ANALYSIS OF SOLITON SOLUTIONS TO THE NONLINEAR SCHRODINGER AND KONOPELCHENKO-DUBROVSKY EQUATIONS

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Dedicated

to

My Parents

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ABSTRACT

Many phenomena in the real world are described by nonlinear evolution equations (NLEEs), which have recently gained popularity. In this dissertation, we have explored two NLEEs to develop the generic and compatible closed form stable wave solutions by applying the (w/g)-expansion methods and the Modified Version of the New Kudryashov (MVNK) method. In this research, the (2+1)-dimensional paraxial nonlinear Schrodinger equation is investigated by the (w/g)-expansion methods. Also, the (2+1)-Konopelchenko–Dubrovsky (KD) equation is investigated via MVNK method. With the help of MATLAB, Wolfram Mathematica and Maple software, the solutions describe many forms of solitons and vary their nature and positions displayed in 3-dimensional and 2-dimensional figures for the values of Kerr nonlinearity, nonlinear coefficient, wave number, wave speed etc. Even so, it is found that the features of the solutions are crucial in parameter selection when comparing our results to existing literature produced using various methodologies and evaluating the solutions by drawing figures for various values of the corresponding parameters. Additionally, we show how the values of the various kinds of parameters relate to the physical justification of the determined solution. We have shown that the main reason why wave profiles behave differently when their associated free parameters change. The impacts of wave velocity and other free parameters on the wave profile are also examined. However, by sketching images of the solutions for different values of the associated parameters and examining the results of these approaches, it is evident that the solutions characteristics are greatly influenced by the parametric values. Although the facing few limitations, the techniques used are trustworthy, clear-cut, useful, and simple to apply.

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Nomenclature

$g \rightarrow$ arbitrary function

$l, m, n \rightarrow$arbitrary constant

$M, N \rightarrow$ arbitrary constant

$p, q \rightarrow$ nonlinear coefficient

$u \rightarrow$ velocity component along x-axis

$v \rightarrow$ velocity component along y-axis

$w \rightarrow$ arbitrary function

Greek Symbols

$\delta_1, \delta_2 \rightarrow$ frequency of the soliton

$\epsilon \rightarrow$ kerr nonlinearity

$\theta \rightarrow$ amplitude function

$\theta_0 \rightarrow$ obliqueness

$\vartheta \rightarrow$ diffraction parameter

$\lambda \rightarrow$ wave number

$\mu \rightarrow$ speed of phase

$\sigma \rightarrow$ wave velocity

Abbreviation

KD→ Konopelchenko-Dubrovsky

KdV→ Korteveg-de-Vries

MVNK→ Modified Version of New Kudryashov

NLEEs→ nonlinear evolution equations

NLS→ nonlinear Schrodinger

ODE→ ordinary differential equation

PDEs→ partial differential equations

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Chapter 1: Introduction

The study of partial differential equations (PDE's) started in the 18^{th} century in the work of Euler, d'Alembert, Lagrange, and Laplace as a central tool in the description of mechanics of continua and more generally, as the principal mode of analytical study of models in the physical science. Beginning in the middle of the 19^{th} century, particularly with the work of Riemann, partial differential equations (PDE's) also became an essential tool in other branches of mathematics. This duality of viewpoints has been central to the study of PDE's through the 19^{th} and 20^{th} century. The nonlinear evolution equations (NLEEs) are a particular kind of partial differential equation (PDE) that have been the focus of extensive research in many fields of nonlinear science and engineering. Evolution equations is the evolution of a system depending on a continuous time variable t described by an equation of the form

$$u_t = f(u)$$

where, u_t denoting the time derivative, $u(t) \in X$ is the state of the system at time t, and f is a given vector field on X. The space X is the statespace of the system; a point in X specifies the instantaneous state of the system. In physics, applied mathematics, and engineering fields, a wave is a propagating dynamic disturbance (change from equilibrium) of one or more quantities, sometimes as illustrated by a wave equation of the following

$$y(x,t) = A\sin(kx - \omega t + \phi)$$

In other words, waves involve the transport of energy without the transport of matter through a medium. Nowadays NLEEs have become most examined subject of all-embracing studies in several branches of nonlinear sciences. A special class of analytical solutions named solitary wave solutions for NLEEs has a lot of importance, because most of the phenomena that arise in mathematical physics and engineering fields can be described by NLEEs. NLEEs are repeatedly used to describe many problems of the wave propagation phenomena, quantum mechanics, fluid mechanics, plasma physics, propagation of shallow water waves, optical fibres, biology, solid state physics, electricity, and so forth.

1.1 Some Nonlinear Waves

1.1.1 Standing wave

A standing wave, also called a stationary wave, is the combination of two waves moving in opposite directions, each having the same amplitude and frequency. The phenomenon is the result of interference; that is, when waves are superimposed, their energies are either added together or canceled out. If two people shake a jump rope, the pattern of waves it forms is quite similar to standing or stationary waves.

1.1.2 Travelling wave

A travelling wave in which the particles of the medium move progressively in the direction of the wave propagation with such a gradation of speeds that the faster overtake the slower and are themselves in turn overtaken compared to a standing wave. A few examples of these waves are water waves, sound waves, seismic waves, etc.

Photo collected from Google

Figure 1.1: Travelling wave

1.1.3 General wave

A wave transfers energy from one place to another without the transfer of particles in the medium. Rather, individual particles vibrate (oscillate) about fixed positions instead. In other words, wave is a disturbance in a medium that carries energy without a net movement of particles. It may take the form of elastic deformation, a variation of pressure, electric or magnetic intensity, electric potential, or temperature.

Photo credit: AGU journal, Wiley,V-113,2008

Figure 1.2: General wave

1.1.4 Rogue wave

Rogue wave (also known as freak wave, monster wave, episodic wave, killer wave,

Collected from Google

Figure 1.3: Rogue wave

extreme wave, abnormal wave) are usually large, unexpected, and suddenly appearing surface wave that can be extremely dangerous, even to large ships such as ocean liners.

1.1.5 Solitary wave

A solitary wave is a wave which propagates without any temporal evolution in shape or size when viewed in the reference frame moving with the group velocity of the wave. Roughly speaking, a solitary wave is a nonsingular solution that travels as a localized packet.

Photo credit: AGU journal, Wiley,V-113,2008

Figure 1.4: Solitary wave

1.2 Balance of dispersion and nonlinearity

The dynamics of water waves in shallow water is described mathematically by the Korteveg-de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = 0$$

where $u = u(x, t)$ measures the elevation at time t and position x, $i.e.$ the height of the water above the equilibrium level. The subscripts denote partial differentiation. The second and the third term in the equation is the dispersive and the nonlinear term respectively.

1.2.1 Dispersion

Let us first investigate the effect of the dispersive term. Thus, we neglect the nonlinear term in the KdV equation. This leaves us with the following:

$$u_t + u_{xxx} = 0$$

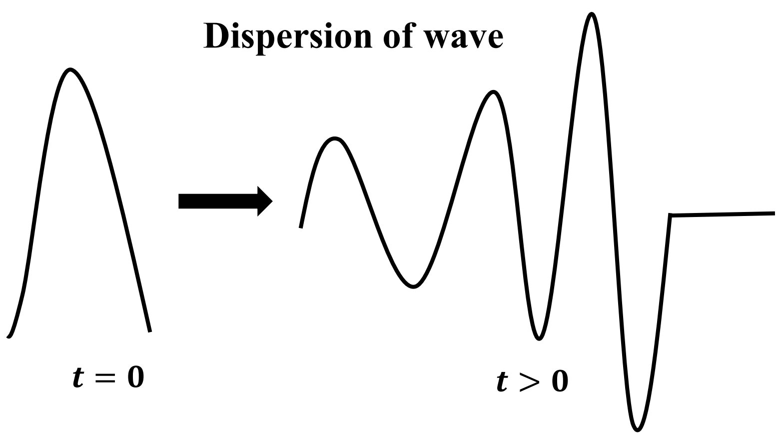


Figure 1.5: Dispersive wave

1.2.2 Nonlinearity

Now let us see the effect of the nonlinear term. We neglect the dispersive term in the KdV equation, which leaves us with the following:

$$u_t + uu_x = 0$$

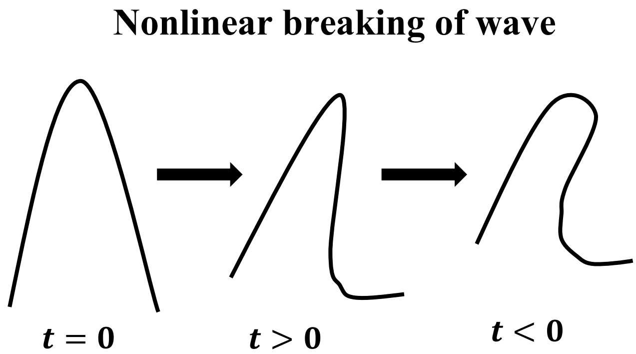


Figure1.6: Nonlinearity

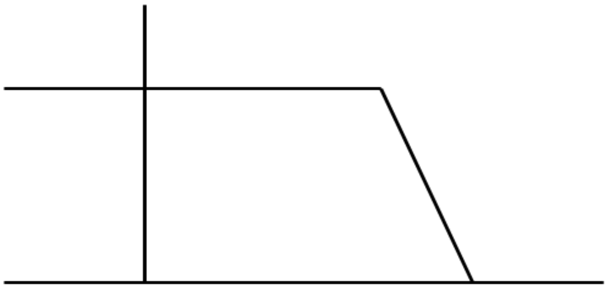
1.3 Soliton

In mathematics and physics, a soliton or solitary wave is a self-reinforcing wave packet that maintains its shape while it propagates at a constant velocity. Solitons are caused by a cancellation of nonlinear and dispersive effects in the medium which have been described through the KdV equation as follows:

$$u_t - 6uu_x + u_{xxx} = 0$$

Nonlinear term

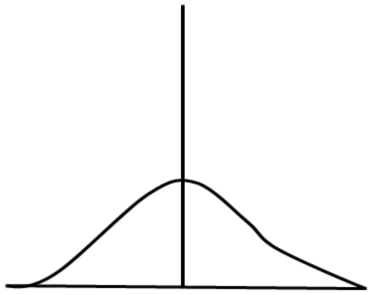
$u_t - 6uu_x$



Steepen

Dispersion term

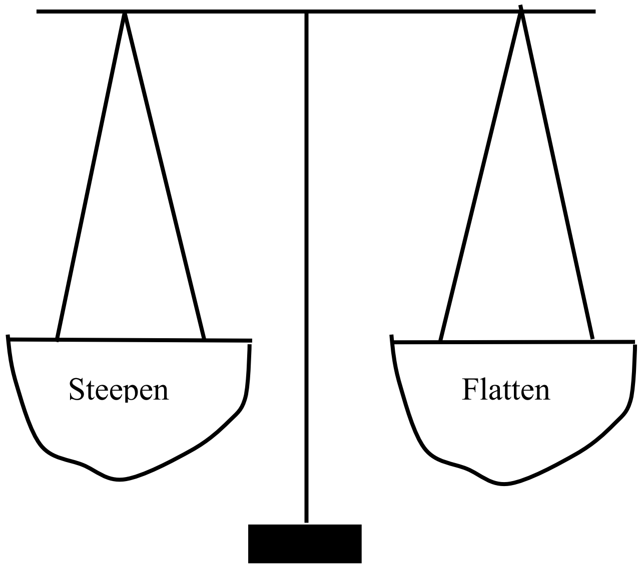
$u_t + u_{xxx}$



Flatten

Profile of the solution curve:

- ✓ Unchanging in shape
- ✓ Bounded
- ✓ Localized



Steepen + Flatten =Stable

When both the dispersive and the nonlinear terms are present in the equation the two effects can neutralize each other. If the water wave has a special shape the effects are exactly counterbalanced, and the wave rolls along undistorted. The soliton shape can be found by direct integration of the KdV equation:

$$u(x,t) \;=\; a\,\mathrm{sech}^2[b(x-\omega t)]$$

with $b = \left(\frac{a}{12}\right)^{\frac{1}{2}}$ and $\omega = 3a$. The constant a is the only free parameter in the solution. It defines the amplitude and the width in such a way that a large (tall) soliton will be narrow, while a low soliton will be broad. The constant ω defines the velocity of the soliton. Since $\omega = 3a$ a tall soliton will move faster than a low one.

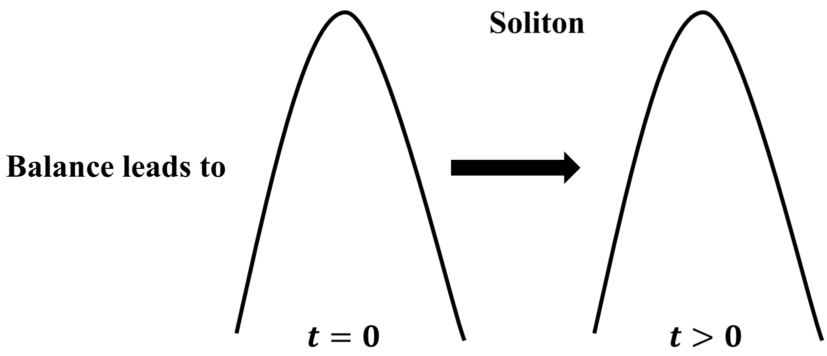


Figure 1.7: Soliton

1.4 Some solitary solutions

Solitary waves are localized travelling waves with constant speeds and shape, asymptotically zero at large distances. Solitons are special kinds of solitary waves. Solitons have a remarkable soliton property in that it keeps its identity upon interacting with other solitons.

The soliton solution is spatially localized solution, hence $u'(\xi), u''(\xi)\ and\ u'''(\xi) \to 0, as\ \xi \to \pm\infty, \xi = x \pm \omega t$.

1.4.1 Periodic Solutions

Periodic solutions are travelling wave solutions that are periodic such as $\cos(x - t)$. The standard wave equation $u_{tt} = u_{xxx}$ gives periodic solutions. As stated before, because this standard wave equation is linear, it admits d'Alembert solution, and components can be superposed. Figure 1.8 shows a periodic solution $u(x, t) = \cos(x - t) - \pi \leq x, t \leq \pi$ for a standard wave equation.

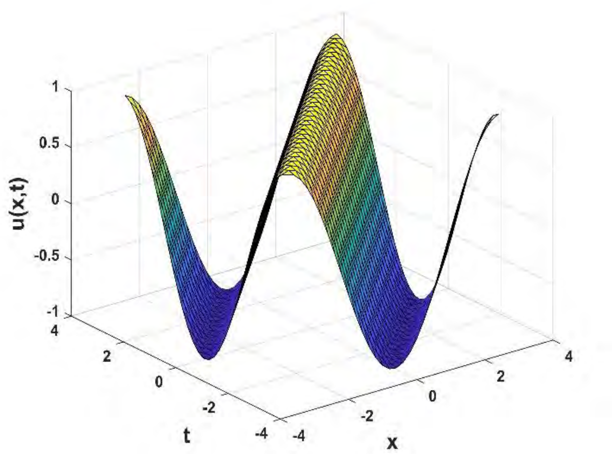


Figure 1. 8: Graph of a periodic solution $u(x, t) = \cos(x - t)\,; -\pi \leq x, t \leq \pi$.

1.4.2 Kink Type Soliton Solution

Kink waves are travelling waves which rise or descend from one asymptotic state to another. The kink solution approaches a constant at infinity. The standard dissipative Burgers equation is

$$u_t + uu_x = vu_{xx}$$

Where ν is the viscosity coefficient, is a well-known equation that gives kink solutions. Figure 1.9 shows a kink soliton Solution $u(x,t) = \tanh(x-t) - 10 \leq x, t \leq 10$ for Burgers equation with $v = \frac{1}{2}$

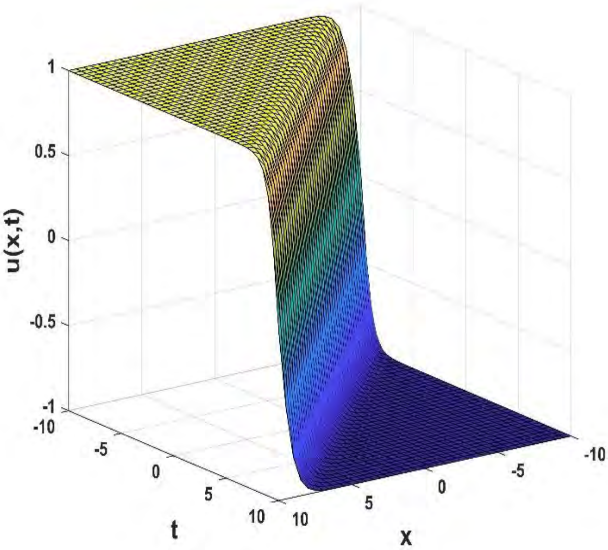


Figure 1.9: Graph of a kink solution $u(x,t) = \tanh\,(x-t) - 10 \leq x, t \leq 10.$

1.4.3 Bell-Shaped Soliton Solution

Bell-shaped soliton solution has infinite wings or infinite tails. This soliton referred to as non-topological solitons. This solution does not depend on the amplitude and high frequency soliton. Figure 1.10 shows the shape of the exact bell-shaped soliton solution $u(x,t) = \operatorname{sech}(x-t) - 4 \leq x, t \leq 4.$

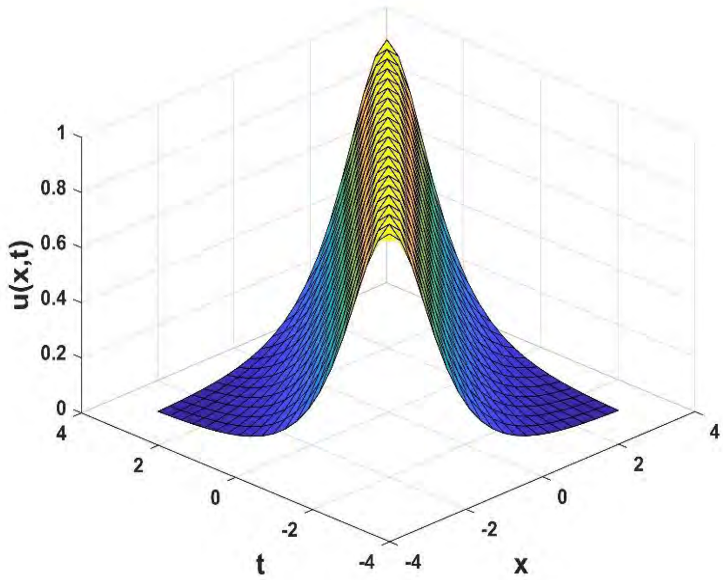


Figure 1.10: Graph of a bell solution $u(x, t) = \text{sech}\,(x - t); -4 \leq x, t \leq 4$.

1.5 Background of Soliton

The soliton phenomenon was first described in 1834 by John Scott Russell (1808–1882) who observed a solitary wave in the Union Canal in Scotland. He reproduced the phenomenon in a wave tank and named it the "Wave of Translation".

In 1834, John Scott Russell describes "Wave of Translation". The discovery is described here in Scott Russell's own words: "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the "Wave of Translation". Scott Russell's solitary "Wave of Translation" is nowadays called a "soliton".

A soliton is a special solution of a special non-linear PDE (wave equation) which:

- ✓ Is localized (“a lump of energy”)
- ✓ Moves with constant shape and velocity in isolation.
- ✓ Is preserved under collisions with other solitons if two solitons collide, they re-emerge with the shapes and velocities.

When a young engineer named John Scott Russell was engaged for a summer job in 1834 to look into how to increase the effectiveness of designs for barges that were intended to ply canals—specifically the Union Canal near Edinburgh, Scotland he made the first known observation of a lone wave. The barge abruptly came to a stop one August day when the tow rope connecting the mules to the barge snapped. However, the water mass in front of the barge's blunt prow “rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth, and well-defined heap of water, which continued its course along the channel without change of form or diminution of speed.” [1].

Photo credit: Hieu D. Nguyen, IEEE Night, 20-05-2003

Figure 1.11: Modern picture of narrow-boat being towed by horse on tow-path which inspired Russell’s observation on solitary waves.

Photo credit: Hieu D. Nguyen, IEEE Night, 20-05-2003

Figure 1.12: Soliton on the Scott Russell Aqueduct on the Union Canal near Heriot-Watt University, 12 July 1995.

Russell pursued this serendipitous observation and "followed it [the launched 'Wave of Translation'] on horseback and overtook it still rolling on at a rate of some eight or nine miles per hour, preserving its original form some thirty feet long and a foot to a foot and a half in height." He then conducted controlled laboratory experiments using a wave tank and quantified the phenomenon in an 1844 publication [1]. He demonstrated four facts:

- ✓ The waves are stable, and can travel over very large distances (normal waves would tend to either flatten out, or steepen and topple over)
- ✓ The speed depends on the size of the wave, and its width on the depth of water.
- ✓ Unlike normal waves they will never merge – so a small wave is overtaken by a large one, rather than the two combining.
- ✓ If a wave is too big for the depth of water, it splits into two, one big and one small.

Scott Russell's experimental work seemed at odds with Isaac Newton's and Daniel Bernoulli's theories of hydrodynamics. George Biddell Airy and George Gabriel Stokes had difficulty accepting Scott Russell's experimental observations because they could not be explained by the existing water wave theories. Their contemporaries spent some time attempting to extend the theory, but it would take until the 1870s before Joseph Boussinesq

[2] and Lord Rayleigh published a theoretical treatment and solutions. In 1895 Diederik Korteweg and Gustav de Vries provided what is now known as the Korteweg–de Vries equation, including solitary wave and periodic cnoidal wave solutions [3].

In 1965 Norman Zabusky of Bell Labs and Martin Kruskal of Princeton University first demonstrated soliton behavior in media subject to the Korteweg–de Vries (KdV) equation in a computational investigation using a finite difference approach [4]. They also showed how this behavior explained the puzzling earlier work of Fermi, Pasta, Ulam, and Tsingou.

