NUMERICAL SOLUTION OF HIGHER ORDER BOUNDARY VALUE PROBLEM BY EXP-FUNCTION METHOD

By

Md. Oliur Rahman

Student No. 1017092511F

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The thesis entitled **"NUMERICAL SOLUTION OF HIGHER ORDER BOUNDARY VALUE PROBLEM BY EXP-FUNCTION METHOD",** submitted by **Md. Oliur Rahman**, Roll no: 1017092511F, Registration No. 1017092511, Session: October 2017 has been accepted as satisfactory in partial fulfillment of the requirement for the degree of Master of Science in Mathematics on 3 March, 2021.

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Department of Applied Mathematics
University of Dhaka, Dhaka-1000

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I, Md. Oliur Rahman, declare that the work contained in this thesis entitled "NUMERICAL SOLUTION OF HIGHER ORDER BOUNDARY VALUE PROBLEM BY EXP-FUNCTION METHOD" was done by me, under the supervision of Dr. Md. Mustafizur Rahman, Professor, Bangladesh University of Engineering and Technology (BUET), Dhaka-1000 for the award of the degree of Master of Science and this work has not been submitted elsewhere for a degree.

Md. Oliur Rahman

Name:

03/03/2021

Date

DEDICATION

This work is dedicated

To

My Parents

Abstract

Higher order boundary value problem which are known to arise in the study of astrophysics, hydrodynamic and hydromagnetic stability, fluid dynamics, astronomy, beam and long wave theory, engineering and applied physics. In this thesis under the title "Numerical Solution of Higher Order Boundary Value Problem by Exp-Function Method", two problems have been studied.

Firstly, we discuss the propagation of nonlinear kinky periodic wave and breather wave for the dominant nonlinear pseudo-parabolic physical models: the one-dimensional Oskolkov equation is explored. By executing Exp-Function method, compilation of disguise adaptation of exact nonlinear wave solutions with some noteworthy parameters for the Oskolkov equations is accessed. The presentation of this technique is reliable, direct, and easy to execute contrasted with other existing strategies.

Secondly, there are many methods to solve Fisher's equation, but each method leads to single special solution. In this thesis, a new method, namely the Exp-Function method, is employed to solve the Fisher's equation. The obtained results are shown graphically. The generalized solution with some free parameters might imply some fascinating meanings hidden in the Fisher's equation.

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(Md. Oliur Rahman)

Date:

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Nomenclature

D	diffusion coefficient
m	rate of chemical reaction
k	constant
u	unknown function
a_0	parameter
b_0	parameter
t	time

Greek symbols

α	arbitrary constant
β	arbitrary constant
ω	wave speed
η	wave variable

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Chapter 1

Introduction

This thesis is concern with numerical solutions of higher order boundary value problem by Exp-function method. In thesis Oskolkov equation and Fisher's equation are solved by Exp-function method.

The mathematical model of physical phenomena usually results in non-linear equations, which may be algebra, ordinary differential, partial differential, integral or combination of these. The non-linear equations may contain one or several independent variables. The solutions of these non-linear systems are dominated by their singularities (if exist). A value of independent variable (or variables) for which the function is undefined is known as a singularity of the function. Singularity plays an important role in many reflect some changes in the nature of the flow and their study is of great practical interest. Sometimes it is very difficult to find out the exact solution of physical problems. Particularly in statistical mechanics, there a large number of problems for which the first few terms of the power series may be obtained exactly while the exact solution is unobtainable. On the other corresponding function is not known, then it becomes difficult to reproduce the function from the given power series. However, one can study their singularities by some power series approximant methods. In order to study these problems many powerful techniques have been used to find the power series coefficients. At the same time a variety of methods have been introduced for getting the required information about the singularities by using a finite number of series coefficients.

1.2 Motivation

There are many methods to solve non-linear higher order differential equations, but each method can only lead to special solution. A new method, namely the Exp-function method, is employed to solve non-linear higher order differential equations. The obtained result includes all solutions in open literature as special cases, and the generalized solution with some free parameters might imply some fascinating meanings latent in the non-linear higher order differential equations.

In the Exp-function method, we can solve non-linear higher order differential equations without their boundary conditions. So in this thesis, I have motivated to solve the Fisher's and Oskolkov equations in the Exp-function method with their free boundary conditions.

1.3 Literature Review

Higher order boundary value problems occur in the study of fluid dynamics, astrophysics, hydrodynamic, hydromagnatic stability, astronomy, beam and long wave theory, induction motors, engineering, and applied physics. The boundary value problems of higher order have been examined due to their mathematical importance and applications in diversified applied science. Boutayeb and Twizell [2] used finite difference method to solve eighth order boundary value problem. Noor and Mohyud Din [25] solved the same order boundary value problem by variational iteration decomposition method. Siddiqi *et al*. [30] used the variational iteration technique for the solution of the eleventh order boundary value problem. Wazwaz [9] used the modified decomposition method for solving linear and nonlinear boundary value problems of tenth-order and twelfth-order. Adeosum *et al.* [32] presented the variational iteration method (VIM) to find the approximate solutions of linear and nonlinear thirteenth order boundary value problems. In the last few decennary, the study of nonlinear evolution equations established much concentration in diverse fields of nonlinear science, such as fluid mechanics, nuclear physics, solid-state physics, plasma physics, chemical physics, optical fiber and geochemistry. Many scholars planned through NEEs to construct traveling wave solution by implement several methods. The procedures that are well established in recent literature such as extended Kudryashov method [14], New extended (G'/G) expansion method [17] trial solution method [3]. Roshid [26] showed exact and explicit traveling wave solutions to two nonlinear evolution equations which describe incompressible viscoelastic Kelvin vogit-fluid. Turgut*et al*. [7] Propagation of nonlinear shock waves for the generalized Oskolkov equation and its dynamic motions in the presence of an external periodic perturbation by implement in this method. Sviridyuk [15] showed on the stability of solutions of the Oskolkov equations on a graph. In 2006, He and Wu [20] have introduced the Exp-function method to obtain the solitary solutions and the periodic solutions of nonlinear wave equations. Wu [34] discussed Exp-function method and its application to nonlinear equations. Chun [11] obtained new solitary wave solutions to nonlinear evolution equations by the Exp-function method. Ebaid [4] showed the possibility of solving Burgers equation by the exp-function method. He also obtained the exact solutions for the KdV equation and the extended KdV equation and exact solitary wave solutions for some nonlinear evolution equations via this method. The method was also used to solve different nonlinear wave equations, generalized Klein–Gordon equation [5], evolution equations with nonlinear terms of any orders [16], (2+1)-dimensional Konopelchenko-Dubrovsky equations [31] the Schwarzian Korteweg-de Vries equation [18], Broer-Kaup-Kupershmidt equations [8] nonlinear evolution equations with variable coefficients [1] fifth order KdV equation and modified Burgers equation [19]. Feng [24] presented the higher-order soliton, breather, and rogue wave solutions of the coupled nonlinear Schrödinger equation by applying the DT method. Al-Khaled [23] presented a sin collocation method to study numerical solutions of nonlinear reaction diffusion Fisher's equation. Rajni and Mittal [29] describes numerical study of reaction diffusion Fisher's equation by fourth order cubic B-spline collocation method. Eisa [13] represented numerical solution of Fisher's equation using finite difference method. Numerous complex phenomena in real life are modeled by nonlinear evolution equations. Pseudo parabolic model is one kind of partial differential equations in which the time derivative emerged in highest order derivative and they have been exploiting for different areas of mathematics and physics such as instance, for fluid flow in fissured rock, consolidation of clay, shear in second-order fluids, thermodynamics and propagation of long waves of small amplitude. Nowadays, much attention has been paid to investigate NEEs such as Pseudo parabolic equation [27]. It is important to note that a completely integerable Pseudo parabolic model provides innovative and explicit different type exact traveling wave solution. Shuimeng [36] obtain N-soliton solutions of the KP equation used Exp-function method. Wu [33] used Exp-function method get solitary solutions, periodic solutions and compacton-like solutions. Chang [22] applied the Exp-function method to solve a system of nonlinear PDEs, and some new exact solitary solutions are obtained with some free parameters. Yildrim [10] solved nonlinear reaction-diffusion equation arising mathematical biology by the application of Exp-function method. Ji-Huan [21] used the Exp-function method to show the generalized solitary solution and compacton-like solution of the Jaulent-Miodek equations. Chun [12] find soliton and periodic solutions for the fifth- order Korteweg-de Vries (KdV) equation with the Exp-function method. Qiuand Sloan [35] solved the Fisher's equation by moving mesh method and showed that moving mesh methods produce much better results if the monitor function is chosen to suit the proper-ties of the differential equation and of the numerical solution. In 2005, Anguelov *et al*. [28] solved the problem $u_t = u_{xx} + u(1-u)$ by using a periodic initial condition with θ-nonstandard method. They concluded that their method is elementarily stable in the limit case of space-independent variable, stable with respect to the boundedness and positivity property and finally stable with respect to the conservation of energy in the stationary case.

1.4 Objectives of The Thesis

The primary objective is to numerically investigate the performance of exp-function method over higher order boundary value problem. The specific aims are:

- To use the exp-function method to solve different higher order boundary value problems.
- To analyze the alternative ways of solving higher order boundary value problem by exp-function method.
- To compare the results found from this study with other related published works to validate the computational procedure.

1.5 Outline of The Thesis

In chapter one, the introductory discussions, motivations, literature review and objectives are discussed. Some basic definitions and some basic formula are given in the chapter two. The idea and method of the Exp-function method and some theorem are also given in this chapter. In chapter three, the analytic results of the Oskolkov and Fisher's equations in the Exp-function method without their boundary conditions are discussed. The numerical result discussion of the Oskolkov and Fisher's equations in the Exp-function method are also discussed in this chapter. And finally, in the chapter four, the conclusion of the thesis and my future work are given.

Chapter 2

Mathematical Preliminaries

2.1 Introduction

Real-life problems are mainly modeled by partial differential equations (PDEs) with applications to engineering, physics, chemistry, ecology, biology, and other related fields of science.

Partial Differential Equation can be of different forms:

(i) linear or nonlinear,

(ii) homogeneous or non-homogeneous,

(iii) elliptic, hyperbolic, or parabolic PDEs

Have some specifications that give the information how smooth the solution is, how rapid information propagates, and what is the impact of initial and boundary conditions (which help to find if a particular approach is suitable to the problem being portrayed by the PDEs).

2.2 Differential Equation

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

For example of differential equations we list the following:

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0 \tag{2.1}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = sint \tag{2.2}$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial z} = v \tag{2.3}$$

From the brief list of differential equations in first example it is clear that the various variables and derivatives involved in a differential equation can occur in a verity of ways.

2.2.1 Ordinary Differential Equation

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

Example:

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0 \tag{2.4}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = sint \quad (2.5$$

In Equation (2.4) the variable x is the single independent variable where as y is dependent variable. In Equation (2.5) the independent variable is t, where as x is dependent.

2.2.2 Partial Differential Equation

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation.

Example:

$$\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = v \tag{2.6}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{2.7}$$

In Equation (2.6) the variables s and t are independent variables and v is a dependent variable. In equation (2.7) there are three independent variables x, y and z, in the equation where as u is dependent.

More about Partial Differential Equation

In Mathematics, a partial differential equation is one of the types of differential equations, in which the equation contains unknown multi variables with their partial derivatives. It is a special case of an ordinary differential equation.

2.2.3 Types of Partial Differential Equation

The different types of partial differential equations are:

(i) First-order Partial Differential Equation

(ii) Linear Partial Differential Equation

(iii) Quasi-Linear Partial Differential Equation

(iv) Homogeneous Partial Differential Equation

Let us discuss these types of PDEs here.

(i) First-Order Partial Differential Equation

In Math's, when we speak about the first-order partial differential equation, then the equation has only the first derivative of the unknown function having 'm' variables. It is expressed in the form of;

$$F(x_1, \cdots\cdots\cdots x_m, u_1, u_{x1}, \cdots\cdots\cdots\cdots u_{xm}) = 0$$

(ii) Linear Partial Differential Equation

If the dependent variable and all its partial derivatives occur linearly in any PDE then such an equation is called linear PDE otherwise a nonlinear PDE.

Example:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

$$(x^2 + y^2)\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x \partial y} - 3u = 0$$

(iii) Quasi-Linear Partial Differential Equation

A PDE is said to be quasi-linear if all the terms with the highest order derivatives of dependent variables occur linearly, that is the coefficient of those terms are functions of only lower-order derivatives of the dependent variables. However, terms with lower-order derivatives can occur in any manner.

Example:

$$ux\frac{\partial^2 u}{\partial x^2} + u^2xy\frac{\partial^2 u}{\partial x \partial y} + uy\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u^3 = 0$$

(iv) Homogeneous Partial Differential Equation

If all the terms of a PDE contain the dependent variable or its partial derivatives then such a PDE is called non-homogeneous partial differential equation or homogeneous otherwise. In the above four examples, Example

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial t} - T \frac{\partial^2 u}{\partial x^2} = 0$$

2.2.4 Initial Value Problem

An Initial Value Problem is a differential equation along with an appropriate of initial conditions. The following is an initial value problem

$$4x^2y'' + 12xy' + 3y = 0; y(4) = 8, y'(4) = -3/64 \tag{2.8}$$

2.2.5 Boundary Value Problem

If the given conditions are given at more than one point and the differential equation is of order two or greater, it is called a boundary value problem. A Boundary Value Problem can have none, one, or many solutions.

Example

$$\frac{d^2y}{dx^2} - y = 0; y(0) = 1, y(\pi/2) = 2. \tag{2.9}$$

2.3 Non Linear Equation

A nonlinear system of equations is a set of equations where one or more terms have a variable of degree two or higher and/or there is a product of variables in one of the equations. Most real-life physical systems are non-linear systems, such as the weather.

2.3.1 Higher Order Non Linear Differential Equation

We shall concerned with sixth-order nonlinear differential equations of the form

$$\frac{d^6x}{dt^6} = F\left(x, \frac{dx}{dt}\right)$$

As a specific example of such an equation we list the important Van der Pol equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$$

Where, μ is a positive constant.

2.3.2 Oskolkov Equation

It's some kind of non-linear partial differential equation used to model the propagation of shock waves in certain materials.

Example:

The (1+1) dimensional Oskolkov equation is in the following from

$$u_t - \beta u_{xxt} - \alpha u_{xx} + u u_x = 0$$

2.3.3 Fisher's Equation

Fisher's equation, which describes a balance between linear diffusion and nonlinear reaction or multiplication.

