

**A NEW QUADRATIC FORMULATION AND ADVANCED  
TRANSFORMATION TECHNIQUE TO SOLVE MULTI  
OBJECTIVE QUADRATIC PROGRAMMING PROBLEM**

by

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IN  
MATHEMATICS



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The thesis titled  
**A NEW QUADRATIC FORMULATION AND ADVANCED  
TRANSFORMATION TECHNIQUE TO SOLVE MULTI OBJECTIVE  
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Submitted by

**Margia Yesmin**


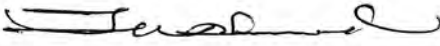
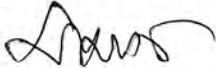


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## DECLARATION

I hereby declare that this thesis work submitted to the Department of Mathematics, Bangladesh University of Engineering and Technology (BUET) has not been submitted elsewhere (Universities or Institutions) for the award of any other degree or diploma.



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Margia Yesmin

## **DEDICATION**

This work is dedicated to my beloved Parents.

## **Acknowledgement**

All praises and gratitude belong to Allah (Most Gracious and Most Merciful) for making things and situations congenial and favorable for me to complete this thesis.

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## **Abstract**

Multi-objective optimization problem (MOOP) is an important class of optimization problem that ensures users to model a large variety of real world applications. A survey of quadratic programming is presented in corporate and economic planning formulation. An Advanced transformation technique has been proposed to solve MOOP. The research aims to find maximum profits diagnostics of a Textile industry by using proposed technique. Chandra Sen's technique, statistical average technique and modified statistical average technique are used to solve multi-objective quadratic programming problem (MOQPP) and Pareto optimal solution analysis has been done. Applying on different types of examples, the result indicates that the proposed method gives better solution than other methods and it is less time consuming. This is easy to calculate which ensures more accuracy. Physical presentation and data analysis represent the worth of the method more compactly.

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## **NOMENCLATURE**

OR	Operations Research
MOO	Multi Objective Optimization
LP	Linear Programming
NLP	Non Linear Programming
LPP	Linear Programming Problem
QPP	Quadratic Programming Problem
MOLPP	Multi Objective Linear Programming Problem
MOQPP	Multi Objective Quadratic Programming Problem
BV	Basic Variables
NBV	Non Basic variable
DM	Decision Maker
DMU	Decision Making Unit
MOMIP	Multi Objective Mixed Integer Programming
MOIP	Multi Objective Integer Programming
SAT	Statistical Average Technique
AMT	Arithmetic Mean Technique
GMT	Geometric Mean Technique
HMT	Harmonic Mean Technique
MSAT	Modified Statistical Average Technique
MAMT	Modified Arithmetic Mean Technique
MGMT	Modified Geometric Mean Technique
MHMT	Modified Harmonic Mean Technique

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# CHAPTER- 1

## Introduction

### 1.1 Preliminary

An exciting area of applied mathematics called Operations Research (OR) combines mathematics, statistics, computer science, physics, engineering, economics, and social sciences to solve real-world business problems. Numerous companies in industry require Operations Research professionals to apply mathematical techniques to a wide range of challenging questions.

Operations Research can be defined as the science of decision-making. It has been successful in providing a systematic and scientific approach to all kinds of government, military, manufacturing, and service operations. Operations Research is a splendid area for graduates of mathematics to use their knowledge and skills in creative ways to solve complex problems and have an impact on critical decisions.

There are many mathematical techniques that were developed specifically for OR applications. These techniques arise from basic mathematical ideas and became major areas of expertise for industrial operations.

One important area of such techniques is Optimization. The process of optimizing systemically or simultaneously a collection of objective function is called Multi-Objective Optimization (MOO). Many problems in industry require finding the maximum or minimum of an objective function of a set of decision variables, subject to a set of constraints on those variables. Typical objectives are maximum profit, minimum cost, or minimum delay. Frequently there are many decision variables and the solution is not obvious. Techniques of mathematical programming for optimization include linear programming (optimization where both the objective function and constraints depend linearly on the decision variables), non-linear programming (non-linear objective function or constraints), integer programming (decision variables restricted to integer solutions), stochastic programming (uncertainty in model parameter values) and dynamic programming (stage-wise, nested, and periodic decision-making).The hypothesis of this research is that the study and evaluation of

fundamental MOO methods has been largely deficient; critical information concerning the significance of method parameters, the effect of formulation variation and guidelines for effective use of methods in general has been unavailable and unpublished. The primary goals of this work are to scrutinize, enhance and create MOO methods and methodologies to provide new understanding and to discover new aspects of solutions. We increase the often unrealized effectiveness of common MOO methods. We investigate the physical and mathematical significance of method parameters.

With MOO, an improvement in one objective often results in detriment to another. Consequently, the idea of a solution is less straightforward than it is with single objective optimization. The predominant solution concept is Pareto Optimality and a point is Pareto optimal if and only if it is impossible to move from that point and reduce at least one objective function without increasing or affecting any other function. A Pareto optimality analysis has been discussed here.

In real life problems, many different situations lead to nonlinear formulation of constraints and objective functions. The importance of nonlinear programming application is growing due to rapidly increasing sophistications of managers and operation researchers in implementing decision oriented mathematical models. Among all nonlinear programming problems, the quadratic formulation is the easiest to solve when one deals with a QP involving linear inequality constraints. In many situations, decision makers want to optimize several different objective functions at the same time under same constraints. This leads to Multi-Objective concept. The MOQPP has been greatly received by user to solve different types of real life problems.

## **1.2 Operation Research and Its Practical Applications**

Though the framework of many processes and some aspects of their running may be extremely differentiated in the elementary operations composing them, they frequently reveal such similarities of behavior as to justify specialized inquiries and analyze. These analyses aim at detecting typical problems with a similar structure and, if possible, at

systemizing approaches to them. This has hastened the development of model or techniques peculiar to certain classes of problems. Some of these techniques were taken from various fields of science, while others are quite new and typical of operations research, and are therefore, susceptible of further developments and improvements. We do not pretend to give here an exhaustive and systematic documentation of them but giving a short survey of the most familiar techniques and of the characteristic problems connected with them.

These techniques are the followings:

- Mathematical programming (linear, nonlinear, dynamic and combinatorial programming)
- Queuing Theory
- Monte Carlo Method
- Theory of Games
- Replacement Theory

### **1.3 Scope and Specific Objectives**

The aim of this research is to analyze several solution techniques to solve multi objective optimization problems. Quadratic objective functions are treated specially. To convert multi-objective function into single objective function a popular technique named Chandra Sen's technique has been discussed. Different types of statistical analysis such as arithmetic, geometric, harmonic methods are presented to solve MOQPP. Also modified statistical average techniques are discussed.

After analyzing different techniques, a new technique named Advanced transformation technique has been proposed which reveal the optimal solutions more accurately. Data collected from real world problems.

The major objectives of this study are-

- To ensure maximum profit of a textile company oriented with buyer's requirement

- A Multi-Objective Quadratic Programming Problem (MOQPP) has been formulated from real industrial data to change the problem into mathematical formula
- Solved the model by different techniques such as Graphical method, AMPL software, Chandra Sen's technique, Statistical (arithmetic, geometric, harmonic) average technique and modified statistical average techniques
- Compare the solutions with Pareto optimality analysis
- Proposed new technique titled 'Advanced Transformation technique' to solve MOQPP
- Different types of data analysis have been done to present the effectiveness of the proposed technique and also expressed the limitations

After comparing, the results show that the new proposed advanced transformation technique is better than others. This research will present a survey of Quadratic Programming Problem in corporate and economic planning formulation. The aim is to find maximum profits diagnostics of a Textile industry by using Quadratic formulation and its solutions procedures.

#### **1.4 Research Background**

Multi-objective optimization (MOO) is an effective technique for studying optimal trade off solutions that balance several criteria. The fundamentals and applications of MOO have been already explored in great detail [1]. The main limitation of MOO is that its computational burden grows in size with the number of objectives. Among all nonlinear programming problems, the quadratic formulation is the easiest to solve when one deals with a QP involving linear inequality constraints. Various types of solution procedure have been already developed for solving MOO problems [2]. Some of them deal with theory and some of them concern with solution methods and applications.

To solve multi-objective linear programming problems, various types of methods have been proposed by various scholars. Such as -Mean and median method by Sulaiman and Sadiq



[3], Optimal transformation technique by Sulaiman and Ameen [4], Harmonic mean by Sulaiman and Mustafa [5], New statistical average method by Nahar and Alim [6].

On the other hand, Linear fractional programming problem has been solved by different researchers by different techniques. For example- A new procedure proposed by Tantawy [7] and by Guzel [8], An improved method by Mehdi, et al. [9], Arithmetic average technique by Sulaiman, et al. [10], A new approach presented by Akter and Modi [11], New geometric average technique proposed by Nahar and Alim [12].

Many research scholars have solved multi objective quadratic programming problem by applying several methods. We include some of them. Optimal cutting plane procedure [13] and Arithmetic average transformation technique [14] had been used by Sulaiman and Rahim. Optimal average maximum- minimum technique and Optimal geometric average technique had been used by Sulaiman & Nawkhass [15] and by Sulaiman et al. [16] gradually to solve multi objective quadratic programming problem.

In multi-objective programming problem, it is difficult to find an optimal solution to achieve the extreme value of every objective function, so that the decision maker is exploring for the compromise solution. Based on this idea, the concepts of Pareto optimal solution and weakly Pareto optimal solution are introduced into multi-objective programming problem [17].

Liu and Peng analyzed the relations among absolutely optimal solutions, effective solutions and weakly effective solutions of multi-objective programming problem [18]. Caramin and Dell'olmo describes Scalarization techniques,  $\epsilon$ -constraints methods, Goal problem, Multilevel programming to solve Multi-objective optimization problem (MOOP ) [19]. A new approach to solve MOOP providing a rapid solution for Pareto set if the objective function involved quadratic are described by Britto, et al. [20]. Zhang and Zuo then analyze the Pareto solution for convex MOLPP [21].

The larger the size of the problem, the greater the number of inefficient solution generated and thus the slower the convergence of the algorithm. To overcome this problem, we propose an advanced transformation technique. We test the capabilities of this proposed technique drawing a comparison with other techniques.

In this paper, we focus our interest on multi-objective quadratic programming problem (MOQPP) where several quadratic objectives are to be optimized subject to a set of linear constraints and nonnegative integer variables. The optimization software package, namely AMPL has been employed in the computation.

Multi-objective linear programming problem (MOLPP) has been solved by many research scholars. Hamad-Amin used arithmetic average technique to solve it [22]. Yesmin and Alim suggested a modified harmonic average technique to get more accurate solution by solving MOQPP [23].

This study has presented MOQPP and proposed an advanced transformation technique to solve it. The result is compared with Pareto optimality and different techniques such as statistical average techniques and modified statistical average techniques. The comparison table shows the effectiveness of the proposed method. The proposed technique is more easy to realize and less time consuming. No matter how complex the problem, this proposed method can be applied. Physical interpretations have been presented and data analysis have been discussed for more convenience.

## **1.5 Outlines of The Thesis**

For the sake of self-contain of the thesis, firstly the multi-objective optimization programming problems are introduced briefly in **Chapter 1**. Also we have given some related definitions and important theorems for better understanding of the research.

The major objective of **Chapter 2** is to describe multi-objective optimization with its variation.

**In Chapter 3**, the usual Graphical method, Pareto Optimality, the computer oriented algorithm AMPL and Chandra Sen's technique, the statistical average technique (SAT) such as arithmetic mean, geometric mean and harmonic mean, the modified statistical average technique (MSAT) for solving linear/nonlinear programming problems are discussed briefly.

The proposed Advanced transformation technique to solve multi objective optimization problem has been presented in **Chapter 4**.

**In Chapter 5**, a numerical example of multi-objective quadratic programming problem (MOQPP) is discussed briefly. Also Data analysis and limitations of the proposed techniques are discussed.

**In Chapter 6**, Practical aspects from real life problem are analyzed for maximum profit of a textile industry in Bangladesh.

The research has been concluded by proper discussion according to the solved outcomes.

## **1.6 Expected Outcomes**

Textile product's quantities and respective prices are formulated as a quadratic function. Individual objective function will be presented for individual product. For different types of products, it forms multi objective function. With concern of company's limitations and resources, it forms constraints. After that the multi-objective quadratic programming problem (MOQPP) will be formed.

It will be seen that optimization results using new proposed technique will be better than that of other techniques. From the analysis, it will be found that MOQPP will be best optimized by using new proposed advanced transformation technique. The conclusion will be focus on the maximum profit of the industry which ensures by using new proposed technique.

MOQPP has attracted the interest of many researchers due to its application in many important fields such as production planning, financial and corporate planning, portfolio selection, monopolist's profit maximization, inequality constrained least squares estimation, spatial equilibrium analysis, goal programming, optimal decision rules etc.

### **✓ Organizational outcomes expected**

The results may have some influences on future production decision and it illustrates an application of quadratic programming since the profit to be maximized may be written as quadratic function of product quantities and prices. Moreover, since the companies are already using software to maximize their profit, this conception may help them to develop the algorithm of the software.

### **✓ National impacts expected**

The importance of the textile industry in the economy of Bangladesh is very high. Bangladesh is world's 2<sup>nd</sup> biggest apparel exporter. Increasing profit in textile industry has great effect in our national economic balance. This research focuses on the maximization of profit of any textile company which adds a new dimension in our national economy.

## 1.7 Fundamental Concept

Most of the fundamental theory concerning multi-objective optimization (MOO) veers from the more familiar paradigms of single objective optimization. This chapter addresses basic concepts that are necessary for method analysis and development.

### Definitions

MOO originally grew out of three areas: economic equilibrium and welfare theories, game theory and pure mathematics. Consequently, many terms and fundamental ideas stem from these fields. For the sake of brevity, only necessary terms are defined below. Many of these terms have multiple definitions in the literature stemming from the differences between engineering and economic jargon and in such cases the most common and most appropriate definitions are used.

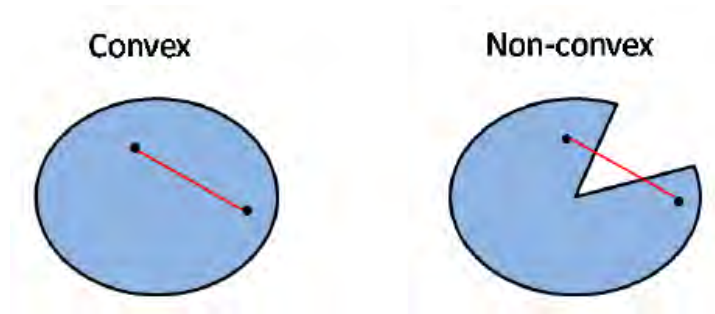
#### Convex Set:

A convex set is a set of elements from a vector space such that all the points on the straight line between any two points of the set are also contained in the set. If  $a$  and  $b$  are points in a vector space the points on the straight line between  $a$  and  $b$  are given by

$$\mathbf{x} = \mathbf{a}\lambda + (\mathbf{1} - \lambda)\mathbf{b} \quad \text{for all } \lambda \text{ from } 0 \text{ to } 1.$$

A set  $S$  is convex if there are no points  $a$  and  $b$  in  $S$  such that there is a point on the line between  $a$  and  $b$  that does not belong to  $S$ . The point of this restatement is to include the empty set within the definition of convexity. The definition also includes singleton sets where  $a$  and  $b$  have to be the same point and thus the line between  $a$  and  $b$  is the same point.

With the inclusion of the empty set as a convex set then it is true that: The intersection of any two convex sets is a convex set.

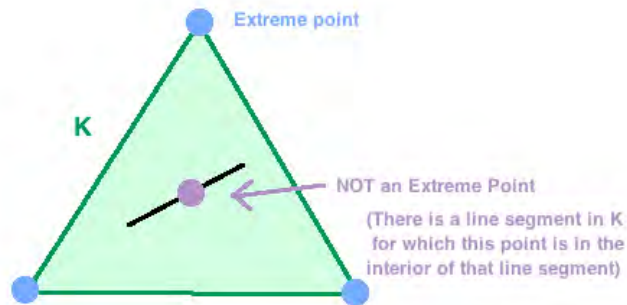


**Figure 1.1:** Convex and Non-convex set

Extreme point:

An extreme point is a point in a convex set which does not lie in any open line segment joining two points in the set.

Let  $C \subseteq \mathbb{R}^n$  be a non-empty closed convex set. Then  $\bar{x}$  is an extreme point of  $C$  if there are no two points  $x_1, x_2 \in C$  and  $\lambda \in (0,1)$  such that  $\bar{x} = \lambda x_1 + (1-\lambda) x_2$ .



**Figure 1.2:** Extreme point

Saddle-point:

Saddle point is a point at which a function of two variables has partial derivatives equal to zero but at which the function has neither a maximum nor a minimum value.

Let  $\bar{x} \in X$  and  $\bar{\delta} \geq 0$ , then  $(\bar{x}, \bar{\delta})$  is a saddle point of  $l(x, \delta)$  if

$$L(\bar{x}, \delta) \leq L(\bar{x}, \bar{\delta}) \leq L(X, \bar{\delta}) \text{ for all } X, \delta \geq 0.$$

Decision variables:

A decision variable is a quantity that the decision-maker controls.

In the context of optimization, decision variables are unknown and controllable parameters of the system which finding their value is the purpose of problem solving effort. The value of decision values determines the system objective function value.

Consider, a linear programming model

$$\max Z = 3x + 4y$$

Here,  $x$  and  $y$  represents decision variables.

Objective function:

Objective function defines the criterion for evaluating the solution. It is a mathematical function of the decision variables that converts a solution in to a numerical evaluation of that solution. For example, the objective function may measure the profit or cost that occurs as a function of the amounts of various products produced. The objective function also specifies a direction of optimization, either to maximize or minimize. An optimal solution for the model is the best solution as measured by that criterion.

The objective function indicates how much each variable contributes to the value to be optimized in the problem. The objective function takes the following general form:

$$\text{maximize or minimize } Z = \sum_{i=1}^n c_i X_i$$

where

$c_i$  = the objective function coefficient corresponding to the  $i^{th}$  variable and  
 $X_i$  = the  $i^{th}$  decision variable

The coefficients of the objective function indicate the contribution to the value of the objective function of one unit of the corresponding variable.

Constraints:

Constraints are a set of functional equalities or inequalities that represent physical, economic, technological, legal, ethical, or other restrictions on what numerical values can be assigned to the decision variables. For example, constraints might ensure that no more input is used than is available. Constraints can be definitional, defining the number of employees at the start of a period  $t+1$  as equal to the number of employees at the start of period  $t$ , plus those added during period  $t$  minus those leaving the organization during period  $t$ . In constrained optimization models we find values for the decision variables that maximize or minimize the objective function and satisfy all constraints.

Linear Programming Problem (LPP):

Linear programming or linear optimization is a process which takes into consideration certain linear relationships to obtain the best possible solution to a mathematical model.

linear programming, mathematical modeling technique in which a linear function is maximized or minimized when subjected to various constraints. This technique has been useful for guiding quantitative decisions in business planning in industrial engineering.

The solution of a linear programming problem reduces to finding the optimum value (largest or smallest, depending on the problem) of the linear expression called the objective function

$$f = c_1x_1 + \dots + c_nx_n$$

subject to a set of constraints expressed as inequalities:

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &\leq b_1 \\ &\vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &\leq b_m \text{ with } \forall x_i \geq 0. \end{aligned}$$

The  $a$ 's,  $b$ 's, and  $c$ 's are constants determined by the capacities, needs, costs, profits, and other requirements and restrictions of the problem.

Linear programming models have three major components: decision variables, objective function, and constraints.

Solution of LPP:

Any set of variables  $\bar{x} = (x_1, x_2, \dots, x_n)$  is called solution of LPP if it satisfies constraints only.

If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible set, S, associated with the problem.

Basic solution and basic variable:

A solution is a basic solution if it is obtained by setting some variables equal to zero and then solving for remaining variables with the determinant of coefficients of these variables is non zero. Usually we call those variables as basic variables (BV) which are used to get identity matrix in solving LPP using simplex method. Of course, the different choices of non-basic variables (NBV) will lead to different basic solutions.

