MATHEMATICAL MODELING AND ANALYSIS OF THE DEPLETION OF DISSOLVED OXYGEN IN LAKE ECOSYSTEM

The thesis submitted

in partial fulfilment of the requirements for the degree of

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IN

MATHEMATICS

By

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The thesis entitled "MATHEMATICAL MODELING AND ANALYSIS OF THE DEPLETION OF DISSOLVED OXYGEN IN LAKE ECOSYSTEM" Submitted by B. M. Rakibul Hasan, Student No. 0419092513F, Registration No. 0419092513, Session: April-2019 has been accepted as satisfactory in partial fulfillment of the requirement for the degree of Master of Science in Mathematics on 22/07/2023

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Declaration of Authorship

I, B. M. Rakibul Hasan, declare that the work contained in this thesis entitled "MATHEMATICAL MODELING AND ANALYSIS OF THE DEPLETION OF DISSOLVED OXYGEN IN LAKE ECOSYSTEM" was done by me, under the supervision of Dr. Mohammed Forhad Uddin, Professor, Department of Mathematics, Bangladesh University of Engineering and Technology (BUET), Dhaka-1000 for the award of the degree of Master of Science and this work has not been submitted elsewhere for a degree.

Signed: Rakibul Hasan

Date: 06/08/2023

DEDICATION

This work is dedicated

To

My Family

Abstract

A non-linear mathematical model is being proposed to study the depletion of dissolved oxygen caused by the excessive discharge of organic pollutant in water body. The interactions among concentration of nutrients, density of algae, density of detritus, density of zooplankton and concentration of dissolved oxygen are considered in this model. The model consists of five coupled non-linear differential equations. To validate the model, the boundedness of the state variables using the theory of differential inequality and positivity of each state variables have been done in this research. The equilibrium points of the proposed model have been demonstrated. The stability of the equilibrium points has been checked by computing the eigen-value and applying Routh's Hurwitz criterion. Finally, the characteristics of the state variables with respect to different values of different parameters such as cumulative rate of discharge of nutrients, natural depletion rate of zooplankton, growth rate of algae due to nutrients, natural depletion rate of algae, and depletion rate of dissolved oxygen due to detritus have been discussed both graphically and analytically. It is speculated that detritus uses dissolved oxygen to supplement the total concentration of nutrients in the water body. It has been demonstrated graphically that the density of algae is strongly influenced by the cumulative rate of nutrient discharge from water or wastes itself. Algae engage in self-purification by eating both organic and inorganic contaminants while also making substances that are advantageous to their surroundings. Alarmingly excess detritus reduces the concentration of dissolved oxygen. It has been demonstrated numerically that the density of zooplankton rises when the rate of algal depletion and the rate of nutrients delivery from diverse sources both increase. In this study, it has also been discussed how the concentration of dissolved oxygen decreases as a consequence of the cumulative rate of nutrient discharge, the rate of algae growth due to nutrients, the rate of dissolved oxygen depletion due to detritus, and the rate of zooplankton growth due to algae. In conclusion, the study highlights the intricate interactions between the state variables and emphasizes the delicate balance found within an aquatic ecosystem. The results highlight how crucial it is to comprehend these processes in order to manage and maintain the health of our aquatic ecosystems.

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NOMENCLATURE

Descriptions	Notations
Concentration of nutrients	T
Density of algae	B
Density of zooplankton	Z
Density of detritus	S
Concentration of dissolved oxygen	C
Cumulative rate of discharge of nutrients	q
Natural Depletion rate of nutrients	α_0
Natural depletion rate of algae	α_1
Natural Depletion rate of Zooplankton	α_2
Natural Depletion rate of dissolved oxygen	α_3
Depletion rate of nutrients due to algae	β_1
Depletion rate of Algae due to Zooplankton	β_2
Growth rate of nutrients due to detritus	π_0
Growth rate of detritus due to algae	π_1
Growth rate of detritus due to Zooplankton	π_2
Depletion rate of detritus due to decomposing	δ
Depletion rate of dissolved oxygen due to detritus	δ_1
Growth rate of algae due to nutrients	θ_1
Growth rate of Zooplankton due to Algae	θ_2
Depletion rate of dissolved oxygen due to algae	η
Increasing rate of dissolved oxygen by various sources	q_c

CHAPTER ONE: INTRODUCTION

1.1 An Overview of the Study

In recent years, water pollution has become an alarming issue. The behavior of contaminants in the aquatic environment depends on many factors: chemical, physical, hydrodynamic and biological. Pollution is a pressure that influence the state of aquatic ecosystems.

A combination of high temperatures, stagnant water and organic pollutants overload can result in different results. Our lake and river water are the living place of different species. It is an ecosystem of innumerable micro-organism and fish population, phytoplankton and zooplankton. Day by day the amount of detritus is depleting the healthy environment of the ecosystem. The organic pollutant in the water plays a key role in it. These so-called increase in organic pollutants can lead to a depletion of oxygen in the water, release of toxins and taste and odor problems. Without treatment, the algae and bacteria will grow more every year, resulting in an unbalanced ecosystem. That is why, it is important to control the discharge of organic pollutants for a healthy ecosystem.

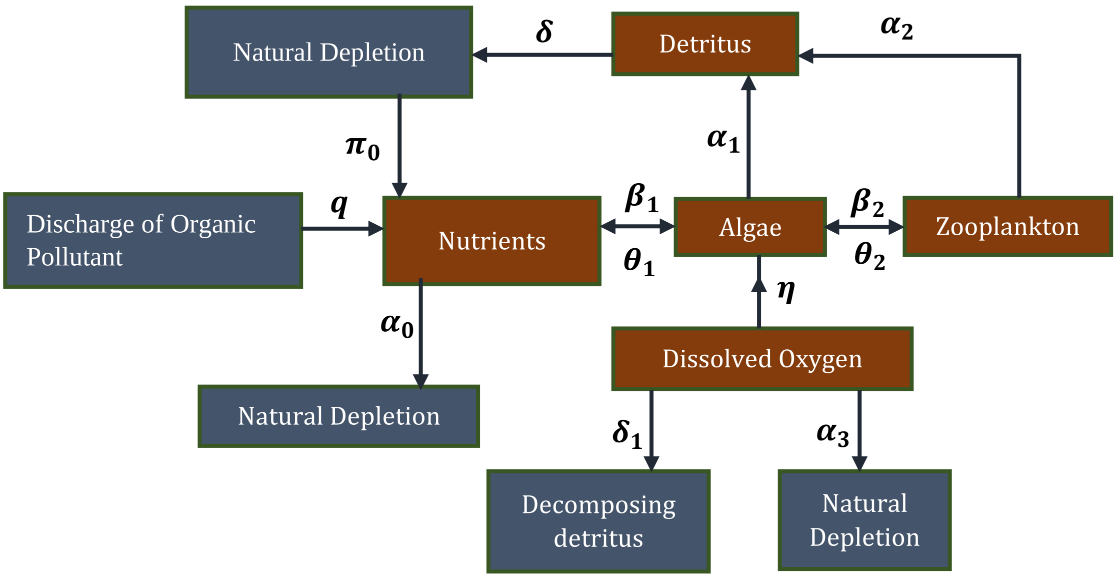


Figure 1.1: A flow diagram of the process in the depletion of DO

A mix of sewage, storm runoff and other that has, in some way, come in contact with us is known as waste water. Wastewater is nonetheless a promising habitat for many a microbe. That's part of what makes waste water dangerous both for us and the environment. Diseases out of bacteria, viruses and parasites can thrive in wastewater, which may also be filled with pollutants like metals or the ingredients of our own pharmaceuticals. The waste water come in contact with different ecosystems, the combined decay of organic matter and excess of nutrients can create a chain of events that depletes oxygen from the water and affects the balance of life previously maintained there. The bits of organic matter are contaminants but to microbes, it's a veritable buffet. Bacteria, coming from the water or the waste itself, shape this self-purification, consuming organic and inorganic matter while producing compounds that benefit their neighbors.

1.2 Definition of Algal Bloom

Before introducing algal bloom, we have to introduce algae first. Algae are a diverse group of aquatic organisms that have the ability to conduct photosynthesis [23]. Certain algae are familiar to most people; for instance, seaweeds (such as kelp or phytoplankton), pond scum or the algal blooms in lakes. In other words, algae are a primitive group of autotrophic plants, i.e., chlorophyll-bearing plants, which originated in sea water more than three billion years ago. In Bangladesh, a large number of algal species occur in freshwater, brackish water and marine habitats. Aquatic algae over 300 species and varieties of freshwater algae have been described from Bangladesh [26].

Now that we have a clear idea about algae, we are going to introduce algal bloom. Algal bloom is a rapid growth of microscopic algae or cyanobacteria in water, often resulting in a colored scum on the surface. In other words, algal bloom or algae bloom is a rapid increase or accumulation in the population of algae in freshwater or marine water systems, and is recognized by the discoloration in the water from their pigments. Cyanobacteria were mistaken for algae in the past, so cyanobacterial blooms are sometimes also called algal blooms [21]. Blooms which can injure animals or the ecology are called "harmful algal blooms" (HAB), and can lead to fish die-offs, cities cutting off water to residents, or states having to close fisheries. Also, a bloom can block out the sunlight from other organisms, and deplete oxygen levels in the water. Also, some algae secrete poisons into the water.

For example [21], we can see some real evidences:

i. In 2005, the Canadian HAB was discovered to have come further south than it has in years prior by a ship called The Oceanus, closing shellfish beds in Maine and

Massachusetts and alerting authorities as far south as Montauk (Long Island, NY) to check their beds. Experts who discovered the reproductive cysts in the seabed warn of a possible spread to Long Island in the future, halting the area's fishing and shellfish industry and threatening the tourist trade, which constitutes a significant portion of the island's economy [25].

ii. In 2008, large blooms of the algae ***Cochlodinium polykrikoid*** were found along the Chesapeake Bay and nearby tributaries such as the James River, causing millions of dollars in damage and numerous beach closures [12].

iii. In 2009, Brittany, France experienced recurring algal blooms caused by the high amount of fertilizer discharging in the sea due to intensive pig farming, causing lethal gas emissions that have led to one case of human unconsciousness and three animal deaths [30].

iv. In 2013, an algal bloom was caused in Qingdao, China, by sea lettuce [27].

Source: Google

Figure 1.2: Algal bloom

1.3 Why is Algal Bloom Harmful?

Nitrogen and phosphorus are nutrients that are natural parts of aquatic ecosystems. Nitrogen is also the most abundant element in the air we breathe. Nitrogen and phosphorus support the growth of algae and aquatic plants, which provide food and habitat for fish, shellfish and smaller organisms that live in water [28].