Example:

We consider the reaction diffusion equation is

$$u_t = u_{xx} + u(1-u)$$

2.4 The Exp-Function Method

In this section we state some basic theorem describing general properties of exp-function method whose material of this section can be found in [6]

The Exp-function method has been widely used to solve different kinds of nonlinear partial differential equations. These nonlinear partial differential equations are transformed first into nonlinear ordinary differential equations and then the ansatz of the Exp-function method is $u(\eta) = \frac{\sum_{n=-c}^{d} a_n exp(n\eta)}{\sum_{m=-p}^{q} b_m exp(m\eta)}$ applied to obtain the solution. However, a part of the solution using this method is to construct the relations between c, p, d and q by balancing the highest order linear term with the highest order nonlinear term. It is proved in this thesis that $\mathrm{c} = \mathrm{d}$ and $\mathrm{p} = \mathrm{q}$ are the only relations that can be obtained by applying this method to any nonlinear ordinary differential equation. Therefore, the additional calculations of balancing the highest order linear term with the highest order nonlinear term are not longer required in future. Hence, the method becomes more straightforward.

2.4.1 Method

Consider the given (1+1)-dimensional nonlinear wave equation

$$N(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots) = 0 \tag{2.10}$$

Or, the (2+1)-dimensional nonlinear PDE

$$N\big(u, u_t, u_x, u_{y,} u_{xx}, u_{tt}, u_{yy}, u_{xy}, \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\big) = 0 \tag{2.11}$$

Seeking for the wave solution for equation (2.10) requires the transformation:

$$u = u(\eta), \qquad \eta = \mu(x - \omega t). \tag{2.12}$$

For equation (2.11), the wave solution requires the transformation:

$$u = u(\xi), \qquad \eta = x + \alpha y + \beta t. \tag{2.13}$$

Consequently, equation (2.10) is reduced to the ODE:

$$N(u, -\mu\omega u', \mu u', \mu^2 u'', \mu^2\omega^2 u'', -\mu^2 u'', \cdots\cdots\cdots) = 0,\ u = u(\eta) \tag{2.14}$$

And, equation (2.11) is reduced to

$$N(u, \beta u', u', \alpha u', u'', \beta^2 u'', \alpha^2 u'', \alpha u'' \cdots\cdots\cdots) = 0,\ u = u(\xi). \tag{2.15}$$

The Exp-function method anstaz is expressed in the form:

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n exp(n\eta)}{\sum_{m=-p}^{q} b_m exp(m\eta)} = \frac{a_{-c} exp(-c\eta) + \cdots\cdots\cdots\cdots + a_p exp(p\eta)}{b_{-d} exp(-d\eta) + \cdots\cdots\cdots\cdots + b_q exp(q\eta)}, \tag{2.16}$$

Which can be applied to solve equation (2.14), a similar anstaz can be also applied to solve equation (2.15).

2.4.2 General formula

On using the ansatz given by equation (2.4.1.4) then the following derivatives are resulted

$$u^{(1)}(\eta)=\frac{\tau_1\exp[-(c+d)\eta]+\cdots\cdots\sigma_1\exp[(p+q)\eta]}{\varrho_1\exp[(-2d)\eta]+\cdots\cdots\Gamma_1\exp[(2q)\eta]},$$
$$u^{(2)}(\eta)=\frac{\tau_2\exp[-(c+3d)\eta]+\cdots\cdots\sigma_2\exp[(p+3q)\eta]}{\varrho_2\exp[(-4d)\eta]+\cdots\cdots\Gamma_2\exp[(4q)\eta]},$$
$$u^{(3)}(\eta)=\frac{\tau_3\exp[-(c+7d)\eta]+\cdots\cdots\sigma_3\exp[(p+7q)\eta]}{\varrho_3\exp[(-8d)\eta]+\cdots\cdots\Gamma_3\exp[(8q)\eta]},$$
$$u^{(4)}(\eta)=\frac{\tau_4\exp[-(c+15d)\eta]+\cdots\cdots\sigma_4\exp[(p+15q)\eta]}{\varrho_1\exp[(-16d)\eta]+\cdots\cdots\Gamma_4\exp[(16q)\eta]}, \quad (2.17)$$

Where,$\tau_i, \sigma_i, \varrho_i$ and Γ_i are all constants. Therefore, the following general derivative formula is obtained

$$u^{(r)}(\eta)=\frac{\tau_r\exp[-(c+(2^r-1)d)\eta]+\cdots\cdots+\sigma_r\exp[(p(2^r-1)q)\eta]}{\varrho_r\exp[(-2^r d)\eta]+\cdots\cdots+\Gamma_r\exp[(2^r q)\eta]}. \quad (2.18)$$

This formula for the r-derivative of $u(\eta)$ will be used in the next section to explore the mathematical aspect of the Exp-function ansatz.

2.4.3 Theorems

Theorem 1: Suppose that $u^{(r)}$and u^{γ} are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r and γ are both positive integers. Then the balancing procedure using the Exp-function ansatz: $u(\eta)=\frac{\sum_{n=-c}^{d} a_n exp(n\eta)}{\sum_{m=-p}^{q} b_m exp(m\eta)}$ leads to $c=d$ and $p=q, \forall\, r\geq 1, \forall \gamma\geq 2$.

Proof. Assuming that γ is a positive integer then have from the Exp-function ansatz:

$$u^{\gamma}=\left(\frac{\sum_{n=-c}^{d} a_n exp(n\eta)}{\sum_{m=-p}^{q} b_m exp(m\eta)}\right)^{\gamma}=\frac{a_{-c}^{\gamma}\exp(-\gamma c\eta)+\cdots\cdots+a_p^{\gamma}\exp(\gamma p\eta)}{b_{-d}^{\gamma}\exp(-\gamma d\eta)+\cdots\cdots+b_q^{\gamma}\exp(\gamma q\eta)} \quad (2.19)$$

In order to balance the linear term of the highest derivative $u^{(r)}$ with the highest nonlinear term u^{γ}, we first rewrite $u^{(r)}$ as $u^{(r)}(\eta)=$

$$\frac{\tau_r\exp[-(c+(2^r-1)d)\eta]+\cdots\cdots+\sigma_r\exp[(p(2^r-1)q)\eta]}{\varrho_r\exp[(-2^r d)\eta]+\cdots\cdots+\Gamma_r\exp[(2^r q)\eta]}\times\frac{b_{-d}^{\gamma}\exp(-\gamma d\eta)+\cdots\cdots+b_q^{\gamma}\exp(\gamma q\eta)}{b_{-d}^{\gamma}\exp(-\gamma d\eta)+\cdots\cdots+b_q^{\gamma}\exp(\gamma q\eta)}$$

$$= \frac{\tau_r b_{-d}^{\gamma} \exp[-(c+(2^r-1+\gamma)d)\eta]+\cdots\cdots+\sigma_r b_q^{\gamma} \exp[(p(2^r-1+\gamma)q)\eta]}{\varrho_r b_{-d}^{\gamma} \exp[(-2^r+\gamma)d\eta]+\cdots\cdots+\Gamma_r b_q^{\gamma} \exp[(2^r+\gamma)q\eta]}. \tag{2.20}$$

Also from equation (2.19) we obtain

$$u^{\gamma} = \frac{a_{-c}^{\gamma} \exp(-\gamma c\eta)+\cdots\cdots+a_p^{\gamma} \exp(\gamma p\eta)}{b_{-d}^{\gamma} \exp(-\gamma d\eta)+\cdots\cdots+b_q^{\gamma} \exp(\gamma q\eta)} \times \frac{\varrho_r \exp[(-2^r d)\eta]+\cdots\cdots+\Gamma_r \exp[(2^r q)\eta]}{\varrho_r \exp[(-2^r d)\eta]+\cdots\cdots+\Gamma_r \exp[(2^r q)\eta]}$$

$$= \frac{\varrho_r a_{-c}^{\gamma} \exp[(-2^r d+\gamma c)\eta]+\cdots\cdots+\Gamma_r a_p^{\gamma} \exp[(2^r q+\gamma p)\eta]}{\varrho_r b_{-d}^{\gamma} \exp[(-2^r+\gamma)d\eta]+\cdots\cdots+\Gamma_r b_q^{\gamma} \exp[(2^r+\gamma)q\eta]} \tag{2.21}$$

On balancing the lowest and the highest order of the Exp-function in Eqs. (2.20) and (2.21), it then follows

$$\begin{aligned} &-(c + (2^r - 1 + \gamma)d) = -(2^r d + \gamma c) \\ &p + (2^r - 1 + \gamma)q = (2^r q + \gamma p). \end{aligned} \tag{2.22}$$

Simplifying the last two equations yields

$$\begin{aligned} &(\gamma - 1)d = (\gamma - 1)c, \\ &(\gamma - 1)p = (\gamma - 1)q. \end{aligned} \tag{2.23}$$

Noting that $\gamma \geq 2$, we find from Eqs. (2.23) that c=d and p=q, and this complete the proof.

Theorem 2: Suppose that $u^{(r)}$and $u^{(s)}u^{(k)}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and k are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q, \forall\ r, s, k \geq 1$.

Proof. Let r, s, and k be positive integers. The nonlinear term $u^{(s)}u^{(k)}$ can be evaluated by using the general formula given by Eq.(2.18) and the Exp-function ansatz as

$$u^{(s)}u^k = \frac{\tau_s \exp[-(c+(2^s-1)d)\eta]+\cdots+\sigma_s \exp[(p(2^s-1)q)\eta]}{\varrho_s \exp[(-2^s d)\eta]+\cdots+\Gamma_s \exp[(2^s q)\eta]} \times \frac{b_{-c}^{k} \exp(-kc\eta)+\cdots+b_p^{k} \exp(kp\eta)}{b_{-d}^{k} \exp(-kd\eta)+\cdots+b_q^{k} \exp(kq\eta)}$$

$$= \frac{\tau_s b_{-c}^{k} \exp[-((k+1)c+(2^s-1)d)\eta]+\cdots+\sigma_s b_p^{k} \exp[((k+1)p+(2^s-1)q)\eta]}{\varrho_s b_{-d}^{k} \exp[(-2^s d)\eta]+\cdots+\Gamma_s b_q^{k} \exp[(2^s+k)q\eta]} \tag{2.24}$$

Multiplying both numerator and denominator of the R.H.S of this equation by

$$(\varrho_r \exp[(-2^r d)\eta] + \cdots\cdots + \Gamma_r \exp[(2^r q)\eta]),$$

We then get

$$u^{(s)}u^k = \frac{\alpha_1 \exp[-((k+1)c+(2^s+2^r-1)d)\eta]+\cdots+\alpha_2 \exp[((k+1)p+(2^s+2^r-1)q)\eta]}{\beta_1 \exp[-(2^s+2^r+k)d\eta]+\cdots+\beta_2 \exp[(2^s+2^r+k)q\eta]}, \tag{2.25}$$

Where $\alpha_1, \alpha_2, \beta_1, and\ \beta_2$ are constants and given as

$$\begin{aligned} \alpha_1 &= \varrho_r \tau_s b_{-c}^k, \qquad \alpha_2 = \Gamma_r b_p^k \sigma_s, \\ \beta_1 &= \varrho_r \varrho_s b_{-d}^k, \qquad \beta_2 = \Gamma_r \Gamma_s b_q^k, \end{aligned} \tag{2.26}$$

On multiplying both numerator and denominator of the R.H.S of $u^{(r)}$ by

$$(\varrho_s b_{-d}^k \exp[(-(2^s+k))d\eta] + \cdots\cdots + \Gamma_s b_q^k \exp[(2^s+k)q\eta]),$$

We get

$$u^{(r)} = \frac{\delta_1 \exp[-(c+(2^s+2^r+k-1)d)\eta]+\cdots\cdots\cdots+\delta_2 \exp[(p+(2^s+2^r+k-1)q)\eta]}{\beta_1 \exp[-(2^s+2^r+k)d\eta]+\cdots\cdots\cdots+\beta_2 \exp[(2^s+2^r+k)q\eta]}, \tag{2.27}$$

Where $\delta_1\ and\ \delta_2$ are given by

$$\delta_1 = \tau_r \varrho_s b_{-d}^k, \qquad \delta_2 = \sigma_r \Gamma_s b_q^k. \tag{2.28}$$

In view of Eqs.(2.25) and (2.27) and balancing the lowest and the highest order of the Exp-function, we get

$$\begin{aligned} -((k+1)c + (2^s+2^r-1)d) &= -(c + (2^s+2^r+k-1)d), \\ (k+1)p + (2^s+2^r-1)q &= p + (2^s+2^r+k-1)q. \end{aligned} \tag{2.29}$$

These equations can be also simplified to give c=d and p=q.

Theorem 3: Let $u^{(r)}$and $(u^{(s)})^{\Omega}$ be respectively the highest order linear term and the highest order nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to c=d and p=q, $\forall r, s \geq 1, \forall \Omega \geq 2$.

Proof. Proceeding as above, the nonlinear term $(u^{(s)})^{\Omega}$ can be evaluated by using the general formula given by Eq.(2.18) as

$$(u^{(s)})^{\Omega} == \frac{\epsilon_1 \exp[-[(c+(2^s-1)d)\Omega+2^r d]\eta+\cdots\cdots+\epsilon_2 \exp[[(p+(2^s-1)q)\Omega+2^r q]\eta]}{\epsilon_3 \exp[-(2^s\Omega+2^r)d\eta]+\cdots\cdots+\epsilon_4 \exp[(2^s\Omega+2^r)q\eta]}, \tag{2.30}$$

Where, $\epsilon_1, \epsilon_2, \epsilon_3$ and ϵ_4 are constants. Also $u^{(r)}$ can be written as

$$\mathrm{u}^{(\mathrm{r})}=\frac{\chi_1\exp[-[(c+(2^r-1)d)+2^s\Omega d)\eta]+\cdots\cdots+\chi_2\exp[(p+(2^r-1)q+2^s\Omega q)\eta]}{\chi_3\exp[-(2^s\Omega+2^r)d\eta]+\cdots\cdots+\chi_4\epsilon\exp[(2^s\Omega+2^r)q\eta]}, \tag{2.31}$$

Where,χ_1,χ_2,χ_3andχ_4 are also constants. The balancing procedure leads to the system:

$$\begin{aligned}&-[(c+(2^s-1)d)\Omega+2^r d=-(c+(2^r-1)d)+2^s\Omega d),\\&(p+(2^s-1)q)\Omega+2^r q=(p+(2^r-1)q+2^s\Omega q.\end{aligned} \tag{2.32}$$

On simplifying this system we then have

$$\begin{aligned}&(\Omega-1)(c-d)=0,\\&(\Omega-1)(p-q)=0.\end{aligned} \tag{2.33}$$

Noting that $\Omega\neq 1$ the system above requires that c=d and p=q, and this complete the proof.