To illustrate, we find all the basic solutions to the following system of two equations in 3 variables:

$$x_1 + x_2 = 3$$

$$-x_2 + x_3 = -1$$

We begin by choosing a set of  $3-2=1$  (3 variables, 2 equations) non basic variables. For example, if  $NBV=\{x_3\}$ , then  $BV=\{x_1, x_2\}$ . We obtain the values of the basic variables by setting  $x_3=0$  and solving

$$x_1 + x_2 = 3$$

$$-x_2 = -1$$

We find that  $x_1 = 2, x_2 = 1$ . Thus  $x_1 = 2, x_2 = 1, x_3 = 0$  is a basic solution to the equation.

However, if we choose  $NBV=\{x_1\}$  and  $BV=\{x_2, x_3\}$ , we obtain the basic solution  $x_1 = 0, x_2 = 3, x_3 = 2$ . If we choose  $NBV=\{x_2\}$ , we obtain the basic solution  $x_1 = 3, x_2 = 0, x_3 = -1$ .

Feasible solution:



Feasible solution is a set of values for the decision variables that satisfies all of the constraints in an optimization problem. It satisfies the entire restrictions foreseen in the optimization problem. When at least one restriction of the problem is not met, the solution is considered not feasible.

Any set of variables  $\bar{x} = (x_1, x_2, \dots, x_n)$  is called feasible solution of LPP if satisfies constraints and must be non-negative.

Basic feasible solution:

A basic solution to a system of  $m$  linear equations in  $n$  unknowns ( $n \geq m$ ) is obtained by setting  $n-m$  variables to 0 and solving the resulting system to get the values of the other  $m$  variables.

The variables set to 0 are called non-basic, the variables obtained by solving the system are called basic.

A basic solution is called feasible if all its basic variables are non-negative.

Example:

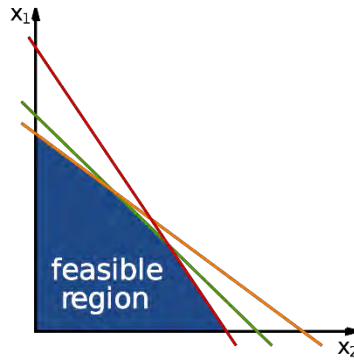
$$x + y + u = 4$$

$$x + 3y + v = 6$$

Then  $(0,0,4,6)$  is basic feasible solution.  $x, y$  are non-basic;  $u, v$  are basic feasible solution.

Feasible region:

A feasible region, feasible set, search space, or solution space is the set of all possible points (sets of values of the choice variables) of an optimization problem that satisfy the problem's constraints, potentially including inequalities, equalities, and integer constraints.



**Figure 1.3:** Feasible region

Unbounded solution:

The solutions of a linear programming problem which is feasible can be classified as a bounded solution and an unbounded solution.

The unbounded solution is a situation when the optimum feasible solution cannot be determined, instead there are infinite many solutions. It is not possible to solve the problem in which this situation occurs. If the value of objective function can be increased or decreased infinitely then such a solution is called as unbounded solution.

Degenerate solution:

A Linear Programming is degenerate if in a basic feasible solution, one of the basic variables takes on a zero value. Degeneracy is caused by redundant constraints.

Non- degenerate basic feasible solution:

A non-degenerate basic feasible solution is a basic feasible solution with exactly all basic variables are positive.

Optimal solution:

The term optimal solution refers to the best solution to solve a problem or achieve its aims. A solution to an optimization problem which minimizes (or maximizes) the objective function. The values of the decision variables that either maximize or minimize the objective function.

Pareto optimal solution:

A solution is called Pareto optimal, if none of the objective functions can be improved without degrading some of the other objective values.

A vector  $x^* \in W$  is said to be Pareto optimal for a multi-objective problem if all other vectors  $x \in W$  have a higher value for at least one of the objective function  $f_i$ , with  $i = 1, \dots, n$ , or have the same value for all the objective functions.

General Multi-Objective Optimization Problem:

A general multi-objective optimization problem is defined as the minimization (or maximization) of the objective function set  $F(x) = (f_1(x), f_2(x), \dots, f_k(x))$  subject to inequality constraints  $g_i(x) \leq 0, i = \{1, 2, \dots, m\}$  and equality constraints  $h_j(x) = 0, j = \{1, \dots, p\}$ . The solution of a multi-objective problem minimizes (or maximizes) the components of a vector  $F(x)$  where  $x$  is a  $n$ -dimensional decision variable vector  $x = (x_1, \dots, x_n)$  from some universe  $\Omega$ .

It is noted that  $g_i(x) \leq 0$  and  $h_j(x) = 0$  represent constraints that must be fulfilled while minimizing (or maximizing)  $F(x)$  and  $\Omega$  contains all possible  $x$  that can be used to satisfy an evaluation of  $F(x)$ . Thus, a multi-objective problem consists of  $k$  objectives reflected in the  $k$  objective functions,  $m + p$  constraints on the objective functions and  $n$  decision variables. The  $k$  objective functions may be linear or nonlinear and continuous or discrete in nature. Of course, the vector of decision variables  $x_i$  can also be continuous or discrete.

**Existence of Optimal Solutions**

Most of the topics of this part are concerned with

- existence of optimal solutions
- characterization of optimal solutions and
- algorithms for computing optimal solutions.

To illustrate the questions arising in the first topic, consider the following optimization problems:

$$\begin{aligned} \min \quad & \frac{1+x}{2x} \\ \text{s. t.} \quad & x \geq 1 \end{aligned}$$

Here there is no optimal solution because the feasible region is unbounded.

$$\begin{aligned} \min \quad & \frac{1}{x} \\ \text{s. t.} \quad & 1 \leq x < 2 \end{aligned}$$

Here there is no optimal solution because the feasible region is not closed.

$$\begin{aligned} \min \quad & f(x) \\ \text{s. t.} \quad & 1 \leq x \leq 2 \end{aligned}$$

where  $f(x) = \begin{cases} \frac{1}{x}, & x < 2 \\ 1, & x = 2 \end{cases}$

Here there is no optimal solution because the function  $f(\cdot)$  is not sufficiently smooth.

### **Related Theorems**

#### Theorem 1:

If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible set,  $S$ , associated with the problem. Furthermore, if the objective function  $P$  is optimized at two adjacent vertices of  $S$ , then it is optimized at every point on the line segment joining these two vertices, in which case there are infinitely many solutions to the problem.

#### Theorem 2:

Suppose we are given a linear programming problem with a feasible set  $S$  and an objective function  $P = ax + by$ . Then, If  $S$  is bounded then  $P$  has both a maximum and minimum value on  $S$ . If  $S$  is unbounded and both  $a$  and  $b$  are nonnegative, then  $P$  has a minimum value on  $S$  provided that the constraints defining  $S$  include the inequalities  $x \geq 0$  and  $y \geq 0$ . If  $S$  is the empty set, then the linear programming problem has no solution; that is,  $P$  has neither a maximum nor a minimum value.

Theorem 3:

Weierstrass' Theorem for functions: Let  $f(x)$  be a continuous real-valued function on the compact nonempty set  $F \subset R^n$ . Then  $F$  contains a point that minimizes (maximizes)  $f(x)$  on the set  $F$ .

Proof: Since the set  $F$  is bounded,  $f(x)$  is bounded below on  $F$ . Since  $F \neq \emptyset$ , there exists  $v = \inf_{x \in F} f(x)$ . By definition, for any  $\varepsilon > 0$ , the set  $F_\varepsilon = \{x \in F: v \leq f(x) \leq v + \varepsilon\}$  is non-empty. Let  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ , and let  $x_k \in F_{\varepsilon_k}$ . Since  $F$  is bounded, there exists a subsequence of  $\{x_k\}$  converging to some  $\bar{x} \in F$ . By continuity of  $f(x)$ , we have  $f(\bar{x}) = \lim_{k \rightarrow \infty} f(x_k)$  and since

$v \leq f(x_k) \leq v + \varepsilon_k$ , it follows that  $f(\bar{x}) = \lim_{k \rightarrow \infty} f(x_k) = v$ .

Theorem 4:

The Fundamental Theorem of Linear Programming

Every LP has the following three properties:

- (i) If it has no optimal solution, then it is either infeasible or unbounded
- (ii) If it has a feasible solution, then it has a basic feasible solution
- (iii) If it is bounded, then it has an optimal basic feasible solution.

Proof: Part (i): Suppose an LP has no solution. This LP is either feasible or infeasible. Let us suppose it is feasible. In this case, the first phase of the two-phase simplex algorithm produces a basic feasible solution. Hence, the second phase of the two-phase simplex algorithm either discovers that the problem is unbounded or produces an optimal basic feasible solution. By assumption, the LP has no solution so it must be unbounded. Therefore, the LP is either infeasible or unbounded.

Part (ii): If an LP has a feasible solution, then the first phase of the two-phase simplex algorithm produces a basic feasible solution.

Part (iii): Suppose an LP is bounded. In particular, this implies that the LP is feasible, and, so by Part(ii), it has a basic feasible solution. The second phase of the two-phase simplex

algorithm either discovers that the problem is unbounded or produces an optimal basic feasible solution. Since the LP is bounded, the second phase produces an optimal basic feasible solution.

## CHAPTER- 2

### Multi-Objective Optimization

The optimal value or the best solution can be found through the optimization process. The optimization problems include looking for maximum or minimum value or using one objective or multi-objective. Problems that have more than one objective is referred to as multi-objective optimization (MOO). This type of problem is found in everyday life, such as mathematics, engineering, social studies, economics, agriculture, aviation, automotive, and many others.

#### 2.1 Multi-Objective Optimization

The foundation of mathematical programming relies on two major scientific works. John von Neumann and Oskar Morgenstern published Theory of Games and Economic Behavior in 1944. George B. Dantzig discovered in 1947 the simplex method. In the same year, John von Neumann developed the theory of duality. The origin of vector optimization goes back to Francis Y. Edgeworth (1881) and Vilfredo Pareto (1896). The two economists developed the theory of indifference curves and defined the basic concept of optimality in multi-objective optimization (MOO).

Multi-objective optimization is an area of multiple criteria decision making that is concerned with mathematical optimization problems involving more than one objective function to be optimized simultaneously.

Mathematically, Multi objective decision making problems can be expressed as:

$$\begin{aligned} & \text{Max /Min}[f_1(x), f_2(x), \dots, f_k(x)] \\ \text{Subject to} & \quad x \in X = \{x \mid g_h(x) : \{\geq, =, \leq\} 0, \quad h = 1, 2, \dots, m \} \end{aligned}$$

Where,  $f_j(x) = \text{Objective for maximization}, j \in J$

$f_i(x) = \text{Objective for minimization}, i \in I$

The problem consists of n decision variables, m constraints and k objectives.  $f_j(x), f_i(x)$  and  $g_h(x) \forall i, j, h$  might be linear or nonlinear.

The primary goals in multi-objective optimization problem solution are:

- To preserve non dominated points in the objective space and associated solution points in the decision space;
- To keep making algorithmic progress toward the Pareto front in the objective function space;
- To maintain diversity of points on the Pareto front and of Pareto optimal solutions (decision space);
- To provide the decision maker, the designer, with a large enough but limited number of Pareto points for selection.

## **2.2 Development of Multi-Objective Optimization**

One should also indicate the fast evolution of a MOO approach in the mid-1980s with the help of evolutionary algorithm. An early attempt to use a genetic search algorithm to solve MOO problems was realized by Ito et al. (1983). Goldberg proposed Pareto-set fitness assignment to solve Schaffer's multi-objective problems, Schaffer and Grefenstette, 1985. In the same period, two books were devoted to the theory and techniques of MOO, such as Chankong and Haimes' book (1983) and that of Sawaragi et al. (1985). The fast expansion of this approach was stimulated by numerous real-world applications from science, technology, management, and finance. Rangaiah's book (2009) was the first publication on MOO with a focus on chemical engineering. The applications in this area are notably in chemical, mineral processing, oil and gas, petroleum, pharmaceutical industries, etc. Lai and Huang (1994) extended the MOO approach to fuzzy decision-making problems.

## **2.3 Classification of Multi-Objective Optimization**

MOO techniques can be ordered in different ways. The classification retained by Miettinen (1999) and Diwekar (2008) is based on two criteria, the number of generated Pareto solutions



and the decision-maker (DM)'s preferences. There are two groups of methods: A first group includes the generating methods and the other the preference-based methods.

- ✓ In the generating methods, the Pareto-optimal solutions are obtained without any action of the DM during the determination process. The role of the DM is the selection of all the generated optimal solutions.
- ✓ On the contrary, preference-based methods integrate DM's preferences at some stage of the resolution process.

Generating methods include three following subgroups such as

- (i) no-preference methods,
- (ii) a posteriori methods using a scalarization transform and
- (iii) a posterior methods using a multi-objective approach.

The no-preference methods include in particular the method of global criterion, the multi-objective proximal bundle method. Posteriori methods using a scalarization transformation include the constraint method and weighting methods. A posteriori methods aim at producing all the Pareto optimal solutions or a representative subset of the Pareto optimal solutions. Most a posteriori methods fall into either one of the following two classes: mathematical programming-based a posteriori methods, where an algorithm is repeated and each run of the algorithm produces one Pareto optimal solution, and evolutionary algorithms where one run of the algorithm produces a set of Pareto optimal solutions.

## **2.4 Different Types of MOO**

### **2.4.1 Multi-objective linear programming problem**

In Multi-Objective Linear Programming (MOLP) we are concerned with a continuum of alternatives demarcated by a finite number of linear constraints in a finite-dimensional space. Furthermore, there is a finite number of linear objective functions, and a single decision maker or a decision making body.

Linear programs are constrained optimization models that satisfy three requirements.

1. The decision variables must be continuous; they can take on any value within some restricted range.
2. The objective function must be a linear function.
3. The left-hand sides of the constraints must be linear functions.

Mathematically, the Multi-Objective Linear Programming Problem (MOLPP) can be defined as:

$$\text{Max } f_i = C_i x + \alpha_i, \quad i = 1, \dots, r$$

$$\text{Min } f_i = C_i x + \alpha_i, \quad i = r + 1, \dots, s$$

Subject to

$$Ax \begin{bmatrix} \leq \\ \equiv \\ \geq \end{bmatrix} B$$

$$x \geq 0$$

where  $x$  is an  $n$ -dimensional vector of decision variables  $c$  is  $n$ -dimensional vector of constants,  $B$  is  $m$ -dimensional vector of constants,  $r$  is the number of objective function to be maximized,  $s$  the number of objective function to maximized plus minimized,  $(s-r)$  is the number of objective that is to be minimized,  $A$  is a  $(m \times n)$  matrix of coefficients all vectors are assumed to be column vectors unless transposed,  $\alpha_i (i = 1, \dots, s)$  are scalar constants,  $C_i x + \alpha_i, \quad i = 1, \dots, s$  are linear factors for all feasible solutions. Most linear programs require that all decision variables be nonnegative.

Linear programs make the following implicit assumptions.

1. **Proportionality:** With linear programs, we assume that the contribution of individual variables in the objective function and constraints is proportional to their value. That is, if we double the value of a variable, we double the contribution of that variable to the objective function and each constraint in which the variable appears. The contribution per unit of the variable is constant. For example, suppose the variable  $x_j$  is the number of units of product  $j$  produced and  $c_j$  is the cost per unit to produce product  $j$ . If doubling the amount of product

j produced doubles its cost, the per unit cost is constant and the proportionality assumption is satisfied.

2. **Additivity:** Additivity means that the total value of the objective function and each constraint function is obtained by adding up the individual contributions from each variable.

3. **Divisibility:** The decision variables are allowed to take on any real numerical values within some range specified by the constraints. That is, the variables are not restricted to integer values. When fractional values do not make a sensible solution, such as the number of flights an airline should have each day between two cities, the problem should be formulated and solved as an integer program.

4. **Certainty:** We assume that the parameter values in the model are known with certainty or are at least treated that way. The optimal solution obtained is optimal for the specific problem formulated. If the parameter values are wrong, then the resulting solution is of little value. In practice, the assumptions of proportionality and additivity need the greatest care and are most likely to be violated by the modeler. With experience, we recognize when integer solutions are needed and the variables must be modeled explicitly

#### 2.4.2 Multi-objective quadratic programming problem

In a non-linear programming problem, either the objective function is non-linear, or one or more constraints have non-linear relationship or both.

Mathematically the multi-objective quadratic programming problem (MOQPP) can be stated as:

$$\begin{aligned} \text{Max } F_r &= \frac{1}{2} x^T P_r x + C_r^T x \\ \text{Min } F_s &= \frac{1}{2} x^T P_s x + C_s^T x \end{aligned} \quad (2.4.2)$$

$$\begin{aligned} \text{Subject to} \quad & Ax \begin{bmatrix} \leq \\ = \\ \geq \end{bmatrix} b \\ & x \geq 0 \end{aligned}$$

Where  $r$  is the number of objective function to be maximized,  $s$  is the number of objective function to be maximized and minimized and  $(s-r)$  is the number of objective function to be minimized. Here  $P$  is a  $(n \times n)$  symmetric matrix of coefficients,  $x$  is an  $n$ - dimensional vector of decision variables,  $C$  is the  $n$ -dimensional vector of constants,  $b$  is  $m$ -dimensional vector of constants.  $A$  is  $(m \times n)$  matrix of coefficients. All vectors are assumed to be column vectors unless transposed.

Classification of quadratic programming problem:

The main basis for classification of quadratic problems comes from the nature of the quadratic matrix  $P$  (from equation [2.4.2]). Based upon this, quadratic problems can be classified as:

1. Bilinear problems: Often, the matrix  $P$  is such that there exists two sub-vectors of distinct variables  $y$  and  $z$  of  $x$  such that the problem is linear when one of these vectors is fixed. Such problems are termed bilinear problems.
2. Concave quadratic problems: When the matrix  $P$  is negative semidefinite (all its eigenvalues are non-positive) the problems reduces to one of concave minimization.
3. Indefinite quadratic problems: These problems arise when the matrix  $P$  has both positive and negative eigenvalues. From the view of point of solution, this class of problems is the most intractable among these subclasses and consequently there are too many methods for solving these problems.

### 2.4.3 Multi-objective integer programming problem

Consider the following multi-objective problem (P):

$$\begin{aligned}
 & \max z_1 = f_1 \\
 & \dots\dots \\
 & \max z_k = f_k \\
 & s. t. \quad x \in X
 \end{aligned}
 \tag{2.4.3}$$

Where  $X \subset R^n$  denotes the non-convex set of feasible solutions defined by a set of functional constraints,  $x \geq 0$  and  $x_j$  integer  $j \in J \subseteq \{1, \dots, n\}$ . It is assumed that  $X$  is compact (closed and bounded) and non-empty. The integer variables can either be binary or take on general integer values. (P) is a multi-objective integer programming (MOIP) problem if all variables are integer.

#### 2.4.4 Multi-objective mixed integer programming problem

Equation (2.4.3) denotes a multi-objective mixed integer programming (MOMIP) problem if all variables are not integer. Models with linear constraints and linear objective functions have been more often considered than nonlinear cases.

In linear multi-objective integer or mixed-integer problems (MOILP/MOMILP), the functional constraints can be defined as  $Ax \leq b$  and the objective functions  $f_i(x) = c^i(x)$ ,  $i = 1, \dots, k$  where  $A$  is  $m \times n$  matrix,  $b$  is a  $m$ -dimensional column vector and  $c^i(x)$ ,  $i = 1, \dots, k$  are  $n$ -dimensional row vectors.

#### 2.4.5 Convex multi-objective programming problem

A convex multi-objective optimization problem can be stated as follows:

$$\text{Minimize } [f_1(x), f_2(x) \dots f_m(x)]$$

$$\text{Subject to } g_j(x) \leq 0; \quad j = 1, 2, \dots, p$$

where  $x$  is an  $n$ -dimensional vector of decision variables,  $f_1(x), f_2(x), \dots, f_m(x)$  are convex functions defined on  $X$ , and  $X = \{x \mid g_j(x) \leq 0; j = 1, 2, \dots, p\}$  is convex set. The given Problem is called convex multi-objective programming problem.