1.6 Literature review

A special category of partial differential equations is designated nonlinear evolution equations (NLEEs). The study of numerous models in engineering, physics, and mathematical physics is significantly impacted by the NLEEs. Due to the practical applications of the analytical solutions in numerous fields, including chaos, optical fibres, nonlinear optics, quantum mechanics, mathematical biology, shallow-water wave propagation, electromagnetic theory, solid-state physics, neural physics, diffusion process, reaction process, and others, the studies are therefore of fundamental importance. Several phenomena are explained by nonlinear evolution equations for large-scale study in various fields of physical engineering and science. Firstly, Gardner et al. [5] developed an inverse scattering transformation for findings the soliton Solutions that describe any finite number of solitons in interaction can be expressed in closed form. two years later, assisting with analytical solutions from the KdV equation. This equation highly used in weakly nonlinear ion-acoustic waves in a magnetized plasma. Consequently, Wang [6] introduced nonlinear transformation to find the exact and explicit solitary wave solutions to the KdV Burgers equation by using a homogeneous balance method that describes the shallow water waves, plasma waves. Also, Camassa et al. [7] peakons introduced by Camassa and Holm the simplified modified Camassa-Holm (SMCH) equation which describes the shallow water waves and complex integrable Hamiltonian systems on Riemann surfaces.
optical soliton is a type of soliton that plays an important role in science and engineering. Nonlinear Schrodinger (NLS) model clarifies the pulses in optical fiber. For example, Kudryashov [8, 9] used the nonlinear Schrodinger (NLS) model to describe the propagation of pulses in an optical fiber and to describe a wave packet envelope for changing the pulse's amplitude and width. Kundu et al. [10] studied the Kundu and Mukherjee equation and obtained optical soliton solutions and its wave events play an important role in the ocean currents of rogue waves through exact dynamical lamp solitons,

high speed data transmission in communication systems and describing of propagation pulses in optical fibers. Konopelchenko and Dubrovsky first got the Konopelchenko–Dubrovsky equations that's describes the propagation of weak shock waves in a fluid and plasma physics [11]. Recently, Liu et al. [12] investigate the (2+1) dimensional Konopelchenko and Dubrovsky equation and obtained the exact dynamical lamp solitons, which play a vital role in the interaction of rough waves, in closed form. Moreover, the nonlinear optics, optical fibers phenomena reported by the the Biswas Arshad (BA) equation that is examined by Biswas and Arshed [13]. Also, the new Hamiltonian amplitude (nHA) equation used in nonlinear optics, optical fibers which is noticed by Wadati et al. [14]. Recently, Bilal et al. [15] observed that the new wave structures to the Gilson–Pickering equation (GPE) for numerical and experimental verification in plasma physics. Biswas [16] purify the generalized Zakharov–Kuznetsov modified equal width (ZKMEW) equation and found the 1-soliton that's describe in the quantum magneto-plasmas. The Maccari system (MS) introduced by Maccari based on Fourier expansion and spatio-temporal rescaling from the Kadomtsev–Petviashvili equation [17]. The (2 + 1)-dimensional couple breaking soliton equation describe in fluid dynamics, the Burgers equation represents the propagation of weak shock waves in a fluid, the disparity in vehicle density in highway traffic [18,19] and so on.

There are several NLEEs through which scientists have expressed various phenomena that occur in nature in relation to the various aspects. Different groups of mathematicians and physicists have successfully examined the (2+1)-dimensional paraxial NLS equation in Kerr media. Arshad et al. [20, 21] studied the non-dispersive and non-diffraction spatiotemporal localized waves of the paraxial NLS equation using a modified extended mapping technique. The modified auxiliary expansion method was applied to the paraxial NLS equation and obtained new optical soliton solutions, stability analysis and also showed constraint conditions of the attained solutions by Gao et al. [22]. Ali et al. [23], the extended trial equation method is applied to the paraxial NLS equation and retrieves the periodic, bright and singular solitons.

Moreover, many researchers find the solitons solutions of the coupled (2+1)-dimensional nonlinear Konopelchenko–Dubrovsky (KD) model by using different techniques. Sachin Kumar et al. [24] scrutinized the Konopelchenko-Dubrovsky (KD) equation, which in mathematical physics represents non-linear waves with weak dispersion via rational function method. Also, they analyzed the bifurcation with the help of first Hamiltonian integral. Hongyan [25] inquired similarity reduction of the Lax pair for Konopelchenko-Dubrovsky (KD) equation aid of Lie symmetry technique. Taghizadeh and Mirzazadeh [26] investigated the Konopelchenko-Dubrovsky (KD) equation by using the first integral method. Pandey et al. [27] explored the instability of the Konopelchenko-Dubrovsky (KD) equation and they demonstrate that these waveforms are longitudinally unstable with regard to two-dimensional perturbations that have long wavelengths in the direction perpendicular and are periodic in both directions. Zhang [28] reported the numerous brand-new, precise non-travelling wave solutions and their physical phenomena to the Konopelchenko-Dubrovsky (KD) equation with the help of ansatz technique.

Due to the NLEEs high success rate in illuminating complex issues across industries. Hence, among researchers, looking for single wave solutions has become more common. Yet, it takes a lot of time and effort to find numerical or theoretical solutions to NLEEs that pertain to real-life situations. Theoretical or numerical analysis of a nonlinear model of a real-world issue is challenging. The occurrence of peaking regimes, many steady states under different conditions, the multiplicity or lack of steady states, and many other complicated nonlinear phenomena can all be explained by stable solutions to nonlinear equations. The consistency and error estimates of various numerical, asymptotic, and approximate analytical approaches can be checked using even the special precise solutions that lack a clear physical meaning. Precise solutions can be used to refine and test computer algebra software that solve NLEEs. To evaluate theoretical, approximative, or numerical solutions to nonlinear models, many researchers studied variety and more sophisticated methods. Irshad et al. [29] applied the Exp-function method to the improved Boussinesq equation for finding the solitary solutions and periodic solutions appearing in mathematical physics. Wazwaz [30] investigated exact solutions via compactons solutions, solitons solutions and plane periodic solutions of mCH equation by using direct anaatze which is simplified from mCH equation. They obtained some exponential solutions that leads to singular soliton. The (G'/G)-expansion method applied to SMCH equation and find out some exact travelling wave solutions by Liu et al. [31]. Hafez et al. [32] through the using novel (G'/G)-expansion method to the (2+1)-dimensional nonlinear complex coupled Maccari equation had been successfully and analysis of distinct physical structures. Again, Hafez et al. [33] noticed importance of couple physical model by using the exp $(-\varphi(\xi))$-expansion method to the coupled Higgs equation and the Maccari system. Akbar et al. [34] examined the time-fractional Kundu–Eckhaus equation in the sense of beta fractional derivative through the $(G'/G, 1/G)$-expansion approach for search out the transmission of data through the optical fiber. Fractional implications of the time-fractional modified equal width equation and examined soliton solutions via $(G'/G, 1/G)$-expansion method by Bashar et al. [35]. Islam et al. [36] studied Benney–Luke equation by using the enhanced (G'/G)-expansion method. They clarify the handling of nonlinear optics phenomena. Islam et al. [37] presented some more and more exact solutions yields from the SMCH equation with modified simple equation method. Lump solutions, lump-soliton solutions, and lump-kink solutions result from special instances obtained employed the Hirota Bilinear scheme by Ma et al.[38]. Li-xin et al. [39] found the exact travelling wave solutions and double soliton solutions of CH equation and he also introduced convex peaked and smooth soliton solutions. By using the modified Kudryashov and new auxiliary equation approaches, new dual-wave soliton solutions to the two-mode Sawada-Kotera (TmSK) equation emerging in fluids are addressed by Kumar et al. [40]. In comparison to the family of tanh function methods, the unified method is not just more general Compared to the more recent members of the G′/G - expansion method family, it provides far more generic solutions. First, unifying the family of tanh function methods and the family of (G'/G)-expansion methods is the unified method's substantial contribution in comparison to previous approaches studied by Akcagil and Aydemir [41]. Fatema et al. [42] investigated symmetric regularized long-wave (SRLW) equation describes the attribute of the nonlinear ion acoustic waves, space charge waves, undular bore in meteorology by using two different scheme namely New auxiliary equation method and Improved Bernoulli sub-equation function method. Gonzalez-Gaxiola et al. [43] have investigated the KMN model by the Laplace Adomian decomposition method, and it's reported some breathers type optical soliton solutions. Li et al. [44] presented the (w/g) -expansion method to construct the exact solutions for the Vakhnenko equation and explained the loop soliton solution. Very earlier Zayed and Arnous [45] applied this method to a modified generalized Vakhnenko equation and obtained the periodic, soliton and rational function solutions. Gepreel [46] explored the solitary wave solutions are derived from the travelling waves solutions when the parameters are taken some special values for Kadomtsev–Petviashvili hierarchy (KPH) equations via the (w/g)-expansion method. Durur and Asif [47] obtained the exact solutions and discussed the paraxial wave equation on diffraction and the dispersion phenomena using the modified $(1/G)$-expansion and modified Kudryashov methods.

Kudryashov et al. [48] investigated the second-order nonlinear differential equation by using modified version of new Kudryashov methods and explained the encountered in nonlinear optics.

Also, the necessity to solve the different types of NLEEs, a variety of methods for exact and explicit stable soliton solutions of nonlinear physical models have been established such as the complex method [49], the extended rational sin-cos and sinh-cosh methods [50], the Jacobi-elliptic function method [51], the Backlund transformation [52] and so on. Since the success rate of NLEEs is high in illustrating multifaced problems in different sectors. Thus, searching solitary wave solutions has gained popularity among the researchers. However, numerical, or theoretical solutions to real-life-related NLEEs are time-consuming and cumbersome. It is not easy to examine a nonlinear model of real-life problems theoretically or numerically. In recent years much attention is paid by researchers to establish better and efficient methods for determining solutions approximate or exact, analytical, or numerical to nonlinear models. In the necessity to solve the different types of NLEEs, variety of methods for exact and explicit stable soliton solutions of nonlinear physical models have been established. Among these methods, (w/g) -expansion method, the Modified Version of the New Kudryashov (MVNK) method is functional, easy to adapt, effective, and provide further generic, advanced and useful travelling wave solution.

1.7 Research Gap

In the nonlinear sciences, shallow water wave propagation investigation has grown in importance. The lashing procedure of the wave solutions, which sorts of waves are common in lakes, rivers, beaches, and oceans, and the mechanisms that create them can be used in ocean engineering, influences the wave propagation of surface waves. The investigations are also fundamentally significant because analytical answers from them can be applied successfully in a variety of domains, including mathematical biology, shallow-water wave propagation, electromagnetic theory, optical fibres, diffusion processes, reaction processes, chaos, neural physics, solid-state physics, plasma physics, and others.

Many phenomena are described as nonlinear evolution equations for large-scale study in various fields of physical sciences and engineering. In solid-state physics, optical fibres for example, the paraxial nonlinear Schrodinger (NLS) equation occurs. Because NLEEs have a high success rate in presenting multifaceted challenges in several areas. Thus, the hunt for solitary wave solutions has grown in support among researchers. Numerical or theoretical solutions to real-life-related NLEEs, on the other hand, are time-consuming and inconvenient. It is difficult to investigate a nonlinear model of real-world situations theoretically or numerically.

1.8 Study Plan

Every model is based primarily on a variety of phenomena that occur in nature. Different scientists have solved this same model in different ways by applying it to the related phenomenon and we have got the benefit of the obtained solutions to express the different problems and it is still going on today. So, despite the fact that some models already have solutions, we will look for more different solutions through different methods that will help to rescue the damaged of nature and some new direction by applying in different phenomena of the solutions.

In recent years, academics have focused on developing better and more efficient methods for obtaining approximate or exact, analytical, or numerical solutions to nonlinear models. To my best knowledge, the (2+1)-dimensional paraxial NLS equation in Kerr media is not examined through the (w/g)-expansion methods. In this section, we established, the (2+1)-dimensional paraxial (NLS) equation in Kerr media by using the (w/g)-expansion methods. Also, the coupled (2+1)-dimensional nonlinear Konopelchenko–Dubrovsky (KD) model is not examined through the Modified Version of the New Kudryashov (MVNK) method. We investigated, (2+1)-dimensional nonlinear Konopelchenko–Dubrovsky (KD) model by using the Modified Version of the New Kudryashov (MVNK) method.

Therefore, in this dissertation, we will put forward the methods to establish the broad ranging travelling wave solution to the stated equations. Also, we will discuss the effect of the value of the associated free parameters of the obtained solutions on wave profile by illustrating the various type of waves. Especially, we will examine the effect of the values of the coefficient of the highest order linear and nonlinear terms of the NLEEs on the wave profile. We will also, compare the obtained results by the studied methods and the other methods that are investigated by other researchers.

1.9 Objectives of the research

The objective of this thesis work is to find out solitary wave solution of the (2+1)-dimensional paraxial NLS equation in Kerr media through the (w/g)-expansion methods and the coupled (2+1)-dimensional nonlinear Konopelchenko–Dubrovsky (KD) model through the Modified Version of the New Kudryashov (MVNK) method. It can be used to fluid dynamics, optical fibers, communication systems and other numerous areas.

The specific objectives of the present research work are as follows:

i. The (w/g)-expansion method and the generalized Kudryashov method are applied to the (2+1)-dimensional paraxial NLS equation and the coupled (2+1)-dimensional nonlinear Konopelchenko–Dubrovsky (KD).
ii. To observe effect of the free parameters, dispersion, and nonlinearity of the obtained soliton solutions of the wave profile.
iii. The obtained solutions are compared with other solutions obtained by various methods available in the literature.

1.10 Organization of the thesis

The dissertation entitled "ANALYSIS OF SOLITON SOLUTIONS TO THE NONLINEAR SCHRODINGER AND KONOPELCHENKO-DUBROVSKY EQUATIONS" has been separated into four parts which are arranged chapter wise as follows:

Chapter 1 contains introduction of the present work including a literature review of the past studies on nonlinear evolution equations through different types of method. Objectives of the present study have also been incorporated in this chapter. The algorithms of two analytical methods are summarized in Chapter 2. Also, we have formulated diverse generic and standard closed form solutions to the (2+1)-dimensional paraxial (NLS) equation in Kerr media and the coupled (2+1)-dimensional nonlinear Konopelchenko–Dubrovsky (KD) model through the instructed methods in this chapter. In Chapter 3 we have illustrated the figures of the obtained soliton solutions and explained the physical significance. Also, comparison of the obtained our solutions via others solution with the merits and demerits have been discussed in this Chapter. Finally, in Chapter 4 the conclusions and future directions of this study are presented.

Chapter 2: Methodology and Applications

In this chapter we discuss about the methodology and its application. The (w/g)-expansion method (where w and g are arbitrary functions to find the analytical solutions.) and the Modified Version of the New Kudryashov (MVNK) method. The new computational approach the (w/g) is a family of some expansion methods which includes (G'/G)-expansion method, tanh-function method, g'-expansion method, and (g'/g^2)-expansion method. In 2009, Li Wen-An and his collaborators suggested g'-expansion method, and (g'/g^2)-expansion method based on the closed form soliton solution of nonlinear evolution equation and applied to the Vakhnenko equation and explained the loop soliton solution [44]. The modern computational approach the Modified Version of the New Kudryashov (MVNK) method, which Kamyar Hosseini initially suggested in 2020 [53], based on the solutions of the nonlinear evolution equations. The Modified Version of the New Kudryashov (MVNK) method is another form of the new Kudryashov method that has proved its potential in handling nonlinear evolution equations. In this thesis paper, we consider two nonlinear evolution model namely the (2+1)-dimensional paraxial nonlinear Schrodinger (NLS) equation in Kerr media and the (2+1)-Konopelchenko–Dubrovsky (KD) equation. According to the literature review we will apply , g'-expansion, (g'/g^2)-expansion methods to the (2+1)-dimensional paraxial nonlinear Schrodinger (NLS) equation in Kerr media and Modified Version of the New Kudryashov (MVNK) method to the (2+1)-Konopelchenko–Dubrovsky (KD) equation respectively.

2.1 The (*w/g*)-expansion methods

In this section we discuss about g'-expansion method, and (g'/g^2)-expansion method which are applied for searching closed form soliton solutions. Now we consider the NLEE in conjunction with three independent variables x, y and t of the form:

$$L(u, u_x, u_y, u_t, u_{xx}, u_{xy}, u_{xt}, \dots \dots \dots \dots) = 0 \tag{2.1}$$

Where $u = u(x, y, t)$ is an unknown function, L is a polynomial of $u(x, y, t)$, and various partial derivatives involve in equation (2.1).

Step-1. By using the wave variable, the NLEE (2.1) is reduced to the nonlinear ordinary differential equation (ODE). Making use of the wave transformations

$$\begin{cases} u(x,t) = u(\xi), \\ \xi = x + y \pm \sigma t \end{cases} \tag{2.2}$$

where σ is the speed of the travelling wave. Inserting equation (2.2) into equation (2.1) yields the following ODE

$$P(u, -\sigma u', u', \sigma^2 u'', -\sigma u'', u'', \cdots\cdots\cdots) = 0 \tag{2.3}$$

where σ will be determined later.

Step-2. Suppose that the trail solution of Ordinary Differential Equation (2.3) is given by

$$u(\xi) = \sum_{i=0}^{N} \beta_i \left(\frac{w}{g}\right)^i \tag{2.4}$$

where $\beta_i (i = 0,1,2,3, \dots\dots\dots\dots N)$ are constants will be determined later. From equation (2.3), using balance principle and yields N.

Step-3. For w and g the following relation is given:

$$\left(\frac{w}{g}\right)' = l + m\left(\frac{w}{g}\right) + n\left(\frac{w}{g}\right)^2 \tag{2.5}$$

where l, m, n are arbitrary constants. This equation can be written as

$$w'g - wg' = lg^2 + mwg + nw^2 \tag{2.6}$$

If we consider $w = g', l = -\mu, m = -\lambda, n = -1$, Then $u(\xi)$ can be expressed as

$$u(\xi) = \sum_{i=0}^{N} \beta_i \left(\frac{g'}{g}\right)^i \tag{2.7}$$

where g satisfies the following relation:

$$g'' + \lambda g' + \mu g = 0, \tag{2.8}$$

This is called (G'/G)-expansion method and this method was proposed by Wang et al. [54]. Again, we consider $w = \tanh(\xi), g = 1, l = 1, m = 0, n = -1$, Then $u(\xi)$ can be expressed as

$$u(\xi) = \sum_{i=0}^{N} \beta_i (\tanh(\xi))^i \tag{2.9}$$

where g satisfies the following relation

$$(\tanh(\xi))' = 1 - (\tanh(\xi))^2 \tag{2.10}$$

This method is called the tanh-function method and proposed by Wazwaz [55].

In this research, we have mentioned another two applications of (w/g) which are more than effective and essay to another technique. Firstly, we consider $w = gg'$, Then $u(\xi)$ can be expressed as

$$u(\xi) = \sum_{i=0}^{N} \beta_i (g')^i \tag{2.11}$$

where g satisfies the following relation

$$g'' = l + mg' + ng'^2 \tag{2.12}$$

This is called g'-expansion method that is proposed by Li et al. [44].

Secondly, we consider $w = g'/g$ and $m = 0$, Then $u(\xi)$ can be expressed as

$$u(\xi) = \sum_{i=0}^{N} \beta_i \left({g'}/{g^2} \right)^i \tag{2.13}$$

where g satisfies the following relation

$$g''g^2 - 2gg'^2 = lg^4 + ng'^2 \tag{2.14}$$

This is called (g'/g^2)-expansion method that is proposed by Li et al. [44].

Step-4. With the help of Eq. (2.12) or Eq. (2.14) substituting Eq. (2.11) or Eq. (2.13) into Eq. (2.3) and equate each coefficient of all powers of $(w/g)^i$ to zero, we obtain a determining system equation. The obtained system includes $\beta_i (i = 0,1, \dots \dots \dots \dots N)$, l, m, n and σ.

Step-5. If we solve the determining equations with the aid of Maple, we get values of $\beta_i (i = 0,1, \dots \dots \dots \dots N)$ and σ. If we put obtained values into Eq. (2.11) or Eq. (2.13), we will get all possible solutions.

we are use two new familiar (w/g)-expansive approaches namely the g'-expansion approach and another the (g'/g^2)-expansion approach which are describing in the below.

2.1.1 The g'-expansion method

The solution of Eq. (2.12) is given by

Case -1: If $4ln - m^2 < 0$,

$$g = \frac{1}{2n}\left[\ln\left(\tanh^2\left(\frac{\xi\sqrt{m^2-4ln}}{2}\right) - 1\right) - m\xi\right]$$

And $$g' = \frac{1}{2n}\left[\sqrt{m^2 - 4ln}\tanh\left(-\frac{\xi\sqrt{m^2-4ln}}{2}\right) - m\right] \tag{2.15}$$

Case -2: If $4ln - m^2 = 0$,

$$g = -\frac{1}{n}\left[\ln(\xi) + \frac{\xi m}{2}\right]$$

And $$g' = -\frac{1}{n}\left[\frac{1}{\xi} + \frac{m}{2}\right] \tag{2.16}$$

Case -3: If $4ln - m^2 > 0$,

$$g = \frac{1}{2n}\left[\ln\left(\tan^2\left(\frac{\xi\sqrt{4ln-m^2}}{2}\right) + 1\right) - m\xi\right]$$

And $$g' = \frac{1}{2n}\left[\sqrt{4ln - m^2}\tan\left(\frac{\xi\sqrt{4ln-m^2}}{2}\right) - m\right] \tag{2.17}$$

2.1.2 The (g'/g^2) expansion method

The solution of Eq. (2.14) is given by

Case -1: If $nl < 0$,

$$g = -\frac{2n}{2\xi\sqrt{|nl|} - \ln\left[\frac{n}{4l}\left(\Delta_1 e^{2\xi\sqrt{|nl|}} - \Delta_2\right)^2\right]}$$

And $\quad {g'}/{g^2} = -\frac{\sqrt{|nl|}}{n}\left(\frac{\Delta_1 \sinh\left(2\xi\sqrt{|nl|}\right)+\Delta_1 \cosh\left(2\xi\sqrt{|nl|}\right)+\Delta_2}{\Delta_1 \sinh\left(2\xi\sqrt{|nl|}\right)-\Delta_1 \cosh\left(2\xi\sqrt{|nl|}\right)-\Delta_2}\right) \qquad (2.18)$

Case -2: If $l = 0, n \neq 0$,

$$g = \frac{n}{\ln[n(\xi\Delta_1+\Delta_2)]}$$

And $\quad {g'}/{g^2} = -\frac{\Delta_2}{n(\xi\Delta_1+\Delta_2)} \qquad (2.19)$

Case -3: If $nl > 0$,

$$g = \frac{2n}{\ln\left[\frac{n}{l}\left(\Delta_1 \sin(\xi\sqrt{nl})-\Delta_2 \sin(\xi\sqrt{nl})\right)^2\right]}$$

And $\quad {g'}/{g^2} = \sqrt{\frac{l}{n}}\left(\frac{\Delta_1 \cos(\xi\sqrt{nl})+\Delta_2 \sin(\xi\sqrt{nl})}{\Delta_1 \sin(\xi\sqrt{nl})-\Delta_2 \cos(\xi\sqrt{nl})}\right) \qquad (2.20)$

where Δ_1 and Δ_2 are arbitrary constant.