But when too much nitrogen and phosphorus enter the environment - usually from a wide range of human activities - the water becomes polluted. Algal bloom has impacted many streams, rivers, lakes, bays and coastal waters for the past several decades, resulting in serious environmental and human health issues, and impacting the economy.

Source: Google

Figure 1.3: Harmful algal bloom (HAB)

Too much nitrogen and phosphorus in the water causes algae to grow faster than ecosystems can handle. Significant increases in algae harm water quality, food resources and habitats, and decrease the oxygen that fish and other aquatic life need to survive. Large growths of algae are called algal blooms and they can severely reduce or eliminate oxygen in the water, leading to illnesses in fish and the death of large numbers of fish. Some algal blooms are harmful to humans because they produce elevated toxins and bacterial growth that can make people sick if they come into contact with polluted water, consume tainted fish or shellfish, or drink contaminated water.

1.4 Where This Occurs?

Algal bloom affects the water around the country. The impacts of algal bloom are found in all types of water bodies. Pollutants often enter upstream waters like creeks and streams and then flow into larger water bodies like lakes, rivers and bays. Excess nitrogen and phosphorus can also travel thousands of miles to coastal areas where the effects of the pollution are felt in the form of massive dead zones, such as those in the Gulf of Mexico and Chesapeake Bay [28]. More than 100,000 miles of rivers and streams, close to 2.5 million acres of lakes, reservoirs and ponds, and more than 800 square miles of bays and estuaries in the United States have poor water quality because of nitrogen and phosphorus pollution [30].

Additionally, nutrients can soak into ground water, which provides drinking water to millions of Americans. And urban areas across the country have hazy skies and air quality problems related to airborne nitrogen pollution.

1.5 Causes of Algal Bloom

Excessive nitrogen and phosphorus that washes into water bodies are often the direct result of human activities. The primary sources of algal bloom are:

i. **Agriculture:** Animal manure, excess fertilizer applied to crops and fields, and soil erosion make agriculture one of the largest sources of nitrogen and phosphorus pollution in the country.

ii. **Storm Water:** When precipitation falls on our cities and towns, it runs across hard surfaces - like rooftops, sidewalks and roads - and carries pollutants, including nitrogen and phosphorus, into local waterways.

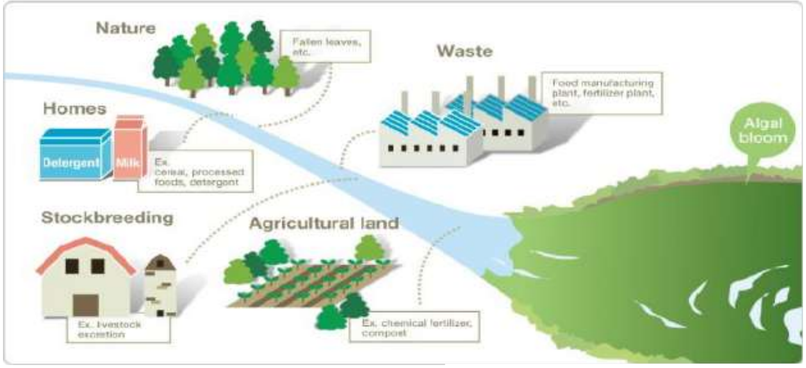
iii. **Wastewater:** Our sewer and septic systems are responsible for treating large quantities of waste, and these systems do not always operate properly or remove enough nitrogen and phosphorus before discharging into waterways.

iv. **Fossil Fuels:** Electric power generation, industry, transportation and agriculture have increased the amount of nitrogen in the air through use of fossil fuels.

v. **High temperatures:** The global world is facing the destruction of the ozone layer caused by global warming. This is one of the main reasons of algal bloom and thriving at a fast rate. Conducive temperature is needed for certain bacteria to survive both in and out of water. The exceedingly high temperatures experienced due to global warming have led to rapid decomposition of the nutrients such as nitrates and ammonia, which are easier forms for bacteria to use up and grow in quantity [22].

vi. **Slow moving water:** Algal blooms need large masses of water which are almost still to thrive. There is less disturbance in their propagation in such waters and this explain their limited growth in rivers and streams with fast flow rate [22].

vii. **Presence of dead organic matter:** Generally, there are many kinds of bacteria present in the atmosphere as well as in water. They are all in search of suitable media for growth and nutrition. Therefore, like other bacteria, the algae bacterium is facilitated by the presence of dead organisms in water. Together with the nutrients present in water, the dead organic matter ends up propagating the growth of algae in water leading to algae bloom [22].



Source: Google

Figure 1.4: Causes of algal bloom

1.6 Algal Bloom in Bangladesh

Algal bloom builds up in our nation's lakes, ponds, and streams. National Lakes Assessment, 2010 found that almost 40 percent of the 700 rivers, lakes had been impacted by nitrogen and phosphorus pollution [18]. The report also showed that poor lake conditions related to nitrogen or phosphorus pollution doubled the likelihood of poor ecosystem health.

When river or stream currents are slow, or when waters are stagnant, nutrients, sediment, and particles accumulate, increasing the chances of harmful pollution and algal growth. In Stream Assessment 2006, 30 percent of streams across the country had high levels of nitrogen or phosphorus [19]. Lakes and rivers are common sources for drinking water supplies. Both algae and high nitrate levels cause problems in sources of drinking water. Nationwide, violations of the nitrate limit in drinking water doubled over a 10-year period.

Source: Google

Figure 1.5: Algal bloom in Bangladesh

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But when too much nitrogen and phosphorus enter the environment - usually from a wide range of human activities - the air and water can become polluted. Nutrient pollution has impacted many streams, rivers, lakes, bays and coastal waters for the past several decades, resulting in serious environmental and human health issues, and impacting the economy.

Too much nitrogen and phosphorus in the water causes algae to grow faster than ecosystems can handle. Significant increases in algae harm water quality, food resources and habitats, and decrease the oxygen that fish and other aquatic life need to survive. Large growths of algae are called algal blooms and they can severely reduce or eliminate oxygen in the water, leading to illnesses in fish and the death of large numbers of fish. Some algal blooms are harmful to humans because they produce elevated toxins and bacterial growth that can make people sick if they come into contact with polluted water, consume tainted fish or shellfish, or drink contaminated water.

1.7 Effects of Algal Bloom

Harmful algal blooms cause major environmental damage as well as serious health problems in people and animals.

i. **Endangerment to human health/life:** Algal blooms produce toxins which reduce the suitability of water for human consumption. Their large presence on water and their well propagating sequences leads to quick contamination of water thus posing a health hazard to humans. Strong irritation, itching and even skin diseases are as well be experienced when such contaminated water comes into contact with the human skin.
ii. **Death of aquatic life:** For any living organism to survive, they need oxygen for respiration. Fishes and other aquatic life depend on the oxygen dissolved in water. Similarly, for the algae bacterium to survive, it needs oxygen for survival. However, plant life high mode of propagation and dense growth in a very short period of time increases competition for oxygen leading to an imbalance in

the aquatic ecosystem and suffocation of aquatic animals like fish. More death of aquatic animals means more food for the algae leading to faster propagation and in the long-run, deterioration of aquatic life.

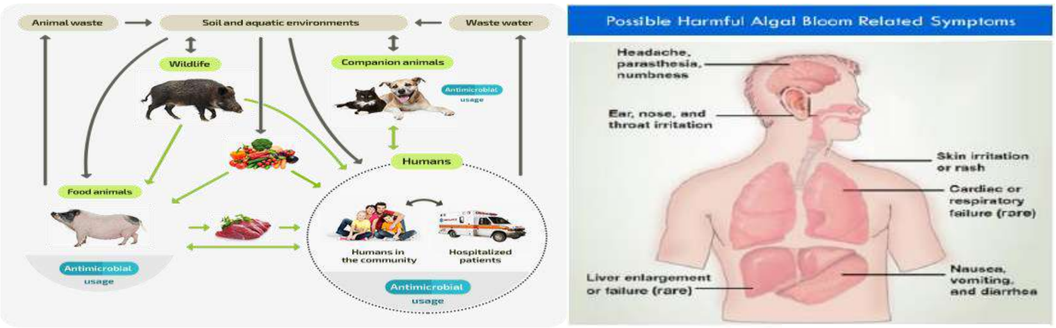
iii. **Dead zones:** The presence of extensive algal blooms can result in the massive deaths of aquatic life. As a result, the area around the algal blooms will be a dead zone with dead animal and plant life alike. The resulting foul smell may affect the rest of the aquatic life, sending them further away from the area.

iv. **Strain on economy:** The presence of algal bloom makes transport on water ways cumbersome leading to more expensive means of transport such as air. Countries that have realized the growth of algal blooms too late have to seek alternative transport routes to engage in trade, resulting in economic losses. Also, since the growth of algal blooms leads to death of aquatic life, there can be widespread losses to fishermen who depend on fishing as an income generating activity. Moreover, the concept and process of treatment of the algal bloom is a costly affair and often requires millions of tax payers' money.

v. **Strain on industries:** Some industries, for example food processing companies, only require clean water from water bodies to drive their production. This means that the presence of algal blooms will cause additional water treatment costs to get clean water leading to increased overhead costs.

vi. **Losses in the tourism industry:** With the dense growth of algal blooms on natural recreational water surfaces, the tourism industry suffers greatly as the resulting foul smell and dead zones means there are no fishes to watch, no available ways to navigate the water, and no swimming or boating activities.

vii. **High water utility bill for domestic consumers:** With algal blooms contamination or not, people still need water for consumption. The municipality will have to invest in water treatment processes that eliminate the toxins caused by algal blooms. In some cases, extensive growth of algal blooms may lead to scarcity of fresh drinking water if the town or community depends on the contaminated source as the only one for distributing consumption water. All these increase the costs of treatment and the demand for water, which eventually dramatically raises water utility bill for domestic consumers.



Source: Google

Figure 1.6: Effects of algal bloom

1.8 Definition of Zooplankton

Zooplankton are the smallest animals in our oceans, ponds, lakes. “Zoo" comes from the Greek word for ‘*animal*’. They are heterotrophic (other-feeding), meaning they cannot produce their own food and must consume instead other plants or animals as food. In particular, this means they eat phytoplankton. Zooplankton is made up of small water invertebrates feeding on phytoplankton. Even though “plankton” means passively floating or drifting, some representatives of zooplankton may be strong swimmers. The yearly plankton cycle consists of various phytoplankton species blooming in response to a particular sequence of changes in temperature, salinity, photoperiod and light intensity, nutrient availability, and a consequent bloom of zooplankton populations. Phytoplankton and zooplankton populations are therefore intimately linked in a continuous cycle of bloom and decline that has evolved and persisted throughout millions of years of evolution.

