A METHODOLOGY FOR SOLVING TWO PERSON GAME UNDER INTERVAL UNCERTAINTY

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ABSTRACT

In this thesis, robust optimization methodologiesare developed for solving two person zero sum and non zero sum games that consider single or multiple interval inputs (i.e., interval-valued payoffs). Real life problems are not always deterministic and in competitive situations, exact information of competitors is not available. In practice, as the sufficient data from historical sources is quite difficult to obtain, this leads to the games in an uncertain environment. Thus deterministic assumptions about inputs in stochastic environments may lead to infeasibility or poor performance. In such situations, conventional methods that usedeterministic payoffs are not appropriate.Therefore, a method is necessary that can incorporate interval data uncertainty in the analysis of competitive situations. In this thesis,methods for two-person games with interval payoffs have been investigated. The proposed approaches are able to aggregate information from multiple sources and thereby result in more realistic outcomes. The robust optimization methods developed in this thesis can be used to solve two person non-cooperative games with interval-valued (single or multiple intervals) payoffs as well as with single-valued payoffs or a combination of both. A decoupled approach isalso proposed in this thesis to un-nest the robustness-based optimization from the analysis of interval variables to achieve computational efficiency. The proposed methodologies are illustrated with several numerical examples including an investment decision analysis problem.The proposed decoupled approach is compared with some previously developed approaches and it is demonstrated that the proposed formulations generate conservative solutions in the presence of uncertainty.

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CHAPTER 1

INTRODUCTION

1.1 Background

Game theory is a collection of mathematical models to study the behaviors of decision makers with interest conflict and has been applied extensively to engineering and economics (Gao, 2011). Classical game theory has many useful applications in business and engineering that include investment decision analysis (Li 2011), capacity expansion analysis (Do et al., 2015), and supply chain analysis (Reyes, 2006; Zhang and Huang, 2010). In classical game theory, it is generally assumed that all inputs (i.e., payoffs) are precisely known and the influence of data or distribution parameter uncertainty on the optimality and feasibility of the solutions is not explicitly considered. However, real-life problems are not deterministic and this deterministic assumption about inputs may lead to infeasibility or poor performance. In real-world situations, the players often lack the information about the other players' (or even their own) payoffs, which leads to the games in uncertain environments (Li 2011). In the real world, interval data frequently occur in many situations such as in measurement, expert opinions, etc.However, entire input and output relation of a game is conclusively determined for a deterministic model which assumes that input data (i.e. payoffs) are precisely known in advance.The current research focuses on developing generalized computational methods of game theory under uncertainty arising from interval data.

There exists an extensive volume of methods and applications of game theory under interval uncertainty. Li (2011) proposed a liner programming approach to solve two-person zero-sum game with interval-valued payoff matrix. Collins and Hu (2008) proposed fuzzy logic-based methodology to solve interval-valued two-person games. Interval-valued two-person games have also been studied extensively by many authors including Sohraiee et al (2010), and Alparslan-Gök et al (2013). Most of the existing methods for interval game theory consider only single interval data. However, in practice, a single interval may not be available due to the availability of data from multiple sources, for example, from experts' opinion. Therefore, an approach to solve two-person games by aggregating information from multiple sources is needed. The current research is intended to develop a general robust optimization formulation for two-person games under interval uncertainty arising from multiple sources, where robustness is achieved by simultaneously optimizing the mean and minimizing the variance of the value of the game.

1.2 Objectives of the Study

The specific objectives of this research are-

- Development of a robust optimization model for two-person zero sum games with interval payoff matrix which consists of all possible outcomes of a game, where each outcome is described by multiple interval data.
- Development of a robust optimization model for two-person nonzero sum games with interval payoff matrix which consists of all possible outcomes of a game, where each outcome is described by multiple interval data.

Therefore, the proposed research develops and demonstrates generalized methodologies for solving two-person games under interval uncertainty arising from multiple sources. The methodologies developed in this research can be used to solve problems in various domains including business and engineering, for example, investment decision problem, capacity expansion problem, supply chain analysis problem, etc.

1.3 Outline of the Methodology

The research methodology is outlined below:

a) First two moments of interval data have been calculated as bounds using moment bounding algorithms (Zaman et al, 2011a).
b) A framework for the representation of interval data uncertainty in game theory analysis model has been developed using the moment bounds obtained in (*a*).
c) A decoupled formulation for robust optimization to solve two-player zero sum games has been proposed based on the uncertainty representation framework developed in (*b*).
d) A decoupled formulation for robust optimization to solve two-player non zero sum games has been proposed based on the uncertainty representation framework developed in (*b*).

e) A computational framework has been proposed to solve the optimization formulations developed in (*c*) and (*d*).
f) The proposed methodologies have been illustrated for several example problems including an investment decision analysis problem.

1.4 Contributions of the Present Study

Life is full of conflict and competition. In competitive situations, a decision does not only depend on one's own strategies but also on opponent's decisions. In many cases to make a decision, we have to rely on interval data from multiple sources. The overall goal of this research is to develop and demonstrate robustness-based optimization methods for two-person games under interval uncertainty. This thesis proposes two decoupled formulations for robust optimization to solve two person games-one for zero sum games and the other for non zero sum games.

The proposed solution approaches can be characterized as follows:

a) The objective function and the constraint functions of the models are expressed in terms of the mean and variance of payoff values.
b) The robustness-based optimization model provides a conservative solution under interval data uncertainty.
c) Proposed decoupled approachfor solving zero sum and non zero sum game can determine game value by considering the variation of multiple interval data.

1.5 Organization of the Thesis

The rest of the thesis is organized as follows. Chapter 2 presents the literature review of all the relevant topics of the thesis. Chapter 3 gives an overview of game theory, non-cooperative game, two person zero sum game and two person non zero sum game.Chapter 4 describes the proposed decoupled approaches for solving two-person games under interval uncertainty. In chapter 5, the proposed methods are illustrated withnumerical examples including an investment decision analysis problem. Chapter 6 provides conclusions and suggestions for future work.

CHAPTER 2

LITERATURE REVIEW

Game theory is a powerful tool for modeling the interactions of independent decision makers. The internal consistency and mathematical foundations make the game theory a prime tool to make decisions in competitive environments where outcomes are interdependent. Game theory, a branch of mathematics, gives mathematical expressions to the strategies of opposing players and has widely been applied in engineering, business, finance and management that include commercial supply chains (Ketchen and Hult, 2007), investment decision (Nanduri et al, 2009), and marketing strategy (Huang and Li, 2001).Moorthy (1985) applied non-cooperative game theory principles to two airline companies and asserted that Nash equilibrium, a solution of a game where each player's strategy is optimal given the strategies of all other players, is necessary for firms to be comfortable with their strategies and the assumptions it foregrounds concerning other players of the game. Nadeau (2002) used game theory in the health and safety sector in order to model conditions for cooperation between managers and workers.Moreover, Smit and Ankum (1993) used the real options approach for project timing and Murphy and Smeers (2005) considered three model of investment in capacity expansion.

Concepts of game theory began with point data. In deterministic games, it is considered that all inputs (i.e., payoffs) are considered fixed point data and precisely known, and the influence of natural variability and uncertainty is ignored to determine the game value. Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent and rational decision-makers (Myerson, 1991). Some ideas of game theory can be traced to the eighteenth century but it was extensively developed mainly in the 20th century with the work of John von Neumann (1903–1957) and Oskar Morgenstern (1902-1977) who established game theory in a more uniformed way and gave the basis for future research. In 1913 Ernst Zermelo published the paper"*Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels*".It proved that a game as complicated as chess is solvable through backward induction. This paved the way for more general theorems for two person non-cooperative games (Screpantiet al, 2005).

Minimax theorem is considered as a useful tool to solve games, especially zero-sum games. In 1928, this theorem was established and published by John von Neumann(1928).Most important and influential was his book *Theory of Games and Economic Behavior* co-authored with Oskar Morgensternpublished in 1944 (Von Neumann and Morgenstern, 1944), which is considered the groundbreaking text that created the interdisciplinary research field of game theory. This foundational work consisted of the method for solving two-person zero-sum games. It introduced axioms for the concept of the individual rational player. Such a player makes consistent decisions in the face of certain and uncertain alternatives. Game theory did not really exist as a unique field until the paper "*On the theory of games of strategy*." was published by Neumann (1959).

In game theory, minimax theorem shows minimax solution of two person games. Minimax solution of a game is named Nash equilibrium after John Forbes Nash, Jr. Antoine Augustin Cournot in 1838 introduced an initial version of Nash equilibrium in his theory of oligopoly (Neumann, 1928). In a non-cooperative game, players try to choose their best payoff, which constitutes anoptimum solution for the game. A solution of a game is called Nash equilibrium when all the players choose their best payoffs (Cachon & Netessine 2004). Nash-equilibrium is a set of strategies that no single player can obtain a higher value of expected utility by deviating unilaterally from that strategies. Nash gave two existence proofs, one (Nash, 1950) was based on Kakutani's Fixed Point Theorem and other (Nash, 1951) was based on the less general Brouwer's Fixed Point Theorem. A Nash player is not only rational but also assumes that all players are rational to such a degree that players can coordinate their strategies so that Nash equilibrium exists (Holler, 2002). Nash in his 1951 article defined mixed strategy Nash equilibrium for any game with a finite set of actions and proved that at least one (mixed-strategy) Nash equilibrium must exist in any game. The Nash equilibrium, a crucial concept in non-cooperative games, helped produce a revolution in the use of game theory in economics and Nash won Nobel Memorial Prize in Economic Sciences in 1994 as a game theorist.The studies mentioned above developed basic concepts of game theory useful to analysis in competitive environments when inputs (payoffs) are deterministic.