2.2 The Modified Version of the New Kudryashov (MVNK) method

In this portion, we provide a brief explanation of the Modified Version of the New Kudryashov (MVNK) method [53]. Now we consider the NLEE in conjunction with two independent variables x and t of the form

$$L(u, u_x, u_t, \dots) = 0 \qquad (2.21)$$

Where,$u = u(x,t)$ is an unknown function, L is a polynomial of $u(x,t)$ and various partial derivatives involves in equation (2.21).

Step-1. By using the wave variable, the NLEE (2.21)is reduced to the nonlinear Ordinary Differential Equation (ODE). Making use of the complex wave transformations

$$\begin{cases} u(x,t) = u(\xi)e^{i(\alpha x+\mu t)}, \\ \qquad \xi = x + vt \end{cases} \qquad (2.22)$$

where v is the speed of the travelling wave. Computing equation (2.22) into equation (2.21) yields the following nonlinear ODE

$$R(u, u', u'', \dots\dots) = 0 \qquad (2.23)$$

Where prime (ʹ) represent derivative of ξ that means $\frac{d}{d\xi}$.

Step-2. Based upon the Modified Version of the New Kudryashov (MVNK) method the trail solution of Eq. (2.23) can be expressed as following:

$$u(\xi) = c_0 + \sum_{j=1}^{T}\left(\frac{k(\xi)}{1+k^2(\xi)}\right)^{j-1}\left(c_j\frac{k(\xi)}{1+k^2(\xi)} + d_j\frac{1-k^2(\xi)}{1+k^2(\xi)}\right), c_T \text{ or } d_T \neq 0 \qquad (2.24)$$

Where $c_0, c_j, j = 1,2, \dots\dots T$ and $d_j, j = 1,2, \dots\dots T$ are determined later. T is the homogeneous balance of number.

Step-3. $k(\xi)$ satisfied the following function.

$$k(\xi) = \frac{1}{(M-N)\sinh(\xi) + (M+N)\cosh(\xi)} \tag{2.25}$$

And satisfying the following Jacobi equation

$$\left(k'(\xi)\right)^2 = k^2(\xi)\left(1 - 4MNk^2(\xi)\right) \tag{2.26}$$

Step-4. By inserting Eq. (2.24) into Eq. (2.23) and rearranging the teams, we attain a system of nonlinear algebraic equations whose solution results in soliton solutions of Eq. (2.21).

2.3 Applications of the (*w/g*)-expansion methods

we apply two new familiar (w/g)-expansive techniques namely g'-expansion technique and another (g'/g^2)-expansion technique to our governing equation. So, we consider the (2+1)-dimensional paraxial nonlinear Schrodinger (NLS) equation in Kerr media is

$$iu_y + \frac{\mu}{2}u_{tt} + \frac{\vartheta}{2}u_{xx} + \epsilon|u|^2u = 0 \tag{2.27}$$

where the function $u(x, y, t)$ represent the complex wave profile and μ, ϑ and ϵ represent dispersion, diffraction and Kerr nonlinearity, respectively. This equation is also called the elliptic nonlinear Schrodinger equation when the product of dispersion and diffraction $(\mu\vartheta)$ is less than zero. i.e. $(\mu\vartheta < 0)$ [20].

Now, we consider the following wave transformation

$$\begin{cases} u(x,y,t) = u(\xi)e^{i\theta(x,y,t)} \\ \xi = x + y - \sigma t \\ \theta(x,y,t) = \delta_1 x + \delta_2 y - \lambda t \end{cases} \tag{2.28}$$

where $u(\xi)$ is the amplitude portion includes wave velocity σ. Also $e^{i\theta(x,y,t)}$ is the phase portion which including frequency of soliton δ_1 and δ_2 also including wave number of solitons λ. Now, inserting equation (2.28) into equation (2.27), we get the following real and imaginary parts:

$$-(\sigma^2\mu + \vartheta)u'' + (\vartheta\delta_1^2 + \mu\lambda^2 + 2\delta_2)u - 2\epsilon u^3 = 0 \tag{2.29}$$

$$(1 + \vartheta\delta_1 + \sigma\mu\lambda)u' = 0 \tag{2.30}$$

If we take $(u' \neq 0)$, we get from Eq. (2.30)

$$\vartheta = \frac{-1-\sigma\mu\lambda}{\delta_1} \tag{2.31}$$

Making use Eq. (2.31), from Eq. (2.29) we achieved Ordinary Differential Equation (ODE) as

$$(1 - \sigma^2\mu\delta_1 + \sigma\mu\lambda)u'' - \delta_1(\delta_1 + \sigma\mu\delta_1\lambda - \mu\lambda^2 - 2\delta_2)u - 2\epsilon\delta_1 u^3 = 0 \tag{2.32}$$

Using the balancing principle between the highest degree nonlinear term u^3 and the highest derivative u'', we get $N = 1$.

2.3.1 Applications of the g'-expansion method to NLS equation

Using the fact $N = 1$, solution (2.11) can be written as

$$u(\xi) = \beta_0 + \beta_1 g' \tag{2.33}$$

Similarly noticeable as

$$g'' = l + mg' + ng'^2 \tag{2.34}$$

$$u'(\xi) = \beta_1\left(l + mg' + ng'^2\right) \tag{2.35}$$

$$u''(\xi) = \beta_1\left(m\left(l + mg' + ng'^2\right)\right) + 2ng'\left(m\left(l + mg' + ng'^2\right)\right) \tag{2.36}$$

Introducing $u''(\xi)$ and $u(\xi)$ into (2.32) we accomplish

$$\begin{aligned}&\left(-2\mu n^2\sigma^2\beta_1\delta_1 + 2\lambda\mu n^2\sigma\beta_1 - 2\epsilon{\beta_1}^3\delta_1 + 2n^2\beta_1\right)g'^3\\&+\left(-3m\mu n\sigma^2\beta_1\delta_1 + 3\lambda m\mu n\sigma\beta_1 - 6\epsilon\beta_0{\beta_1}^2\delta_1 + 3mn\beta_1\right)g'^2\\&+\left(-2l\mu n\sigma^2\beta_1\delta_1 - m^2\mu\sigma^2\beta_1\delta_1 + 2l\lambda\mu n\sigma\beta_1 + \lambda m^2\mu\sigma\beta_1 - \lambda\mu\sigma c - 6\epsilon{\beta_0}^2\beta_1\delta_1\right.\\&\left.+\lambda^2\mu\beta_1\delta_1 + 2ln\beta_1 + m^2\beta_1 - \beta_1{\delta_1}^2 + 2\beta_1\beta_1\delta_1\delta_2\right)g' - lm\mu\sigma^2\beta_1\delta_1 + l\lambda m\mu\sigma\beta_1\\&-\lambda\mu\sigma\beta_0{\delta_1}^2 - 2\epsilon{\beta_0}^3\delta_1 + \lambda^2\mu\beta_0\delta_1 + lm\beta_1 - \beta_0{\delta_1}^2 + 2\beta_0\delta_1\delta_2\\&= 0\end{aligned} \tag{2.37}$$

It is possible to collect all term with same order of g' and will get the following system of algebraic equations,

$$\text{Constant: } -lm\mu\sigma^2\beta_1\delta_1 + l\lambda m\mu\sigma\beta_1 - \lambda\mu\sigma\beta_0{\delta_1}^2 - 2\epsilon{\beta_0}^3\delta_1 + \lambda^2\mu\beta_0\delta_1 + lm\beta_1 - \beta_0{\delta_1}^2 + 2\beta_0\delta_1\delta_2 = 0$$

$$g': -2l\mu n\sigma^2\beta_1\delta_1 - m^2\mu\sigma^2\beta_1\delta_1 + 2l\lambda\mu n\sigma\beta_1 + \lambda m^2\mu\sigma\beta_1 - \lambda\mu\sigma c - 6\epsilon{\beta_0}^2\beta_1\delta_1 + \lambda^2\mu\beta_1\delta_1 + 2ln\beta_1 + m^2\beta_1 - \beta_1{\delta_1}^2 + 2\beta_1\beta_1\delta_1\delta_2 = 0$$

$$g'^2: \ -3m\mu n\sigma^2\beta_1\delta_1 + 3\lambda m\mu n\sigma\beta_1 - 6\epsilon\beta_0{\beta_1}^2\delta_1 + 3mn\beta_1 = 0$$

$$g'^3: \ -2\mu n^2\sigma^2\beta_1\delta_1 + 2\lambda\mu n^2\sigma\beta_1 - 2\epsilon{\beta_1}^3\delta_1 + 2n^2\beta_1 = 0$$

Now, solving these equations with the help of package software Maple, the following values of the unknown parameters are attained

Set-1:

$$\mu = -\frac{(4\epsilon ln\beta_1^2 - \epsilon m^2\beta_1^2 - 2n^2\delta_1 + 4n^2\delta_2)^2}{2n^2\lambda^2(4\epsilon ln\beta_1^2 - \epsilon m^2\beta_1^2 + 2\epsilon\beta_1^2\delta_1^2 - 4n^2\delta_1 + 4n^2\delta_2)},$$

$$\sigma = -\frac{2\lambda(\epsilon\beta_1^2\delta_1 - n^2)}{4\epsilon ln\beta_1^2 - \epsilon m^2\beta_1^2 - 2n^2\delta_1 + 4n^2\delta_2},$$

$$\beta_0 = \frac{\beta_1 m}{2n},$$

$$\beta_1 = \beta_1.$$

Set-2:

$$\mu = -\frac{4ln - m^2 - 2\delta_1^2 + 4\delta_1\delta_2}{2\lambda^2\delta_1},$$

$$\sigma = 0,$$

$$\beta_0 = \pm\frac{m}{2\sqrt{\epsilon\delta_1}},$$

$$\beta_1 = \pm\frac{n}{\sqrt{\epsilon\delta_1}}.$$

By using Set-1 along with Eq. (2.15), Eq. (2.16), Eq. (2.17) and Eq. (2.33), We are rapidly reducing the travelling waves solution of Eq. (2.27) as below:

Family -1: when $4ln - m^2 < 0$

$$u_1(\xi) = e^{i(\delta_1 x + \delta_2 y - \lambda t)}\,\frac{-\beta_1\sqrt{m^2-4ln}\tanh\left(\frac{\sqrt{m^2-4ln}}{2}\xi\right)}{2n} \tag{2.38}$$

where $\xi = x + y - \sigma t$, $\sigma = -\frac{2\lambda(\epsilon\beta_1^2\delta_1 - n^2)}{4\epsilon ln\beta_1^2 - \epsilon m^2\beta_1^2 - 2n^2\delta_1 + 4n^2\delta_2}$.

Family -2: when $4ln - m^2 = 0$

$$u_2(\xi) = -e^{i(\delta_1 x + \delta_2 y - \lambda t)}\,\frac{\beta_1}{n\xi} \tag{2.39}$$

where $\xi = x + y - \sigma t, \sigma = -\frac{2\lambda(\epsilon\beta_1^2\delta_1 - n^2)}{4\epsilon ln\beta_1^2 - \epsilon m^2\beta_1^2 - 2n^2\delta_1 + 4n^2\delta_2}$.

Family -3: when $4ln - m^2 > 0$

$$u_3(\xi) = e^{i(\delta_1 x + \delta_2 y - \lambda t)}\,\frac{\beta_1\sqrt{4ln-m^2}\tan\left(\frac{\xi\sqrt{4ln-m^2}}{2}\right)}{2n} \tag{2.40}$$

where $\xi = x + y - \sigma t, \sigma = -\frac{2\lambda(\epsilon\beta_1^2\delta_1 - n^2)}{4\epsilon ln\beta_1^2 - \epsilon m^2\beta_1^2 - 2n^2\delta_1 + 4n^2\delta_2}$.

Similarly, by using Set-2 along with Eq. (2.15), Eq. (2.16), Eq. (2.17) and Eq. (2.33), We are rapidly reducing the travelling waves solution of Eq. (2.27) as below

Family -4: when $4ln - m^2 < 0$

$$u_4(\xi) = e^{i(\delta_1 x + \delta_2 y - \lambda t)} \times \frac{\sqrt{m^2-4ln}\tanh\left(\frac{\xi\sqrt{m^2-4ln}}{2}\right)}{2\sqrt{\epsilon\delta_1}} \tag{2.41}$$

where $\xi = x + y - \sigma t, \sigma = 0$.

Family -5: when $4ln - m^2 = 0$

$$u_5(\xi) = e^{i(\delta_1 x + \delta_2 y - \lambda t)} \times \frac{1}{\xi\sqrt{\epsilon\delta_1}} \tag{2.42}$$

where $\xi = x + y - \sigma t, \sigma = 0$.

Family -6: when $4ln - m^2 > 0$

$$u_6(\xi) = e^{i(\delta_1 x + \delta_2 y - \lambda t)} \times \frac{\sqrt{4ln - m^2}\tan\left(\frac{\xi\sqrt{4ln-m^2}}{2}\right)}{2\sqrt{\epsilon\delta_1}} \tag{2.43}$$

where $\xi = x + y - \sigma t, \sigma = 0$.

2.3.2 Applications of the (g'/g^2)-expansion method to NLS equation

Using the fact $N = 1$, solution (2.13) can be written as

$$u(\xi) = \beta_0 + \beta_1 \left({g'}/{g^2}\right)' \tag{2.44}$$

Similarly, satisfied following relation as

$$g''g^2 - 2gg'^2 = lg^4 + cg'^2 \tag{2.45}$$

$$u'(\xi) = \frac{a_1\left(ag^2(\xi) + \frac{cg'^2(\xi)}{g^2(\xi)} + \frac{2g'^2(\xi)}{g(\xi)}\right)}{g^2(\xi)} - \frac{2a_1 g'^2(\xi)}{g^3(\xi)} \tag{2.46}$$

$$u''(\xi) = \frac{1}{g^2(\xi)}\left(a_1\left(\begin{array}{c} 2ag(\xi)g'(\xi) + \frac{2cg'(\xi)\left(ag^2(\xi)+\frac{cg'^2(\xi)}{g^2(\xi)}+\frac{2g'^2(\xi)}{g(\xi)}\right)}{g^2(\xi)} \\ -\frac{2cg'^3(\xi)}{g^3(\xi)} + \frac{4g'(\xi)\left(ag^2(\xi)+\frac{cg'^2(\xi)}{g^2(\xi)}+\frac{2g'^2(\xi)}{g(\xi)}\right)}{g(\xi)} - \frac{2g'^3(\xi)}{g^2(\xi)}\end{array}\right)\right)$$

$$-\frac{6a_1\left(ag^2(\xi) + \frac{cg'^2(\xi)}{g^2(\xi)} + \frac{2g'^2(\xi)}{g(\xi)}\right)g'(\xi)}{g^3(\xi)} + \frac{6a_1 g'^3(\xi)}{g^4(\xi)} \tag{2.47}$$

Introducing $u''(\xi)$ and $u(\xi)$ into (2.32) we accomplish

$$\begin{aligned}(-2\alpha k a_1 c^2 m^2 + 2\alpha c^2 m v a_1 - 2\gamma k {a_1}^3 + 2a_1 c^2)\left(\frac{g'}{g^2}\right)^3 - 6\gamma k a_0 {a_1}^2\left(\frac{g'}{g^2}\right)^2 \\ + (-2\alpha k a_1 c a m^2 + 2a\alpha c m v a_1 - a_1 m\alpha v k^2 - 6\gamma k {a_0}^2 a_1 + \alpha k a_1 v^2 \\ + 2a_1 ca - a_1 k^2 + 2ka_1\omega)\left(\frac{g'}{g^2}\right) - a_0 m\alpha v k^2 + ka_0\alpha v^2 - 2\gamma k {a_0}^3 \\ - a_0 k^2 + 2ka_0\omega = 0\end{aligned} \tag{2.48}$$

It is possible to collect all term with same order of (g'/g^2) and will get the following system of algebraic equations,

Constant: $a_0 m\alpha v k^2 + ka_0\alpha v^2 - 2\gamma k {a_0}^3 - a_0 k^2 + 2ka_0\omega = 0$

$\left(\frac{g'}{g^2}\right)$: $-2\alpha k a_1 c a m^2 + 2a\alpha c m v a_1 - a_1 m\alpha v k^2 - 6\gamma k {a_0}^2 a_1 + \alpha k a_1 v^2 + 2a_1 ca - a_1 k^2 + 2ka_1\omega = 0$

$\left(\frac{g'}{g^2}\right)^2 : -6\gamma k a_0 {a_1}^2 = 0$

$\left(\frac{g'}{g^2}\right)^3 : -2\alpha k a_1 c^2 m^2 + 2\alpha c^2 m v a_1 - 2\gamma k {a_1}^3 + 2 a_1 c^2 = 0$

Now, solving these equations with the help of package software Maple, the following values of the unknown parameters are attained

Set -1:

$$\mu = -\frac{(2l\epsilon\beta_1^2 - n\delta_1 + 2n\delta_2)^2}{\lambda^2(2ln\epsilon\beta_1^2 + \epsilon\delta_1^2\beta_1^2 - 2n^2\delta_1 + 2n^2\delta_2)},$$

$$\sigma = -\frac{\lambda(\epsilon\delta_1\beta_1^2 - n^2)}{n(2l\epsilon\beta_1^2 - n\delta_1 + 2n\delta_2)},$$

$$\beta_0 = 0,$$

$$\beta_1 = \beta_1$$

Set -2:

$$\mu = -\frac{2ln - \delta_1^2 + 2\delta_1\delta_2}{\delta_1\lambda^2},$$

$$\sigma = 0,$$

$$\beta_0 = 0,$$

$$\beta_1 = \pm\frac{n}{\sqrt{\epsilon\delta_1}}$$

By using Set-1 along with Eq. (2.18), Eq. (2.19), Eq. (2.20) and Eq. (2.44), We are rapidly reducing the travelling waves solution of Eq. (2.27) as below

Family -7: when $ln < 0$

$$u_7(\xi) = -e^{i(\delta_1 x + \delta_2 y - \lambda t)} \times \frac{\sqrt{|ln|}\left(\Delta_1 \sinh\left(2\xi\sqrt{|ln|}\right) + \Delta_1 \cosh\left(2\xi\sqrt{|ln|}\right) + \Delta_2\right)\beta_1}{n\left(\Delta_1 \sinh\left(2\xi\sqrt{|ln|}\right) + \Delta_1 \cosh\left(2\xi\sqrt{|ln|}\right) - \Delta_2\right)} \quad (2.49)$$

where $\xi = x + y - \sigma t, \sigma = -\frac{\lambda(\epsilon\delta_1\beta_1^2 - n^2)}{n(2l\epsilon\beta_1^2 - n\delta_1 + 2n\delta_2)}$.

Family -8: when $l = 0, n \neq 0$

$$u_8(\xi) = -e^{i(\delta_1 x + \delta_2 y - \lambda t)} \times \frac{\Delta_1\beta_1}{n(\Delta_1\xi + \Delta_2)} \quad (2.50)$$

where $\xi = x + y - \sigma t, \sigma = -\frac{\lambda(\epsilon\delta_1\beta_1^2 - n^2)}{n(2l\epsilon\beta_1^2 - n\delta_1 + 2n\delta_2)}$.

Family -9: when $ln > 0$

$$u_9(\xi) = e^{i(\delta_1 x+\delta_2 y-\lambda t\,)} \times \frac{\sqrt{\frac{l}{n}}\left(\Delta_1\cos(\xi\sqrt{ln}) + \Delta_2\sin(\xi\sqrt{ln})\right)\beta_1}{\Delta_1\sin(\xi\sqrt{ln}) - \Delta_2\cos(\xi\sqrt{ln})} \quad (2.51)$$

where $\xi = x + y - \sigma t, \sigma = -\frac{\lambda(\epsilon\delta_1\beta_1^2-n^2)}{n(2l\epsilon\beta_1^2-n\delta_1+2n\delta_2)}$.

By using Set-2 along with Eq. (2.18), Eq. (2.19), Eq. (2.20) and Eq. (2.44), We are rapidly reducing the travelling waves solution of Eq. (2.27) as below

Family 10: when $ln < 0$

$$u_{10}(\xi) = e^{i(\delta_1 \mathrm{x}+\delta_2 y-\lambda \mathrm{t}\,)} \times \frac{\sqrt{|ln|}\left(\Delta_1\sinh\left(2\xi\sqrt{|ln|}\right)+\Delta_1\cosh\left(2\xi\sqrt{|ln|}\right)+\Delta_2\right)}{\sqrt{\epsilon\delta_1}\left(\Delta_1\sinh\left(2\xi\sqrt{|ln|}\right)+\Delta_1\cosh\left(2\xi\sqrt{|ln|}\right)-\Delta_2\right)} \quad (2.52)$$

where $\xi = x + y - \sigma t, \sigma = 0$.

Family -11: when $l = 0, n \neq 0$

$$u_{11}(\xi) = e^{i(\delta_1 x+\delta_2 y-\lambda t\,)} \times \frac{\Delta_1}{\sqrt{\epsilon\delta_1}(\Delta_1\xi + \Delta_2)} \quad (2.53)$$

where $\xi = x + y - \sigma t, \sigma = 0$.

Family -12: when $ln > 0$

$$u_{12}(\xi) = e^{i(\delta_1 x+\delta_2 y-\lambda t\,)} \times \frac{\sqrt{ln}\left(\Delta_1\cos(\xi\sqrt{ln}) + \Delta_2\sin(\xi\sqrt{ln})\right)}{\sqrt{\epsilon\delta_1}\left(\Delta_1\sin(\xi\sqrt{ln}) - \Delta_2\cos(\xi\sqrt{ln})\right)} \quad (2.54)$$

where $\xi = x + y - \sigma t, \sigma = 0$.

