ANALYSIS OF HEAT ABSORPTION ON NATURAL CONVECTIONFLOW ALONG A VERTICAL WAVY SURFACE WITH VISCOUS DISSIPATION UNDER VARIABLE VISCOSITY

A dissertation submitted to the
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Submitted by
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The dissertation titled "**ANALYSIS OF HEAT ABSORPTION ON NATURAL CONVECTION FLOW ALONG A VERTICAL WAVY SURFACE WITH VISCOUS DISSIPATION UNDER VARIABLE VISCOSITY**"

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DEDICATIONS

This work is dedicated to my

Family
&
respectable
Teacher (Professor Dr. Nazma Parveen)

CERTIFICATE OF RESEARCH

This is to certify that the work presented in this thesis is carried out by the author under the supervision of **Dr. Nazma Parveen**, Professor, Department of Mathematics, Bangladesh University of Engineering and Technology (BUET), Dhaka-1000.

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Author's Declaration

I hereby declare that the work which is being presented in this dissertation entitled **"ANALYSIS OF HEAT ABSORPTION ON NATURAL CONVECTION FLOW ALONG A VERTICAL WAVY SURFACE WITH VISCOUS DISSIPATION UNDER VARIABLE VISCOSITY"** was being carried out in accordance with the regulations of Bangladesh University of Engineering and Technology (BUET), Dhaka, Bangladesh. This is an authentic record of my own work.

This dissertation has not been submitted elsewhere (University or institution) for the award of any other degree in home or abroad.

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At first all praise belongs to***Almighty Allah***

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Abstract

Viscous incompressible fluid in a steady two-dimensional natural convection flow considering viscous dissipation along a uniformly heated vertical wavy surface in presence of internal heat absorption and variable viscosity has been investigated in this thesis. Using the appropriate transformations of variables of the basic governing equations are changed to non-dimensional boundary layer equations, which are solved numerically by employing the implicit finite difference method together with Keller-box scheme. The program code of this method has been developed in FORTRAN.

Attention is focused on the evolution of the surface shear stress in terms of local skin friction, rate of heat transfer in terms of local Nusselt Number, velocity, temperature, isotherms as well as the streamlines for a selection of parameter sets consisting of viscosity parameter ε, heat absorption parameter$Q$, Eckert NumberEc, Prandtl number $Pr$and the amplitude of wavinessof the surface α.The results obtained from the numerical study have been discussed emphasizing the physical prospects and shown graphically by utilizing the visualizing software TECHPLOT.

The skin friction coefficient C_{fx}, the rate of heat transfer in terms of Nusselt number Nu_x, the velocity, the temperature, the streamlines as well as the isotherms are shown graphically in figures for different values of the viscosity variation parameter ε (= 0.0 to 60.0), heat absorption parameter Q(= -0.40 to 0.0), Eckert number Ec (= 0.0 to 8.0), the amplitude of waviness of the surface α(= 0.0 to 0.4)and Prandtl number Pr (= 0.73 , 3.0, 7.0, 15.5) which correspond to the air at 2100°K, water at20□, 60□ and 100□respectively.

The results of the present investigations for heat absorption parameter Q, the rate of heat transfer and velocity increases and the skin friction coefficient and temperature decreases. For increasing Eckert number Ec, the skin friction coefficient, velocity and temperature are increases, but the significant decreases over the whole boundary layer for the rate of heat transfer. The comparisons of the present numerical results with previous published works performed and the results show excellent agreement.

Contents

Nomenclature

C_p : Specific heat at constant pressure

C_{fx} : Local skin friction coefficient

N_{ux} : Local Nusselt number

Gr : Grashof number

f : Dimensionless stream function

g : Acceleration due to gravity

q_w : Heat flux at the surface

L : Wave length associated with wavy surface

K : Thermal conductivity

Q : Heat absorption parameter

Q_0 : heat generation constant

Pr : Prandtl number

$\overline{p}$: Pressure of the fluid

p : Dimensionless pressure function

T : Fluid temperature in the boundary layer

T_∞ : Temperature of the ambient fluid

T_w : Temperature at the surface

(U, V) : Velocity component along x and y

(u, v) : Dimensionless velocity component

(X,Y) : Axis in the direction along and normal to the tangent of the surface

(x, y) : Non-dimensional coordinate system

$\Delta\, \psi$: Distance between two streamlines

$\Delta\, \theta$: Distance between two isotherms

n : Wave number indicator

Greek Symbols

α : Amplitude of the wavy surface

β : Volumetric coefficient of thermal expansion

ν : Kinematic coefficient of viscosity

ψ : Stream function

η : Dimensionless similarity variable

τ_ω : Shearing stress

μ : Dynamic coefficient of viscosity

ρ : Density of the fluid

θ : Dimensionless temperature function

σ_x : Non-dimensional surface profile function

$\overline{\sigma}$: Surface profile function

Subscripts

W : Wall conditions

∞ : Ambient Condition

Superscripts

/ : Differentiation with respect to η

_ : Dimensional quantity

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Chapter One

Introduction

Natural Convection is the mode of energy transfer between a solid surface and the adjacent liquid or gas that is in motion and it involves the combined effects of conduction and fluid motion. The faster the fluid motion, the greater the heat transfers due to convection. Natural convection flow is often encountered in cooling of nuclear reactors or in the study of the structure of stars and planets. In natural convection, fluid surrounding a heat source receives heat and by thermal expansion becomes less dense and rises. The surrounding, cooler fluid then moves to replace it. It has attracted a great deal of attention from researchers because of its presence both in nature and engineering applications. In engineering applications, convection is commonly visualized in the formation of microstructures during the cooling of molten metal, and fluid flows around shrouded heat-dissipation fins and solar ponds. A very common industrial application of natural convection is free air cooling without the aid of fans: this can happen on small scales (computer chips) to large scale process equipment.

The surrounding fluid then moves to replace it. This cooler fluid is then heated and the process continues, forming convection current. Since there is no external fan to accelerate the heat transfer, the design of the heat sink should be thermally efficient to dissipate maximum amount of heat. The driving force for natural convection is buoyancy, a result of differences in fluid density. Because of this, the presence of a proper acceleration such as arises from resistance to gravity, or an equivalent force (arising from acceleration, centrifugal force or Coriolis effect), is essential for natural convection. For example, natural convection essentially does not operate in free-fall (inertial) environments, such as that of the orbiting International Space Station, where other heat transfer mechanisms are required to prevent electronic components from overheating. The natural convection procedures are governed essentially by three features namely the body force, the temperature difference in the flow field and the fluid density variations with temperature. The manipulation of natural convection heat transfer can be deserted in the case of large Reynolds number and very small Grashof number. Alternately, the natural convection should be the governing aspect for large Grashof number and small Reynolds number. The analysis of natural convection has been considerable interest to engineers and scientists since it is important in many industrial and natural problems. There are many physical processes in which buoyancy forces resulting from thermal diffusion play an important role in the convection transfer of heat.

Few examples of the heat transfer by naturalconvection can be found in geophysics and energy related engineering problems such as natural circulation in geothermal reservoirs, refrigerator coils, hot radiator used for heating a room, transmission line, porous insulations, solar power collectors, spreading of pollutants etc.In nature, convection cells formed from air raising above sunlight-warmed land or water are a major feature of all-weather systems. Convection is also seen in the rising plume of hot air from fire, oceanic currents, and sea-wind formation (where upward convection is also modified by Coriolis forces).

It is also necessary to study the heat transfer from an irregular surface because irregular surfaces are often present in many applications. It is often encountered in heat transfer devices to enhance heat transfer. Laminar natural convection flow from irregular surfaces can be used for transferring heat in several heat transfer devices, for examples, flat- plate solar collectors, flat-plat condensers in refrigerators, heat exchanger, functional clothing design, geothermal reservoirs and other industrial applications. They are widely used in space heating, refrigeration, air conditioning, power plants, chemical plants, petrochemical plants, petroleumrefineries and natural gas processing. One common example of a heat exchanger is the radiator used in vehicles, in which the heat generated from engine transferred to air flowing through the radiator. Heat exchanger also widely used in industry both for cooling and heating large scale industrial process.

It is a model problem for the investigation of heat transfer from roughened surfaces in order to understand heat transfer enhancement. The sinusoidal wavy surface can be viewed as an approximation too much practical geometries in heat transfer. A good example is a cooling fin. Since cooling fins have a larger area than a flat surface, they are better heat transfer devices. Another example is a machine-roughened surface for heat transfer enhancement. The interface between concurrent or countercurrent two-phase flow is another example remotely related to this problem. Such an interface is always wavy and momentum transfer across it is by no means similar to that across a smooth, flat surface, and neither is the heat transfer. Also, a wavy interface can have an important effect on the condensation process.

Viscosity is a term used to describe resistance to flow at a particular temperature. A liquid of a high internal resistance to flow is described as viscosity (such as honey atroomtemperature). A liquid with a low internal resistance to flow is described as having a low viscosity (Such as water at room temperature). The internal resistance being referred to is related to the abilityfor molecules torearrange and move past each other. This rearrangement is necessary for flow. Liquids make up of small molecules have a low viscosity, and liquids with long chain molecules (such as plastics) have a much higher viscosity.

The viscosity of materials generally decreases with increasing temperature. This is true of plastics. Plastics also generally decrease in viscosity with increasing shear. Shear is created when twisting or sliding motion is imposed on a material; such as when plastic is being melted in extrusion or injection molding by the rotation of the screw while the barrel remains stationary. In injection molding, shear is also created as plastic moves past itself and the walls of the sprue, runner and cavity walls during injection. Faster injection promotes a lower material viscosity.

Viscosity, thermal conductivity etc. are physical properties which may be changed significantly with temperature. Temperature decreases the viscosity of liquids and increases theviscosity of gases. When the temperature is 100℃, 500℃ and 800℃, the viscosity of air is 1.3289, 2.671 and 3.625 kg$m^{-1}s^{-1}$ respectively. The viscosity of water is 1006.523, 471.049, 282.425 and 138.681 kg$m^{-1}s^{-1}$ respectively at 20℃, 60℃, 100℃ and 200℃ temperature.

The viscous dissipation effect plays an important role in natural convection in various devices which are subjected to large deceleration or which operate at high rotational speeds and also in strong gravitational field processes on large scales(on large planets),in geological process and in nuclear engineering in connection with the cooling of reactors. The irreversible process by means of which the work done by a fluid on adjacent layers due to the action of shear forces is transformed into heat is defined as viscous dissipation. It is also important in the flow of fluids having high viscosities. Temperature of the fluid increases because of it.

In any type of fluid flow there must present variable viscosity. Due to temperature, viscosity varies in fluid. In any simple flow such as in hot water flow through duct water molecules must make collisions among them which create temperature as a result viscosity varies. We can say about outer surface of the Sun. There occur many reactions simultaneously as a result temperature creates as well as viscosity also varies. Generally, viscous dissipation is taken to be theirreversible transfer of mechanical energy to heat by the flow working against the viscous stresses.

The study of temperature and heat transfer is of great importance to the engineers because of its almost universal occurrence in many branches of science and engineering. Heat generation is a volumetric phenomenon. That is, it occurs throughout the body of a medium. Therefore, the rate of heat generation in a medium is usually specified per unit volume. Heat generation is the ability to emit greater-than-normal heat from the body. The amount of heat generated or absorbed per unit volume is defined as $Q_o(T - T_\infty)$, where $Q_o$being a constant, which may take either positive or negative. The source term represents the heat generation when Q_o▪ 0 and the heat absorption when Q_o▪ 0.

1.1 Literature Review

Now a days, viscosity is playing a vital role in case of natural convection flows. The two-dimensional laminar natural convection of a Newtonian fluid and heat transfer problem has been presented by many investigators because of its considerable practical applications.Keller [15] investigatednumerical methods in boundary layer theory. The effect of significant viscosity variation on convective heat transport in water-saturated porous media investigated by Gray et al.[6]. Yao [32]studied natural convection along a vertical wavy surface. He has found that the frequency of the local heat transfer rate was twice then that of the wavy surface.Ithasfound that the investigation results for Physical and computational aspects of convective heat transfer by Cebeci and Bradshaw [5]. Yao and Moulic[20]added uniform heat flux parameter on natural convection along a vertical wavy surface. Heat transfer on natural convection was converted into sinusoidal wavy surface by Bhavnani andBergles[4].Mehta and Sood[18]analyzed transient free convection flow with temperature dependent viscosity in a fluid saturated porous media.Munir et al. [21] proposed natural convection of a viscous fluid with viscosity inversely proportional to linear function of temperature from a vertical wavy cone.Heat transfer in a viscous fluid over a stretching sheet with viscous dissipation and internal heat generation studied by Vejravelu and Hadjinicolaou[31]. Rees and Pop [29] presented a note of free convection along a vertical wavy surface in a porous medium. Kafoussius and Williams[12]researched effect of temperature dependent viscosity on the free convective laminar boundary layer flow past a vertical isothermal flat plate.Hossainand Pop [7] explored magnetohydrodynamic boundary layer flow and heat transfer on a continuous moving wavy surface. Natural convection along a wavy vertical plate to non-Newtonian fluids investigated by Kim[16]. Rees and Kafoussius studied numerically on the combined free and forced convective laminar boundary layer flow past a vertical isothermal flat plate with temperature dependent viscosity [13]. Again,Rees and Hossain[8]researched combined heated and mass transfer in natural convection flow from a vertical wavy surface. They also [9] investigated flow of viscous incompressible fluid with temperature dependent viscosity and thermal conductivity past a permeable wedge with uniform surface heat flux.Chamkha and Ramadan [28] considered analytical solutions for Hydromagnetic free convection of a particulate suspension from an inclined plate with heat absorption. Hossain et al.[22] investigated natural convection with variable viscosity and thermal conductivity from a vertical wavy cone. They found that natural convection of a viscous fluid with viscosity inversely proportional to linear function of temperature from a vertical wavy cone. On the other hand, Kabir et al.[11] introduced natural convection of fluid with temperature dependent viscosity from heated vertical wavy surface.Jang et al. [10] performed natural convection heat and mass transfer along a vertical wavy surface.Natural convection flow along a vertical wavy surface with uniform surface temperature in presence of heat generation/absorption investigated by Molla et al.[19]. Alam et al. [2]studied viscous dissipation effects on MHD natural convection flow over a sphere in the presence of heat generation.MHD free convection flow along a heated vertical wavy surface with heat generation presented by Ahmed [1].Nasrin and Alim [23] consideredMHD free convection flow along a vertical flat plate with thermal conductivity and viscosity depending on temperature. Parveen and Alim [25]investigated that effect of temperature-dependent variable viscosity on magneto hydrodynamic natural convection flow along a vertical wavy surface. They also investigated [26] Joule heating and MHD free convection flow along a vertical wavy surface with viscosity and thermal conductivity dependent on temperature.Laminar free convection over a vertical wavy surface embedded in a porous medium saturated with a nanofluid were studiedby Mahdy and Ahmed [17].Ali [3]analyzed transition of free convection boundary layer flow.Nath and Parveen [24]proposed effects of viscous dissipation and heat generation on natural convection flow along a vertical wavy surface.Parveen and Alim [27] examined numerical solution of temperature dependent thermal conductivity on MHD free convection flow with Jouleheating along a vertical wavy surface.Tajul and Parveen [30] considered natural convection flow along a vertical wavy surface with the effect of viscous dissipation and magnetic field in presence of Joule heating.

In the light of above literatures, none of the above investigations considered the effect of heat absorption, viscous dissipation on natural convection flow with variable viscosity along a vertical wavy surface. The present study addresses the natural convection flow along a vertical wavy surface with the effects of heat absorption and viscous dissipation under variable viscosity.

The results will be obtained for different values of relevant physical parameters (viscosity variation parameter ε, heat absorption parameter Q, Eckert number Ec, the amplitude of the waviness α of the surface and Prandtl number Pr) and will be shown in graphs as well as in tables.

1.2 Main Objectives of the Present Works

The aim of this research is to investigate numerically the effects of temperature dependent variable viscosity, viscous dissipation and heat absorption on natural convection flow viscous incompressible fluid along a vertical wavy surface.The stream is assumed to flow in the upward vertical direction. Here the surface temperature $T_w$is higher than the ambient temperature T_∞. Solutions will be obtained and analyzed for the velocity and temperature profiles, the streamlines and isotherms patterns, the surface shear stress in terms of the local skin friction coefficient and the rate of heat transfer in terms of local Nusselt number over the whole boundary layer for a selection of parameters set consisting of viscosity variation parameter, Eckert number, heat absorption parameter, the amplitude of the waviness of the surface and Prandtl number.

The major objectives of the present study are:

- To develop the mathematical model regarding the proposed study.
- To reduce the mathematical model into a system of non-dimensional ordinary differential equations using suitable transformations.
- To solve the system of ordinary differential equations numerically with the help of implicit finite difference method together with the Keller-Box scheme.
- To investigate the effects of dimensionless parameters namely viscosity variation parameter, Eckert number, heat absorption parameter, Prandtl number and amplitude-to-length ratio of the wavy surface on local heat transfer rate, skin friction coefficient, velocity, temperature with streamlines and isotherms.
- To present the numerical results graphically for different values of the parameters entering into the present study.
- To compare the present results with other published works.

1.3 Outline of Methodology

There are generally three types of numerical techniques depending on the types of problem to be solved. They are (i) Finite Element (ii) Finite Difference and (iii) Finite Volume Method. The Finite Difference Method is very efficient for programming and rapid convergence among of the three methods. The transformed boundary layer equations are solved numerically with the help of implicit finite difference method together with the Keller-box scheme [15], which has been in details by Cebeci and Bradshaw [5]. The momentum and energy equations are first converted into a system of first order differential equations. Then these equations are expressed in finite difference forms by approximating the functions and their derivatives in terms of the center differences. Denoting the mesh points in the x and η-plane by x_i and η_j where i = 1, 2,...,M and j = 1, 2,…,N, central difference approximations are made, such that those equations involving x explicitly are centered at $(x_{i-1/2}, \eta_{j-1/2})$ and the remainder at $(x_i, \eta_{j-1/2})$, where $\eta_{j-1/2} = 1/2(\eta_j + \eta_{j-1})$ etc. The above central difference approximations reducethe system of first order differential equations to a set of non-linear difference equations for the unknown at x_i in terms of their values at x_{i-1}. The resulting set of non-linear difference equations are solved by using the Newton's quasi-linearization method. The Jacobian matrix has a block-tridiagonal structure and the difference equations are solved using a block-matrix version of the Thomas algorithm. The whole procedure namely reduction to first order followed by central difference approximations, Newton's Quasi-linearization method and the block Thomas algorithm, is well known as Keller-box method.

Effects of various parameters on the velocity and temperature profiles, the surface shear stress in terms of the skin friction coefficient, the rate of heat transfer in terms of local Nusselt number, the streamlines as well as the isotherms are shown graphically for different values of parameters entering into the problem using the post processing software TECPLOT and also in tabular form.

In the program test, a finer axial step size is tried and find to give acceptable accuracy. A uniform grid of 201 points is used in x- direction with Δx =0.05, while a non-uniform grid of 76 points lying between η = 0.0 and 10.0 is chosen. Grid points are concentrated towards the heated surface in order to improve resolution and accuracy of the computed values of the surface shear stress and rate of heat transfer. During the program test, the convergent criteria for the relative errors between two iterations are less 10^{-5}. It means that iterative procedure is stopped when the maximum change between successive iterates is less than 10^{-5}.

1.4 Outline of this Thesis

In chapter one, a brief introduction is presented with aim and objectives. This chapter also consists of a literature review of the past studies regarding natural convection, temperature dependent physical properties like viscosity, viscous dissipation and heat transfer in various irregular surfaces.

The basic governing equations for this work are shown in standard vector form and mathematical modeling of the problem are discussed and the governing equations associated with boundary conditions for this work are converted to dimensionless form using suitable transformations in chapter two.

In chapter three, effects of viscous dissipation with variable viscosity has been investigated on natural convection flow along a vertical wavy surface. The skin friction coefficient C_{fx}, the rate of heat transfer in terms of Nusselt number Nu_x, the velocity, the temperature, the streamlines as well as the isotherms have been exhibited graphically in figures for different values of the viscosity variation parameter ε , heat absorption parameter Q, Eckert number Ec , the amplitude of waviness of the surface α and Prandtl number Pr for 0.73, 3.0, 7.0, 15.5which correspond to the air at 2100°K, water at 100℃, 60℃ and 20℃ respectively.Inthis chapter, comparison of present numerical result with previous work represented in tabular form. The value of skin friction coefficient C_{fx}, the rate of heat transfer in terms of Nusselt number Nu_x are represented for different values of viscosity variation parameter ε, the amplitude of waviness of the surface $\alpha$and the Eckert number Ec, the velocity and temperature are represented for different values of the amplitude of waviness of the surface α and Prandtl number Prin tabular form.

Overall Conclusions and extension of the present work have been presented and relevant references have been quoted at the end of the thesis paper and the implicit finite difference method of solving the problems numerically is displayed in Appendix.

Chapter Two

Mathematical modeling of the flow problem

This chapter describes the effects of viscous dissipation under temperature dependent variable viscosity on natural convection flow of viscous incompressible fluid along a uniformly heated vertical wavy surface. Using the appropriate transformations, the governing equations with associated boundary conditions are converted to non-dimensional boundary layer equations, which are solved numerically by employing the implicit finite difference methods, known as Keller-box Scheme.

2.1 Governing equations of the flow

The continuity equation

Thecontinuity equation for viscous incompressible electrically conducting fluid remains same as that of usual continuity equation $\nabla \cdot \vec{q} = 0$ (2.1.1)

The Navier-Stokes equation

The motion of the conducting fluid across the magnetic field generates electric currents, which change the magnetic field and the action of the magnetic field on these current give rises to mechanical forces, which modify the flow of the fluid. Thus, the fundamental equation of the magneto-fluid combines the equations of the motion from fluid mechanics with Maxwell's equations from electrodynamics.

Then the Navier-stokes equation for a viscous incompressible fluid may be written in the following form:

$$\rho(\vec{q} \cdot \nabla)\vec{q} = -\nabla P + \mu \nabla^2 \vec{q} + \vec{F} + \vec{J} \times \vec{B} \qquad (2.1.2)$$

where ρ is the fluid density, μ is the viscosity and P is the pressure. The first term on the right-hand side of equation (2.1.2) is the pressure gradient, second term is the viscosity, third term is the body force per unit volume and last term is the electromagnetic force due to motion of the fluid.