However, a deterministic assumption about inputs may result in infeasibility and poor performance in supporting decision making, because real-life problems are not always deterministic (Sim, 2004). There exists a large volume of work for games under interval uncertainty that developed many useful models for analyzing competitive situations in engineering and business. This thesis specifically focuses on developing robust optimization approach for two-person non cooperative games under interval uncertainty arising as single and multiple interval data.

Real-world optimization problems, such as those arising from optimal design of physical, medical or engineering systems, often contain parameters whose values cannot be exactly determined because of various technical difficulties. Therefore, assuming the availability of point data for input variables is infeasible in many cases. Uncertainty arises in engineering problems from different sources, quantification with accuracy is necessary for analysis. It is challenging to quantify uncertainty arising from different sources. The sources and characterization of uncertainties are essential in engineering modeling for risk and reliability analysis. Sources of uncertainty may be divided into two types: aleatory and epistemic (Oberkampf et. al., 2004). Aleatory uncertainty is inherent randomness of a system and irreducible. Examples include phenomena that exhibit natural variation like environmental conditions (temperature, wind speed, etc.). System failure due to a natural disaster andthe government policy are examples of aleatory uncertainty in investment decision problem. In contrast, epistemic uncertainty can be reduced by gathering more data or by refining models and results.Liu (2011) introduced the importance of uncertainty analysis and the situations where the theory of uncertainty is applicable. Liu (2013) proposed the concepts of chance distribution, expected value, and variance of uncertain random variables. In practical problems, uncertain data may be available in the form of interval due to the availability of data from multiple sources.

Interval data are frequently occurred in engineering analysis and decision making, for example, structural design (Du et al, 2005) and aerospace vehicle design (Zaman et al, 2011b). A number of methods (e.g. Zaman et al, 2011c; Ferson et al, 2007; Zaman et al, 2011a) were developed that can deal with interval variables. Du et al. (2005) proposed a method of reliability-based design optimization (RBDO) where the uncertain variables characterized by the mixture of probability distributions and intervals. Zaman et al, (2010) developed a probabilistic approach for representation and propagation of uncertainty in system analysis. Sampling-based and optimization-based methods for the propagation of both probabilistic and interval uncertainty are represented and compared in term of accuracy and computational expense. Zaman et al. (2011a) developed a method using nonlinear programming for calculating the moments from the different combination of multiple interval data. Zaman and Kritee (2014) proposed an optimization algorithm for construction of confidence interval on mean for interval data with different numbers of intervals and type of overlap. A Likelihood-based methodology has been developed to estimate the epistemic uncertainty from the interval data for any distribution (Dey and Zaman, 2014).

In order to make a decision, we have to rely on expert's opinion when sufficient historical data are not available. In many situations, data are available in form of multiple intervals. When dataarises from multiple sources, it is needed to evaluate the mean, variance, and other high moments for interval data. Finding moments with interval data has been generally considered an NP hard problem, because it includes a search among the combinations of multiple values of the variables, including interval end points. Zaman et al. (2011b) developed a methodology for uncertainty representation in system analysis when input parameters are available in either probability distribution or simply interval form. Uncertainty in interval data was represented through a flexible family of probability distributions. Zaman et al. (2011a) proposed a probabilistic approach to represent interval data for input variables in reliability and uncertainty analysis problems, using flexible families of continuous Johnson distributions. This is a unified framework for representing both aleatory and epistemic uncertainty. Moments of the interval data are represented in the form of intervals which have upper and lower bounds. Algorithms to evaluate the bounds on second and higher moments of interval data were developed based on continuous optimization.

In the real-world competitive situations, the payoffs may be unknown or available as intervals.Therefore, the outcome of a game is not feasible under deterministic assumptions about inputs in the uncertain environment. There are different types of games based on the number of players, the number of strategies and the structure of payoff matrix (Von Neumann and Morgenstern, 1953). After1944 when a book published by Von Neumann and Morgenstern, the researchers have been working to add new features such as data uncertainty, interval data and so on to make broader application areas of non-cooperative games. On the other hand, some work on game theory are primarily focused on cooperative games. The differences between cooperative and non-cooperative games were introduced by Aumarm (1959) and Shubik (2002).In the present thesis, a methodology for two-person non-cooperativegames is developed where the payoffs represent by interval data. In the rest of the chapter, we focus on two-person non-cooperative games under uncertain interval inputs.

There exists a large volume of work for non-cooperative games that consider interval payoffs. Within the framework of uncertainty theory, Gao (2011) examined the uncertain-payoffs of two player nonzero-sum games. He introduced three decision criteria to define the behaviors of players, which lead to three types of games. For each type, he presented a new definition of Nash equilibrium as well as one sufficient and necessary conditions that provide a way to find such Nash equilibrium. In order to determine the probability distribution or density function of a random variable, it needs enough historical data for probabilistic reasoning. However, things often go contrary to our wishes. People resort to the concept of fuzzy set that was initiated by Zadeh (1965) when enough historical data is not available. Fuzzy games have been studied by many authors including Garagic and Cruz (2003), Russell and Lodwick (2002) and Wu and Soo (1998). Collins and Hu (2005) extended the strategies for classical strictly determined matrix into fuzzily determined interval matrix games by defining fuzzy binary comparison. Loganthan and Christi (2012) considered payoffs as interval values in two-person zero-sum game.

Liu and Kao (2009) developed a methodology for solving two-person zero-sum games where the payoffs were expressed with intervals (i.e., data having lower and upper bounds). A pair of mathematical models was used for obtaining upper and lower bounds of the value of the game. Based on the duality theorem and by applying a variable substitution technique, the pair of two-level mathematical programs had been transformed into a pair of ordinary one-level linear programs. Solving the pair of linear programs produces the value of the game as an interval. Li (2011) developed a simple and effective linear programming method in which payoffs were imprecise and represented by intervals instead of deterministic point values. The value (interval) of the two-player zero-sum interval-valued matrix game is obtained by using lower and upper bounds of the payoff intervals. This method was also compared with other established methods such as Liu and Kao (2009) to check validity.Most of the existing approaches determine game value by using upper and lower bounds on separate models. Those models don't consider variance of interval data.

A few papers (e.g., Kuhn, 1961; Lemke and Howson, 1964; Mangasarian and Stone, 1964) specifically dealt with an actual numerical method for finding equilibrium points of two-person non-zero sum games. Lemke and Howson (1964) gave an algebraic proof of the equilibrium point which is Nash equilibrium point for two-person non-zero-sum (bimatrix) games.Two-person non-zero sum game under interval and unknown payoffs is investigated based on Linear Complementarity Problem (LCP) by Sohraiee et al, (2010). They showed that the two-person games with interval data can be transformed to LCP. Mangasarian and Stone (1964) proposed a methodology to solve two-person non-zero-sum games with a finite number of strategies. They developed a quadratic programming model with linear constraints and a quadratic objective function (not concave) that has a global maximum of zero. Meng and Zhan (2014) introduced a two-step method for constrained bimatrix games. In this thesis, thequadratic programming model (Mangasarian and Stone, 1964)is converted into a robust optimization problem that considers uncertainty in payoff values.

Game theory is a useful tool to analyze real-life competitive situations in order to make decisions. A deterministic model results in low-performance efficiency and infeasible solution under the condition of uncertainty. Variations are common in manufacturing processes, service processes, and users' environment. Effects of the variations are considered explicitly in robust optimization in order to minimize their consequences without eliminating their sources. The origins of robust optimization date back to the establishment of modern decision theory in the 1950s and the use of the worst case analysis and Wald's maximin model as a tool for the treatment of severe uncertainty. Taguchi (1924-2012) has played a vital role in popularizing the notion of robust design by introducing his well-known statistical or robust design method. He considered noise and control factors and their effect in order to minimize their effects in manufacturing processes. Taguchi (1993) proposed robust design method that is insensitive to noise to achieve product and process quality. The method focuses on design and development to create efficient, reliable products. Taguchi method has been used in Operations Research, Control Theory, Finance, Portfolio Management, Logistics, Manufacturing Engineering, Chemical Engineering that include process improvement (Rosa et al, 2009) and production (Rao et al, 2004).

Although Taguchi's methods have extensive applications in engineering, the statistics community pointed to inefficiencies in the method (Box, 1988) and these methods cannot solve problems with multiple measures of performances and design constraints (Wei et al, 2009). There is now an extensive volume of literature for robust optimization methods and applications. Du and Chen (2000) examined several feasibility modeling methods for robust optimization under the effect of uncertainties. Although many real-life optimization problems are nonlinear and nonconvex, most studies in robust optimization focused on convex programming with linear performance function. To overcome these drawbacks, Zhang (2007) introduced a nonlinear robust optimization method(the first order method) with uncertain parameters involving both equality and inequality constraints. However, the method is applicable when variations are moderate. Hale and Zhang (2007) solved some robust nonlinear programming problems in order to assess the effectiveness of the first order method. They showed that the method is inexpensive and produces reasonable solutions when the level of uncertainty is small to moderate.