In lakes and ponds, the most common groups of zooplankton include Cladocera and Copepods (which are both micro-crustaceans), rotifers and protozoans. Most lakes will have 40 or more species of zooplankton common to them.

Zooplankton occupy the center of the open-water food web of most lakes. They eat bacteria and algae that form the base of the food web and, in turn, are heavily preyed upon by fish, insects and other zooplankton. Many zooplankton have clear shells to avoid being seen by visual feeders, such as fish.

1.9 Eating Habits

In keeping with their taxonomic diversity, zooplankton use a variety of feeding strategies, and they may eat bacteria, algae, other zooplankton and can even be parasites. Some zooplankton, like many Cladocera, are indiscriminate grazers, using their feeding appendages like rakes to filter particles from the water. Other zooplankton, such as many Copepods, are more selective and pick out individual particles or zooplankton prey based on their size, shape and taste. Zooplankton feed on microscopic plants known as phytoplankton. Zooplankton are generally larger than phytoplankton, mostly still microscopic but some can be seen with the naked eye. Some start small and stay small. Others are microscopic larvae that will someday turn into bigger animals like crabs, octopi and jellyfish. Many protozoans (single-celled protists that prey on other microscopic life) are zooplankton.

Zooplankton is a categorization spanning a range of organism sizes including small protozoans and large metazoans. It includes holoplanktonic organisms whose complete life cycle lies within the plankton, as well as meroplanktonic organisms that spend part of their lives in the plankton before graduating to either the nekton or a sessile, benthic existence. Although zooplankton are primarily transported by ambient water currents, many have locomotion, used to avoid predators (as in diel vertical migration) or to increase prey encounter rate.

Because zooplankton eat algae, it has been proposed that it may be possible to control algal blooms by increasing zooplankton grazing. This method is called "biomanipulation" and is usually done by reducing predation on zooplankton by planktivorous fish either by directly removing these fish or adding a fish predator such as pike.

Figure 1.7: Zooplankton

1.10 Why Zooplankton is Important?

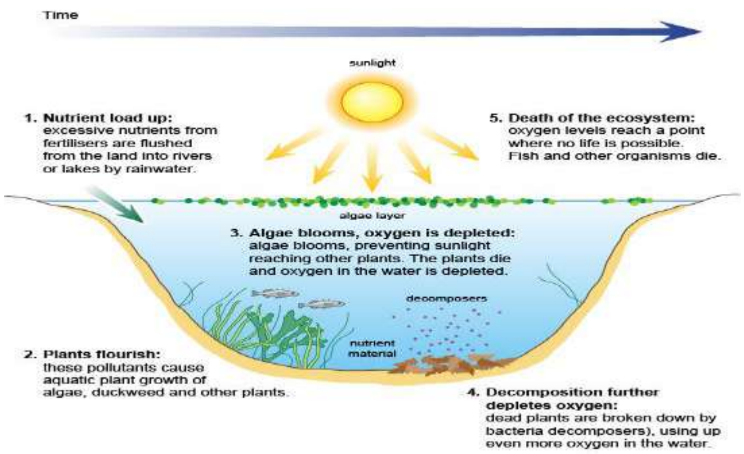
As a result of their central position in lake food webs, zooplankton can strongly affect water quality, algal densities, fish production, and nutrient and contaminant cycling. Zooplankton are commonly included in biomonitoring programs because their densities and species composition can be sensitive to changes in environmental conditions. In recent years, many species of zooplankton have been accidentally introduced to Canadian lakes and rivers from Europe and elsewhere, including the spiny water flea (*Bythotrephes*) and the larval stages of zebra mussels. Occasionally, some species of zooplankton, such as *Mysis*, have been deliberately introduced to lakes to enhance fish production.

1.11 Effect on Human Life

i. Zooplankton absorb carbon dioxide which slow down climate change
ii. Precious oil made from plankton. Single celled creature like diatoms and dinoflagellates die and sink to the sea floor where they crushed and transformed over millions of years into oil and natural gas.

1.12 Definition of Eutrophication

Eutrophication (from Greek *eutrophos*, "well-nourished") is when a body of water becomes overly enriched with minerals and nutrients which induce excessive growth of plants and algae. This process may result in oxygen depletion of the water body and also becoming anoxic in the end. This whole process is known as Eutrophication [29].



Source: Wikipedia

Figure 1.8: Eutrophication

1.13 Causes of Eutrophication

Eutrophication is most often the result of human activity. Farms, golf courses, lawns and other fields tend to be heavily fertilized by people. These fertilizers are the perfect type of nutrients to feed hungry algae and plankton, and when it rains, these fertilizers run off into lakes, streams, rivers and oceans. Concentrated animal feeding operations (CAFOs) are also a major source of polluting nutrients. Eutrophication can also come from natural events. If a stream, river or lake floods, it may wash away any excess nutrients off the land and into the water. However, eutrophication is less likely to occur in areas that are not surrounded by fertilized lands [22].

1.14 Effects of Eutrophication

Eutrophication is most often the result of human activity. Farms, golf courses, lawns and other fields tend to be heavily fertilized by people. These fertilizers are the perfect type of nutrients to feed hungry algae and plankton, and when it rains, these fertilizers run off into lakes, streams, rivers and oceans. Concentrated animal feeding operations (CAFOs) are also a major source of polluting nutrients [29].

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1.15 Literature Review of the Model

It has been studied by several researchers about the effects of discharge of organic pollutants in water bodies, such as a lake, ponds. Nijboer and Verdonschot [9] carried out review work on the processes involved with the effects of eutrophication on stream and river ecosystems. The differences in local stream characteristics and effects on the biota were included. They discussed the relationships between nutrient input, nutrient uptake versus transport, and their effects on the biotic community combined in stream eutrophication models. Stream eutrophication models included the relationships between regional stream characteristics and the expected extent of the eutrophication effects (the sensitivity of a stream for eutrophication). The sensitivity of a stream to eutrophication depended on the regional stream characteristics. That type of model contained four main aspects: nutrient input, nutrient uptake versus transport, in-stream eutrophication processes, and local stream characteristics. They also reviewed to extract the major variables related to the enlarged input of nutrients in streams and their effects on the stream ecosystem, to build a predictive community-based eutrophication model.

Shukla et al. [13] proposed a nonlinear mathematical model for algal bloom in a lake caused by the excessive flow of nutrients from domestic drainage and water runoff from agricultural fields. They considered interactions of cumulative concentration of nutrients, density of algal population, density of detritus, and concentration of dissolved oxygen in the lake. If the supply of nutrients iAn the water body increased, the cumulative density of the algal population would increase, and algal bloom would occur causing eutrophication. They also showed that due to a decrease in the concentration of dissolved oxygen (as the net production of oxygen, floating algae does not affect the concentration of dissolved oxygen caused by photosynthesis), the density of detritus increases and also threatens the survival of the fish population. They used MAPPLE 7.0 for solving the mathematical model. From the numerical solution, they found that when the cumulative rate of input of nutrients, and the density of detritus were increased, the algal population also increased.

Khare et al. [6] presented a model analyzing the effect of depleting dissolved oxygen on interacting planktonic populations. The mathematical model is formulated with four state variables such as nutrient concentration, density of algae, density of the zooplankton population, and concentration of dissolved oxygen. They observed that all the feasible equilibria had been locally stable under certain conditions from the stability analysis of the system of equations. Here also applied Lyapunov's direct method for nonlinear stability analysis. It had been shown in the numerical simulation part, while the cumulative rate of input of nutrients increased, the concentration of dissolved oxygen was unchanged.

Destania et al. [2] modified a nonlinear mathematical model for plankton ecosystems. They analyzed the stability of plankton ecosystems affected by oxygen deficit. They used Lyapunov's direct method to test the stability of the system of equations. Four equilibrium points were obtained from a system of equations. They observed from the numerical simulations, that ecosystem would reach a stable condition.

Misra et al. [8] proposed a nonlinear mathematical model for the depletion of dissolved oxygen caused by interactions of organic pollutants with bacteria using dissolved oxygen in a water body and the subsequent growth of bacteria, which depends explicitly on the concentration of dissolved oxygen. Three dependent variables, namely, the cumulative concentration of organic pollutants, the density of bacteria, and the concentration of dissolved oxygen was assumed for the system. The model analyzed that if the coefficient of interaction depended upon dissolved oxygen explicitly, the decrease in its concentration would be more than the case when the interaction did not depend on dissolved oxygen and consequently the depletion of organic pollutants was also more in such a case.

Tiwari et al. [14] proposed a five-dimensional nonlinear mathematical model by considering the interactions among organic pollutants, inorganic pollutants, bacteria, dissolved oxygen, and fish population in the system. They studied the effects of pollutants from various sources on fish survival in water bodies. Numerical simulations and analytical solutions were also performed. From the result, they suggested that to maintain water quality and to save fish life, the global community has to limit the release of organic and inorganic pollutants into the aquatic system.

Babitha et al. [1] have focused on mathematical modeling research papers to analyze the survival of aquatic species in the presence of pollutants. They have also discussed the stability analysis of all models which were referred to in the paper. Further, they have justified the numerical solutions to water pollution and its effects on the survival of aquatic species.

Kalra and Tangri [5] have presented a model to study the effects of toxicants and acidity on oxygen-dependent aquatic populations. From the stability analysis, they have observed that the dissolved oxygen and aquatic population exhibit a decrease with the rise in toxicity and acidity of water. They have also discussed the sensitivity analysis of the model and determined the equilibrium points. Numerical simulations are performed by using MATLAB.

Omar [11] developed a feed-forward neural network (FFNN) model with a back-propagation learning algorithm to predict the dissolved oxygen from water temperature and 5 days-biological oxygen demand in the Tigris River, Baghdad-Iraq. The FFNN model demonstrated the capability of ANN in predicting the values of dissolved oxygen in river water courses. He used several statistical measures to evaluate the performance of trained artificial neural networks such as- R, MSE, MAE, and NS. He confirmed that with adequate accuracy from only a small data set utilizing feed-forward backpropagation, dissolved oxygen in the Tigris River could be forecasted. He thought that the acquired results could be utilized by water quality controllers to be applied in water treatment and water management plans.

Henderson et al. [4] have presented a mathematical model for algal bloom due to the dramatic growth of confined animal feeding operations. The dramatic growth of Confined Animal Feeding Operations (CAFOs) has produced voluminous quantities of untreated waste. The bifurcation process has discussed in this model.