The energy equation

The energy equation for a viscous incompressible fluid is obtained by adding the electromagnetic energy term into the classical gas dynamic energy equation. This equation can be written as

$$\rho C_P (\vec{q} \cdot \nabla) T = \nabla \cdot (k \nabla T) + \alpha \nabla^2 u + \mu \nabla^2 \vec{q} \tag{2.1.3}$$

Where, *k*is the thermal conductivity, C_P is the specific heat with constant pressure. In the physical problem of temperature variation, u(x,y,z,t) is the temperature and α is the thermal diffusivity. For the mathematical treatment it is sufficient to consider the case α = 1. The left side of equation (2.1.3) represents the net energy transfer due to mass transfer, the first term on the right-hand side represents conductive heat transfer and second term is heat absorption term and third term is for viscous dissipation term.

Where $\vec{q} = (U, V)$, U and V are the velocity components along the X and Y axes respectively, $\vec{F}$ is the body force per unit volume which is defined as $-\rho g$, the terms $\vec{J}$ and $\vec{B}$ are respectively the current density and magnetic induction vector and the term $\vec{J} \times \vec{B}$ is the force on the fluid per unit volume produced by the interaction of the current and magnetic field in the absence of excess charges, T is the temperature of the fluid in the boundary layer, g is the acceleration due to gravity, k is the thermal conductivity and C_P is the specific heat at constant pressure and μ is the viscosity of the fluid.

Here, ∇ is the vector differential operator and is defined for two-dimensional case as

$$\nabla = \hat{i}_x \frac{\partial}{\partial x} + \hat{i}_y \frac{\partial}{\partial y}$$

Where $\hat{i}_x$ and $\hat{i}_y$ are the unit vector along x and y axes respectively. When the external electric field is zero and the induced electric field is negligible.

2.2Physical Model of the Problem

The steady two-dimensional laminar free convection boundary layer flow of a viscous incompressible fluid along a vertical wavy surface in presence of temperature dependent variable viscosity is considered. It is assumed that the wavy surface is electrically insulated and is maintained at a uniform temperature T_w. Far above the wavy plate, thefluid is stationary and is kept at atemperature T_∞, where$T_w > T_\infty$. The boundary layer analysis outlined below allows $\bar{\sigma}(\bar{x})$ being arbitrary, but our detailed numerical work will assume that the surface exhibits sinusoidal deformations. The wavy surface may be described by

$$\bar{y}_w = \bar{\sigma}(\bar{x}) = \alpha \sin\left(\frac{n\pi\bar{x}}{L}\right) \quad (2.2.1)$$

where Lis the characteristic length associated with the wavy surface.

The geometry of the wavy surface and the two-dimensional Cartesian coordinate system are shown in Fig. 2.1.

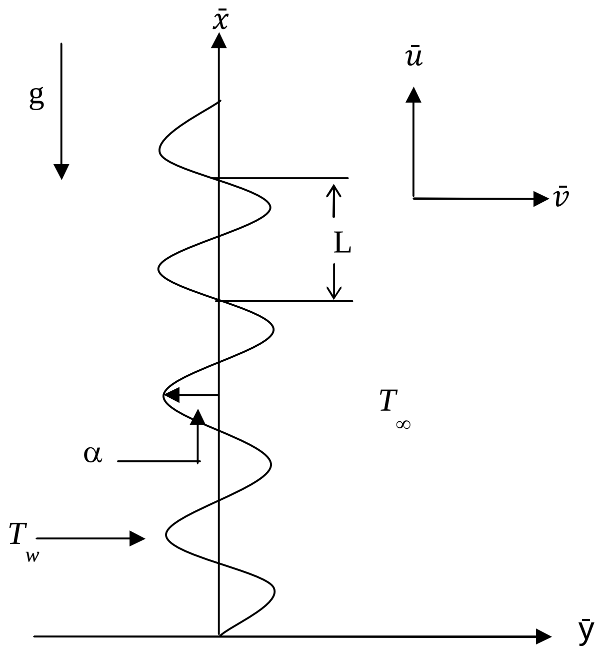


Figure 2.1: The coordinate system and the physical model.

2.3 Formulation of the problem

Under the usual Boussinesq approximation, the equations governing the flow can be written as:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{2.3.1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{\rho}\nabla.(\mu \nabla \bar{u}) + g\beta(T - T_\infty) \tag{2.3.2}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{1}{\rho}\nabla.(\mu \nabla \bar{v}) \tag{2.3.3}$$

$$\bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \frac{k}{\rho C_p}\nabla^2 T + \frac{Q_0}{\rho C_p}(T - T_\infty) + \frac{\nu}{C_p}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 \tag{2.3.4}$$

where $(\bar{x}, \bar{y})$ are the dimensional coordinates along and normal to the tangent of the surface and $(\bar{u}, \bar{v})$ are the velocity components parallel to $(\bar{x}, \bar{y})$, $\nabla^2 (= \partial^2/\partial x^2 + \partial^2/\partial y^2)$ is the Laplacian operator, $g$is the acceleration due to gravity, $\bar{p}$ is the dimensional pressure of the fluid, ρ is the density,k is the thermal conductivity, β is the coefficient of thermal expansion, μ(T) is the viscosity of the fluid depending on temperature T of the fluidin the boundary layer region and C_p is the specific heat due to constant pressure and ν (= μ/ρ) is the kinematic viscosity.

The boundary conditions relevant to the above problem are

$$\bar{u} = 0, \bar{v} = 0, T = T_w \quad at\ \bar{y} = \bar{y}_w = \bar{\sigma}(\bar{x}) \tag{2.3.5a}$$

$$\bar{u} = 0, \quad T = T_\infty, \quad \bar{p} = p_\infty \quad as \quad \bar{y} \to \infty \tag{2.3.5b}$$

where T_w is the surface temperature, $T_\infty$is the ambient temperature of the fluid and $P_\infty$is the pressure of fluid outside the boundary layer.

There are very few forms of viscosity variation available in the literature. Among them we have considered that one which is appropriate for liquid introduced by Hossain et al. [9] as follows:

$$\mu = \mu_\infty[1 + \varepsilon^*(T - T_\infty)] \tag{2.3.6}$$

where μ_∞ is the viscosity of the ambient fluid and $\varepsilon^* = \frac{1}{\mu_f}\left(\frac{\partial \mu}{\partial T}\right)_f$ is a constant evaluated at the film temperature of the flow $T_f = 1/2\,(T_w + T_\infty)$.

2.4 Transformation of the Governing equations

Using Prandtl's transposition theorem to transform the irregular wavy surface into a flat surface as extended by Yao [33] and boundary-layer approximation, the following dimensionless variables are introduced for non-dimensioning the governing equations,

$$x = \frac{\bar{x}}{L}, \quad y = \frac{\bar{y} - \bar{\sigma}}{L} Gr^{\frac{1}{4}}, \; p = \frac{L^2}{\rho v^2} Gr^{-1} \bar{p}$$

$$u = \frac{\rho L}{\mu_\infty} Gr^{-1/2} \bar{u}, \quad v = \frac{\rho L}{\mu_\infty} Gr^{-1/4} \left(\bar{v} - \sigma_x \bar{u}\right), \theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{2.4.1}$$

$$\sigma_x = \frac{d\bar{\sigma}}{d\bar{x}} = \frac{d\sigma}{dx}, \qquad Gr = \frac{g\beta(T_w - T_\infty)}{v^2} L^3$$

where θ is the dimensionless temperature function and (u, v) are the dimensionless velocity components parallel to (x, y). Here (x, y) are not orthogonal, but a regular rectangular computational grid can be easily fitted in the transformed coordinates. It is also worthwhile to point out that (u, v) are the velocity components parallel to(x, y) which are not parallel to the wavy surface and v $(= \mu/\rho)$ is the kinematic viscosity.

The conservation equations for the flow characterized with steady, laminar and two-dimensional boundary layer; under the usual Boussinesq approximation, dimensionless form of the continuity, momentum and energy equations can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2.4.2)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial x} + Gr^{\frac{1}{4}}\sigma_x \frac{\partial p}{\partial y} + \left(1+\sigma_x^2\right)\left(1+\varepsilon\theta\right)\frac{\partial^2 u}{\partial y^2} + \varepsilon\left(1+\sigma_x^2\right)\frac{\partial \theta}{\partial y}\frac{\partial u}{\partial y} + \theta \qquad (2.4.3)$$

$$\sigma_x\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\right) = -Gr^{-\frac{1}{4}}\frac{\partial p}{\partial y} + \sigma_x\left(1+\sigma_x^2\right)\left(1+\varepsilon\theta\right)\frac{\partial^2 u}{\partial y^2} + \varepsilon\sigma_x\left(1+\sigma_x^2\right)\frac{\partial \theta}{\partial y}\frac{\partial u}{\partial y} - \sigma_{xx}u^2 \qquad (2.4.4)$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\Pr}\left(1+\sigma_x^2\right)\frac{\partial^2 \theta}{\partial y^2} + Q\theta + Ec\left(\frac{\partial u}{\partial y}\right)^2 \qquad (2.4.5)$$

Where $\Pr = \frac{C_p \mu_\infty}{k}$ is the Prandtl number, $Q = \frac{Q_0 L^2}{\mu C p Gr^{1/2}}$ is the heat absorption parameter, $Ec = \frac{v^2 Gr}{L^2 C_p (T_w - T_\infty)}$ is the Eckert number, $\varepsilon = \varepsilon^*(T_w - T_\infty)$

It can easily be seen that the convection induced by the wavy surface is described by Eqns. (2.4.2)–(2.4.5). We further notice that, Eq. (2.4.4) indicates that the pressure gradient along the y-direction is $O(Gr^{-1/4})$, which implies that lowest order pressure gradient along x -direction can be determined from the inviscid flow solution. For the present problem this pressure gradient ($\partial p/\partial x = 0$) is zero. Eq. (2.4.4) further shows that $Gr^{-1/4}\partial p/\partial y$ is $O(1)$ and is determined by the left-hand side of this equation. Thus, the elimination of $\partial p/\partial y$ from Eqns. (2.4.3) and (2.4.4) leads to

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \left(1+\sigma_x^2\right)\left(1+\varepsilon\theta\right)\frac{\partial^2 u}{\partial y^2} - \frac{\sigma_x \sigma_{xx}}{1+\sigma_x^2}u^2 + \varepsilon\left(1+\sigma_x^2\right)\frac{\partial u}{\partial y}\frac{\partial \theta}{\partial y} + \frac{1}{1+\sigma_x^2}\theta \qquad (2.4.6)$$

The corresponding boundary conditions for the present problem then turn into

$$\left.\begin{aligned} u = v = 0, \quad \theta = 1 \qquad & at \quad y = 0 \\ u = \theta = 0, \quad p = 0 \qquad & as \quad y \to \infty \end{aligned}\right\} \qquad (2.4.7)$$

Now we introduce the following transformations to reduce the governing equations to a convenient form:

$$\psi = x^{3/4} f(x,\eta), \quad \eta = y x^{-1/4}, \quad \theta = \theta(x,\eta) \text{ (2.4.8)}$$

where η is the pseudo similarity variable and ψ is the stream function that satisfies the Eq. (2.4.2) and is defined by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \text{ .(2.4.9)}$$

Introducing the transformations given in Eqn. (2.4.8) and into Eqns. (2.4.6) and (2.4.5) then gets,

$$\left(1+\sigma_x^2\right)\left(1+\varepsilon\theta\right) f''' + \frac{3}{4} ff'' - \left(\frac{1}{2} + \frac{x\sigma_x \sigma_{xx}}{1+\sigma_x^2}\right) f'^2 + \frac{1}{1+\sigma_x^2}\theta \quad (2.4.10)$$
$$+\,\varepsilon\left(1+\sigma_x^2\right)\theta' f'' = x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right)$$

$$\frac{1}{\Pr}\left(1+\sigma_x^2\right)\theta'' + \frac{3}{4} f\theta' + x^{\frac{1}{2}} Q\theta + Ecx\; f''^2 = x\left(f'\frac{\partial \theta}{\partial x} - \theta'\frac{\partial f}{\partial x}\right)$$

(2.4.11)

The boundary conditions (2.4.7) now take the following form:

$$\left.\begin{array}{ll} f(x,o) = f'(x,o) = 0, & \theta(x,o) = 1 \\ f'(x,\infty) = 0, & \theta(x,\infty) = 0 \end{array}\right\} \text{(2.4.12)}$$

In the above equations prime denote the differentiation with respect to η.

In practical applications, the physical quantities of principle interest are the shearing stress τ_w and the rate of heat transfer in terms of the skin-friction coefficients Cf_x and Nusselt number Nu_x respectively, which can be written as

$$C_{fx} = \frac{2\tau_w}{\rho U_\infty^2} \quad \text{and} \quad Nu_x = \frac{q_w x}{k(T_w - T_\infty)} \quad (2.4.13)$$

$$\text{where} \quad \tau_w = (\mu\, \overline{n}.\nabla \overline{u})_{y=0} \quad \text{and} \quad q_w = -k(\overline{n}.\nabla T)_{y=0} \quad (2.4.14)$$

Using the transformations (2.4.8) into Eq. (2.4.13), the local skin friction coefficient, Cf_x and the rate of heat transfer in terms of the local Nusselt number, Nu_x takes the following form:

$$C_{fx}(Gr/x)^{1/4}/2 = (1+\varepsilon)\sqrt{1+\sigma_x^2}\; f''(x,o) \quad (2.4.15)$$

$$Nu_x(Gr/x)^{-1/4} = -\sqrt{1+\sigma_x^2}\,\theta'(x,o) \quad (2.4.16)$$

Forthe computational purpose the period of oscillations in the waviness of this surface has been considered to be π.

Chapter Three

Effects of Viscous Dissipation with Variable Viscosity on Natural Convection Flow along a Vertical Wavy Surface in presence of Heat Absorption

Introduction

This chapter describes the effects of viscous dissipation under variable viscosity on natural convection flow along a vertical wavy surface in presence of heat absorption. The governing boundary layer equations with associated boundary conditions are converted to non-dimensional boundary layer equations using the appropriate transformation and the resulting non-linear system of partial differential equations are reduced to local non-similarity equations which are solved numerically by employing the implicit finite difference method, known as Keller-Box Scheme.

The effects of pertinent parameters, such as the heat absorption parameter (Q) where the amount of heat absorption constant $Q_0<0$, the Eckert number (Ec), the Prandtl number (Pr), the viscosity variation parameter (ε) and the amplitude of the wavy surface ranging from (α) on the surface shear stress in terms of the skin friction coefficient Cf_x , the rate of heat transfer in terms of Nusselt number Nu_x, the velocity, the temperature, the streamlines as well as the isotherms are shown graphically.

Numerical values of local shearing stress and the rate of heat transfer are calculated from equations (2.4.15) and (2.4.16) in terms of the skin friction coefficient Cf_x and Nusselt number Nu_x respectively for a wide range of the axial distance x.

3.1 Result and Discussions on Variation of Viscosityparameter*ε*

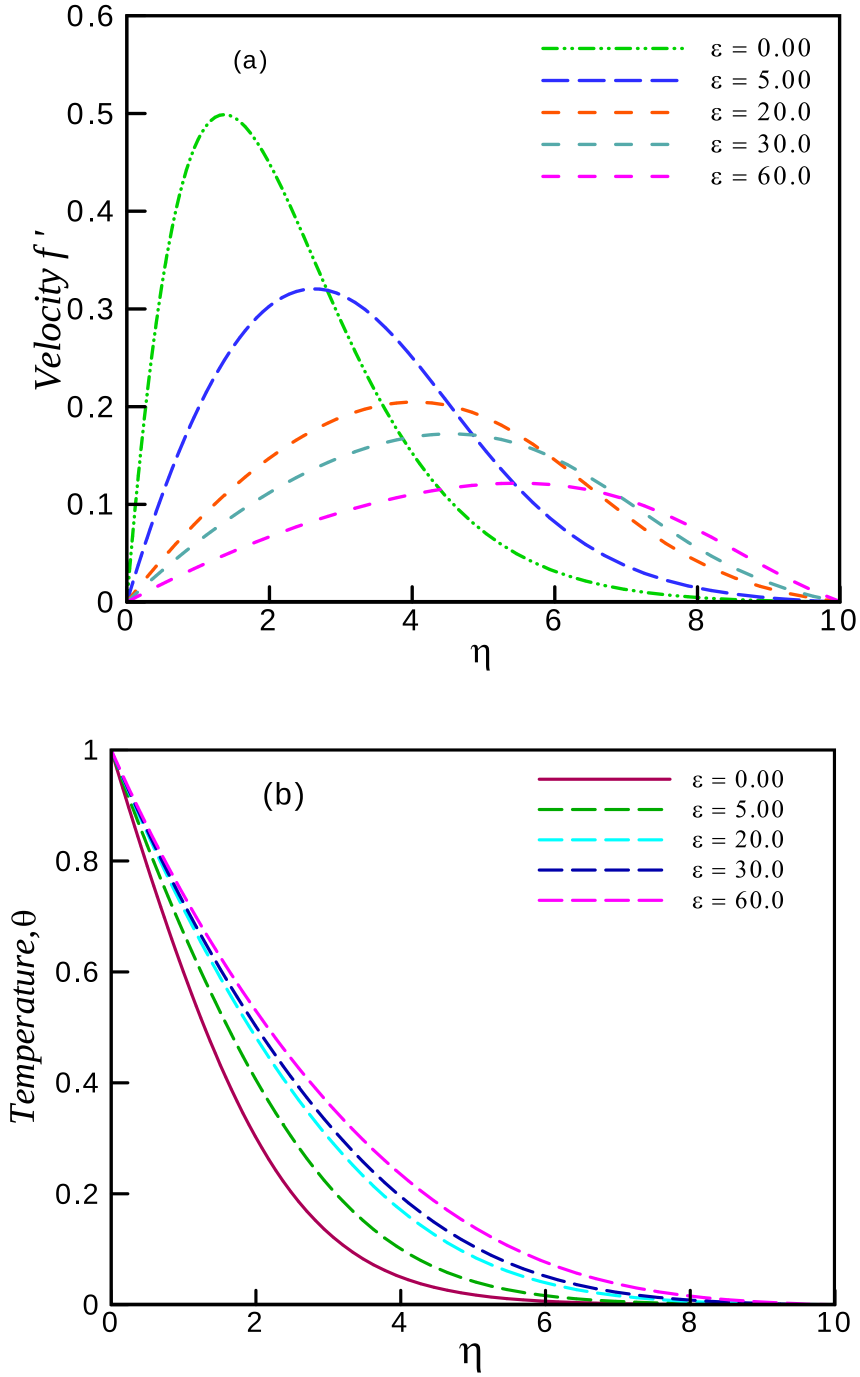


Figure 3.1:(a) Velocity and (b) temperatureprofiles for different values of viscosityvariation parameter ε while Pr = 1.0, ε = 0.3, Q = -0.1, Ec = 0.02.

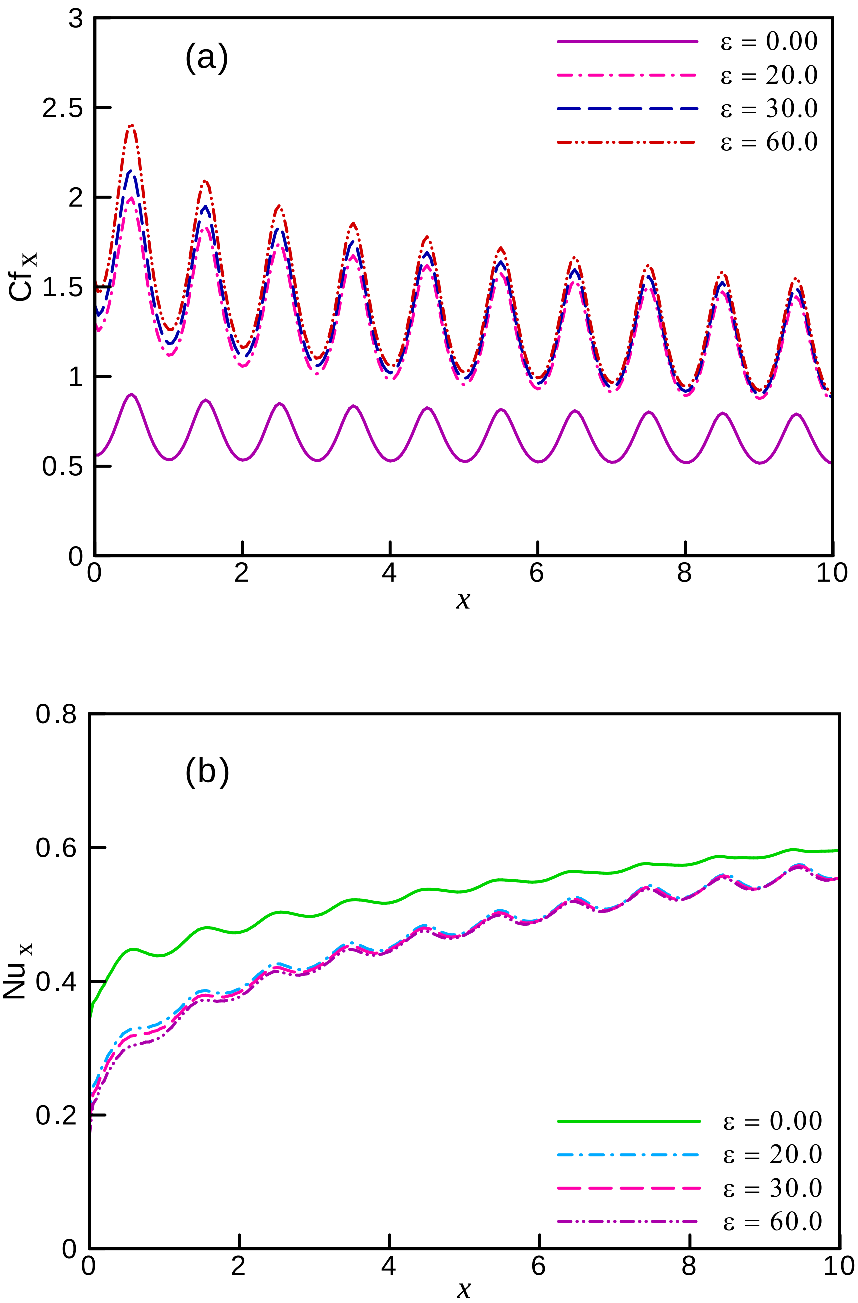


Figure 3.2: Variation of (a) skin friction coefficient (Cfx) and (b) rate of heat transfer (Nux) for different values of viscosity variation parameter ε while Pr = 1.0, α = 0.3, Q = -0.1, Ec = 0.02.

Figures 3.1 (a) and (b) display results of velocity and temperature, for different values of viscosity variation parameter ε = 0.0, 5.0, 20.0, 30.0, 60.0 while the Prandtl Number Pr = 1.0, the amplitude of the wavy surface α = 0.3, the heat absorption parameter Q = -0.1 and the Eckert number Ec = 0.02. It is noted from Figures 3.1 (a) and (b) that as the viscosity variation parameter ε increases, the velocity decreases and the temperature increases.