Over the years, robust optimization techniques have been used in many areas, such as operations research (Bertsimas and Sim, 2004), control theory (Khargonekar et al, 1990), logistics (Yu and Li, 2000), finance (Fabozzi et al, 2007), and chemical engineering (Bernardo and Saraiva, 1998).These methods had only been studied with respect to physical or natural variability represented by probability distributions. Uncertainty in system design also arises from other contributing factors. A few studies on robust design optimization to deal with epistemic uncertainty arises due to limited data and knowledge. A possibility-based method was integrated with robust optimization under epistemic uncertainty by Youn et al, (2007) to redefine the performance measure of robust design using the most likely values of fuzzy random variables.Dai and Mourelatos (2003) proposed two-step methods for robust design optimization to achieve robustness under both aleatory and epistemic uncertainty using a range method and a fuzzy sets approach. Zaman et al. (2011b) proposed robustness-based design optimization formulations that work under both aleatory and epistemic uncertainty using probabilistic representations of different types of uncertainty; it deals with both sparse and interval data without any assumption about probability distributions of random variables. Dey and Zaman(2015) proposed the maximum likelihood estimation based robust optimization model that is computationally inexpensive.

Zaman (2010) and Zaman et al. (2011b) mentioned that the essential elements of robust design optimization are: (1) maintaining robustness in the objective function (objective robustness); (2) maintaining robustness in the constraints (feasibility robustness); (3) estimating mean and measure of variation (variance) of the performance function; and (4) multi-objective optimization.

Objective robustness

Robustness in the objective function can be achieved by simultaneously optimizing mean and minimizing variance of the objective function. Two major robustness measures are available in the literature: one is the variance, which is extensively discussed in the literature (Du and Chen, 2000; Lee and Park, 2001 and Doltsinis and Kang, 2004) and the other is based on the percentile difference (Du et al, 2004).

Feasibility Robustness

Constraints robustness or feasibility robustness means satisfying the constraints in the presence of uncertainty. The methods of feasibility robustness classified by Du and Chen (2000) fall into two categories: methods that use probabilistic and statistical analysis, and methods that do not require them. A probabilistic feasibility formulation (Du and Chen, 2000 and Lee et al, 2008) and a moment matching formulation (Parkinson et al, 1993) have been proposed for the methods that require probabilistic and statistical analysis. Du and Chen (2000) proposed a sampling method based on most probable point (MPP) to achieve computational efficiency. The moment matching formulation is a simplified approach that assumes the performance function is normally distributed. The feasible region reduction method (Park et al, 2006) is more general and does not require the normality assumption.This method requires the mean and variance of the performance function and a user-defined constant. On the other hand, Worst case analysis (Parkinson et al, 1993), corner space evaluation (Sundaresan et al, 1995), and manufacturing variation patterns (MVP) (Yu and Ishii, 1998) are also available for the methods that do not require probabilistic and statistical analysis. A comparison study of the different constraint feasibility methods can be found in Du and Chen (2000).

Estimating mean and variance of the performance function

A number of methods are developed to estimate mean and variance of a function, those methods can be classified into three major classes: (i) Taylor series expansion methods, (ii) sampling-based methods and (iii) point estimate methods (Huang and Du, 2007).

The Taylor series expansion method (Haldar and Mahadevan, 2000; Du and Chen, 2000; and Lee et al, 2001) is a simple approach to estimate mean and variance. However, when the function is nonlinear and the variances of random variables are large, this approximation may result in large errors (Du et al., 2004). A second-order Taylor series expansion is generally more useful than the first-order approximation in terms of accuracy, but it is computationally expensive.

Sampling based methods are expensive and require distributions information. Importance sampling, Latin hypercube sampling, etc. (Robert and Cesalla, 2004) are efficient sampling techniques and may be used to reduce the computational effort, but prohibitive in the context of optimization. Surrogate models (Ghanem and Spanos 1991; Bichon etal., 2008; Cheng and Sandu, 2009) are also sampling techniques and can be used to further reduce computational effort.

A point estimation method (Rosenlblueth, 1975) was proposed to overcome the difficulties associated with the computation of derivatives required in Taylor series expansion to compute the first few moments of the performance function. Although point estimate methods are easier to implement, the accuracy may be low and may generate points that lie outside the domain of the random variable.

Multi objective optimization

In robust optimization, the robust solution is achieved by optimizing the mean and minimizing the variance of the performance function. Among the available methods, weighted sum approach is the simplest approach and probably the most widely used method to multi-objective optimization and has been extensively used in robust optimization (Lee and Park, 2001; Doltsinis and Kang, 2004; Zou and Mahadevan, 2006). In weighted sum method, all objectives functions convert into a single objective function. The weighted sum strategy converts the multi-objective problem of minimizing/maximizing the vector into a scalar problem by constructing a weighted sum of all the objectives. Although a simple method, the weighted sum method may not obtain potentially desirable solutions (Park et al, 2006).

Another useful method is ε-constraint methodthat involves minimizing/maximizing a primary objective, and solves multiobjective optimization problem by transforming the other objectives in the form of constraints. This approach is able to identify a number of non-inferior solutions that are not obtainable using the weighted sum technique. In spite of its advantages over weighted sum method, the ε-constraint method is computational expensive for more than two objective functions.

Other methods include goal programming (Zou and Mahadevan, 2006), compromise decision support problem (Bras and Mistree,1993, 1995; Chen et al, 1996), compromise programming (CP) (Zelney, 1973; Zhang, 2003; Chen et al, 1999) and physical programming (Messac, 1996; Messac et al, 2001; Messac and Ismail-Yahaya, 2002; Chen et al, 2000). Each method has its own advantages and limitations.

Robustness based optimization model proposed by Zaman et al. (2011b) can be used in game theory. In present research, a robustness-based optimization formulation is proposed for two-person games under interval uncertainty arising from multiple sources. First two moments of interval data are determined by moment bounding algorithmsproposed by Zaman et al. (2011a). The resulted mean and standard deviation are used in multi-objective robust optimization algorithms to get possible solution space of a two-person game.We use variance as a measure of variation of the performance function in order to achieve objective robustnessand the feasible region reduction method to achieve feasibility robustness. Mean and variance of functions with independent variables are estimated by first order Taylor series expansion.In the robust optimization of games, the weighted sum method and the ε-constraint method are used to obtain game values. In the following chapters, this study develops and demonstrates generalized methodologies and tools for solving two-person games under interval uncertainty that will provide support to make decisions in competitive environments.

CHAPTER 3

THEORETICAL FRAMEWORK

3.1 Game Theory

Game theory is a collection of mathematical models to study competitive situations and cooperation between decision-makers. Competitive situations are ubiquitous; numerous examples involving adversaries in conflict include investment decision, supply chain analysis, advertising and marketing campaigns by competing business firms, and so forth. An important characteristic in thesesituations is that the final outcome depends primarily on the combination of strategies selected by the adversaries. Interdependent decisions are everywhere, potentially including almost any endeavor in which self-interested agents cooperate and/or compete and game theoretic concepts apply whenever the actions of several agents are interdependent. A game (in strategic or normal form) consists of the following three elements: a set of players, a set of actions available to each player, and a payoff (or utility) function for each player (Cachon and Netessine, 2004). The payoff functions represent each player's preferences over action profiles, where an action profile is simply a list of actions, one for each player (Pereira and Ferreira, 2011). It is assumed that all adversaries behave rationally and desire to win. Before the game begins, each player knows the strategies she or he has available. The actual play of the game consists of each player simultaneously choosing a strategy without knowing the opponent's choice. The two branches of game theory differ in how they formalize interdependence among the players. In the non-cooperative theory, a game is a detailed model of all the moves available to the players. In contrast, the cooperative theory abstracts away from this level of detail and describes only the outcomes that result when the players come together in different combinations. In this thesis, a robust optimization algorithm for the non-cooperative game is developed in order to incorporate interval uncertainty in making the decision in competitive environments.

3.2 Non-Cooperative Game

The theory of non-cooperative games studies and models conflict situations among players (agents). It studies situations where the profits (gains, utility or payoffs) of each player (agent) depend not only on his/her own acts but also on the acts of the other agents. A fundamental characteristic of non-cooperative games is that it is not possible to sign contracts among players. That is, there is no external institution (for example, courts of justice) capable of enforcing the agreements. In this context, co-operation among players only arises as an equilibrium or solution proposal if the players find it in their best interest. The equilibrium is called Nash equilibrium where none of the players cannot improve her payoff by a unilateral move. The game theory provides an appropriate solution to a problem if its conditions are properly satisfied. These conditions are often termed as the assumptions of the game theory. Some of these assumptions are as follows:

- The number of players (competitors) is finite.
- A player can adopt multiple strategies for solving a game.
- There is a set of pre-defined outcomes.
- All players act rationally and intelligently.
- There is a conflict of interest among the players.

A non-cooperative game is called a two person game when the number of players is limited to two. Each player tries to maximize (a utility function or benefit function) or minimize (a cost function or a loss function) his/her objective function. The objective function of a player depends on the strategies of the other player and a player cannot simply optimize her own objective function because it depends onthe strategies of the other player. Here, a decoupled approach for solving two person non-cooperative games is proposed that can aggregate information from multiple sources.

3.3 Two-Person Zero-Sum Game

In game theory, a non-cooperativetwo person zero-sum game is a mathematical representation of a competitive situation in which one person's gain (or loss) is equivalent to another's loss (or gain), so the net change in benefit or loss is zero. In the game, players are rational and choose their strategies solely to promote their own welfare. Deterministic payoff matrix for two person zero sum game $(n \times m)$ is given below.