1.16 Objectives of the Study

i. To formulate a model including concentration of nutrients, density of algae, density of zooplankton, density of detritus, and concentration of dissolved oxygen as variables

ii. To find out the points of equilibrium of the proposed model

iii. To analyze the stability of the points of equilibrium

iv. To find out the characteristics of the concentration of dissolved oxygen and the density of algae with respect to different parameters

1.17 Possible Outcomes

i. The model will be able to demonstrate the depletion of dissolved oxygen in a eutrophic lake

ii. The study could be helpful to indicate the effects of different parameters of real data

iii. The outcome could be implemented for further research to control the depletion of dissolved oxygen and algal bloom to aid in agricultural aspects

1.18 Significance of the Study

i. This study may be useful to the people all around the world to aware of the effects of the inappropriate discharge of different organic pollutant.

ii. The study may help the farmers to aware about the bad effects of this threat in aquatic ecosystem.

iii. This study introduces several aspects of this complex phenomenon by knowing the reasons of their occurrence, the expanding behavior of algae, zooplankton and at the same time the effect on the dissolved oxygen.
iv. It shows the toxic sides in an aquatic environment which reflects the phenomenon that has been occurring beyond our imagination.

1.19 Process of the Study

i. Study the consequential background of mathematical model. Analyze the equilibrium points of the system of differential equations.
ii. Use numerical methods to show the behavior of the characteristics of the state variables.
iii. Executes computer Matlab programs to show the behavior of the state variables.

1.20 Organization

In this study, the introduction of the mathematical model, its significance, explanation of special terminologies, and literature review of the model are discussed in chapter one. In chapter two, the mathematical preliminaries are discussed elaborately. The analytical analysis of the model including the positivity analysis, equilibrium analysis, stability analysis, and characteristics of the state variables are discussed in chapter three. The numerical analysis and results discussion are included in chapter four. Lastly, chapter five contains the conclusion and future scopes of the study.

CHAPTER TWO: MATHEMATICAL PRELIMINARIES

2.1 Mathematical Modeling

Modeling is a process of application of fundamental knowledge or experience to simulate or describe the performance of a real system to achieve certain goals. It is also a process of producing a model; a model is a representation of the construction and working of some system of interest. A model is similar to but simpler than the system it represents. One purpose of a model is to enable the analyst to predict the effect of changes to the system. On one hand, a model should be a close approximation to the real system and incorporate most of its salient features. On the other hand, it should not be so complex that it is impossible to understand and experiment with it.

Mathematical modeling, in essence, involves the transformation of the system under study from its natural environment to mathematical environment in terms of abstract symbols and equations. Mathematical modeling (or mechanistic modeling) is based on the deductive or theoretical approach. Here, fundamental theories and principles governing the system along with simplifying assumptions are used to derive mathematical relationships between the variables known to be significant. The resulting model can be calibrated using historical data from the real system and can be validated using additional data. Predictions can then be made with predefined confidence. In contrast to empirical models, mathematical models reflect how changes in system performance are related to changes in inputs. The emergence of mathematical techniques to model real systems has alleviated many of the limitations of physical and empirical modeling.

Nowadays, Environmental modeling has become a center topic in Modeling. It deals with different topics of our environment. For example- Atmosphere, surface water, Groundwater, Subsurface, ocean etc. Our environment is changing every year which leads to different problems all around the world. At this point of view, Environmental modeling is the only logical way to get a proper solution regarding those topics.

2.1.1 Historical background

"Modeling" is a word that came from '*modellus*' which is a Latin word [24]. It describes a typical human way of coping with the reality. Although abstract representations of real-world objects have been in use since the stone age, a fact backed up by cavemen paintings, the real breakthrough of modeling came with the cultures of the Ancient Near East and with the Ancient Greek. The first recognizable models were numbers; counting and "writing" numbers (e.g., as marks on bones) is documented since about 30.000 BC [18]. Astronomy and Architecture were the next areas where models played a role, already about 4.000 BC. It is well known that by 2.000 BC at least three cultures (Babylon, Egypt, India) had a decent knowledge of mathematics and used mathematical models to improve their every-day life. Most mathematics was used in an algorithmic way, designed for solving specific problems.

2.1.2 Classifications of mathematical model

Models in mathematics often have a set of interconnected relations and several independent variables. Operators, such as those found in algebra, functions, differential operators, etc., are useful for describing relationships. In a system, variables are stand-ins for the measurable quantities of interest. Variables are optional for operators' operations. Several types of models include [16]:

i. **Linear vs. Nonlinear:** If all the operators in a mathematical model exhibit linearity, the resulting mathematical model is defined as linear. A model is considered to be nonlinear otherwise. The definition of linearity and nonlinearity depends on context, and linear models may have nonlinear expressions. For example, in a linear statistical model, a relationship is assumed to be linear in the parameters but may be nonlinear in the predictor variables. Similarly, a differential equation is considered linear if written with linear differential operators, but it can still have nonlinear expressions. In a mathematical programming model, if the objective functions and constraints are represented entirely by linear equations, then the model is considered linear. If one or more objective functions or constraints are represented with a nonlinear equation, then the model is known as a nonlinear model.

Nonlinearity, even in fairly simple systems, is often associated with chaos and irreversibility. Although there are exceptions, nonlinear systems and models are more difficult to study than linear ones. A common approach to nonlinear problems is linearization, but this can be problematic if one tries to study aspects such as irreversibility, which is strongly tied to nonlinearity.

ii. **Static vs. Dynamic:** A dynamic model accounts for time-dependent changes in the system's state, while a static (Or steady-state) model calculates the system in equilibrium and thus is time-invariant.
   Dynamic models are typically represented by differential equations.

iii. **Explicit vs. Implicit:** If all of the input parameters of the overall model are known, and the output parameters can be calculated by a finite series of computations (known as linear programming, not to be confused with linearity as described above), the model is said to be explicit. Nevertheless, the output parameters are sometimes known, and the corresponding inputs must be solved by an iterative procedure, such as Newton's method (if the model is linear) or Broyden's method (if nonlinear). For example, a jet engine's physical properties, such as turbine and nozzle throat areas, can be explicitly calculated given a thermodynamic design cycle (air and fuel flow rates, pressures, and temperatures) at a specific flight condition and power setting. However, the engine's operating cycles at other flight conditions and power settings cannot be explicitly calculated from the constant physical properties.

iv. **Discrete vs. Continuous:** A discrete model treats objects as discrete, such as the particles in a molecular model or the states in a statistical model. In contrast, a continuous model represents the objects continuously, such as the velocity field of fluid in pipe flows, temperatures and stresses in a solid, and the electric field that applies continuously over the entire model due to a point charge.

v. **Deterministic vs. probabilistic (stochastic):** A deterministic model is one in which every set of variable states is uniquely determined by parameters in the model and by sets of previous states of these variables. Therefore, deterministic models perform similarly for a given set of initial conditions. Conversely, in a stochastic model, randomness is present, and variable states are not described by unique values but by probability distributions.

vi. **Deductive, Inductive, or Floating:** A deductive model is a logical structure based on a theory. An inductive model arises from empirical findings and generalization from them. The floating model rests on neither theory nor observation but is merely the invocation of expected structure. Applying mathematics in social sciences outside economics has been criticized for unfounded models. The application of catastrophe theory in science has been characterized as a floating model.

2.1.3 Formulation of a mathematical model

When we speak about dynamical systems, we often refer to an abstract mathematical model instead of the empirical phenomena whose dynamics we seek to characterize. We begin by identifying the physical characteristics that we feel are responsible for the phenomenon's observed behavior. Then, we may build an equation or system that reflects the relationship between our assumed variables.

A fundamental goal of modeling is to provide light on real-world phenomena' underlying mechanisms, whether biological, chemical, physical, or economic. Achieving findings consistent with real-world phenomena is a required but not sufficient quality of a good model, and as we will see, there are other criteria by which an appropriate model must be maintained.

The first stage in formulating a model is to outline the primary determinants of the real-world scenario being replicated [16]. A solid strategy for formulating a testable model is to conceptualize the issue such that all essential variables are accounted for insofar as they represent the mechanics of the observed occurrence. Early drawings of a model may be created using a flowchart diagram or pseudocode that explains state variables and the nature of their relationships.

Gilpin and Ayala [16] propose the following criteria by which a good model should uphold:

i. **Simplicity:** By Occam's razor, simple models are favorable over-complicated models "Because their empirical content is greater and better testable" [16]. Incorporating the minimum possible number of parameters to account for the observed results is always favorable. As Albert Einstein famously stated, "Everything should be made as simple as possible, but not simpler."
ii. **Reality:** All of the model's parameters should be biologically relevant and represent the mechanics of the questioned biological system. Thus, the modeler should have a solid understanding of the modeled phenomena. Mechanistic models explain a

phenomenon from “first principles” or the bottom up. They recognize that a biological phenomenon is the sum of numerous separate but interrelated processes, and therefore they strive to characterize the phenomenon in terms of its main mechanisms (in ecology, often at the level of the individual). On the other hand, models that describe a phenomenon are phenomenological. The top-down empirical determination of a phenomenological model’s structure from a population’s features precludes its ability to predict behaviors independent of the original data. Hence, the parameters utilized in phenomenological models are aggregate summations of several lower-level processes; (Schoener) refers to them as “mega parameters” [16].

Schoener [16] describes the mechanistic method in ecological modeling, recommending it above the phenomenological approach by proposing a “mechanistic ecologists’ paradise.” However, mechanistic and phenomenological modeling techniques have benefits and downsides in certain situations. Virtually all of the models we have chosen to examine in this paper is phenomenological because they give a relatively easy mathematical form and analytical flexibility.

iii. **Generality:** Utilizing dimensionless variables enables magnitudes to take on a universal meaning, hence facilitating scaling. Thus, the generic model may become more detailed for certain instances to account for individual circumstances.

iv. **Accuracy:** The model should deviate as little as possible from the observed data. Thus, a model with little predictive or explanatory value should be revised. Before beginning the step-by-step procedures used to formulate and analyze continuous population models, we will examine population dynamics modeling from a historical perspective, providing insights into the key figures associated with the field of population ecology as well as the methods they developed to understand population systems mathematically. In addition, we will use this occasion to introduce new vocabulary and ideas.

2.2 Process of Mathematical Modeling

Process of mathematical modeling is a series of steps taken to convert an idea first into a conceptual model and then into quantitative model. A conceptual model represents our ideas about how the system works. The main stages in modeling problems in the real world are illustrated in the figure (2.1).