2.4 Applications of the MVNK method to KD equation

In this section, by using the Modified Version of the New Kudryashov (MVNK) method we perform the general, pertinent, and widespread transparent soliton solutions to the (2+1)-Konopelchenko–Dubrovsky (KD) equation. In 1984, B.G. Konopelchenko and V.G. Dubrovsky constructed the (2+1)-dimensional Konopelchenko-Dubrovsky (KD) equation [11]. The (2+1)-dimensional nonlinear Konopelchenko–Dubrovsky (KD) model is

$$u_t - u_{xxx} - 6quu_x + \frac{3}{2}p^2u^2u_x - 3v_y + 3pu_xv = 0 \quad (2.55)$$

$$u_y = v_x \quad (2.56)$$

where u and v represent the velocity components along the x-axis and y-axis, respectively. In equation (2.55) u_t is the time evaluation term, uu_x and u^2u_x are non-linear term which include the non-linear coefficient p and q which works as the amplitude of the wave, others are the dispersive terms. The (2+1)-dimensional (KD) equation covers the Gardner equation (if $u_y = 0$), Kadomtsev–Petviashvili (KP) equation (if $p = 0$), the modified KP equation (if $q = 0$). These equations are also used to simulate shallow coastal waves, irregular waves in paramagnetic media, super-fluids, and ion-acoustic wave propagation in a plasma with quasi electrons and liquid crystals. Substituting the wave transformation

$$\begin{cases} u(x, y, t) = u(\xi), \\ \xi = \alpha x + \beta y + \mu t + \theta_0 \end{cases} \tag{2.57}$$

Putting the value of Eq. (2.57) into Eq. (2.55) and (2.56) as a result we obtained the system of non-linear ordinary differential equations.

$$\mu u' - \alpha^3 u''' - 6q\alpha u u' + \frac{3}{2}p^2 \alpha u^2 u' - 3\beta v' + 3p\alpha u' v = 0 \tag{2.58}$$

$$\beta u' = \alpha v' \tag{2.59}$$

Integrating Eq. (2.59) with respect to v once and integrating constant zero then it reduces to

$$v = \frac{\beta}{\alpha} u \tag{2.60}$$

Substituting Eq. (2.60) into Eq. (2.58) and we get

$$\mu u' - \alpha^3 u''' + \frac{3}{2}p^2 \alpha u^2 u' - \frac{3}{\alpha}\beta^2 u' + \left(\frac{3p\beta - 6q\alpha}{2}\right) 2\mu u' = 0 \tag{2.61}$$

Integrating Eq. (2.61) with respect to v once and integrating constant zero then it reduces to

$$\mu u - \alpha^3 u'' + \frac{1}{2}p^2 \alpha u^3 - \frac{3}{\alpha}\beta^2 u + \left(\frac{3p\beta - 6q\alpha}{2}\right) u^2 = 0 \tag{2.62}$$

Balancing the highest order derivative term u'' with the highest power of nonlinear term u^3, gives $T = 1$.Through the proposed method, using the value of T with the help of Eq. (2.24), we achieved.

$$u(\xi) = c_0 + c_1 \frac{K(\xi)}{1 + K^2(\xi)} + c_1 \frac{1 - K^2(\xi)}{1 + K^2(\xi)} \tag{2.63}$$

Where c, c_1 and c_2 are constants to be determined latter, such that $c_1 \neq 0$ or $c_2 \neq 0$ and $k(\xi)$ satisfies the Eq. (2.25) and Eq. (2.26).

$$u'(\xi) = \frac{c_1 \sqrt{k^2(\xi)\left(1 - 4MNk^2(\xi)\right)}}{1 + k^2(\xi)} - \frac{2c_1 k^2(\xi) \sqrt{k^2(\xi)\left(1 - 4MNk^2(\xi)\right)}}{\left(1 + k^2(\xi)\right)^2}$$

$$- \frac{2c_2 k(\xi) \sqrt{k^2(\xi)\left(1 - 4MNk^2(\xi)\right)}}{1 + k^2(\xi)}$$

$$-\frac{2c_2\big(1-k^2(\xi)\big)K(\xi)\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}}{\big(1+k^2(\xi)\big)^2} \tag{2.64}$$

$$u''(\xi)=\frac{c_1\begin{pmatrix}2k(\xi)\big(1-4MNk^2(\xi)\big)\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}\\ -8k^3(\xi)MN\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}\end{pmatrix}}{2\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}\big(1+k^2(\xi)\big)}$$

$$-\frac{6c_1k^3(\xi)\big(1-4MNk^2(\xi)\big)}{\big(1+k^2(\xi)\big)^2}+\frac{8k^5(\xi)\big(1-4MNk^2(\xi)\big)}{\big(1+k^2(\xi)\big)^3}$$

$$-\frac{\begin{matrix}c_1k^2(\xi)\Big(2k(\xi)\big(1-4MNk^2(\xi)\big)\Big)\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}\\ -8k^3(\xi)MN\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}\end{matrix}}{\big(1+k^2(\xi)\big)^2\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}}$$

$$-\frac{2C_2k^2(\xi)\big(1-4MNk^2(\xi)\big)}{1+k^2(\xi)}$$

$$-\frac{\begin{matrix}c_2k(\xi)\Big(2k(\xi)\big(1-4MNk^2(\xi)\big)\Big)\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}\\ -8k^3(\xi)MN\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}\end{matrix}}{\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}\big(1+k^2(\xi)\big)}$$

$$+\frac{8C_2k^4(\xi)\big(1-4MNk^2(\xi)\big)}{\big(1+k^2(\xi)\big)^2}+\frac{8C_2\big(1-k^2(\xi)\big)k^4(\xi)\big(1-4MNk^2(\xi)\big)}{\big(1+k^2(\xi)\big)^3}$$

$$-\frac{2C_2\big(1-k^2(\xi)\big)k^2(\xi)\big(1-4MNk^2(\xi)\big)}{\big(1+k^2(\xi)\big)^2}$$

$$-\frac{1}{\big(1+k^2(\xi)\big)^2\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}}$$

$$\Bigg(C_2\big(1-k^2(\xi)\big)k(\xi)\Bigg(2k(\xi)\big(1-4MNk^2(\xi)\big)\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}$$

$$-8k^3(\xi)MN\sqrt{k^2(\xi)\big(1-4MNk^2(\xi)\big)}\Bigg)\Bigg) \tag{2.65}$$

Introducing $u''(\xi)$ and $u(\xi)$ into (2.62) we accomplish

$$\begin{aligned}
&(32Mc_2N\alpha^4 + \alpha^2p^2{c_0}^3 - 3\alpha^2p^2{c_0}^2c_2 + 3\alpha^2p^2c_0{c_2}^2 - \alpha^2p^2{c_2}^3 - 6\alpha^2q{c_0}^2 \\
&\quad + 12\alpha^2qc_0c_2 - 6\alpha^2q{c_2}^2 + 3\alpha p\beta{c_0}^2 - 6\alpha c_2p\beta c_0 + 3\alpha\beta p{c_1}^2 \\
&\quad + 3\alpha{c_2}^2p\beta + 2\alpha\mu c_0 - 2\alpha c_2\mu - 6\beta^2c_0 + 6\beta^2c_2)K^6(\xi) \\
&\quad + (-48MN\alpha^4c_1 + 3\alpha^2p^2{c_0}^2c_1 - 6\alpha^2p^2c_0c_1c_2 + 3\alpha^2p^2c_1{c_2}^2 - 2\alpha^4c_1 \\
&\quad - 12\alpha^2qc_0c_1 + 12\alpha^2qc_1c_2 + 6\alpha\beta pc_0c_1 - 6\alpha\beta pc_1c_2 + 2\alpha\mu c_1 \\
&\quad - 6\beta^2c_1)k^5(\xi) \\
&\quad + (-96Mc_2N\alpha^4 + 3\alpha^2p^2{c_0}^3 - 3\alpha^2p^2{c_0}^2c_2 + 3\alpha^2p^2c_0{c_1}^2 \\
&\quad - 3\alpha^2p^2c_0{c_2}^2 - 3\alpha^2p^2{c_1}^2c_2 + 3\alpha^2p^2{c_2}^3 - 16\alpha^4c_2 - 18\alpha^2q{c_0}^2 \\
&\quad + 12\alpha^2qc_0c_2 - 6\alpha^2q{c_1}^2 + 6\alpha^2q{c_2}^2 + 9\alpha\beta p{c_0}^2 - 6\alpha c_2p\beta c_0 \\
&\quad + 3\alpha\beta p{c_1}^2 - 3\alpha{c_2}^2p\beta + 6\alpha\mu c_0 - 2\alpha c_2\mu - 18\beta^2c_0 + 6\beta^2c_2)k^4(\xi) \\
&\quad + (16MN\alpha^4c_1 + 6\alpha^2p^2{c_0}^2c_1 + \alpha^2p^2{c_1}^3 - 6\alpha^2p^2c_1{c_2}^2 + 12\alpha^4c_1 \\
&\quad - 24\alpha^2qc_0c_1 + 12\alpha\beta pc_0c_1 + 4\alpha\mu c_1 - 12\beta^2c_1)k^3(\xi) \\
&\quad + (3\alpha^2p^2{c_0}^3 + 3\alpha^2p^2{c_0}^2c_2 + 3\alpha^2p^2c_0{c_1}^2 - 3\alpha^2p^2c_0{c_2}^2 + 3\alpha^2p^2{c_1}^2c_2 \\
&\quad - 3\alpha^2p^2{c_2}^3 + 16\alpha^4c_2 - 18\alpha^2q{c_0}^2 - 12\alpha^2qc_0c_2 - 6\alpha^2q{c_1}^2 \\
&\quad + 6\alpha^2q{c_2}^2 + 9\alpha\beta p{c_0}^2 + 6\alpha c_2p\beta c_0 + 3\alpha\beta p{c_1}^2 - 3\alpha{c_2}^2p\beta + 6\alpha\mu c_0 \\
&\quad + 2\alpha c_2\mu - 18\beta^2c_0 - 6\beta^2c_2)k^2(\xi) \\
&\quad + (3\alpha^2p^2{c_0}^2c_1 + 6\alpha^2p^2c_0c_1c_2 + 3\alpha^2p^2c_1{c_2}^2 - 2\alpha^4c_1 - 12\alpha^2qc_0c_1 \\
&\quad - 12\alpha^2qc_1c_2 + 6\alpha\beta pc_0c_1 + 6\alpha\beta pc_1c_2 + 2\alpha\mu c_1 - 6\beta^2c_1)k(\xi) \\
&\quad + \alpha^2p^2{c_0}^3 + 3\alpha^2p^2{c_0}^2c_2 + 3\alpha^2p^2c_0{c_2}^2 + \alpha^2p^2{c_2}^3 - 6\alpha^2q{c_0}^2 \\
&\quad - 12\alpha^2qc_0c_2 - 6\alpha^2q{c_2}^2 + 3\alpha p\beta{c_0}^2 + 6\alpha c_2p\beta c_0 + 3\alpha{c_2}^2p\beta + 2\alpha\mu c_0 \\
&\quad + 2\alpha c_2\mu - 6\beta^2c_0 - 6\beta^2c_2 \\
&\quad = 0 \qquad (2.66)
\end{aligned}$$

It is possible to rearranging all term with same order of $k(\xi)$ and we reached a system of algebraic equations

Constant: $\alpha^2p^2{c_0}^3 + 3\alpha^2p^2{c_0}^2c_2 + 3\alpha^2p^2c_0{c_2}^2 + \alpha^2p^2{c_2}^3 - 6\alpha^2q{c_0}^2 - 12\alpha^2qc_0c_2 - 6\alpha^2q{c_2}^2 + 3\alpha p\beta{c_0}^2 + 6\alpha c_2p\beta c_0 + 3\alpha{c_2}^2p\beta + 2\alpha\mu c_0 + 2\alpha c_2\mu - 6\beta^2c_0 - 6\beta^2c_2 = 0$

$k(\xi)$: $3\alpha^2p^2{c_0}^2c_1 + 6\alpha^2p^2c_0c_1c_2 + 3\alpha^2p^2c_1{c_2}^2 - 2\alpha^4c_1 - 12\alpha^2qc_0c_1 - 12\alpha^2qc_1c_2 + 6\alpha\beta pc_0c_1 + 6\alpha\beta pc_1c_2 + 2\alpha\mu c_1 - 6\beta^2c_1 = 0$

$$k^2(\xi): 3\alpha^2p^2{c_0}^3 + 3\alpha^2p^2{c_0}^2c_2 + 3\alpha^2p^2c_0{c_1}^2 - 3\alpha^2p^2c_0{c_2}^2 + 3\alpha^2p^2{c_1}^2c_2 - 3\alpha^2p^2{c_2}^3 + 16\alpha^4c_2 - 18\alpha^2q{c_0}^2 - 12\alpha^2qc_0c_2 - 6\alpha^2q{c_1}^2 + 6\alpha^2q{c_2}^2 + 9\alpha\beta p{c_0}^2 + 6\alpha c_2p\beta c_0 + 3\alpha\beta p{c_1}^2 - 3\alpha{c_2}^2p\beta + 6\alpha\mu c_0 + 2\alpha c_2\mu - 18\beta^2c_0 - 6\beta^2c_2 = 0$$

$$k^3(\xi): 16MN\alpha^4c_1 + 6\alpha^2p^2{c_0}^2c_1 + \alpha^2p^2{c_1}^3 - 6\alpha^2p^2c_1{c_2}^2 + 12\alpha^4c_1 - 24\alpha^2qc_0c_1 + 12\alpha\beta pc_0c_1 + 4\alpha\mu c_1 - 12\beta^2c_1 = 0$$

$$k^4(\xi): -96Mc_2N\alpha^4 + 3\alpha^2p^2{c_0}^3 - 3\alpha^2p^2{c_0}^2c_2 + 3\alpha^2p^2c_0{c_1}^2 - 3\alpha^2p^2c_0{c_2}^2 - 3\alpha^2p^2{c_1}^2c_2 + 3\alpha^2p^2{c_2}^3 - 16\alpha^4c_2 - 18\alpha^2q{c_0}^2 + 12\alpha^2qc_0c_2 - 6\alpha^2q{c_1}^2 + 6\alpha^2q{c_2}^2 + 9\alpha\beta p{c_0}^2 - 6\alpha c_2p\beta c_0 + 3\alpha\beta p{c_1}^2 - 3\alpha{c_2}^2p\beta + 6\alpha\mu c_0 - 2\alpha c_2\mu - 18\beta^2c_0 + 6\beta^2c_2 = 0$$

$$k^5(\xi): -48MN\alpha^4c_1 + 3\alpha^2p^2{c_0}^2c_1 - 6\alpha^2p^2c_0c_1c_2 + 3\alpha^2p^2c_1{c_2}^2 - 2\alpha^4c_1 - 12\alpha^2qc_0c_1 + 12\alpha^2qc_1c_2 + 6\alpha\beta pc_0c_1 - 6\alpha\beta pc_1c_2 + 2\alpha\mu c_1 - 6\beta^2c_1 = 0$$

$$k^6(\xi): 32Mc_2N\alpha^4 + \alpha^2p^2{c_0}^3 - 3\alpha^2p^2{c_0}^2c_2 + 3\alpha^2p^2c_0{c_2}^2 - \alpha^2p^2{c_2}^3 - 6\alpha^2q{c_0}^2 + 12\alpha^2qc_0c_2 - 6\alpha^2q{c_2}^2 + 3\alpha p\beta{c_0}^2 - 6\alpha c_2p\beta c_0 + 3\alpha\beta p{c_1}^2 + 3\alpha{c_2}^2p\beta + 2\alpha\mu c_0 - 2\alpha c_2\mu - 6\beta^2c_0 + 6\beta^2c_2 = 0$$

Now, solving these equations with the help of package software Maple, the following values of the unknown set of parameters are attained

Set-1: $c_0 = -c_2, c_1 = 0, c_2 = c_2, N = \frac{-c_2^2p^2+4\alpha^2}{16\alpha^2M}, \alpha = \alpha$

$$\beta = \frac{\alpha(c_2^2p^2+4\alpha^2+4qc_2)}{2pc_2},$$

$$\mu = \frac{\alpha(3p^4c_2^4 + 40\alpha^2p^2c_2^2 + 24p^2qc_2^3 + 48\alpha^4 + 96\alpha^2qc_2 + 48q^2c_2^2)}{4p^2c_2^2}.$$

Set-2: $c_0 = c_0, c_1 = 0, c_2 = c_2, N = \frac{-c_0(c_0-c_2)}{2M(2c_0^2+c_0c_2-c_2^2)}, \alpha = \pm p\Delta,$

$$\Delta = \sqrt{-\frac{c_2^3 - 2c_0^2c_2 - c_0c_2^2}{12c_0 - 4c_2}},$$

$$\beta = \mp\frac{2\Delta(p^2c_0^2 - 3qc_0 + qc_2)}{3c_0 - c_2}$$

$$\mu = \pm\frac{\Delta\begin{pmatrix}33p^4c_0^4 - 4p^4c_0^2c_2^2 + 4p^4c_0c_2^3 - p^4c_2^4 - 144p^2qc_0^3 + \\ 48p^2qc_0^2c_2 + 216q^2c_0^2 - 144q^2c_0c_2 + 24q^2c_2^2\end{pmatrix}}{2p(3c_0 - c_2)^2}.$$

Set-3: $c_0 = 0, c_1 = \pm\frac{4i\alpha}{p}, c_2 = 0, N = 0,\ \beta = \frac{2\alpha q}{p}, \alpha = \alpha,\ \mu = \frac{\alpha(\alpha^2p^2+12q^2)}{p^2}.$

Set-4: $c_0 = \frac{\sqrt{2}i\alpha}{p}, c_1 = \pm\frac{4i\alpha}{p}, c_2 = 0, N = 0,\ \beta = \sqrt{2}\alpha\left(\frac{q}{p} - i\alpha\right), \alpha = \alpha,$

$$\mu = \frac{-12\alpha}{p^2}\left(\sqrt{2}i\alpha pq + \frac{2\alpha^2p^2}{3} - q^2\right).$$

Set-5: $c_0 = -\frac{\sqrt{2}i\alpha}{p}, c_1 = \pm\frac{4i\alpha}{p}, c_2 = 0, N = 0, \alpha = \alpha,$

$$\beta = \sqrt{2}\alpha\left(\frac{q}{p} + i\alpha\right),$$

$$\mu = \frac{-12\alpha}{p^2}\left(\sqrt{2}i\alpha pq + \frac{2\alpha^2p^2}{3} - q^2\right).$$

Set-6: $c_0 = -c_2, c_1 = \pm 2ic_2, c_2 = c_{2,}\ N = -\frac{\alpha^2 - c_2^2p^2}{4\alpha^2M}, \alpha = \alpha,$

$$\beta = \frac{\alpha(\alpha^2 + 2qc_2)}{pc_2},$$

$$\mu = \frac{\alpha(\alpha^2p^2c_2^2 + 3\alpha^4 + 12\alpha^2qc_2 + 12q^2c_2^2)}{p^2c_2^2}.$$

Set-7: $c_0 = c_0, c_1 = \pm 2ic_2, c_2 = c_{2,}\ \alpha = \Delta p,$

$$N = \frac{-c_0(c_0 - c_2)}{4M(c_0^2 + 2c_0c_2 + c_2^2)},$$

$$\Delta = \sqrt{\frac{c_2}{c_2 + 3c_0}}(c_0 + c_2),$$

$$\beta = -\frac{2\Delta(p^2c_0^2 + p^2c_0c_2 - 3qc_0 - qc_2)}{3c_0 + c_2},$$

$$\mu = \frac{\Delta(33p^4c_0^4 + 66p^4c_0^3c_2 + 32p^4c_0^2c_2^2 - 2p^4c_0c_2^3 - p^4c_2^4 - 144p^2qc_0^3 - 192p^2qc_0^2c_2 - 48p^2qc_0c_2^2 + 216q^2c_0^2 + 144q^2c_0c_2 + 24q^2c_2^2)}{2p(3c_0 + c_2)^2}$$

Set-8: $c_0 = c_0, c_1 = \pm 2ic_2, c_2 = c_{2,}$

$$N = \frac{-c_0(c_0 - c_2)}{4M(c_0^2 + 2c_0c_2 + c_2^2)},$$

$$\Delta = \sqrt{\frac{c_2}{c_2 + 3c_0}}(c_0 + c_2),$$

$$\alpha = -\Delta p,$$

$$\beta = \frac{2\Delta(p^2c_0^2 + p^2c_0c_2 - 3qc_0 - qc_2)}{3c_0 + c_2},$$

$$\mu = -\frac{\begin{array}{c}\Delta(33p^4c_0^4 + 66p^4c_0^3c_2 + 32p^4c_0^2c_2^2 - 2p^4c_0c_2^3 - p^4c_2^4 - 144p^2qc_0^3 \\ -192p^2qc_0^2c_2 - 48p^2qc_0c_2^2 + 216q^2c_0^2 + 144q^2c_0c_2 + 24q^2c_2^2)\end{array}}{2p(3c_0 + c_2)^2}.$$

By using Set-1 along with Eq. (2.25) and Eq. (2.63), We are rapidly reducing the travelling waves solution of Eq. (2.58) and Eq. (2.59) as below

Collection-1:

$$u_{1_0} = \frac{-512\alpha^4M^2c_2}{\left(\begin{array}{c}((512M^2+32)\alpha^4-16\alpha^2p^2c_2{}^2+2p^4c_2{}^2)\cosh^2(\xi)+ \\ ((512M^2-32)\alpha^4-16\alpha^2p^2c_2{}^2+2p^4c_2{}^2) \\ \sinh(\xi)\cosh(\xi)+(-256M^4+128M^2-16)\alpha^4+32p^2\left(M^2+\frac{1}{4}\right)c_2{}^2\alpha^2-p^2c_2{}^4\end{array}\right)} \tag{2.67}$$

$$v_{1_0} = \frac{\beta}{\alpha}\left(\frac{-512\alpha^4M^2c_2}{\left(\begin{array}{c}((512M^2+32)\alpha^4-16\alpha^2p^2c_2{}^2+2p^4c_2{}^2)\cosh^2(\xi)+ \\ ((512M^2-32)\alpha^4-16\alpha^2p^2c_2{}^2+2p^4c_2{}^2) \\ \sinh(\xi)\cosh(\xi) \\ +(-256M^4+128M^2-16)\alpha^4+32p^2\left(M^2+\frac{1}{4}\right)c_2{}^2\alpha^2-p^2c_2{}^4\end{array}\right)}\right) \tag{2.68}$$

For arbitrary $M = \pm\frac{1}{2}, c_0 = -1, c_1 = 0, c_2 = 1$

$$u_{1_1}(\xi) = -\frac{128\alpha^4}{\begin{array}{c}(64\alpha^4 - 16\alpha^2p^2 + 2P^4)\cosh^2(\xi) + (16\alpha^2p^2 - 2P^4)\sinh(\xi)\cosh(\xi) \\ +16\alpha^2p^2 - P^4\end{array}} \tag{2.69}$$

$$v_{1_1}(\xi) = -\frac{128\beta\alpha^3}{\begin{array}{c}(64\alpha^4 - 16\alpha^2p^2 + 2P^4)\cosh^2(\xi) + (16\alpha^2p^2 - 2P^4)(\sinh(\xi)\cosh(\xi)) \\ +16\alpha^2p^2 - P^4\end{array}} \tag{2.70}$$

Where $\xi = \alpha x + \beta y + \mu t + \theta_0, \mu = \frac{\alpha(3p^4+40\alpha^2p^2+24p^2q+48\alpha^4+96\alpha^2q+48q^2)}{4p^2}$ and

$$\beta = \frac{\alpha(p^2+4\alpha^2+4q)}{2p}.$$

By using Set-2 along with Eq. (2.25) and Eq. (2.63), We are rapidly reducing the travelling waves solution of Eq. (2.58) and Eq. (2.59) as below