Theorem 4: Suppose that $u^{(r)}$and$(u^{(s)})^{\Omega}u^{\lambda}$are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r,s,Ω and λ are all positive integers. Then the balancing procedure used the Exp-function leads to c=d and p=q, $\forall r,s,\Omega,\lambda\geq 1$.

Proof. On using (2.4.1.4) and (2.4.2.2) we can rewrite $u^{(r)}$and $(u^{(s)})^{\Omega}u^{\lambda}$as

$$(u^{(s)})^{\Omega}u^{\lambda}=\frac{\epsilon_5\exp[-[(c+(2^s-1)d)\Omega+2^r d+\lambda c]\eta+\cdots\cdots+\epsilon_6\exp[[(p+(2^s-1)q)\Omega+2^r q+\lambda p]\eta]}{\epsilon_7\exp[-(2^s\Omega+2^r+\lambda)d\eta]+\cdots\cdots+\epsilon_8\epsilon\exp[(2^s\Omega+2^r+\lambda)q\eta]} \tag{2.34}$$

And

$$u^{(r)}=\frac{\chi_5\exp[-[(c+(2^r+2^s\Omega+\lambda-1)d)\eta]+\cdots\cdots+\chi_6\exp[(p+(2^r+2^s\Omega+\lambda-1)q)\eta]}{\chi_7\exp[-(2^s\Omega+2^r+\lambda)d\eta]+\cdots\cdots+\chi_8\exp[(2^s\Omega+2^r+\lambda)q\eta]} \tag{2.35}$$

Where, ϵ_i and χ_i $i=5,\cdots\cdots\cdots\cdots 8$, are constants. From Eqs. (2.34) and (2.35) we obtain

$$\begin{aligned}&-[(c+(2^s-1)d)\Omega+2^r d+\lambda c]=-[(c+(2^r+2^s\Omega+\lambda-1)d),\\&(p+(2^s-1)q)\Omega+2^r q+\lambda p=\ p+(2^r+2^s\Omega+\lambda-1)q,\end{aligned} \tag{2.36}$$

Which can be simplified as

$$\begin{aligned}&(\lambda+\Omega-1)(c-d)=0,\\&(\lambda+\Omega-1)(c-d)=0.\end{aligned} \tag{2.37}$$

Noting that $\lambda+\Omega-1\neq 0,\forall\ \lambda,\ \Omega\geq 1$, we obtain c=d and p=q, and this complete the proof.

Chapter 3

Solution of Higher Order Boundary Value Problems

3.1 Introduction

The problems of Oskolkov and Fisher's equation have been solved by Exp-function method. Oskolkov and Fisher's equation are most widely studied problems in which are mostly used in modeling transport of air, adsorption of pollutants in soil, diffusion of neutrons, food processing, modeling of biological and ecological systems, modeling of semiconductors, oil reservoir flow transport, fluid mechanics, nuclear physics, solid-state physics, plasma physics, chemical physics, optical fiber and geochemistry. There are many methods to solve Fisher's and Oskolkov equations, but each method can only lead to special single solution. In this thesis, a new method, namely the Exp-function method, is employed to solve the Fisher's and Oskolkov equation. The obtained result includes all solutions in open literature as special cases, and the generalized solution with some free parameters might imply some fascinating meanings latent in the Fisher's and Oskolkov equations. For this reason, we can solve a higher order boundary value problem without its boundary conditions. So, in this thesis, we have solved the Fisher's and Oskolkov equations with their free boundary conditions. In this thesis, we also implement such figures by solving the Fisher's and Oskolkov equations we have computer software to get different shape of figure.

3.2 Oskolkov Equation

In this subsection we implement the Exp-function method for (1+1) Dimensional Oskolkov Equation in the following form

$$u_t - \beta u_{xxt} - \alpha u_{xx} + u u_x = 0 \tag{3.2.1}$$

Where α, β are arbitrary constant and $u(x,t)$ is an unknown function . Used the traveling wave variable $u(x,t) = u(\eta)$ and $\eta = kx - \omega t$ where k is a constant & ω is wave speed. Now we convert the equation (3.2.1) into the following ordinary differential equation

$$2k^2\omega\beta u'' - 2\alpha k^2 u' - 2\omega u + ku^2 = 0 \tag{3.2.2}$$

Where the prime denote the derivative with respect to η

We know, the Exp-function method is based on the postulate that traveling wave solutions could expressed in the following form

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n exp(n\eta)}{\sum_{m=-p}^{q} b_m exp(m\eta)} \tag{3.2.3}$$

$$= \frac{a_c exp(c\eta) + \cdots\cdots\cdots + a_{-d} exp(-d\eta)}{b_p exp(p\eta) + \cdots\cdots\cdots + b_{-q} exp(-q\eta)} \tag{3.2.4}$$

Where c, d, p and q are positive integers which are unknown to be further determined a_n & b_m are unknown constants. To determine the values of c & p. We balanced the linear term of highest order in equation with the highest order nonlinear term. Similarly to determine the values of d & q, we balanced the linear term of lowest order in equation with the lowest order nonlinear term.

Now differentiate equation (3.2.4) with respect to η and both side squaring we get

$$u' = \frac{b_p e^{p\eta} a_c c e^{c\eta} - b_p p e^{p\eta} a_c e^{c\eta}}{b_p^2 e^{2p\eta}} \tag{3.2.5}$$

$$= \frac{e^{p\eta} e^{c\eta}[b_p a_c (c-p)]}{b_p^2 e^{2p\eta}} \tag{3.2.6}$$

$$= \frac{e^{(p+c)\eta}}{e^{2p\eta}} \tag{3.2.7}$$

Here $\frac{b_p a_c (c-p)}{b_p^2} = \text{constant}$

Again,

$$u'' = \frac{e^{2p\eta} e^{(p+c)\eta}.(p+c) - e^{(p+c)\eta}.2p.e^{2p\eta}}{(e^{2p\eta})^2} \tag{3.2.8}$$

$$\Rightarrow u'' = \frac{e^{2p\eta} e^{(p+c)\eta}\{(p+c) - 2p\}}{e^{4p\eta}} \tag{3.2.9}$$

$$\therefore u'' = \frac{c_1 e^{(3p+c)\eta}}{c_2 e^{4p\eta}} \tag{3.2.10}$$

And

$$u^2 = \frac{a_c^2.e^{2c\eta}}{b_p^2.e^{2p\eta}} \tag{3.2.11}$$

$$u'u^2 = \frac{e^{p\eta}e^{c\eta}[b_p a_c(c-p)]}{b_p^2 e^{2p\eta}} \cdot \frac{a_c^2.e^{2c\eta}}{b_p^2.e^{2p\eta}} \tag{3.2.12}$$

$$= \frac{(c-p).b_p.a_c.a_c^2.e^{3c\eta+p\eta}}{b_p^4.e^{4p\eta}} \tag{3.2.13}$$

$$= \frac{c_3.e^{(3c+p)\eta}}{c_4.e^{4p\eta}} \tag{3.2.14}$$

Balanced the highest order of Exp-function in equation (3.2.10) & (3.2.14), we have $3p + c = 3c + p$, and we obtain $p = c$. Used the same method, we can also obtain that $q = d$.
For solved, we put $p = c = 1$ and $q = d = 1$, so equation (3.2.4) reduce to

$$u(\eta) = \frac{a_1 exp(\eta) + a_0 + a_{-1} exp(-\eta)}{b_1 exp(\eta) + b_0 + b_{-1} exp(-\eta)} \tag{3.2.15}$$

Substituting equation (3.2.15) into equation (3.2.2) and by the help of computer program we get

$$\begin{aligned} c_1 = &-2\omega\beta k^2 a_0 b_1 b_0 + 2\omega\beta k^2 a_1 b_0^2 + 2\alpha k^2 a_0 b_1 b_0 - 2\alpha k^2 a_1 b_0^2 - 4\omega a_0 b_1 b_0 + 2k a_1 a_0 b_0 \\ &+ 4\alpha k^2 a_{-1} b_1^2 - 4\omega a_1 b_1 b_{-1} + 2k a_1 a_{-1} b_1 + k a_1^2 b_{-1} - 2\omega a_{-1} b_1^2 - 2\omega a_1 b_0^2 \\ &+ k a_0^2 b_1 - 8\beta\omega a_1 b_1 b_{-1} + 8\beta\omega k^2 a_{-1} b_1^2 - 4\alpha k^2 a_1 b_1 b_{-1} \end{aligned}$$

$$\begin{aligned} c_{-1} = &\, 2\alpha k^2 a_0 b_{-1}^2 - 4\omega a_{-1} b_0 b_{-1} + 2k a_0 a_{-1} b_{-1} + k a_{-1}^2 b_0 - 2\omega a_0 b_{-1}^2 - 2\omega\beta k^2 a_{-1} b_0 b_{-1} \\ &+ 2\omega\beta k^2 a_0 b_{-1}^2 + 2\alpha k^2 a_{-1} b_0 b_{-1} \end{aligned}$$

$$\begin{aligned} c_2 = &\, 2\alpha k^2 a_0 b_1^2 - 4\omega a_1 b_1 b_0 + 2k a_1 a_0 b_1 + k a_1^2 b_0 - 2\omega a_0 b_1^2 - 2\omega\beta k^2 a_1 b_1 b_0 + 2\omega\beta k^2 a_0 b_1^2 \\ &- 2\alpha k^2 a_1 b_1 b_0 \end{aligned}$$

$$\begin{aligned} c_{-2} = &-6\alpha k^2 a_1 b_0 b_{-1} + 6\alpha k^2 a_{-1} b_1 b_0 + 6\omega\beta k^2 a_1 b_0 b_{-1} + 6\omega\beta k^2 a_{-1} b_1 b_0 \\ &- 12\omega\beta k^2 a_0 b_1 b_{-1} - 2\omega a_0 b_0^2 + k a_0^2 b_0 + 2k a_1 a_0 b_{-1} + 2k a_1 a_{-1} b_0 \\ &+ 2k a_0 a_{-1} b_1 - 4\omega a_1 b_0 b_{-1} - 4\omega a_0 b_1 b_{-1} - 4\omega a_{-1} b_1 b_0 \end{aligned}$$

$$c_3 = -2\omega a_1 b_1^2 + k a_1^2 b_1$$

$$c_{-3} = k a_{-1}^2 b_{-1} - 2\omega a_{-1} b_{-1}^2$$

Now solved the above equations used by computer program get the following set of solutions are

Case-1: $k = \sqrt{\frac{-1}{6\beta}}, \omega = \frac{\alpha}{5\beta}, a_{-1} = 0, a_0 = 0, a_1 = -\frac{3b_0^2\alpha}{5b_{-1}} \cdot \sqrt{\frac{-1}{6\beta}}, b_{-1} = b_{-1}, b_0 = b_0, b_1 = \frac{b_0^2}{4b_{-1}}$

Now substituting all values of case -1 in equation (3.2.15) and yielding the following solution of equation (3.2.1)

$$u(\eta) = \frac{3}{5}\frac{b_0^2\alpha\left(1/\sqrt{-6\beta}\right)e^{-\eta}}{b_1\left(b_1e^{\eta}+b_o+\frac{b_0^2e^{-\eta}}{4b_1}\right)} \tag{3.2.16}$$

Similarly get solution of equation (3.2.1) for the case of 2, 3, & 4

Case-2: $k = \sqrt{\frac{1}{6\beta}}, \omega = \frac{-\alpha}{5\beta}, a_{-1} = 0, a_0 = a_0, a_1 = -\frac{5a_0^2\beta}{8\alpha b_{-1}} \cdot \sqrt{\frac{1}{6\beta}}, b_{-1} = b_{-1}, b_0 = -\frac{5a_{0\beta}}{\alpha} \cdot \sqrt{\frac{-1}{6\beta}}, b_1 = \frac{25a_0^2\beta}{96\alpha^2 b_{-1}}$

$$u(\eta) = \frac{\frac{-5a_0^2.e^{\eta}.\sqrt{6\beta}}{48.\alpha.b_{-1}}+a_0}{\frac{25.a_0^2.\beta e.^{\eta}}{96.\alpha^2.b_{-1}}+\frac{5.\sqrt{6\beta}.a_0}{12.\alpha}+b_{-1}e^{-\eta}} \tag{3.2.17}$$

Case-3: $k = \sqrt{\frac{-1}{6\beta}}, \omega = \frac{-\alpha}{5\beta}, a_{-1} = \frac{3b_0^2\alpha}{5b_1} \cdot \sqrt{\frac{-1}{6\beta}}, a_0 = 0, a_1 = 0, b_{-1} = \frac{b_0^2}{4b_1}, b_0 = b_0, b_1 = b_1$

$$u(\eta) = \frac{3}{5}.\frac{b_0^2.\alpha.e^{-\eta}}{\sqrt{-6\beta}.\left(b_1 \cdot e^{\eta}+b_0+\frac{b_0^2.e^{-\eta}}{4b_1}\right)} \tag{3.2.18}$$

Case-04: $k = \sqrt{\frac{1}{6\beta}}, \omega = \frac{\alpha}{5\beta}, a_{-1} = \frac{5a_0^2\beta}{8\alpha b_1} \cdot \sqrt{\frac{1}{6\beta}}, a_0 = a_0, a_1 = 0, b_{-1} = \frac{25a_0^2\beta}{96\alpha^2 b_1}, b_0 = \frac{5a_{0\beta}}{2\alpha} \cdot \sqrt{\frac{1}{6\beta}}, b_1 = b_1$

$$u(\eta) = \frac{a_0+\frac{5}{48}.\frac{a_0^2}{\alpha b_1}.\sqrt{6\beta}.e^{-\eta}}{b_1 \cdot e^{\eta}+\frac{5\ a_0}{12\ \alpha}.\sqrt{6\beta}+\frac{25a_0^2\beta}{96\alpha^2 b_1}.e^{-\eta}} \tag{3.2.19}$$

3.3 Fisher's Equation

The nonlinear reaction-diffusion equation

$$u_t = Du_{xx} + mu(1-u) \tag{3.3.1}$$

This equation was first introduced by Fisher as a model for the propagation of a mutant gene.

Here $u(x,t)$ is the concentration of the reactant, D represents its diffusion coefficient, and m represents the rate of chemical reaction. In media of other natures, u might be temperature or electric potential, D might be the thermal conductivity or specific electrical conductivity. Equation (3.3.1) becomes

$$u_t = u_{xx} + u(1-u) \tag{3.3.2}$$

Here the independent variables $x \,\&\, t$ into wave variable $\eta = kx + \omega t$ to carry out a partial differential equation into two independent variables.