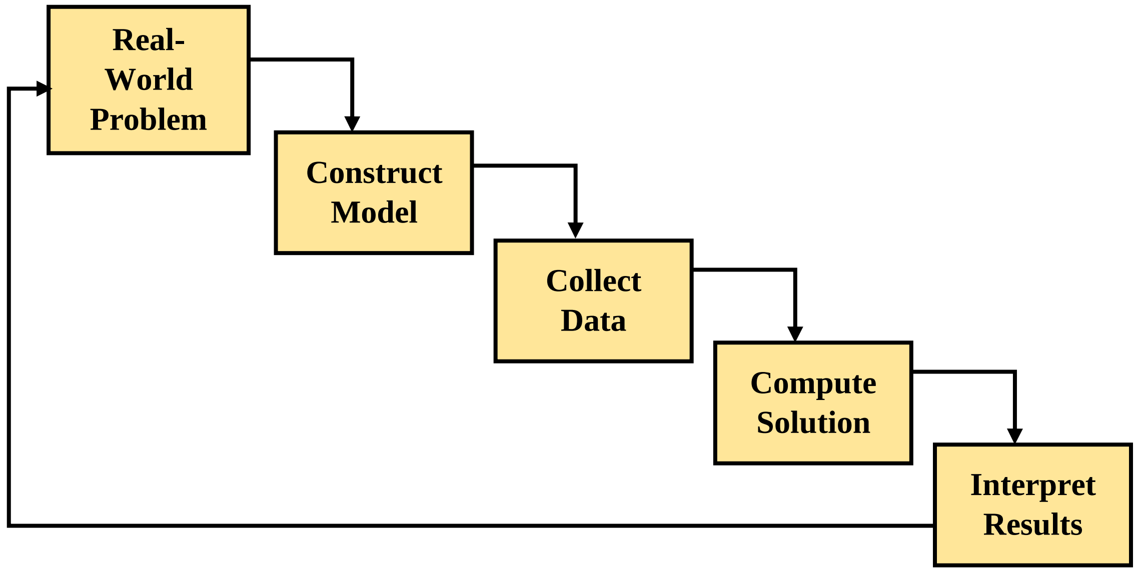

Figure 2.1: Process of mathematical modeling

i. **Variables:** These represent unknown or changing parts of the model, e.g., whether to take a decision or not (decision variable), how much of a given product is being produced, the thickness of a beam in the design of a ceiling, an unknown function in a partial differential equation, an unknown operator in some equation in infinite dimensional spaces as they are used in the formulation of quantum field theory, etc.

ii. **Relations:** Different parts of the model are not independent of each other, but connected by relations usually written down as equations or inequalities. E.g., the amount of a product manufactured has influence on the number of trucks needed to transport it, and the size of the trucks has an influence on the maximal dimensions of individual pieces of the product.

iii. **Data:** All numbers needed for specifying instances of the model. e.g., the maximal forces on a building, the prices of the products, and the costs of the resources.

2.3 Advantage of Mathematical Modeling

i. They can be analyzed in a precise way by means of mathematical theory and algorithms.
ii. They can simplify a more complex situation.
iii. They can help us improve our understanding of the real world as certain variables can readily be changed.
iv. They enable predictions to be made

v. They are quick and easy to produce

2.4 Environmental Models

Mathematical models in the environmental field can be traced to back to the 1900s, the pioneering work of Streeter and Phelps on dissolved oxygen being the most cited [20]. Today, driven mainly by regulatory forces, environmental studies have to be multidisciplinary, dealing with a wide range of pollutants undergoing complex biotic and abiotic processes in the soil, surface water, groundwater, and atmospheric compartments of the ecosphere. In addition, environmental studies also encompass equally diverse engineered reactors and processes that interact with the natural environment through pathways. Consequently, modeling large scale environmental systems is often a complex and challenging task.

The economic activity of society brings negative changes in aquatic systems for example: changing the chemical composition of water and disrupting aquatic systems. Most human activities are carried out using water from rivers, which is lately steadily declining. Water must meet quality standards in order to be used. The term "water quality" is defined in several ways:

Depending on the intended use of the water is a set of chemicals, physical and biological characteristics, concerning its capacity for a particular case application.

From the point of view of environmentalists - the state of an aquatic system referred to the physicochemical conditions of this system, which could support a healthy community to the aquatic biota in balance in local conditions.

2.5 Ordinary Differential Equations

In Mathematics, an ordinary differential equation [17] is an equation containing a function of one independent variable and its derivatives. The function generally represents physical quantities, the derivatives represent their rates of change and the equation defines a relationship between them.

Example: (i) $\frac{d^2y}{dx^2} = x^2 + y^2$

(ii) $\frac{d^2y}{dx^2}+7\frac{dy}{dx}+12y=0$

2.6 Linear Differential Equations

A differential equation is said to be linear if every dependent variable and every derivative involved occurs in the first degree only [17], no products of dependent variables and derivatives. A linear first order differential equation is in the form.

$$a_0(x)y+a_1(x)y^{'}+a_2(x)y^{''}+\cdots+a_n(x)y^n+b\left(x\right)=0$$

Example: (i) $y^{''}+ay^{'}+by=0$

(ii) $\frac{dy}{dx}=x+e^x$

2.7 Non-Linear Differential Equations

A differential equation involving the product of the dependent variable or its derivatives or the transcendental function of the dependent variable is called non-linear differential equation [17].

Example: (i) $\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by^2=0$

(ii) $\frac{dy}{dx}=cy^2+dx$

2.8 Functions

i. **Bounded function:** A function that is not bounded is said to be unbounded. Sometimes, if f(x) ≤ A for all x in X, then the function is said to be bounded above by A. On the other hand, if f(x) ≥ B for all x in X, then the function is said to be bounded below by B. Bounded function is a function whose values are bounded to a limit. For example, f(x) = 1 means the function is neither bigger nor smaller than 1.

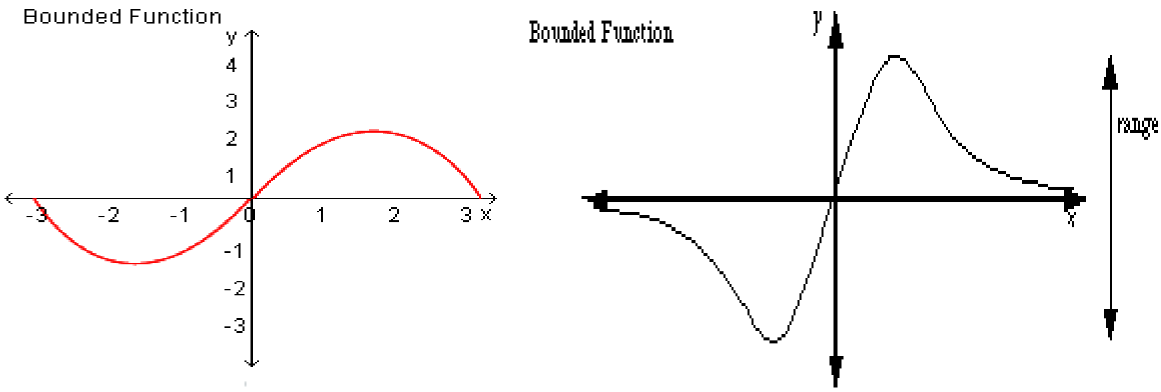

Figure 2.2: Bounded function

ii. **Closed function:** Closed function is a function such that image of every closed set is closed. It is relatively easy to see that, for any, every -discrete subset of real line is closed.

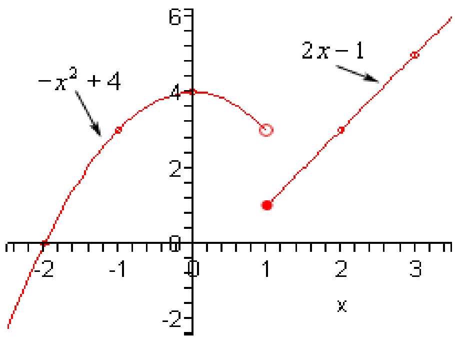

Figure 2.3: Closed function

iii. **Continuous Function:** A function f is continuous at $x=a$ provided that $f(a)$ and $\lim_{x\to a} f(x)$ exists a $\lim_{x\to a} f(x) = f(a)$.

Let us begin by constructing functions that are not continuous. Let us sketch a graph of a function that is not continuous at $x=2$

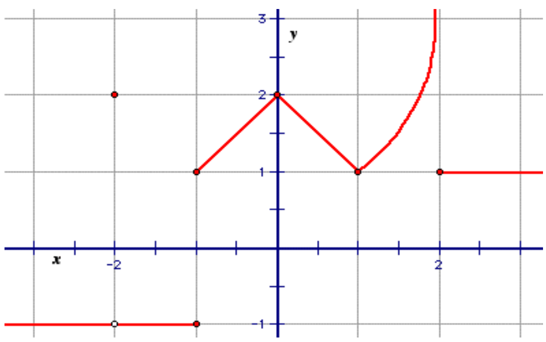

Figure 2.4: An illustration of a continuous and discontinuous function

iv. **Differentiable function:** A differentiable function of one real variable is a function whose derivative exists at each point in its domain. As a result, the graph of a differentiable function must have a (non-vertical) tangent line at each point in its domain, be relatively smooth, and cannot contain any breaks, bends, or cusps.

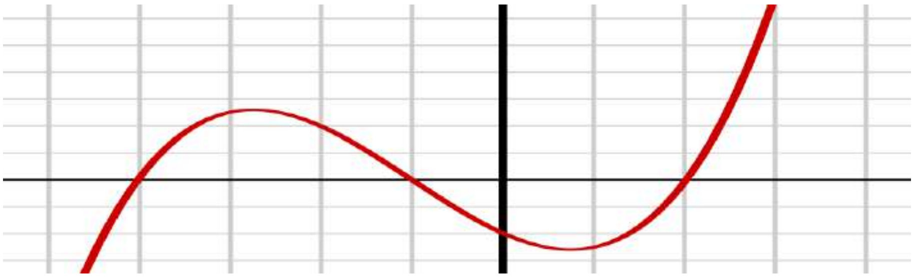

Figure 2.5: Differentiable function

More generally, if x_0 is a point in the domain of a function f, then is said to be differentiable *at* if x_0 the derivative $f'(x_0)$ exists. This means that the graph of f has a non-vertical tangent line at the point $(x_0, f(x_0))$. The function f may also be called locally linear at x_0, as it can be well approximated by a linear function near this point.

v. **Continuously differentiable function:** A function f is said to be continuously differentiable if the derivative $f'(x)$ exists and is itself a continuous function. Although the derivative of a differentiable function never has a jump discontinuity, it is possible for the derivative to have an essential discontinuity. Or, the space of continuously differentiable functionis denoted as C^1 and corresponds to the k= 1 case of a C^k function

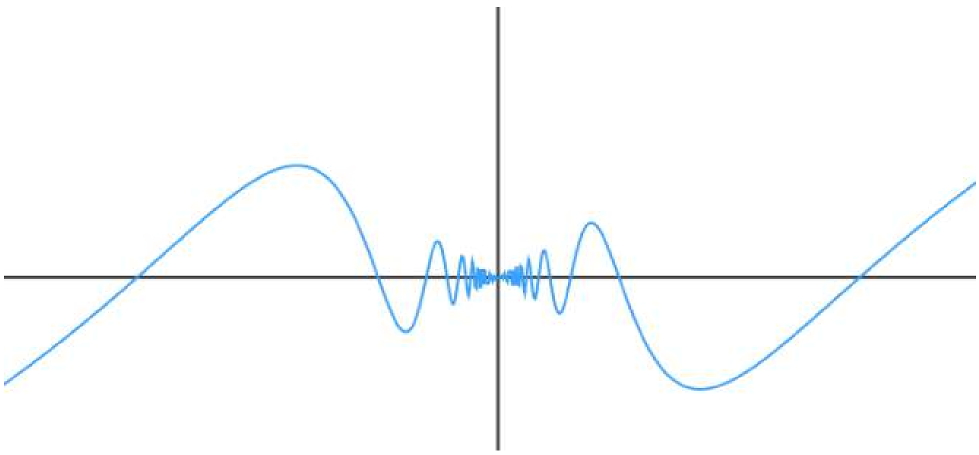

Figure 2.6: Continuously differentiable function

When we say f∈C1, we mean that f is continuously differentiable. Isn't the continuity a redundant word? I mean, we have a theorem that says if f is differentiable then it is continuous.