The physical nature of the viscosity depends on shear stress of the fluid, where shear stress also depends on velocity gradient. Here, from figure 3.1 (a) the velocity is zero at the boundary wall then the velocity increases to the peak value as η increases and then decreases, finally the velocity approaches to zero (the asymptotic value). The maximum values of the velocities are recorded as 0.49896 and 0.12168 for viscosity variation parameter ε = 0.0 and 60.0 respectively which occurs at the position of η = 1.36929 and η = 5046625. So, the maximum velocity decreases approximately by 77% when ε increases from 0.0 to 60.0.

When temperature dependent viscosity variation parameter is increasing, temperature is also increasing. The changes of temperature in the η direction also shows the typical temperature for natural convection boundary layer flow that is the value of temperature is 1.0 (one) at the boundary wall the temperature decreases gradually η direction to the asymptotic values.

From figures 3.2 (a) and (b), the surface shear stress in terms of the local skin friction Cf_x and the rate of heat transfer in terms of the local Nusselt number Nu_x are depicted graphically for the different values of viscosity variation parameter ε = 0.0, 20.0, 30.0, 60.0 when amplitude of wavy surface α = 0.3, the value of Prandtl number Pr = 1.0, the heat absorption parameter Q = -0.1 and the Eckert number Ec = 0.02.The skin friction coefficient Cf_x increases approximately by 63% and the local rate of heat transfer Nu_x decreases approximately by 5% from the different values of viscosity variation parameter ε = 0.0 to 60.0.

Figure 3.3 and 3.4 illustrate the effect of variation of different values of viscosity variation parameter ε = 0.0, 20.0, 30.0, 60.0 on the streamlines and isotherms respectively while the Prandtl Number Pr = 1.0, the amplitude of the wavy surface α = 0.3, the heat absorption parameter Q = -0.1and the Eckert number Ec = 0.02. Figure 3.3 depicts that the maximum values of ψ decreases while the values of ε increases that is ψ_{max} are 7.00, 3.56, 3.00, and 1.86 for ε = 0.0, 20.0, 30.0, 60.0 *r*espectively. It is noted from figure 3.4 the values of ε increases the thermal boundary layer becomes thickergradually that the isotherms increase while the values of ε increase.

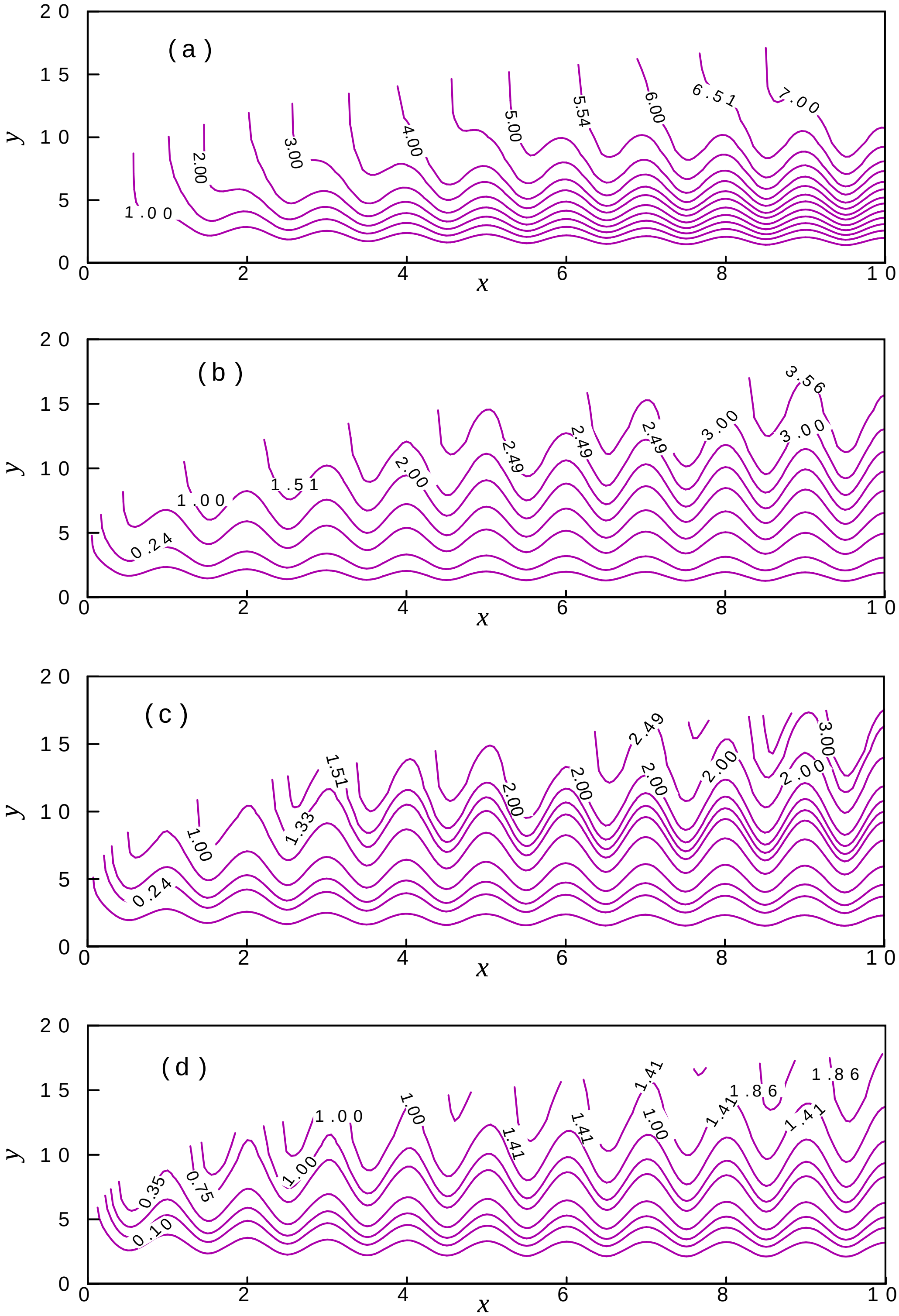


Figure 3.3 : Streamlines for (a) ε = 0.0, (b) ε = 20.0, (c) ε = 30.0 and (d) ε = 60.0 while Pr = 1.0, α = 0.3, Q= -0.1, Ec = 0.02.

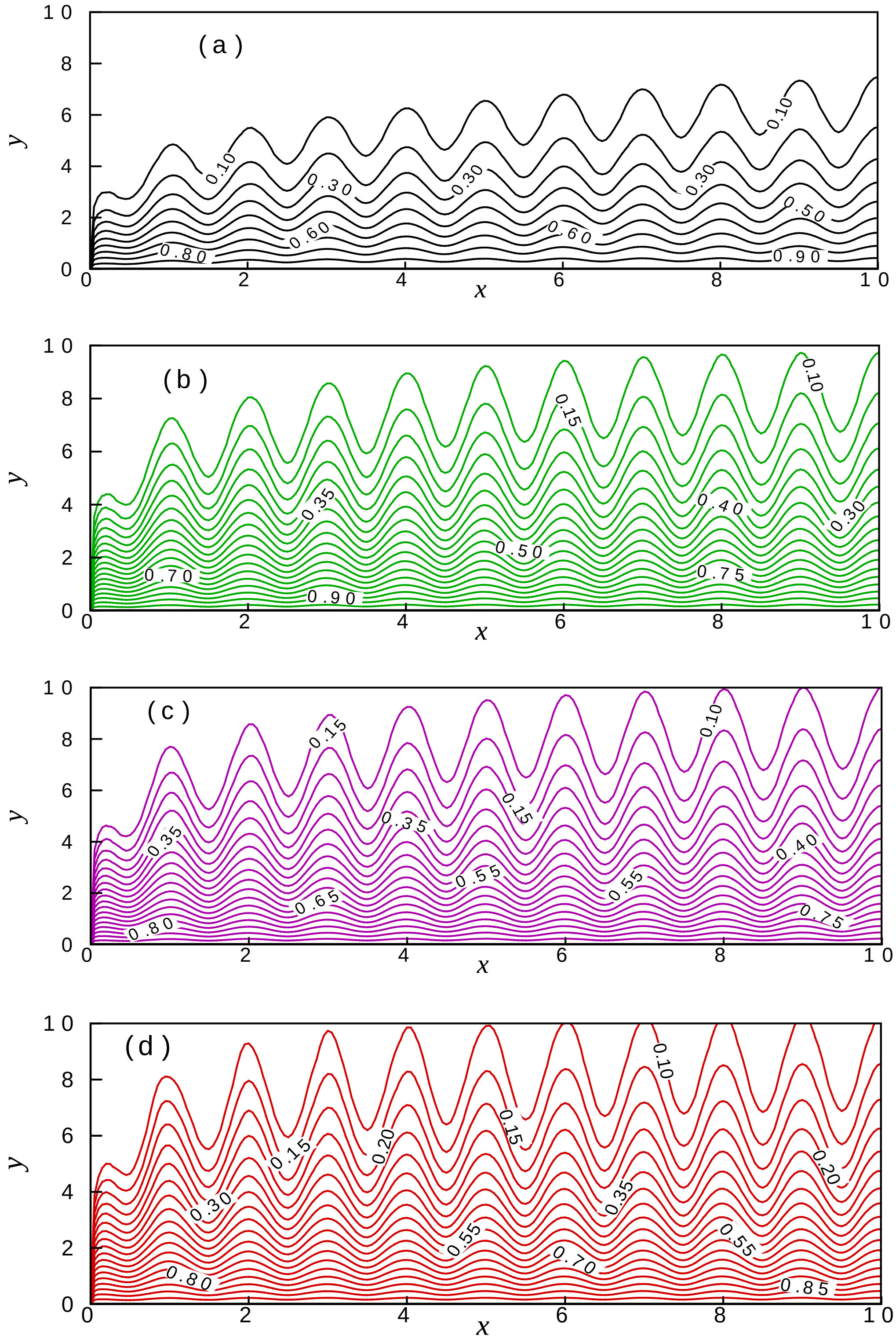
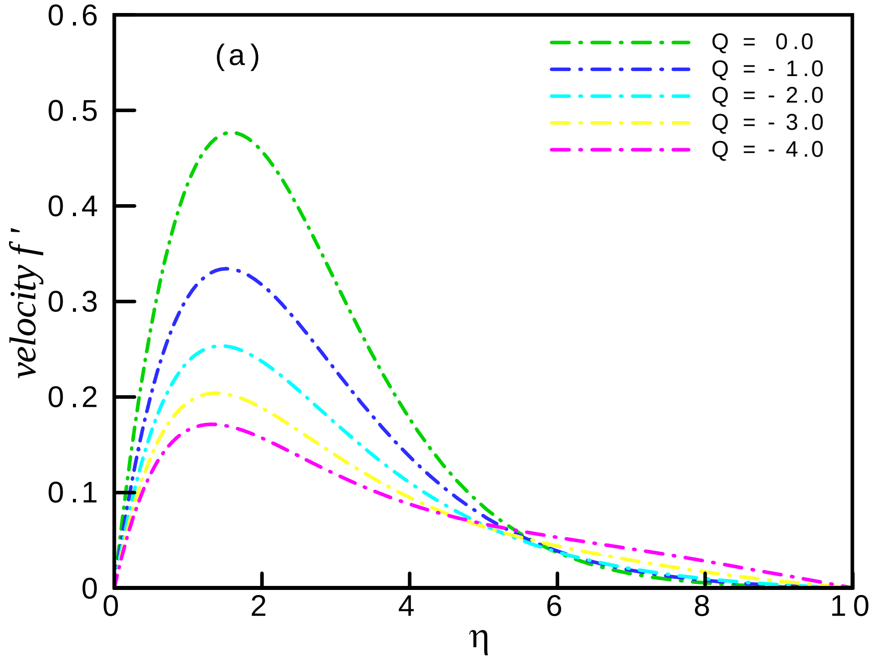


Figure 3.4 : Isotherms for (a) $\varepsilon = 0.0$, (b) $\varepsilon = 20.0$, (c) $\varepsilon = 30.0$ and (d) $\varepsilon = 60.0$ while $Pr = 1.0$, $\alpha = 0.3$, $Q = -0.1$, $Ec = 0.02$.

3.2 Result and Discussions on Variation of Heat Absorption parameter *Q*



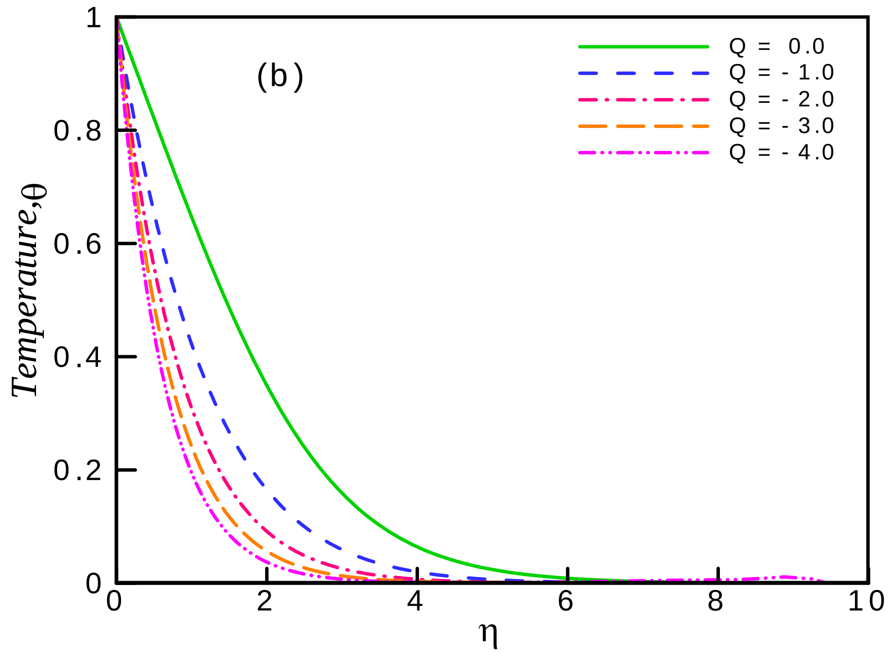


Figure 3.5 : (a) Velocity and (b) temperatureprofiles fordifferent values of heat absorption parameter Q while $Pr = 1.0$, $\alpha = 0.3$, $\varepsilon = 0.5$, $Ec = 0.02$.

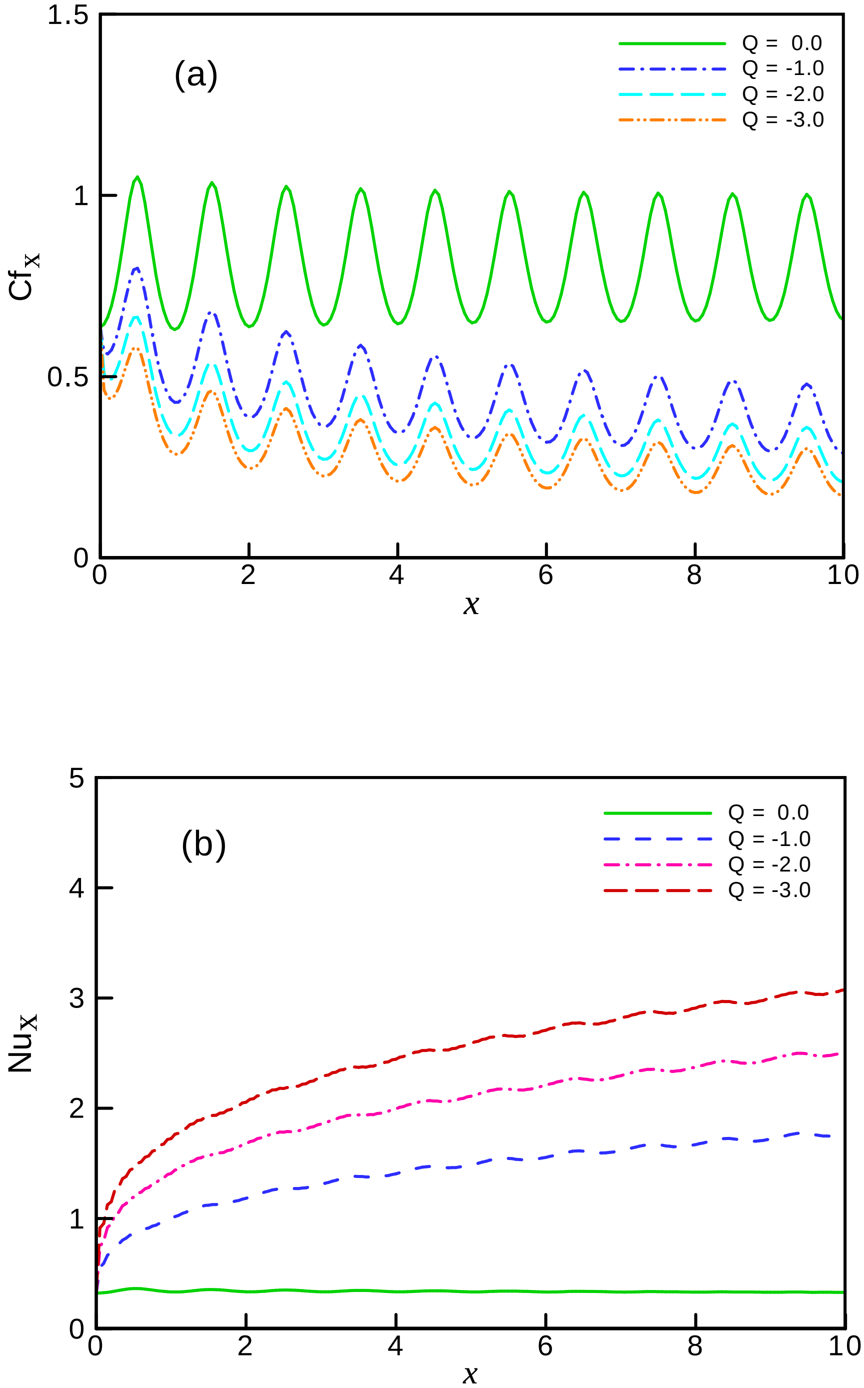


Figure 3.6: Variation of(a) skin friction coefficient (*Cfx*) and (b) rate ofheat transfer (*Nux*) for different values of heat absorption parameter *Q* while *Pr* =1.0, α = 0.3, ε = 0.5, *Ec* = 0.02.

Showing results of velocity and temperature in figures 3.5 (a) and (b), for different values of heat absorption parameter Q = 0.0, -0.1, -2.0, -3.0, -4.0 while the Prandtl Number Pr = 1.0, the amplitude of the wavy surface α = 0.3, the viscosity variation parameter ε = 0.5 and the Eckert number Ec = 0.02.

It is displayed from figures 3.5 (a) that the heat absorption parameter Q increases the velocity decrease up to the position of η = 5.69294. It is also observed from the figure that as the changes of velocity in the η direction reveals the typical velocity for natural convection boundary layer flow. i.e. The velocity is zero at the boundary wall then the velocity increases to the peak value as η increases and finally the velocity approaches to zero (the asymptotic value). But observing from this figure that all the velocity meet together at the position of η = 5.69294 and cross the side and increasing with the heat absorption parameter Q.

This is because of the velocity having lower peak values for higher values of heat absorption parameter Q tends to decrease comparatively slower along η-direction than velocity with higher peak values for lower values of heat absorption parameter Q. The maximum values of velocities are recoded as 0.47685, 0.33430, 0.25353, 0.20412, 0.17142 at the position of η = 1.58311, 1.50946, 1.43822, 1.36929, 1.30254 for heat absorption parameter Q = 0.0, -0.1, -2.0, -3.0, -4.0 respectively. The velocity 0.47685 is maximum at η = 1.58311 for Q = 0.0. Here, it is observed that the velocity decreases by approximately 64% as the heat absorption parameter Q changes from 0.0 to -4.0. From figure 3.5 (b), as the heat absorption parameter Q increases, the temperature decrease. It observed that the temperature is 1.0 (one) at the boundary wall then the temperature decreasing gradually to η direction.

It is found on figures 3.6 (a) and (b) an increase in the heat absorption parameter Q = 0.0, -1.0, -2.0, -3.0 leads to decrease the local skin friction coefficient Cf_x and increase the local rate of heat transfer Nu_x at different position of x. These are happened, since the increasing values of Q leads to decrease temperature of the fluid flow. Decreasing temperature decrease the viscosity of the fluid. Hence the corresponding shearing stress in terms of local skin friction coefficient decreases. To decrease the local skin friction coefficient Cf_x and the local rate of heat transfer Nu_x increases approximately 66% and 87% respectively.

Explanation of streamlines and Isotherms on figures 3.7 and 3.8,the effect of variation of heat absorption parameter Q = 0.0, -0.1, -2.0, -3.0 while the Prandtl Number Pr = 1.0, the amplitude of the wavy surface α = 0.3, the viscosity variation parameter ε = 0.5 and the Eckert number Ec = 0.02. Figure 3.7 depicts that the maximum values of ψ decreases while the values of Q increase that is ψ_{max} are 8.50, 3.08, 1.92, 1.42 for Q = 0.0, -0.1, -2.0, -3.0 respectively. It is noted from figure 3.8 that as the value of Q increases the thermal boundary layer becomes thinner gradually. So, the isotherms decrease while the values of Q increase.