Table 1: Two person zero sum game payoff matrix

		Player 2				
Strategy	$i \backslash j$	1	2	3	……	m
Player 1	1	a_{11}	a_{12}	a_{13}	……	a_{1m}
	2	a_{21}	a_{22}	a_{23}	……	a_{2m}
	3	a_{31}	a_{32}	a_{33}	……	a_{3m}
		$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	n	a_{n1}	a_{n2}	a_{n3}	……	a_{nm}

Here, the intention of player 1 is to maximize the game value, but players 2 tries to minimize it. By applying probability theory, the definition of expected game value of player 1 is given below.

$$\text{Expected game value for player 1} = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}x_i y_j \tag{1}$$

x_i represents the probability that player 1 uses pure strategy i , y_j represents the probability that player 2 uses pure strategy j , n and m are the numbers of available strategies for player 1 and player 2 respectively and a_{ij} is the payoff value for player-1where player 1 uses pure strategy i and player 2 uses pure strategy j . Nash equilibrium is the solution of zero-sum games. Any game can be solved by transferring the problem to a linear programming problem. Whenever a game does not possess a saddle point, game theory advises each player to assign a probability distribution over his/her set of strategies.To find equilibrium point(s) and the probability of selecting strategies for zero sum game, it is necessary and sufficient to solve the following linear programming problems(Hillier and LieBerman, 1982).

$$\max_{x_i,\alpha} \quad \alpha \tag{2}$$

$$s.t.\ \alpha - \sum_{i=1}^{n} a_{ij} x_i \leq 0$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \geq 0 \ \ for \quad all \quad i \quad and \quad j$$

$$\min_{y_j,\beta} \quad \beta \tag{3}$$

$$s.t.\ \sum_{j=1}^{m} a_{ij} y_j - \beta \leq 0$$

$$\sum_{j=1}^{m} y_j = 1$$

$$y_j \geq 0 \ \ for \quad all \quad i \quad and \quad j$$

where, α and β are the same and the game value of player 1 obtained by solving maximizing and minimizing the linear programming problems respectively. Equations (2) and (3) are used to determine x_i and y_j respectively. Then, by putting these values in Eq. (1), we can calculate the game value of player 1.

In the real world, a payoff matrix might be uncertain and the game value of the problem is sensitive to the variations. Robustness-based optimization method considers the variation of input variables. In this thesis, a decoupled approach for robustness-based optimization is developed by considering the mean and variance of payoff values. In the following chapter, the description of the proposed decoupled approach for robust optimization is given.

3.4Two-Person Non-Zero Sum Game

A non-cooperative two-person non-zero sum game is a mathematical representation of competitive situations where one player's gain (or loss) is not necessarily the other player's loss (or gain). In other words, the profits and losses of all players do not sum up to zero and everyone can gain/loss. Payoff matrix for non-zero sum game $(n \times m)$ is shown in the following table.

Table 2: Two person non zero sum game payoff matrix

Player 1: Payoff matrix	Player 2: Payoff matrix
a_{11} a_{12} a_{13} $\ldots\ldots$ a_{1m}	b_{11} b_{12} b_{13} $\ldots\ldots$ b_{1m}
a_{21} a_{22} a_{23} $\ldots\ldots$ a_{2m}	b_{21} b_{22} b_{23} $\ldots\ldots$ b_{2m}
a_{31} a_{32} a_{33} $\ldots\ldots$ a_{3m}	b_{31} b_{32} b_{33} $\ldots\ldots$ b_{3m}
$\vdots$ $\vdots$ $\vdots$ $\ddots$ $\vdots$	$\vdots$ $\vdots$ $\vdots$ $\ddots$ $\vdots$
a_{n1} a_{n2} a_{n3} $\ldots\ldots$ a_{nm}	b_{n1} b_{n2} b_{n3} $\ldots\ldots$ b_{nm}

In non zero sum games, both players intention is to maximize their game values. The expected game values of both players are given below.

$$\text{Expected game value for player 1 } = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} x_i y_j \tag{4}$$

$$\text{Expected game value for player 2} = \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij} x_i y_j \tag{5}$$

where, n and m are the number of available strategies for player 1 and player 2, respectively,$a_{ij}$and b_{ij} are the payoff values of the 1st and 2nd playersrespectively when player 1 uses pure strategy i and player 2 uses pure strategy j. The game values of two person non zero sum games can be obtained by solving a quadratic programming problem (Mangasarian and Stone 1964) with linear constraint functions and quadratic objective function. Also, the probability of selecting strategies is obtained by solving the quadratic programming problem proposed by Mangasarian and Stone (1964), which is given below. In the following optimization problem, α and β are variables whose maximum values are the game values of player 1 and 2, respectively.

$$\max_{x_i, y_j, \alpha, \beta} \quad \sum_{i=1}^{n} \sum_{j=1}^{m} (a_{ij} + b_{ij}) x_i y_j - \alpha - \beta \tag{6}$$

$$s.t. \sum_{i=1}^{n} b_{ij} x_i - \beta \leq 0$$

$$\sum_{j=1}^{m} a_{ij} y_j - \alpha \leq 0$$

$$\sum_{i=1}^{n} x_i = 1$$

$$\sum_{j=1}^{m} y_j = 1$$

$$x_i \geq 0$$

$$y_j \geq 0 \;\; for \quad all \quad i \quad and \quad j$$

In many practical situations, payoff values may be available in the form of intervals (or a combination of point and interval data) instead of the deterministic payoff values. Robustness-based optimization method considers the uncertainty of interval inputs. A decoupled approach for solving two-person nonzero-sum games is proposed that incorporates the mean and variance of interval-valued payoffs.The description of the proposed decoupled approach for robust optimization to solve two person non zero sum games is given in the following chapter.

CHAPTER 4

PROPOSED ROBUST OPTIMIZATION METHODOLOGY

This research proposes decoupled approaches for robust optimization of two person games. Proposed methods can be used to solve two-person zero sum and non-zero gameswith interval-valued payoffs. In this chapter, details of the proposed decoupled approaches for solving two-person games under interval uncertainty are given.

4.1 Robust Optimization Algorithm for Two Person Zero Sum Game

This research develops a decoupled approach for robust optimization of two person zero sum games. This algorithm requires that the mean and the variance of the interval data be available as bounds. We use variance as a measure of variation of the performance function in order to achieve objective robustness and the feasible region reduction method to achieve feasibility robustness. Mean and variance of functions with independent variables are estimated by the first order Taylor series expansion. Finally, a weighted sum method is used to solve multiple objectives (mean and standard deviation) optimization problem. The proposed robust optimization problem can be formulated as follows:

$$x_i^*, \mu_\alpha^*, \sigma_\alpha^* = \underset{x_i, \mu_\alpha, \sigma_\alpha}{\arg\max} \left(w\mu_\alpha - (1-w)\sigma_\alpha \right) \tag{7}$$

$$s.t. \qquad E\left(\alpha - \sum_{i=1}^{n} a_{ij}^* x_i \right) + k\sigma\left(\alpha - \sum_{i=1}^{n} a_{ij}^* x_i \right) \le 0$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \ge 0 \quad for \quad all \quad i \quad and \quad j$$

$$y_j^*, \mu_\beta^*, \sigma_\beta^* = \underset{y_j, \mu_\beta, \sigma_\beta}{\arg\min} \left(w\mu_\beta - (1-w)\sigma_\beta \right) \tag{8}$$

$$s.t. \qquad E\left(\sum_{j=1}^{m} a_{ij}^* y_j - \beta \right) + k\sigma\left(\sum_{j=1}^{m} a_{ij}^* y_j - \beta \right) \le 0$$

$$\sum_{j=1}^{m} y_j = 1$$

$$y_j \ge 0 \quad for \quad all \quad i \quad and \quad j$$

$$\mu^*_{a_{ij}} = \arg\min_{\mu_{a_{ij}}} \left(w * E\left(\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i^* y_j^* \right) - (1-w) * \sigma\left(\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i^* y_j^* \right) \right) \tag{9}$$

$$s.t.\ LB \le \mu_{a_{ij}} \le UB \ \ for \ \ all \ \ i \ \ and \ \ j$$

In Eqs. (7)-(9), $E(x)$ and $\sigma(x)$ are the vectors of the mean values and standard deviations of the performance function, w is weight coefficient $\sum w = 1, w \ge 0$. $\mu_{a_{ij}}$ is the mean vectorof the payoffs, and,**LB** and **UB** are the vectors of lower and upper bounds of $\mu_{a_{ij}}$ .In Eq. (7), μ_α and σ_α are the mean and standard deviation of the variable, α and in Eq. (8), μ_β and σ_β are the mean and standard deviation of the variable, β . k is a constant that adjusts the robustness of the method against the level of conservatism of the solution. It reduces the feasible region by accounting for the variations and is related to the probability of constraint satisfaction. For example, if a constraint function is normally distributed, $k = 1$ corresponds to the probability 0.8413, $k = 2$ to the probability 0.9772, etc.Eqs (7)- (9) are solved iteratively until convergence.In the optimization problems in Eqs. (7) and (8), $\mu^*_{a_{ij}}$ is a set of fixed quantities. The optimization in Eq. (9) is the analysis for the variables $\mu_{a_{ij}}$, where the optimizer searches among the possible values of $\mu_{a_{ij}}$ for a conservative game value. Therefore, the proposed decoupled approach gives the conservative game value for player 1.The objective function in Eq. (9) consists of the mean and variance of player 1 game value. The method proposed in Zaman et al. (2011a) is used to compute moments (e.g. mean and variance) of interval data. After computing moments of interval data, the upper bounds of variances are used in the decoupled approaches of two-person games. Therefore, the game value becomes least sensitive to the uncertainty of interval-valued payoffs. Figure 1 illustrates the decoupled approach for two person zero sum game.