#### 2.4.6 Fuzzy multi-objective optimization

Often, constraint limits and objectives are not in absolute terms; they may be fuzzy rather than crisp and such constraints and objectives may be modeled mathematically using fuzzy theory. Fuzzy theory is a field of mathematics that enables one to model systems that involve non-quantitative human reason, perception and interpretation. With fuzzy optimization,

constraints and objectives are treated equivalently, a condition that lends itself to use with multiple objectives. Each constraint and objective is modeled with what is called a membership function which is developed based on the insight and experience of the user. The literature on general fuzzy optimization and fuzzy MOO is incorporated into weighted methods, goal programming and global criterion methods.

#### **2.4.7 Genetic multi-objective optimization**

There are many approaches to solve MOO problems by using global methods. However, genetic algorithms are by far the most common global optimization technique that is used with multi objective problems. Consequently, the amount of literature concerning genetic algorithm for MOO and genetic algorithm is vast. The algorithm is loosely based on Darwin's theory of natural selection, although deficiencies in the comparison between biology and optimization algorithms are highlighted in the review.

In addition, the basic components of general genetic algorithms are outlined. Genetic algorithm work with a population of points and consequently can converge on the Pareto optimal set as a whole rather than finding one point at a time. In this respect, they are most appropriate as a method for a posteriori articulation of preferences.

### **2.5 Formulation of Multi-Objective Quadratic Programming Problem**

Multi-objective quadratic programming problem (MOQPP) involves a decision-making unit (DMU) with multiple objective functions. The objective functions of the DMU are quadratic in nature and the constraints are linear function. Mathematically the multi-objective quadratic programming problem (MOQPP) can be stated as:

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$$\text{Max } F_1 = \frac{1}{2} x^T P_1 x + C_1^T x$$

$$\text{Max } F_2 = \frac{1}{2} x^T P_2 x + C_2^T x$$

.....

.....

$$\text{Max } F_r = \frac{1}{2} x^T P_r x + C_r^T x$$

$$\text{Max } F_{r+1} = \frac{1}{2} x^T P_{r+1} x + C_{r+1}^T x$$

$$\text{Min } F_{r+2} = \frac{1}{2} x^T P_{r+2} x + C_{r+2}^T x$$

.....

.....

$$\text{Min } F_s = \frac{1}{2} x^T P_s x + C_s^T x$$

subject to

$$Ax \begin{bmatrix} \leq \\ = \\ \geq \end{bmatrix} b$$

$$x \geq 0$$

Where r is the number of objective function to be maximized, s is the number of objective function to be maximized and minimized and (s-r) is the number of objective function to be minimized. Here P is a  $(n \times n)$  symmetric matrix of co-efficient, x is an n-dimensional vector of decision variables, C is the n-dimensional vector of constants, b is m-dimensional vector of constants. A is  $(m \times n)$  matrix of coefficients. All vectors are assumed to be column vectors unless transposed.

The procedure for mathematical formulation of quadratic programming problem consists of the following major steps:

**Step 1:** Identify the unknown variables to be determined (decision variables) and represent them in terms of an algebraic symbol.

**Step 2:** Formulate the other conditions of the problem such as resource limitations, market constraints and inter-relation between variables etc. as quadratic equations in terms of decision variables.

**Step 3:** Identify the objective or criterion and represent it as linear or quadratic function of the decision variables, which is to be maximized or minimized.

**Step 4:** Add the 'non-negativity' constraint from the consideration that negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraints and the non-negative constraints together form a quadratic programming problem.

The quadratic objective function may be convex - which makes the problem easy to solve or non-convex, which makes it very difficult to solve.

The best QPs have Hessians that are positive definite (in a minimization problem) or negative definite (in a maximization problem). A QP with a semi-definite Hessian is still convex. A QP with an indefinite Hessian has a "saddle" shape - a non-convex function. Its true minimum or maximum is not found in the "interior" of the function but on its boundaries with the constraints, where there may be many locally optimal points. Optimizing an indefinite quadratic function is a difficult global optimization problem and is outside the scope of most specialized quadratic solvers.

Note:  $x^T Px$  represents a quadratic form. This quadratic form is said to be positive-definite (negative definite) if  $x^T Px > 0 (< 0)$  for  $x \neq 0$  and positive-semi-definite (negative-semi-definite) if  $x^T Px \geq 0 (\leq 0)$  for all  $x$  such that there is one  $x \neq 0$  satisfying  $x^T Px = 0$ .

It can easily be shown that

1. If  $x^T Px$  is positive-semi-definite (negative-semi-definite) then it is convex (concave) in  $x$  over all of  $R^n$  and
2. If  $x^T Px$  is positive-definite (negative-definite) then it is strictly convex (strictly concave) in  $x$  over all of  $R^n$ .



These results help in determining whether the quadratic objective function is concave (convex) and the implication of the same on the sufficiency of the Karush-Khun-Tucker condition for constrained maxima (minima) of  $f(x)$ .

Constrained quadratic programming problems:

A special case of the NLP arises when the objective functional  $f$  is quadratic and the constraints  $h, g$  are linear in  $x \in R^n$ . Its general form is

$$\begin{aligned} \min f(x) &= \frac{1}{2}x^T Bx - x^T b \\ \text{over } x &\in R^n \\ \text{s.t. } A_1 x &= c \\ A_2 x &\leq d \end{aligned} \tag{2.5}$$

Where  $B \in R^{n \times n}$  is symmetric,  $A_1 \in R^{m \times n}$ ,  $A_2 \in R^{p \times n}$  and  $b \in R^n, c \in R^m, d \in R^p$ .

The above QP is known as Constrained quadratic programming problem.

Equality constrained quadratic programming:

If only equality constraints are imposed, the QP (2.5) reduces to

$$\begin{aligned} \min f(x) &= \frac{1}{2}x^T Bx - x^T b \\ \text{over } x &\in R^n \\ \text{s.t. } Ax &= c \end{aligned}$$

Where  $A \in R^{m \times n}, m \leq n$ . For the time being we assume that  $A$  has full row rank  $m$ . The KKT conditions for the solution  $x \in R^n$  of the QP give rise to the following linear system

$$\begin{pmatrix} B & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x^* \\ \gamma^* \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}$$

Where  $\gamma^* \in R^m$  is the associated Lagrange multiplier. We denote by  $Z \in R^{n \times (n-m)}$  the matrix whose column span  $\text{Ker } A$ , i.e.,  $AZ=0$ .



## **CHAPTER- 3**

### **Solution Procedure of MOQPP**

Quadratic programming is a special form of nonlinear programming which has special characteristics; that is, the objective function is in quadratic forms and constraint functions are linear form.

Although the quadratic programming is part of nonlinear programming, the completion is still adopting some linear programming problem solving methods such as graphical solution, graphical analysis, simplex method, Wolfe method etc.

The linear or quadratic programming problem with two variables can be solved graphically. The graphical method of solving linear or quadratic programming problem is of limited application in the business problems as the number of variables is substantially large. If the problem has larger number of variables, the suitable method for solving is simplex method. The simplex method is an iterative process through which it reaches ultimately to the minimum or maximum value of the objective function.

Wolfe method transforms the quadratic programming problem into a linear programming problem. Wolfe modified the simplex method to solve quadratic programming problem by adding conditions of the Karush-Kuhn-Tucker (KKT) and changing the objective function of quadratic forms into a linear form.

A Mathematical Programming Language (AMPL) is an algebraic modeling language to describe and solve high-complexity problems for large-scale mathematical computing such as large-scale optimization problems. AMPL can be used to solve QPP.

At first Chandra Sen presented the Multi-objective programming (MOP) method to solve Multi-objective optimization problems. Different types of statistical average (mean) techniques and modified statistical average techniques can be used to solve MOQPP, which gives much better solutions.

Here we use the below techniques to solve MOQPP.

- Graphical Method

- Pareto Optimal Solution
- AMPL
- Chandra Sen's Technique
- Statistical Average Techniques (Arithmetic, Geometric, Harmonic)
- Modified Statistical Average Techniques

These all techniques are briefly described below.

### **3.1 Graphical Method**

Owing to the importance of programming models in various industries, many types of algorithms have been developed over the years to solve them. Here we are going to concentrate on one of the most basic methods to handle a quadratic programming problem i.e. the graphical method.

In principle, this method works for almost all different types of problems but gets more and more difficult to solve when the number of decision variables and the constraints increases. Therefore, we'll illustrate it in a simple case i.e. for two variables only.

We will first discuss the steps of the algorithm:

**Step 1:** Formulate the quadratic programming problem

We have already understood the mathematical formulation of an QPP problem. Note that this is the most crucial step as all the subsequent steps depend on our analysis here.

**Step 2:** Construct a graph and plot the constraint lines

The graph must be constructed in 'n' dimensions, where 'n' is the number of decision variables. This should give an idea about the complexity of this step if the number of decision variables increases.

One must know that one cannot imagine more than 3-dimensions. The constraint lines can be constructed by joining the horizontal and vertical intercepts found from each constraint equation.

**Step 3:** Determine the valid side of each constraint line

This is used to determine the domain of the available space, which can result in a feasible solution. A simple method is to put the coordinates of the origin (0,0) in the problem and determine whether the objective function takes on a physical solution or not. If yes, then the side of the constraint lines on which the origin lies is the valid side. Otherwise it lies on the opposite one.

**Step 4:** Identify the feasible solution region

The feasible solution region on the graph is the one which is satisfied by all the constraints. It could be viewed as the intersection of the valid regions of each constraint line as well. Choosing any point in this area would result in a valid solution for our objective function.

**Step 5:** Plot the objective function on the graph

It will clearly be a straight line since we are dealing with linear equations here. One must be sure to draw it differently from the constraint lines to avoid confusion. Choose the constant value in the equation of the objective function randomly, just to make it clearly distinguishable.

**Step 6:** Find the optimum point

An optimum point always lies on one of the corners of the feasible region. Place a ruler on the graph sheet, parallel to the objective function. Be sure to keep the orientation of this ruler fixed in space. We only need the direction of the straight line of the objective function. Now begin from the far corner of the graph and tend to slide it towards the origin.

- If the goal is to minimize the objective function, find the point of contact of the ruler with the feasible region, which is the closest to the origin. This is the optimum point for minimizing the function.
- If the goal is to maximize the objective function, find the point of contact of the ruler with the feasible region, which is the farthest from the origin. This is the optimum point for maximizing the function.

Example:

Calculate the maximal and minimal value of  $z = 5x^2 + 3y^2$  for the following constraints.

$$x + 2y \leq 14$$

$$3x - y \geq 0$$

$$x - y \leq 2$$

Solution:

The three inequalities indicate the constraints. The optimization equation  $z = 5x^2 + 3y^2$ .

We have to find the  $(x, y)$  corner points that give the largest and smallest values of  $z$ .

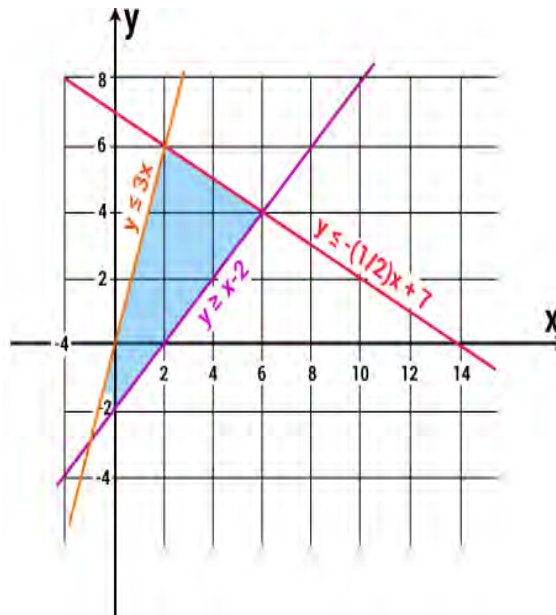
To begin with, first solve each inequality.

$$x + 2y \leq 14 \Rightarrow y \leq -(1/2)x + 7$$

$$3x - y \geq 0 \Rightarrow y \leq 3x$$

$$x - y \leq 2 \Rightarrow y \geq x - 2$$

The area of the plane that will be marked is the feasible region.



**Figure 3.1:** Feasible region for the given plane

Now pair the lines to form a system of linear equations to find the corner points.

Here,  $y = -(\frac{1}{2})x + 7$  and  $y = 3x$

Solving the above equations, we get the corner points as (2, 6)

Again,  $y = -1/2 x + 7$  and  $y = x - 2$

Solving the above equations, we get the corner points as (6, 4)

Also,  $y = 3x$  and  $y = x - 2$

Solving the above equations, we get the corner points as (-1, -3)

For quadratic systems, the maximum and minimum values of the optimization equation lie on the corners of the feasibility region. Therefore, to find the optimum solution, we only need to plug these three points (2,6), (6,4) and (-1,-3) in the objective function  $z = 5x^2 + 3y^2$ .

At (2, 6):  $z = 5(2)(2) + 3(6)(6) = 20+108=128$

At (6, 4):  $z = 5(6)(6) + 3(4)(4) = 180+48=228$

At (-1, -3):  $z = 5(-1)(-1) + 3(-3)(-3) = 5+27=32$

Hence, the maximum of  $z = 228$  lies at (6, 4) and the minimum of  $z = 32$  lies at (-1, -3).

In multi-objective programming problem, it is difficult to find an optimal solution to achieve the extreme value of every objective function, so that the decision maker is exploring for the compromise solution. Based on this idea, the concepts of Pareto optimal solution and weakly Pareto optimal solution are introduced into multi- objective programming problem.

A Pareto outcome is an action that harms no one and helps at least one. The aim of this chapter is to define a solution concept of Pareto optimality for a Multi-Objective Quadratic Programming Problem (MOQPP) and design different methods to extract Pareto optimal solution of MOQPP.



### 3.2 Pareto Optimality

A vector  $x^* \in W$  is said to be Pareto optimal for a multi-objective problem if all other vectors  $x \in W$  have a higher value for at least one of the objective function  $f_i$ , with  $i = 1, \dots, n$  or have the same value for all the objective functions.

We have the following definitions:

- A point  $x^*$  is said to be a weak Pareto optimum or a weak efficient solution for the multi-objective problem if and only if there is no  $x \in W$  such that  $f_i(x) < f_i(x^*)$  for all  $i \in \{1, \dots, n\}$ .
- A point  $x^*$  is said to be a strict Pareto optimum or a strict efficient solution for the multi-objective problem if and only if there is no  $x \in W$  such that  $f_i(x) \leq f_i(x^*)$  for all  $i \in \{1, \dots, n\}$  with at least one strict inequality.
- Construct a function for convex multi-objective programming problem(MOPP)

$$\alpha_i(f_i(x)) = \frac{f_i(x) - f_i^1}{f_i^* - f_i^1}, \quad i = 1, 2, \dots, m; \quad f_i^* = \min_{x \in X} f_i(x), \quad f_i^1 = \max_{x \in X} f_i(x)$$

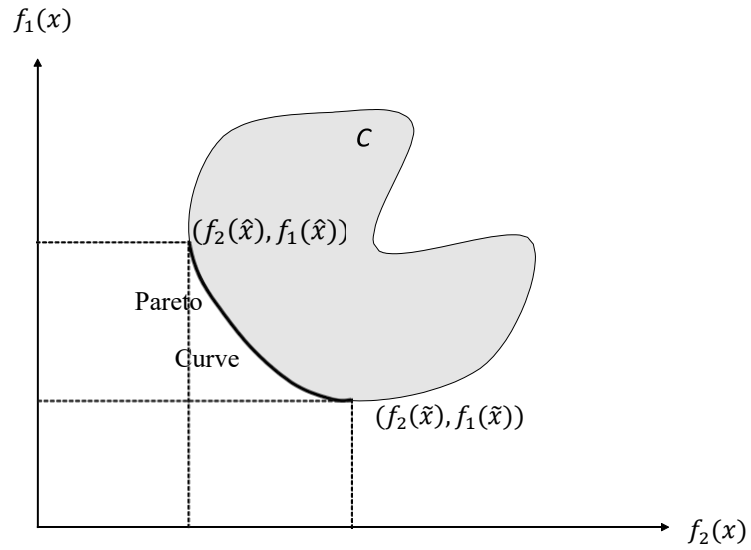
$x^* \in X$  is said to be a M-Pareto optimal solution of convex MOPP, if and only if there does not exist another  $x \in X$  such that  $\alpha_i(f_i(x)) \geq \alpha_i(f_i(x^*))$ ,  $i = 1, 2, \dots, m$  with strict inequality holding for at least one  $i$ .

- $x^* \in X$  is said to be a Weakly M-Pareto optimal solution of convex MOPP, if and only if there does not exist another  $x \in X$  such that

$$\alpha_i(f_i(x)) > \alpha_i(f_i(x^*)), \quad i = 1, 2, \dots, m$$

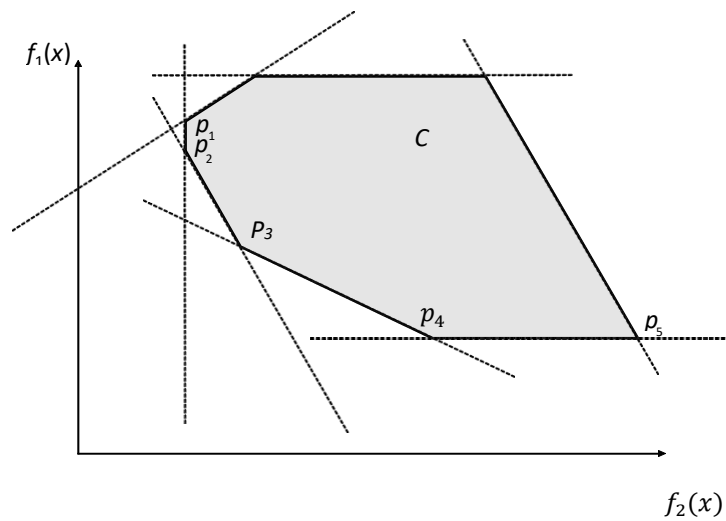
The image of all the efficient solutions is called Pareto front or Pareto curve or surface. The shape of the Pareto surface indicates the nature of the trade-off between the different objective functions. An example of a Pareto curve is showed in Figure 3.2, where all the points between  $(f_2(\hat{x}), f_1(\hat{x}))$  and  $(f_2(\tilde{x}), f_1(\tilde{x}))$  define the Pareto front. Those

Points are called non-inferior or non-dominated points.



**Figure 3.2:** Pareto curve

An example of weak and strict Pareto optima is shown in Figure 3.3: Points  $p_1$  and  $p_5$  are weak Pareto optima; Points  $p_2, p_3$  and  $p_4$  are strict Pareto optima.



**Figure. 3.3:** weak and strict Pareto optima

### Techniques to Solve MOOP Using Pareto Optimality:

Pareto curves cannot be computed efficiently in many cases. Even if it is theoretically possible to find all these points exactly, they are often of exponential size; a straightforward reduction from the knapsack problem shows that they are hard to compute. Thus, approximation methods for them are frequently used. However, approximation does not represent a secondary choice for the decision maker. Indeed, there are many real-life problems for which it is quite hard for the decision maker to have all the information to correctly and/or completely formulate them.

Approximating methods can have different goals: representing the solution set when the latter is numerically available (for convex multi-objective problems). Approximating the solution set when some but not all the Pareto curve is numerically available, approximating the solution set when the whole efficient set is not numerically available (for discrete multi-objective problems).

There are several techniques to solve multi-objective optimization problem:

- Scalarization Technique
- $\varepsilon$  – constraints Method
- Goal Programming
- Multi-Level Programming
- The Norm-Ideal Point Method
- The Membership-Function Method.

Here, we discuss about the above mentioned methods.

#### 3.2.1 The Norm-Ideal point method

For the convex MOPP, firstly give ideal value  $\bar{f}_i$  for every objective function  $f_i(x)$ , which satisfies  $\bar{f}_i \leq \min_{x \in X} f_i(x), i = 1, 2, \dots, m$ ,  $\bar{f} = (\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m)$  is called Ideal point, after then introduce the norm  $\|\cdot\|$ , finally get the feasible solution which is having the nearest distance with the given ideal point  $\bar{f}$  in the norm.