Now,

$$\eta = kx + \omega t \tag{3.3.3}$$

Differentiation equation (3.3.3) with respect to *t* we get

$$\Rightarrow \frac{\partial \eta}{\partial t} = \omega \tag{3.3.3a}$$

$$\therefore u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} \tag{3.3.3b}$$

$$\therefore u_t = u'.\omega \tag{3.3.3c}$$

Again,

$$\frac{\partial \eta}{\partial x} = k \tag{3.3.3d}$$

$$\therefore u_x = \frac{\partial u}{\partial \eta}.\frac{\partial \eta}{\partial x} \tag{3.3.3e}$$

$$= k \cdot u' \tag{3.3.3g}$$

$$\therefore u^2 u_x = k u^2 u' \tag{3.3.3h}$$

Again differentiate equation (3.3.3g) with respect to x we get

$$u_{xx} = k^2 \cdot u'' \tag{3.3.3i}$$

Now used the equation (3.3.3) to (3.3.3i) convert the Fisher's equation (3.3.2) to the Ordinary Differential Equation is

$$-\omega u' + k^2 u'' + u(1-u) = 0 \tag{3.3.4}$$

Where the prime denote the derivative with respect to η. We know, the Exp-function method is based on the postulate that traveling wave solutions can be expressed in the following form

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n exp(n\eta)}{\sum_{m=-p}^{q} b_m exp(m\eta)} \tag{3.3.5}$$

$$= \frac{a_c exp(c\eta) + \cdots\cdots\cdots\cdots + a_{-d} exp(-d\eta)}{b_p exp(p\eta) + \cdots\cdots\cdots\cdots + b_{-q} exp(-q\eta)} \tag{3.3.5a}$$

Where c, d, p and q are positive integers which is unknown to be further determined a_n & b_m are unknown constants. To determine the values of c & p. We balance the linear term of highest order in equation with the highest order nonlinear term. Similarly to determine the values of d & q, we balance the linear term of lowest order in equation with the lowest order nonlinear term.

Now differentiate equation (3.3.5a) with respect to η we get

$$u' = \frac{b_p e^{p\eta} a_c c e^{c\eta} - b_p p e^{p\eta} a_c e^{c\eta}}{b_p^2 e^{2p\eta}} \tag{3.3.5b}$$

$$= \frac{e^{p\eta} e^{c\eta} [b_p a_c (c-p)]}{b_p^2 e^{2p\eta}} \tag{3.3.5c}$$

$$\therefore u' = \frac{e^{(p+c)\eta}}{e^{2p\eta}} \tag{3.3.5d}$$

Here $\frac{b_p a_c (c-p)}{b_p^2} =$ constant.

Again, from equation (3.3.5d) we get

$$u'' = \frac{e^{2p\eta} e^{(p+c)\eta}.(p+c) - e^{(p+c)\eta}.2p.e^{2p\eta}}{(e^{2p\eta})^2} \tag{3.3.5e}$$

$$\Rightarrow u'' = \frac{e^{2p\eta} e^{(p+c)\eta}\{(p+c)-2p\}}{e^{4p\eta}} \tag{3.3.5f}$$

$$\therefore u'' = \frac{c_1 e^{(3p+c)\eta}}{c_2 e^{4p\eta}} \tag{3.3.5g}$$

And squaring equation (3.3.5a) obtain,

$$u^2 = \frac{a_c^2.e^{2c\eta}}{b_p^2.e^{2p\eta}} \tag{3.3.5h}$$

Now multiplying equation (3.3.5c) & (3.3.5h). Then

$$u'u^2 = \frac{e^{p\eta} e^{c\eta} [b_p a_c (c-p)]}{b_p^2 e^{2p\eta}} . \frac{a_c^2.e^{2c\eta}}{b_p^2.e^{2p\eta}} \tag{3.3.5i}$$

$$= \frac{(c-p).b_p.a_c.a_c^2.e^{3c\eta+p\eta}}{b_p^4.e^{4p\eta}} \tag{3.3.5j}$$

$$\therefore u'u^2 = \frac{c_3.e^{(3c+p)\eta}}{c_4.e^{4p\eta}} \tag{3.3.5k}$$

Balancing the highest order of Exp-function in equation (3.3.5g) & (3.3.5k) & we have $3p + c = 3c + p$, and we obtain $p = c$. Used the same method, we can also obtain that $q = d$.

For solved, we put $p = c = 1$ & $q = d = 1$, so equation (3.3.5a) reduce to

$$u(\eta) = \frac{a_1 exp(\eta)+a_0+a_{-1}exp(-\eta)}{b_1 exp(\eta)+b_0+b_{-1}exp(-\eta)} \tag{3.3.6}$$

Substituting equation (3.3.6) into equation (3.3.4) and by the help of computer program we get

$$\begin{aligned} c_1 = {} & \omega a_0 b_1 b_0 - k^2 a_0 b_1 b_0 + a_1 b_0^2 + a_0^2 b_1 + 2a_0 b_1 b_0 - \omega a_1 b_0^2 + k^2 a_1 b_0^2 - 2a_0 a_1 b_0 \\ & + 2\omega a_{-1} b_1^2 + 4k^2 a_{-1} b_1^2 - 2a_1 a_{-1} b_1 + 2a_1 b_1 b_{-1} - 2\omega a_1 b_1 b_{-1} - 4k^2 a_1 b_1 b_{-1} \\ & - a_1^2 b_{-1} + a_{-1} b_1^2 \end{aligned}$$

$$\begin{aligned} c_{-1} = {} & -\omega a_0 b_{-1} b_0 - k^2 a_0 b_{-1} b_0 + a_{-1} b_0^2 + a_0^2 b_{-1} + 2a_0 b_{-1} b_0 + \omega a_{-1} b_0^2 + k^2 a_{-1} b_0^2 \\ & - 2a_0 a_{-1} b_0 + 2a_{-1} b_1 b_{-1} - 2\omega a_1 b_{-1}^2 + 4k^2 a_{-1} b_{-1}^2 - 2a_1 a_{-1} b_{-1} \\ & + 2\omega a_{-1} b_1 b_{-1} - 4k^2 a_{-1} b_1 b_{-1} - a_{-1}^2 b_1 + a_1 b_{-1}^2 \end{aligned}$$

$$\begin{aligned} c_2 = {} & -a_{-1}^2 b_0 + \omega a_{-1} b_0 b_{-1} - k^2 a_{-1} b_0 b_{-1} + k^2 a_0 b_{-1}^2 + 2a_{-1} b_0 b_{-1} - \omega a_0 b_{-1}^2 + a_0 b_{-1}^2 \\ & - 2a_0 a_{-1} b_{-1} \end{aligned}$$

$$c_{-2} = a_0 b_1^2 - a_1^2 b_0 + k^2 a_0 b_1^2 + 2a_1 b_1 b_0 + \omega a_0 b_1^2 - k^2 a_1 b_1 b_0 - 2a_1 a_0 b_1 - \omega a_1 b_1 b_0$$

$$c_3 = -a_1^2 b_1 + a_1 b_1^2$$

$$c_{-3} = a_{-1} b_{-1}^2 - a_{-1}^2 b_{-1}$$

$$\begin{aligned} c_0 = {} & -3\omega a_1 b_0 b_{-1} + 3\omega a_{-1} b_1 b_0 + 3k^2 a_1 b_0 b_{-1} + 3k^2 a_{-1} b_1 b_0 - 6k^2 a_0 b_1 b_{-1} + 2a_1 b_0 b_{-1} \\ & + 2a_{-1} b_1 b_0 + 2a_0 b_1 b_{-1} - 2a_1 a_0 b_{-1} - 2a_1 a_{-1} b_0 - 2a_0 a_{-1} b_1 \end{aligned}$$

Now solve the above equations by using computer program get the following sets of solutions are

Case-01: $k = \sqrt{\frac{1}{6}}\omega = \frac{5}{6}a_{-1} = 0a_0 = 0a_1 = \frac{b_0^2}{4b_{-1}}b_{-1} = b_{-1}b_0 = b_0 b_1 = \frac{b_0^2}{4b_{-1}}$

Now substituting all values of case-1 in equation (3.3.6) and yielding the following solution of equation (3.3.1)

$$u(\eta) = \frac{1}{4} \cdot \frac{b_0^2 e^{\eta}}{b_{-1}\left(\frac{b_0^2 e^{\eta}}{4b_{-1}} + b_0 + b_{-1}e^{-\eta}\right)} \tag{3.3.7}$$

Similarly get solution of equation (3.3.1) for the case of 2, 3, & 4

Case-02: $k = \sqrt{\frac{-1}{6}}, \omega = \frac{5}{6}, a_{-1} = 0, a_0 = b_0, a_1 = 0, a_1 = \frac{b_0^2}{4b_1}, b_{-1} = b_{-1}, b_0 = b_0, b_1 = \frac{b_0^2}{4b_{-1}}$

$$u(\eta) = \frac{1/4 \frac{b_0^2 e^{\eta}}{b_{-1}} + b_0}{1/4 \frac{b_0^2 e^{\eta}}{b_{-1}} + b_0 + b_{-1}e^{-\eta}} \tag{3.3.8}$$

Case-03: $k = \sqrt{\frac{1}{6}}, \omega = -\frac{5}{6}, a_{-1} = \frac{b_0^2}{4b_1}, a_0 = 0, a_1 = 0, b_{-1} = \frac{b_0^2}{4b_1} b_0 = b_0, b_1 = b_1$

$$u(\eta) = \frac{1}{4} \frac{b_0^2 e^{-\eta}}{b_1\left(b_1 e^{\eta} + b_0 + \frac{b_0^2 e^{-\eta}}{4b_1}\right)} \tag{3.3.9}$$

Case-04: $k = \sqrt{\frac{-1}{6}}, \omega = -\frac{5}{6}, a_{-1} = \frac{b_0^2}{4b_1}, a_0 = b_0, a_1 = 0, b_{-1} = \frac{b_0^2}{4b_1}, b_0 = b_0, b_1 = b_1$

$$u(\eta) = \frac{b_0 + \frac{b_0^2 e^{-\eta}}{4b_1}}{b_1 e^{\eta} + b_0 + \frac{b_0^2 e^{-\eta}}{4b_1}} \tag{3.3.10}$$

3.4. Result Discussion

3.4.1 Numerical Result Discussion of Oskolkov Equation

In this area, we have discussed about the physical portrayal of the acquired exact and solitary wave solution to the (1+1) Dimensional Oskolkov equation. We speak to these solutions in graphical and check about the sort of solution. Now we pictorial some obtain solutions realize by Exp-function method for the Oskolkov equation.

Case-1:

The real and imaginary part of solution (3.2.16) is shown in figure-1(a) and figure-1(b) which is the rogue wave solution for the values $b_0 = -1, \alpha = 1, \beta = 1, b_{-1} = -1/10$. In these figure it can be seen that lower density plot appears in the 3D plot. If we increase the values of α then we analyze a dynamics behave of all solution. Here if we increase the values of α then we can seen that the rogue wave solution deform in kinky rogue wave solution as shown in the fig-1(c) to fig -1(f). From fig-1(g), we seen that this graph embodies the rogue wave solution of the imaginary part of solution (3.2.16) whose 3D plot lower density plot below for the values of the parameters $b_0 = -1, \alpha = -1, \beta = 1, b_{-1} = -1/10$. The fig-1(h) behave kink shape solution of the solution (3.2.16) for the values of parameter $b_0 = -1, \alpha = 1, \beta = -2, b_{-1} = -1/10$ we get this type solutions for the condition $\alpha > 0$. Fig-1(i) represent anti-kink shape solution of (3.2.16) for the parametric values $b_0 = -1, \alpha = -1, \beta = -2, b_{-1} = -1/10$. We estimate anti type solutions for the condition $\alpha < 0$.

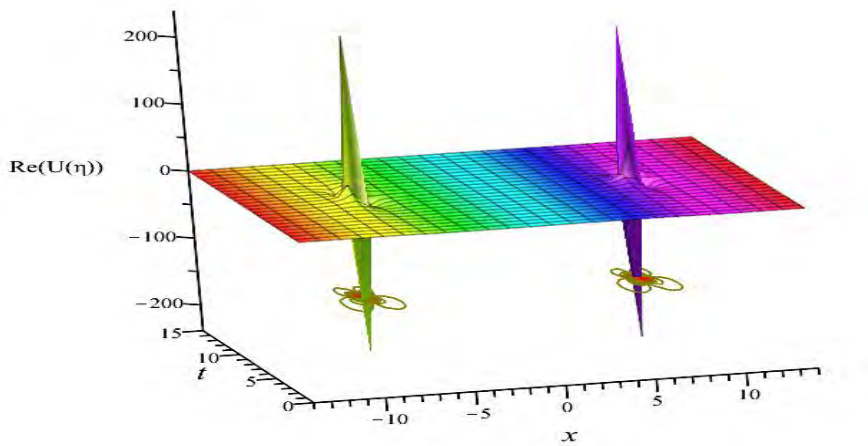

Figure-1(a): The real part of the solution (3.2.16) for the values of the parameters

$b_0 = -1, \alpha = 1, \beta = 1, b_{-1} = -1/10$

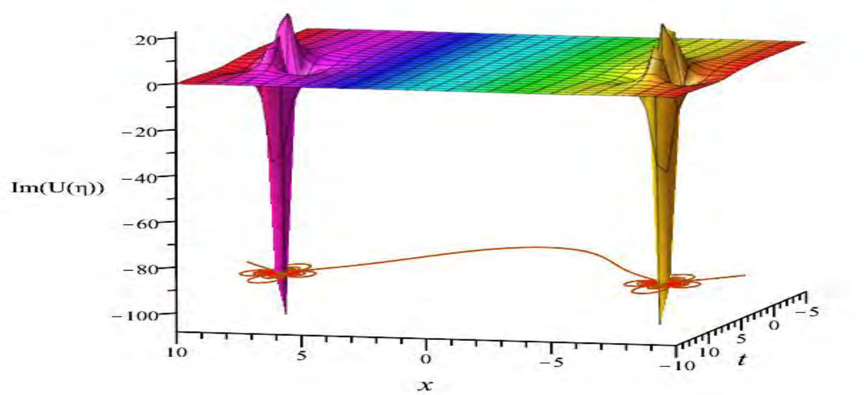

Figure-1(b): The imaginary part of the solution (3.2.16) for the values of the parameters