Collection -2:

$$u_{2_0} = c_0 + \frac{c_2\left(\frac{-1}{\left(M+\frac{(c_0-c_2)c_0}{2M(2c_0{}^2+c_0c_0-c_2{}^2)}\right)\sinh(\xi)+\left(\left(M-\frac{(c_0-c_2)c_0}{2M(2c_0{}^2+c_0c_0-c_2{}^2)}\right)\cosh(\xi)\right)^2+1}\right)}{\frac{1}{\left(M+\frac{(c_0-c_2)c_0}{2M(2c_0{}^2+c_0c_0-c_2{}^2)}\right)\sinh(\xi)+\left(\left(M-\frac{(c_0-c_2)c_0}{2M(2c_0{}^2+c_0c_0-c_2{}^2)}\right)\cosh(\xi)\right)^2+1}} \tag{2.71}$$

$$v_{2_0}=\frac{\beta}{\alpha}\left(c_0+\frac{c_2\left(\frac{-1}{\left(\left(M+\frac{(c_0-c_2)c_0}{2M(2c_0{}^2+c_0c_0-c_2{}^2)}\right)\sinh(\xi)+\left(\left(M-\frac{(c_0-c_2)c_0}{2M(2c_0{}^2+c_0c_0-c_2{}^2)}\right)\cosh(\xi)\right)\right)^2}+1\right)}{\frac{1}{\left(\left(M+\frac{(c_0-c_2)c_0}{2M(2c_0{}^2+c_0c_0-c_2{}^2)}\right)\sinh(\xi)+\left(\left(M-\frac{(c_0-c_2)c_0}{2M(2c_0{}^2+c_0c_0-c_2{}^2)}\right)\cosh(\xi)\right)\right)^2}+1}\right) \tag{2.72}$$

For arbitrary $M=\pm1, c_0=1, c_1=0, c_2=1$

$$u_{2_1}(\xi)=1+\tanh(\xi) \tag{2.73}$$

$$v_{2_1}(\xi)=\frac{\beta}{\alpha}(1+\tanh(\xi)) \tag{2.74}$$

For arbitrary $M=\pm i, c_0=1, c_1=0, c_2=1$

$$u_{2_2}(\xi)=1+\coth(\xi) \tag{2.75}$$

$$v_{2_2}(\xi)=\frac{\beta}{\alpha}(1+\coth(\xi)) \tag{2.76}$$

Where $\xi=\alpha x+\beta y+\mu t+\theta_0, \alpha=\pm\frac{p}{2}, \beta=\mp\left(\frac{-p^2}{2}+q\right)$ and

$$\mu=\pm\left(\frac{(32p^4-96p^2q+96q^2)}{16p}\right).$$

By using Set-3 along with Eq. (2.25) and Eq. (2.63), We are rapidly reducing the travelling waves solution of Eq. (2.58) and Eq. (2.59) as below

Collection -3:

$$u_{3_0}=\frac{\pm4I\alpha M\big(\sinh(\xi)+\cosh(\xi)\big)}{p(2M^2\cosh^2(\xi)+2M^2\sinh(\xi)\cosh(\xi)-M^2+1)} \tag{2.77}$$

$$v_{3_0}=\frac{\beta}{\alpha}\left(\frac{\pm4I\alpha M\big(\sinh(\xi)+\cosh(\xi)\big)}{p(2M^2\cosh^2(\xi)+2M^2\sinh(\xi)\cosh(\xi)-M^2+1)}\right) \tag{2.78}$$

For arbitrary $M=\pm1, c_0=0, c_1=\frac{4i\alpha}{p}, c_2=0$

$$u_{3_{1,2}}(\xi)=\pm\frac{2i\alpha}{p}\operatorname{sech}(\xi) \tag{2.79}$$

$$v_{3_{1,2}}(\xi)=\pm\frac{2i\beta}{p}\operatorname{sech}(\xi) \tag{2.80}$$

For arbitrary $M=\pm i, c_0=0, c_1=\frac{4i\alpha}{p}, c_2=0$

$$u_{3_{3,4}}(\xi)=\pm\frac{2\alpha}{p}\operatorname{csch}(\xi) \tag{2.81}$$

$$v_{3_{3,4}}(\xi)=\pm\frac{2\beta}{p}\operatorname{csch}(\xi) \tag{2.82}$$

Where $\xi=\alpha x+\beta y+\mu t+\theta_0, \beta=\frac{2\alpha q}{p}$ *and* $\mu=\frac{\alpha(\alpha^2p^2+12q^2)}{p^2}$.

By using Set-4 along with Eq. (2.25) and Eq. (2.63), We are rapidly reducing the travelling waves solution of Eq. (2.58) and Eq. (2.59) as below

Collection -4:

$$u_{4_0} = \frac{I\alpha\left(\left(M^2\sinh(\xi) + M^2\cosh(\xi) - \sinh(\xi) + \cosh(\xi)\right)\sqrt{2} \pm 4M\right)}{p\left(M^2\sinh(\xi) + M^2\cosh(\xi) - \sinh(\xi) + \cosh(\xi)\right)} \quad (2.83)$$

$$v_{4_0} = \frac{\beta}{\alpha}\left(\frac{I\alpha\left(\left(M^2\sinh(\xi) + M^2\cosh(\xi) - \sinh(\xi) + \cosh(\xi)\right)\sqrt{2} \pm 4M\right)}{p\left(M^2\sinh(\xi) + M^2\cosh(\xi) - \sinh(\xi) + \cosh(\xi)\right)}\right) \quad (2.84)$$

For arbitrary $M = \pm 1; c_0 = \frac{\sqrt{2}i\alpha}{p}; c_1 = \frac{4i\alpha}{p}; c_2 = 0$

$$u_{4_{1,2}}(\xi) = \frac{\sqrt{2}i\alpha}{p}\left(1 \pm \sqrt{2}\,\mathrm{sech}(\xi)\right) \quad (2.85)$$

$$v_{4_{1,2}}(\xi) = \frac{\sqrt{2}i\beta}{p}\left(1 \pm \sqrt{2}\,\mathrm{sech}(\xi)\right) \quad (2.86)$$

For arbitrary $M = \pm i, c_0 = \frac{\sqrt{2}i\alpha}{p}, c_1 = \frac{4i\alpha}{p}, c_2 = 0$

$$u_{4_{3,4}}(\xi) = \frac{\sqrt{2}\alpha}{p}\left(i \pm \sqrt{2}\,\mathrm{csch}(\xi)\right) \quad (2.87)$$

$$v_{4_{3,4}}(\xi) = \frac{\sqrt{2}\beta}{p}\left(i \pm \sqrt{2}\,\mathrm{csch}(\xi)\right) \quad (2.88)$$

Where $\xi = \alpha x + \beta y + \mu t + \theta_0, \beta = \sqrt{2}\alpha\left(\frac{q}{p} - i\alpha\right),\ \mu = \frac{-12\alpha}{p^2}\left(\sqrt{2}i\alpha pq + \frac{2\alpha^2 p^2}{3} - q^2\right)$.

By using Set-5 along with Eq. (2.25) and Eq. (2.63), We are rapidly reducing the travelling waves solution of Eq. (2.58) and Eq. (2.59) as below

Collection -5:

$$u_{5_0} = \frac{-I\alpha\left(\left(M^2\cosh(\xi) + M^2\sinh(\xi) + \cosh(\xi) - \sinh(\xi)\right)\sqrt{2} \mp 4M\right)}{p\left(M^2\cosh(\xi) + M^2\sinh(\xi) + \cosh(\xi) - \sinh(\xi)\right)} \quad (2.89)$$

$$v_{5_0} = \frac{\beta}{\alpha}\left(\frac{-I\alpha\left(\left(M^2\cosh(\xi) + M^2\sinh(\xi) + \cosh(\xi) - \sinh(\xi)\right)\sqrt{2} \mp 4M\right)}{p\left(M^2\cosh(\xi) + M^2\sinh(\xi) + \cosh(\xi) - \sinh(\xi)\right)}\right) \quad (2.90)$$

For arbitrary $M = \pm 1,\ c_0 = -\frac{\sqrt{2}i\alpha}{p}, c_1 = \pm\frac{4i\alpha}{p}, c_2 = 0$

$$u_{5_{1,2}}(\xi) = \frac{\sqrt{2}i\alpha}{p}\left(\pm\sqrt{2}\,\mathrm{sech}(\xi) - 1\right) \quad (2.91)$$

$$v_{5_{1,2}}(\xi) = \frac{\sqrt{2}i\beta}{p}\left(\pm\sqrt{2}\,\mathrm{sech}(\xi) - 1\right) \quad (2.92)$$

For arbitrary $M = \pm i,\ c_0 = \frac{\sqrt{2}i\alpha}{p}, c_1 = \pm\frac{4i\alpha}{p}, c_2 = 0$

$$u_{5_{3,4}}(\xi) = \frac{\sqrt{2}\alpha}{p}\left(\pm\sqrt{2}\,\mathrm{csch}(\xi) - i\right) \quad (2.93)$$

$$v_{5_{3,4}}(\xi) = \frac{\sqrt{2}\beta}{p}\left(\pm\sqrt{2}\,\mathrm{csch}(\xi) - i\right) \tag{2.94}$$

Where $\xi = \alpha x + \beta y + \mu t + \theta_0, \beta = \sqrt{2}\alpha\left(\frac{q}{p} + i\alpha\right), \mu = \frac{-12\alpha}{p^2}\left(\sqrt{2}i\alpha pq + \frac{2\alpha^2 p^2}{3} - q^2\right)$.

By using Set-6 along with Eq. (2.25) and Eq. (2.63), We are rapidly reducing the travelling waves solution of Eq. (2.58) and Eq. (2.59) as below

Collection -6:

$$u_{6_0} = \pm\frac{\left(32\alpha^2 M c_2\left(\left(\left(IM^2-\frac{I}{4}\right)\alpha^2+\frac{Ip^2c_2^2}{4}\right)\cosh(\xi)+\left(\left(IM^2+\frac{I}{4}\right)\alpha^2-\frac{Ip^2c_2^2}{4}\right)\sinh(\xi)-\alpha^2 M\right)\right)}{\left(\begin{array}{c}((32M^4+2)\alpha^4-4\alpha^2p^2c_2^2+2p^4c_2^4)\cosh^2(\xi)+\\((32M^4-2)\alpha^4+4\alpha^2p^2c_2^2-2p^4c_2^4)\sinh(\xi)\cosh(\xi)+\\(-16M^4+8M^2-1)\alpha^4+8\left(M^2+\frac{1}{4}\right)p^2c_2^2\alpha^2-p^4c_2^4\end{array}\right)} \tag{2.95}$$

$$v_{6_0} = \pm\frac{\beta}{\alpha}\frac{\left(32\alpha^2 M c_2\left(\left(\left(IM^2-\frac{I}{4}\right)\alpha^2+\frac{Ip^2c_2^2}{4}\right)\cosh(\xi)+\left(\left(IM^2+\frac{I}{4}\right)\alpha^2-\frac{Ip^2c_2^2}{4}\right)\sinh(\xi)-\alpha^2 M\right)\right)}{\left(\begin{array}{c}\left((32M^4+2)\alpha^4-4\alpha^2p^2c_2^2+2p^4c_2^4\right)\cosh^2(\xi)+\\\left((32M^4-2)\alpha^4+4\alpha^2p^2c_2^2-2p^4c_2^4\right)\sinh(\xi)\cosh(\xi)+\\(-16M^4+8M^2-1)\alpha^4+8\left(M^2+\frac{1}{4}\right)p^2c_2^2\alpha^2-p^4c_2^4\end{array}\right)} \tag{2.96}$$

For arbitrary $M = \pm 1, c_0 = -1, c_1 = \pm 2i, c_2 = 1$

$$u_{6_{1,2}}(\xi) = -\frac{8\alpha^2\left((5i\alpha^2 - ip^2)\sinh(\xi) + (3i\alpha^2 + ip^2)\cosh(\xi) \mp 4\alpha^2\right)}{\left(\begin{array}{c}(34\alpha^4 - 4\alpha^2p^2 + 2P^4)\cosh^2(\xi) + (30\alpha^4 + 4\alpha^2p^2 - 2P^4)\\\sinh(\xi)\cosh(\xi) - 9\alpha^4 + 10\alpha^2p^2 - P^4\end{array}\right)} \tag{2.97}$$

$$v_{6_{1,2}}(\xi) = -\frac{8\alpha\beta\left((5i\alpha^2 - ip^2)\sinh(\xi) + (3i\alpha^2 + ip^2)\cosh(\xi) \mp 4\alpha^2\right)}{\begin{array}{c}(34\alpha^4 - 4\alpha^2p^2 + 2P^4)\cosh^2(\xi) + (30\alpha^4 + 4\alpha^2p^2 - 2P^4)\\\sinh(\xi)\cosh(\xi) - 9\alpha^4 + 10\alpha^2p^2 - P^4\end{array}} \tag{2.98}$$

Where $\xi = \alpha x + \beta y + \mu t + \theta_0,\ \beta = \frac{\alpha(\alpha^2+2q)}{p}$ and $\mu = \frac{\alpha(\alpha^2p^2+3\alpha^4+12\alpha^2q+12q^2)}{p^2}$.

By using Set-7 along with Eq. (2.25) and Eq. (2.63), We are rapidly reducing the travelling waves solution of Eq. (2.58) and Eq. (2.59) as below

Collection -7:

$$u_{7_0} = \frac{4(c_0+c_2)\left(\begin{array}{c}\left(\left(-M^2+\frac{1}{4}\right)c_0^2+\left(-M^2c_2-\frac{1}{4}c_2\right)c_0-M^2c_2^2\right)\cosh(\xi)+\\\left(\left(-M^2-\frac{1}{4}\right)c_0^2+\left(-2M^2c_2+\frac{1}{4}c_2\right)c_0-M^2c_2^2\right)\sinh(\xi)+I(c_0-c_2)M(c_0+c_2)\end{array}\right)}{\left(\begin{array}{c}((4M^2-1)c_0^2+(8M^2c_2+c_2)c_0+4M^2c_2^2)\cosh(\xi)+\\((4M^2+1)c_0^2+(8M^2c_2-c_2)c_0+4M^2c_2^2)\sinh(\xi)+4IM(c_0+c_2)^2\end{array}\right)} \tag{2.99}$$

$$v_{7_0} = \frac{\beta}{\alpha}\frac{4(c_0+c_2)\left(\begin{array}{c}\left(\left(-M^2+\frac{1}{4}\right)c_0^2+\left(-M^2c_2-\frac{1}{4}c_2\right)c_0-M^2c_2^2\right)\cosh(\xi)+\\\left(\left(-M^2-\frac{1}{4}\right)c_0^2+\left(-2M^2c_2+\frac{1}{4}c_2\right)c_0-M^2c_2^2\right)\sinh(\xi)+I(c_0-c_2)M(c_0+c_2)\end{array}\right)}{\left(\begin{array}{c}((4M^2-1)c_0^2+(8M^2c_2+c_2)c_0+4M^2c_2^2)\cosh(\xi)+\\((4M^2+1)c_0^2+(8M^2c_2-c_2)c_0+4M^2c_2^2)\sinh(\xi)+4IM(c_0+c_2)^2\end{array}\right)} \tag{2.100}$$

For arbitrary $M = i, c_0 = i, c_1 = \pm 2i, c_2 = 1$

$$u_{7_{1,2}}(\xi) = \frac{(6i-8)\cosh(\xi) + (10i-8)\sinh(\xi) \pm 8i \pm 8}{(7i-1)\cosh(\xi) + (9i+1)\sinh(\xi) \mp 8i} \tag{2.101}$$

$$v_{7_{1,2}}(\xi) = \frac{\beta}{\alpha}\left(\frac{(6i-8)\cosh(\xi) + (10i-8)\sinh(\xi) \pm 8i \pm 8}{(7i-1)\cosh(\xi) + (9i+1)\sinh(\xi) \mp 8i}\right) \tag{2.102}$$

For arbitrary $M = -i, c_0 = i, c_1 = \pm 2i, c_2 = 1$

$$u_{7_{3,4}}(\xi) = \frac{(6i-8)\cosh(\xi) + (10i-8)\sinh(\xi) \mp 8i \mp 8}{(7i-1)\cosh(\xi) + (9i+1)\sinh(\xi) \pm 8i} \tag{2.103}$$

$$v_{7_{3,4}}(\xi) = \frac{\beta}{\alpha}\left(\frac{(6i-8)\cosh(\xi) + (10i-8)\sinh(\xi) \mp 8i \mp 8}{(7i-1)\cosh(\xi) + (9i+1)\sinh(\xi) \pm 8i}\right) \tag{2.104}$$

Where $\xi = \alpha x + \beta y + \mu t + \theta_0$, $\alpha = \left(\frac{i}{10} + \frac{1}{10}\right)\sqrt{10-30i}p$, $\beta = \left(\frac{i}{25} - \frac{2}{25}\right)\left((i-3)p^2 - (3i+1)q\right)\sqrt{10-30i}$ and $\mu = \frac{1}{p}\left(\frac{7i}{1000} - \frac{1}{1000}\right)(-68ip^4 + 96ip^2q + 192p^2q - 192q^2 + 144iq^2)$.

By using Set-8 along with Eq. (2.25) and Eq. (2.63), We are rapidly reducing the travelling waves solution of Eq. (2.58) and Eq. (2.59) as below

Collection -8:

$$u_{8_0} = \frac{4(c_0+c_2)\left(\begin{array}{c}\left(\left(M^2-\frac{1}{4}\right)c_0{}^2+\left(2M^2c_2+\frac{1}{4}c_2\right)c_0+M^2c_2{}^2\right)\cosh(\xi)+ \\ \left(\left(M^2+\frac{1}{4}\right)c_0{}^2+\left(2M^2c_2-\frac{1}{4}c_2\right)c_0+M^2c_2{}^2\right)\sinh(\xi)+IM(c_0+c_2)(c_0-c_2)\end{array}\right)}{\left(\begin{array}{c}\left((4M^2-1)c_0{}^2+(8M^2c_2+c_2)c_0+4M^2c_2{}^2\right)\cosh(\xi)+ \\ \left((4M^2+1)c_0{}^2+(8M^2c_2-c_2)c_0+4M^2c_2{}^2\right)\sinh(\xi)+4IM(c_0+c_2)^2\end{array}\right)} \tag{2.105}$$

$$v_{8_0} = \frac{\beta}{\alpha}\frac{4(c_0+c_2)\left(\begin{array}{c}\left(\left(M^2-\frac{1}{4}\right)c_0{}^2+\left(2M^2c_2+\frac{1}{4}c_2\right)c_0+M^2c_2{}^2\right)\cosh(\xi)+ \\ \left(\left(M^2+\frac{1}{4}\right)c_0{}^2+\left(2M^2c_2-\frac{1}{4}c_2\right)c_0+M^2c_2{}^2\right)\sinh(\xi)+IM(c_0+c_2)(c_0-c_2)\end{array}\right)}{\left(\begin{array}{c}\left((4M^2-1)c_0{}^2+(8M^2c_2+c_2)c_0+4M^2c_2{}^2\right)\cosh(\xi)+ \\ \left((4M^2+1)c_0{}^2+(8M^2c_2-c_2)c_0+4M^2c_2{}^2\right)\sinh(\xi)+4IM(c_0+c_2)^2\end{array}\right)} \tag{2.106}$$

For arbitrary $M = \pm 1, c_0 = \frac{1}{2}, c_1 = \pm i, c_2 = \frac{1}{2}$

$$u_{8_{1,2}}(\xi) = \frac{\cosh(\xi) + \sinh(\xi)}{\cosh(\xi) + \sinh(\xi) \mp i} \tag{2.107}$$

$$v_{8_{1,2}}(\xi) = \frac{\beta}{\alpha}\left(\frac{\cosh(\xi) + \sinh(\xi)}{\cosh(\xi) + \sinh(\xi) \mp i}\right) \tag{2.108}$$

$$u_{8_{3,4}}(\xi) = \frac{\cos(i\xi) - i\sin(i\xi)}{\cos(i\xi) - \sin(i\xi) \mp i} \tag{2.109}$$

$$v_{8_{3,4}}(\xi) = \frac{\beta}{\alpha}\left(\frac{\cos(i\xi) - i\sin(i\xi)}{\cos(i\xi) - i\sin(i\xi) \mp i}\right) \tag{2.110}$$

Where $\xi = \alpha x + \beta y + \mu t + \theta_0$, $\alpha = \frac{-p}{2}$, $\beta = \frac{p^2}{4} - q$ and $\mu = \frac{(48p^2q - 8p^4 - 96q^2)}{16p}$.

Chapter 3: Result and Discussion

In this chapter, we will present physical illustrations of the obtained closed-form solutions of the stated equations cited in chapter four using various methods. Furthermore, we discuss the impact of the values of the various types of parameters on determining solutions on the wave profile. Furthermore, we will discover the distinction between solutions obtained through different methods, as well as the limitations of the implemented methods, which will be elaborately described in two parts namely Comparisons of the investigated methods and Benefits and drawbacks of the investigated methods. Rather, we analyze the physical description and then display the graphs of the gained solutions of the (2+1)-dimensional paraxial nonlinear Schrodinger (NLS) equation and the (2+1)-dimensional nonlinear Konopelchenko–Dubrovsky (KD) equation.

3.1 Graphical and physical discussion of the wave profile

This section contains three subsections that locate the well-delineated representations and discussions of innumerable solitary waves of the ascertained solutions for (2+1)-dimensional paraxial nonlinear Schrodinger (NLS) equation through the g'- expansion and (g'/g^2)-expansion methods. In addition, the (2+1)-dimensional nonlinear (KD) model via Modified Version of New Kudryashov (MVNK) Method. Various types such as 3D, and combined 2D plots represented of the real, imaginary and modulus graph are obtainable by using of MATLAB and Wolfram Mathematica software, our gained travelling wave solutions of the proposed equations are represented in the figures and discuss the characteristics of those waves for disagreeable values of the free parameters.

3.1.1 Wave profile analysis of paraxial nonlinear Schrodinger model

With the help of the Mathematical software Wolfram Mathematica, the obtained solutions to the g' method is picturized in 3D, and 2D plots with suitable values of the related parameters in this sub-section. By applying the g' technique obtainment to the paraxial NLS equation, we attain the families of well-known and standard soliton solutions such as smooth kink, idea kink, singular kink, peakon, periodic, anti-bell, bell shape. By applying the (g'/g^2)-expansion technique on the paraxial NLS equation we have achieved trigonometric, hyperbolic, and rational solutions. In general, depending on the kerr nonlinearity ϵ, the resulting $u_1(\xi)$, $u_3(\xi)$ and $u_4(\xi)$ exhibit different sorts of solitary waves with the coefficient of the higher degree of nonlinear term. Also, the outcomes $u_7(\xi)$, $u_9(\xi)$ and $u_{10}(\xi)$ shows various types of solitary waves depending on wave numbers λ with the coefficient of the higher order of nonlinear term. In this section, we will show the wave shape those changes as a result of the effect of free parameters which will play an important role in the field of engineering modern physics. Here, we have shown a change in the graph due to the non-linearity effect and explain the graphical representation with some fixed parametric values.

