\rho} \bar{u}$$

$$\frac{1}{T} \int_0^T \bar{\rho} u' dt = 0$$

$$\frac{1}{T} \int_0^T \bar{\rho} u' dt = 0$$
\[ \frac{1}{T} \int_0^T \bar{\rho} \bar{v} \, dt = \bar{\rho} \bar{v}, \frac{1}{T} \int_0^T \bar{\rho} v' \, dt = 0, \frac{1}{T} \int_0^T \rho' \bar{v} \, dt = 0 \]

\[ \frac{1}{T} \int_0^T \bar{\rho} \bar{w} \, dt = \bar{\rho} \bar{w}, \frac{1}{T} \int_0^T \bar{\rho} w' \, dt = 0, \frac{1}{T} \int_0^T \rho' \bar{w} \, dt = 0 \]

\[ \frac{1}{T} \int_0^T \bar{\rho} \, dt = \bar{\rho}, \frac{1}{T} \int_0^T \rho' \, dt = 0, \frac{1}{T} \int_0^T \rho' u' \, dt = \bar{\rho}' u' \]

\[ \frac{1}{T} \int_0^T \rho' v' \, dt = \bar{\rho}' v', \frac{1}{T} \int_0^T \rho' w' \, dt = \bar{\rho}' w' \]

Putting the above results in equation (A.12) we obtain mean continuity equation for unsteady compressible flow as follows

\[ \frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x} (\bar{\rho} \bar{u} + \bar{\rho}' u') + \frac{\partial}{\partial y} (\bar{\rho} \bar{v} + \bar{\rho}' v') + \frac{\partial}{\partial y} (\bar{\rho} \bar{w} + \bar{\rho}' w') = 0 \]

(A.13)
Appendix B

PROGRAM outline

1. Read problem name from command line

2. CALL INIT: Read input parameters and grid data

3. Start Main time loop

   3.1 Start Relaxations loop

      3.1.1 CALL CALCUV: Build and iterate $u, v$ momentum equations.

         3.1.1.1 CALL GRADFI: Compute $u, v$ gradients

         3.1.1.2 CALL FLUXUV: Compute momentum fluxes

         3.1.1.3 CALL SIPSOL: Call Solver for $u, v$

      3.1.2 Build $p'$ equation. Call Solver for $p'$

         3.1.2.1 CALL GRADFI: Compute $p'$ gradients

         3.1.2.2 CALL FLUXM: Compute mass fluxes

         3.1.2.3 CALL SIPSOL: Call Solver for $p'$

         3.1.2.4 Correct pressure, velocities and mass fluxes

      3.1.3 CALL CALCSC: Build and iterate $k, \varepsilon$ equations.

         3.1.3.1 CALL GRADFI: Compute $k, \varepsilon$ gradients

         3.1.3.2 CALL FLUXSC: Compute fluxes of $k, \varepsilon$

         3.1.3.3 CALL SIPSOL: Call Solver for $k, \varepsilon$

      3.1.4 Check Convergence

4. Print out final results
Explanations for Grid.f

NGR  number of grid

NICV  number of cells in I direction

NJCV  number of cells in J direction

XLS, YLS  coordinate of starting point of line

XLE, YLE  coordinate of ending point of line.

DXI  size of segment at line beginning

EXP  expansion factor

NI, NJ  number of nodes in I, J directions (including boundary nodes)

NIJ  total number of nodes. NIJ=NI*NJ

NINL  number of boundary CV faces assigned with inlet boundary condition

NOUT  number of boundary CV faces assigned with outlet boundary condition

NSYM  number of boundary CV faces assigned with symmetry boundary condition.

NWAL  number of boundary CV faces assigned with wall boundary condition.

NOC  number of boundary CV faces in O-C cuts.

NWALI  number of boundary CV faces assigned with Isothermal wall boundary condition

NWALA  number of boundary CV faces assigned with adiabatic wall boundary condition

X (I), Y (I)  coordinates of grid corners

XC (I),YC (I)  coordinates of cell centers
FX (I), FY (I)  
interpolation factors

VOL (I)  
volume of each cell

AR  
cell face area

DN  
normal distance from cell face center to cell center

SRDW  
AR/DN at the wall

SRDS  
AR/DN at the symmetry plane

FIS  
angle at line start

FIE  
angle at line end

**Explanations for CAFFA.f**

RESAB  
stored as RESOR (IFI) and used to check the convergence of outer iterations

MAXIT  
maximum allowed number of outer iterations per time step

IMON/JMON  
grid indices of the monitoring point

IPR/JPR  
grid indices of the point at which the pressure is kept constant

SORMAX/SLARGE  
limits for the absolute sums of residuals which define convergence and divergence,

ALFA  
relaxation parameter in the SIP-solver

URPP  
under-relaxation parameter in the pressure-correction

UIN/VIN/PIN/TIN  
initial values of velocity, pressure and temperature

DT  
time step size

LCAL(I), I=1,NFI  
logical variable identifier where NFI is the number of variables

SOR(I)  
convergence criterion for inner iterations

NSW(I)  
maximum allowed number of inner iterations for each variable

GDS(I)  
blending factor, defining the proportion of central differencing
scheme used (1.0 - pure central; 0.0 - pure upwind).

RESMAX  residual level at which iterations are stopped
DEN      density
VEL      velocity
ITSTEP   maximum number of time steps
UDS/CDS  upwind differencing scheme/ central differencing scheme
GRAVX    gravity vector in the x-direction
GRAVY    gravity vector in the Y-direction
F1(IJ)/ F2(IJ) mass fluxes
DPXEL    pressure gradient at CV face
LREAD, LWRITE,
LTEST, LOUTS,
LOUTE and LTIME logical variables