Use the absolute value norm to structure the corresponding single objective programming  $(S_{\bar{f}})$ :

$$\min_{x \in X} \sum_{i=1}^m w_i |f_i(x) - \bar{f}_i| \text{ where } w = (w_1, w_2, \dots, w_m)^T \in R_+^m \setminus \{0\}$$

Because  $\bar{f}_i \leq \min_{x \in X} f_i(x), i = 1, 2, \dots, m$ ;  $(S_{\bar{f}})$  can be simplified to  $\min_{x \in X} \sum_{i=1}^m w_i (f_i(x) - \bar{f}_i)$ .

For the given ideal point  $\bar{f}$  and weights  $w \in R_+^m \setminus \{0\}$ , the optimal solution of  $S_{\bar{f}}$  is weakly Pareto optimal solution.

There states a theorem that: If  $x^* \in X$  is weakly M-Pareto optimal solution of convex MOPP then there exist  $w \in R_+^m \setminus \{0\}$  such that  $x^*$  is the optimal solution of the corresponding single objective programming problem.

Attention should be paid that Pareto optimal solution must be weakly Pareto optimal solution, which implies that if all weakly Pareto optimal solutions can be obtained, then all Pareto optimal solutions can be obtained. This also shows that theoretically all Pareto optimal solutions can be obtained through changing weights.

### 3.2.2 Membership function method

Firstly structure membership function  $\alpha_i (f_i (x))$  for every objective function  $f_i(x)$ , then use  $\alpha_i (f_i (x))$  as the new objective functions to structure the new multi objective programming problem and then turn the new multi objective programming problem to single objective programming problem through some appropriate methods, finally solve the single objective programming problem to get the optimal solution, which is also the M-Pareto optimal solution of the original multi objective programming problem.

Now, Structure the membership function for every objective function as follows:

$$\alpha_i(f_i(x)) = \frac{f_i(x) - f_i^1}{f_i^* - f_i^1}, \quad i = 1, 2, \dots, m; \quad f_i^* = \min_{x \in X} f_i(x), \quad f_i^1 = \max_{x \in X} f_i(x)$$

Without loss of generality  $f_i^* < f_i^1, i = 1, 2, \dots, m$

There states two necessary theorems:

Theorem 1:  $x^*$  is M-Pareto optimal solution of problem convex MOPP, if and only if  $x^*$  is Pareto optimal solution of problem convex MOPP.

Theorem 2: If  $x^*$  is weakly M-Pareto optimal solution of problem convex MOPP, then there exists  $w \in R^m \setminus \{0\}$  such that  $x^*$  is the optimal solution of the corresponding problem  $S_\alpha$ .

The membership function of objective function is set by using simple linear function. According to the properties of the composition of convex function, if the membership function of  $f_i(x)$  is non-increasing and concave about  $f_i(x)$ , then the above calculation still holds.

### 3.3 AMPL- A Mathematical Programming Language

Practical large-scale mathematical programming involves more than just the application of an algorithm to minimize or maximize an objective function subject to constraints equations and inequalities. AMPL is a new language designed to solve those problems easily and with less error-prone.

By itself, AMPL can only be employed to specify classes of mathematical programming models. For the language to be useful, it must be incorporated into a system that manages data, models and solutions.

#### Basic Definitions:

**Sets:** A set can be any unordered collection of objects pertinent to a model. Two unordered sets in our example are the set P of final products and the set R of raw materials. They are declared by AMPL statements

*set prd; # products*

*set raw; # raw materials*

#### **Parameters:**

A parameter is any numerical value pertinent to a model. The simplest kind of parameter is a single, independent value, such as the number of periods or the maximum total production in our example.

Most AMPL statements that declare parameters also specify certain restrictions on them.

For instance, the number of periods is declared by

$$\text{param } T > 0 \text{ integer};$$

### **Variables:**

A linear program's variables are declared much like its parameters. The only substantial difference is that the values of the variables are to be determined through optimization, whereas the values of the parameters are data given in advance.

A typical declaration of variables is the one for raw material in storage:

$$\text{var Store } \{raw, 1..T+1\} \geq 0;$$

### **Objective:**

An objective function can be any linear expression in the parameters and variables. The AMPL representation of the given objective is transcribed from the algebraic objective expression. AMPL representations of indexed sums appear for the first time in our example's objective. The sum of the estimated profits for period  $t$  is typical:

$$\text{sum } \{j \text{ in } prd\} \text{ profit}[j,t] * \text{Make}[j,t]$$

The identifier  $j$  is a dummy index that has exactly the same purpose and meaning as its counterpart in the algebraic notation. It is defined for the scope of the sum, which extends to the end of the following term.

### **Constraints:**

A constraint may be any linear equality or inequality in the parameters and variables. Thus a model's constraints use all the same kinds of expressions as its objective. Whereas the objective is a single expression, however, the constraints come in collections indexed over sets. There is one production-limit constraint, for example, in each period.

The AMPL representation for a collection of constraints must specify two things: the set over which the constraints are indexed, and the expression for the constraints. Thus the production limits look like this:

$$\text{subject to limit } \{t \text{ in } 1..T\}: \text{sum } \{j \text{ in } \textit{prd}\} \textit{Make}[j,t] \leq \textit{max\_prd};$$

Following the keywords *subject to* and the identifier *limit*, the expression in braces gives  $1, \dots, T$  as the indexing set.

### **Data:**

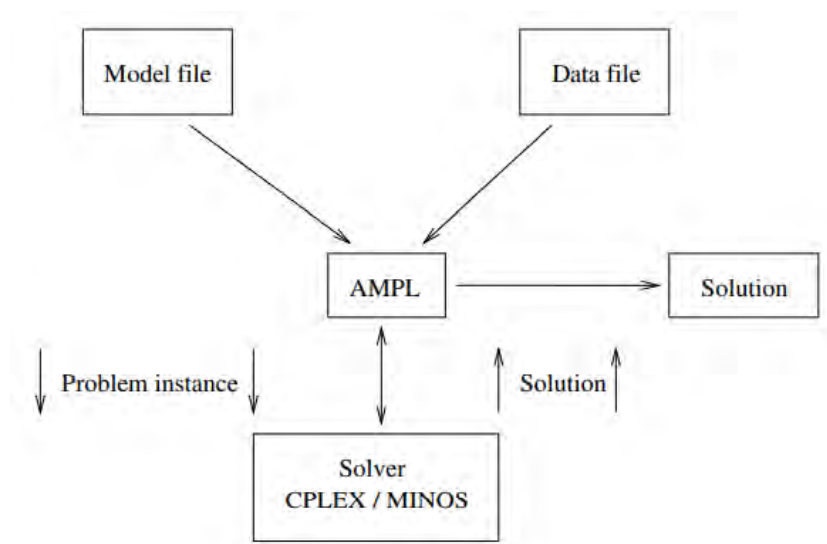
Once the AMPL translator has read and processed the contents, it is ready to read the data. Strictly speaking, the rules for the data are not a part of AMPL; each implementation of an AMPL translator may accept data in whatever formats its creators deem appropriate. As a practical matter, however, we wish to have a standard data format that all versions of the translator will accept. A small data set for the sample LP in our standard format; it is a largely self-explanatory transcription.

The components are fundamental to all models:

- sets- like the products
- parameters- like the production and profit rates
- variables- whose values the solver is to determine
- an objective- to be maximized or minimized
- constraints- that the solution must satisfy

### **Model file, Data file:**

This is exactly what AMPL does. It gives us a way to express the algebraic representation of the model and these value for the parameters separately. It does this using two separate files, a model file and a data file. AMPL reads the model from the *.mod* file, data from the *.dat* file and puts them together into a format that the solver understands. Then, it hands over this problem instance to the solver, which in turn, solves the instance, and hands back the solution to AMPL.



**Figure 3.4:** AMPL model

**Algorithm for AMPL:**

Step 1: Once the file is created and we have checked the file name, run the AMPL program by selecting it from appropriate start menu. After some text appears, we will see the AMPL prompt, which is

***AMPL:***

Step 2: We have to select a solver, which is the program that AMPL calls to solve the math programs. We will be using the solver called cplex. So type:

***option solver cplex;***

Step 3: Type in the following command, which tells AMPL to read in the file we have created:

***model problem.mod;***

Step 4: If we type in the model correctly, we will get no error messages. If we made a mistake, correct the model file and first reset AMPL by typing:

***reset;***



To reload the model, type in: ***model problem.mod;***

Step 5: For data file do the same process and type:

***data problem.dat;***

Step 6: The files are successfully loaded, tell AMPL to run the model by typing:

***solve;***

Once AMPL has solved the model, it displays the results.

**Example:**

We have the following linear program:

$$\begin{aligned} & \text{Maximize } 25 \text{ XB} + 30 \text{ XC} \\ & \text{Subject to } (1/200) \text{ XB} + (1/140) \text{ XC} \leq 40 \\ & \quad 0 \leq \text{XB} \leq 6000 \\ & \quad 0 \leq \text{XC} \leq 4000 \end{aligned}$$

This is a very simple linear program. Solving this linear program with AMPL can be as simple as typing AMPL's description of the linear program:

```
var XB;
var XC;

maximize Profit: 25 * XB + 30 * XC;
subject to Time: (1/200) * XB + (1/140) * XC <= 40;
subject to B_limit: 0 <= XB <= 6000;
subject to C_limit: 0 <= XC <= 4000;
```

into a file — call it prod0.mod — and then typing a few AMPL commands:

```
ampl: model prod0.mod;
ampl: solve;
```

MINOS 5.5: optimal solution found.

2 iterations, objective 192000

AMPL: **display XB, XC;**

XB = 6000

XC = 1400

AMPL: **quit;**

The invocation and appearance of an AMPL session will depend on operating environment and interface, but there will always have the option of typing AMPL statements in response to the *AMPL:* prompt, until we leave AMPL by typing quit.

Example: Consider a basic production model in algebraic form:

Given: P, a set of products

$a_j$  = tons per hour of product  $j$ , for each  $j \in P$

$b$  = hours available at the mill

$c_j$  = profit per ton of product  $j$ , for each  $j \in P$

$u_j$  = maximum tons of product  $j$ , for each  $j \in P$

Define variables:  $X_j$  = tons of product  $j$  to be made, for each  $j \in P$

Maximize:  $\sum_{j \in P} c_j X_j$

Subject to:  $\sum_{j \in P} \left( \frac{1}{a_j} \right) X_j \leq b$

$0 \leq X_j \leq u_j$ , for each  $j \in P$

Basic production model file in AMPL (file prod.mod).

*set P;*

*param a {j in P};*

*param b;*

*param c {j in P};*

```

param u {j in P};

var X {j in P};

maximize Total_Profit: sum {j in P} c[j] * X[j];

subject to Time: sum {j in P} (1/a[j]) * X[j] <= b;

subject to Limit {j in P}: 0 <= X[j] <= u[j];

```

Basis production data file in AMPL (file prod.dat).

```

set P := bands coils;

param:   a       c       u :=
bands   200     25     6000
coils   140     30     4000 ;

param b := 40;

```

If we put all of the model declarations into a file called prod.mod and the data specification into a file prod.dat, then as before a solution can be found and displayed by typing just a few statements:

```
ampl: model steel.mod;
```

```
ampl: data steel.dat;
```

```
ampl: solve;
```

MINOS 5.5: optimal solution found.

2 iterations, objective 192000

```
ampl: display Make;
```

```
Make [*] :=
```

```
bands 6000
```

```
coils 1400
```

```
;
```

The model and data commands each specify a file to be read, in this case the model from prod.mod, and the data from prod.dat. The use of two file-reading commands encourages a clean separation of model from data.

### Nonlinear Program:

Many mathematical programs include nonlinear functions in the constraints and for the objective. Fortunately, AMPL allows us to model and solve nonlinear programs, as well. First, you have to change the solver, since *cplex* does not solve nonlinear programming problems. For nonlinear problems, we shall use the solver *minos*. So type:

```
option solver minos;
```

If the objective function is non-linear, more specifically if the objective function is quadratic such as

$$\begin{aligned} &\text{Maximize } x_1^2 + x_2^2 \\ &\text{Subject to } x_1 + 2x_2 \leq 7 \end{aligned}$$

Then it can be expressed in model file as:

```
var x{1..2}>=0;
maximize x[1]^2 + x[2]^2;
s.t. m: x[1]+2*x [2]<=7;
```

In this section, we have just scratched the surface of the things we can do with AMPL.

### 3.4 Chandra Sen's Technique

Mathematically the multi objective quadratic programming problem (MOQPP) can be stated as:

Chapter-3, Solution Procedure of MOQPP

$$\text{Max } Z_1 = \frac{1}{2} x^T P_1 x + C_1^T x$$

$$\text{Max } Z_2 = \frac{1}{2} x^T P_2 x + C_2^T x$$

.....

.....

$$\text{Max } Z_r = \frac{1}{2} x^T P_r x + C_r^T x$$

$$\text{Min } Z_{r+1} = \frac{1}{2} x^T P_{r+1} x + C_{r+1}^T x$$

$$\text{Min } Z_{r+2} = \frac{1}{2} x^T P_{r+2} x + C_{r+2}^T x$$

.....

.....

$$\text{Min } Z_s = \frac{1}{2} x^T P_s x + C_s^T x$$

subject to

$$Ax \begin{bmatrix} \leq \\ \geq \end{bmatrix} b$$

$$x \geq 0$$

In Sen's technique, all objective functions need to be maximized or minimized firstly by Simplex method. Each objective function is solved and following equations are obtained:

$$\text{Max } Z_1 = \varphi_1$$

$$\text{Max } Z_2 = \varphi_2$$

.....

.....

$$\text{Max } Z_r = \varphi_r$$

$$\text{Min } Z_{r+1} = \varphi_{r+1} \quad (3.4)$$

$$\text{Min } Z_{r+2} = \varphi_{r+2}$$

.....

$$\text{Min } Z_s = \varphi_s$$

Where  $\varphi_1, \varphi_2 \dots \dots \varphi_s$  are optimal values of the objective functions of given equations.

By Chandra Sen's technique, these values are used to form a single objective function by adding (for maximization) and subtracting (for minimization). Mathematically,

$$\text{Max } Z = \sum_{i=1}^r \frac{Z_i}{|\varphi_i|} - \sum_{i=r+1}^s \frac{Z_i}{|\varphi_i|}$$

$$\text{Where } |\varphi_i| \neq 0$$

Subject to constraints of the given equation remains same. The optimum value  $\varphi_i$  may be positive or negative.

Example 1: Here is an example of MOLPP with constraints.

$$\begin{aligned} \max z_1 &= x_1 + 2x_2 \\ \max z_2 &= x_1 \\ \min z_3 &= -2x_1 - 3x_2 \\ \min z_4 &= -x_2 \\ \text{s. t. } 6x_1 + 8x_2 &\leq 48 \\ x_1 + x_2 &\geq 3 \\ x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The values of these objective functions can be obtained by using AMPL algorithm first.

Thus the optimum values of the objective functions with same constraints are given in Table 3.4.

Table 3.4

	$\varphi_i$	$x_i$	$AX =  \varphi_i $	$AY =  \varphi_i $
1	10	(4,3)	10	
2	4	(4,3)	4	
3	-17	(4,3)		-17
4	-3	(4,3)		-3

AX stands for maximization problems and AY stand for minimization problems.

Then with these values, a single objective function is developed using Chandra Sen's technique.

$$\begin{aligned}
 \text{Max } Z &= \sum_{i=1}^2 \frac{Z_i}{|\varphi_i|} - \sum_{i=3}^4 \frac{Z_i}{|\varphi_i|} \\
 &= \frac{x_1+2x_2}{10} + \frac{x_1}{4} - \frac{-2x_1-3x_2}{-17} - \frac{-x_2}{-3} \\
 &= 0.4676 x_1 + 0.7098 x_2
 \end{aligned}$$

By Chandra Sen's technique, the multi-objective optimization problem is converted into single objective problem. The system becomes,

$$\begin{aligned}
 \text{Max } Z &= 0.4676 x_1 + 0.7098 x_2 \\
 \text{s. t. } \quad &6x_1 + 8x_2 \leq 48 \\
 &x_1 + x_2 \geq 3 \\
 &x_1 \leq 4 \\
 &x_2 \leq 3 \\
 &x_1, x_2 \geq 0
 \end{aligned}$$

After solving this applying AMPL, we get the optimal solution

$$Z_{\max} = 3.9998 \text{ with } x_1 = 4 \text{ and } x_2 = 3.$$

### 3.5 Statistical Average Technique

Statistics is a branch of applied mathematics that involves the collection, description, analysis, and inference of conclusions from quantitative data. The two major areas of statistics are descriptive and inferential statistics. Descriptive statistics mostly focus on the central tendency, variability and distribution of sample data. Central tendency means the estimate of the characteristics, a typical element of a sample or population and includes descriptive statistics such as mean, median, and mode.

The mean, also referred to by statisticians as the average, is the most common statistic used to measure the center of a numerical data set. There are different types of mean, viz. arithmetic mean (AM), weighted mean (WM), geometric mean (GM) and harmonic mean (HM), root mean square (RMS), contra-harmonic mean (CHM).

There are majorly three different types of mean value that we will discuss for our present research.

1. Arithmetic Mean
2. Geometric Mean
3. Harmonic Mean

In case of multi objective optimization problems, all the objectives are optimized individually. The mean, median and optimal averages of optimal values for maximization and minimization objectives are estimated separately. The combined objective function is constructed by weighting the objective functions by the inverse of the mean, median and optimal average as given below:

$$\text{Max } Z = \sum_{i=1}^r \frac{Z_i}{\text{mean, median, optimal average}} - \sum_{i=r+1}^s \frac{Z_i}{\text{mean, median, optimal average}}$$



### 3.5.1 Arithmetic mean technique

Arithmetic Mean: The arithmetic mean of a set of observed data is defined as being equal to the sum of the numerical values of each and every observation divided by the total number of observations.

If any data set consisting of the values  $x_1, x_2, \dots, x_n$  then the arithmetic mean A.M is defined as:

A.M = (Sum of all observations)/ (Total number of observation)

$$= \frac{1}{n} \sum_{i=1}^n x_i = \frac{x_1+x_2+\dots+x_n}{n}$$

Properties of Arithmetic Mean

- The sum of deviations of the items from their arithmetic mean is always zero.
- The sum of the squared deviations of the items from Arithmetic Mean (A.M) is minimum, which is less than the sum of the squared deviations of the items from any other values.
- If each item in the arithmetic series is substituted by the mean, then the sum of these replacements will be equal to the sum of the specific items.

Merits of Arithmetic Mean

- The arithmetic mean is simple to understand and easy to calculate.
- It is influenced by the value of every item in the series.
- A.M is rigidly defined.
- It has the capability of further algebraic treatment.
- It is a measured value and not based on the position in the series.

Demerits of Arithmetic Mean

- It is changed by extreme items such as very small and very large items.
- It can rarely be identified by inspection.

- In some cases, A.M. does not represent the original item. For example, average patients admitted to a hospital are 10.7 per day.
- The arithmetic mean is not suitable in extremely asymmetrical distributions.

In order to convert multi objective function in equation (3.4) into single objective function, we can use arithmetic mean technique (AMT) as follows:

$$(AMT) \quad Max \ Z = \sum_{i=1}^r \frac{Z_i}{A.M_1} - \sum_{i=r+1}^s \frac{Z_i}{A.M_2}$$

Where  $A.M_1 = A.M(|\varphi_1|, |\varphi_2|, \dots, |\varphi_r|)$  &  $A.M_2 = A.M(|\varphi_{r+1}|, |\varphi_{r+2}|, \dots, |\varphi_s|)$  and A.M is arithmetic mean.

### 3.5.2 Geometric mean technique

Geometric Mean: The geometric mean is defined as the nth root of the product of n numbers. The Geometric Mean (G.M) of a series containing n observations is the nth root of the product of the values.