$b_0 = -1, \alpha = 1, \beta = 1, b_{-1} = -1/10$

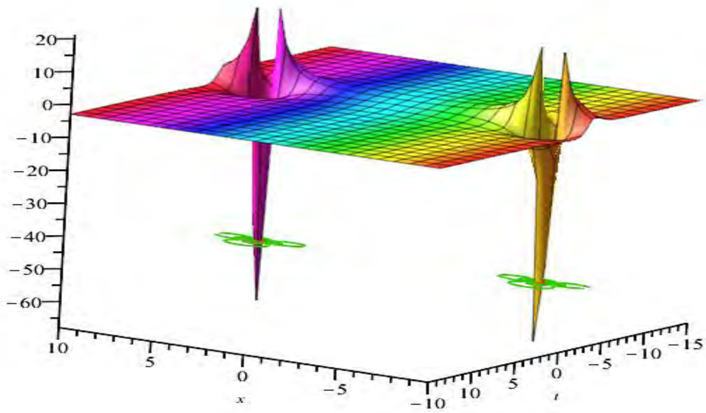

Figure-1(c): The 3D plot of the solution (3.2.16) for the values of the parameters

$b_0 = -1, \alpha = 3, \beta = 1, b_{-1} = -1/10$

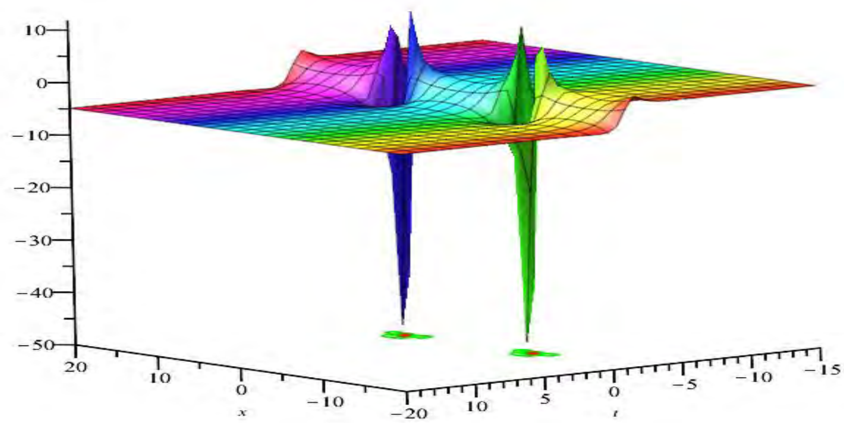

Figure-1(d): The 3D plot of the solution (3.2.16) for the values of the parameters

$b_0 = -1, \alpha = 5, \beta = 1, b_{-1} = -1/10$

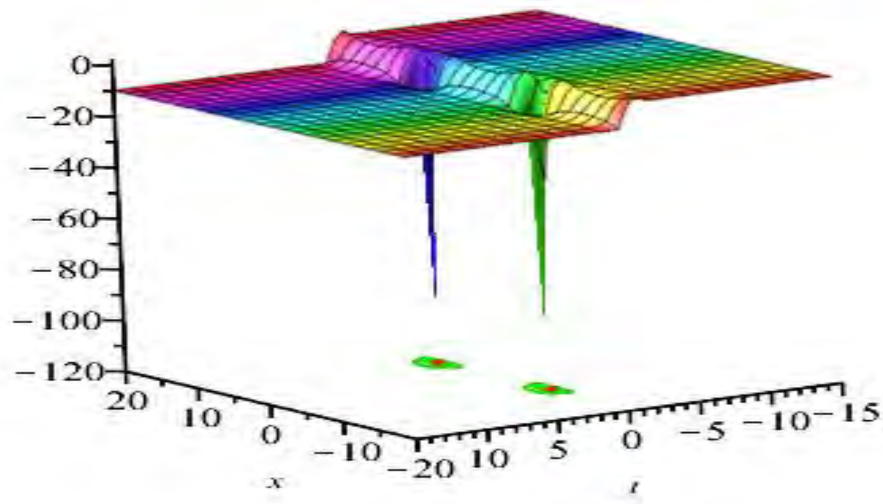

Figure-1(e): The 3D plot of the solution (3.2.16) for the values of the parameters
$b_0 = -1, \alpha = 7, \beta = 1, b_{-1} = -1/10$

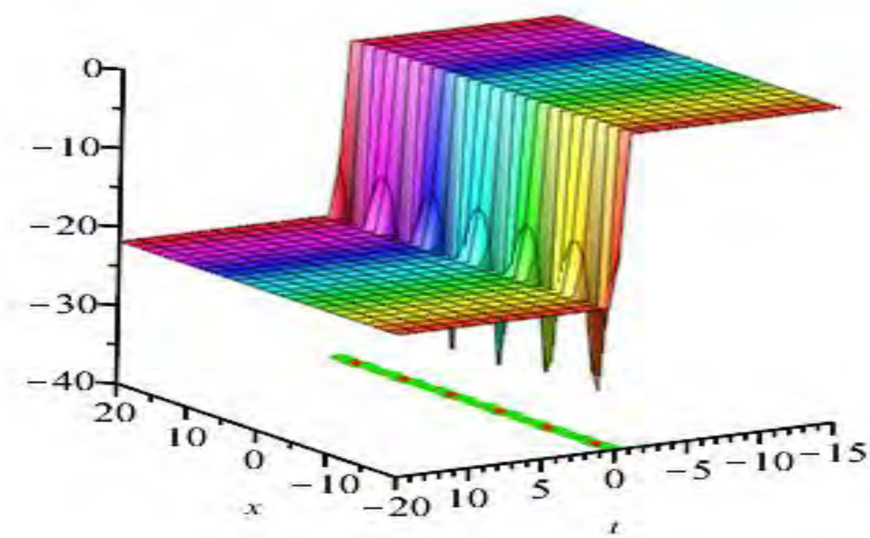

Figure-1(f): The 3D plot of the solution (3.2.16) for the values of the parameters
$b_0 = -1, \alpha = 10, \beta = 1, b_{-1} = -1/10$

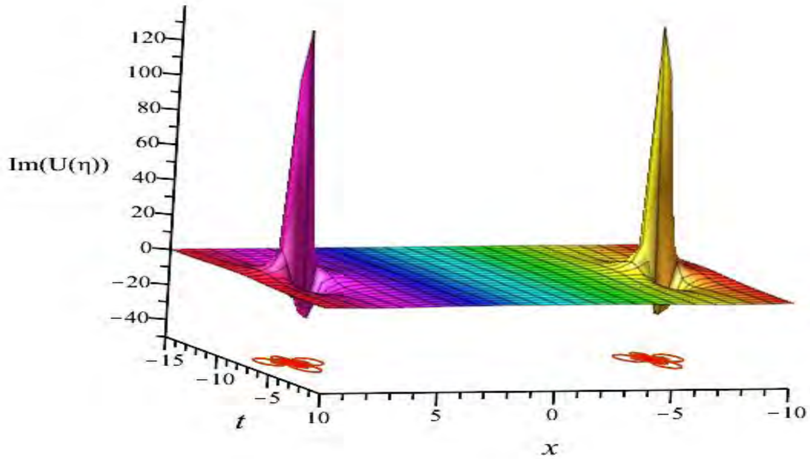

Figure-1(g): The imaginary part of the solution (3.2.16) for the values of the parameters

$b_0 = -1, \alpha = -1, \beta = 1, b_{-1} = -1/10$

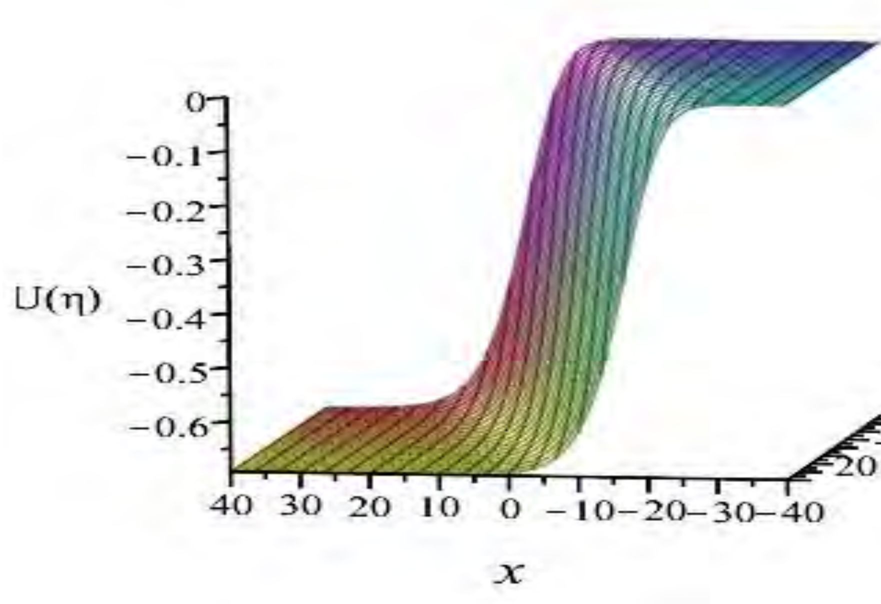

Figure-1(h): The 3D plot of the solution (3.2.16) for the values of the parameters

$b_0 = -1, \alpha = 1, \beta = -2, b_{-1} = -1/10$

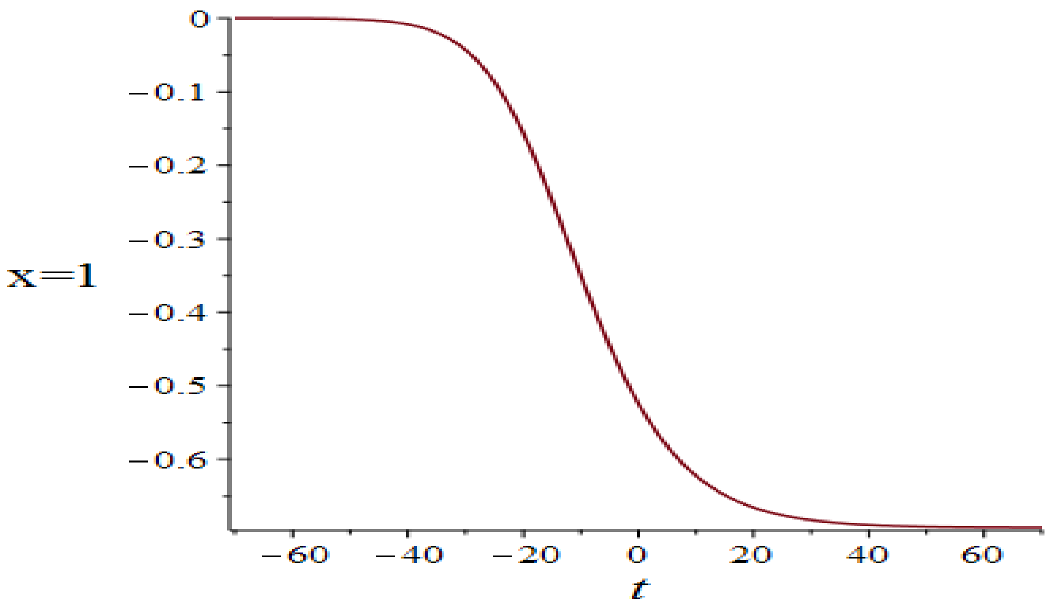

Figure-1(h(i)): The 2D plot the solution (3.2.16) for the value of

$x = 1$

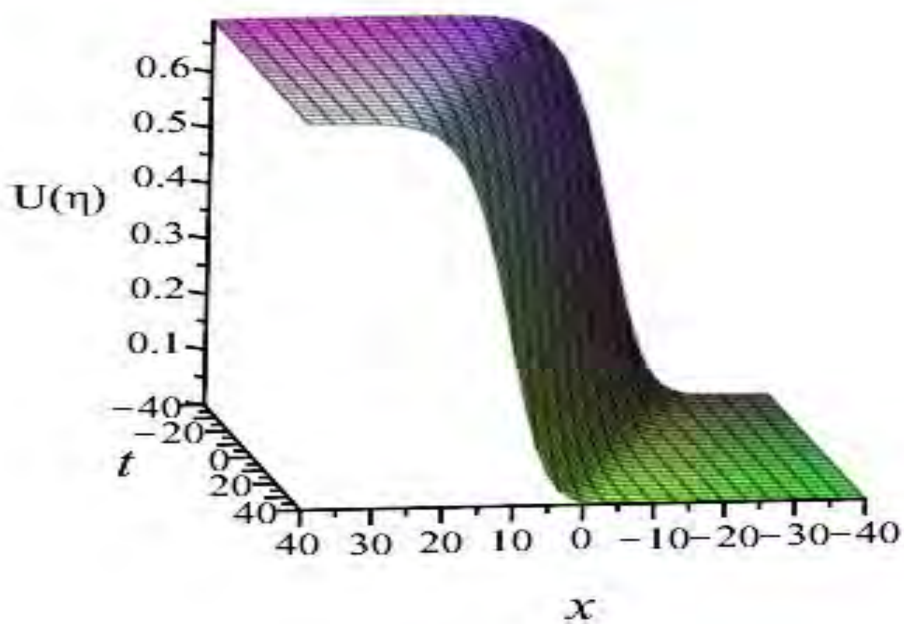

Figure-1(i): The 3D plot of the solution (3.2.16) for the values of the parameters

$b_0 = -1, \alpha = -1, \beta = -2, b_{-1} = -1/10$

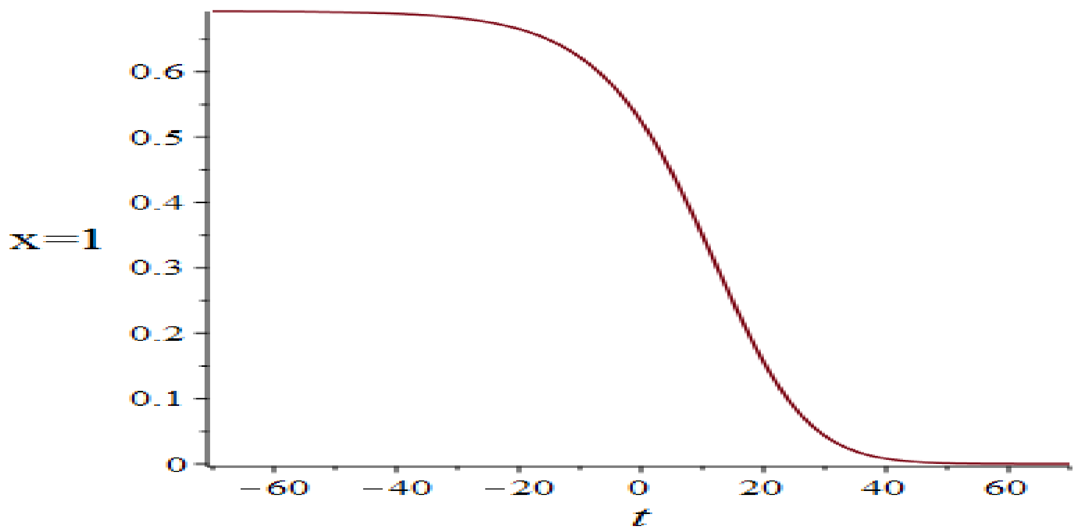

Figure 1(i(a)): The 2D plot of the solution (3.2.16) for the value of the $x = 1$.