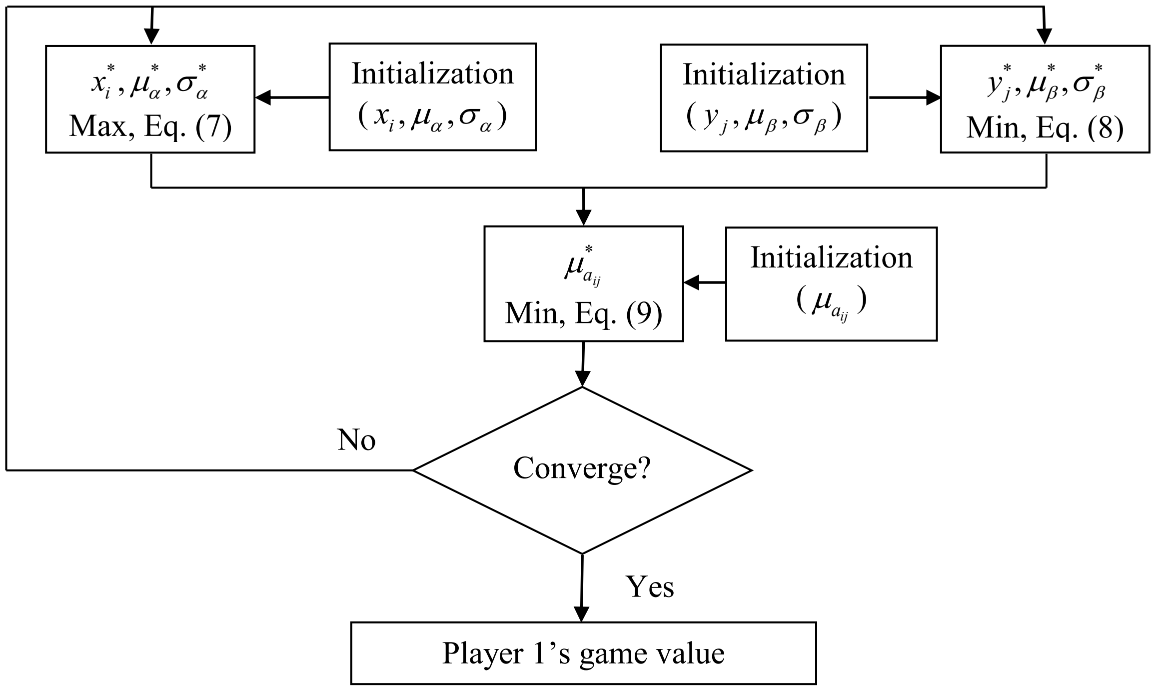

Figure 1: Proposed decoupled approach for two person zero sum game

4.2 Robust Optimization Algorithm for Two Person Non Zero Sum Game

This research also proposes a decoupled approach for robust optimization of two person non-zero sum games. Theproposed algorithm for robustness-based optimization under interval uncertainty can be formulated as follows:

$$x_i^*, y_j^*, \mu_\alpha^*, \sigma_\alpha^*, \mu_\beta^*, \sigma_\beta^* = \underset{x_i, y_j, \mu_\alpha, \sigma_\alpha, \mu_\beta, \sigma_\beta}{\arg\max} \left(w * E\left(\sum_{i=1}^{n} \sum_{j=1}^{m} \left(a_{ij}^* + b_{ij}^*\right) x_i y_j - \alpha - \beta \right) - (1-w) * \sigma\left(\sum_{i=1}^{n} \sum_{j=1}^{m} \left(a_{ij}^* + b_{ij}^*\right) x_i y_j - \alpha - \beta \right) \right)$$

$$s.t. \qquad E\left(\sum_{i=1}^{n} b_{ij}^* x_i - \beta \right) + k\sigma\left(\sum_{i=1}^{n} b_{ij}^* x_i - \beta \right) \le 0 \tag{10}$$

$$E\left(\sum_{j=1}^{m} a_{ij}^* y_j - \alpha \right) + k\sigma\left(\sum_{j=1}^{m} a_{ij}^* y_j - \alpha \right) \le 0$$

$$\sum_{i=1}^{n} x_i = 1$$

$$\sum_{j=1}^{m} y_j = 1$$

$$x_i \ge 0$$

$$y_j \ge 0 \;\; for \quad all \quad i \quad and \quad j$$

$$\mu^*_{a_{ij}}, \mu^*_{b_{ij}} = \underset{\mu_{a_{ij}}, \mu_{b_{ij}}}{\arg\min} \left(w * E\left(\sum_{i=1}^{n} \sum_{j=1}^{m} (a_{ij} + b_{ij}) x^*_i y^*_j - \alpha^* - \beta^* \right) - (1-w) * \sigma\left(\sum_{i=1}^{n} \sum_{j=1}^{m} (a_{ij} + b_{ij}) x^*_i y^*_j - \alpha^* - \beta^* \right) \right)$$

$$s.t.\ LB \le \mu_{a_{ij}} \le UB \tag{11}$$

$$LB \le \mu_{b_{ij}} \le UB \ \ for \ \ all \ \ i \ \ and \ \ j$$

In Eqs. (10) and (11), $\mu_{a_{ij}}$ and $\mu_{b_{ij}}$ are the mean vector of payoffs, and **LB**and**UB**are the vectors of lower and upper bounds of $\mu_{a_{ij}}$ and $\mu_{b_{ij}}$. The optimization problems in Eqs (10) and (11) are solved iteratively until convergence. It should be noted that $\mu^*_{a_{ij}}$ and $\mu^*_{b_{ij}}$ are fixed quantities in the optimization in Eq.(10), and the fixed quantities in Eq. (11) are x^*_i, y^*_j, α and β .After convergence, the mean and standard deviation of the game values for both players can be determined from Eqs. (12)-(15) asgiven below by using optimum values of x_i, y_j, $\mu_{a_{ij}}$ and $\mu_{b_{ij}}$.

For Player 1's game value,

$$\text{Mean} = E\left(\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i y_j \right) \tag{12}$$

$$\text{Standard deviation} = \sigma\left(\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i y_j \right) \tag{13}$$

For Player 2's game value,

$$\text{Mean} = E\left(\sum_{i=1}^{n} \sum_{j=1}^{m} b_{ij} x_i y_j \right) \tag{14}$$

$$\text{Standard deviation} = \sigma\left(\sum_{i=1}^{n} \sum_{j=1}^{m} b_{ij} x_i y_j \right) \tag{15}$$

The decoupled approach for two person non zero sum games is illustrated in Figure 2.

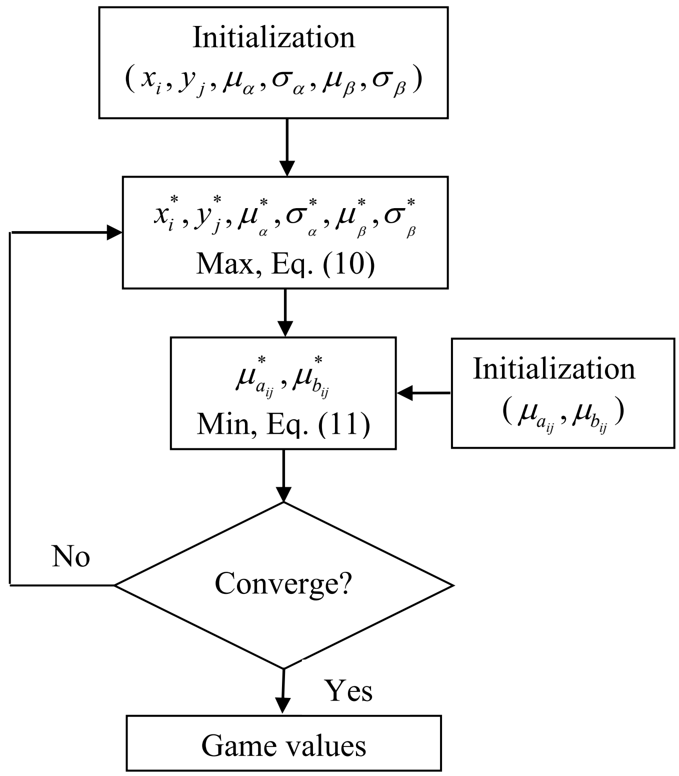

Figure 2: Proposed decoupled approach for two person non zero sum game

In the decouple approaches for robust optimization the weighted sum method is used for multiobjective optimization. Other optimization methods such as ε-constraint method can be used for the purpose. The mean and variance of performance function and first two moments of interval payoffsare needed to determine the game value. A first-order Taylor series approximation for estimating mean and variance of performance function and the methods for calculating first two moments of interval data are given in the following two sections.