Consider, if  $x_1, x_2, \dots, x_n$  are the observation, then the G.M is defined as:

$$\begin{aligned} G.M &= \sqrt[n]{\text{Product of all observations}} \\ &= \sqrt[n]{\prod_{i=1}^n x_i} = (x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}} \end{aligned}$$

Properties of Geometric Mean

- The G.M for the given data set is always less than the arithmetic mean for the data set
- If each object in the data set is substituted by the G.M, then the product of the objects remains unchanged.
- The ratio of the corresponding observations of the G.M in two series is equal to the ratio of their geometric means
- The products of the corresponding items of the G.M in two series are equal to the product of their geometric mean.

Merits of geometric mean

- It is rigidly determined.
- The calculation is based on all the terms of the sequence.
- It is suitable for further mathematical analysis.
- Fluctuation in sampling will not affect the geometric mean.
- It gives relatively more weight to small observations.

Demerits of geometric mean

- One of the main drawbacks of the geometric mean is that if any one of the observations is negative, then the geometric mean value will be imaginary despite the quantity of the other observations.
- Due to complex numerical character, it is not easy to understand and to calculate for a non-mathematics person.

In order to convert multi objective function in equation (3.4) into single objective function, we can use geometric mean technique (GMT) as follows:

$$(GMT) \quad Max \ Z = \sum_{i=1}^r \frac{Z_i}{G.M_1} - \sum_{i=r+1}^s \frac{Z_i}{G.M_2}$$

Where  $G.M_1 = G.M(|\varphi_1|, |\varphi_2|, \dots, |\varphi_r|)$  &  $G.M_2 = G.M(|\varphi_{r+1}|, |\varphi_{r+2}|, \dots, |\varphi_s|)$  and G.M is geometric mean.

### 3.5.3 Harmonic mean technique

Harmonic Mean: Harmonic mean of a set of observations is defined as the reciprocal of the arithmetic average of the reciprocal of the given values. If  $x_1, x_2, \dots, x_n$  are n observations then

$$H.M = \frac{\text{Total number of observation}}{\sum_{\text{observation values}} \frac{1}{x}}$$

$$= \frac{n}{\sum_{i=1}^n \left(\frac{1}{x_i}\right)} = \frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}}$$

#### Properties of Harmonic Mean

- If all the observations taken are constant say c, then the harmonic mean of the observation is also c.
- The harmonic mean has the least value as compared to the geometric and the arithmetic mean. i.e; AM>GM>HM.

#### Merits of Harmonic Mean

- It is rigidly defined.
- It is defined on all observations.
- It is amenable to further algebraic treatment.
- It is the most suitable average when it is desired to give greater weight to smaller observations and less weight to the larger ones.

#### Demerits of Harmonic Mean

- It is not easily understood.
- It is difficult to compute.
- It is only a summary figure and may not be the actual item in the series
- It gives greater importance to small items and is therefore, useful only when small items have to be given greater weightage.
- It is rarely used in grouped data.

In order to convert multi objective function in equation (3.4) into single objective function, we can use harmonic mean technique (HMT) as follows:

$$(HMT) \quad \text{Max } Z = \sum_{i=1}^r \frac{Z_i}{H.M_1} - \sum_{i=r+1}^s \frac{Z_i}{H.M_2}$$

Where  $H.M_1 = H.M(|\varphi_1|, |\varphi_2|, \dots, |\varphi_r|)$  &  $H.M_2 = H.M(|\varphi_{r+1}|, |\varphi_{r+2}|, \dots, |\varphi_s|)$  and H.M is harmonic mean.

**Example:** Consider from previous example (Example-1 and Table 3.4), with these values, a single objective function is developed by using Statistical average techniques (AMT, GMT, HMT) as follows:

Using Arithmetic average technique (AMT), from table 3.4 we get

$$A.M_1 = (10,4) = 7, \quad A.M_2 = (17,3) = 10$$

Using Geometric average technique (GMT),

$$G.M_1 = \sqrt{10 \times 4} = 6.324, \quad G.M_2 = \sqrt{17 \times 3} = 7.1414$$

Using Harmonic average technique (HMT),

$$H.M_1 = (10,4) = 5.7143, \quad H.M_2 = (17,3) = 5.1$$

By AMT, GMT and HMT, the multi objective optimization problem is converted into single objective problem. Thus we obtain,

$$\begin{aligned} \text{AMT:} \quad \text{Max } Z &= \sum_{i=1}^2 \frac{Z_i}{A.M_1} - \sum_{i=3}^4 \frac{Z_i}{A.M_2} = \frac{1}{7} (Z_1 + Z_2) - \frac{1}{10} (Z_3 + Z_4) \\ &= 0.4857 x_1 + 0.6857 x_2 \end{aligned}$$

$$\text{GMT:} \quad \text{Max } Z = \sum_{i=1}^2 \frac{Z_i}{G.M_1} - \sum_{i=3}^4 \frac{Z_i}{G.M_2} = 0.5963 x_1 + 0.8763 x_2$$

$$\text{HMT:} \quad \text{Max } Z = \sum_{i=1}^2 \frac{Z_i}{H.M_1} - \sum_{i=3}^4 \frac{Z_i}{H.M_2} = 0.7421 x_1 + 1.1342 x_2$$

After solving, we get the optimal solution,

$$\text{AMT:} \quad Z_{max} = 3.9999 \text{ with } x_1 = 4 \text{ and } x_2 = 3.$$

$$\text{GMT:} \quad Z_{max} = 5.0141 \text{ with } x_1 = 4 \text{ and } x_2 = 3.$$

$$\text{HMT:} \quad Z_{max} = 6.37103 \text{ with } x_1 = 4 \text{ and } x_2 = 3.$$

### 3.6 Modified Statistical Average Technique

Modified statistical average technique (MSAT) is an updated version of statistical average technique. Some operation researcher has been discussed about it and they have applied modified technique to get better solution than statistical average technique.

Modified Statistical average technique can be classified as:

1. Modified Arithmetic Average Technique
2. Modified Geometric Average Technique
3. Modified Harmonic Average Technique

#### 3.6.1 Modified arithmetic mean technique

Consider the multi-objective optimization problem in equation (3.4) and solve it by using the modified arithmetic mean technique (MAMT),

Let,  $m_1 = \min(|\varphi_1|, |\varphi_2|, \dots, |\varphi_r|)$ ; for maximization,

$$\varphi_i \text{ is maximum value of } Z_i, i = 1, \dots, r$$

$m_2 = \min(|\varphi_{r+1}|, |\varphi_{r+2}|, \dots, |\varphi_s|)$ , for minimization,

$$\varphi_i \text{ is minimum value of } Z_i, i = r + 1, \dots, s$$

Modified Arithmetic Mean technique:  $AM.AV = (m_1 + m_2)/2$

$$Max Z = \frac{1}{AM.AV} [\sum_{i=1}^r Z_i - \sum_{i=r+1}^s Z_i]$$

#### 3.6.2 Modified geometric mean technique

Consider the multi-objective optimization problem in equation (3.4) and solve it by using the modified geometric mean technique (MGMT),

Let,  $m_1 = \min(|\varphi_1|, |\varphi_2|, \dots, |\varphi_r|)$ ; for maximization,

$$\varphi_i \text{ is maximum value of } Z_i, i = 1, \dots, r$$

$m_2 = \min(|\varphi_{r+1}|, |\varphi_{r+2}|, \dots, |\varphi_s|)$ , for minimization,

$\varphi_i$  is minimum value of  $Z_i, i = r + 1, \dots, s$

Modified Geometric Mean technique:  $GM.AV = \sqrt{m_1 m_2}$

$$Max Z = \frac{1}{GM.AV} \left[ \sum_{i=1}^r Z_i - \sum_{i=r+1}^s Z_i \right]$$

### 3.6.3 Modified harmonic mean technique

Consider the multi-objective optimization problem in equation (3.4) and solve it by using the modified Harmonic mean technique (MHMT),

Let,  $m_1 = \min(|\varphi_1|, |\varphi_2|, \dots, |\varphi_r|)$ ; for maximization,

$\varphi_i$  is maximum value of  $Z_i, i = 1, \dots, r$

$m_2 = \min(|\varphi_{r+1}|, |\varphi_{r+2}|, \dots, |\varphi_s|)$ , for minimization,

$\varphi_i$  is minimum value of  $Z_i, i = r + 1, \dots, s$

Modified Harmonic Mean technique:  $HM.AV = \frac{2}{\frac{1}{m_1} + \frac{1}{m_2}}$

$$Max Z = \frac{1}{HM.AV} \left[ \sum_{i=1}^r Z_i - \sum_{i=r+1}^s Z_i \right]$$

**Example:** Consider the same example, a single objective function is developed by using modified statistical average techniques (MAMT, MGMT, MHMT) as follows:

Using Arithmetic average technique (MAMT), from table 3.4 we get

$$m_1 = \min\{10,4\} = 4, \quad m_2 = \{17,3\} = 3$$

$$AM.AV = \frac{m_1 + m_2}{2} = 3.5$$

Using Geometric average technique (MGMT),

$$GM.AV = \sqrt{m_1 m_2} = 3.4641$$

Using Harmonic average technique (MHMT),

$$HM.AV = \frac{2}{\frac{1}{m_1} + \frac{1}{m_2}} = 3.4483$$

By MAMT, MGMT and MHMT, the multi objective optimization problem is converted into single objective problem. Thus we obtain,

$$\begin{aligned} \text{MAMT:} \quad \text{Max } Z &= \frac{1}{AM.AV} [\sum_{i=1}^2 Z_i - \sum_{i=3}^4 Z_i] = \frac{1}{3.5} [Z_1 + Z_2 - Z_3 - Z_4] \\ &= 1.142 x_1 + 1.714 x_2 \end{aligned}$$

$$\text{MGMT:} \quad \text{Max } Z = \frac{1}{GM.AV} [\sum_{i=1}^2 Z_i - \sum_{i=3}^4 Z_i] = 1.1547x_1 + 1.732 x_2$$

$$\text{MHMT:} \quad \text{Max } Z = \frac{1}{HM.AV} [\sum_{i=1}^2 Z_i - \sum_{i=3}^4 Z_i] = 1.1599x_1 + 1.7399 x_2$$

After solving, we get the optimal solution,

$$\text{MAMT:} \quad Z_{max} = 9.7141 \text{ with } x_1 = 4 \text{ and } x_2 = 3.$$

$$\text{MGMT:} \quad Z_{max} = 9.8142 \text{ with } x_1 = 4 \text{ and } x_2 = 3.$$

$$\text{MHMT:} \quad Z_{max} = 9.8593 \text{ with } x_1 = 4 \text{ and } x_2 = 3$$



## CHAPTER-4

### Advanced Transformation Technique

The aim of this chapter is to present a new method for finding an optimal solution to quadratic programming problems.

#### 4.1 Preamble

The principle of the method is based on calculating the value of average point. It should be noted here that the objective function may be convex or non-convex. Moreover, the search for the optimal solution is to find the hyper plane separating the convex and the critical point. Notice that one does not need to transform the quadratic problem into an equivalent linear one as in the numerical methods; the method is purely analytical and avoids the usage of initial solution. An algorithm computing the optimal solution of the convex function has given. It also can be applied on linear programming problems. In reality a manager meets problems involving simultaneous changes in several variables. Quadratic programming is used to meet these problems. Dynamic programming is used to solve sequential problems.

Multi objective optimization problem can be defined as:

$$\begin{array}{l} \text{Max /Min}[z_1, z_2, \dots, z_s] \\ \text{Subject to} \quad Ax\{\geq, =, \leq\} b, \quad x \geq 0 \end{array}$$

Multi-objective programming is used in application for many real world problems including problems in the fields of engineering, industrial and finance. In multi-objective programming there are multiple conflicting objectives whereby improving one objective will reduce the value of others, leading to a trade-off between solutions.

It is assumed that no single solution will optimize all objectives simultaneously because this would be a trivial case. The main aim of multi-objective programming is to assist a decision maker (DM) to choose a preferred solution among all the trade-offs. In this case, it is not necessary to generate all solutions when the DM is involved in the process since some solutions may be eliminated at each stage. However, we will focus on non-interactive exact methods that do not involve the DM

in order to generate the entire solution set. Multi-objective problems take the form of linear, quadratic, integer (MOIP), and mixed integer (MOMIP) which have continuous, discrete, and both continuous and discrete solutions respectively.

## **4.2 Feature Of The Technique**

The proposed technique has following features:

- (a) This technique is prescriptive in nature.
- (b) The technique looks at the whole problem.
- (c) It makes use of a wide range of techniques. They based on quantitative data to develop mathematical model as the logical presentation of the problem.
- (d) The technique facilitates rapid evaluation by helping the management to know how a given process will respond to select changes in the factors.
- (e) It facilitates in the development of alternatives.
- (f) It makes use of new techniques.
- (g) Its objective is to achieve optimum utilization of resources.
- (h) It is an inter-disciplinary approach to problem solving.
- (j) It studies the problem at actual working conditions.

## **4.3 Formulation**

This is a process of building, testing, and operating commercial world phenomena through the use of mathematical relationships that exist among critical factors.

The steps involved in formulation are:

- i. To define a problem
- ii. Construction of a model.
- iii. Specify the values of the variables and the boundaries.

This is basically a technique which has a wide range of applications. This can be applied both inside and outside the factory. This is used in complicated waiting lines, arrangement of offices, physical distribution systems, alternate corporate strategies and product demand patterns.

Multi objective optimization problem:

$$\begin{aligned} & \text{Max /Min}[z_1, z_2, \dots, z_s] \\ \text{Subject to} & \quad Ax\{\geq, =, \leq\} b, \quad x \geq 0 \end{aligned}$$

Suppose we optimize all the objective functions individually and obtain the values

$$\begin{aligned} & \text{Max } z_1 = \alpha_1 \\ & \text{Max } z_2 = \alpha_2 \\ & \quad \dots \\ & \quad \dots \\ & \text{Max } z_r = \alpha_r \\ & \text{Min } z_{r+1} = \alpha_{r+1} \\ & \quad \dots \\ & \text{Min } z_s = \alpha_s \\ \text{Subject to} & \quad Ax\{\geq, =, \leq\} b, \quad x \geq 0 \end{aligned}$$

Where  $\alpha_i$  are the values of objective functions.

We require the common set of decision variables to be the best compromising optimal solution. Here we can determine the common set of decision variables from the following combined objective function. By our proposed Advanced transformation technique, we can obtain the single objective function as follows:

$$\text{Max } Z = \frac{\sum_{i=1}^r z_i - \sum_{i=r+1}^s z_i}{O_{AT}}$$

Where  $O_{AT} = \frac{1}{(1/m)}$  and  $m = \min \{m_1, m_2\}$

and  $m_1 = \min\{\alpha_i\}, \forall i = 1, \dots, r$  &  $m_2 = \min\{\alpha_i\}, \forall i = r + 1, \dots, s$ . Subject to the same constraints.

## 4.4 Advantages, Disadvantages

### Advantages

#### 1. Enhanced productivity

It helps in improving the productivity of the organizations. Operations controls provide significant information to the DM before making an important decision. Effective and accurate decision making helps in improving the productivity of the organization.

#### 2. Improved coordination

It improves the coordination between different fields.

#### 3. Lower risks of failure

It lowers the chances of failure as with the help of less calculation and get to know about all the alternative solutions for a single problem.

#### 4. Less time consuming

This technique saves time. It needs comparatively less time for calculation.

### Disadvantages

#### 1. Unquantifiable factors

The effectiveness of solutions developed by using the technique largely depends on the various factors. It is easy to measure quantifiable factors but the problem arises when important factors are in unquantifiable form. Unquantifiable factors result in inaccurate solutions.

#### 2. Difficult to implement

The solutions obtained from this technique are difficult to implement as most of them are usually unrealistic. Some modifications are required to make to implement the solutions which hamper the effectiveness of the solution.

### 3. Restricted

It perfectly works on mixed type of multi-objective optimization problems. In case of single type (only maximization or only minimization) of MOOP it does not have any effect.

## 4.5 Algorithm

Step 1: Find the value of each objective function which is to be maximize or minimize.

Step 2: Solve the first objective function by mathematical programming language AMPL or by any other solution method.

Step 3: Check the feasibility of the solution in step 2. If it is feasible then go to step 4. Otherwise, use different methods to remove infeasibility.

Step 4: Assign a name to the optimum value of first objective function  $z_1$  by  $\alpha_1$ .

Step 5: Repeat step -2 for  $i = 2, 3, \dots, s$ .

Step 6: Select  $m_1 = \min\{\alpha_i\}, \forall i = 1, \dots, r$  and  $m_2 = \min\{\alpha_i\}, \forall i = r + 1, \dots, s$ .

Step 7: Select  $m = \min \{m_1, m_2\}$  and calculate  $O_{AT} = \frac{1}{(1/m)}$ .

Step 8: Optimize the combined objective function as  $Max Z = \frac{\sum_{i=1}^r z_i - \sum_{i=r+1}^s z_i}{O_{AT}}$

Under the same constraints by repeating steps 2 & 3.

4.6 Flow Chart

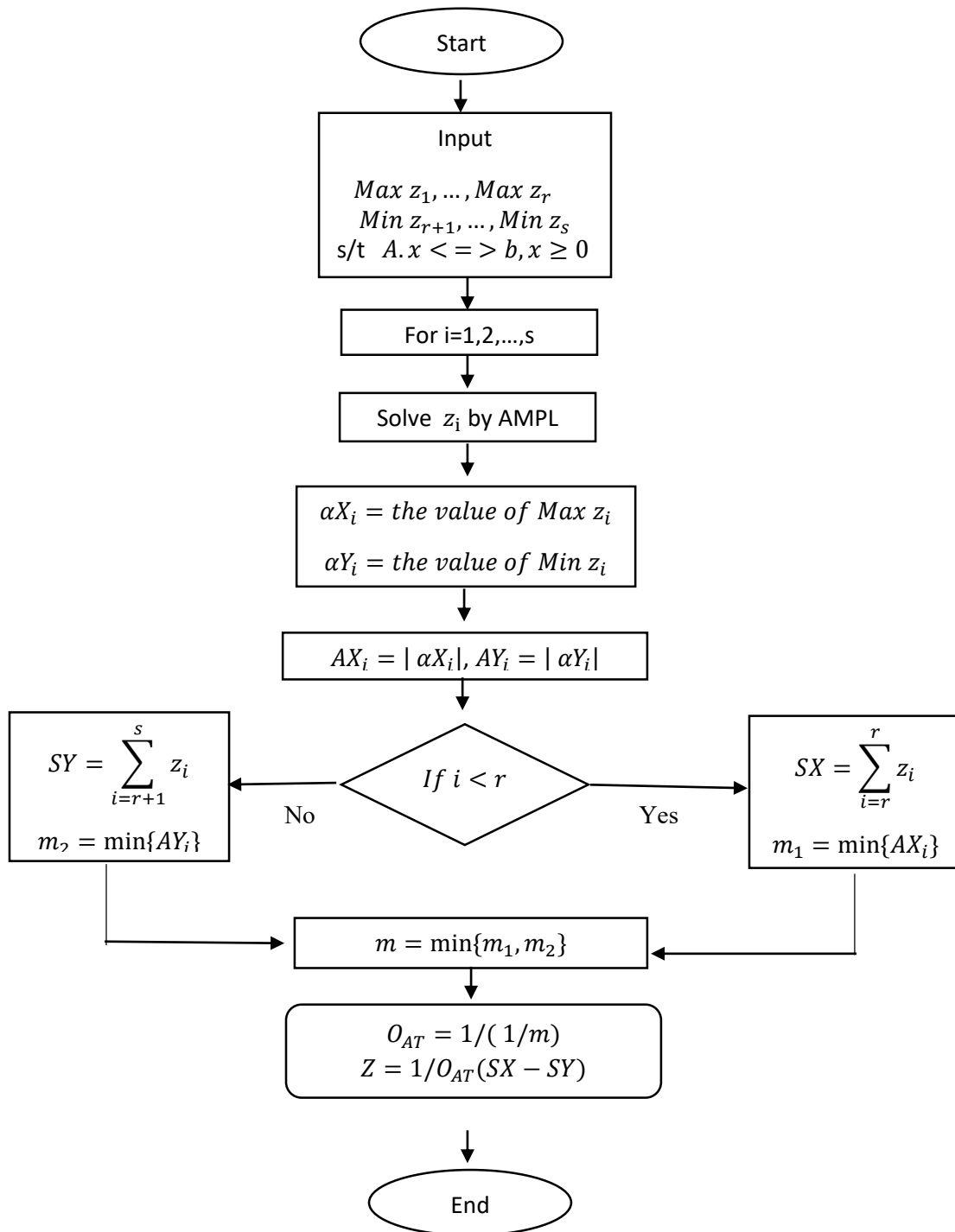


Figure 4.6: Flow chart of the proposed technique

## CHAPTER-5

### Example and Analysis

We use numerical example to solve MOQPP.