So, these are all equivalent:

- f∈C1
- f is continuously differentiable
- f′ exists

2.9 Bifurcation

A Bifurcation of a dynamical system is a qualitative change in its dynamics produced by varying parameters.

Consider an autonomous system of ODEs

$$\dot{x} = f(x, \lambda), x \in \Re^n, \lambda \in \Re^p$$

where f is smooth. A bifurcation occurs at parameter $\lambda = \lambda_0$ if there are parameter values λ_1 arbitrarily close to λ_0 with dynamics topologically inequivalent from those at λ_0.

For Example, the number of stability of equilibria or periodic orbits of f may change with perturbations of λ from λ_0. One goal of bifurcation theory is to produce parameter space maps or bifurcation diagrams that divide the λ parameter space into regions of topologically equivalent systems. Bifurcations occur at points that do not lie in the interior of one of these regions.

2.10 Critical Point

Let $\frac{dx}{dy} = P(x, y)$, $\frac{dx}{dy} = P(x, y)$ be the autonomous system.

Here, a point (x_0, y_0) at which both $P(x_0, y_0) = 0$ and $Q(x_0, y_0) = 0$ is called a critical point of the autonomous system.

There are 4 types of critical point:

i. Centre
ii. Saddle point
iii. Spiral Point
iv. Node

2.10.1 Saddle Point

If $\frac{dx}{dy} = P(x, y)$ and $\frac{dx}{dy} = P(x, y)$ be the autonomous system. Here, a point (x_0, y_0) at which both $P(x_0, y_0) = 0$ and $Q(x_0, y_0) = 0,$ the critical point of the autonomous system is called saddle point (figure 2.9). Such a point may be characterized as follows:

1. It is approached and entered by two half line paths as $t \to +\infty$, these two paths forming the geometric curve AB
2. It is approached and entered by two half line paths as $t \to +\infty$, these two paths forming the geometric curve CB
3. Between the four half-line paths described in (1) and (2) there are four domains R_1 , R_2, R_3 and R_4, each containing an infinite family of semi-hyperbolic-like paths which do not approach O as $t \to +\infty$ or as $t \to -\infty$, but which become asymptotic to one or another of the four half-line paths as $t \to +\infty$ and as $t \to -\infty$

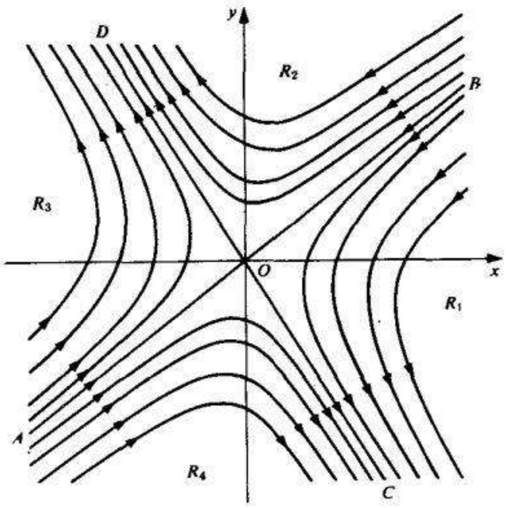

Figure 2.7: Saddle Point

2.11 Jacobian Matrix

If f be the function of independent variable t then the Jacobian matrix is defined to be matrix of partial derivatives.

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial t_1}(t) & \frac{\partial f_1}{\partial t_2}(t) & \frac{\partial f_1}{\partial t_3}(t) & \cdots & \frac{\partial f_1}{\partial t_n}(t) \\ \frac{\partial f_2}{\partial t_1}(t) & \frac{\partial f_2}{\partial t_2}(t) & \frac{\partial f_2}{\partial t_3}(t) & \cdots & \frac{\partial f_2}{\partial t_n}(t) \\ \frac{\partial f_3}{\partial t_1}(t) & \frac{\partial f_3}{\partial t_2}(t) & \frac{\partial f_3}{\partial t_3}(t) & \cdots & \frac{\partial f_3}{\partial t_n}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial t_1}(t) & \frac{\partial f_m}{\partial t_2}(t) & \frac{\partial f_m}{\partial t_3}(t) & \cdots & \frac{\partial f_m}{\partial t_n}(t) \end{pmatrix}$$

2.12 Eigenvalues and Eigenvectors

If A is a square matrix, then a non-zero vector V in R^n is called a eigenvector of A, if V is a scalar multiplied by A, and V is a scalar multiplier with λ which two are equal, such that $AV = V\lambda$, then the number λ is called eigenvalue.

2.13 Characteristic Matrix and Characteristic Polynomial

Let, A be a square matrix, then we can write $AV = \lambda V$ which is equivalently $(\lambda I - A)V = 0$.

The matrix $\lambda I - A$ where I represents identity matrix is called the characteristic matrix of A. The determinant of the characteristic matrix $\lambda I - A$ is a polynomial in and which is called the characteristic polynomial of A.

2.14 Runge-Kutta Method

This method was derived by Runge about the year 1894 and extended by Kutta a few years later. It is one of the most widely used methods and it is particularly suitable in case when the computation of higher derivatives is complicated. Here the increments of the functions are calculated once for all by means of a definite set of formulas [15]. We consider the differential equation $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$. Let h be the interval between equidistant values of x. Then the first increment Δy of y is computed from the following formulae:

$$k_1 = hf\left(x_0, y_0\right)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(x_0 + \mathrm{h}, y_0 + k_3\right)$$

$$\Delta y = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

If we take interval in the given order, then $x_1 = x_0 + h$ and $y_1 = y_0 + \Delta y$. The increment in y for the second interval is computed by means of the formulae:

$$k_1 = hf\left(x_1, y_1\right)$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(x_1 + \mathrm{h}, y_1 + k_3\right)$$

$$\Delta y = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Similarly, we calculate the next intervals. It is noted that calculations for the first increment of x are exactly the same as for the increment. The change in the formulae for the different intervals is only in the values x and y to be substituted. Hence, we obtain Δy for the n^{th} interval we substitute x_{n-1}, y_{n-1} in the expression for k_1, k_2 etc.

2.15 Autonomous and Non-autonomous System

Let us consider a system of differential equation

$$\frac{dx}{dt} = P(x, y)$$

$$\frac{dy}{dt} = Q(x, y)$$

where, $P(x, y)$ and $Q(x, y)$ have continuous first partial derivatives for all (x, y), then the above system is called autonomous system if it does not depend explicitly on the independent variable t. Since the variable t denotes time, the system also called a time invariant system. When the above system depends on time t .then it is known as a non-autonomous system.

2.16 Equilibrium Point

A point $x^* \in \Re^n$ is an equilibrium point (or stationary point or singular point or critical point or fixed point) of the differential equation, $\frac{dx}{dt} = f(t, x)$ If there exist a finite t^* time such that $f(t, x^*) = 0 \,\forall t \geq t^*$ [16].

In the special case of an autonomous system in which f is a function of x only then, it is called an equilibrium point of the system $\frac{dx}{dt} = f(x) \forall t \geq t^*$.

2.17 Stability of the Dynamical System

An equilibrium point x^* of $\frac{dx}{dt} = f(t, x)$ is stable if $\forall \varepsilon > 0$ and any $t_0 \epsilon R^+$ there is $\omega(\varepsilon, t_0)$ such that $|U(t, t_0, \alpha) - x^*| < \varepsilon \,\forall t \geq t_0$ whenever $|\alpha - x^*| < \omega(\varepsilon, t_0)$ where $U(t, t_0, \alpha)$ a solution of $\frac{dx}{dt} = f(t, x)$ with the initial condition $x(t_0) = \alpha$ [16].

2.18 Criteria of Negativity of the Real Parts of All Roots of a Characteristic Equation

In the preceding section, the problem of the stability of a trivial solution of a systems of differential equations was reduced to investigate the signs of the real parts of the roots of the characteristic equation.

If the characteristic equation has a high degree, then its solution is complicated, for this reason, very important are methods permit establishing whether all its roots have a negative real part or not.

2.19 Routh's Hurwitz Criterion

Theorem: A necessary and sufficient condition for the negativity of the reals of all the roots of the polynomial $z^n + a_1 z^{n-1} + \cdots\cdots + a_{n-1} z + a_n$ with real coefficients is the possibility of all the principal diagonals of the minors of the Hurwitz matrix,

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & \cdots \\ a_7 & a_6 & a_5 & a_4 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

The principal diagonal of the Hurwitz matrix exhibits the coefficients of the polynomial under consideration in the order of their numbers of form a_1 to a_n the column alternately consist of coefficients with odd only indices, including the coefficient $a_0 = 1$. Hence the matrix element $b_{ik} = a_{2i-k}$. All missing coefficients are replaced by zeros.

Principal diagonal minor of the Hurwitz matrices is given below:

$$\Delta_1 = [a_1], \Delta_2 = \begin{bmatrix} a_1 & 0 \\ a_3 & a_2 \end{bmatrix}, \Delta_2 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}, \cdots\cdots \Delta_n = \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & \cdots \\ a_5 & a_4 & a_3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

Observed that since $\Delta_n = \Delta_{n-1} a_n$, the last of the Hurwitz conditions $\Delta_1 > 0, \Delta_2 > 0, \cdots\cdots, \Delta_n > 0,$ may be replaced by the demand that $a_n > 0$.