4.3 Mean and Variance of Performance Function

For a functional relationship, the mean and variance of the function can be estimated approximately as a function of mean and variance of random variables. The mean and variance of a performance function can be estimated by using a first-order Taylor series approximation (Haldar, A., & Mahadevan, 2000). If there is a response variable Y which is represented by a non-linear performance function $f$ofa set of random variables $x_1, x_2, x_3, ...x_n$, then the response variable can be represented as, Performance Function: $Y = f(x_1, x_2, x_3, ...x_n)$ (16)

First-order approximate mean of Y : $E(Y') \approx f(\mu_{x_1}, \mu_{x_2}, \mu_{x_3}, ...\mu_{x_n})$ (17)

First-order variance of Y : $Var(Y') = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 Var(x_i) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} Cov(x_i, x_j)$ (18)

Using Eqs. (17) and (18), based on approximation method, the moments of the performance function can be estimated. Covariance, $Cov(x_i, x_j)$, is zero when x_i and x_j are independent. These procedures are followed to estimate the mean and the standard deviation of the performance function in the robustness-based optimization.

4.4Moments of Interval Data

This section presents algorithms that estimate lower and upper bounds of the mean and variance for single and multiple interval data. Zaman et al. (2011a) proposed the methods to compute the bounds of moments for both single and multiple interval data. The methods for calculating bounds of the first two moments for interval data are summarized in Table 3 below:

Table 3: Methods for calculating moment bounds for interval data

Moment	Formula	
	Single Interval	Multiple Interval
1	$M_1 = \sum_{i=1}^{2} x_i p(x_i)$ $\underline{M}$: PMF=1 at lower end point and 0 elsewhere $\overline{M}$: PMF=1 at upper end point and 0 elsewhere	$[\underline{M}, \overline{M}] = \left[\frac{1}{n} \sum_{i=1}^{n} lb_i, \frac{1}{n} \sum_{i=1}^{n} ub_i \right]$
2	$M_2 = \sum_{i=1}^{2} x_i^2 p(x_i) - \left(\sum_{i=1}^{2} x_i p(x_i) \right)^2$ $\underline{M}$: PMF = 1 at any end point and 0 elsewhere $\overline{M}$: PMF = 0.5 at each endpoint	$\max_{x_1, x_2, ... x_n} / \min_{x_1, x_2, ... x_n} \frac{1}{n} \sum_{i=1}^{n} \left(x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right)^2$ $s.t., lb_i \leq x_i \leq ub_i$ $i = \{1,2,3,...n\}$

Note: $[lb_i, ub_i] =$ Set of intervals, $n =$Number of interval

In the following chapter, the proposed approaches for robust optimization are illustrated with numerical examples.

CHAPTER5

NUMERICAL EXAMPLES

5.1 Numerical Examples

In this thesis, the proposed methodologies are illustrated withthree (03) numerical examples. All interval inputs (entries of payoff matrix) are overlapping or non-overlapping in nature and with different number of intervals. Each example of zero sum and non-zero sum games is solved by decoupled approaches for robust optimization. An investment decision analysis problem is also solved by the proposed decoupled approach for non-zero sum game. We compare our result with the result of some existing methods to check the validity of our developed algorithms.

5.2 Example1:Two Person Zero Sum Game

In this thesis, the proposed methodology for solving zero-sum game is illustrated with a two person zero sum game. All inputs (entries of payoff matrix) are single interval data as given in Table 4.This problem is solved by the proposed decoupled approach for robust optimization.

Table 4: Payoff matrix of the two person zero sum game

Two-Person Zero Sum Game (3×3)	
a_{11}	[39,43]
a_{12}	[21,24]
a_{13}	[21,23]
a_{21}	[49,52]
a_{22}	[35,38]
a_{23}	[14,17]
a_{31}	[5,9]
a_{32}	[77,80]
a_{33}	[35,36]

The moment bounding approach for single interval data given in Table 3 is used to estimate upper and lower bounds of the means and variances for interval data. We note here that we have assumed independence among the uncertain input variables and thereby ignored the covariance terms in Eq. (18) to estimate the variance of the functions.For illustration, we consider the constraint functions are normally distributed and $k = 1.96$ for the example problem. Eqs. (7)-(9) are used to calculate the robust game value of the two-person zero sum game.We use the Matlab solver *fmincon*, which implements a sequential quadratic programming (SQP) algorithm for the optimization of performance function for different values of weights, w. The optimization problem in Eqs (7)-(9) are solved iteratively until convergence by the Matlab solver 'fmincon' for different value of w ranging from 0 to 1. In this case of decoupled formulation, the optimization problems converged in 3 iterations. The results obtained by the decoupled approach is shown in Table 5.

Table 5:Game values of the zero sum gameat optimal solutions

Two-Person Zero Sum Game (3×3)								
w	Player 1					Player 2		
	Game Value		Probability			Probability		
	Mean	SD	x_1	x_2	x_3	y_1	y_2	y_3
0.0	25.4563	0.5391	0.3333	0.3333	0.3333	0.2769	0.0000	0.7231
0.2	26.0352	0.5515	0.5215	0.0983	0.3802	0.2817	0.0000	0.7183
0.4	26.0352	0.5515	0.5215	0.0983	0.3802	0.2817	0.0000	0.7183
0.6	26.0352	0.5515	0.5215	0.0983	0.3802	0.2817	0.0000	0.7183
0.8	26.0352	0.5515	0.5215	0.0983	0.3802	0.2817	0.0000	0.7183
1.0	26.0352	0.5515	0.5215	0.0983	0.3802	0.2817	0.0000	0.7183

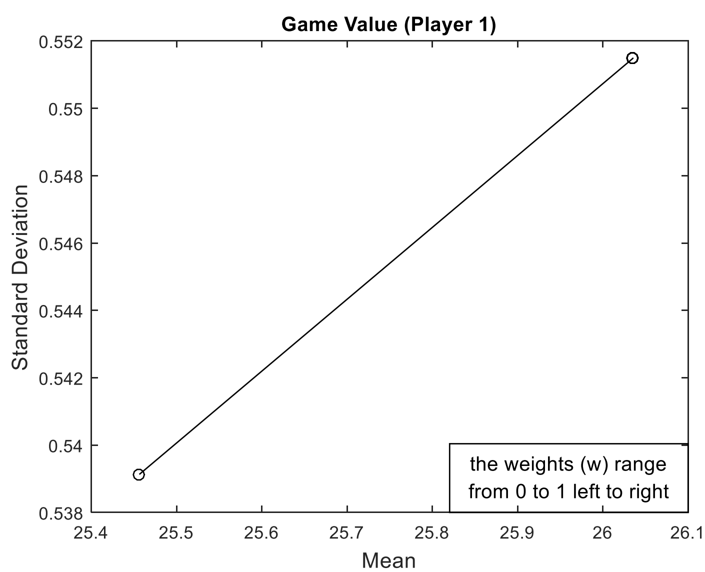

Figure3: The game value(player 1) of the zero sum game

The trade-off between the mean and the standard deviation of the game value of player 1 obtained by using the upper bounds for the variances of the random variables is presented in the Figure3. It is seen that, the standard deviation increases with the increase of the mean. We get only two different points by using weighted sum method of multi-objectives optimization and it may occur with this method. Because, in this method, uniformly distributed set of weights does not guarantee a uniformly distributed set of Pareto-optimal solutions and two different set of weight vectors not necessarily lead to two different Pareto-optimal solutions.

Comparison among Different Approaches (Zero Sum Game)

Li (2011) formulated a pair of linear programming models to obtain the upper and lower bounds of the game value of a two person zero sum game with interval payoff matrix by using the upper and the lower bounds of the payoff values. The algorithms ignored the variance of interval data. A zero sum game (3×4) was solved by Li (2011)using the linear programming models and the resulted interval game value of player 1 was [0.068, 0.14]. We also solved the problem by the proposed decoupled approach without considering variances and the resulted game value for player 1 is [0.068, 0.096]. The intention of player 1 is to maximize the value of a game. Our robustness-based optimization model provides a conservative solution. Our conservative bound of game value is narrower than that obtained by his method. The lower bound (0.068) of the game value is same as that of their models and the upper bound (0.096) is less than upper bound of their resulted value. This is expected because the proposed model provides a conservative solution.

5.3Example 2: Two Person Non Zero Sum Game

A numerical example of two-person nonzero-sum game is solved by the proposed decoupled approach. We considered non-overlapping andoverlapping interval payoffs with a different number of intervals. Interval payoff matrix of the game (2×3) is given in Table 6.

Table 6: Payoff matrix of the two person non zero sum game

Two-Person Non Zero Sum Game (2×3)	
a_{11}	[4,8.5;5,7.5;6.5,10;5,10;7,8;4.5,9;6,9]
a_{12}	[5.35,7.3;5.8,6.35;5,7.1;4.5,7.4;5,8;6,6]
a_{13}	[3.45,5.25;3.5,5.8;4.2,6.8;3,6;2.8,6]
a_{21}	[4.5,6.5;6,7;5,8;6,9;7,8;6.5,7;4.5,9]
a_{22}	[8,11.8;9,10.5;9.25,10.60;7.5,10;7.9,12.5;8,13]
a_{23}	[4.45,7.3;5.2,6.95;6.55,7.5;4.25,8.3;5.1,9]
b_{11}	[5.6,8.8;7,10.5;6.25,10.10;5.5,10;5,10.5;5,9]
b_{12}	[4.5,6.8;4,7.25;5,7.3;5,8.3;5.5,7.4;4,6.8;6.25,8.8]
b_{13}	[4.45,7.3;5.32,6.95;6.6,7.25;6.25,8.3;5.1,7]
b_{21}	[5.35,9.44;4.7,10;6.8,7.8;5.8,6.9;5,8.35;4.5,9.1]
b_{22}	[8,9.8;7.59,11.6;6.9,10.1;9,9.7;7.95,10.5;8,8.8]
b_{23}	[4.5,9.4;4.8,7.25;5,8.9;5,6.3;5.5,7.4;4,8.8;6.25,9.45]

For multiple interval data, the approaches given in Table 3 are used to determine means and variances of the payoffs. The covariance terms are ignored to estimate the variance of the performance functions. For illustration, we assumed that the constraint functions be normally distributed with $k = 1.96$ for the example problem. After calculating bounds on the parameters, Eqs. (10) and (11) are used to calculate the game value of the two-person non zero sum game.The optimization problems in Eqs (10)-(11) are solved iteratively until convergence by the Matlab solver 'fmincon' for different values of w. Then, the game values of both players are determined using Eqs. (12)-(15). For each case of two-person non-zero-sum games, the optimization problems converged in 2 iterations. The results obtained by the decoupled approach ofnon zero sum games are shown in Table 7.