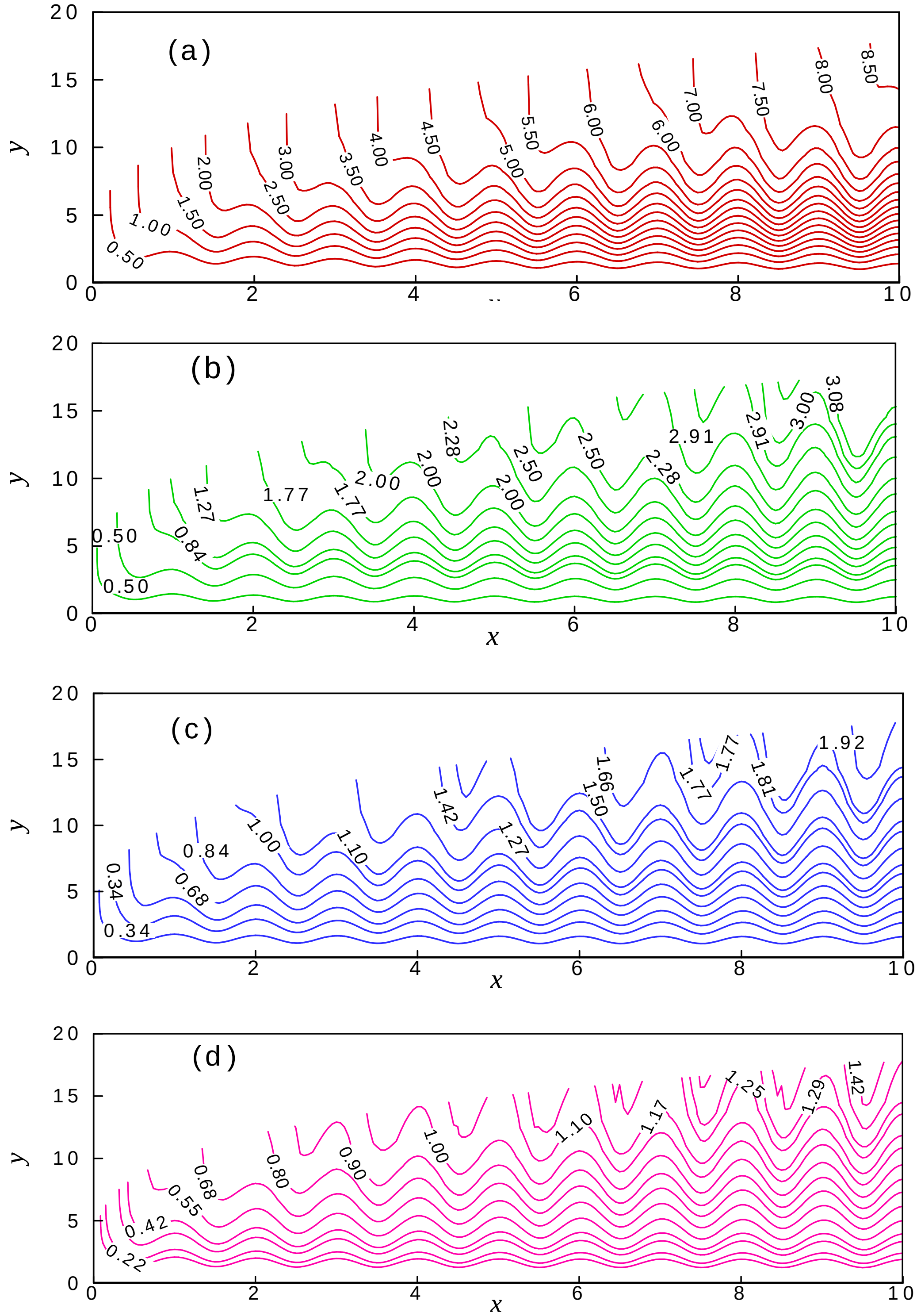


Figure 3.7 : Streamlines for (a) Q = 0.0, (b) Q = -1.0, (c) Q = -2.0 and (d) Q = - 3.0 while Pr = 1.0, α = 0.3, ε = 0.5, Ec = 0.02.

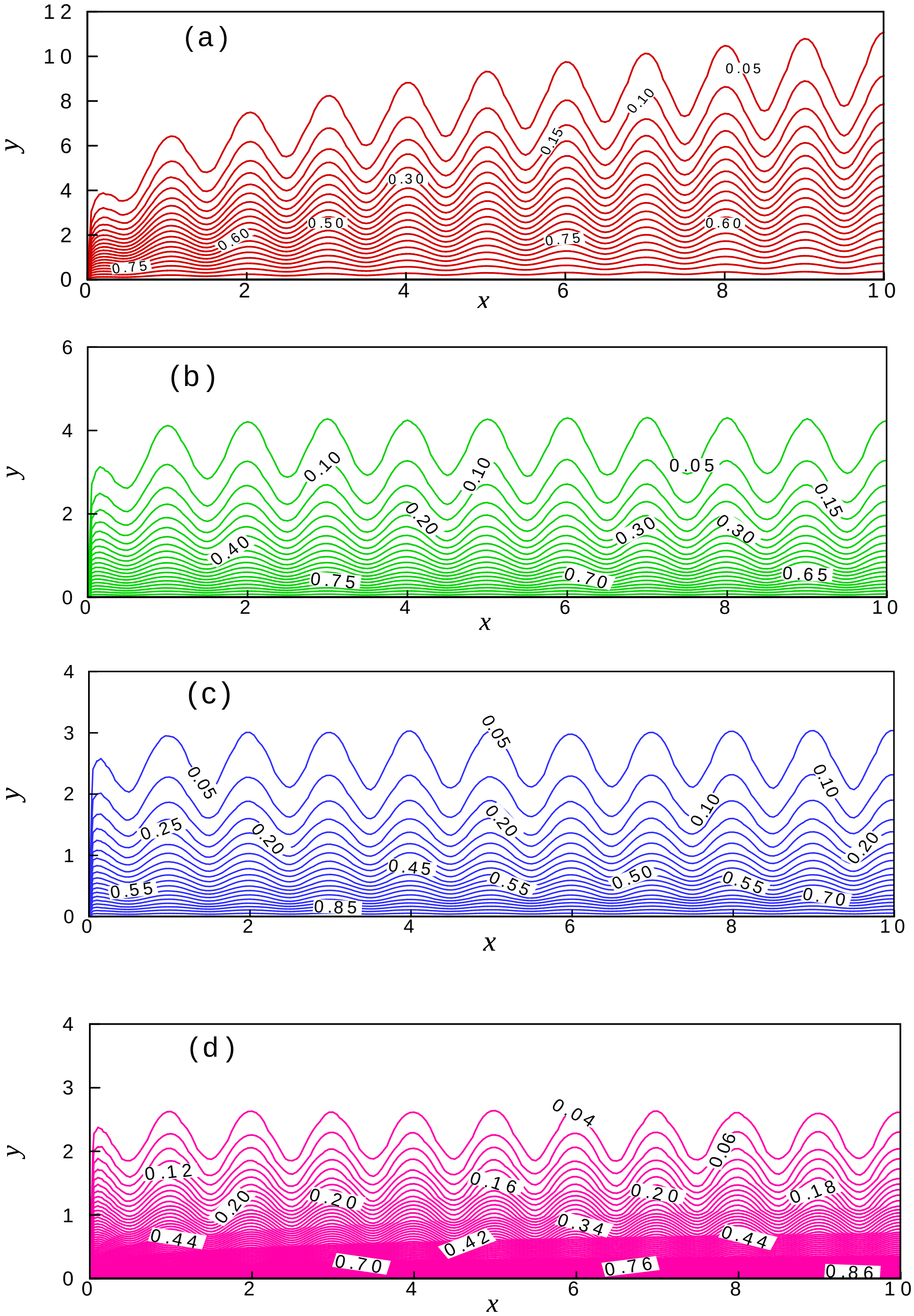


Figure 3.8 : Isotherms for (a) Q = 0.0, (b) Q = -1.0, (c) Q = -2.0 and (d) Q = -3.0 while Pr = 1.0, α = 0.3, ε = 0.5, Ec = 0.02.

3.3 Result and Discussions on Variation of Eckert Number *Ec*

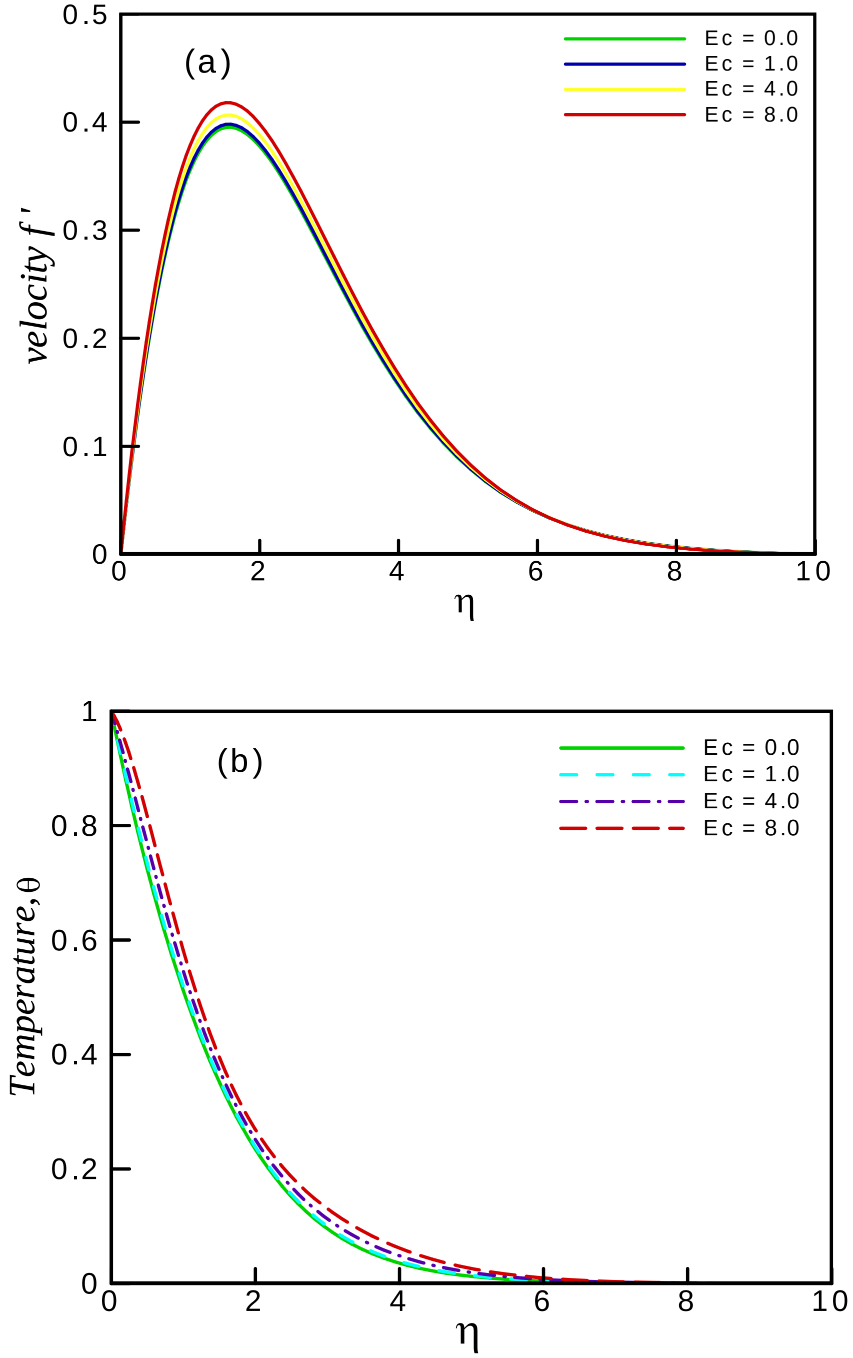


Figure 3.9 : (a) Velocity and (b) temperature profile for different values of Eckert number *Ec* while $Pr = 1.0$, $\alpha = 0.3$, $Q = -0.5$, $\varepsilon = 0.5$.

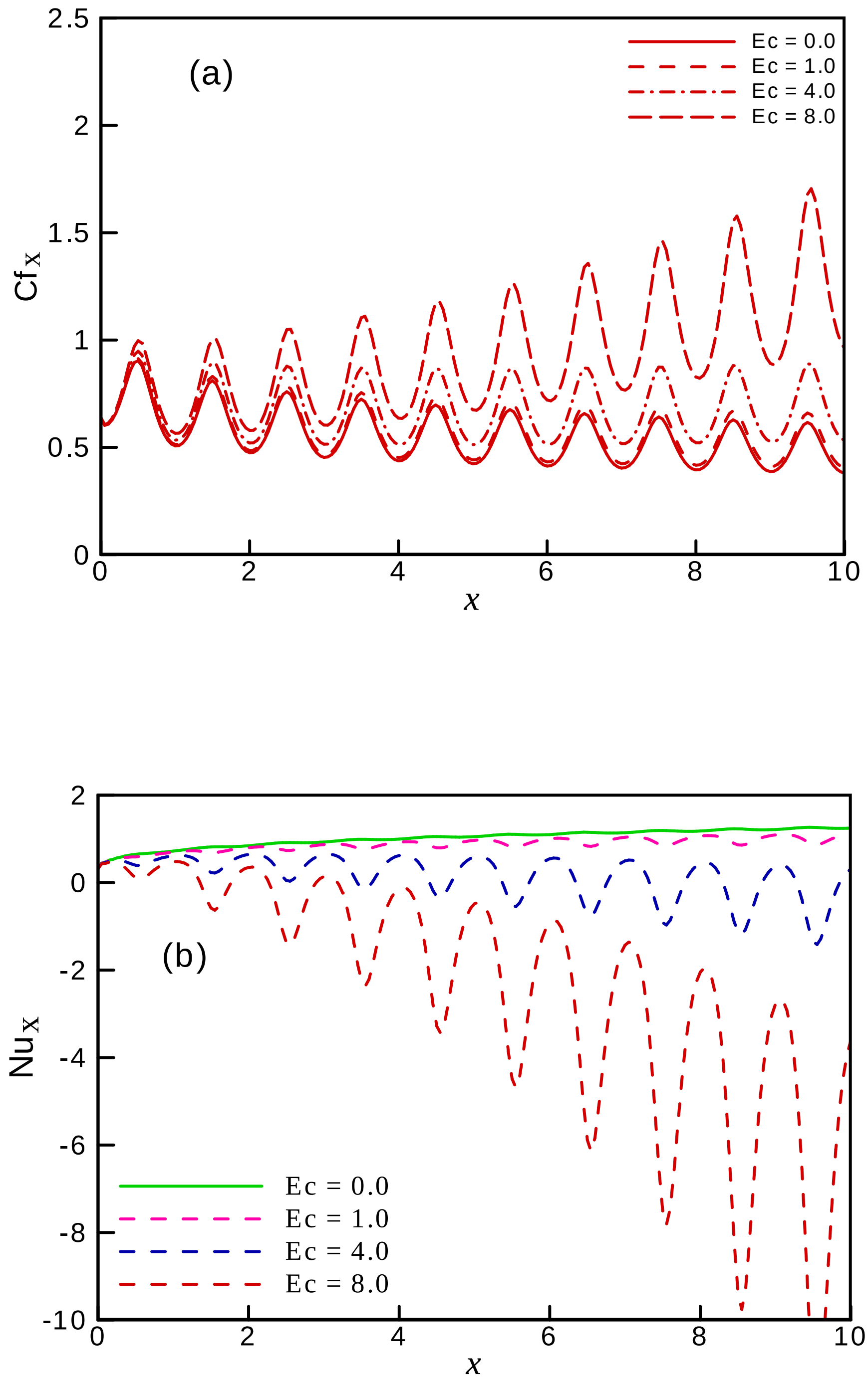


Figure 3.10 : Variation of(a) Skin friction coefficient (*Cfx*) and (b) rate of heat transfer (*Nux*) for different values of Eckert number *Ec* while $Pr = 1.0$, $\alpha = 0.3$, $Q = -0.5$, $\varepsilon = 0.5$.

Figures 3.9 (a) and (b), the effects for different values of the Eckert number *Ec* = 0.0, 1.0, 4.0, 8.0 on the velocity and temperature with the Prandtl Number *Pr* = 1.0, the amplitude of the wavy surface α = 0.3, the heat absorption parameter *Q* = -0.5 and the viscosity variation parameter ε = 0.5 have been shown graphically. It is shown from figure 3.9 (a) that as the Eckert number *Ec* increases, the velocities rising up to the position of η = 1.58311 for Eckert number *Ec* = 0.0, 1.0, 4.0, 8.0 and from that position of η velocities fall down slowly and finally approaches to zero.

It is expected because increasing value of *Ec* increases thermal energy inside the boundary layer due to fluid friction which is obviously increase convection and ultimately increases velocity. In the Eckert number process heat is automatically generated which increases temperature of the fluid flow.

The maximum values of the velocities are recorded as 0.39525, 0.39807, 0.40658, 0.41796 for Eckert number *Ec* = 0.0, 1.0, 4.0, 8.0 respectively which occur at the same position of η = 1.58311 and the maximum velocity increases approximately 91% (Lowest 0.39525; Highest 0.41796 at η = 1.58311).

It is also observed from figure 3.9 (b) that as the Eckert number *Ec* increases, the temperature increase. Temperatures are recorded as 0.42992, 0.43653, 0.45690, 0.48525 for Eckert number *Ec* = 0.0, 1.0, 4.0, 8.0 respectively which occur at the same position of η = 1.23788 and the temperature increases by approximately 11 %.

The effect of the different values of the Eckert number *Ec* on the local skin friction Cf_x and the rate of heat transfer in terms of the local Nusselt number Nu_x are shown in figures 3.10 (a) and (b) respectively while the values of Prandtl Number *Pr* = 1.0, the heat absorption parameter *Q* = -0.5, the amplitude of the wavy surface α = 0.3 and viscosity variation parameter ε = 0.5. It is noted that an increase in the values of the Eckert number *Ec* = 0.0,1.0, 4.0, 8.0 leads to enhance in the results of local skin friction Cf_x increases along the upstream direction of the surface and to decrease of the heat transfer rates. It is noted that the skin friction coefficient increases by approximately 47% for the increasing value of *Ec* from 0.0 to 8.0.

Shownin figure 3.11 and figure 3.12 the effect of the Eckert number *Ec* = 0.0, 1.0, 4.0, 8.0 on the streamlines and isotherms with the Prandtl Number *Pr* = 1.0, the amplitude of the wavy surface $\alpha = 0.3$, the heat absorption parameter Q = -0.5and the viscosity variation parameter $\varepsilon = 0.5$. It is found that for *Ec* = 0.0 the value of ψ_{max} is 4.40, for *Ec* = 1.0 the value of ψ_{max} is 4.65, for *Ec* = 4.0 the value of ψ_{max} is 4.79 and for *Ec* = 8.0 the value of ψ_{max} is 6.76. From figure 3.12, it is seen that the effect of theEckert number *Ec*, the flow rate in the boundary layer increases. It is also observed that due to the effect of *Ec*, the thermal state of the fluid increases. Finally, the thermal boundary layer becomes higher.

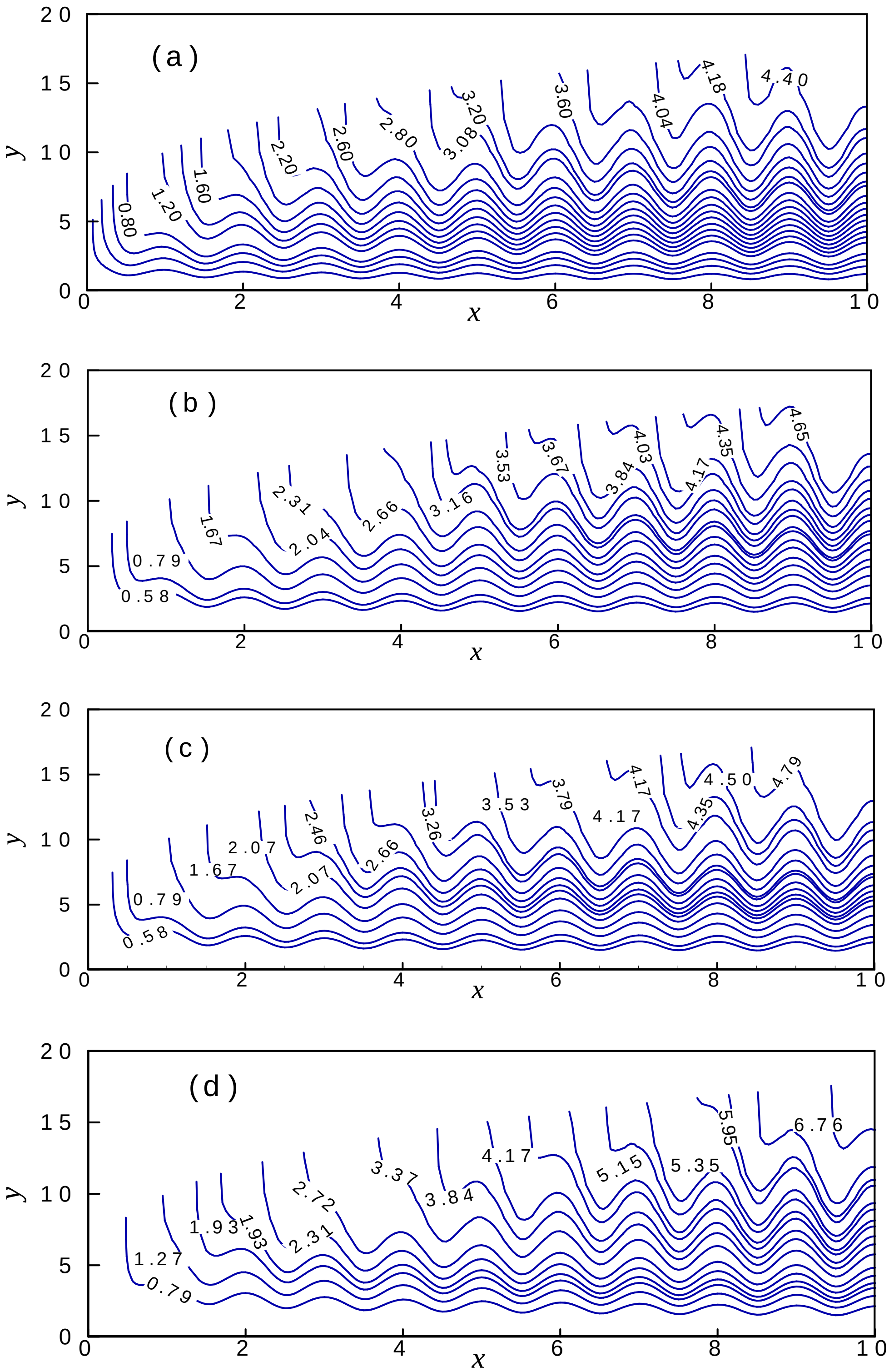


Figure 3.11 : Streamlines for (a) Ec = 0.0, (b) Ec = 1.0, (c) Ec = 4.0 and (d) Ec = 8.0 while Pr = 1.0, α = 0.3, Q = -0.5, ε = 0.5.