Example 1: Solve the following MOQPP by using Pareto optimality, Chandra Sen's techniques, Statistical average techniques, Modified average techniques and Advanced transformation technique.

Consider the following Multi Objective Quadratic Programming problem with linear constraints:

$$\text{Max } Z_1 = 4x_1 + 2x_2 - x_1^2 - x_2^2 + 5$$

$$\text{Max } Z_2 = 2x_1 + x_2 - x_1^2$$

$$\text{Min } Z_3 = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$$

$$\text{Min } Z_4 = 2x_1 + 3x_2 - 2x_1^2$$

$$s/t \quad x_1 + 4x_2 \leq 9$$

$$x_1 + x_2 \leq 3$$

$$3x_1 + 2x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

### 5.1 Using Pareto Optimality

Convert the system into convex MOQPP:

$$\text{Min } Z_1 = -4x_1 - 2x_2 + x_1^2 + x_2^2 - 5$$

$$\text{Min } Z_2 = -2x_1 - x_2 + x_1^2$$

$$\text{Min } Z_3 = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2 \quad (5.1)$$

$$\text{Min } Z_4 = 2x_1 + 3x_2 - 2x_1^2$$

$$s/t \quad x_1 + 4x_2 \leq 9$$

$$x_1 + x_2 \leq 3$$

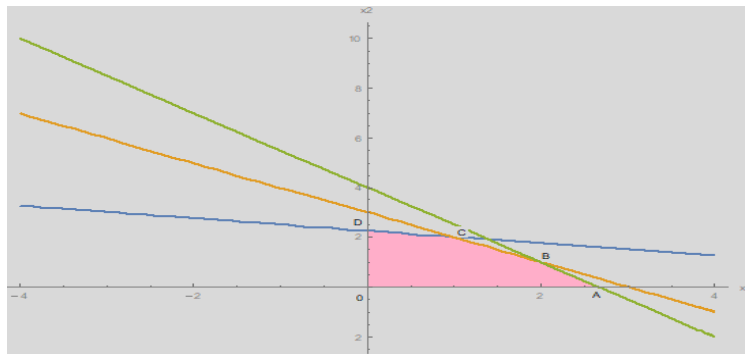
$$3x_1 + 2x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

Now we will find Pareto optimal solution of this system by using Norm-Ideal Point method and Membership Function Method:

▪ Using Norm-Ideal Point method

For convenience, the feasible region of the given problem is:



**Figure 5.1:** Graph of feasible region for Norm-Ideal point method

The feasible region is 0ABCD. The vertices of feasible region formed by constraints are 0(0,0), A(2.6,0), B(2,1), C(1,2), D(0,2.25) and the function values of the objective functions in vertices are given in the following table I:

Table I: function values at vertices

	$Z_1$	$Z_2$	$Z_3$	$Z_4$
0(0,0)	-5	0	6	0
A(2.6,0)	-8.64	1.56	3.92	-8.32
B(2,1)	-10	-1	0	-1
C(1,2)	-8	-3	6	6
D(0,2.25)	-4.4	-2.25	16.12	6.75



It can easily say that,  $Min \{ Z_1 \} = -10, Min \{ Z_2 \} = -3, Min \{ Z_3 \} = 0, Min \{ Z_4 \} = -8.32$ .

All Pareto optimal solutions of problem (5.1) is :  $A(2.6,0), B(2,1), C(1,2)$

The Ideal point method is used to solve the given MOQPP and illustrate that there exists weights  $w$  such that each Pareto optimal solution of (5.1) is the optimal solution of corresponding single objective function. In the Ideal point method, we take the Ideal point  $\bar{p} = (\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4)$  where

$$\begin{aligned} \bar{p}_1 &= \min_{x \in X} \{-4x_1 - 2x_2 + x_1^2 + x_2^2 - 5\} = -10, & \bar{p}_2 &= \min_{x \in X} \{-2x_1 - x_2 + x_1^2\} = -3, \\ \bar{p}_3 &= \min_{x \in X} \{6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2\} = 0, & \bar{p}_4 &= \min_{x \in X} \{2x_1 + 3x_2 - 2x_1^2\} = -8.32 \end{aligned}$$

So the MOQPP (5.1) can be turned into the following single objective programming problem:

$$Min \quad w_1[z_1 + 10] + w_2[z_2 + 3] + w_3[z_3 - 0] + w_4[z_4 + 8.32]$$

$$\Rightarrow \quad Min \quad w_1[-4x_1 - 2x_2 + x_1^2 + x_2^2 - 5 + 10] + w_2[-2x_1 - x_2 + x_1^2 + 3] + w_3[6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2 - 0] + w_4[2x_1 + 3x_2 - 2x_1^2 + 8.32]$$

$$\begin{aligned} \text{s/t} \quad & x_1 + 4x_2 \leq 9 \\ & x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{5.2}$$

The objective function can be written as:

$$\begin{aligned} & x_1[-4w_1 - 2w_2 - 6w_3 + 2w_4] + x_2[-2w_1 - w_2 + 3w_4] + x_1^2[w_1 + w_2 + 2w_3 - 2w_4] \\ & + x_2^2[w_1 + 2w_3] - 2w_3x_1x_2 + [5w_1 + 3w_2 + 6w_3 + 8.32w_4] \end{aligned}$$

In order to take Minimum in point  $B(2,1)$  according to the characteristic of linear programming, the slope of objective function only need to satisfy

$$-4w_1 - 2w_2 - 6w_3 + 2w_4 > 0 \ \& \ \frac{4w_1+2w_2+6w_3-2w_4}{-2w_1-w_2+3w_4} < -1 \quad (5.3)$$

$$\text{Or} \quad -4w_1 - 2w_2 - 6w_3 + 2w_4 > 0 \ \& \ \frac{4w_1+2w_2+6w_3-2w_4}{-2w_1-w_2+3w_4} < -3/2 \quad (5.4)$$

Taking,  $w_1 = \frac{1}{4}, w_2 = \frac{1}{4}, w_3 = \frac{1}{4}, w_4 = \frac{1}{4}$

The multi objective programming problem (5.2) can be converted into the following single objective programming problem:

$$\begin{aligned} \text{Min} \quad & -\frac{5}{2}x_1 + \frac{1}{2}x_1^2 + \frac{3}{4}x_2^2 - \frac{1}{2}x_1x_2 + \frac{279}{50} \\ \text{s/t} \quad & x_1 + 4x_2 \leq 9 \\ & x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned} \quad (5.5)$$

After solving we get,  $x_1 = 2.29$  &  $x_2 = 0.9$

The optimal solution of single objective programming problem (5.5) is  $(2.29, 0.9)$  which is also the Pareto optimal solution (approximately) of problem (5.2). So there exist weights

$$w_1 = \frac{1}{4}, w_2 = \frac{1}{4}, w_3 = \frac{1}{4}, w_4 = \frac{1}{4}$$

such that  $B(2,1)$  is the optimal solution of the corresponding single objection programming problem (5.5).

In order to take Minimum in point  $C(1,2)$  according to the characteristic of linear programming, the slope of objective function only need to satisfy

$$-4w_1 - 2w_2 - 6w_3 + 2w_4 > 0 \ \& \ \frac{4w_1+2w_2+6w_3-2w_4}{-2w_1-w_2+3w_4} < -1 \quad (5.6)$$

$$\text{Or } -4w_1 - 2w_2 - 6w_3 + 2w_4 > 0 \ \& \ \frac{4w_1+2w_2+6w_3-2w_4}{-2w_1-w_2+3w_4} < -1/4 \quad (5.7)$$

Taking,  $w_1 = \frac{2}{5}, w_2 = \frac{1}{5}, w_3 = \frac{1}{5}, w_4 = \frac{1}{5}$

The multi objective programming problem (5.2) can be converted into the following single objective programming problem:

$$\begin{aligned} \text{Min } & -2.8x_1 - 0.4x_2 + 0.6x_1^2 + 0.8x_2^2 - 0.4x_1x_2 + 5.46 \\ \text{s/t } & x_1 + 4x_2 \leq 9 \\ & x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned} \quad (5.8)$$

After solving we get,  $x_1 = 2.23$  &  $x_2 = 0.91$

The optimal solution of single objective programming problem (5.8) is (2.23, 0.91) which is also the Pareto optimal solution (approximately) of problem (5.2). So there exist weights

$$w_1 = \frac{2}{5}, w_2 = \frac{1}{5}, w_3 = \frac{1}{5}, w_4 = \frac{1}{5}$$

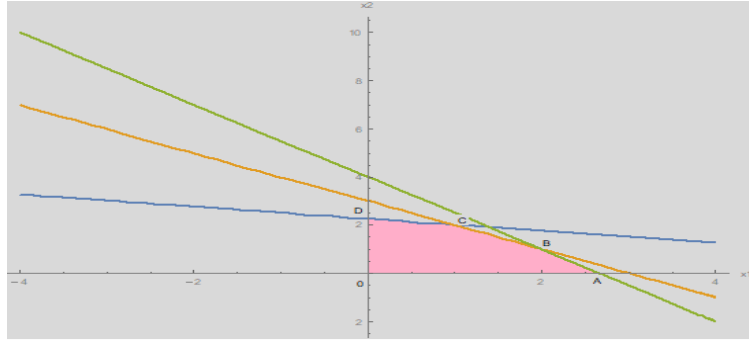
such that  $B(2,1)$  is the optimal solution of the corresponding single objection programming problem (5.8).

So for both cases, the Pareto optimal solution is (2,1).

For all Pareto optimal solutions of multi objective programming problem, there exists weights such that the Pareto optimal solution is the optimal solution of the corresponding single objective programming problem.

- Using Membership Function method:

For convenience, the feasible region of the given problem is:



**Figure 5.2:** Graph of feasible region for Membership-Function method

For convenience denote that,  $f_1 = -4x_1 - 2x_2 + x_1^2 + x_2^2 - 5$ ,  $f_2 = -2x_1 - x_2 + x_1^2$

$$f_3 = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2, \quad f_4 = 2x_1 + 3x_2 - 2x_1^2$$

The feasible region is 0ABCD. The vertices of feasible region formed by constraints are 0(0,0), A(2.6,0), B(2,1), C(1,2), D(0,2.25) and the function values of the objective functions in vertices are given in the following table II:

Table II: function values at vertices

	$f_1$	$f_2$	$f_3$	$f_4$
0(0,0)	-5	0	6	0
A(2.6,0)	-8.64	1.56	3.92	-8.32
B(2,1)	-10	-1	0	-1
C(1,2)	-8	-3	6	6
D(0,2.25)	-4.4	-2.25	16.12	6.75

Due to the particularity of linear programming, it is obvious that,

$$\text{Min } f_1 = -10, \quad \text{Max } f_1 = -4.4$$

$$\text{Min } f_2 = -3, \quad \text{Max } f_2 = 1.56$$

$$\text{Min } f_3 = 0, \quad \text{Max } f_3 = 16.12$$

$$\text{Min } f_4 = -8.32, \quad \text{Max } f_4 = 6.75$$

For each objective function, membership function can be structured as follows:

$$\alpha_1(f_1) = \frac{-f_1 - 4.4}{5.6}, \alpha_2(f_2) = \frac{-f_2 + 1.56}{4.56}, \alpha_3(f_3) = \frac{-f_3 + 16.12}{16.12}, \alpha_4(f_4) = \frac{-f_4 + 6.75}{15.07}$$

The membership function values in vertices are given in the following table III:

Table III: Membership function values at vertices

	$\alpha_1(f_1)$	$\alpha_2(f_2)$	$\alpha_3(f_3)$	$\alpha_4(f_4)$
0(0,0)	3/28	13/38	253/403	675/1507
A(2.6,0)	53/70	0	305/403	1
B(2,1)	1	32/57	1	775/1507
C(1,2)	9/14	1	253/403	75/1507
D(0,2.25)	0	127/152	0	0

From the definition of M-Pareto optimal solution, The M-Pareto optimal solutions of problem (5.1) are: 0(0,0), A(2.6,0), B(2,1), C(1,2), D(0,2.25).

Next, membership function method is used to solve the multi objective programming problem (5.1) and illustrates that there exist weights  $w$  such that the M-Pareto optimal solution of problem (5.1) is the optimal solution of the corresponding single objective programming problem.

According to model ( $S_\alpha$ ), the multi objective programming problem (5.1) is converted to the single objective programming problem:

$$\begin{aligned} \text{Max} \quad & w_1\alpha_1(f_1) + w_2\alpha_2(f_2) + w_3\alpha_3(f_3) + w_4\alpha_4(f_4) \\ \text{s/t} \quad & x_1 + 4x_2 \leq 9 \\ & x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{5.9}$$

Due to the way of structuring membership function of objective functions  $f_1, f_2$  and  $f_3$ , the above method can be written as:

$$\begin{aligned}
 \text{Max} \quad & w_1 \left( \frac{-f_1 - 4.4}{5.6} \right) + w_2 \left( \frac{-f_2 + 1.56}{4.56} \right) + w_3 \left( \frac{-f_3 + 16.12}{16.12} \right) + w_4 \left( \frac{-f_4 + 6.75}{15.07} \right) \\
 \text{s/t} \quad & x_1 + 4x_2 \leq 9 \\
 & x_1 + x_2 \leq 3 \\
 & 3x_1 + 2x_2 \leq 8 \\
 & x_1, x_2 \geq 0
 \end{aligned} \tag{5.10}$$

This can be written as,

$$\begin{aligned}
 \text{Max} \quad & x_1 \left[ \frac{4}{5.6} w_1 + \frac{2}{4.5} w_2 + \frac{6}{16.12} w_3 - \frac{2}{15.07} w_4 \right] + x_2 \left[ \frac{2}{5.6} w_1 + \frac{1}{4.56} w_2 - \frac{3}{15.07} w_4 \right] + \\
 & x_1^2 \left[ -\frac{1}{5.6} w_1 - \frac{1}{4.56} w_2 - \frac{2}{16.12} w_3 + \frac{2}{15.07} w_4 \right] + x_2^2 \left[ -\frac{1}{5.6} w_1 - \frac{2}{16.12} w_3 \right] + \frac{2}{16.12} x_1 x_2 w_3 + \\
 & \left[ \frac{.6}{5.6} w_1 + \frac{1.56}{4.5} w_2 + w_3 + \frac{6.75}{15.07} w_4 \right] \\
 \text{s/t} \quad & x_1 + 4x_2 \leq 9 \\
 & x_1 + x_2 \leq 3 \\
 & 3x_1 + 2x_2 \leq 8 \\
 & x_1, x_2 \geq 0
 \end{aligned} \tag{5.11}$$

In order to take maximum in point  $B(2,1)$ , the slope of objective function only need to satisfy

$$\begin{aligned}
 & \frac{4}{5.6} w_1 + \frac{2}{4.5} w_2 + \frac{6}{16.12} w_3 - \frac{2}{15.07} w_4 > 0 \quad \text{and} \quad -\frac{\frac{4}{5.6} w_1 + \frac{2}{4.5} w_2 + \frac{6}{16.12} w_3 - \frac{2}{15.07} w_4}{\frac{2}{5.6} w_1 + \frac{1}{4.56} w_2 - \frac{3}{15.07} w_4} < -1 \\
 \text{Or} \quad & \frac{4}{5.6} w_1 + \frac{2}{4.5} w_2 + \frac{6}{16.12} w_3 - \frac{2}{15.07} w_4 > 0 \quad \text{and} \quad -\frac{\frac{4}{5.6} w_1 + \frac{2}{4.5} w_2 + \frac{6}{16.12} w_3 - \frac{2}{15.07} w_4}{\frac{2}{5.6} w_1 + \frac{1}{4.56} w_2 - \frac{3}{15.07} w_4} < -\frac{3}{2}
 \end{aligned}$$

If we take,  $w_1 = \frac{1}{2}, w_2 = \frac{1}{4}, w_3 = \frac{1}{8}, w_4 = 1/16$ , the multi-objective programming problem (5.1) can be converted into single objective programming problem (5.11) and putting values we get,

$$\begin{aligned}
 \text{Max} \quad & 0.5064 x_1 + 0.2209 x_2 - 0.1513 x_1^2 - 0.3126 x_2^2 + 0.015 x_1 x_2 + 0.692 \\
 \text{s/t} \quad & x_1 + 4x_2 \leq 9 \\
 & x_1 + x_2 \leq 3 \\
 & 3x_1 + 2x_2 \leq 8 \\
 & x_1, x_2 \geq 0
 \end{aligned} \tag{5.12}$$

After solving we get,  $x_1 = 2$  and  $x_2 = 1$ .

The optimal solution of (5.12) is  $x_1 = 2$  and  $x_2 = 1$ , which is also the M-Pareto optimal solution of problem (5.1). So there exist weights  $w_1 = \frac{1}{2}, w_2 = \frac{1}{4}, w_3 = \frac{1}{8}, w_4 = 1/16$ , such that  $B(2,1)$  is the optimal solution of the corresponding single objective programming problem (5.12).

Similarly, we can show that for point  $C(1,2)$ .

Therefore, for all M-Pareto optimal solution of multi objective programming problem (5.1), there exist weights, such that M-Pareto optimal solution is the optimal solution of the corresponding single objective programming problem.

For both case Norm-Ideal method and Membership function method, the Pareto optimal solution is

$$x_1 = 2 \text{ and } x_2 = 1$$

## 5.2 Using AMPL

After finding the value of each of individual objective functions of example (1) by using AMPL software, the numerical results are given below in table 5.2:

Table 5.2

$i$	$Z_i$	$x_i$	$\varphi_i$	$OA_i$	$OL_i$
1	10	(2,1)	10	10	
2	3.0156	(.875, 2.0313)	3.0156	3.0156	
3	15	(0,2.25)	15		15
4	6.9453	(0.3125, 2.1719)	6.9453		6.9453

To convert multi-objective function into single objective function we apply different types of techniques such as Chandra Sen's technique, statistical average techniques, modified statistical average techniques. Also the proposed technique can be applied.

### 5.3 Using Chandra Sen's Technique

By Chandra Sen's Approach,

$$\begin{aligned}
 \text{Max } Z &= \sum_{k=1}^r \frac{Z_k}{|\varphi_k|} - \sum_{k=r+1}^s \frac{Z_k}{|\varphi_k|} \\
 \text{Max } Z &= \frac{Z_1}{\varphi_1} + \frac{Z_2}{\varphi_2} - \frac{Z_3}{\varphi_3} - \frac{Z_4}{\varphi_4} \\
 &= \frac{4x_1 + 2x_2 - x_1^2 - x_2^2 + 5}{10} + \frac{2x_1 + x_2 - x_1^2}{3.0156} - \frac{6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2}{15} - \frac{2x_1 + 3x_2 - 2x_1^2}{6.9453} \\
 &= 1.1734x_1 + 0.0968x_2 - 0.2751x_1^2 - 0.2333x_2^2 + 0.1333x_1x_2 + 0.1
 \end{aligned}$$

Using Chandra Sen's Approach, the system becomes,

$$\text{Max } Z = 1.1734x_1 + 0.0968x_2 - 0.2751x_1^2 - 0.2333x_2^2 + 0.1333x_1x_2 + 0.1$$

$$\text{Subject to } x_1 + 4x_2 \leq 9$$

$$x_1 + x_2 \leq 3$$

$$3x_1 + 2x_2 \leq 8$$

$$x_1, x_2 \geq 0$$



### 5.4 Using Statistical Average Techniques

Here three types of statistical average techniques have been discussed. Such as- Arithmetic mean technique (AMT), Geometric mean technique (GMT), Harmonic mean technique (HMT).