Table 7: Game values of the non-zero sum gameat optimal solutions

Two person non-zero sum game (2×3)									
w	Player 1				Player 2				
	Game Value		Probability		Game Value		Probability		
	Mean	SD	x_1	x_2	Mean	SD	y_1	y_2	y_3
0.0	5.8414	0.6372	0.4500	0.5500	5.9242	0.7443	0.2395	0.4677	0.2928
0.2	6.1444	0.6537	0.3708	0.6292	6.0938	0.7503	0.2668	0.5113	0.2219
0.4	6.3690	0.6786	0.3204	0.6796	6.2405	0.7669	0.2662	0.5567	0.1771
0.6	6.6032	0.7160	0.2729	0.7271	6.4105	0.7946	0.2503	0.6106	0.1391
0.8	6.9378	0.7899	0.2102	0.7898	6.6760	0.8537	0.2136	0.6904	0.0959
1.0	8.2900	1.2800	0.0000	1.0000	7.9100	1.3200	0.0000	1.0000	0.0000

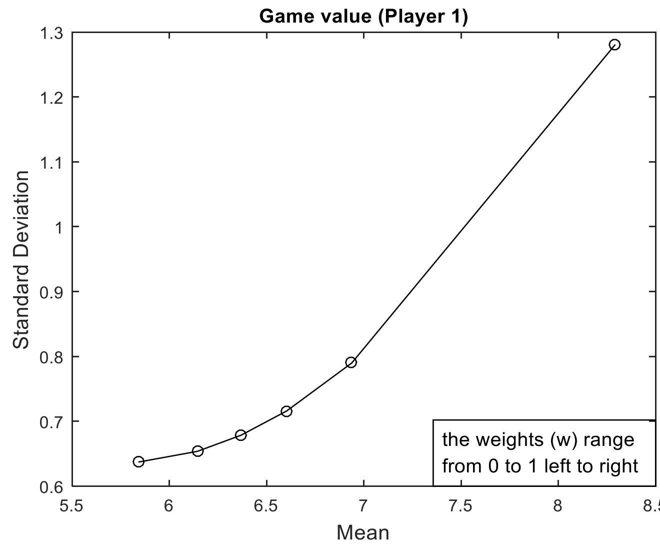

Figure4: The game value (player 1) of the non zero sum game

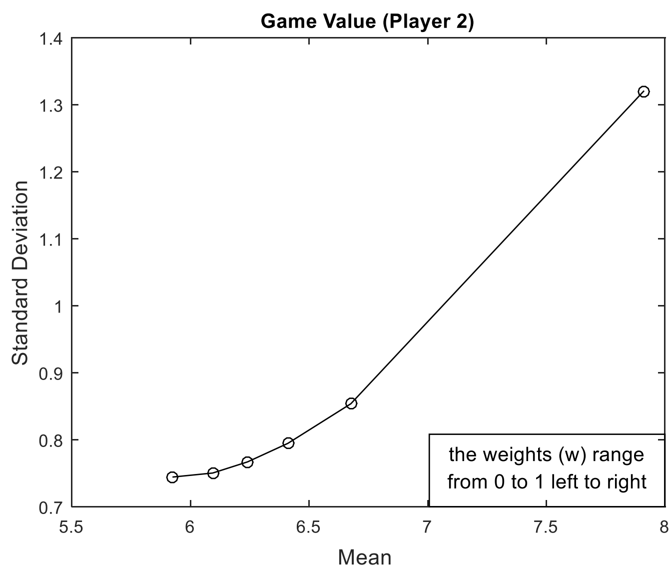

Figure5: The game value (player 2) of the non zero sum game

The trade-offs between the mean and standard deviation of the game values of player 1 and player 2 are illustrated in Figures 4 and 5, respectively. For both players, the standard deviations of game values increase with the increase of means of the game values. The decoupled approach for robustness-based optimization generates conservative solutions under interval-valued payoffs. It is also seen from Figures 4 and 5 that as the value of w increases, both the standard deviation and mean of game values increase.

5.4 Example 3: Investment Decision Analysis Problem

In this section, we chose a 5-bus network from Powerworld software package to illustrate the proposed decoupled approach for solving two person non zero sum game (Nanduri et al, 2009). In the network, there are 3 generators to supply power. Generators compete against each other and want to maximize their profits. Among three generators, Gen 3 accepted the price set by the market, while the two other generators (Gen 1 and Gen 2) submitted strategic bids aimed at maximizing individual profits. Nanduri et al. (2009) converted this problem into a two person (Gen1 and Gen 2) non-zero sum game. They projected 4 years plan to make investment (capacity expansion) decision. In this thesis, we solve only oneyear expansion plan by the proposed robust optimization algorithm. In this example, there are three investment alternatives for both Gen 1 and Gen 2 that give rise to an investment matrix game with nine (3×3) elements, each of which is a potential expansion alternative. The expansion alternatives considered for Gen 1 are: do nothing (DN), expand the natural gas plant (EGP), and expand the coal plant (ECP). Gen 2 considers the following investment alternatives: do nothing (DN), expand natural gas plant (EGP), and expand the petroleum fired plant (EPP). Payoff matrix for the game is given below in Table 8.

Table 8: Deterministic investment decision analysis payoff matrix

	Gen 1			Gen 2		
	DN	EGP	EPP	DN	EGP	EPP
DN	692	748	698	697	758	721
EGP	709	749	692	739	794	697
ECP	908	958	940	609	649	608

It was assumed in Ref that each projected payoff values(a_{ij} and b_{ij}) be available aspoint data. We solved the problem by the quadratic programming model (Mangasarian and Stone, 1964)and obtained the equilibrium expansion plan as: Gen 1 expands the coal plant and Gen 2 expands the natural gas plant. This is the exact same results reported in Ref. However, in real life, projected values are not deterministic, for example we cannot forecast exact potential profit. For this, we consider interval payoffs matrix instead of the given deterministic one. We convert this investment decision analysis problem into two non zero sum investment games (matrices-1 and 2) with interval payoffs. Interval payoff matrices and solutions of the gameswith interval-valued payoffs are given in the following two subsections.

5.4.1 Interval-valuedpayoffmatrix-1

In the first problem (matrix-1), we assume that the payoffs a_{ij} and b_{ij} given in Table 8 are the mean values of the interval-valued payoffs. Intervals around these mean values are constructed as $a_{ij} \pm 2$ and $b_{ij} \pm 2$. The resulted payoff matrix with interval data is given in Table 9.

Table 9: Investment decision analysis interval-valuedpayoff matrix-1

Generator Payoff Matrix (3×3)			
Gen 1		Gen 2	
a_{11}	[690,694]	b_{11}	[695,699]
a_{12}	[746,750]	b_{12}	[756,760]
a_{13}	[696,700]	b_{13}	[719,723]
a_{21}	[707,711]	b_{21}	[737,741]
a_{22}	[747,751]	b_{22}	[792,796]
a_{23}	[690,694]	b_{23}	[695,699]
a_{31}	[906,910]	b_{31}	[607,611]
a_{32}	[956,960]	b_{32}	[647,651]
a_{33}	[938,942]	b_{33}	[606,610]

For single interval data, the methods given in Table 3 are used to determine the bounds on mean and variance of each payoff. We note here that we have assumed independence among the uncertain input variables and therefore covariance is assumed zero to estimate the variance of the performance functions.We assumed that the constraint functions be normally distributed with $k = 1.96$corresponds to the probability 0.9750for the example problem. Here, *fmincon* uses an SQP algorithm. We use ε-constraint method to solve the investment decision problem.The solutions of the investment decision problem by the decoupled approachare given inTable 10.