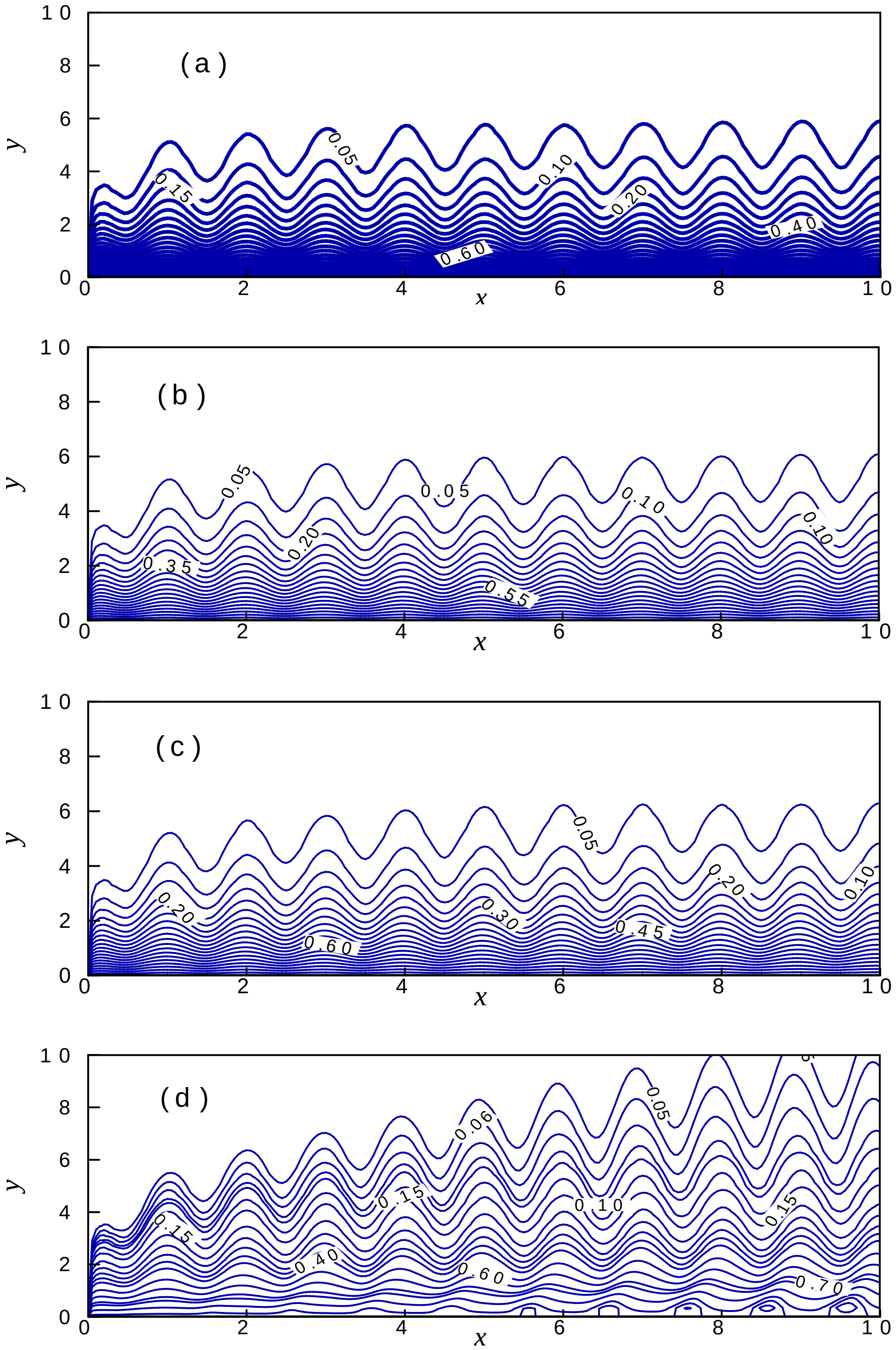


Figure 3.12 : Isotherms for (a) Ec = 0.0, (b) Ec = 1.0, (c) Ec = 4.0 and (d) Ec = 8.0 while Pr = 1.0, α = 0.3, Q = -0.5, ε = 0.5.

3.4 Result and Discussions on Variation of Prandlt Number*Pr*

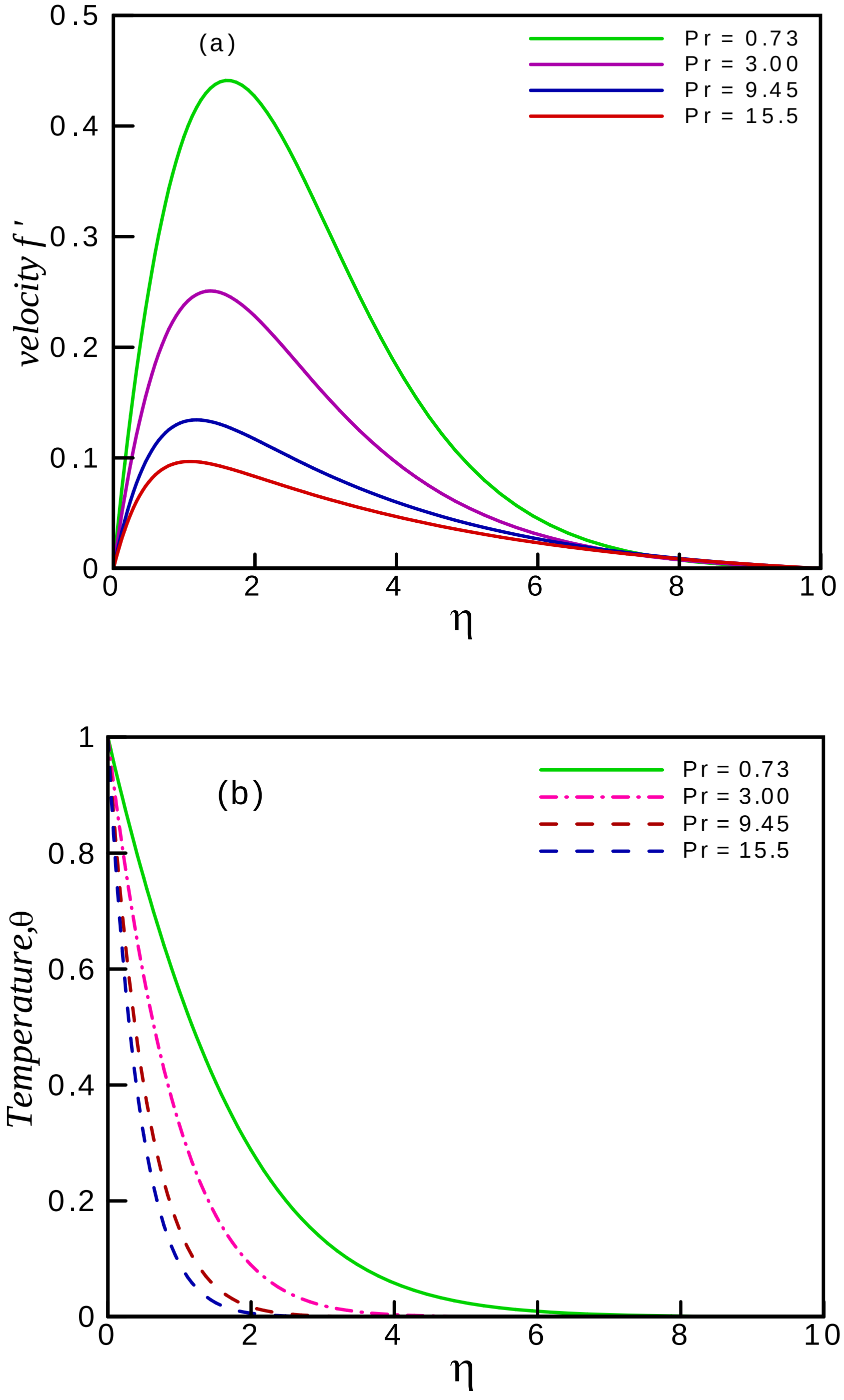
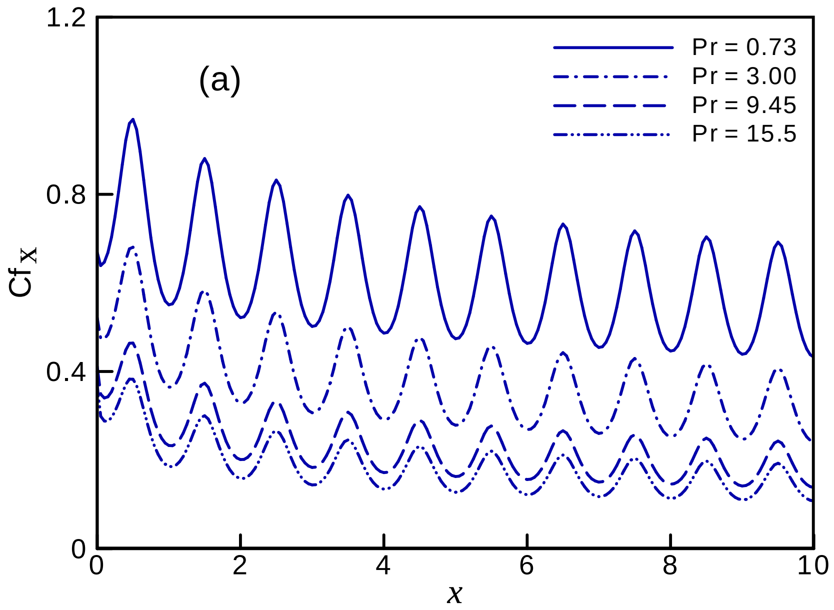


Figure 3.13 : (a) Velocity and (b) temperature profiles for different values of *Pr* while Q = - 0.1, α = 0.3, ε = 0.5, Ec = 0.02.



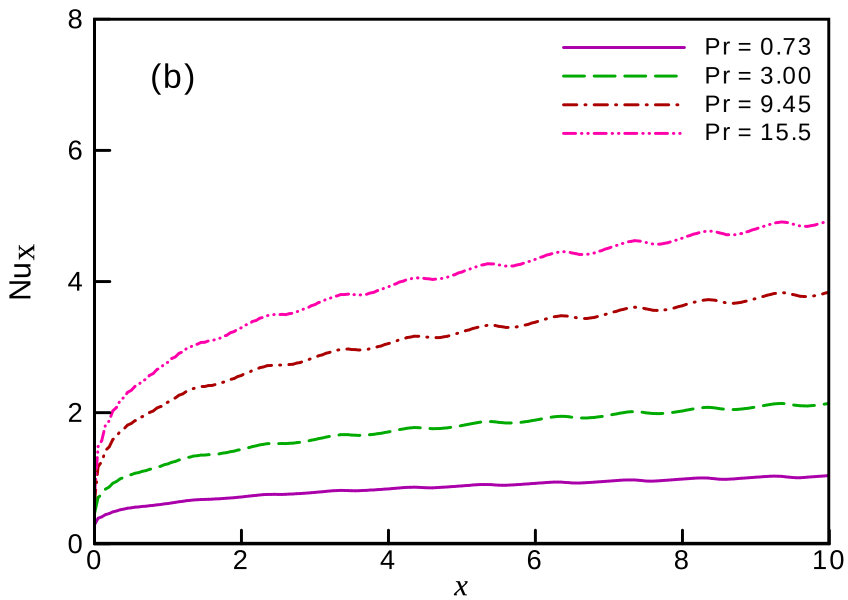


Figure 3.14 : Variation of(a) skin friction coefficient (*Cfx*) and (b) rate of heat transfer (*Nux*) for different values of *Pr* while $Q = -0.1$, $\alpha = 0.3$, $\varepsilon = 0.5$, $Ec = 0.02$.

To noticed the results of velocity and temperature in figures 3.13 (a) and (b), for different values of Prandtl Number *Pr* = 0.73, 3.00, 9.45, 15.5 while the heat absorption parameter *Q* = -0.1, the amplitude of the wavy surface α = 0.3, viscosity variation parameter ε = 0.5 and the Eckert number *Ec* = 0.02. It can be observed from figure 3.13 (a) that the velocity of the fluid decreases as well as its position moves toward the interface with the increasing values of Prandtl number *Pr*.

It is known that Prandtl number *Pr* is the ratio of viscosity force and thermal force. So, increasing values of *Pr* increases viscosity and decreases thermal action of the fluid. Because of this fact, figure is seen that velocity of the fluid decreases with the increasing values of Prandtl number *Pr*. It is found that the velocity decreases by approximately 78% when *Pr* increases from 0.73 to 15.5 for different value of η. It is noticed from figure 3.13 (b) that the temperature shift downward with the increasing values of*Pr*.

In the figures 3.14 (a) and (b), the surface shear stress in terms of the local skin friction Cf_x and the rate of heat transfer in terms of the local Nusselt number Nu_x are depicted graphically for the different values of Prandtl Number *Pr* = 0.73, 3.00, 9.45, 15.5 while the heat absorption parameter *Q* =-0.1, the amplitude of the wavy surface α =0.3, viscosity variation parameter ε =0.5and the Eckert number*Ec* =0.02. It is observed that the local skin friction coefficient decreases monotonically along the downward direction of the surface for increasing value of Prandtl number *Pr*.

From figures 3.14 (b) it is noted that the effect is more pronounced with the Prandtl number *Pr*. Increasing values of *Pr* increases viscosity. If viscosity increases, then fluid does not move freely. Increasing the values of Prandtl Number *Pr*, speed up the decay of temperature field away from the heat surface with a consequent increase in the heat transfer.

The maximum values of local skin friction coefficient Cf_x are 0.70401 and 0.19809 and the rate of heat transfer in terms of the local Nusselt number Nu_x are 1.00538 and 4.75503 for Pr = 0.73 and 15.5 at the position of x = 8.50 and x = 8.25 respectively. It is seen that the local skin friction coefficient Cf_x decreases by approximately 72% and the local Nusselt number Nu_x increases by approximately 79% as Pr increases from 0.73 to 15.5.

The effect of variation of the surface roughness on the streamlines and isotherms for the values of Prandtl Number Pr = 0.73, 3.00, 9.45, 15.5 while the heat absorption parameter Q = -0.1, the amplitude of the wavy surface α = 0.3, viscosity variation parameter ε = 0.5and the Eckert number Ec = 0.02 in figures 3.15 and 3.16.

In figures 3.15 depicts that the maximum values of streamline decrease steadily while the values of Pr increases. The maximum values of streamlines are 5.50, 2.36, 0.91 and 0.55 for Pr = 0.73, 3.00, 9.45 and 15.5. It is observed in figure 3.16 that as the values of Pr decreases the thermal boundary layer thickness becomes thinner gradually.

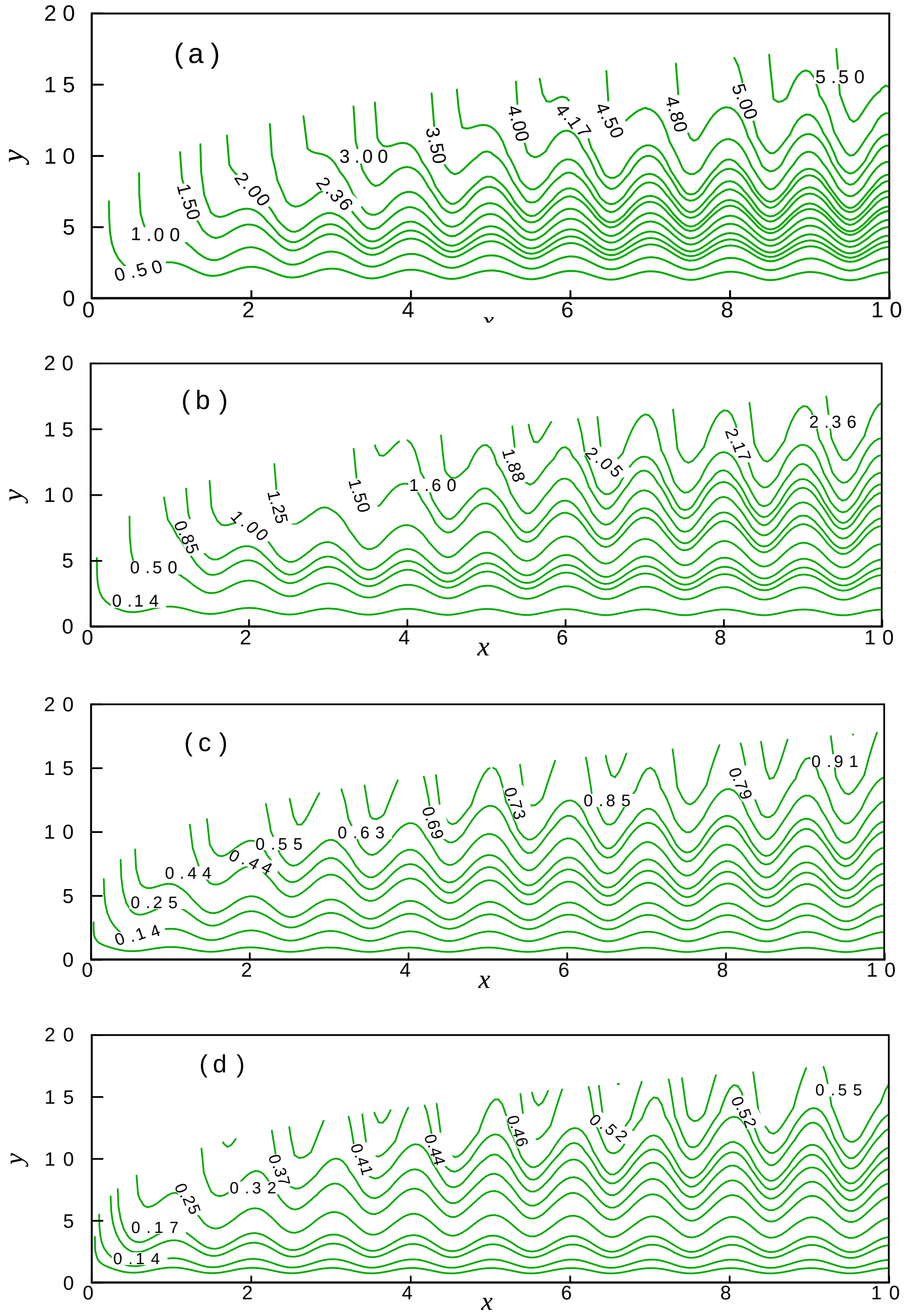


Figure 3.15 : Streamlines for (a) Pr = 0.73, (b) Pr = 3.00, (c) Pr = 9.45 and (d) Pr = 15.5 while ε = 0.5, α = 0.3, Q = -0.5, Ec = 0.02.

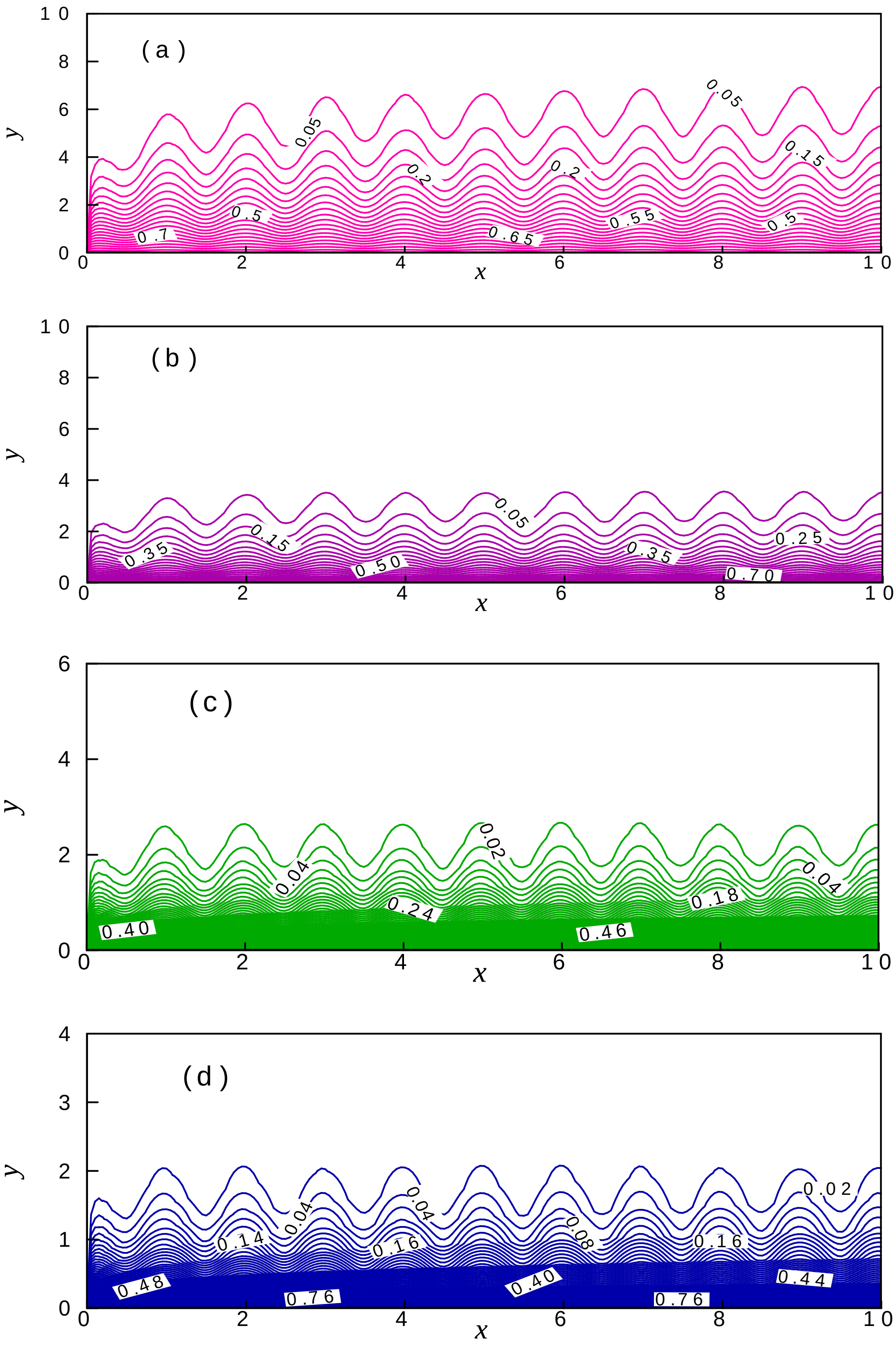


Figure 3.16 : Isotherms for (a) Pr = 0.73, (b) Pr = 3.00, (c) Pr = 9.45 and (d) Pr = 15.5 while ε = 0.5, α = 0.3, Q = -0.5, Ec = 0.02.