Applying AMT,

$$A.M_1(10, 3.0157) = 6.5, \quad A.M_2(15, 6.9) = 10.95$$

$$\begin{aligned} \text{Now } \text{Max } Z &= \frac{1}{6.5}(Z_1 + Z_2) - \frac{1}{10.95}(Z_3 + Z_4) \\ &= 1.288x_1 + 0.1876x_2 - 0.3077x_1^2 - 0.6105x_2^2 + 0.1826x_1x_2 + 0.2213 \end{aligned}$$

Using Arithmetic Mean Technique, the system becomes,

$$\text{Max } Z = 1.288x_1 + 0.1876x_2 - 0.3077x_1^2 - 0.6105x_2^2 + 0.1826x_1x_2 + 0.2213$$

$$\begin{aligned} \text{Subject to } \quad x_1 + 4x_2 &\leq 9 \\ x_1 + x_2 &\leq 3 \\ 3x_1 + 2x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Using GMT,

$$G.M_1(10, 3.0156) = \sqrt{10 \times 3.0156} = 5.4914$$

$$G.M_2(15, 6.9) = \sqrt{15 \times 6.9} = 10.1735$$

$$\begin{aligned} \text{Now } \text{Max } Z &= \frac{1}{5.4914}(Z_1 + Z_2) - \frac{1}{10.1735}(Z_3 + Z_4) \\ &= 1.4858x_1 + 0.2514x_2 - 0.3642x_1^2 - 0.6735x_2^2 + 0.1966x_1x_2 + 0.3207 \end{aligned}$$

Using Geometric Mean Technique, the system becomes,

$$\text{Max } Z = 1.4858x_1 + 0.2514x_2 - 0.3642x_1^2 - 0.6735x_2^2 + 0.1966x_1x_2 + 0.3207$$

$$\text{Subject to } \quad x_1 + 4x_2 \leq 9$$

$$x_1 + x_2 \leq 3$$

$$3x_1 + 2x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

Using HMT,

$$H.M_1(10, 3.0156) = \frac{2}{\frac{1}{10} + \frac{1}{3.0156}} = 4.6338$$

$$H.M_2(15, 6.9) = \frac{2}{\frac{1}{15} + \frac{1}{6.9}} = 9.4521$$

$$\begin{aligned} \text{Max } Z &= \frac{1}{4.6338}(Z_1 + Z_2) - \frac{1}{9.4521}(Z_3 + Z_4) \\ &= 1.718x_1 + 0.33x_2 - 0.4316x_1^2 - 0.7448x_2^2 + 0.2116x_1x_2 + 0.4442 \end{aligned}$$

Using Harmonic Average Technique, the system becomes,

$$\text{Max } Z = 1.718x_1 + 0.33x_2 - 0.4316x_1^2 - 0.7448x_2^2 + 0.2116x_1x_2 + 0.4442$$

Subject to  $x_1 + 4x_2 \leq 9$

$$x_1 + x_2 \leq 3$$

$$3x_1 + 2x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

### 5.5 Using Modified Statistical Average Techniques

Here three types of modified statistical average techniques have been discussed. Such as- modified arithmetic mean technique (MAMT), modified geometric mean technique (MGMT), modified harmonic mean technique (MHMT).

Applying MAMT,

From table 5.2, we consider  $m_1 = 3.0156$  &  $m_2 = 6.9$

Then  $AM.AV = \frac{m_1+m_2}{2} = 4.9578$

$$\begin{aligned} Max Z &= \frac{1}{AM.AV} (Z_1 + Z_2 - Z_3 - Z_4) \\ &= 2.017x_1 - 0.4034x_1^2 - 1.2102x_2^2 + 0.4034x_1x_2 - 0.2017 \end{aligned}$$

Thus the QPP becomes,

$$Max Z = 2.017x_1 - 0.4034x_1^2 - 1.2102x_2^2 + 0.4034x_1x_2 - 0.2017$$

Subject to  $x_1 + 4x_2 \leq 9$

$$x_1 + x_2 \leq 3$$

$$3x_1 + 2x_2 \leq 8$$

Using MGMT,

$$GM.AV = \sqrt{m_1m_2} = \sqrt{3.0156 \times 6.9} = 4.5615$$

$$\begin{aligned} Max Z &= \frac{1}{GM.AV} (Z_1 + Z_2 - Z_3 - Z_4) \\ &= 2.1923x_1 - 0.4385x_1^2 - 1.3154x_2^2 + 0.4358x_1x_2 - 0.2192 \end{aligned}$$

Thus the QPP becomes,

$$Max Z = 2.1923x_1 - 0.4385x_1^2 - 1.3154x_2^2 + 0.4358x_1x_2 - 0.2192$$

Subject to  $x_1 + 4x_2 \leq 9$

$$x_1 + x_2 \leq 3$$

$$3x_1 + 2x_2 \leq 8$$

Using MHMT,

$$HM.AV = \frac{2}{\frac{1}{m_1} + \frac{1}{m_2}} = \frac{2}{\frac{1}{3.0156} + \frac{1}{6.9}} = 4.1969$$

$$\begin{aligned} \text{Max } Z &= \frac{1}{HM.AV} (Z_1 + Z_2 - Z_3 - Z_4) \\ &= 2.3827x_1 - 0.4765x_1^2 - 1.4296x_2^2 + 0.4765x_1x_2 - 0.2383 \end{aligned}$$

Thus the QPP becomes,

$$\text{Max } Z = 2.3827x_1 - 0.4765x_1^2 - 1.4296x_2^2 + 0.4765x_1x_2 - 0.2383$$

Subject to

$$\begin{aligned} x_1 + 4x_2 &\leq 9 \\ x_1 + x_2 &\leq 3 \\ 3x_1 + 2x_2 &\leq 8 \end{aligned}$$

## 5.6 Advanced Transformation Technique

Using proposed technique, select

$$m_1 = \min\{10, 3.0156\} = 3.0156 \quad \text{and} \quad m_2 = \min\{15, 6.9453\} = 6.9453$$

$$\text{then } m = \min\{m_1, m_2\} = 3.0156$$

$$\text{Now, we get, } O_{AT} = \frac{1}{(1/m)} = 3.0156$$

Thus the QPP becomes,

$$\begin{aligned} \text{Max } Z &= \frac{1}{O_{AT}} (Z_1 + Z_2 - Z_3 - Z_4) \\ &= 3.3x_1 - 0.66x_1^2 - 0.99x_2^2 + 0.66x_1x_2 - 0.33 \end{aligned}$$

The system,

$$\text{Max } Z = 3.3x_1 - 0.66x_1^2 - 0.99x_2^2 + 0.66x_1x_2 - 0.33$$

Subject to

$$\begin{aligned} x_1 + 4x_2 &\leq 9 \\ x_1 + x_2 &\leq 3 \\ 3x_1 + 2x_2 &\leq 8 \end{aligned}$$

### 5.7 Graphical Method

The single objective quadratic programming problems with same constraints which are achieved by using Chandra Sen's technique, Statistical Average technique, Modified Statistical Average technique and Advanced Transformation technique can be solved by using Graphical method.

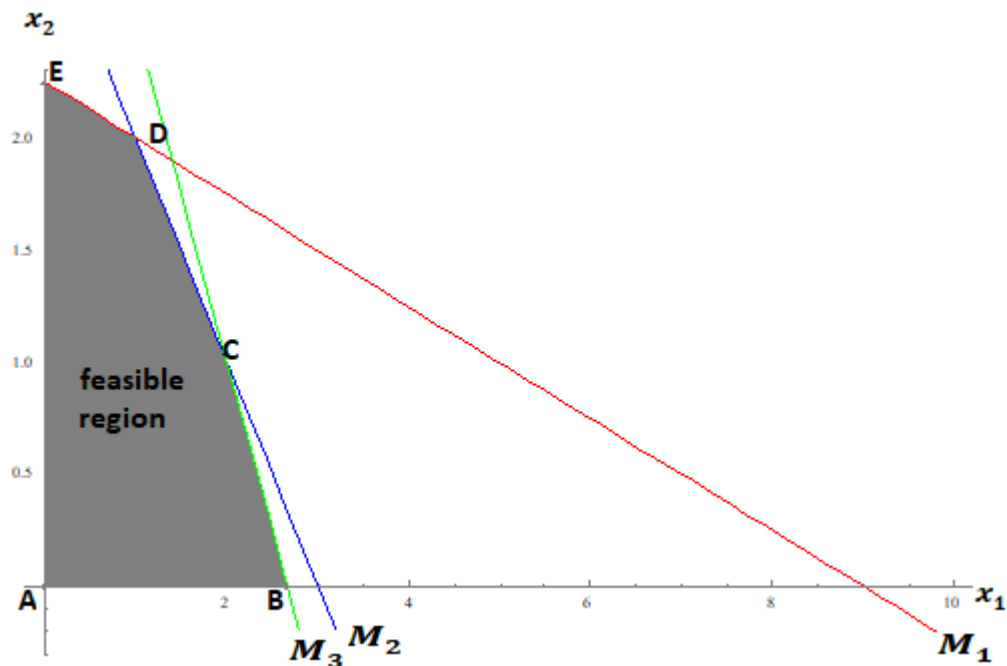
Consider the constraints of the system as:

$$M_1 : x_1 + 4x_2 \leq 9$$

$$M_2 : x_1 + x_2 \leq 3$$

$$M_3 : 3x_1 + 2x_2 \leq 8$$

According to Graphical Method, the feasible region of the system is as follows:



**Figure 5.7:** Feasible region for given MOQPP

Corner points are  $A(0, 0)$ ,  $B(2.6, 0)$ ,  $C(2, 1)$ ,  $D(1, 2)$ ,  $E(0, 2.25)$ .

By Chandra Sen's Technique, the objective function of the system is,

$$\text{Max } Z = 1.1734x_1 + 0.0968x_2 - 0.2751x_1^2 - 0.2333x_2^2 + 0.1333x_1x_2 + 0.1$$

By Graphical method, corner points are used to get maximum value of Z.

Corner point	Z value
<b>A(0, 0)</b>	<b>0.1</b>
<b>B(2.6, 0)</b>	<b>1.29</b>
<b>C(2, 1)</b>	<b>1.47</b>
<b>D(1, 2)</b>	<b>0.5253</b>
<b>E(0, 2.25)</b>	<b>-0.86</b>

So,  $Z_{max} = 1.47$  at the point (2,1).

Similarly, using Graphical method, the maximum value of the respective objective functions for different techniques can be found.

By Modified Harmonic Average Technique, the objective function of the system is,

$$Max Z = 2.3827x_1 - 0.4765x_1^2 - 1.4296x_2^2 + 0.4765x_1x_2 - 0.2383$$

Corner point	Z value
<b>A(0, 0)</b>	<b>-0.2383</b>
<b>B(2.6, 0)</b>	<b>2.1445</b>
<b>C(2, 1)</b>	<b>2.7355</b>
<b>D(1, 2)</b>	<b>1.1913</b>
<b>E(0, 2.25)</b>	<b>-7.475</b>

By Advanced Transformation Technique, the objective function of the system is,

$$Max Z = 3.3x_1 - 0.66x_1^2 - 0.99x_2^2 + 0.66x_1x_2 - 0.33$$

Corner point	Z value
<b>A(0, 0)</b>	<b>-0.33</b>
<b>B(2.6, 0)</b>	<b>3.78</b>
<b>C(2, 1)</b>	<b>3.96</b>

$D(1, 2)$	<b>-0.33</b>
$E(0, 2.25)$	<b>-5.34</b>

### 5.8 Comparison Table

This table summarizes the solutions of the MOQPP using different approaches:

Table 5.8 (a): Comparison Table I

Using Chandra Sen's Approach	Using Arithmetic Ave. Technique	Using Geometric Ave. Technique	Using Harmonic Ave. Technique	Using Modified Arithmetic Ave. Technique	Using Modified Geometric Ave. Technique	Using Modified Harmonic Ave. Technique	Using Advanced Transformation Technique
$Z = 1.47$ $x_1 = 2$ $x_2 = 1$	$Z = 1.51$ $x_1 = 2$ $x_2 = 1$	$Z = 1.81$ $x_1 = 2$ $x_2 = 1$	$Z = 2.16$ $x_1 = 2$ $x_2 = 1$	$Z = 2.32$ $x_1 = 2$ $x_2 = 1$	$Z = 2.52$ $x_1 = 2$ $x_2 = 1$	$Z = 2.74$ $x_1 = 2$ $x_2 = 1$	$Z = 3.96$ $x_1 = 2$ $x_2 = 1$

It shows that the solution of the objective functions improved when we used the proposed Advanced transformation technique.

We are going to compare different techniques with Pareto Optimality. In case of statistical average technique and modified statistical average technique, we consider only harmonic average technique. Then we obtain comparison table II.

Table 5.8 (b): Comparison Table-II

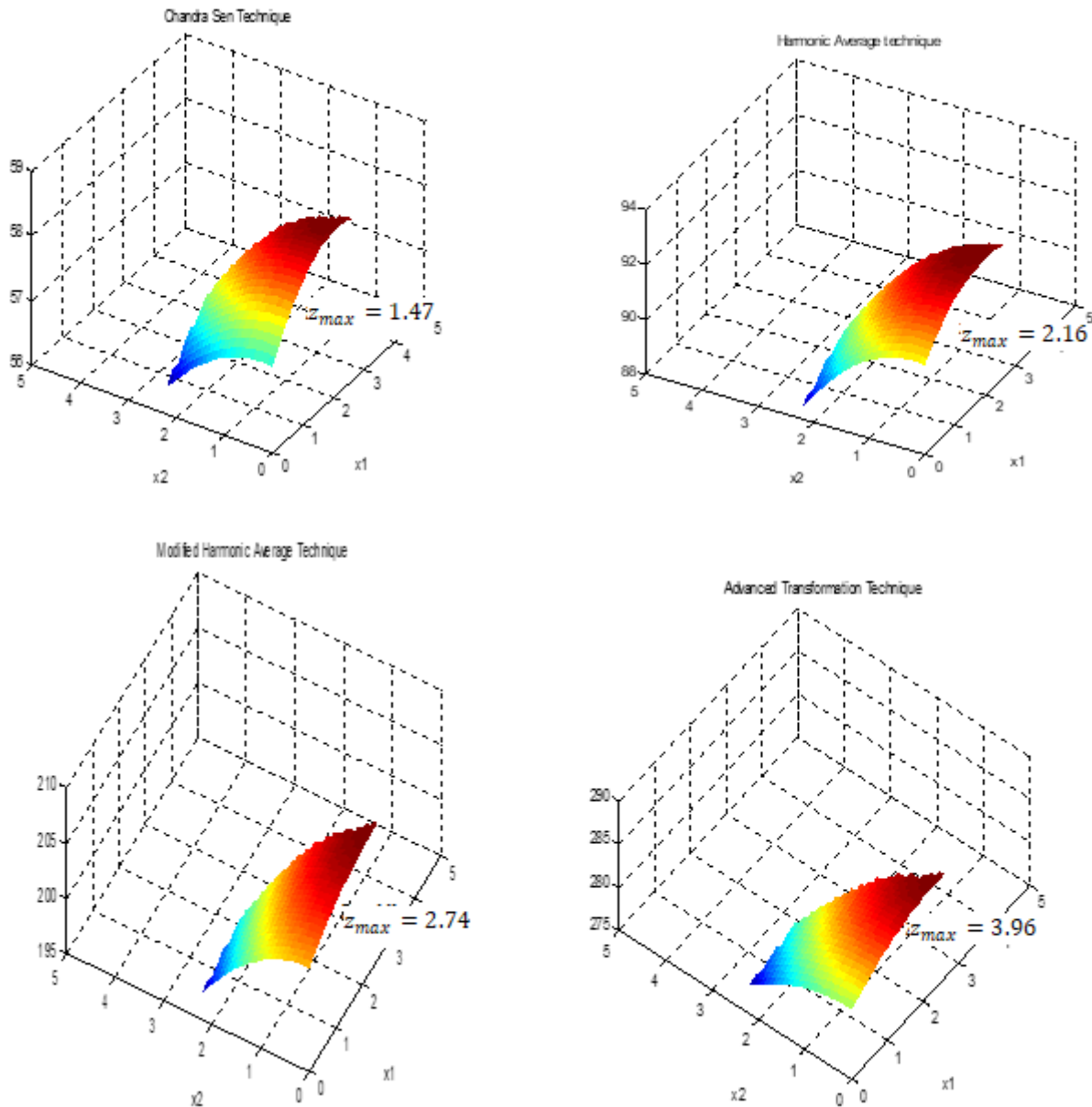
Using Pareto Optimality	Using Chandra Sen's Approach	Using Harmonic Average Technique	Using Modified Harmonic Average Technique	Using Advanced Transformation Technique
$Z = 0.91$ $x_1 = 2$ $x_2 = 1$	$Z = 1.47$ $x_1 = 2$ $x_2 = 1$	$Z = 2.16$ $x_1 = 2$ $x_2 = 1$	$Z = 2.74$ $x_1 = 2$ $x_2 = 1$	$Z = 3.96$ $x_1 = 2$ $x_2 = 1$

Pareto optimality provides a weak but widely accepted standard outcome. This comparison table shows that the results obtained by different techniques are approximately near that of Pareto solutions. After comparing we can easily say that, Advanced transformation technique is more appropriate to solve MOQPP.

## 5.9 Physical Interpretation

In this optimization problem, a process is going to search a better procedure to find maximum value of a given MOQPP.

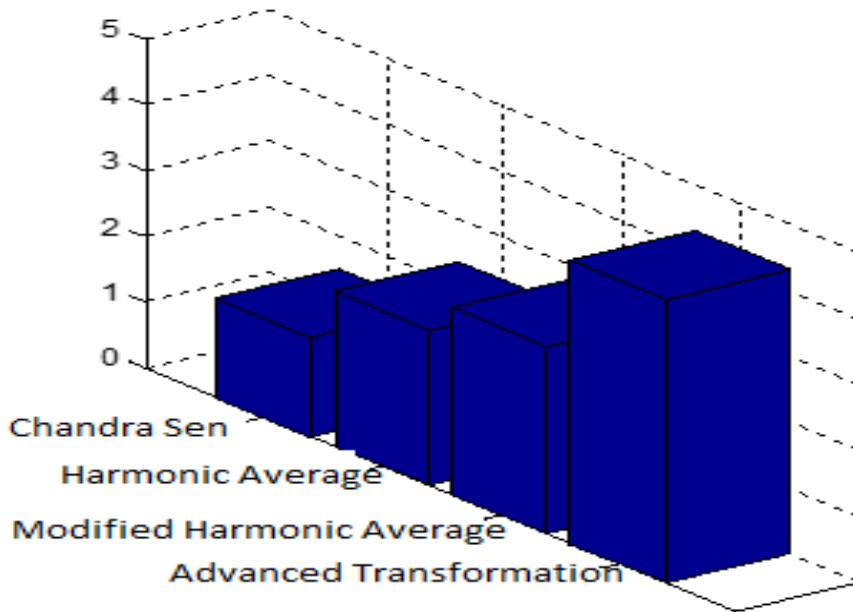
Figure shows that, how the optimized results has improved after applying different techniques. Physical interpretation of the given MOQPP after applying different techniques are shown below:



**Figure 5.9 (a):** Physical interpretation of the given MOQPP after applying different techniques



For more convenience, it can be shown as follows:



**Figure 5.9 (b):** Bar diagram of the given MOQPP comparing different techniques

### 5.10 Data Analysis

The main purpose of data analysis is to find meaning in data so that the derived knowledge can be used to make informed decisions. In simple words, data analysis is the process of collecting and organizing data in order to draw helpful conclusions from it. The process of data analysis uses analytical and logical reasoning to gain information from the data.