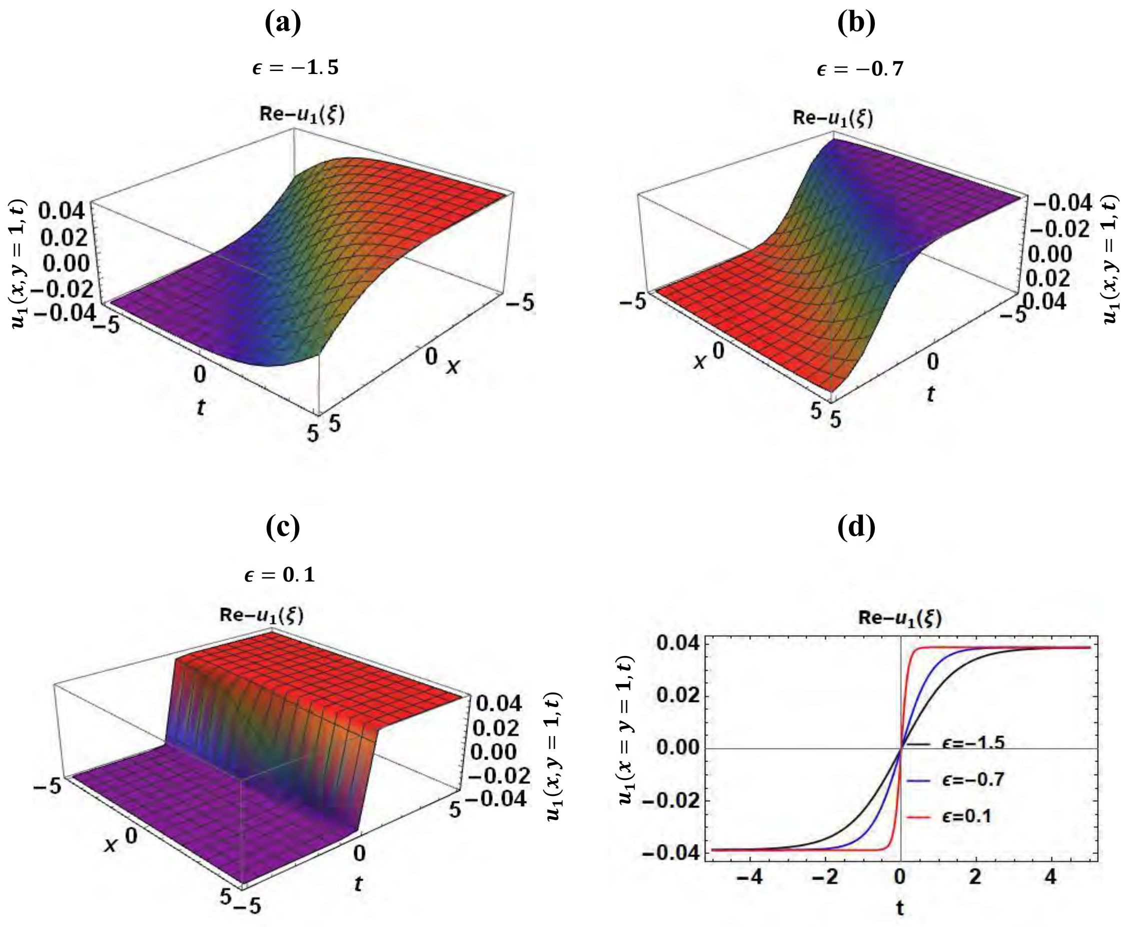


Figure 3.1: The graph of the real part of the solution $u_1(\xi)$ is depicted for selecting parameters within the interval $-5 \leq x, t \leq 5$.

The real part of the result $u_1(\xi)$ represent the flat kink soliton for the free parameter $l = 0.1, m = 1, n = 1, \lambda = 0.01, \delta_1 = \delta_2 = 0.001, \beta_1 = 0.1, \epsilon = -1.5$ shown in Fig-3.1(a). Now, increasing the Kerr nonlinearity ϵ at -0.7 and 0.1 the solution $u_1(\xi)$ represent smooth kink and ideal kink shape respectively portrayed in Fig-3.1(b) and Fig-3.1(c). Also, we illustrate the corresponding line graph for $\epsilon = -1.5, -0.7$ and 0.1 shown in Fig-3.1(d).The soliton solution is stable since the figure cannot change from its original shape for very large values of ϵ, such as $\epsilon = 30, 40, 50$. This type of soliton changes their asymptotic line from one location to another location and is finally stable at $t \longrightarrow \infty$.

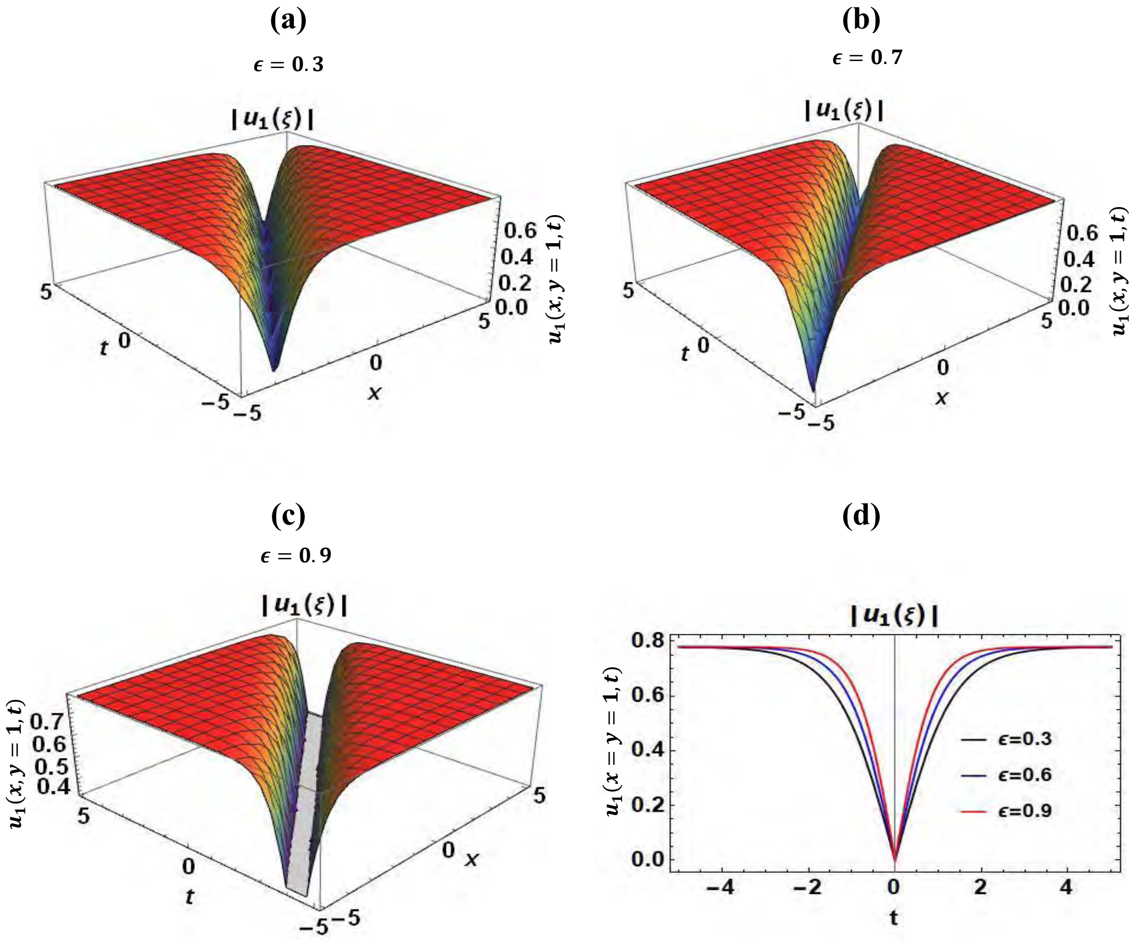


Figure 3.2: The modulus plots of the solution $u_1(\xi)$ is depicted for selecting parameters within the interval $-5 \leq x, t \leq 5$.

Again, the modulus plot of the result $u_1(\xi)$ represent the peakon type soliton for the free parameter $l = 0.01, m = 2, n = 5, \delta_1 = -1, \delta_2 = 2, \beta_1 = 4, \lambda = 3, \epsilon = 0.3$ shown in Fig-3.2(a). A peakon is a type of soliton in which each peakon is a soliton and confined to a finite centre. Peakons are characterized by solitary waves with noticeable soliton properties that, after being consecutively destroyed in various compacts, they reemerge with precise shapes. Now, increasing $\epsilon = 0.7$ and 0.9 the solution $u_1(\xi)$ represent singular kink soliton portrayed in Fig-3.2(b) and Fig-3.2(c) respectively. Also, we illustrate the corresponding line graph for $\epsilon = 0.3$, 0.6 and 0.9 shown in Fig-3.2(d). For fixed-parameter $l = 1.01, m = 2, n = 1.01, \lambda = 0.5, \delta_1 = 1.4, \delta_2 = 1.4, \beta_1 = 0.3, \epsilon = 0.4$ behavior of the result $u_3(\xi)$ shown in Fig-3.3(a). Increases the value of Kerr nonlinearity such as $\epsilon = 1.4$ and 2.6 with keeps the value of all the other

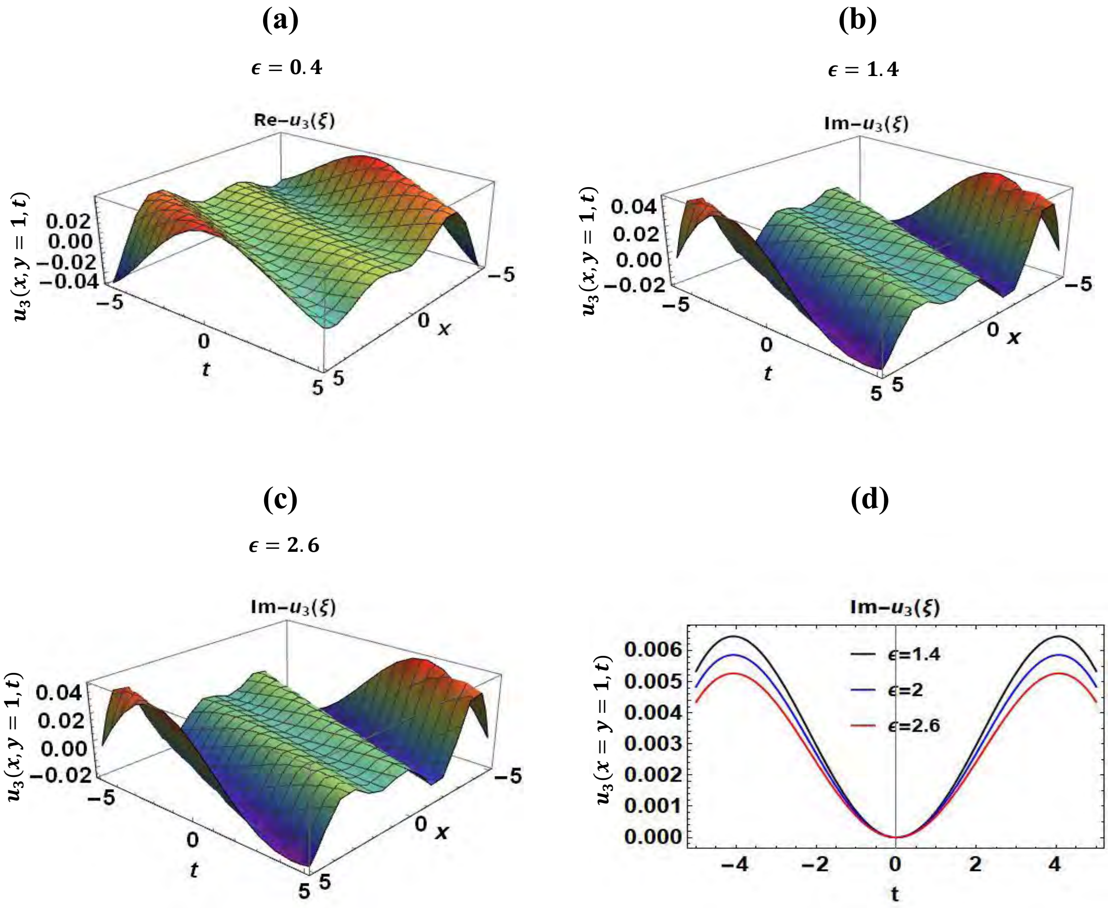


Figure 3.3: The graph of the imaginary part of the solution $u_3(\xi)$ is depicted for selecting parameters within the interval $-5 \le x, t \le 5$.

parameters the same as the structure solution $u_3(\xi)$ has been not changed all are looks like as short wave portrayed in Fig-3.3(b) and Fig-3.3(c). The singular soliton exists for very large values of ϵ, such as $\epsilon = 40$, and 45 that are not shown. The line graphics as portrayed for same values shown in Fig-3.3(d). In addition, the 3D depiction real part of the solution $u_4(\xi)$ represent anti-bell soliton for the fixed values parameter $l = -0.1, m = 1, n = 0.9, \lambda = 0.01, \delta_1 = 0.35, \epsilon = -0.05$ illustrates in Fig-3.4(a). For large value of $\epsilon = 0.01$ and 0.09 the nature of the result $u_4(\xi)$ is parabolic shape and bell shape soliton shown in Fig-3.4(b) and Fig-3.4(c) respectively. The line graphics as portrayed for same values shown in Fig-3.4(d). The 3D bell shape soliton is sketched for the imaginary part of the solution $u_7(\xi)$ for arbitrary real values $l = -0.02, n = 0.01, \lambda = 0.3, \delta_1 = 0.01, \delta_2 = 1.5, \beta_1 = 0.5, \epsilon = 3.5, \Delta_1 = 1, \Delta_2 = -1$ shown in Fig-3.5(a). The bell shape soliton rises from both sides. Here obtained waves are symmetric about the vertical axis and propagated along the x- axis.

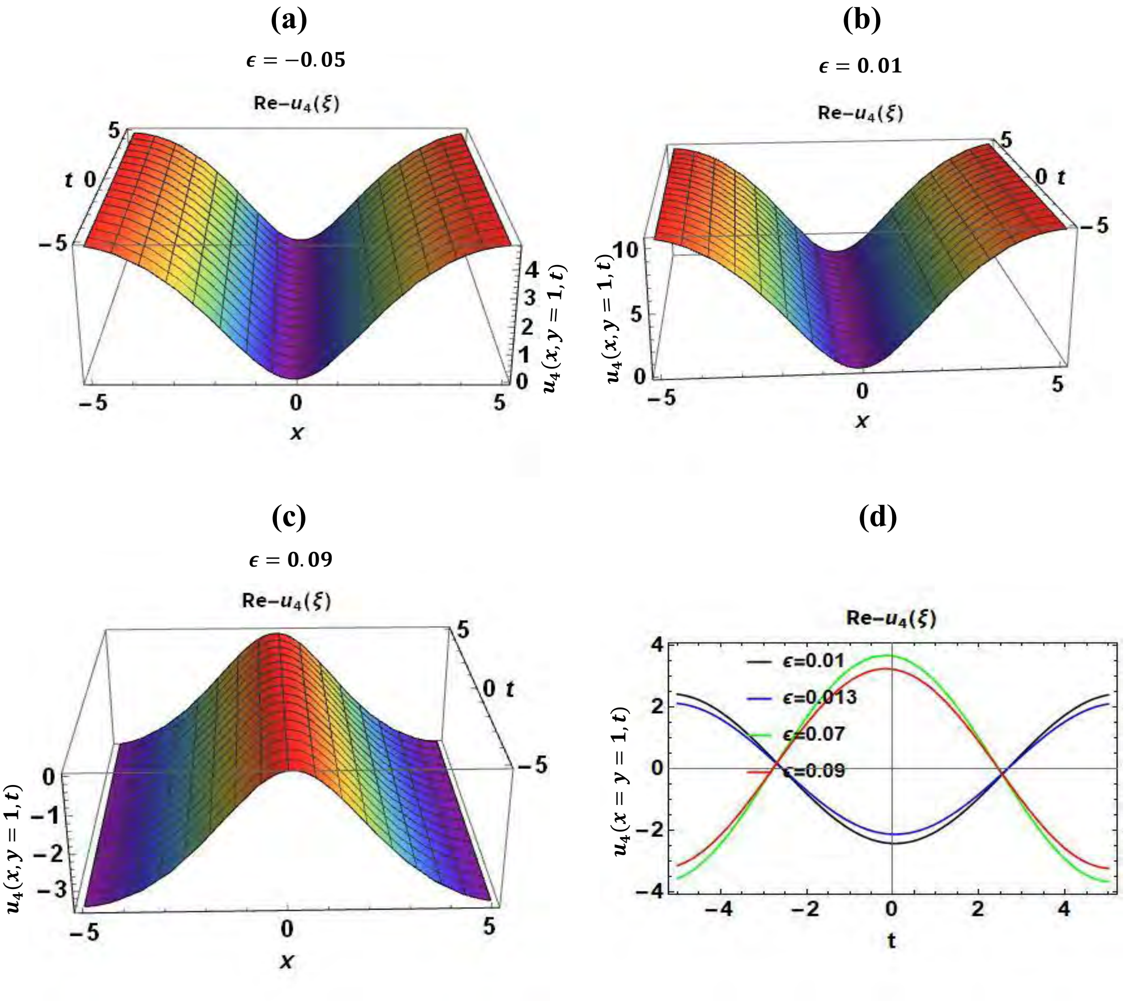


Figure 3.4: The graph of the real part of the solution $u_4(\xi)$ is depicted for selecting parameters within the interval $-5 \leq x, t \leq 5$.

Increasing wave number $\lambda = 0.4$ and 0.9 with rest at all arbitrary values the solitary wave presented at bell shape and W-shape soliton shown in Fig-3.5(b) and Fig-3.5(c) respectively. Also, corresponding combined line graphs are illustrated in Fig-3.5(d).

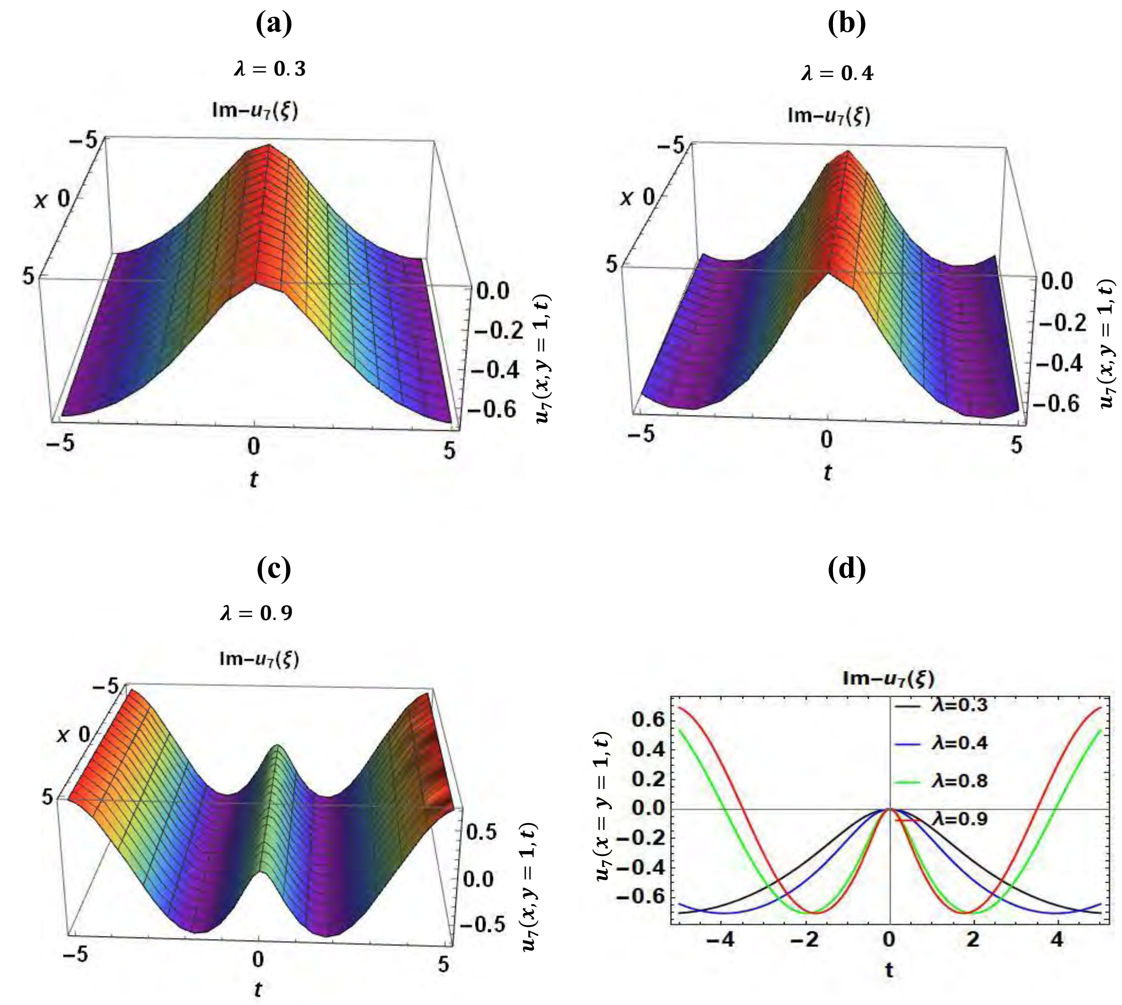
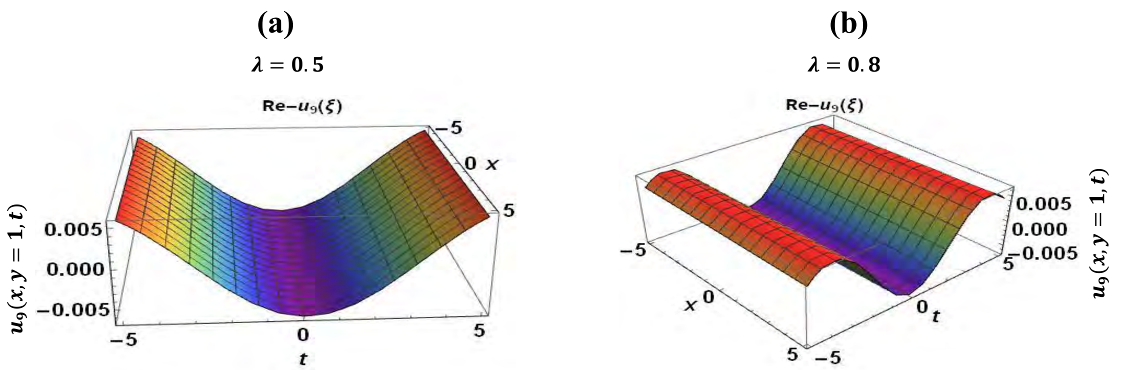


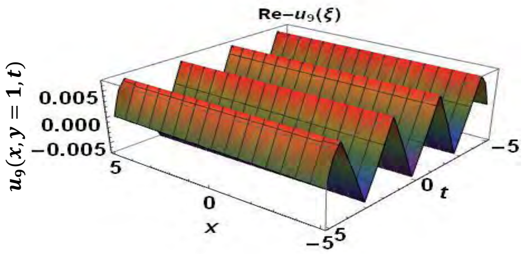
Figure 3.5: The graph of the imaginary part of the solution $u_7(\xi)$ is depicted for selecting parameters within the interval $-5 \leq x, t \leq 5$.