3.5 Result and Discussions on Variation of Surface Amplitude α

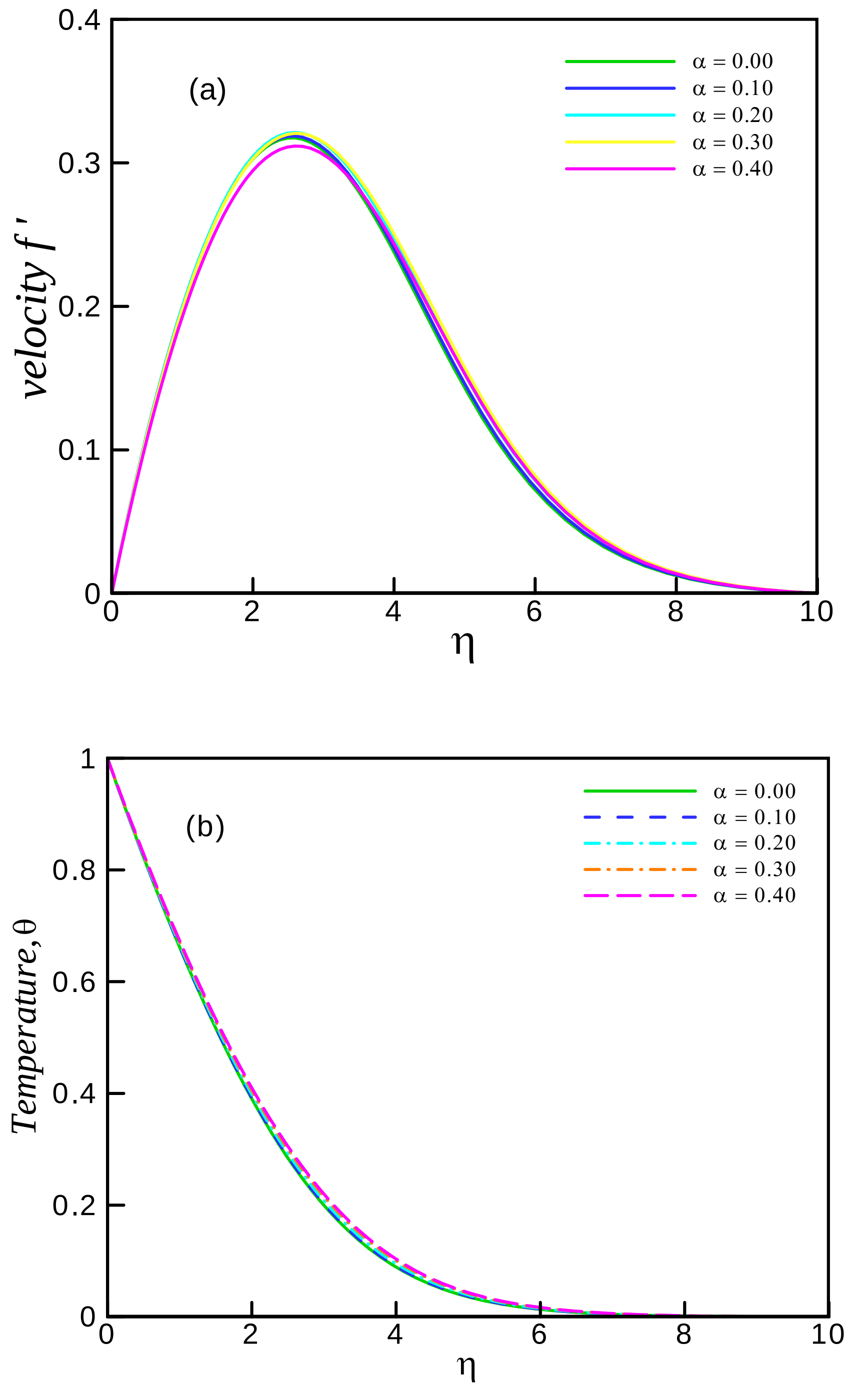


Figure 3.17: (a) Velocity and (b) temperature profiles for different values of α while Pr = 1.0, Q = - 0.1, ε = 0.5, Ec = 0.02.

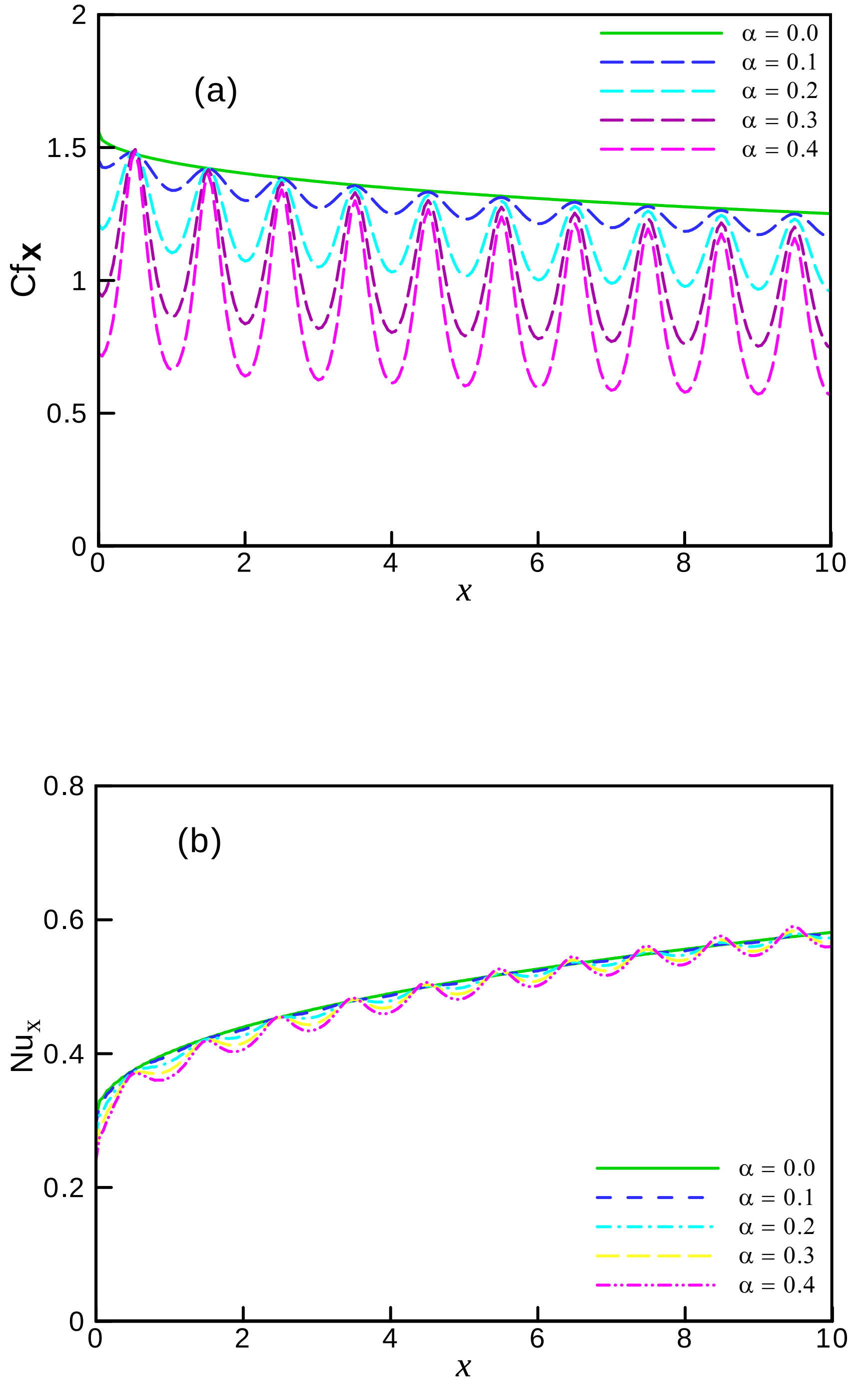


Figure 3.18 : Variation of(a) skin friction coefficient (Cfx) and (b) rate of heat transfer (Nux) for different values of α while Pr = 1.0, Q = -0.1, ε = 0.5, Ec = 0.02.

Figures 3.17 (a) and (b), the effects for different values of the amplitude of the wavy surface α = 0.0, 0.1, 0.2, 0.3, 0.4 on the velocity and temperature with the Prandtl Number Pr = 1.0, the heat absorption parameter Q = -0.5, the Eckert number Ec = 0.02 and the viscosity variation parameter ε = 0.5 have been shown. Figure 3.17 (a) shows the small increment on the velocity $f'(x,\eta)$ for increasing values of α.It is seen that the velocity increases 2% for same values of η when αincreases from 0.0 to 0.4 (Lowest 0.39525; Highest 0.41796 at η = 2.58959). Figure 3.17 (b) depicts the temperature $\theta(x,\eta)$, which increases slowly with the increase of the amplitude of wavy surface α from 0.0 to 0.4.

In figures 3.18 (a) and (b), the surface shear stress in terms of the local skin friction $Cf_x$and the rate of heat transfer in terms of the local Nusselt number Nu_x are depicted graphically for the different values of amplitude of wavy surface α = 0.0, 0.1, 0.2, 0.3, 0.4 when values of viscosity variation parameter ε = 0.5, the value of Prandtl number Pr = 1.0, the heat absorption parameter Q = -0.1 andthe Eckert number Ec= 0.02. Since for increasing the surface waviness, the velocity force decreases at local points so the figures show that increase in the value of amplitude of wavy surface α from 0.0 to 0.4 lends to decrease the value of skin friction coefficient Cf_x and the rate of heat transfer in terms of the local Nusselt number Nu_x. It is seen that the local skin friction coefficient Cf_x decreases 5% for different values of x when α increases from 0.0 to 0.4.

The effect of variation of the surface roughness on the streamlines and isotherms for the values of amplitude of the wavy surface α = 0.0, 0.1, 0.2 and 0.3 are depicted by the figure 3.19 and 3.20 respectively while Prandtl number Pr = 1.0, heat absorption parameter Q = -0.1, the viscosity variation parameter ε = 0.5and the Eckert number Ec = 0.02.It is observed from the figure 3.19 that, the maximum values of streamline are ψ_{max}= 6.28, 6.24, 6.21, 6.14 for the values of α = 0.0, 0.1, 0.2, 0.3 respectively. Here it can be concluded that for increasing values of amplitude to the length ratio of the wavy surfaceα, the roughness of the wavy surface increases so the velocity boundary layer thickness decreases gradually. Similar result is observed for thermal boundary layer thickness. So, isotherms increase for increasing values of amplitude to the length ratio of the wavy surface α.

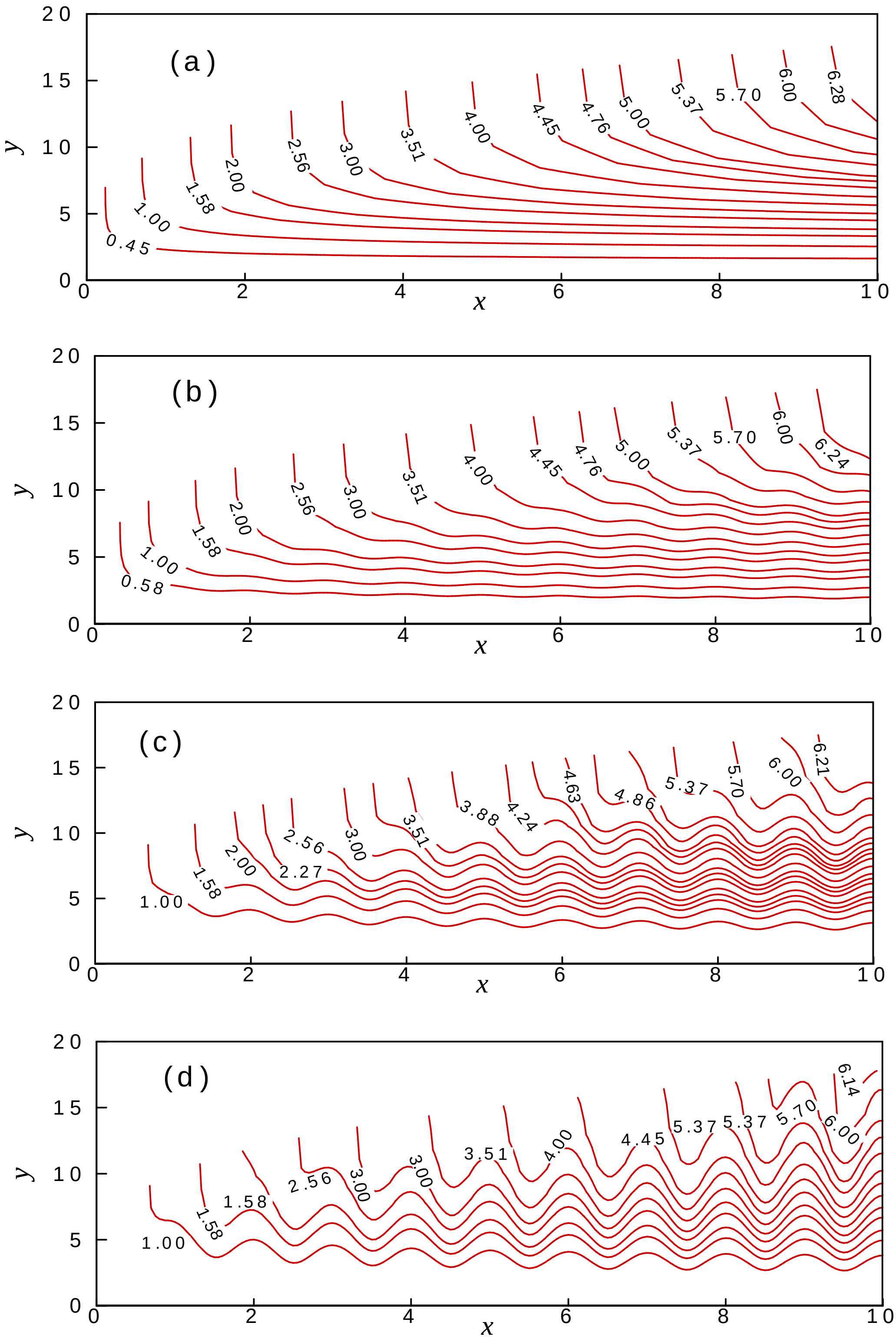


Figure 3.19 : Streamlines for (a) α = 0.0, (b) α = 0.1, (c) α = 0.2 and (d) α = 0.3 while Pr = 1.0, Q = - 0.1, Ec = 0.02, ε = 0.5.

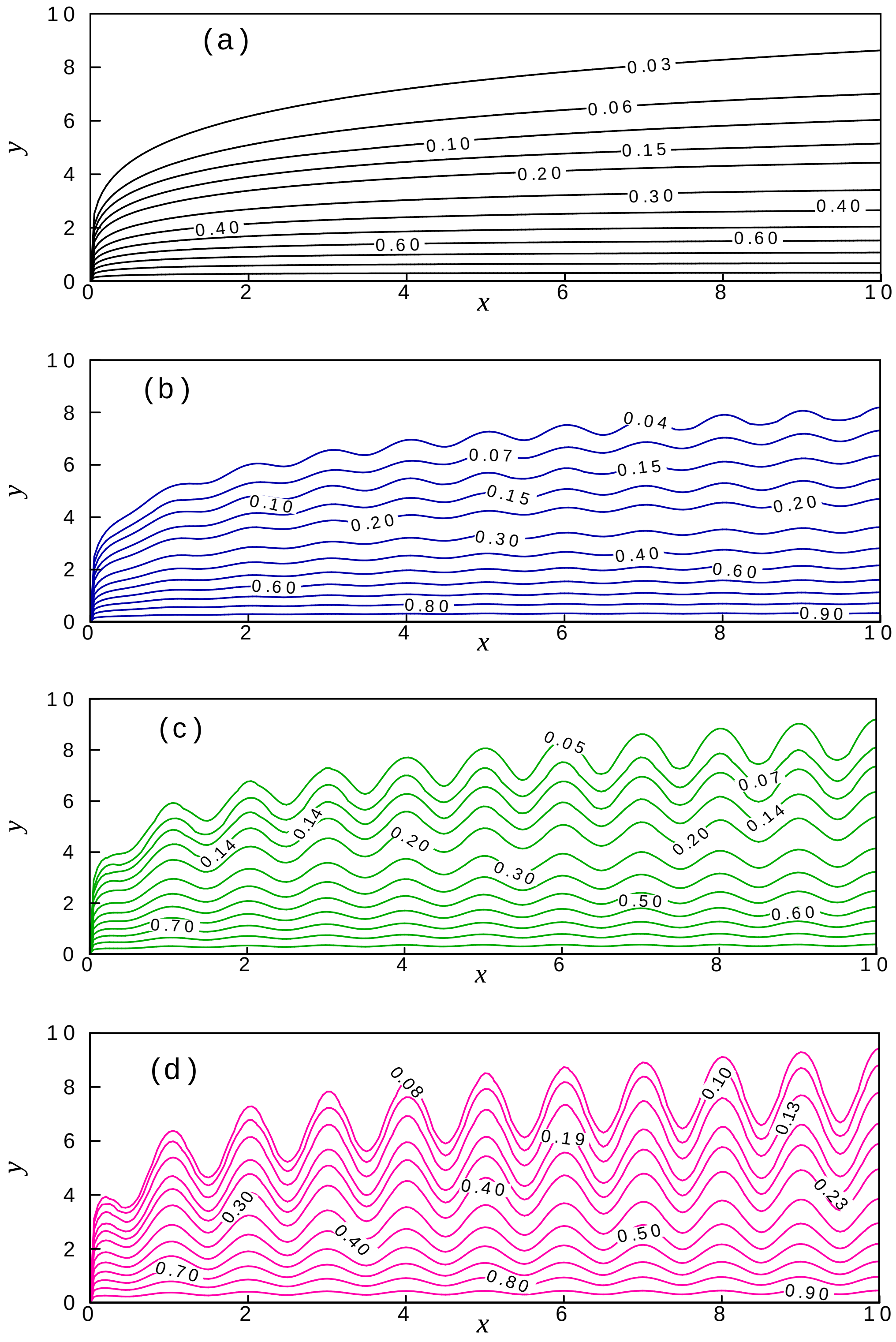


Figure 3.20 : Isotherms for (a) α = 0.0, (b) α = 0.1, (c) α = 0.2 and (d) α = 0.3 while Pr = 1.0, Q = -0.1, Ec = 0.02, ε = 0.5.

3.6Comparison of the results

The problem of heat absorption on natural convection flow along a vertical wavy surface with viscous dissipation under variable viscosity has been investigated. Because of the physical property viscous dissipation and viscosity must have a significant change with temperature, it is necessary to take into account of variation of viscosity and Eckert number to obtain a better estimation of the flow and heat transfer behavior.

Table 3.6.1: Comparison of the present numerical results of skin friction coefficient,$f''(x,0)$ and the heat transfer, $-\theta'(x,0)$ with Hossain et al. (2002) for the variation of Prandtl number Pr while Ec = 0.0, ε= 0.0, Q = 0.0 and α= 0.1.

Pr	$f''(x,0)$		$-\theta'(x,0)$	
	Hossian et al. (2002)	Present Work	Hossian et al. (2002)	Present Work
1.0	0.908	0.90814	0.401	0.39914
10.0	0.591	0.59269	0.825	0.82663
25.0	0.485	0.48733	1.066	1.06847
50.0	0.485	0.41880	1.066	1.28351
100.0	0.352	0.35640	1.542	1.54198

A comparison of the present numerical results of skin friction coefficient, $f''(x,0)$ and the heat transfer, $-\theta'(x,0)$ with the result obtained by Hossain et al. (2002) depicted the table 3.6.1. Here the viscosity variation parameter ε, the Eckert number Ec and the heat absorption parameter Q ignored while different values of prandtl number Pr (= 1.0, 10.0, 25.0, 50.0, 100.0) are chosen with amplitude-to-length ratio α = 0.1. Form table 3.6.1, it is clearly seen that the present results are excellent agreement with the solution of Hossain et al. (2002).

3.7Tables of Numerical Discussions

Table3.7.1: Skin friction coefficient and rate of heat transfer for different values of viscosity variation parameter ε while, Prandtl number Pr = 1.0, heat absorption Q = -0.1, amplitude of wavy surface α = 0.3 and Eckert number Ec = 0.02.

x	Skin friction coefficient			Rate of heat transfer		
	ε=0.0	ε=20.0	ε=60.0	ε=0.0	ε=20.0	ε=60.0
0.00	0.56194	1.29626	1.53359	0.34186	0.20205	0.16754
0. 50	0.90154	1.99589	2.41116	0.44491	0.32507	0.29947
1.00	0.53558	1.11876	1.26186	0.43894	0.34042	0.32028
1.50	0.86948	1.83825	2.09961	0.47900	0.38545	0.37153
2.00	0.53397	1.05732	1.16159	0.47300	0.38874	0.37654
2.50	0.85010	1.74273	1.95470	0.50286	0.42618	0.41444
3.00	0.53151	1.01468	1.10202	0.49747	0.42256	0.41473
3.50	0.83652	1.67393	1.85525	0.52175	0.45745	0.44745
4.00	0.52889	0.98138	1.05837	0.51726	0.44975	0.44446
4.50	0.82588	1.61967	1.77932	0.53767	0.48329	0.47472
5.00	0.52632	0.95408	1.02279	0.53413	0.47278	0.46911
5.50	0.81703	1.57453	1.71819	0.55156	0.50555	0.49805
6.00	0.52385	0.93095	0.99257	0.54898	0.49293	0.49037
6.50	0.80941	1.53577	1.66724	0.56395	0.52521	0.51853
7.00	0.52149	0.91085	0.96635	0.56231	0.51092	0.50916
7.50	0.80268	1.50179	1.62366	0.57517	0.54289	0.53686
8.00	0.51923	0.89307	0.94326	0.57446	0.52723	0.52607
8.50	0.79665	1.47157	1.58566	0.58547	0.55901	0.55350
9.00	0.51708	0.87711	0.92271	0.58565	0.54219	0.54149
9.50	0.79116	1.44439	1.55198	0.59500	0.57385	0.56880
10.0	0.51502	0.86264	0.90426	0.59605	0.55603	0.55569

Table 3.7.1 represents the values of skin-friction and the rate of heat transfer for the computational domain. It is to be pointed out that the complete cycle of the wavy surface is from x = 0.0 to 2.0. The skin friction and the rate of heat transfer increases for the first quarter of the cycle ($x\approx$ 0 to $x\approx$ 0.50) and decreases in the second quarter ($x\approx$ 0.50 to $x\approx$ 1.0). From $x\approx$ 1.0 to $\approx$ 1.5 (i.e., third quarter) the skin friction again increases, whereas the last quarter ($x\approx$ 1.5 to $x\approx$ 2.0) it decreases. The skin friction and the rate of heat transfer shown similar characteristics throughout the domain. But the rate of heat transfer for ε = 20.0and ε = 60.0 only increasesfor x = 0 to 10. The percentage of changes of skin friction coefficient increases for x = 0.5 by approximately 62.6 %. Again,The percentage of changes of rate of heat transfer coefficient decreases for x = 1.5 by approximately 22.43 %.