Let us apply the Hurwitz theorem to polynomials of second, third and fourth degree.

a) $z^2 + a_1 z + a_2$

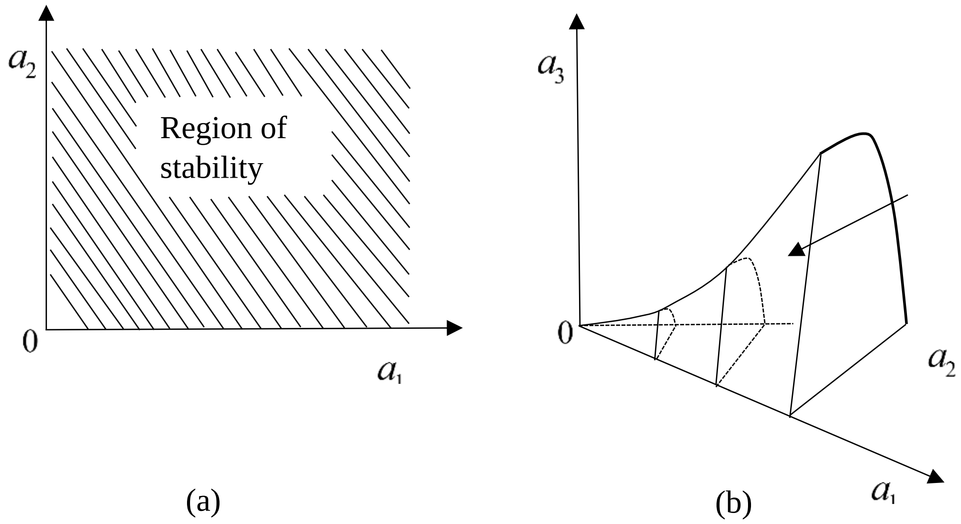

Figure 2.8: Routh's Hurwitz Criterion

The Hurwitz conditions reduce to $a_1 > 0, a_2 > 0$. In figure 2.10 (a), these inequalities define the first quadrant in the space of the conditions a_1 to a_2. Figure (b) depicts the region of asymptotic stability of a trivial solution of some system of differential equations that satisfies the conditions provided that $z^2 + a_1 z + a_2$ is its characteristic polynomial.

b) $z^3 + a_1 z^2 + a_2 z + a_3$

The Hurwitz conditions reduce to $a_1 > 0,\ a_1 a_2 - a_3 > 0,\ a_3 > 0$.

The region defined by this inequality in the coefficient space is depicted in figure (b).

c) $z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$

The Hurwitz conditions reduce to

$$a_1 > 0,\ a_1 a_2 - a_3 > 0,\ (a_1 a_2 - a_3) a_3 - a_1^2 a_4 > 0,\ a_4 > 0$$

The Hurwitz conditions are very convenient and readily verifiable for the polynomials we have just considered. But the Hurwitz conditions rapidly become complicated as the degree of the polynomial increases and it is often more convenient to apply other criteria for the negativity of the real parts of the roots of a polynomial.

2.20 Asymptotical Stability

The equilibrium point x^* of the system $\frac{dx}{dt} = f(t, x)$ is asymptotically stable [16] if, for every $t_0 > 0$ there is a $\mu(t_0) > 0$ such that $\lim_{t \to 0} U(t, t_0, \alpha) = x^*$ whenever $|\alpha - x^*| < \mu(t_0)$.

2.21 Numerical Analysis with Applications

Mathematicians seek accurate approximations of problems whose precise solution is either unattainable or impractical. In addition to the approximate solution, a realistic constraint for the related error with the approximate solution is required. Constructing a mathematical model for a given issue, often consisting of mathematical equations with constraint conditions, is the responsibility of experts in the subject's domain.

Numerical analysis is a branch of mathematics and computer science concerned with developing, analyzing, and implementing techniques for getting numerical solutions to problems involving continuous variables. Similar issues emerge in the scientific sciences, social sciences, engineering, medicine, and the business sector. Since the middle of the 20th century, the rising power and accessibility of digital computers have resulted in a rise in the usage of realistic mathematical models in research and engineering. More sophisticated numerical analysis is required to solve these more intricate models of the world. The formal academic field of numerical analysis goes from highly theoretical mathematics research to computer science problems. Throughout the 1980s and 1990s, the new field of scientific computing or computational science evolved in response to the increased availability of computers. Using numerical analysis, symbolic mathematical calculations, computer graphics, and other computer science disciplines, the field simplifies the creation, solution, and interpretation of complex mathematical models of the actual world.

Naturally, numerical analysis has applications in all branches of engineering and the physical sciences, but in the twenty-first century, the biological sciences and even the arts have incorporated parts of scientific calculations. Ordinary differential equations are found in celestial mechanics (planets, stars, and galaxies); numerical linear algebra is crucial for data processing; stochastic differential equations and Markov chains are necessary for modeling live cells in medicine and biology.

Numerical analysis is concerned with devising methods for approximating the solution to the model and analyzing the results for stability, speed of implementation, and appropriateness. In other words, Numerical Analysis is the study of algorithms that use numerical approximation (as opposed to general symbolic manipulations) for the problems of mathematical analysis (as distinguished from discrete mathematics) [15].

2.22 Common Perspectives in Numerical Analysis

Numerical analysis is concerned with all aspects of the numerical solution of a problem, from the theoretical development and understanding of numerical methods to their practical implementation as reliable and efficient computer programs.

Most numerical analysts specialize in small subfields but share some common concerns, perspectives, and mathematical analysis methods. These include the following:

i. When presented with a problem that cannot be solved directly, they try to replace it with a “nearby problem” that can be solved more easily. Examples are the use of interpolation in developing numerical integration methods and root-finding methods.
ii. There is the widespread use of the language and results of linear algebra, real analysis, and functional analysis (with its simplifying notation of norms, vector spaces, and operators).
iii. There is a fundamental concern with error, size, and analytic form. When approximating a problem, it is prudent to understand the nature of the error in the computed solution. Moreover, understanding the form of the error allows the

creation of extrapolation processes to improve the convergence behavior of the numerical method.

iv. Numerical analysts are concerned with stability, a concept referring to the sensitivity of the solution of a problem to small changes in the data or the parameters of the problem.

v. Numerical analysts are very interested in the effects of using finite precision computer arithmetic. This is especially important in numerical linear algebra, as large problems contain many rounding errors.

vi. Numerical analysts are generally interested in measuring an algorithm's efficiency (or "cost").

2.23 Numerical Techniques for Solving ODE & PDEs

Some numerical techniques to solve ODE and PDEs are listed below [15]:

i. **Finite Difference Method:** In this method, functions are represented by their values at certain grid points, and derivatives are approximated through differences in these values.

ii. **Method of Lines:** The method of lines (MOL, NMOL, NUMOL) solves partial differential equations (PDEs) with discretized dimensions in all but one dimension. MOL permits using techniques and software designed for the numerical integration of ordinary differential equations (ODEs) and algebraic differential equations (DAEs). Over the years, several integration procedures have been written in various programming languages, and some have been released as open-source resources.

iii. **Finite Element Method:** The finite element method (FEM) in mathematics is a numerical methodology for finding approximate solutions to boundary value issues for partial differential equations. It employs variational techniques (the calculus of variations) to generate a stable solution by minimizing an error function.

iv. **Finite Volume Method:** In order to express and evaluate partial differential equations, the finite-volume technique uses algebraic equations [25]. Several CFD software programs use this technique.

v. **Spectral Method:** In applied mathematics and scientific computing, spectral methods are used to solve specific differential equations numerically, often incorporating the Fast Fourier Transform. The concept is to describe the solution of the differential equation as a sum of certain "basic functions" (such as a Fourier series, which is a sum of sinusoids) and then to pick the coefficients in the sum such that the differential equation is satisfied and feasible.

vi. **Meshfree Methods:** Meshfree techniques do not need a mesh linking the simulation domain’s data points. Meshfree approaches provide the modeling of certain otherwise challenging situations at the expense of more computational time and programming effort.

vii. **Domain Decomposition Methods:** Domain decomposition approaches resolve a boundary value issue by decomposing it into smaller boundary value problems on subdomains and coordinating the solution amongst nearby subdomains.

viii. **Multigrid Methods:** In numerical analysis, multigrid (MG) techniques are a set of algorithms for solving differential equations utilizing a hierarchy of discretizations. MG techniques may be used as both solvers and preconditioners.

2.24 Generation and Propagation of Errors

The study of errors is a crucial component of numerical analysis. Many methods exist for introducing an error to address an issue [15].

i. **Round-off:** Since it is difficult to precisely represent all real numbers on a system with limited memory, rounding mistakes occur (which is what all practical digital computers are).

ii. **Truncation and Discretization error:** When an iterative approach is stopped, a mathematical procedure is approximated, and the estimated answer varies from the actual solution, truncation errors are committed. Likewise, discretization results in a discretization error since the solution of the discrete issue does not coincide with the solution of the continuous problem.

What does it imply to indicate that approximating a mathematical technique result in the truncation error? We know that perfect integration of a function entails finding the sum of an unlimited number of trapezoids. Nevertheless, only finite trapezoids may be added numerically, thus the approximation of the mathematical approach. Similarly, the differential element approaches zero to differentiate a function, but we can only choose a finite value for the differential element numerically.

2.25 Numerical Stability and Well-posed Problems

Numerical stability is an important notion in numerical analysis. An algorithm is considered numerically stable if a mistake, regardless of its source, does not amplify significantly throughout the computation. This is the case if the issue is well-conditioned, which means the solution changes little if the problem data are altered slightly. In contrast, minor data inaccuracy will snowball into a significant error if a task is ill-conditioned. The original problem and the technique used to solve it may be well-conditioned or ill-conditioned; any combination is conceivable. A well-conditioned problem-solving method may thus be numerically stable or numerically unstable.

The general objective of numerical analysis is constructing and developing methods that provide approximate but correct solutions to difficult problems, such as those listed below [15].

i. Advanced numerical algorithms are required for numerical weather forecasting to be practicable.
ii. Compounding a spacecraft's trajectory needs an exact numerical solution to a set of ordinary differential equations.
iii. Automobile manufacturers may enhance the crashworthiness of their automobiles via the use of computer simulations of automobile collisions. These simulations rely mostly on numerically solving partial differential equations.
iv. Hedge funds (private investment funds) aim to determine the value of stocks and derivatives more precisely than other market players by using methods from all domains of numerical analysis.
v. Airlines utilize complex optimization algorithms to determine ticket pricing, aircraft and crew allocations, and fuel requirements. In the past, these algorithms were created within operations research.
vi. Insurance firms do actuarial analysis using numerical programs.