Case-2:

The imaginary parts of solution (3.2.17) are shown in fig-2(a), fig -2(b), fig-2(c) and fig -2(d) which is the kinky rogue wave solution for the values $a_0 = 1, \alpha = -1, \beta = 1/10\,, b_{-1} = -1$. For increased values of α, we get an interaction between kink and rogue wave solution. Here in the fig.-2(b) the value of $\alpha = -1/2$, in fig-2(c) the value of $\alpha = 1/2$; in fig-2(d) the value of $\alpha = 1$.

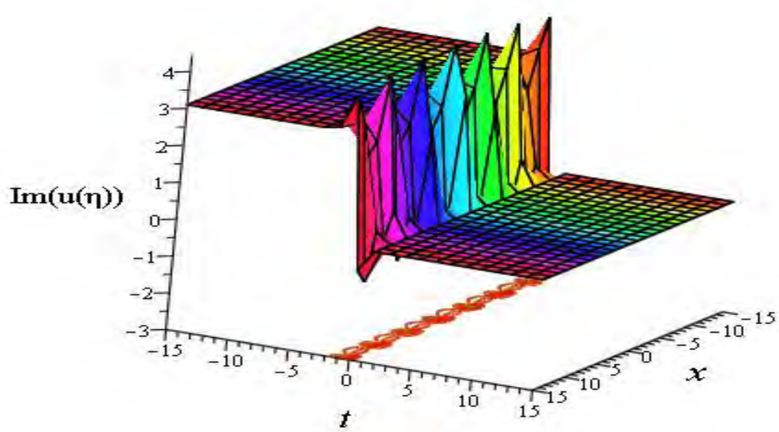

Figure-2(a): The 3D plot of the solution (3.2.17) for the values of the parameters

$a_0 = 1, \alpha = -1, \beta = 1/10\,, b_{-1} = -1$

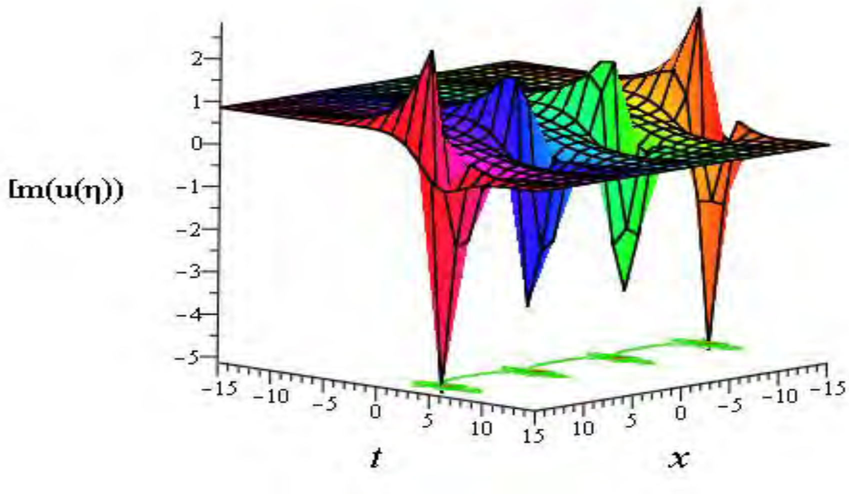

Figure-2(b): The 3D plot of the solution (3.2.17) for the values of the parameters

$a_0 = -1, \alpha = -1/2\,, \beta = 1/3\,, b_{-1} = -1/10$

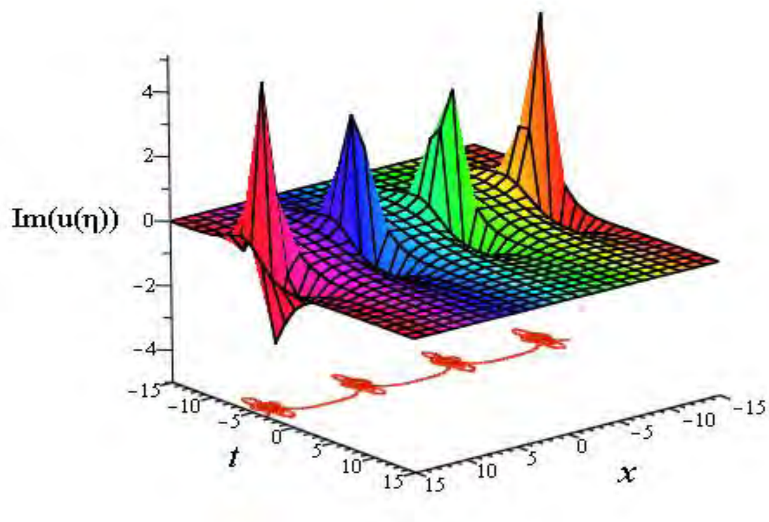

Figure-2(c): The 3D plot of the solution (3.2.17) for the values of the parameters

$a_0 = -1, \alpha = 1/2\,, \beta = 1/3\,, b_{-1} = -1/10$

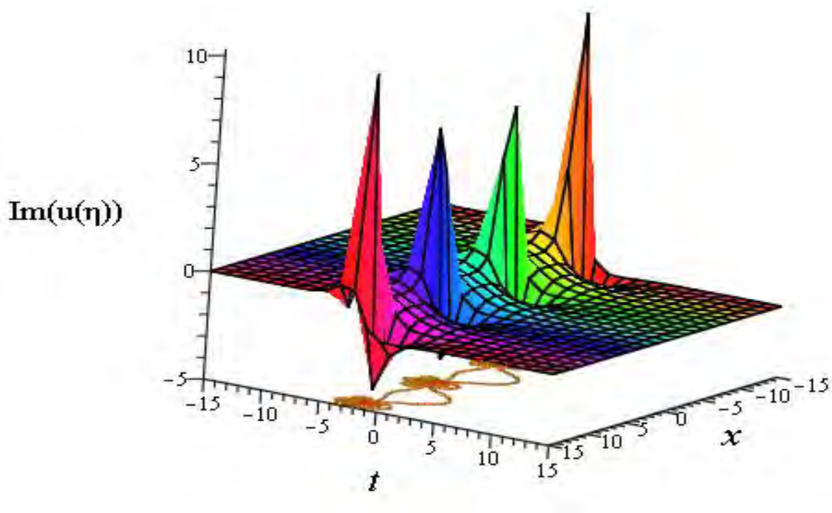

Figure-2(d): The 3D plot of the solution (3.2.17) for the values of the parameters $a_0 = -1, \alpha = 1, \beta = 1/3, b_{-1} = -1/10$.

Case-3:

The imaginary part of solution (3.2.18) is shown in fig-3(a), fig-3(b), fig-3(c) and fig-3(d) which is the anti-kinky rogue wave solution for the values $b_0 = -1/10, \alpha = 1, \beta = 1/3, b_1 = -1/10$. For increasing values of α, we get an interaction between anti-kinky and rogue wave solution. Here in the fig-3(b) the value of $\alpha = -1$ in fig-3(c) the value of $\alpha = -3$. We added that for increase the value of α the height of kink shape increasing & decreasing the value of α the height kink shape decreasing in among fig-3(a) and fig-3(b) to fig-3(c).

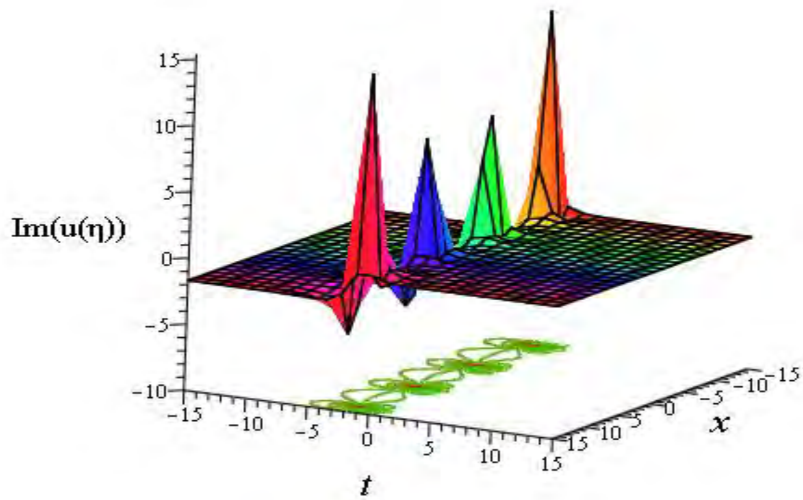

Figure-3(a): The 3D plot of the solution (3.2.18) for the values of the parameters

$b_0 = -1/10\,, \alpha = 1, \beta = 1/3\,, b_1 = -1/10$

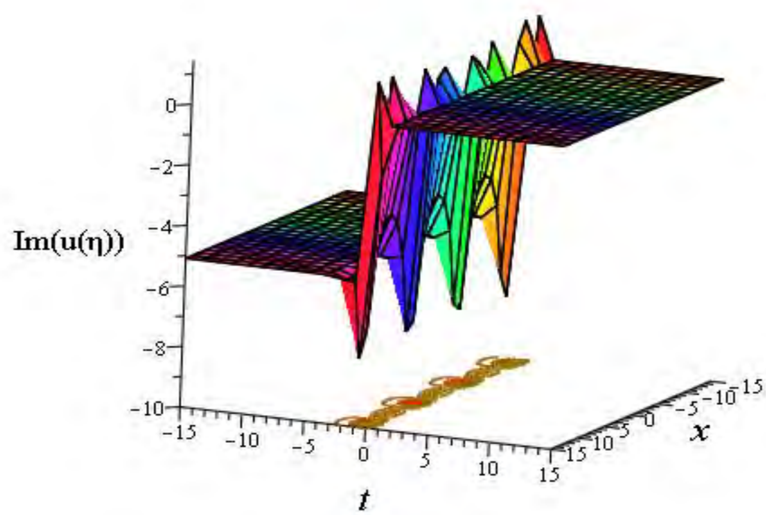

Figure-3(b): The 3D plot of the solution (3.2.18) for the values of the parameters

$b_0 = -1/10\,, \alpha = 3, \beta = 1/3\,, b_1 = -1/10$

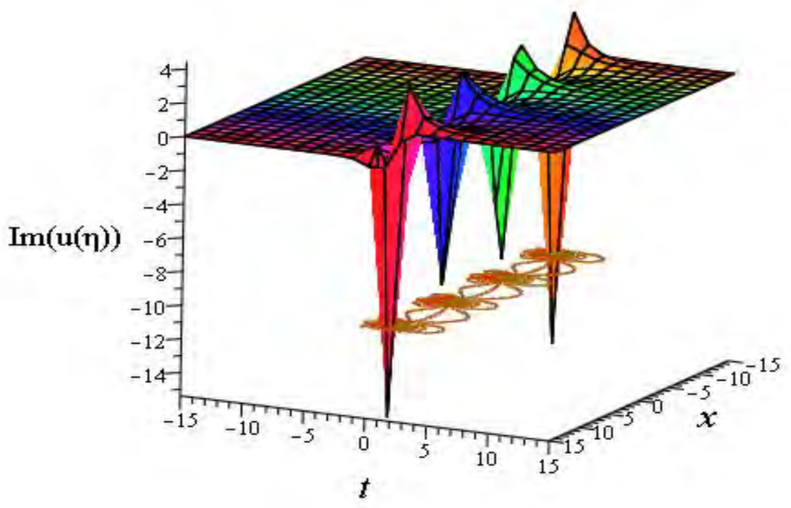

Figure-3(c): The 3D plot of the solution (3.2.18) for the values of the parameters $b_0 = -1/10\,, \alpha = -1, \beta = 1/3\,, b_1 = -1/10$

Case-4:

The real and imaginary part for the solution of equation (3.2.19) is represented different type periodic solution. Here we shown the behave of these solution with 3D and 2D plot as below: fig-4(a) & fig-4(b) represent the periodic solution for these values of parameters $a_0 = -3, \alpha = 1, \beta = -2, b_1 = -1/10$. In this fig-4(c) represent periodic wave solutions of real part for values are $a_0 = -3, \alpha = 1, \beta = 2, b_1 = -1/10$. From fig-4(d) & fig-4(e) we can appeared that the same nature of periodic wave of imaginary and absolute part for these different values of parameters are $a_0 = -3, \alpha = -1, \beta = -2, b_1 = -1/10$ and $a_0 = -4, \alpha = 1, \beta = -2, b_1 = -1/10$.