The real part of the result $u_9(\xi)$ represent the parabolic soliton for the free parameter $l = 0.001, n = 0.001, \lambda = 0.5, \delta_1 = 0.1, \delta_2 = 1, \beta_1 = 0.01, \epsilon = 0.2, \Delta_1 = 0.2, \Delta_2 = 0.3$ shown in Fig-3.6(a) can be noticed. If the value of λ is increased to 0.8 and 2.1 by



(c) **(d)**

$\lambda = 2.1$



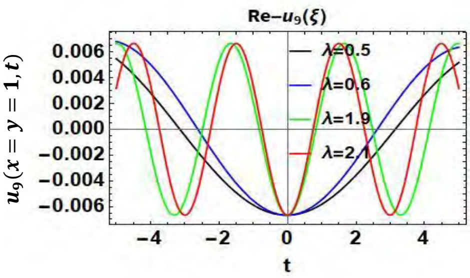
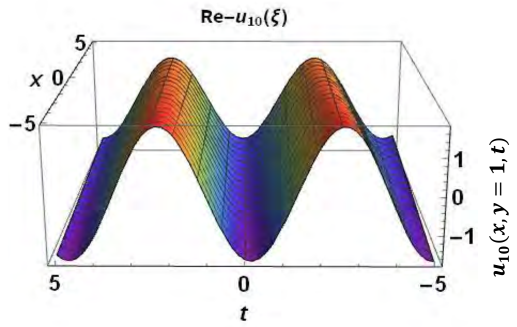


Figure 3.6: The graph of the real part of the solution $u_9(\xi)$ is depicted for selecting parameters within the interval $-5 \leq x, t \leq 5$.

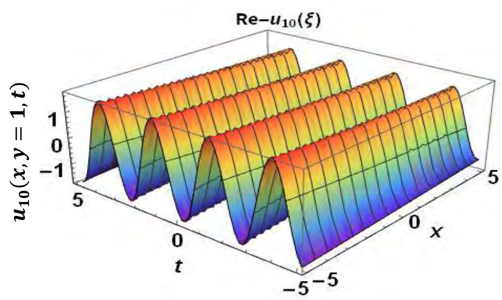
keeping the value of all the parameters the same, the change of propagation wave Fig-3.6(b) and Fig-3.6(c) respectively. Each 3D soliton is shown in a similar line picturized Fig-3.6(d).

(a) **(b)**

$\lambda = 1.3$

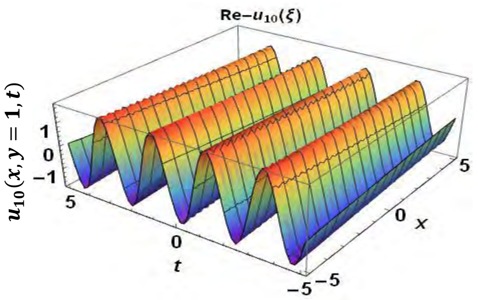


$\lambda = 1.9$



(c) **(d)**

$\lambda = 2.9$



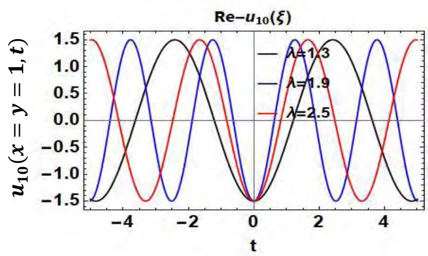


Figure 3.7: The graph of the real part of the solution $u_{10}(\xi)$ is depicted for selecting parameters within the interval $-5 \leq x, t \leq 5$.

For arbitrary fixed parameters $l = 0.01, n = -0.01, \lambda = 1.3, \delta_1 = 0.04, \epsilon = 0.01, \Delta_1 = -2, \Delta_2 = -1$, the real part of the result $u_{10}(\xi)$ represent M-shape soliton in Fig-3.7(a). Now rising λ to 1.9 and 2.5 by the rest of other parameters the wave profile changed shown in Fig-3.7(b) and Fig-3.7(c) respectively. This profile is called the periodic wave profile which is repeated over a period of time. Also, each 3D soliton is shown in a similar line graph Fig-3.7(d).In this section, we attain the families of well-known and standard soliton such as bell shape, W-shape, M-shape, anti-parabolic, periodic for the (2+1)-dimensional paraxial NLS equation in Kerr media. The effect of the parameters (for example ϵ and λ) are studied successfully to observe the change of the behaviour of wave propagation. The above outcomes of the (2+1)-dimensional paraxial NLS equation in Kerr media explain surface waves in signal transmission, hydro-magnetic cold plasma, non-linear optics, quantum mechanics, sound waves in harmonious crystals, and so on.

3.1.2 Wave profile analysis of Konopelchenko–Dubrovsky model

In this segment, we discuss the physical clarifications and graphical representations of the obtained solutions of the (2+1)-Konopelchenko–Dubrovsky (KD) equation via Modified Version of New Kudryashov Method (MVNK), as detailed in chapter 2. The nature of the travelling wave profile changes as the unknown parameters of the accomplished solutions, which are related to the linear and nonlinear terms of the nonlinear equation. Obtained stable wave solutions to of the (2+1)-Konopelchenko–Dubrovsky (KD) equations are illustrated through figures and speculated the nature of these waves for dissimilar values of parameters using MATLAB software.

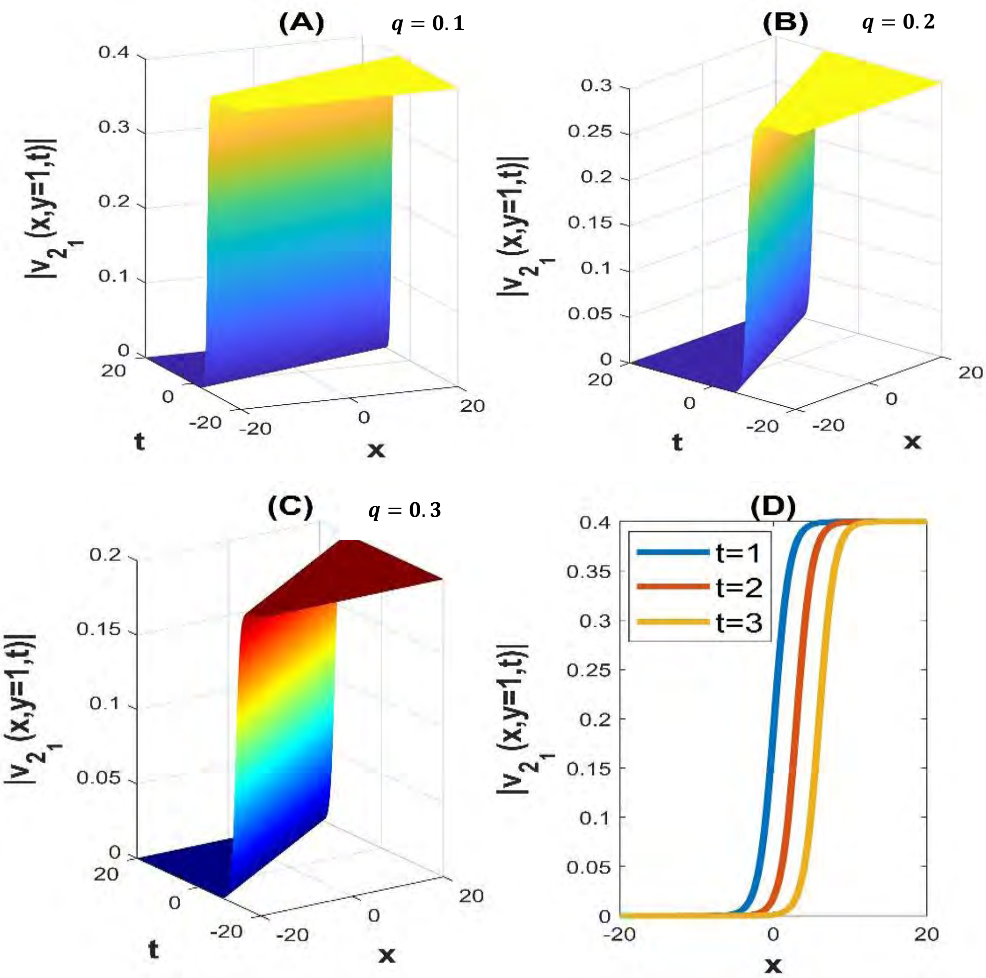


Figure 3.8: The modulus plots of the solution $v_{2_1}(\xi)$ is depicted for selecting parameters within the interval $-20 \le x, t \le 20$.

The obtained solution $v_{2_1}(x, y, t)$, travelling in the upward direction of the x-axis with speeds of phase $\mu = 1.46, 1.04\ and\ 0.74$. Fig-3.8(A) represents the 3D kink soliton profile of $v_{2_1}(x, 1, t)$for the values of $p = 1, q = 0.1\ and\ \theta_0 = 1$. Fig-3.8(B) and Fig-3.8(C) for $q = 0.2\ and\ 0.3$ respectively and others parameter are fixed. Fig-3.8(D) depicts the progression of the wave motion as time passes, with snapshots taken at $t = 1,2,3$. Fig-3.8(D) depicts the temporal evolution of the solution $v_{2_1}(x, 1, t)$ indicating that the solitary wave is travelling in the positive direction of the x-axis. In addition, we clearly noticed from 2D figure that the asymptotic line moves from left to right.

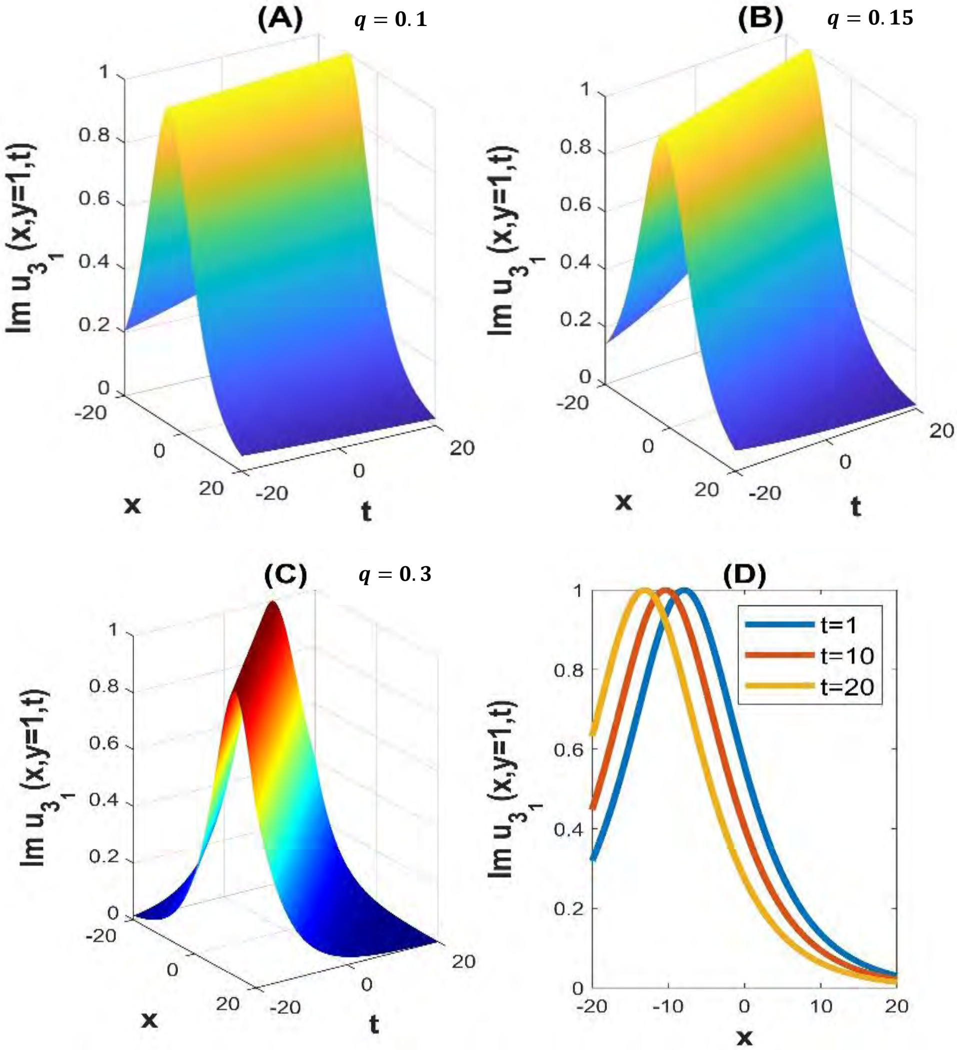


Figure 3.9: The graph of the imaginary part of the solution $u_{3_1}(\xi)$ is depicted for selecting parameters within the interval $-20 \le x, t \le 20$.

The imaginary wave profile of the resultant $u_{3_1}(x, y, t)$ is travelling in the backward direction of the x-axis with speeds of phase $\mu = 0.203375, 0.453375$ *and* 1.803375. Fig-3.9(A) represents the 3D bell shape soliton wave profile of $u_{3_1}(x, 1, t)$for the values of $p = 0.3, q = 0.1$ *and* $\theta_0 = 1$. Fig-3.9(B) and Fig-3.9(C) for $q = 0.15$ *and* 0.3 respectively and others parameter are fixed. Fig-3.9(D) depicts the evolution of the wave motion throughout time, with photograph taken at $t = 1,10$ *and* 20. Fig-3.9(D) depicts the temporal evolution of the solution $u_{3_1}(x, 1, t)$ indicating that the solitary wave is travelling in the negative direction of the x-axis. The bell shape soliton is non topological soliton. Also, it is stable bounded and the initial boundary condition at infinity.

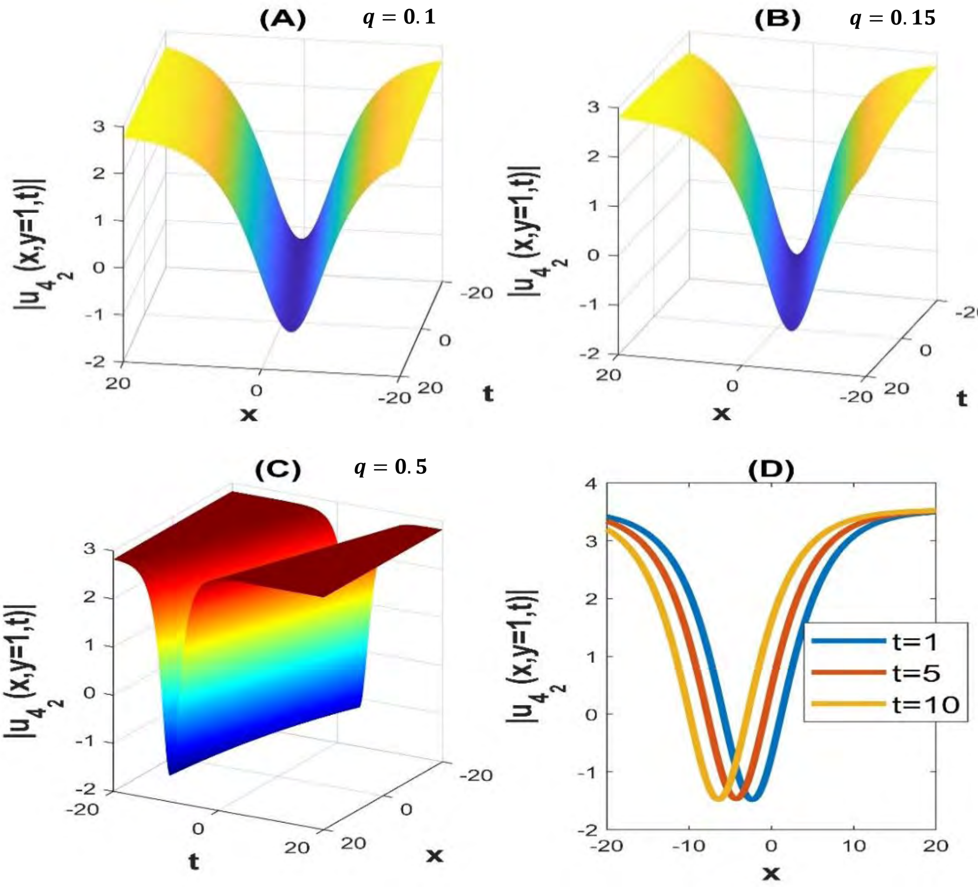


Figure 3.10: The modulus plot of the solution $u_{4_2}(\xi)$ is depicted for selecting parameters within the interval $-20 \leq x, t \leq 20$.

The modulus plot of the result $u_{4_2}(x, y, t)$ represent the anti-bell type soliton for the with speeds of phase $\mu = -0.6788225099i + 2.384, -1.018233765i + 5.384$ and $-3.39411255i + 59.984$ and free parameter $p = 0.1, q = 0.1, \alpha = 0.2$ and $\theta_0 = 0.1$. shown in Fig-3.10(A). An anti-bell is a type of and confined to a finite centre. anti-bell are solitary waves with distinct soliton characteristics that, after being repeatedly annihilated in different compacts, emerge with precise shapes. Fig-3.10(B) and Fig-3.10(C) for $q = 0.15$ and 0.5 respectively and others parameter are fixed. Fig-3.10(D) depicts the progression of the wave motion as time passes, with snapshots taken at $t = 1, 5$ and 10. Fig-3.10(D) depicts the temporal evolution of the solution $u_{4_2}(x, 1, t)$ indicating that the solitary wave is travelling in the negative direction of the x-axis.

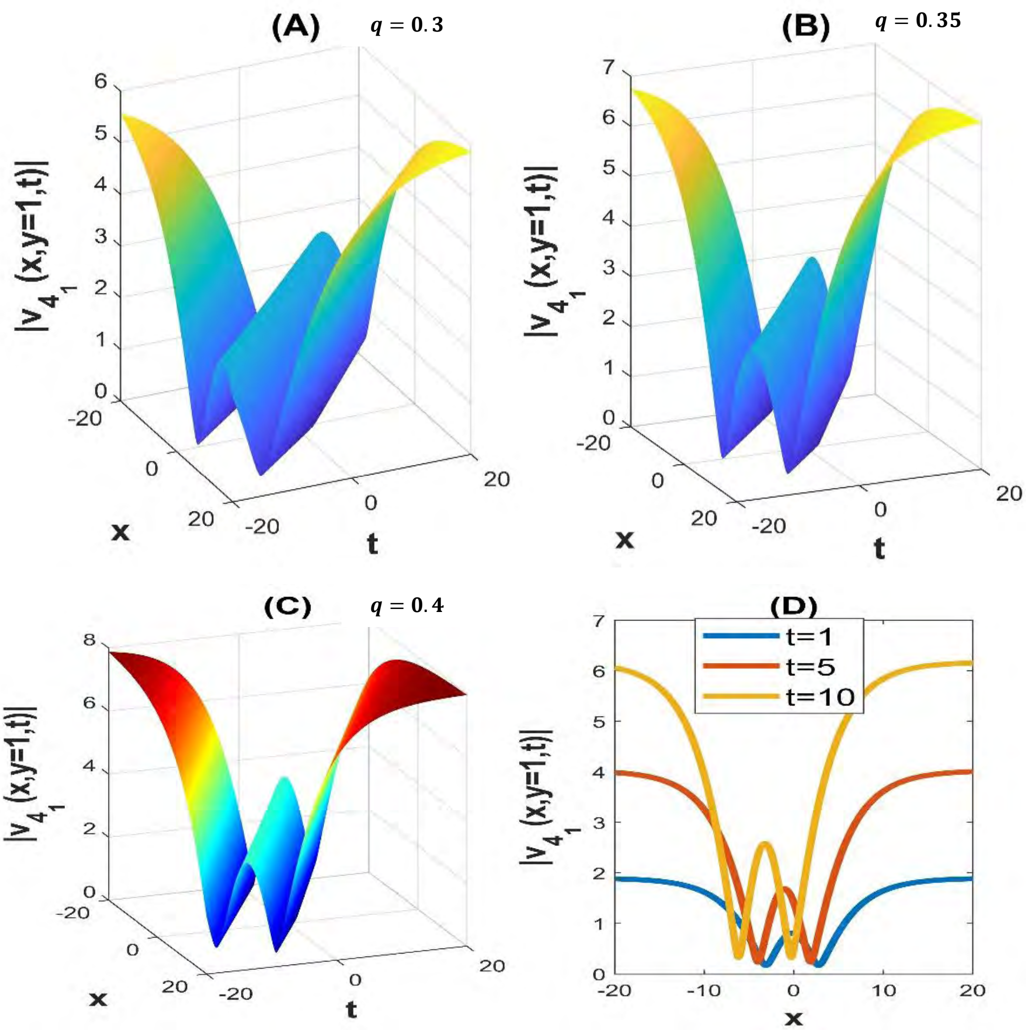


Figure 3.11: The modulus plots of the solution $v_{4_1}(\xi)$ is portrayed for selecting parameters within the interval $-20 \leq x, t \leq 20$.

The modulus plot of the result $v_{4_1}(x, y, t)$ represent the W-shape soliton for the with speeds of phase $\mu = -0.509116882i + 10.792, -0.593969696i + 14.692$ and $-0.678822509i + 19.192$ and free parameter $p = 0.1, q = 0.3, \alpha = 0.1$ and $\theta_0 = 0.1$. shown in Fig-3.11(A). For increasing the value of the nonlinear coefficient $q = 0.35$ and 0.4 and other parameters that are fixed, we clearly see that the left wings of the W-shape soliton also increased, as shown in Fig-3.11(B) and Fig-3.11(C) respectively. Fig-3.11(D) depicts the progression of the wave motion as time passes, with snapshots taken at $t = 1, 5$ and 10. Fig-3.11(D) depicts the temporal evolution of the solution $v_{4_1}(x, 1, t)$ indicating that the solitary wave is travelling in the negative direction of the x-axis.