Table3.7.2: Skin friction coefficient and rate of heat transfer for different values of Eckert number *Ec* while, Prandtl number *Pr* = 1.0, heat absorption *Q* = -0.1, amplitude of wavy surfaceα = 0.3 and viscosity variation parameter ε = 0.5.

x	Skin friction coefficient			Rate of heat transfer		
	Ec =0.0	*Ec =4.0*	*Ec =8.0*	*Ec =0.0*	*Ec =4.0*	*Ec =8.0*
0.00	0.63663	0.63663	0.63663	0.32374	0.32374	0.32374
0. 50	0.90323	0.94808	0.99706	0.64635	0.39588	0.10543
1.00	0.50716	0.53386	0.56527	0.71987	0.61245	0.47690
1.50	0.81046	0.89671	1.00604	0.81425	0.23045	-0.58016
2.00	0.47517	0.52021	0.57913	0.84267	0.64271	0.34656
2.50	0.75968	0.87806	1.04826	0.91562	0.05036	-1.35942
3.00	0.45370	0.51458	0.60393	0.92781	0.63968	0.14091
3.50	0.72450	0.86971	1.10435	0.99145	-0.13335	-2.25175
4.00	0.43735	0.51277	0.63647	0.99463	0.61914	-0.13494
4.50	0.69758	0.86669	1.17179	1.05309	-0.32078	-3.28448
5.00	0.42412	0.51319	0.67538	1.05030	0.58682	-0.48408
5.50	0.67580	0.86708	1.25037	1.10552	-0.51308	-4.48639
6.00	0.41302	0.51509	0.72027	1.09838	0.54541	-0.91318
6.50	0.65755	0.86993	1.34014	1.15142	-0.71123	-5.88457
7.00	0.40346	0.51809	0.77110	1.14091	0.49637	-1.43073
7.50	0.64186	0.87465	1.44124	1.19243	-0.91623	-7.50465
8.00	0.39508	0.52197	0.82797	1.17920	0.44039	-2.04593
8.50	0.62812	0.88089	1.55384	1.22960	-1.12866	-9.37143
9.00	0.38763	0.52658	0.89098	1.21410	0.37803	-2.76838
9.50	0.61592	0.88843	1.67810	1.26369	-1.34946	-11.50832
10.0	0.38092	0.53184	0.96023	1.24625	0.30934	-3.60737

Table 3.7.2 represents the values of skin-friction and the rate of heat transfer for the computational domain. It is to be pointed out that the complete cycle of the wavy surface is from *x* = 0.0 to 2.0. The skin friction and the rate of heat transfer increases for the first quarter of the cycle ($x \approx$ 0 to $x \approx$ 0.50) and decreases in the second quarter ($x \approx$ 0.50 to $x \approx$ 1.0). From $x \approx$ 1.0 to $\approx$ 1.5 (i.e., third quarter) the skin friction again increases, whereas the last quarter ($x \approx$ 1.5 to $x \approx$ 2.0) it decreases. The skin friction and the rate of heat transfer shown similar characteristics throughout the domain. It is Exceptional to *Ec* = 0.0 from *x* = 0.0 to 4.5, the rate of heat transfer is increasing. The percentage of changes of skin friction coefficient increases for x = 0.5 by approximately 10.38 %. Again,the percentage of changes of rate of heat transfer coefficient decreases for x = 2.0 by approximately 58.83 %.

Table 3.7.3: Skin friction coefficient and rate of heat transfer for different values of amplitude of wavy surface α while, Prandtl number Pr = 1.0, heat absorption Q = -0.1, viscosity variation parameter ε= 5.0 and Eckert number Ec = 0.02.

x	Skin friction coefficient			Rate of heat transfer		
	α =0.00	α =0.20	α =0.40	α =0.00	α =0.20	α =0.40
0.00	1.55828	1.21115	0.72446	0.30288	0.27851	0.24135
0.50	1.47529	1.48730	1.48405	0.37400	0.37137	0.36971
1.00	1.44411	1.10384	0.66216	0.40176	0.38709	0.36401
1.50	1.42105	1.42193	1.39711	0.42241	0.42111	0.41926
2.00	1.40220	1.07272	0.64015	0.43941	0.42635	0.40600
2.50	1.38602	1.37935	1.34080	0.45409	0.45387	0.45533
3.00	1.37175	1.04996	0.62523	0.46714	0.45525	0.43639
3.50	1.35890	1.34705	1.29963	0.47895	0.47956	0.48318
4.00	1.34718	1.03136	0.61336	0.48978	0.47880	0.46128
4.50	1.33638	1.32065	1.26660	0.49983	0.50114	0.50635
5.00	1.32634	1.01543	0.60330	0.50921	0.49897	0.48258
5.50	1.31694	1.29816	1.23885	0.51804	0.51997	0.52647
6.00	1.30811	1.00138	0.59443	0.52638	0.51676	0.50132
6.50	1.29976	1.27849	1.21491	0.53429	0.53678	0.54439
7.00	1.29184	0.98877	0.58644	0.54184	0.53277	0.51814
7.50	1.28431	1.26096	1.19385	0.54905	0.55203	0.56063
8.00	1.27712	0.97731	0.57914	0.55596	0.54738	0.53345
8.50	1.27024	1.24511	1.17506	0.56261	0.56604	0.57553
9.00	1.26365	0.96678	0.57241	0.56901	0.56086	0.54755
9.50	1.25731	1.23062	1.15808	0.57518	0.57903	0.58933
10.0	1.25120	0.95703	0.56616	0.58115	0.57340	0.56063

Table 3.7.3 represents the values of skin-friction and the rate of heat transfer for the computational domain. It is to be pointed out that the complete cycle of the wavy surface is from x = 0.0 to 2.0. The skin friction and the rate of heat transfer increases for the first quarter of the cycle ($x\approx$ 0 to $x\approx$ 0.50) and decreases in the second quarter ($x\approx$0.50 to $x\approx$ 1.0). From $x\approx$ 1.0 to $\approx$ 1.5 (i.e., third quarter) the skin friction again increases, whereas the last quarter ($x\approx$ 1.5 to $x\approx$2.0) it decreases. The skin friction and the rate of heat transfer shown similar characteristics throughout the domain. The percentage of changes of skin friction coefficient increases for x = 0.5 by approximately 53.5 %. Again,the percentage of changes of rate of heat transfer coefficient decreases for x = 2.0 by approximately 7.60 %.

Table3.7.4: Velocity and Temperature for different values of Eckert number *Ec* while, Prandtl number *Pr* = 1.0, heat absorption *Q* = -0.1, amplitude of wavy surface *α*= 0.3 and viscosity variation parameter *ε* = 0.5.

x	Velocity			Temperature		
	Ec =0.0	*Ec* =4.0	*Ec* =8.0	*Ec* =0.0	*Ec* =4.0	*Ec* =8.0
0.00	0.00000	0.00000	0.00000	1.00000	1.00000	1.00000
0.54375	0.24763	0.25639	0.26545	0.70288	0.74710	0.79551
1.00	0.36376	0.37527	0.38697	0.49184	0.52406	0.55826
1.50	0.39505	0.40650	0.41803	0.34929	0.37042	0.39241
2.00	0.37121	0.38871	0.39911	0.21868	0.25308	0.27066
2.50	0.31858	0.32764	0.34976	0.13992	0.15707	0.18913
3.00	0.25986	0.26785	0.29196	0.08946	0.10656	0.13616
3.50	0.21082	0.21775	0.22449	0.06000	0.07580	0.09229
4.00	0.16181	0.16737	0.17268	0.03774	0.05102	0.06484
5.03870	0.07839	0.08068	0.08260	0.01189	0.01845	0.02508
5.69294	0.04859	0.04949	0.05004	0.00585	0.00958	0.01322
6.17407	0.03355	0.03379	0.03372	0.00344	0.00579	0.00799
7.00	0.01751	0.01721	0.01671	0.00141	0.01721	0.00329
8.52867	0.00391	0.00361	0.00329	0.00022	0.00037	0.00047
9.24368	0.00146	0.00130	0.00115	0.00007	0.00012	0.00015
10.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

In the above table the values of velocity and temperature are recoded to be 0.03355, 0.03379, 0.03372 and .00344, 0.00579, 0.00799 for *Ec* = 0.0, 4.0, 8.0 respectively at the same point of *x* = 6.17407. Here, it observed that the velocities and temperature almost similar at this point.

Table3.7.5: Velocity and Temperature for different values of amplitude of wavy surfaceα while, Prandtl number Pr = 1.0, heat absorption Q = -0.1, viscosity variation parameterε= 5.0 and Eckert numberEc = 0.02.

x	Velocity			Temperature		
	α =0.00	α =0.20	α =0.40	α =0.00	α =0.20	α =0.40
0.00	0.00000	0.00000	0.00000	1.00000	1.00000	1.00000
1.00	0.20777	0.20797	0.20090	0.64478	0.64967	0.65690
1.50946	0.26433	0.26513	0.25623	0.51427	0.52089	0.53059
2.00	0.30676	0.30885	0.29420	0.37141	0.37953	0.41177
2.58959	0.31743	0.32119	0.31174	0.26763	0.27622	0.28854
3.00	0.30574	0.31145	0.30753	0.18966	0.19792	0.22876
3.93977	0.24328	0.25227	0.24884	0.09379	0.10011	0.10865
4.64344	0.17599	0.18585	0.18530	0.04999	0.05436	0.05992
5.00	0.14000	0.14941	0.14970	0.03448	0.03787	0.04199
5.69294	0.09028	0.09794	0.09855	0.01825	0.02035	0.02268
6.17407	0.06287	0.06893	0.06936	0.01128	0.01269	0.01414
7.00	0.03250	0.04560	0.03616	0.00496	0.00751	0.00627
7.50	0.01904	0.02135	0.02121	0.00264	0.00304	0.00334
8.00	0.01001	0.01131	0.01112	0.00128	0.00149	0.00162
8.50	0.00682	0.00773	0.00756	0.00085	0.00098	0.00106
9.62308	0.00103	0.00118	0.00113	0.00012	0.00014	0.00015
10.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

In the above table the values of velocity are recoded to be 0.00682, 0.00773, 0.00756 for α= 0.0,0.2, 0.4 respectively at the point of x = 8.52867. Again, form table the values of temperatures 0.00012, 0.00014, 0.00015 for α= 0.0, 0.2, 0.4 respectively at the point of x = 9.62308. Here, it observed that the velocities and temperature almost similar at this point.

Conclusion

The aim of this study is to investigate numerically the effects of temperature dependent variable viscosity, viscous dissipation and heat absorption on natural convection flow of viscous incompressible fluid along a uniformly heated vertical wavy surface. The resulting nonlinear system of partial differential equations are mapped into the domain of a vertical flat plate and then solved numerically employing the implicit finite difference method.

The effects of different values of the viscosity variation parameter ε, the heat absorption parameter Q, the Eckert number Ec, the Prandtl number Pr and the amplitude of the wavy surface α on natural convection flow of viscous incompressible fluid along a vertical wavy surface has been studied numerically. From the present investigations may got the following conclusions:

- The rate of heat transfer and the velocity decreases significantly while the skin friction coefficient and the temperature increases for increasing values of the viscosity variation parameterε.
- For increasing values of the heat absorption parameter$Q$,the rate of heat transfer and the velocity increasesbutthe skin friction coefficient and the temperature decreases.
- The skin friction coefficient, the temperature distribution and the velocity increase for an increasing values of the Eckert numberEc, over the whole boundary layer but the significant decreases the rate of heat transfer.
- Increased value of the Prandtl number Pr, leads to decrease in the skin friction coefficient, the velocity distribution as well as temperature where the rate of heat transfer increases.
- The rate of heat transfer and skin friction coefficient decreases significantly while the velocity and temperature increasedwith the increasing values of the amplitude of the wavy surface α.
- Streamlines and Isotherms has been changed slightly with the increasing value of the amplitude of the wavy surface α but the thermal boundary layer becomes thickner when amplitude increases.

- Velocity boundary layer becomes thinner for different values of viscosity variation parameter ε, heat absorption parameter Q and the Prandtl number Pr, while the Eckert number Ec increasing the velocity boundary layer also grows up.
- Thermal boundary layer becomesthicker with the different values of viscosity variation parameterε and theEckert numberEc, but the heat absorption parameter$Q$ and the Prandtl number Pr has been thinner.

Major findings can be summarized as per the following conclusions:

Significant effects of heat absorption parameter Q and viscosity variation parameter ε on velocity and temperature profiles as well as on skin friction coefficient $Cf_x$and rate of heat transfer Nu_x have been found in this investigation but the effect of heat absorption parameter Q and viscosity variation parameter εon rate of heat transfer is more significant. An increase in the values of viscosity variation parameter ε leads to the velocity decrease and the temperature increase,the local skin friction coefficient Cf_x increase and the local rate of heat transfer Nu_x decreases at different position of η for Pr =1.0.

For increasing fluid temperature, the temperature difference between fluid and surface decreases and the corresponds the rate of heat transfer decreases. Besides the thermal state of the fluid increases, so the thermal boundary layer becomes thicker.

It is also shown that the variable viscosity, viscous dissipation and heat absorption leads to change the fluid flow and temperature.

Extension of this work

The present work can be extended in different ways. Some of these are:

- Constant viscosity considering to extend the present work.
- The thermal conductivity as a function of temperature can be considered to extend the present work.
- Including Joule heating with magneto-hydrodynamics (MHD) can be considered to extend the present work.
- Complex wavy surface can be considered as a combination of two sinusoidal functions.
- The problem can be extended considering the radiation effect.
- The problem can be extended considering the Micro and Nano fluids.
- The problem can be also extended considering the non-Newtonian fluids.
- Stress work and pressure work can be considered to extend the present work.
- Forced convection may be studied with the same geometry.
- Mixed convection may be studied with the same geometry.
- Forced and mixed convection may be applied with the same geometry.

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Appendix

Implicit Finite Difference Method (IFDM)

To apply the aforementioned method, equations (2.4.10) and (2.4.11) their boundary condition (2.4.12) are first converted into the following system of first order equations. For this purpose, we introduce new dependent variables $u(\xi,\eta)$, $v(\xi,\eta)$, $p(\xi,\eta)$ and $g(\xi,\eta)$ so that the transformed momentum and energy equations can be written as:

$$f' = u \tag{A.1}$$

$$u' = v \tag{A.2}$$

$$g' = p \tag{A.3}$$

$$P_1 v' + P_2 f v - P_3 u^2 + P_4 g + P_5 vp = \xi(u\frac{\partial u}{\partial \xi} - v\frac{\partial f}{\partial \xi}) \tag{A.4}$$

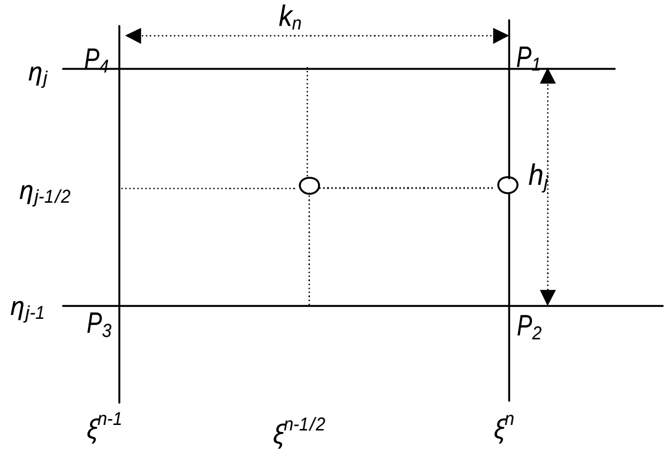
$$\frac{1}{\Pr} P_1 p' + P_2 f\ p + P_6 v^2 + P_7 g = \xi\ (u\ \frac{\partial g}{\partial \xi} - p\ \frac{\partial f}{\partial \xi}) \tag{A.5}$$

where x = ξ, θ = g and

$$P_1 = \left(1+\sigma_x^2\right)\left(1+\varepsilon\theta\right),\ P_2 = \frac{3}{4},\ \ P_3 = \frac{1}{2} + \frac{x\sigma_x\sigma_{xx}}{1+\sigma_x^2},\ \ P_4 = \frac{1}{1+\sigma_x^2},\ \ P_5 = \varepsilon\left(1+\sigma_x^2\right),$$
$$P_6 = Ecx\,,\ P_7 = x^{1/2}Q$$

and the boundary conditions (2.4.12) are

$$\begin{aligned} &f(\xi,0) = 0,\ u(\xi,0) = 0, g\left(\xi,0\right) = 1 \\ &u(\xi,\infty) = 0,\ g\ (\xi,\infty) = 0 \end{aligned} \tag{A.6}$$



FigureA-1 : Net rectangle of difference approximationsfor the Box scheme.

Now consider the net rectangle on the (ξ,η) plane shown in the figure A-1 and denote the net points by

$$\begin{aligned}&\xi^0 = 0,\ \ \xi^n = \xi^{n-1} + k_n,\ \ n = 1,2,\cdots\cdots\cdots\cdots, N\\&\eta_0 = 0,\ \ \eta_j = \eta_{j-1} + h_{j,}\ \ \ j = 1,2,\cdots\cdots\cdots\cdots, J\end{aligned} \tag{A.7}$$

Here n and j are just sequence of numbers on the (ξ,η) plane, k_n and h_j are the variable mesh widths. Approximate the quantities f,u,v and$p$ at the points (ξ^n,η_j) of the net by $f_j^n,\ u_j^n, v_j^n,\ p_j^n$ which call net function. It is also employed that the notation P_j^n for the quantities midway between net points shown in figure A-1 and for any net function as

$$\xi^{n-1/2} = \frac{1}{2}(\xi^n + \xi^{n-1}) \tag{A.8}$$

$$\eta_{j-1/2} = \frac{1}{2}(\eta_j + \eta_{j-1}) \tag{A.9}$$

$$g_j^{\ n-1/2} = \frac{1}{2}(g_j^n + g_j^{n-1}) \tag{A.10}$$

$$g_{j-1/2}^n = \frac{1}{2}(g_j^n + g_{j-1}^n) \tag{A.11}$$

The finite difference approximations according to box method to the three first order ordinary differential equations (A.1) – (A.3) are written for the midpoint $(\xi^n,\eta_{j-1/2})$ of the segment P_1P_2 shown in the figure (A.1) and the finite difference approximations to the two first order differential equations (A.4) and (A.5) are written for the midpoint $(\xi^{n-1/2},\eta_{j-1/2})$ of the rectangle $P_1P_2P_3P_4$. This procedure yields

$$\frac{f_j^n - f_{j-1}^n}{h_j} = u_{j-1/2}^n = \frac{u_{j-1}^n + u_j^n}{2} \tag{A.12}$$

$$\frac{u_j^n - u_{j-1}^n}{h_j} = v_{j-1/2}^n = \frac{v_{j-1}^n + v_j^n}{2} \tag{A.13}$$

$$\frac{g_j^n - g_{j-1}^n}{h_j} = p_{j-1/2}^n = \frac{p_{j-1}^n + p_j^n}{2} \tag{A.14}$$

$$\begin{aligned}&\frac{1}{2}(P_1T)_{j-1/2}^n\left(\frac{v_j^n - v_{j-1}^n}{h_j}\right) + \frac{1}{2}(P_1T)_{j-1/2}^{n-1}\left(\frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) + \left(P_2 fv\right)_{j-1/2}^{n-1/2} - (P_3u^2)_{j-1/2}^{n-1/2} \\ &+ (P_4g)_{j-1/2}^{n-1/2} - (P_5vp)_{j-1/2}^{n-1/2} = \xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^{n-1/2}\frac{u_{j-1/2}^n - u_{j-1/2}^{n-1}}{k_n} - v_{j-1/2}^{n-1/2}\,\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right)\end{aligned} \tag{A.15}$$

$$\begin{aligned}&\frac{1}{2\Pr}\left\{(P_1)_{j-1/2}^n\right\}\left(\frac{p_j^n - p_{j-1}^n}{h_j}\right) + \frac{1}{2\Pr}\left\{(P_1)_{j-1/2}^{n-1}\right\}\left(\frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right) + \left(P_2 fp\right)_{j-1/2}^{n-1/2} \\ &+ (P_6v^2)_{j-1/2}^{n-1/2} + (P_7g)_{j-1/2}^{n-1/2} = \xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^{n-1/2}\frac{g_{j-1/2}^n - g_{j-1/2}^{n-1}}{k_n} - p_{j-1/2}^{n-1/2}\,\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right)\end{aligned} \tag{A.16}$$