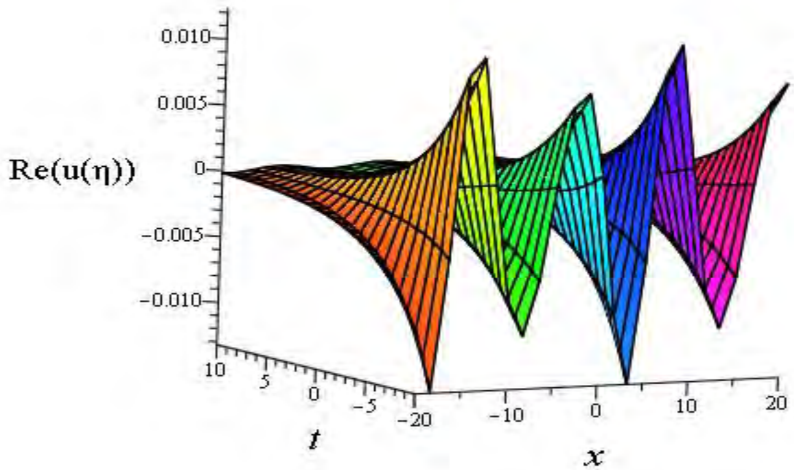

Figure-4(a): The 3D plot of the solution (3.2.19) for the values of the parameters

$a_0 = -3, \alpha = 1, \beta = -2, b_1 = -1/10$

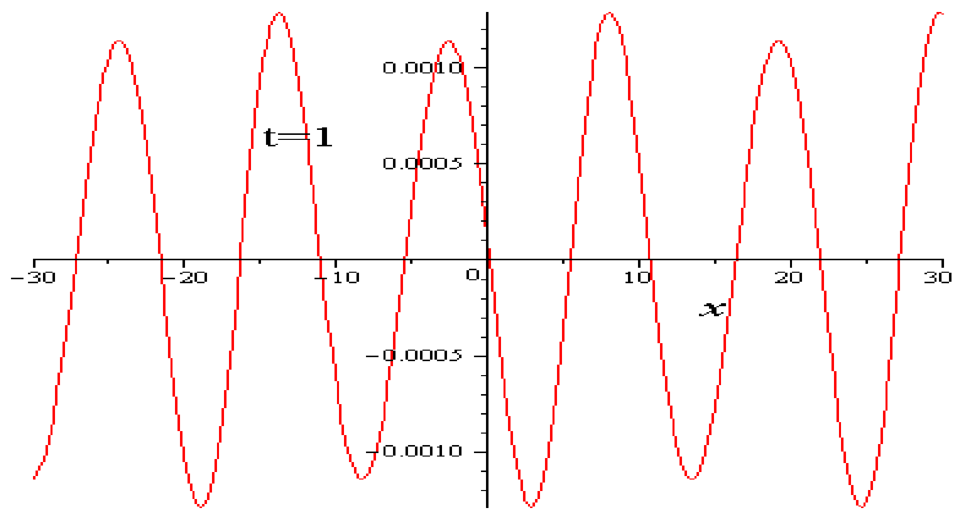

Figure-4(a (i)): The 2D plot of the solution (3.2.19) for the value of $t = 1$

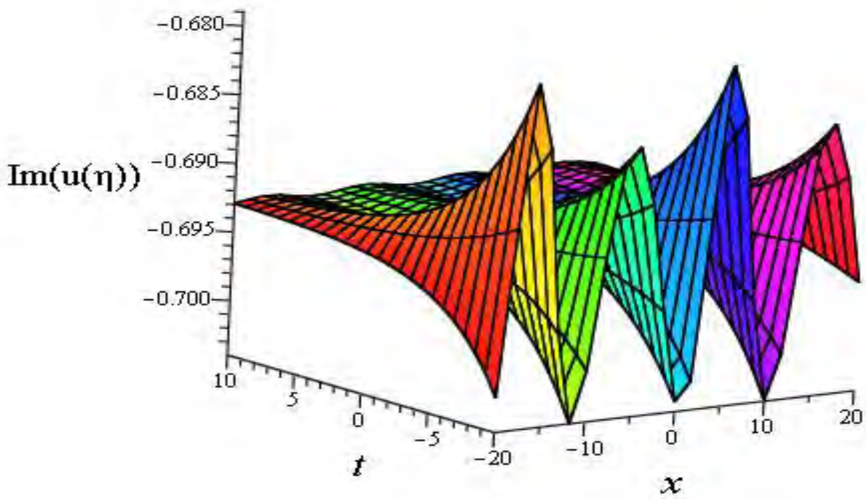

Figure-4(b): The 3D plot of the solution (3.2.19) for the values of the parameters

$a_0 = -3, \alpha = 1, \beta = -2, b_1 = -1/10$

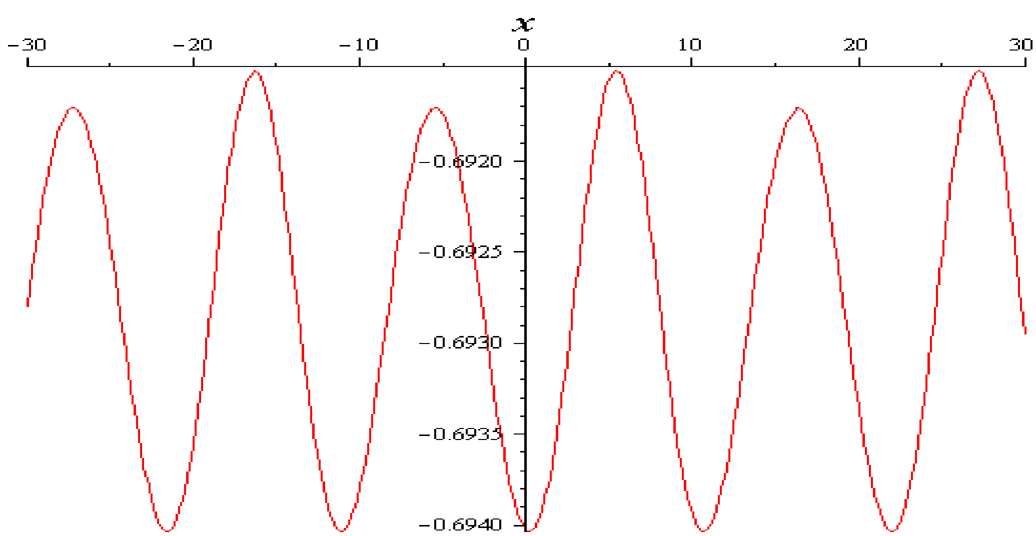

Figure-4(b(i)): The 2D plot of the solution (3.2.19) for the value $t = 1$

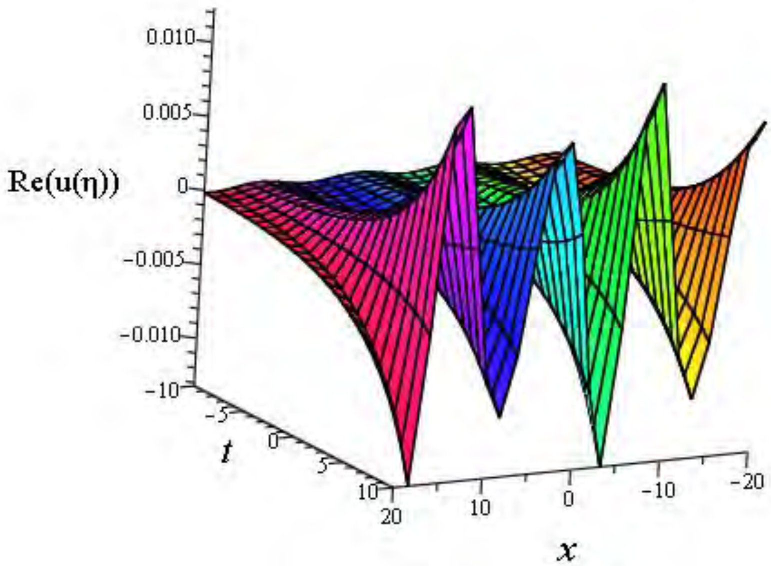

Figure-4(c): The 3D plot of the solution (3.2.19) for the values of the parameters

$a_0 = -3, \alpha = -1, \beta = 2, b_1 = -1/10$

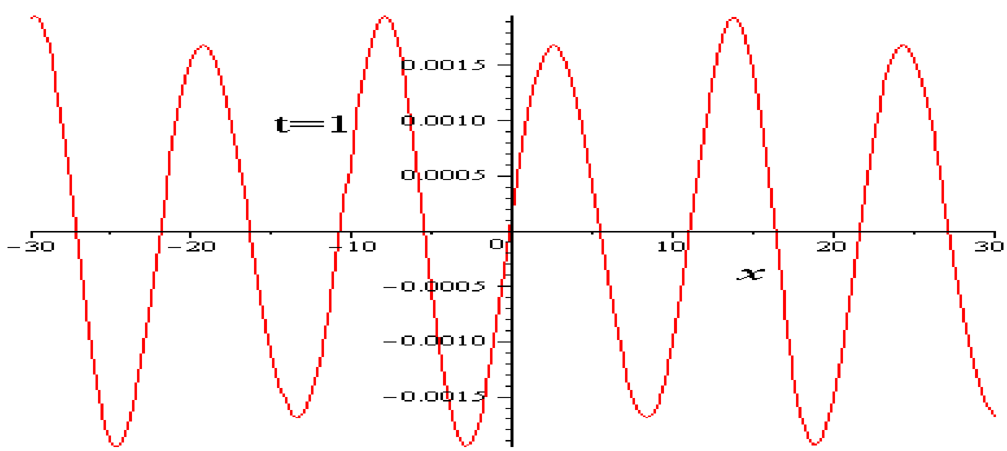

Figure-4(c(i)): The 2D plot of the solution (3.2.19) for the value $t = 1$

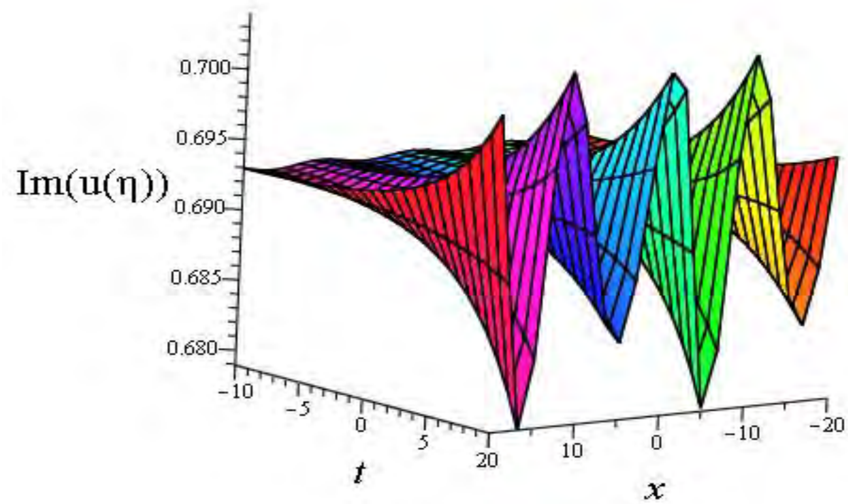

Figure-4(d): The 3D plot of the solution (3.2.19) for the values of the parameters $a_0 = -3, \alpha = -1, \beta = -2, b_1 = -1/10$

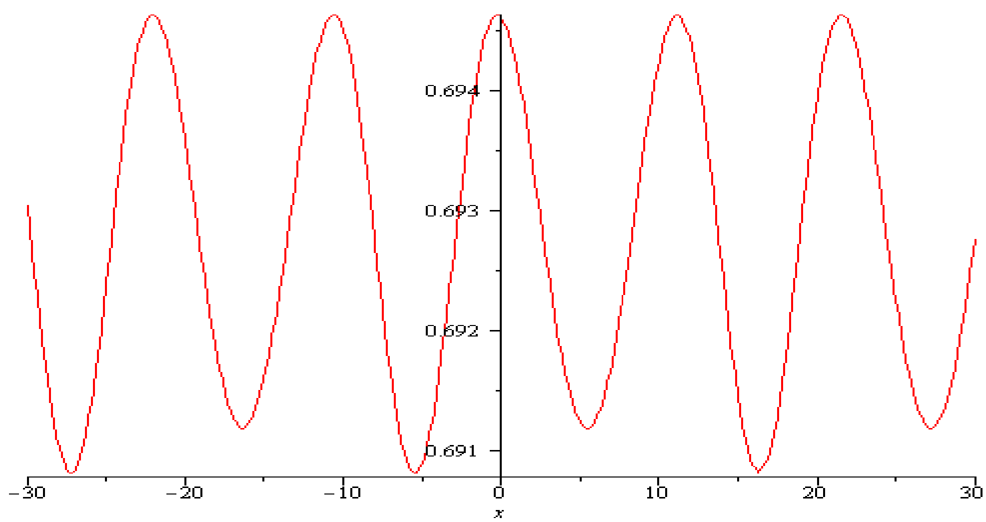

Figure-4(d(i)): The 2D plot of the solution (3.2.19) for the value $t = 1$

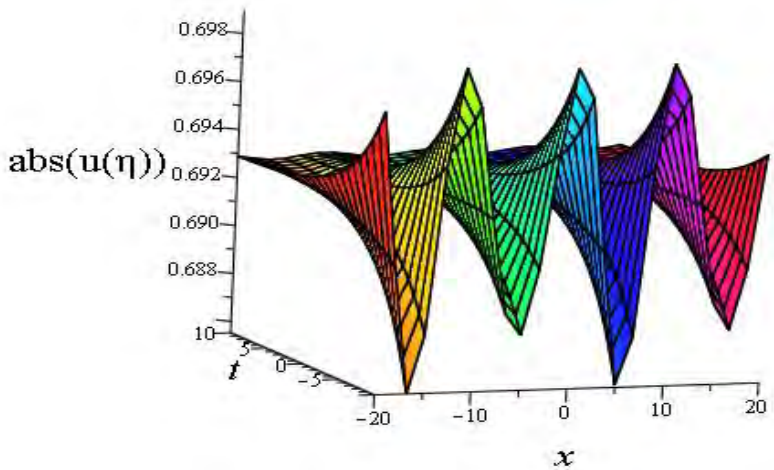

Figure-4(e): The 3D plot of the solution (3.2.19) for the values of the parameters

$a_0 = -4, \alpha = 1, \beta = -2, b_1 = -1/10$

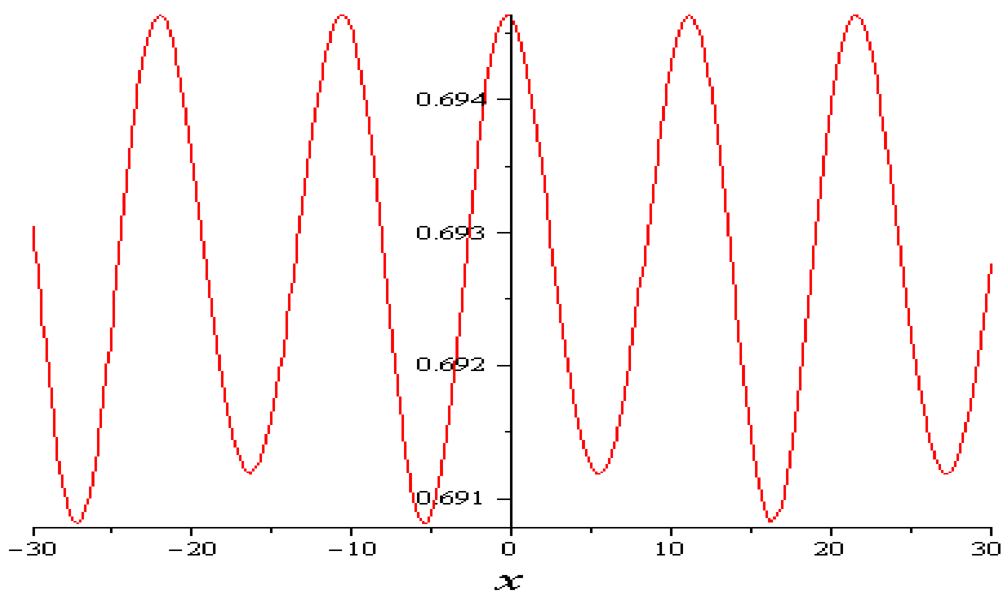

Figure-4(e(i)): The 2D plot of the solution (3.2.19) of for the value $t = 1$

3.4.2 Numerical Result Discussion of Fisher's Equation

In this area, we have wanted to shed light about the physical phenomenon of the obtained exact and solitary wave solution to the (1+1) Dimensional Fisher's equation. We speak to these solutions in graphical and check about the sort of solution. Here we have gotten some figures for the Fisher's equation by putting different arbitrary values.