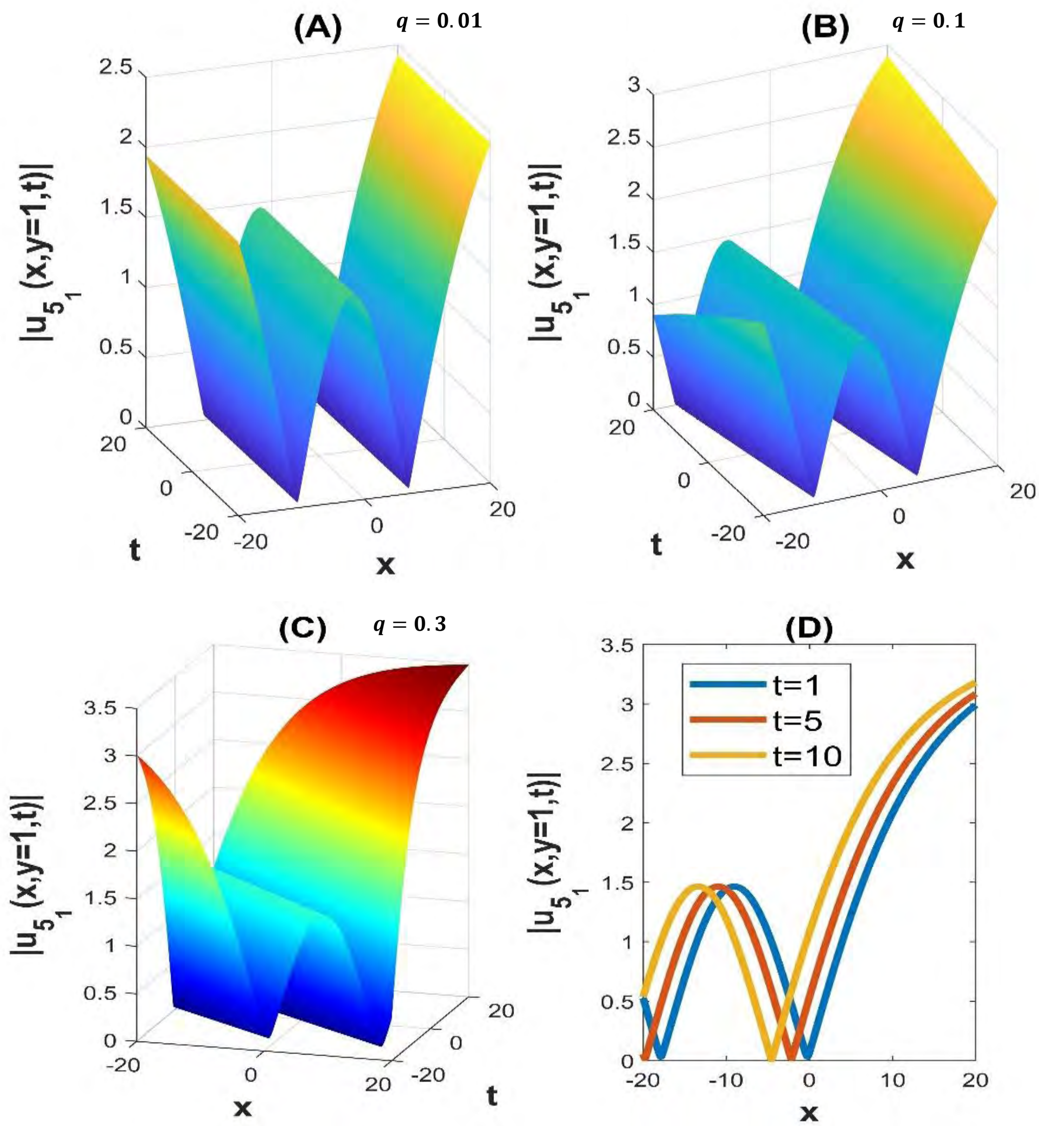


Figure 3.12:The modulus plots of the solution $u_{5_1}(\xi)$ is portrayed for selecting parameters within the interval $-20 \le x, t \le 20$.

The absolute wave pattern of $u_{5_1}(x, y, t)$ is the double anti-bell type soliton for the with speeds of phase $\mu = -0.04242640687i + 0.067$, $-0.4242640687i + 7.492$ and $-1.272792206i + 67.492$ and free parameter $p = 0.04, q = 0.01, \alpha = 0.1$ and $\theta_0 = 0.15$ shown in Fig-3.12(A). Fig-3.12(B) and Fig-3.12(C) for $q = 0.1$ and 0.3 respectively and others parameter are fixed. Fig-3.12(D) illustrates how the flow patterns change over time by displaying snapshots of the wave motion taken at time at $t = 1, 5$ and 10. Fig-3.12(D) depicts the time evolution of the solution $u_{5_1}(x, y, t)$, and it reveals that the solitary wave is travelling away from the x-axis.

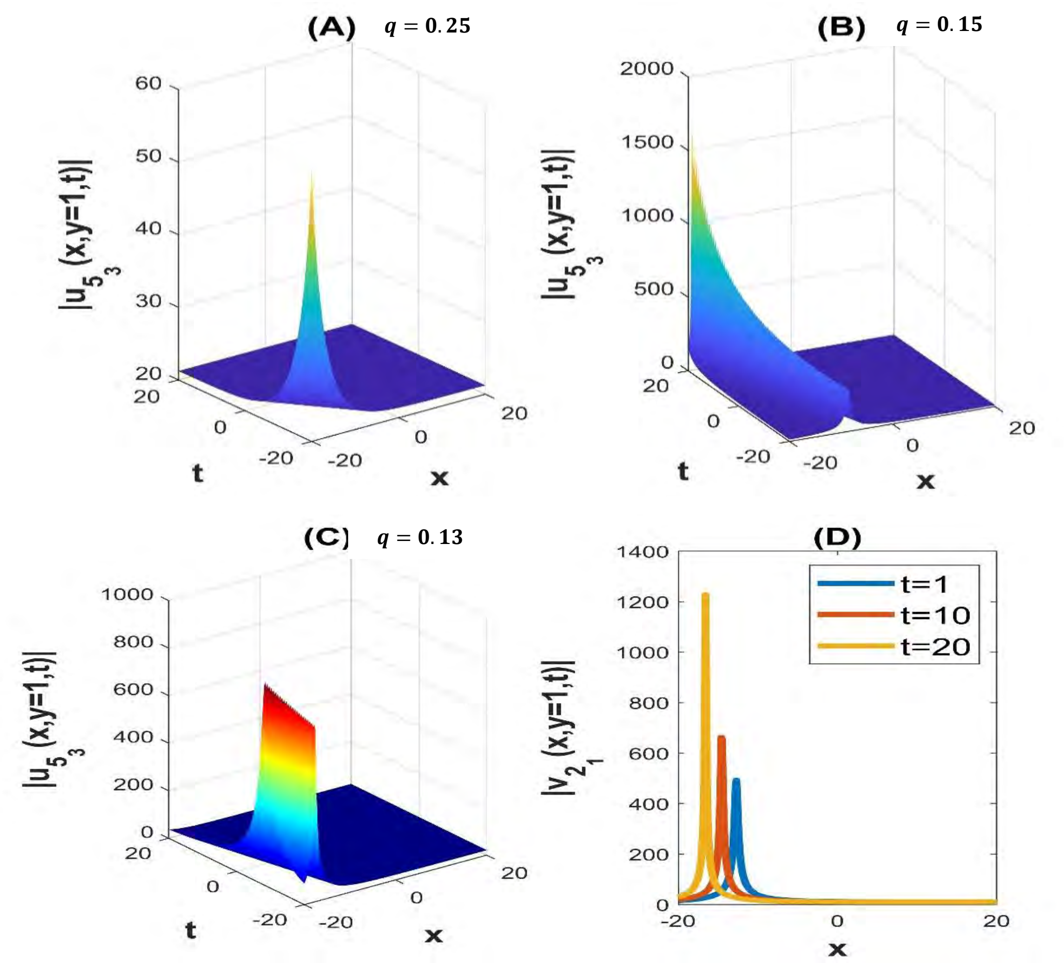


Figure 3.13: The modulus plots of the solution $u_{5_3}(\xi)$ is portrayed for selecting parameters within the interval $-20 \leq x, t \leq 20$.

The modulus graph of the obtained solution $u_{5_3}(x, y, t)$ is travelling in the backward direction of the x-axis with speeds of phase $\mu = -9.545941546i + 1124.973$, $-5.727564928i + 404.973$ and $-4.963889604i + 304.973$. Fig-3.13(A) represents the 3D wave profile of $u_{5_3}(x, 1, t)$for the values of $p = 0.01, q = 0.25, \alpha = 0.15$ and $\theta_0 = 0.5$. Fig-3.13(B) and Fig-3.13(C) for $q = 0.15$ and 0.13 respectively and others parameter are unchanged. Fig-3.13(D) illustrates the propagation of the wave motion as time passes, with snapshots taken at $t = 1, 10$ and 20. The actual development of the solution $u_{5_3}(x, 1, t)$ is shown in Fig-3.13(D), which shows that the solitary wave is moving away from the horizontally.

In this segment, all obtained soliton solutions are expressed as trigonometric and hyperbolic, function solutions. Using dispersion, nonlinearity, and free parametric to the specified equations, physical causes of some of the soliton solutions studied by the suggested method are also visually described.

Also, the description makes it abundantly evident that the soliton solutions of the KD equation, when solved using the MVNK approach, give a wide variety of wave types. These wave types include a soliton kink soliton, bell shape soliton, anti-bell shape soliton, w-shape soliton, and many others soliton solutions. It can be seen that the soliton profile changes their shape depending on the values of the nonlinear coefficient q. This particular solo wave focuses on the application of common scientific, natural, and technology concerns in our daily lives. Although comparable numbers are available for all other alternatives, we have not mentioned minimalism here.

3.2 Comparison

Using the proposed methods, we discovered several general and some new solitary wave solutions to the (2+1)-Konopelchenko–Dubrovsky (KD) equation and the (2+1)-dimensional paraxial NLS equation. The resulting solutions are compared below with other authors solutions [21,56].

3.2.1 For the (2+1)-dimensional paraxial NLS equation

In this section, we will discuss comparison between attained solutions and Arshad et al. [21] solutions. Arshad et al. [21] studied the paraxial NLS equation in Kerr media by different techniques via the improve simple equation, the $\exp(-\phi(\zeta))$-expansion and the modified extended direct algebraic methods. Using the improve simple equation method, Arshad et al. [21] have explored eight precise optical soliton solutions from the paraxial NLS equation in Kerr media. On the other hand, the (w/g)-expansion method have been used to generate many wave solutions for paraxial NLS equation. Both methods have some common solutions as shown in Table 1. The number of wave solutions we have developed is more than the number of solutions found using the improve simple equation method.

Table 1: Comparison between attained solutions with improve simple equation method solutions [21]

Obtained solutions	Arshad et al. [21] solutions
In Eq. (2.40) taking $l = n = \beta_1 = \delta_1 = \delta_2 = 1, m = 0, \lambda = 5, \epsilon = 2$ and $u_3(x, y, t) = k(x, y, t)$, then the solution becomes $k(x, y, t) = e^{i(x+y-5t)} \tan(x + y + t)$	$Q_{13}(y, z, t) = \frac{\sqrt{\delta_0\delta_2(\alpha\omega^2+\beta{k_1}^2)}}{\sqrt{\gamma}} \tan\left(\sqrt{\delta_0\delta_2}\zeta\right) e^{i(\mu_1 y+\mu_2 z+\tau t+\theta)}, \delta_0\delta_2 > 0$ where $\zeta = k_1 y + k_2 z + \omega t$. $\mu_2 = \frac{-\alpha\tau^2}{2} - \frac{\beta\mu_1^2}{2} + \delta_0\delta_2(\alpha\omega^2 + \beta k_1^2)$ Taking $\delta_0 = \delta_2 = \alpha = \mu_1 = k_1 = k_2 = 1,$

	$\omega=1,\beta=25,\gamma=26,\theta=0,\ \ \tau=-5$ and $Q_{13}(y,z,t)=k(x,y,t)$, then the solution becomes $k(x,y,t)=e^{i(x+y-5t)}\tan(x+y+t)$
In Eq. (2.38) taking $n=\delta_1=\delta_2=1,m=0,\beta_1=l=-1\quad\lambda=3,\epsilon=2$ and $u_1(x,y,t)=k(x,y,t)$, then the solution becomes $k(x,y,t)=e^{i(x+y-3t)}\tanh(x+y-t)$	$Q_{14}(y,z,t) = \frac{\sqrt{-\delta_0\delta_2\left(\alpha\omega^2+\beta {k_1}^2\right)}}{\sqrt{\gamma}}\tanh\left(\sqrt{-\delta_0\delta_2}\zeta\right) e^{i(\mu_1 y+\mu_2 z+\tau t+\theta)},\delta_0\delta_2<0$ where $\zeta=k_1y+k_2z+\omega t$. $\mu_2=\frac{-\alpha\tau^2}{2}-\frac{\beta\mu_1^2}{2}+\delta_0\delta_2(\alpha\omega^2+\beta k_1^2)$ Taking $\delta_0=\omega=-1,\delta_2=\mu_1=k_1=k_2=1,\alpha=2,\beta=-8,\gamma=6,\theta=0,\ \tau=-3$ and $Q_{14}(y,z,t)=k(x,y,t)$, then the solution becomes $k(x,y,t)=e^{i(x+y-3t)}\tanh(x+y-t)$

In addition, Arshad et al. [21] have also explored eight precise optical soliton solutions from the paraxial NLS equation in Kerr media through the $\exp(-\phi(\zeta))$-expansion method, As opposed to generate many wave solutions from the stated equation by the mentioned method in this research. Both approaches share some potential solutions, which are compared and contrasted in Table 2. The number of wave solutions that we have generated is more than the number of solutions that can be obtained by employing the $\exp(-\phi(\zeta))$-expansion method.

Table 2: Comparison between attained solutions with the $\exp(-\phi(\zeta))$-expansion method solutions [21]

Obtained solutions	Arshad et al. [21] solutions
In Eq. (2.38) taking $n=\delta_1=\delta_2=1,m=0,\beta_1=l=-1\ \lambda=3,\epsilon=2$ and $u_1(x,y,t)=k(x,y,t)$, then the solution becomes $k(x,y,t)=e^{i(x+y-3t)}\tanh(x+y-t)$	$Q_5(\zeta)=\frac{\sqrt{(\delta^2-4\eta)(\alpha\omega^2+\beta{k_1}^2)}\tanh\left(\frac{\sqrt{\delta^2-4\eta}}{2}(\zeta+\zeta_0)\right)}{2\sqrt{\gamma}}\ e^{i\Pi}$, $\eta\neq 0\ \&\ \delta^2-4\eta>0$ where $\zeta=k_1y+k_2z+\omega t$ and $\Pi=\mu_1y+\mu_2z+\tau t+\Theta$ $\mu_2=\frac{1}{4}(-\alpha\omega^2(\delta^2-4\eta)-2\alpha\tau^2-2\beta\mu_1^2-\beta k_1^2(\delta^2-4\eta))$ Taking $\mu_1=k_1=k_2=1,\omega=\eta=-1,\alpha=2,\beta=-8,\gamma=-6,\delta=\zeta_0=\Theta=0,\tau=-3$ and $Q_5(y,z,t)=k(x,y,t)$, then the solution becomes $k(x,y,t)=e^{i(x+y-3t)}\tanh(x+y-t)$

In Eq. (2.40) taking $l = n = \beta_1 = \delta_1 = \delta_2 = 1, m = 0, \lambda = 5, \epsilon = 2$ and $u_3(x, y, t) = k(x, y, t)$, then the solution becomes $k(x, y, t) = e^{i(x+y-5t)} \tan(x + y + t)$	$Q_6(\zeta) = \frac{\sqrt{(4\eta-\delta^2)(\alpha\omega^2+\beta k_1{}^2)} \tan(\frac{\sqrt{4\eta-\delta^2}}{2}(\zeta+\zeta_0)}{2\sqrt{\gamma}} e^{i\Pi}$, $\eta \neq 0$ & $\delta^2 - 4\eta < 0$ where $\zeta = k_1 y + k_2 z + \omega t$ and $\Pi = \mu_1 y + \mu_2 z + \tau t + \Theta$. $\mu_2 = \frac{1}{4}(-\alpha\omega^2(\delta^2 - 4\eta) - 2\alpha\tau^2 - 2\beta\mu_1^2 - \beta k_1^2(\delta^2 - 4\eta))$ Taking $\alpha = \mu_1 = k_1 = k_2 = \omega = \eta = 1$, $\beta = 25, \gamma = 26, \delta = \zeta_0 = \Theta = 0, \tau = -5$ and $Q_6(y, z, t) = k(x, y, t)$, then the solution becomes $k = e^{i(x+y-5t)} \tan(x + y + t)$

3.2.2 For the (2+1)- Konopelchenko–Dubrovsky (KD) equation

In this part, we will analogy the solutions derived in this article to the analytic wave solutions available in the literature for the KD equation. These solutions can be found in the literature. Many authors have studied the early literature to analyze the KD equation with different techniques. Wazwaz [56] have also explored optical soliton solutions from the KD equation through the tanh technique. On the other hand, by using Modified Version of the New Kudryashov (MVNK) approach, we got many soliton solutions for the stated equation. Some common solutions are shown in Table 3. The hyperbolic function and the trigonometric function solutions are both represent these potential solutions. Finally, the proposed method obtained us a lot of wave solutions than the tanh method.

Table 3: Comparison between attained solutions with the tanh method solutions [56].

Obtained solutions	Wazwaz [56] solutions
In Eq. (2.73) and (2.74) taking $p = q = 1$, $\theta_0 = 0$ and $u_{2_1}(x, y, t) = v_{2_1}(x, y, t) = k(x, y, t)$, then the solution becomes. $k(x, y, t) = 1 + \tanh\left(\frac{x}{2} + \frac{y}{2} + 2t\right)$	$u(x, y, t) = \frac{2b - a}{a^2}\left(1 + \tanh\left[\frac{2b - a}{2a}\left(x + y - \frac{4(ab - a^2 - b^2)}{a^2} t\right)\right]\right)$ Taking $a = b = 1$ and $u(x, y, t) = k(x, y, t)$, then the solution becomes. $k(x, y, t) = 1 + \tanh\left(\frac{x}{2} + \frac{y}{2} + 2t\right)$

In Eq. (2.75) and (2.76) taking $p = q = 1$, $\theta_0 = 0$ and $u_{2_2}(x,y,t) = v_{2_2}(x,y,t) = k(x,y,t)$, then the solution becomes. $k(x,y,t) = 1 + \coth\left(\frac{x}{2} + \frac{y}{2} + 2t\right)$	$u(x,y,t) = \frac{2b-a}{a^2}\left(1 + \coth\left[\frac{2b-a}{2a}\left(x + y - \frac{4(ab-a^2-b^2)}{a^2}\,t\right)\right]\right)$ Taking $a = b = 1$ and $u(x,y,t) = k(x,y,t)$, then the solution becomes. $k(x,y,t) = 1 + \coth\left(\frac{x}{2} + \frac{y}{2} + 2t\right)$

3.3 Merits and Demerits of the investigated methods

In this section, we will outline the Merits and Demerits of the applied methods to examine NLEEs in mathematical physics.

3.3.1 Merits of the *(w/g)*-expansion method

- ✓ The (w/g) approach is straightforward, simple, and approachable.
- ✓ This technique successfully resolves NLEEs with any balance number.
- ✓ Each solution contains several free parameters that aid in illuminating the soliton's nature.

3.3.2 Demerits of the *(w/g)*-expansion method

- ✓ We get singular solitary wave which doesn't belong to real world.
- ✓ The (w/g)-technique is not appropriate for some NLEEs that require fractional balancing numbers.

3.3.3 Merits of the modified new Kudryashov method

- ✓ The modified new Kudryashov method is sleek, simple, and straightforward to use.
- ✓ The solutions consist of several free parameters that are helpful to explain the nature of the soliton.
- ✓ This method is able to give a suitable solution for any NLEEs.

3.3.4 Demerits of the modified new Kudryashov method

- ✓ This method is not able to provide a suitable solution for many equations.
- ✓ For certain NLEEs that must have fractional balancing numbers, the MVNK approach is not applicable.

Chapter 4: Conclusion and Future Work

4.1 Summary of the major outcome

We have investigated one equation to find the closed form travelling wave solutions through the Modified Version of the New Kudryashov (MVNK) approach and we also investigated NLS equations to find the closed form travelling wave solutions through the (w/g)-expansion methods. According to the study, the MVNK approach contains more parameters than other methods, which makes it simpler to use those numbers to explain the wave profile in detail. We have discovered some novel universal and comprehensive closed form travelling wave solutions connected with various types of free parameters of the aforesaid numerous NLEEs by applying these methods to the various equational genres.

Because of the more parameterized approaches we utilized, the solutions have varied dimensions, which makes it easier to illustrate the solitons in great detail. Based on the mention methods, the kink soliton, singular kink soliton, peakon soliton, bell shape soliton, anti-bell shape soliton, periodic wave soliton, double anti-bell soliton, w-shape soliton, parabolic and anti-parabolic shape soliton solutions wave profile obtained from the stated equations. It can be seen that a lot of wave profiles are new to the literature. The linear component of the studied model's coefficient is more successful than the nonlinear part in changing the nature of the wave profile by looking at the values of the parameters found in the solutions to the equations for dispersion, linear, nonlinear, and free parameter terms. By going over the parameter values in the solutions, we can see how they relate to the linear and nonlinear terms of the corresponding equation. As a result, we have been able to determine that the effect of these components on the wave profile is that the linear element of the KD model's coefficient is more successful at changing the nature of the wave profile.

Additionally, we discovered that the wave velocity value on the wave profile performs better than the wave number value. The findings obtained by considering the various values of the free parameters show numerous wave types. Few waves are obtained because it is so harsh and dangerous to nature. By reducing and decaying these kinds of waves based on the values of the linked parameters, the nature will be protected from the risky real-world events.

4.2 Further work

There will be plenty of opportunities for study in this area in the future because of the properties and widespread occurrence of nonlinear evolution models across a wide variety of applications in nature. So, the future prospect of the nonlinear fractional model.
For additional research on the current subject, the following suggestions might be made.

- ✓ For various NLEEs, we have discovered some methodological shortcomings. In order to make these methods appropriate for those equations, the opportunity exists.
- ✓ Additionally, we are aware that all mathematical equations result from a single or a collection of distinct natural facts. We solved those models and got the answers using a variety of parameters connected to the procedure or the principal equations.
- ✓ Our main goal in the future is to determine how much the desire equation solutions can be used to protect nature from its harmful effects.

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