Consider, Mixed type Multi-objective linear programming problem (MOLPP):

**Example 2.**

$$\begin{aligned}
 \max z_1 &= x_1 + 2x_2 \\
 \max z_2 &= x_1 \\
 \min z_3 &= -2x_1 - 3x_2 \\
 \min z_4 &= -x_2 \\
 \text{s/t} \quad &6x_1 + 8x_2 \leq 48
 \end{aligned}$$

$$x_1 + x_2 \geq 3$$

$$x_1 \leq 4$$

$$x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

Solving the example by using Chandra Sen's technique, harmonic average technique, modified harmonic average technique and Advanced transformation technique we get the following table:

Techniques Example	Chandra Sen's technique	Harmonic average technique	Modified harmonic average technique	Advanced transformation technique
2	$z_{max} = 3.99$ $x_1 = 4$ $x_2 = 3$	$z_{max} = 6.37$ $x_1 = 4$ $x_2 = 3$	$z_{max} = 9.86$ $x_1 = 4$ $x_2 = 3$	$z_{max} = 11.32$ $x_1 = 4$ $x_2 = 3$

Again consider, Single type Multi-objective linear programming problem (Maximization):

Example 3.

$$\max z_1 = x_1$$

$$\max z_2 = 2 + x_1 + 2x_2$$

$$\max z_3 = 3 + x_2$$

s/t

$$2x_1 + 3x_2 \leq 6$$

$$x_1 + 2x_2 \leq 2$$

$$x_1 \leq 4$$

$$x_1, x_2 \geq 0$$

After solving by different techniques, we obtain the table:

Techniques Examples	Chandra Sen's technique	Harmonic average technique	Modified harmonic average technique	Advanced transformation technique
3	$z_{max} = 2.75$ $x_1 = 2$ $x_2 = 0$	$z_{max} = 3.01$ $x_1 = 2$ $x_2 = 0$	$z_{max} = 4.5$ $x_1 = 2$ $x_2 = 0$	$z_{max} = 4.5$ $x_1 = 2$ $x_2 = 0$

Integrating results of example 1 (MOQPP), example 2 (Mixed type MOLPP) and example 3 (Single type MOLPP), the following table can be obtained:

Table 5.10: Comparison Table-III

Techniques Examples	Chandra Sen's technique	Harmonic average technique	Modified harmonic average technique	Advanced transformation technique
1 (MOQPP)	$z_{max} = 1.5$	$z_{max} = 2.35$	$z_{max} = 2.85$	$z_{max} = 4.3$
2 (MOLPP-Mixed type)	$z_{max} = 3.99$	$z_{max} = 6.37$	$z_{max} = 9.86$	$z_{max} = 11.32$
3 (MOLPP-Max. type)	$z_{max} = 2.75$	$z_{max} = 3.01$	$z_{max} = 4.5$	$z_{max} = 4.5$

Analysis:

Applying different techniques on MOQPP, mixed type MOLPP and maximization type MOLPP, we reach in a conclusion. After analyzing, the result can be synthesized as:

1. For different techniques applied on MOLPP, the result variation is approximately same whereas for MOQPP, the variation is more for applying Advanced transformation technique.

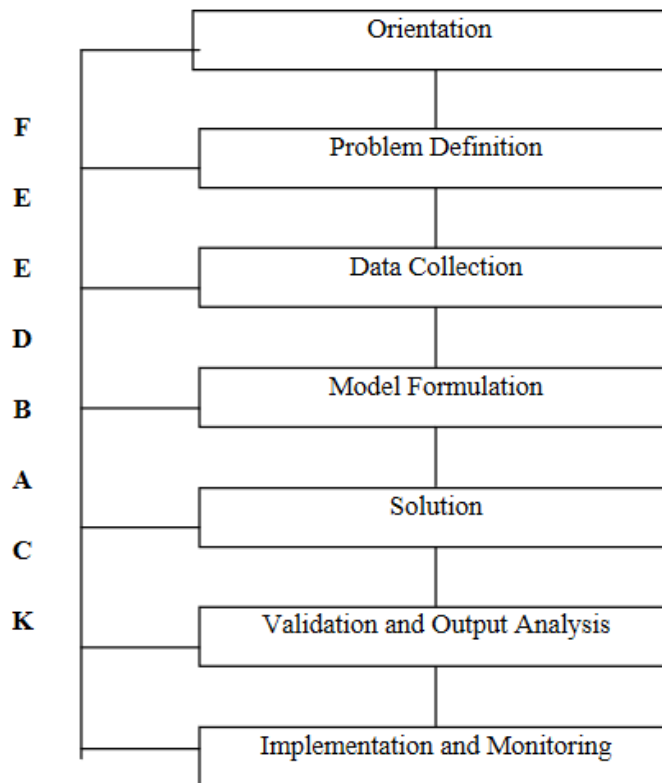
2. For single type MOLPP, modified harmonic average technique and advanced transformation technique give identical output.

## CHAPTER-6

### Practical Aspects

Real world examples demonstrate the complexity and unpredictability of real issues and as such, can stimulate critical thinking. They also highlight the need for an inter- and multi-disciplinary approach to problem solving. Further, using examples from the real world demonstrates that, oftentimes, there is no perfect solution to a given problem. But, in doing so, we think more about solutions, rather than just focusing on problems.

Any kind of real life problems can be model as a mathematical formulation by following step:



**Figure 6:** Mathematical model approach

## Chapter-6, Practical Aspects

In Bangladesh, there are many textile industries. Several buyers purchase different kinds of products from them. The company produces different types of products such as t-shirt, ladies-tops, jeans etc. They use several raw materials like cotton, silk, wool, polyester, viscous etc.

Example: Consider a textile factory, one of their buyer purchases only Polo t-shirt and Ladies tops. To provide that two products the factory needs raw materials, mainly cotton, polyester and viscous. Estimate for the material according to their monthly order from that buyer.

Table 6.1: Raw material list

Raw Material	Yarn (ton)	Finished fabric(ton) (after processing)
Cotton	50	16
Polyester	7	5
Viscous	10	7

Finance report of respective 3 months:

Table 6.2: Monthly Finance Report

Month	Product	Quantity (pc)	Profit/pc (tk)
January	t-shirt	70,000	1
	ladies-tops	50,000	2
February	t-shirt	68000	1
	ladies-tops	55000	4
March	t-shirt	71000	2.5
	ladies-tops	53000	3

Formulate the quadratic programming model to find the maximum profit and optimization output according to buyer's requirement based on product and price.

Solution:

Monthly, average quantity of t-shirt =  $(70000+68000+71000)/3 \cong 70000$

and average quantity of tops =  $(50000+55000+53000)/3 \cong 53000$

These quantities have been produced from finished fabric. For each piece of t-shirt, cotton required =  $16/70000 = .00029$  (ton)

Similarly, the amount of raw materials to produce per piece t-shirt and tops are given below: (multiplied by 1000)

Table 6.3: Converted list

	Cotton (in ton)	Polyester(in ton)	Viscous(in ton)
t-shirt (per pc)	0.29	0.07	0.1
tops(per pc)	0.3	0.09	0.13

Quadratic Formulation:

Let, the quantity of t-shirt =  $x_1$

and tops =  $x_2$

For maximum profit,

profit per piece of t-shirt =  $P_1$

and tops =  $P_2$

The demand functions are based on the three sets of data.

- ✓ The actual quantities of product, say  $\widetilde{x}_1, \widetilde{x}_2$ .
- ✓ Their average profit per piece say  $\widetilde{P}_1, \widetilde{P}_2$ .
- ✓ Price elasticity say  $\varepsilon$ .

These values have been based on existing studies of consumer's behavior. The functions are assumed to be linear. For large positive deviation from the equilibrium prices, this will probably imply an underestimation of demand and hence revenue.

$$\begin{aligned}\frac{x_1 - \widetilde{x}_1}{\widetilde{x}_1} &= -\varepsilon \frac{P_1 - \widetilde{P}_1}{\widetilde{P}_1} \\ \frac{x_2 - \widetilde{x}_2}{\widetilde{x}_2} &= -\varepsilon \frac{P_2 - \widetilde{P}_2}{\widetilde{P}_2}\end{aligned}\quad (6.1)$$

For the price elasticity, let the value is specified as follows:  $\varepsilon = 0.3$

For month 1:

Given  $\widetilde{x}_1 = 70000$ ,  $\widetilde{P}_1 = 1$  and  $\widetilde{x}_2 = 50000$ ,  $\widetilde{P}_2 = 2$

Consider as,  $\widetilde{x}_1 = 70$ ,  $\widetilde{P}_1 = .001$  and  $\widetilde{x}_2 = 50$ ,  $\widetilde{P}_2 = .002$

Using (6.1)

$$\begin{aligned}\frac{x_1 - 70}{70} &= -3 \times 10^{-4} \times \frac{P_1 - .001}{.001} \\ \Rightarrow P_1 &= 3.5 - 0.5 x_1\end{aligned}\quad (6.2)$$

$$\begin{aligned}\text{Also, } \frac{x_2 - 50}{50} &= -3 \times 10^{-4} \times \frac{P_2 - .002}{.002} \\ \Rightarrow P_2 &= 6.5 - 0.13 x_2\end{aligned}\quad (6.3)$$

Now the function to be maximized is  $\sum P_i x_i$ .

Thus the objective function for this case is,

$$\begin{aligned}\text{Max } Z_1 &= P_1 x_1 + P_2 x_2 \\ &= -0.5x_1^2 - 0.13x_2^2 + 3.5x_1 + 6.5x_2\end{aligned}\quad (6.4)$$

For month 2:

Given  $\widetilde{x}_1 = 68000$ ,  $\widetilde{P}_1 = 2$  and  $\widetilde{x}_2 = 55000$ ,  $\widetilde{P}_2 = 4$



Consider as,  $\bar{x}_1 = 68$ ,  $\bar{P}_1 = .002$  and  $\bar{x}_2 = 55$ ,  $\bar{P}_2 = .004$

Using (6.1)

$$\begin{aligned} \frac{x_1-68}{68} &= -3 \times 10^{-4} \times \frac{P_1-.002}{.002} \\ \Rightarrow P_1 &= 6.12 - 0.09 x_1 \end{aligned} \quad (6.5)$$

Also,

$$\begin{aligned} \frac{x_2-55}{55} &= -3 \times 10^{-4} \times \frac{P_2-.004}{.004} \\ \Rightarrow P_2 &= 13.204 - 0.24 x_2 \end{aligned} \quad (6.6)$$

Now the function to be maximized is  $\sum P_i x_i$ .

Thus the objective function for this case is,

$$\begin{aligned} \text{Max } Z_2 &= P_1 x_1 + P_2 x_2 \\ &= -0.09 x_1^2 - 0.24 x_2^2 + 6.12 x_1 + 13.204 x_2 \end{aligned} \quad (6.7)$$

For month 3:

Given  $\bar{x}_1 = 71000$ ,  $\bar{P}_1 = 2.5$  and  $\bar{x}_2 = 53000$ ,  $\bar{P}_2 = 3$

Consider as,  $\bar{x}_1 = 71$ ,  $\bar{P}_1 = .0025$  and  $\bar{x}_2 = 53$ ,  $\bar{P}_2 = .003$

Using (6.1)

$$\begin{aligned} \frac{x_1-71}{71} &= -3 \times 10^{-4} \times \frac{P_1-.0025}{.0025} \\ \Rightarrow P_1 &= 8.5 - 0.12 x_1 \end{aligned} \quad (6.8)$$

Also,

$$\begin{aligned} \frac{x_2-53}{53} &= -3 \times 10^{-4} \times \frac{P_2-.003}{.003} \\ \Rightarrow P_2 &= 10.073 - 0.19 x_2 \end{aligned} \quad (6.9)$$

Now the function to be maximized is  $\sum P_i x_i$ .

Thus the objective function for this case is,

$$\text{Max } Z_3 = P_1 x_1 + P_2 x_2$$

$$= -0.12x_1^2 - 0.19x_2^2 + 8.5x_1 + 10.73x_2 \quad (6.10)$$

Finally using (6.4), (6.7) and (6.10) we get MOQPP with some constraints:

$$\text{Max } Z_1 = -0.5x_1^2 - 0.13x_2^2 + 3.5x_1 + 6.5x_2$$

$$\text{Max } Z_2 = -0.09x_1^2 - 0.24x_2^2 + 6.12x_1 + 13.204x_2$$

$$\text{Max } Z_3 = -0.12x_1^2 - 0.19x_2^2 + 8.5x_1 + 10.73x_2$$

Subject to

$$.29x_1 + 0.3x_2 \leq 16000$$

$$.07x_1 + 0.09x_2 \leq 5000$$

$$.1x_1 + 0.13x_2 \leq 7000$$

$$x_1, x_2 \geq 0$$

After solving by AMPL software, we obtain the result as:

Table 6.4

$i$	$x_i$	$Z_i$	$ \varphi_i $
1	(4, 25)	87.38	87.38
2	(34, 27.5)	285.54	285.54
3	(35, 28.2)	302.01	302.01

Using Chandra Sen's Technique:

The multi objective function is converted into single objective function by this technique,

$$\begin{aligned} \text{Max } Z &= \frac{Z_1}{\varphi_1} + \frac{Z_2}{\varphi_2} + \frac{Z_3}{\varphi_3} \\ &= \frac{1}{87.38} [-0.5x_1^2 - 0.13x_2^2 + 3.5x_1 + 6.5x_2] + \frac{1}{285.54} [-0.09x_1^2 - 0.24x_2^2 + \\ &\quad 6.12x_1 + 13.204x_2] + \frac{1}{302.01} [-0.12x_1^2 - 0.19x_2^2 + 8.5x_1 + 10.73x_2] \\ &= -0.00643 x_1^2 - 0.00296 x_2^2 + 0.089 x_1 + 0.156 x_2 \end{aligned}$$

The system becomes,

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$$\text{Max } Z = -0.00643 x_1^2 - 0.00296 x_2^2 + 0.089 x_1 + 0.156 x_2$$

Subject to

$$\begin{aligned} .29x_1 + 0.3x_2 &\leq 16000 \\ .07x_1 + 0.09x_2 &\leq 5000 \\ .1x_1 + 0.13x_2 &\leq 7000 \\ x_1, x_2 &\geq 0 \end{aligned}$$

After solving we get,

$$Z = 2.36 \text{ with } x_1 = 6.9, x_2 = 26.3$$

Using modified harmonic mean technique:

$$HM.AV = \frac{2}{\frac{1}{m_1} + \frac{1}{m_2}}$$

Here,  $m_1 = 302.01$  and  $m_2 = 87.38$

Then  $HM.AV = 135.54$

Thus,

$$\begin{aligned} \text{Max } Z &= \frac{1}{HM.AV} [Z_1 + Z_2 + Z_3] \\ &= -0.00524 x_1^2 - 0.00413 x_2^2 + 0.134 x_1 + 0.225 x_2 \end{aligned}$$

The system becomes,

$$\text{Max } Z = -0.00524 x_1^2 - 0.00413 x_2^2 + 0.134 x_1 + 0.225 x_2$$

Subject to

$$\begin{aligned} .29x_1 + 0.3x_2 &\leq 16000 \\ .07x_1 + 0.09x_2 &\leq 5000 \\ .1x_1 + 0.13x_2 &\leq 7000 \\ x_1, x_2 &\geq 0 \end{aligned}$$

After solving we get,

$$Z = 3.9 \text{ with } x_1 = 12.78, x_2 = 27.24$$

Using Advanced transformation technique:

$$m = \min\{m_1, m_2\} = 87.38$$

We get, 
$$O_{AT} = \frac{1}{(1/m)} = 87.38$$

Thus, 
$$\begin{aligned} \text{Max } Z &= \frac{1}{O_{AT}} [Z_1 + Z_2 + Z_3] \\ &= \frac{1}{87.38} [-0.71 x_1^2 - 0.559 x_2^2 + 18.16 x_1 + 30.49 x_2] \\ &= -0.008 x_1^2 - 0.006 x_2^2 + 0.208 x_1 + 0.349 x_2 \end{aligned}$$

The system becomes,

$$\text{Max } Z = -0.008 x_1^2 - 0.006 x_2^2 + 0.208 x_1 + 0.349 x_2$$

Subject to

$$\begin{aligned} .29x_1 + 0.3x_2 &\leq 16000 \\ .07x_1 + 0.09x_2 &\leq 5000 \\ .1x_1 + 0.13x_2 &\leq 7000 \\ x_1, x_2 &\geq 0 \end{aligned}$$

After solving we get,

$$Z = 6 \text{ with } x_1 = 13, x_2 = 29$$

Table 3 summarizes the solutions of the MOQPP:

Table-6.5

Chandra Sen's Approach	Modified Harmonic Average Technique	Advanced transformation technique
$Z = 2.36$ $x_1 = 7, \quad x_2 = 26$	$Z = 3.9$ $x_1 = 13, x_2 = 27$	$Z = 6$ $x_1 = 13, x_2 = 29$

The result gives an assumption about the ratio of the production and expresses maximum profit for each product.

Therefore, it can easily be seen that the maximum profit is ensured by using the proposed technique.

## Result and Discussion

A survey has been presented on the use of quadratic programming in solving business planning problem.

We carried out two Pareto optimal methods to solve a multi objective optimization problem and found approximately same results. MOQPP is given to illustrate that for any Pareto optimal solution there exist weights such that Pareto optimal solution is the optimal solution of the corresponding single objective programming problem. Since weights are not unique, all Pareto optimal solution and M- Pareto optimal solution can be obtained through taking out different weights. Results obtained by Pareto optimality are compared with different solution techniques.

It has been seen that the optimization result using proposed advanced transformation technique is better than that of other techniques. In case of statistical average technique and modified statistical average technique, we consider only harmonic mean technique. As among them, harmonic mean technique gives more effective solution.

The quadratic programming formulations are more realistic than is linear programming. Here, a real life problem is converted into quadratic programming problem. The optimization model discussed here is more consistent with the decision maker's preference postulates. Finally, for maximum profit (per piece product), the ratio of quantities of products can be fixed with the help of advanced transformation technique.

### **Limitations:**

In each case, we obtain different single objective functions and observe variations in each single objective function although derived from same set of data. The optimal values are obtained at the same optimum point and the difficulty arise by having a set of optimal solutions instead of a single optimum solution. The result variation occurs due to coefficient varies which may lead complication.

The solutions obtained from this technique is difficult to implement as most of them are usually unrealistic. It is only summary figure which may not be the actual item in the series as these solutions vary more than Pareto optimal solutions. The effectiveness of this technique is conflicting.

**Future Study:**

Many real world search and optimization problems are naturally posed as nonlinear programming problems having multiple objectives. Due to the lack of suitable solution techniques, such problems are artificially converted into a single objective problem and solved.

The complexity arose because such problems give rise to a set of trade-off optimal solutions instead of single optimum solution.

Our next target is to discuss on different Pareto optimality such as NSGA-I, NSGA-II, Evolutionary Algorithm etc. and will check which one suits the problem best.

The goal may be to find a representative set of Pareto optimal solutions, quantify the trade-offs in satisfying the different objectives and finding a single solution that satisfies the subjective preferences of human decision maker (DM).

## Conclusion

In this research, we have defined and discussed a number of techniques which we have used in order to get optimal solution of the MOQPP. This paper performed structural analysis of Pareto optimal solution and M- Pareto optimal solution for convex MOQPP. Two methods discussed here are very effective to obtain Pareto optimal solution. It is observed that to solve MOQPP the statistical average techniques result better than Chandra Sen's approach. We have introduced a modified statistical (A.M, G.M, H.M) average technique. From the analysis, it has been found that MOQPP are best optimized using modified harmonic average technique. Also we have formulated a new quadratic formulation that can be added a new dimension for industrial aspects. We have proposed a new technique to solve MOQPP named Advanced transformation technique.

This research paper discusses the importance of the proposed advanced transformation techniques to solve multi objective optimization problems. This is a quick safe technique which makes large and complex problems more tractable and accurate. The structural analysis of three different techniques for finding a basic feasible solution are compared throughout performed numerical test examples. The study shows that the proposed method has better performance than other methods. Different data analysis and visual presentation ensure its perfection.



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