Case-1:

The real and imaginary parts of solution (3.3.7) are shown in fig-5(a) and fig-5(b) which is the rogue wave solution for the values $b_0 = 15,\ b_{-1} = -1/5$. In these figure it can be seen that lower density plot appears in the 3D plot.

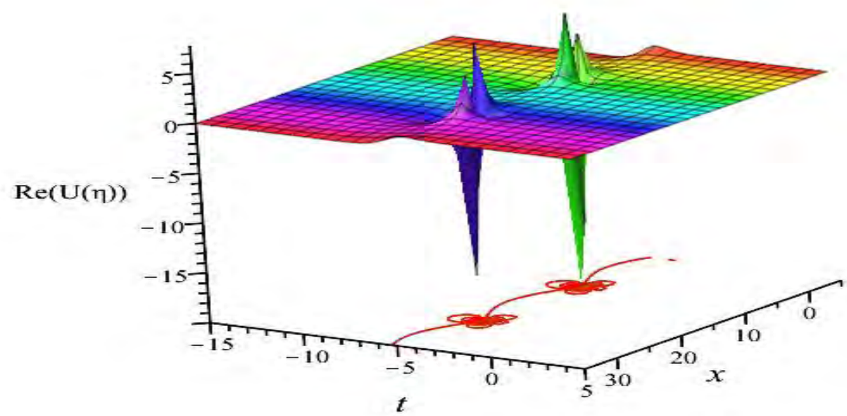

Figure-5(a): The real part of the solution of (3.3.7) for the values

$b_0 = -15, b_{-1} = -1/5$

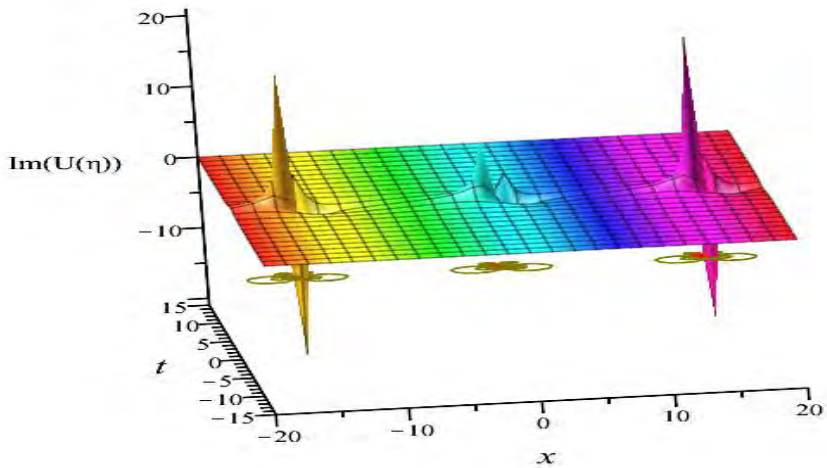

Figure-5(b): The imaginary part of the solution of (3.3.7) for the values

$b_0 = -15, b_{-1} = -1/5$

Case-2:

The imaginary part of solution (3.3.8) is shown in fig-6(a) and fig-6(b) which is the rogue wave and the periodic rogue wave solution for the values $b_0 = 2,\ b_{-1} = -1/10$. and $b_0 = -2, b_{-1} = 1/10$. respectively.

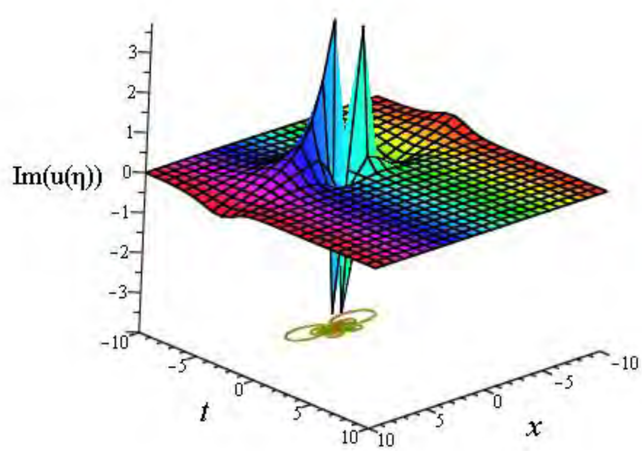

Figure-6(a): The imaginary part of the solution of (3.3.8) for the values

$b_0 = 2, b_{-1} = -1/10$

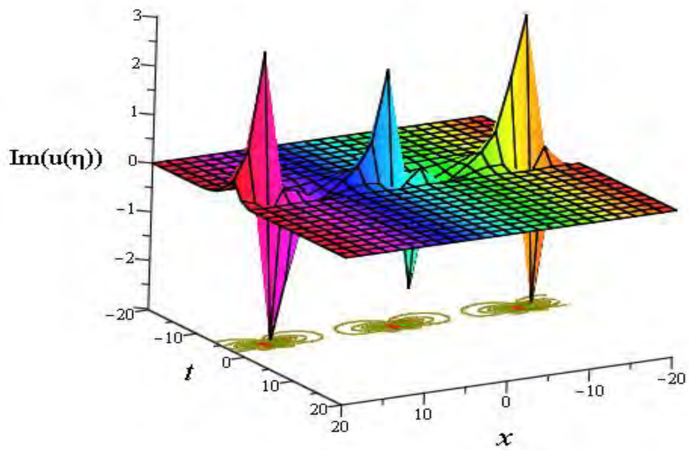

Figure-6(b): The imaginary part of the solution of (3.3.8) for the values

$b_0 = -2, b_{-1} = 1/10$

Case-3:

The real and imaginary part of solution (3.3.9) are shown in fig-7(a) and fig-7(b) which is the kink shape and the soliton graph for the values $b_0 = -2,\ b_1 = 1/10$. and $b_0 = -15, b_1 = 15$.

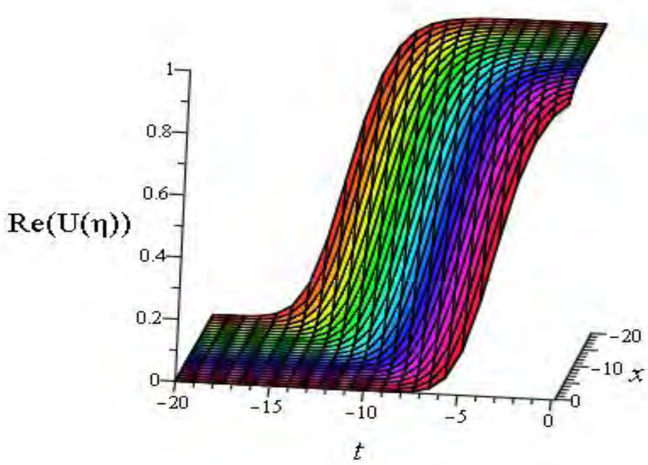

Figure-7(a): The imaginary part of the solution of (3.3.9) for the values

$b_0 = -2, b_1 = 1/10$

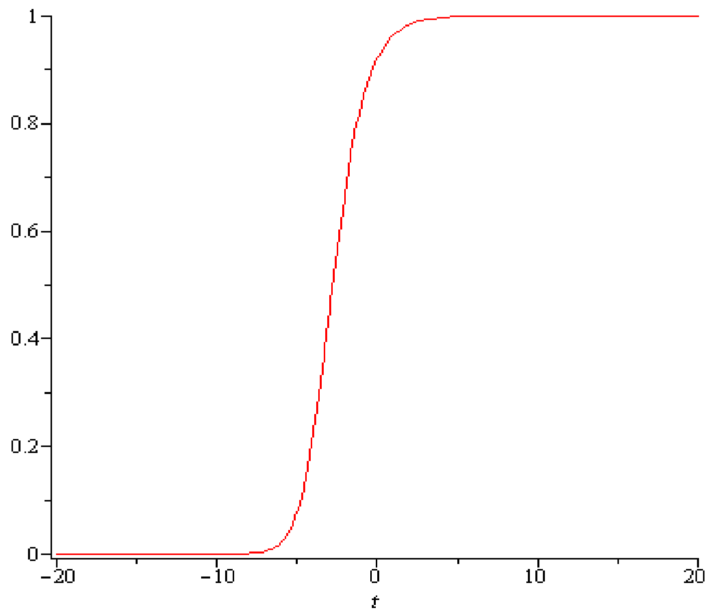

Figure-7(a (i)): The 2D plot of the solution of (3.3.9) for the value $x = 1$

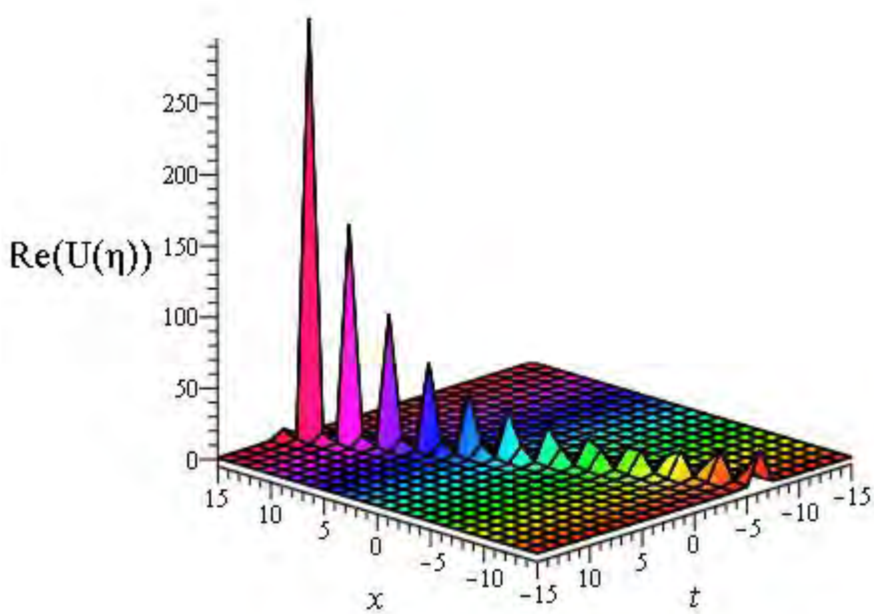

Figure-7(b): The imaginary part of the solution of (3.3.8) for the values

$b_0 = -15, b_1 = 15$

Case-4:

The real and imaginary part of solution (3.3.10) are shown in fig-8(a) and fig-8(b) which are represented the kinky rogue wave solution for the values $b_0 = -15,\ b_1 = 15$. In these figure it can be seen that lower density plot appears in the 3D plot.

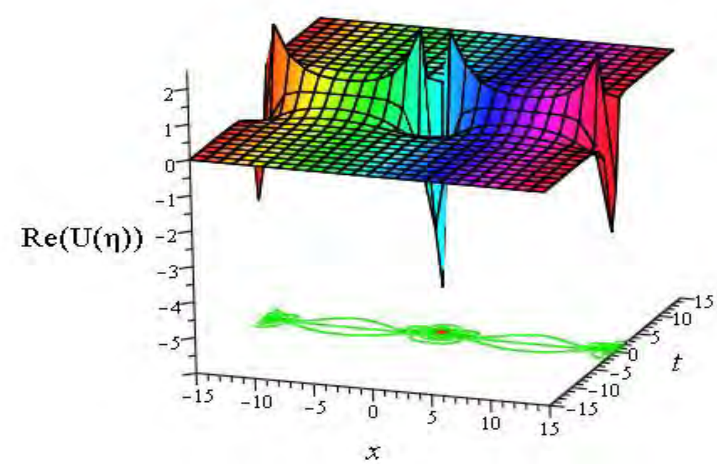

Figure-8(a): The real part of the solution of (3.3.10) for the values

$b_0 = -15, b_1 = 15$

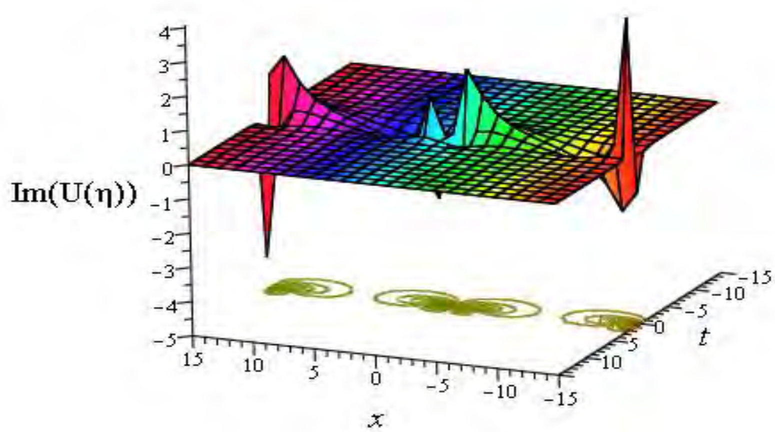

Figure-8(b): The imaginary part of the solution of (3.3.10) for the values

$b_0 = -15, b_1 = 15$

Chapter 4

Conclusion and Future Work

4.1 Conclusion

In this thesis, we obtain exact traveling wave solutions for the Oskolkov and Fisher's equation by using the Exp-function method. The obtained solutions show that the Exp-function method is promising and powerful mathematical tool for solving nonlinear evolution equations. It is hoped that the method can be effectively used for further studies to many nonlinear evolution equations. In this thesis, the principle exertion is to discover, test and break down the new voyaging wave arrangements and physical properties of the nonlinear Oskolkov equation by applying dependable scientific procedures. The Exp-function scheme performance a substantial trick to find traveling wave solutions in-terms of exponential, trigonometric and hyperbolic function from which we could build specially kinky periodic wave, rogue wave solution, solitary and periodic wave solutions. This technique offers arrangements with free parameters that may be essential to clarify some unpredictable nonlinear physical marvels. We give a very simple and straightforward method called Exp-function method for nonlinear wave equations. The used method has some pronounced merits:

(1) The method leads to both the generalized soliton solutions and periodic solutions;

(2) The solution procedure, by the help of computer program, is of utter simplicity, and can be easily extended to all kinds of non-linear equations.

4.2 Future Work

In future, we will solve more non-linear evaluation equations such as beam equation, Calogero-Bogoyavlenskii-Schiff (CBS) equations, Kadomtsev–Petviashvili (KP) equation, Burger equation by Exp-function method.

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