Now the equation (A.15) can be written as

$$\begin{aligned}&\Rightarrow \frac{1}{2}(P_1T)_{j-1/2}^n\left(\frac{v_j^n - v_{j-1}^n}{h_j}\right) + \frac{1}{2}(P_1T)_{j-1/2}^{n-1}\left(\frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) + \frac{1}{2}\left\{(P_2 fv)_{j-1/2}^n + (P_2 fv)_{j-1/2}^{n-1}\right\} \\ &- \frac{1}{2}\left\{(P_3u^2)_{j-1/2}^n + (P_3u^2)_{j-1/2}^{n-1}\right\} + \frac{1}{2}\left\{\left(P_4g_{j-1/2}^n\right) + \left(P_4g_{j-1/2}^{n-1}\right)\right\} + \frac{1}{2}\left\{\left(P_5vp_{j-1/2}^n\right) + \left(P_5\,vp_{j-1/2}^{n-1}\right)\right\} \\ &= \frac{1}{2k_n}\xi_{j-1/2}^{n-1/2}\left(u_{j-1/2}^n + u_{j-1/2}^{n-1}\right)\left(u_{j-1/2}^n - u_{j-1/2}^{n-1}\right) - \frac{1}{2k_n}\xi_{j-1/2}^{n-1/2}\left(v_{j-1/2}^n + v_{j-1/2}^{n-1}\right)\left(f_{j-1/2}^n - f_{j-1/2}^{n-1}\right)\end{aligned}$$

$$\begin{aligned}&\Rightarrow (P_1T)_{j-1/2}^n\left(\frac{v_j^n - v_{j-1}^n}{h_j}\right) + (P_1T)_{j-1/2}^{n-1}\left(\frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) + (P_2)_{j-1/2}^n\,(fv)_{j-1/2}^n + (P_2)_{j-1/2}^{n-1}\,(fv)_{j-1/2}^{n-1} \\ &- \left(P_3\right)_{j-1/2}^n\,(u^2)_{j-1/2}^n - \left(P_3\right)_{j-1/2}^{n-1}\,(u^2)_{j-1/2}^{n-1} + \left(P_4\right)_{j-1/2}^n g_{j-1/2}^n + \left(P_4\right)_{j-1/2}^{n-1} g_{j-1/2}^{n-1} + \left(P_5\right)_{j-1/2}^n (vp)_{j-1/2}^n \\ &+ \left(P_5\right)_{j-1/2}^{n-1}(vp)_{j-1/2}^{n-1} = \alpha_n\left\{(u^2)_{j-1/2}^n - (u^2)_{j-1/2}^{n-1} - (fv)_{j-1/2}^n + f_{j-1/2}^{n-1}v_{j-1/2}^n - f_{j-1/2}^n v_{j-1/2}^{n-1} + (fv)_{j-1/2}^{n-1}\right\}\end{aligned}$$

$$\Rightarrow (P_1T)^n_{j-1/2}\left(\frac{v^n_j - v^n_{j-1}}{h_j}\right) + \left\{(P_2)^n_{j-1/2} + \alpha_n\right\}(fv)^n_{j-1/2} - \left\{(P_3)^n_{j-1/2} + \alpha_n\right\}(u^2)^n_{j-1/2}$$
$$+ (P_4)^n_{j-1/2} g^n_{j-1/2} + (P_5)^n_{j-1/2}(vp)^n_{j-1/2} = \alpha_n\left[\left\{-(u^2)^{n-1}_{j-1/2} + v^n_{j-1/2} f^{n-1}_{j-1/2} - f^n_{j-1/2} v^{n-1}_{j-1/2} + (fv)^{n-1}_{j-1/2}\right\}\right]$$
$$-(P_2)^{n-1}_{j-1/2}(fv)^{n-1}_{j-1/2} + (P_3)^{n-1}_{j-1/2}(u^2)^{n-1}_{j-1/2} - (P_4)^{n-1}_{j-1/2} g^{n-1}_{j-1/2} + (P_5)^{n-1}_{j-1/2}(vp)^{n-1}_{j-1/2}$$
$$-(P_1T)^{n-1}_{j-1/2}\left(\frac{v^{n-1}_j - v^{n-1}_{j-1}}{h_j}\right)$$

$$R^{n-1}_{j-1/2} = \alpha_n\left\{(fv)^{n-1}_{j-1/2} - (u^2)^{n-1}_{j-1/2}\right\} - L^{n-1}_{j-1/2} \tag{A.17}$$

where $\alpha_n = \dfrac{1}{k_n}\xi^{n-1/2}_{j-1/2}$ (A.18)

$$\Rightarrow (P_1T)^n_{j-1/2} h_j^{-1}\left(v^n_j - v^n_{j-1}\right) + \left\{(P_2)^n_{j-1/2} + \alpha_n\right\}(fv)^n_{j-1/2}$$
$$-\left\{(P_3)^n_{j-1/2} + \alpha_n\right\}(u^2)^n_{j-1/2} + (P_4)^n_{j-1/2} g^n_{j-1/2} - (P_5)^n_{j-1/2}(u)^n_{j-1/2} \tag{A.19}$$
$$+\alpha_n\left(f^n_{j-1/2} v^{n-1}_{j-1/2} - v^n_{j-1/2} f^{n-1}_{j-1/2}\right) = R^{n-1}_{j-1/2}$$

$$\Rightarrow (P_1T)^{n-1}_{j-1/2} h_j^{-1}\left(v^{n-1}_j - v^{n-1}_{j-1}\right) + (P_2)^{n-1}_{j-1/2}(fv)^{n-1}_{j-1/2} - (P_3)^{n-1}_{j-1/2}(u^2)^n_{j-1/2}$$
$$+(P_4)^{n-1}_{j-1/2} g^n_{j-1/2} - (P_5)^{n-1}_{j-1/2}(u)^n_{j-1/2} = L^{n-1}_{j-1/2} \tag{A.20}$$

Again, from the equation (A.16) then

$$\frac{1}{\Pr}\left[\left\{(P_1)^n_{j-1/2}\right\}\left(\frac{p^n_j - p^n_{j-1}}{h_j}\right) + \left\{(P_1)^{n-1}_{j-1/2}\right\}\left(\frac{p^{n-1}_j - p^{n-1}_{j-1}}{h_j}\right)\right] + \left\{(P_2\,fp)^n_{j-1/2} + (P_2\,fp)^{n-1}_{j-1/2}\right\}$$
$$+\{(P_6v^2)^n_{j-1/2} + (P_6v^2)^{n-1}_{j-1/2}\} + \{(P_7g)^n_{j-1/2} + (P_7g)^{n-1}_{j-1/2}\}$$
$$= \alpha_n\left[\left(u^n_{j-1/2} + u^{n-1}_{j-1/2}\right)\left(g^n_{j-1/2} - g^{n-1}_{j-1/2}\right) - \left(p^n_{j-1/2} + p^{n-1}_{j-1/2}\right)\left(f^n_{j-1/2} - f^{n-1}_{j-1/2}\right)\right]$$

$$\Rightarrow \frac{1}{\Pr}\left\{(P_1)^n_{j-1/2}\right\}\left(\frac{p^n_j - p^n_{j-1}}{h_j}\right) + \frac{1}{\Pr}\left\{(P_1)^{n-1}_{j-1/2}\right\}\left(\frac{p^{n-1}_j - p^{n-1}_{j-1}}{h_j}\right) + (P_2)^n_{j-1/2}(fp)^n_{j-1/2}$$
$$+(P_2)^{n-1}_{j-1/2}(fp)^{n-1}_{j-1/2} + (P_6)^n_{j-1/2}(v^2)^n_{j-1/2} + (P_6)^{n-1}_{j-1/2}(v^2)^{n-1}_{j-1/2} + (P_7)^n_{j-1/2}(g)^n_{j-1/2}$$
$$+(P_7)^{n-1}_{j-1/2}(g)^{n-1}_{j-1/2} = \alpha_n\left[\begin{array}{l}(ug)^n_{j-1/2} - u^n_{j-1/2} g^{n-1}_{j-1/2} + g^n_{j-1/2} u^{n-1}_{j-1/2} - (ug)^{n-1}_{j-1/2} - (fp)^n_{j-1/2} \\ + p^n_{j-1/2} f^{n-1}_{j-1/2} - f^n_{j-1/2} p^{n-1}_{j-1/2} + (fp)^{n-1}_{j-1/2}\end{array}\right]$$

$$\Rightarrow \frac{1}{\Pr}\left\{(P_1)^n_{j-1/2}\right\}\left(\frac{p_j^n - p_{j-1}^n}{h_j}\right) + \left\{(P_2)^n_{j-1/2} + \alpha_n\right\}(fp)^n_{j-1/2} + (P_6)^n_{j-1/2}(v^2)^n_{j-1/2} + (P_7)^n_{j-1/2}(g)^n_{j-1/2}$$
$$-\alpha_n(ug)^n_{j-1/2} + \alpha_n\left[u^n_{j-1/2}\, g^{n-1}_{j-1/2} - g^n_{j-1/2}\, u^{n-1}_{j-1/2} - p^n_{j-1/2}\, f^{n-1}_{j-1/2} + f^n_{j-1/2}\, p^{n-1}_{j-1/2}\right]$$
$$= -\frac{1}{\Pr}\left\{(P_1)^{n-1}_{j-1/2}\right\}\left(\frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right) - (P_2)^{n-1}_{j-1/2}(fp)^{n-1}_{j-1/2} - (P_6)^{n-1}_{j-1/2}(v^2)^{n-1}_{j-1/2}$$
$$-(P_7)^{n-1}_{j-1/2}(g)^{n-1}_{j-1/2} + \alpha_n\left[(fp)^{n-1}_{j-1/2} - (ug)^{n-1}_{j-1/2}\right]$$

where $T^{n-1}_{j-1/2} = -M^{n-1}_{j-1/2} + \alpha_n\left\{(fp)^{n-1}_{j-1/2} - (ug)^{n-1}_{j-1/2}\right\}$ (A.21)

where, $\alpha_n = \frac{1}{k_n}\xi^{n-1/2}_{j-1/2}$ (A.22)

$$M^{n-1}_{j-1/2} = \frac{h_j^{-1}}{\Pr}(P_1)^{n-1}_{j-1/2}\left\{p_j^{n-1} - p_{j-1}^{n-1}\right\} + (P_2)^{n-1}_{j-1/2}(fp)^{n-1}_{j-1/2}$$
$$+(P_6)^{n-1}_{j-1/2}(v^2)^{n-1}_{j-1/2} + (P_7)^{n-1}_{j-1/2}(g)^{n-1}_{j-1/2} \quad \text{(A.23)}$$

$$\Rightarrow T^{n-1}_{j-1/2} = \frac{1}{\Pr}(P_1)^n_{j-1/2} h_j^{-1}\left(p_j^n - p_{j-1}^n\right) + \left\{(P_2)^n_{j-1/2} + \alpha_n\right\}(fp)^n_{j-1/2}$$
$$+(P_6)^n_{j-1/2}(v^2)^n_{j-1/2} + (P_7)^n_{j-1/2}(g)^n_{j-1/2} - \alpha_n(ug)^n_{j-1/2} \quad \text{(A.24)}$$
$$+\alpha_n\left(u^n_{j-1/2}\, g^{n-1}_{j-1/2} - g^n_{j-1/2}\, u^{n-1}_{j-1/2}\right) - \alpha_n\left(p^n_{j-1/2}\, f^{n-1}_{j-1/2} - f^n_{j-1/2}\, p^{n-1}_{j-1/2}\right)$$

The boundary condition becomes

$$f_0^n = 0, \quad u_0^n = 0, \; g_0^n = 1$$
$$u_j^n = 0, \quad g_j^n = 0 \quad \text{(A.25)}$$

It is assumed that $f_j^{n-1}, u_j^{n-1}, v_j^{n-1}, g_j^{n-1}, p_j^{n-1}$, for $0 \le j \le J$ are known. The equation (A.9) to (A.17) and (A.18) to (A.19) from a system of *(5 J + 5)* non linear equations for the solutions of the *(5J + 5)* unknowns $(f_j^n, u_j^n, v_j^n, g_j^n, p_j^n)$, $j = 0,1,2,3,.....J$. These non-linear systems of algebraic equations are to be linearized by Newton's Quassy linearization method. Theiterates $(f_j^i, u_j^i, v_j^i, g_j^i, p_j^i)$, $i = 0,1,2,3,.....N$ are defined with initial values equal those at the previous x- station (which is usually the initial available). For the higher iterates the following forms can be written

$f_j^{(i+1)} = f_j^i + \delta f_j^i$ (A.26)

$u_j^{(i+1)} = u_j^i + \delta u_j^i$ (A.27)

$v_j^{(i+1)} = v_j^i + \delta v_j^i$ (A.28)

$$g_j^{(i+1)} = g_j^i + \delta g_j^i \tag{A.29}$$

$$p_j^{(i+1)} = p_j^i + \delta p_j^i \tag{A.30}$$

Now by substituting the right hand sides of the above equations in place of f_j^n, u_j^n, v_j^n and g_j^n in equations (A.13) –(A.15) and in equations (A.20) and (A.22) dropping the terms that are quadratic in $\delta f_j^n, \delta u_j^n, \delta v_j^n$ and δp_j^n, then take the following linear system of algebraic form

$$\frac{f_j^i - f_{j-1}^i}{h_j} + \frac{\delta f_j^i - \delta f_{j-1}^i}{h_j} = u_{j-1/2}^i + \delta u_{j-1/2}^i = \frac{u_j^i + \delta u_j^i + u_{j-1}^i + \delta u_{j-1}^i}{2}$$

$$f_j^i + \delta f_j^i - f_{j-1}^i - \delta f_{j-1}^i = \frac{h_j (u_j^i + \delta u_j^i + u_{j-1}^i + \delta u_{j-1}^i)}{2}$$

$$\delta f_j^i - \delta f_{j-1}^i - \frac{h_j (\delta u_j^i + \delta u_{j-1}^i)}{2} = (r_1)_j$$

$$\delta u_j^i - \delta u_{j-1}^i - \frac{h_j (\delta v_j^i + \delta v_{j-1}^i)}{2} = (r_4)_j$$

$$\delta g_j^i - \delta g_{j-1}^i - \frac{h_j (\delta p_j^i + \delta p_{j-1}^i)}{2} = (r_5)_j$$

Where, $(r_1)_j = f_{j-1}^i - f_j^i + h_j u_{j-1/2}^i$ (A.31)

$$(r_4)_j = u_{j-1}^i - u_j^i + h_j v_{j-1/2}^i \tag{A.32}$$

$$(r_5)_j = g_{j-1}^i - g_j^i + h_j p_{j-1/2}^i \tag{A.33}$$

Equation (A.20) becomes

$$(P_1)_{j-1/2}^n h_j^{-1}\left(v_j^i + \delta v_j^i - v_{j-1}^i - \delta v_{j-1}^i\right) + \left\{(P_2)_{j-1/2}^n + \alpha_n\right\}\{(fv)_{j-1/2}^i + \delta(fv)_{j-1/2}^i\}$$
$$-\left\{(P_3)_{j-1/2}^n + \alpha_n\right\}\{(u^2)_{j-1/2}^i + \delta(u^2)_{j-1/2}^i\} + (P_4)_{j-1/2}^n\{g_{j-1/2}^i + \delta g_{j-1/2}^i\} - (P_5)_{j-1/2}^n\{u_{j-1/2}^i + \delta u_{j-1/2}^i\}$$
$$+\alpha_n\{f_{j-1/2} + \delta f_{j-1/2}\}v_{j-1/2}^{n-1} - \alpha_n\{v_{j-1/2} + \delta v_{j-1/2}^i\}f_{j-1/2}^{n-1}$$
$$= R_{j-1/2}^{n-1}$$

$$\Rightarrow (P_1)^n_{j-1/2} h_j^{-1}\left(v_j^i + \delta v_j^i - v_{j-1}^i - \delta v_{j-1}^i\right) + \left\{(P_2)^n_{j-1/2} + \alpha_n\right\}\{(fv)^i_{j-1/2} + \frac{1}{2}(f_j^i \delta v_j^i + v_j^i \delta f_j^i + f_{j-1}^i \delta v_{j-1}^i$$
$$+ v_{j-1}^i \delta f_{j-1}^i\} - \left\{(P_3)^n_{j-1/2} + \alpha_n\right\}\{(u^2)^i_{j-1/2} + u_j^i \delta u_j^i + u_{j-1}^i \delta u_{j-1}^i\} + (P_4)^n_{j-1/2}\{g^i_{j-1/2} + \frac{1}{2}(\delta g_j^i + \delta g_{j-1}^i)\}$$
$$-(P_5)^n_{j-1/2}\{u^i_{j-1/2} + \frac{1}{2}(\delta u_j^i + \delta u_{j-1}^i)\} + \alpha_n\{f^i_{j-1/2} + \frac{1}{2}(\delta f_j^i + \delta f_{j-1}^i)\} v^{n-1}_{j-1/2} - \alpha_n\{v^i_{j-1/2} +$$
$$\frac{1}{2}(\delta v_j^i + \delta v_{j-1}^i)\} f^{n-1}_{j-1/2} = R^{n-1}_{j-1/2}$$

(A.34)

$$\Rightarrow (s_1)_j \delta v_j^i + (s_2)_j \delta v_{j-1}^i + (s_3)_j \delta f_j^i + (s_4)_j \delta f_{j-1}^i + (s_5)_j \delta u_j^i + (s_6)_j \delta u_{j-1}^i + (s_7)_j \delta g_j^i$$
$$+ (s_8)_j \delta g_{j-1}^i (s_9) \delta p_j^i + (s_{10}) \delta p_{j-1}^i = R^{n-1}_{j-1/2}$$

(A.35)

$$(r_2)_j = R^{n-1}_{j-1/2} - (P_1)^n_{j-1/2} h_j^{-1}\left(v_j^i - v_{j-1}^i\right) - \left\{(P_2)^n_{j-1/2} + \alpha_n\right\}(fv)^i_{j-1/2} + \left\{(P_3)^n_{j-1/2} + \alpha_n\right\}(u^2)^i_{j-1/2}$$
$$-(P_4)^n_{j-1/2} g^i_{j-1/2} + (P_5)^n_{j-1/2} u^i_{j-1/2} - \alpha_n (f^i_{j-1/2} v^{n-1}_{j-1/2} - v^i_{j-1/2} f^{n-1}_{j-1/2})$$

Thus, the coefficients of momentum equation are

$$(s_1)_j = h_j^{-1}(P_1)^n_{j-1/2} + \frac{(P_2)^n_{j-1/2} + \alpha_n}{2} f_j^i - \frac{\alpha_n}{2} f^{n-1}_{j-1/2} \tag{A.36}$$

$$(s_2)_j = -h_j^{-1}(P_1)^n_{j-1/2} + \frac{(P_2)^n_{j-1/2} + \alpha_n}{2} f_{j-1}^i - \frac{\alpha_n}{2} f^{n-1}_{j-1/2} \tag{A.37}$$

$$(s_3)_j = \frac{(P_2)^n_{j-1/2} + \alpha_n}{2} v_j^i + \frac{\alpha_n}{2} v^{n-1}_{j-1/2} \tag{A.38}$$

$$(s_4)_j = \frac{(P_2)^n_{j-1/2} + \alpha_n}{2} v_{j-1}^i + \frac{\alpha_n}{2} v^{n-1}_{j-1/2} \tag{A.39}$$

$$(s_5)_j = -\{(P_3)^n_{j-1/2} + \alpha_n\} u_j^i - \frac{(P_5)^n_{j-1/2}}{2} \tag{A.40}$$

$$(s_6)_j = -\{(P_3)^n_{j-1/2} + \alpha_n\} u_{j-1}^i - \frac{(P_5)^n_{j-1/2}}{2} \tag{A.41}$$

$$(s_7)_j = \frac{(P_4)^n_{j-1/2}}{2} \tag{A.42}$$

$$(s_8)_j = \frac{(P_4)^n_{j-1/2}}{2} \tag{A.43}$$

$$(s_9)_j = 0 \tag{A.44}$$

$$(s_{10})_j = 0 \tag{A.45}$$

Similarly, by using the equations (A.51) to (A.55), then the equation (A.49) can be written as

$$\frac{1}{\Pr}(P_1)^n_{j-1/2}h_j^{-1}\left(p_j^i+\delta p_j^i-p_{j-1}^i-\delta p_{j-1}^i\right)+\left\{(P_2)^n_{j-1/2}+\alpha_n\right\}\{(fp)^i_{j-1/2}+\delta(fp)^i_{j-1/2}\}$$
$$+(P_6)^n_{j-1/2}\{(v^2)^i_{j-1/2}+\delta(v^2)^i_{j-1/2}\}+(P_7)^n_{j-1/2}\{(g)^i_{j-1/2}+\delta(g)^i_{j-1/2}\}-\alpha_n\{(ug)^i_{j-1/2}+\delta(ug)^i_{j-1/2}\}$$
$$+\alpha_n\{(u^i_{j-1/2}+\delta u^i_{j-1/2})g^{n-1}_{j-1/2}-u^{n-1}_{j-1/2}(g^i_{j-1/2}+\delta g^i_{j-1/2})\}-\alpha_n\{(p^i_{j-1/2}+\delta p^i_{j-1/2})f^{n-1}_{j-1/2}-p^{n-1}_{j-1/2}(f^i_{j-1/2}$$
$$+\delta f^i_{j-1/2})\}=T^{n-1}_{j-1/2}$$

$$(t_1)_j\delta p_j^i+(t_2)_j\delta p_{j-1}^i+(t_3)_j\delta f_j^i+(t_4)_j\delta f_{j-1}^i+(t_5)_j\delta u_j^i+(t_6)_j\delta u_{j-1}^i+(t_7)_j\delta g_j^i$$
$$+(t_8)_j\delta g_{j-1}^i+(t_9)_j\delta v_j^i+(t_{10})\delta v_{j-1}^i=(r_3)_j \tag{A.46}$$

$$\Rightarrow (r_3)_j=T^{n-1}_{j-1/2}-\frac{1}{\Pr}(P_1)^n_{j-1/2}h_j^{-1}\left(p_j^i-p_{j-1}^i\right)-\left\{(P_2)^n_{j-1/2}+\alpha_n\right\}(fp)^i_{j-1/2}$$
$$-(P_6)^n_{j-1/2}(v^2)^i_{j-1/2}-(P_7)^n_{j-1/2}(g)^i_{j-1/2}+\alpha_n(ug)^i_{j-1/2} \tag{A.47}$$
$$+\alpha_n\left(g^i_{j-1/2}u^{n-1}_{j-1/2}-g^{n-1}_{j-1/2}u^i_{j-1/2}\right)+\alpha_n\left(p^i_{j-1/2}f^{n-1}_{j-1/2}-p^{n-1}_{j-1/2}f^i_{j-1/2}\right)$$

The coefficients of energy equation are

$$(t_1)_j=\frac{1}{\Pr}h_j^{-1}(P_1)^n_{j-1/2}+\frac{(P_2)^n_{j-1/2}+\alpha_n}{2}f_j^i-\frac{\alpha_n}{2}f^{n-1}_{j-1/2} \tag{A.48}$$

$$(t_2)_j=-\frac{1}{\Pr}h_j^{-1}(P_1)^n_{j-1/2}+\frac{(P_2)^n_{j-1/2}+\alpha_n}{2}f_{j-1}^i-\frac{\alpha_n}{2}f^{n-1}_{j-1/2} \tag{A.49}$$

$$(t_3)_j=\frac{(P_2)^n_{j-1/2}+\alpha_n}{2}p_j^i+\frac{\alpha_n}{2}p^{n-1}_{j-1/2} \tag{A.50}$$

$$(t_4)_j=\frac{(P_2)^n_{j-1/2}+\alpha_n}{2}p_{j-1}^i+\frac{\alpha_n}{2}p^{n-1}_{j-1/2} \tag{A.51}$$

$$(t_5)_j=-\frac{\alpha_n}{2}g_j^i+\frac{\alpha_n}{2}g^{n-1}_{j-1/2} \tag{A.52}$$

$$(t_6)_j=-\frac{\alpha_n}{2}g_{j-1}^i+\frac{\alpha_n}{2}g^{n-1}_{j-1/2} \tag{A.53}$$

$$(t_7)_j=-\frac{\alpha_n}{2}u_j^i-\frac{\alpha_n}{2}u^{n-1}_{j-1/2}-\frac{1}{2}(P_6)^n_{j-1/2} \tag{A.54}$$

$$(t_8)_j=-\frac{\alpha_n}{2}u_{j-1}^i-\frac{\alpha_n}{2}u^{n-1}_{j-1/2}-\frac{1}{2}(P_6)^n_{j-1/2} \tag{A.55}$$

$$(t_9)_j = (P_5)^n_{j-1/2} v^i_j \qquad \text{(A.56)}$$

$$(t_{10})_j = (P_5)^n_{j-1/2} v^i_{j-1} \qquad \text{(A.57)}$$

The boundary conditions (A.25) becomes

$$\delta f_0^n = 0, \quad \delta u_0^n = 0, \delta g_0^n = 1$$
$$\delta u_j^n = 0, \quad \delta g_j^n = 0$$

Which just express the requirement for the boundary conditions to remain during the iteration process.