Table 10: Solutions of the interval-valuedpayoff matrix-1

Gen1	Mean		799.18	890.07	937.89	945.60	950.81	956
	SD		0.67	0.94	1.20	1.46	1.73	2
	Probability	DN	0.3077	0.1200	0	0	0	0
		EGP	0.3030	0.0833	0	0	0	0
		ECP	0.3893	0.7967	1	1	1	1
Gen 2	Mean		689.4	644.71	625.53	634.88	641.20	647
	SD		0.67	0.94	1.20	1.46	1.73	2
	Probability	DN	0.3308	0.3154	0.2677	0.1566	0.0814	0
		EGP	0.3484	0.3825	0.4697	0.7005	0.8566	1
		EPP	0.3207	0.3021	0.2626	0.1430	0.0620	0

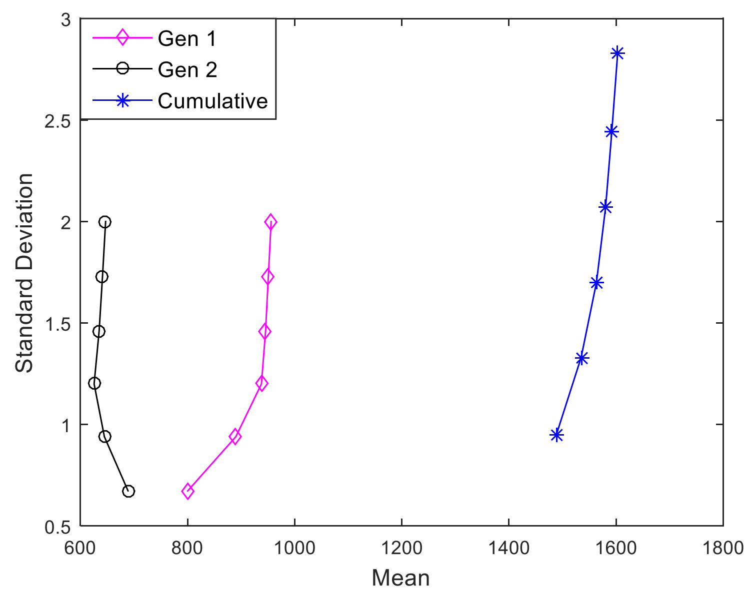

Figure6: Investment decision analysis interval-valuedmatrix-1 solutions

The trade-offs between the means and standard deviations of the game values are shown in Figure6. The cumulative game value is the summation of the game values (profit) of both players. The standard deviations of Gen-1's game valueand the cumulative game value increase when the means increase. In the case of Gen-2, the standard deviation decreases with the increase of the mean.

5.4.2 Intervaldata payoffmatrix-2

In the second problem (matrix-2), we assume that the payoffs a_{ij} and b_{ij} given in Table 8 are the mean values of the interval-valued payoffs. Intervals around these mean values are constructed as $a_{ij} \pm 3$ and $b_{ij} \pm 3$. The interval-valued payoff matrix is given in the following table.

Table 11: Investment decision analysis interval-valued payoff matrix-2

Generator Payoff Matrix (3×3)			
a_{11}	[689,695]	b_{11}	[694,700]
a_{12}	[745,751]	b_{12}	[755,761]
a_{13}	[695,701]	b_{13}	[718,724]
a_{21}	[706,712]	b_{21}	[736,742]
a_{22}	[746,752]	b_{22}	[791,797]
a_{23}	[689,695]	b_{23}	[694,700]
a_{31}	[905,911]	b_{31}	[606,612]
a_{32}	[955,961]	b_{32}	[646,652]
a_{33}	[937,943]	b_{33}	[605,611]

The solutions of the problem obtained by the proposed decoupled approach for two person non zero sum game are given in Table 12.

Table 12: Solutions of the interval-valuedpayoff matrix-2

Gen1	Mean		794.70	886.45	936.42	944.44	948.32	955
	SD		1	1.40	1.79	2.19	2.58	3
	Probability	DN	0.3146	0.1255	0	0	0	0
		EGP	0.3111	0.0899	0	0	0	0
		ECP	0.3742	0.7847	1	1	1	1
Gen 2	Mean		689.90	645.11	623.94	633.68	639.97	646
	SD		1	1.40	1.79	2.19	2.58	3
	Probability	DN	0.3315	0.3156	0.2740	0.1588	0.0831	0
		EGP	0.3445	0.3831	0.4552	0.6958	0.8529	1
		EPP	0.3239	0.3014	0.2708	0.1454	0.0639	0

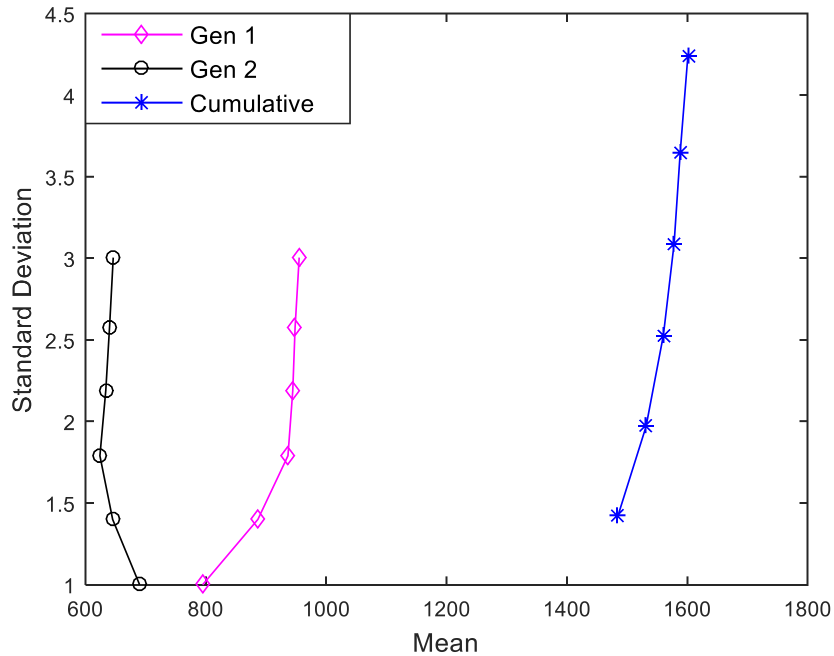

Figure7: Investment decision analysis interval-valuedmatrix-2solutions

The solutions of the two person non zero sum investment decision game are shown in Figure 7.It illustrates the change of standard deviations with the change in mean values.

InFigures 6 and 7, the higher standard deviation (risk) provides the higher mean value (profit) for Gen 1. In the decoupled approach of non zero sum game, the objective functions are not the mean and standard deviation of the game value of a player. For this reason, the standard deviation (SD) may not increase with the increase in the mean for both players, but the cumulative standard deviation will increase if the cumulative mean increases because the objective function consists of the summation of two players' game values.The cumulative game values (profit) with SDs are shown in Figures6 and 7. It is seenthe cumulative SD is increasing with the increase of the cumulative profit. However, Gen 2 shows opposite trend. But the first example of non zero game given in Table 6 shows that game values (mean) of both playersincrease when the standard deviations increase.

Comparison among Different Approaches (Non Zero Sum Game)

Sohraiee et al. (2010) solved a non-zero sum game (2×3) with interval payoff matrix by a procedure which is based on Linear Complementary Problem (LCP). They determined the game values for player 1 and 2 as [1.125, 3.5] and [2.333, 3.333], respectively. The problem is also solved by the proposed decoupled approach of non zero sum game. By decoupled approach, player 1's game value is obtained as [1.3411, 1.4426] with a standard deviation of [0.3612, 0.3982] and player 2's game value is obtained as [2.3084,2.3117] with a standard deviation of [0.2473,0.2604].The bounds of game values obtained by the proposed method are narrower than their bounds and the game values are close to the lower bounds of their game value which is expected because the proposed approach generates a conservative solution. We observe a little disagreement between results obtained by the decoupled approach and the LCP based approach. The LCP based approach was an approximation as the probability structure of equilibrium pair of strategies had been modified by the probability structure of equilibrium pair of primary strategies. Therefore, our outcome may not be the same as the approximate result but there is a little disagreement.In the investment decision analysis problem, interval valued payoff matrix-1, the conservative game values of Gen-1 and Gen-2 are 956 and 647, respectively at maximum standard deviation, 2. Those game values are 2 units less than the deterministic game values at maximum standard deviation, 2. Similarly, in interval valued payoff matrix-2, the conservative game values of both generators are 3 units less than the deterministic game values at maximum standard deviation, 3. This shows that the solutions of the proposed approach are the conservative solution and reflects the variation of payoffs under interval data.

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

6.1 Conclusions

The major contribution of this thesis is to developdecoupled approaches for robust optimization of two person games under interval uncertainty arising from multiple sources. The decoupled approach is computationally efficient and quantifies uncertainty through iterative analysis. Two types of interval data, single and multiple, are considered in this research. In this thesis,the uncertainty is represented using moment bounding approach.In this thesis, we have used the weighted sum approach and ε -constraint method for the aggregation of multiple objectives and to examine the trade-offs among multiple objectives. Other multi-objective optimization techniques can also be explored within the proposed formulations. The proposed robust optimization algorithm is illustrated for numerical examples with different numbers of intervals and types of overlaps. This study proposed two algorithms, one for zero-sum games and the other for nonzero-sum games, which can be used to solve real life problems such as investment decision and marketing strategy selection under data uncertainty in order to make decisions in competitive environments.An investment decision analysis problem is solved by the proposed decoupled approach of two person non zero sum games.

The major advantage of the proposed decoupled approaches is that it can handle uncertainty without any assumption about the probability distributions of payoffs.In the presence of interval uncertainty, the results of the proposed decoupled approaches are valuable to decision makers as it aggregates information from multiple sources, for instance, from experts' opinion. The proposed methodology is quite general and it may be expected that it could be successfully applied to any two person non-cooperative games with single or/and multiple interval-valued payoffs.

6.2 Future Work

Competitive situations are ubiquitous. In order to maximize (or minimize) profit (or loss), a decision maker considers the decision of opponents to make a decision. The methods developed in this research are applicable for robust optimization of two-person non-cooperative games under interval uncertainty. However, in real life situations, the number of players can be more than two. Therefore, this research can be expanded for multiple players non-cooperative games. The proposed methodologies are helpful to make a decision, such as launch a new technology, where historical data is not available.

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