2.26 Software

Since the late 20th century, most algorithms have been implemented in diverse programming languages. The Netlib repository provides several Fortran and C numerical problem-solving software routines. The IMSL and NAG libraries are commercial packages implementing several numerical algorithms; the GNU Scientific Library is a free alternative. Popular numerical computing software includes MATLAB, TK Solver, S-PLUS, LabVIEW, and IDL, as well as free and open-source equivalents include FreeMat, Scilab, GNU Octave (similar to Matlab), IT++ (a C++ library), R (similar to S-PLUS), and certain forms of Python. Although vector and matrix operations are typically quick, the performance of scalar loops may vary by more than an order of magnitude. Several computer algebra systems, including Mathematica, benefit from the availability of arbitrary precision arithmetic, which may provide more precise answers. Additionally, any spreadsheet application may tackle elementary numerical analytic issues.

CHAPTER THREE: MATHEMATICAL FORMULATION OF THE MODEL

3.1 Introduction

Modern agriculture depends on chemical fertilizers, pesticides etc. Some amount of these fertilizers, pesticides reach to the nearest lake through water runoff. These chemicals contain a large amount of nutrients. Some amount of nutrients come from domestic drainage. Due to the presence of these nutrients in the lake, algae grow faster and causes algal bloom by eutrophication. When these algae die out, a large amount of oxygen is utilized to decompose the dead algae or detritus. In this way, the depletion of dissolved oxygen in water bodies occurs. As these are part of a food chain involving zooplankton and other biological populations in a water body, the level of dissolved oxygen decreases further due to various interactive biochemical and biodegradation process. It may be noted that the input of the dissolved oxygen in the water body is mainly due to atmospheric diffusion through the water surface and to a certain extent due to its production by photosynthesis. If the depletion rate is increasing at a high rate, then the water becomes anoxic. Then the zooplanktons, fish population in the water die out for the lack of oxygen. By this process, water is polluted severely and becomes uncongenial and harmful for human and animal health.

In Bangladesh, this situation is very dangerous. Water is being polluted by means of domestic usage, industrial wastes. The rivers are getting polluted severely, that is why fish population is decreasing. It is harmful for economy as well as human health.
There prevails a number of research that show the harmful effect and outcomes of this water pollution, in presence of algae and macrophytes. But here in this research, algae and zooplankton have been used at the same time.

However, a mathematical model will be constructed to analyze the effect on depletion of dissolved oxygen with the variables named cumulative concentration of nutrients, density of algae, zooplankton, density of detritus, and concentration of dissolved oxygen. Equilibrium points, stability of the equilibrium points, and characteristics of the state variables with respect to parameters and numerical results have been explained with proper calculation.

3.2 Formulation of Mathematical Model

We consider here a lake which is being affected due to the overgrowth of algae caused by discharge of nutrients from domestic drainage as well as from water runoff, etc. and also from nutrients that has been formed due to detritus. The bilinear interactions of variables such as the cumulative concentration of nutrients, density of algae, density of zooplankton, density of detritus, and concentration of dissolved oxygen are considered. Through different sources nutrients assemble. Due to domestic drainage as well as run off from agriculture fields various nutrients are supplied into the water body. We assume that the algae population is wholly dependent on nutrients produced from nutrients and is being used as a food by its predator zooplankton population. It is assumed further that the level of dissolved oxygen in the water body increases by diffusion with a constant rate as well as by photosynthesis. We are going to consider cumulative concentration of nutrients, density of algae, density of zooplankton, density of detritus and concentration of dissolved oxygen.

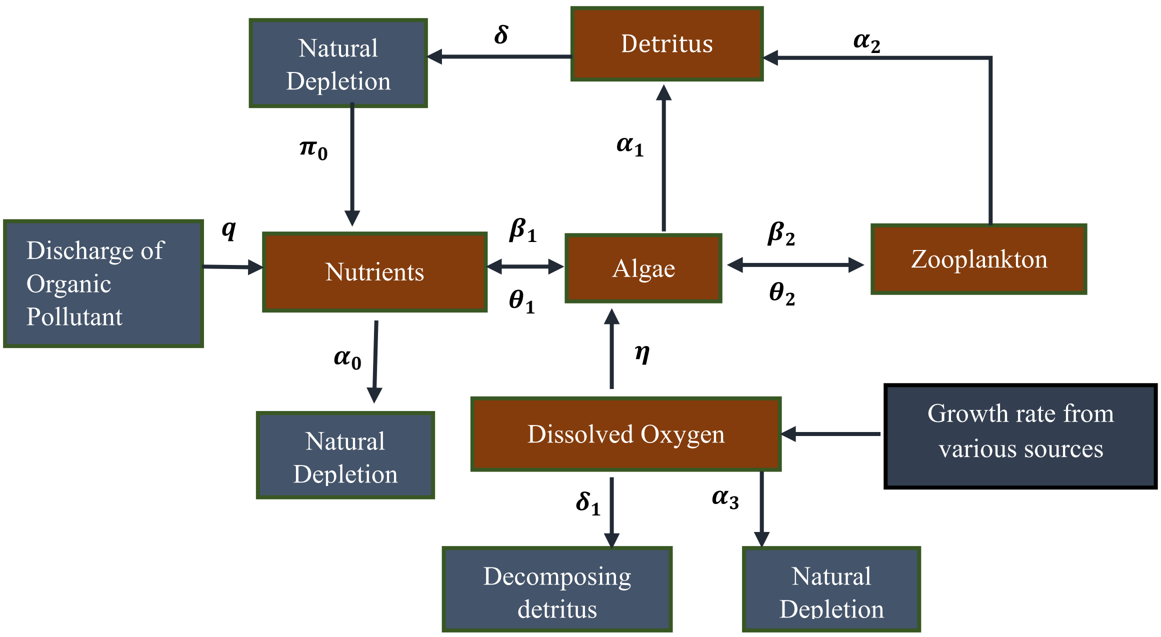

Figure 3.1: Flow diagram of the model

Let us consider T be the cumulative concentration of various nutrients, B be the density of algae, Z be the density of zooplankton, S be the density of detritus and C be the concentration of dissolved oxygen (DO). We assume that the cumulative rate of discharge of nutrients into the aquatic system from outside into water bodies is q, due to natural factors which is depleted with rate $\alpha_0 T$. It is further assumed that the cumulative growth rate of nutrients due to detritus is $\pi_0 \delta S$ and the cumulative rate of depletion of organic pollutant T due to algae

$$\frac{dT}{dt} = q + \pi_0 \delta S - \alpha_0 T - \beta_1 TB \tag{3.1}$$

Thus, the growth rate of algae is proportional to (TB) as it is assumed to be wholly depended on the nutrients. The natural depletion rate of algae is assumed to be proportional to its density B. The depletion rate of algae by zooplankton is proportional to (BZ) which has been considered as a constant rate β_2.

$$\frac{dB}{dt} = \theta_1 \beta_1 TB - \alpha_1 B - \beta_2 BZ \tag{3.2}$$

The growth rate of zooplankton is proportional to (BZ) as it is assumed to be wholly depended on the algae. The natural depletion rate of zooplankton is assumed to be proportional to its density Z.

$$\frac{dZ}{dt} = \theta_2 \beta_2 BZ - \alpha_2 Z \tag{3.3}$$

Since, the natural depletion of algae is converted into detritus, it is assumed to be proportional to $\alpha_1 B$ and natural depletion of zooplankton is converted into detritus as well. Here, natural depletion rate of detritus is assumed to be proportional to S.

$$\frac{dS}{dt} = \pi_1 \alpha_1 B + \pi_2 \alpha_2 Z - \delta S \tag{3.4}$$

We consider that the rate of growth of dissolved oxygen by various sources q_c is assumed to be constant and its natural depletion rate is proportional to its concentration C. It is further assumed that the rate of growth of dissolved oxygen by algae is proportional to B and the depletion of dissolved oxygen caused by decomposing the detritus is proportional to its concentration S.

$$\frac{dC}{dt} = q_c - \alpha_3 C + \eta B - \delta_1 S \tag{3.5}$$

3.3 Basic Assumptions of the Model

Following assumptions are considered from the model:

i. The density of algae increases if too much nutrients and oxygen is present in the water bodies.

ii. Algal population increases absorbing the nutrients present in water bodies. That is why the inter-connection of algae and nutrients are considered.

iii. Growth rate of nutrients is proportional to the density of detritus which comes from outside like water runoff enriched with organic substance and the detritus produced from the death of algae.

iv. Growth rate of detritus proportional to the density of the dead algae and outer sources.

v. The value of all parameters which are used in the model are positive.

With the above assumptions, the following model is proposed in formulation of a nonlinear ordinary differential equation system as follows:

$$\frac{dT}{dt} = q + \pi_0 \delta S - \alpha_0 T - \beta_1 TB = f\left(T, B, Z, S, C\right) \tag{3.6}$$

$$\frac{dB}{dt} = \theta_1 \beta_1 TB - \alpha_1 B - \beta_2 BZ = g\left(T, B, Z, S, C\right) \tag{3.7}$$

$$\frac{dZ}{dt} = \theta_2 \beta_2 BZ - \alpha_2 Z = k\left(T, B, Z, S, C\right) \tag{3.8}$$

$$\frac{dS}{dt} = \pi_1 \alpha_1 B + \pi_2 \alpha_2 Z - \delta S = h\left(T, B, Z, S, C\right) \tag{3.9}$$

$$\frac{dC}{dt} = q_c - \alpha_3 C + \eta B - \delta_1 S = p\left(T, B, Z, S, C\right) \tag{3.10}$$

with non-negative initial conditions:

$$T\left(0\right) \geq 0, \mathrm{B}\left(0\right) \geq 0,\ \mathrm{Z}\left(0\right) \geq 0,\ S\left(0\right) \geq 0 \text{ and } C\left(0\right) \geq 0$$

Here, the positive coefficients $\alpha_i; i = 0,1,2$ are the natural depletion rate coefficients. $\beta_1, \beta_2, \theta_1, \theta_2, \delta, \delta_1$ are proportionality constants which are positive. The positive λ is the proportional constant for the decrease rate of dissolved oxygen due to algae and π_0, π_1 *and* π_2 are positive proportionality constant.

3.4 Model Validation and Mathematical Analysis

We have tested the boundedness of state variables, equilibrium points, analyzed stability at the equilibrium points. Besides, we have done the analysis of the characteristics of state variables with respect to various parameters i.e., sensitivity analysis and analyzed the numerical results of the system (3.6) - (3.10).

3.4.1 Boundedness of state variables

Let, $P = T + B + Z + S + C$

Differentiating both side with respect to t

$$\frac{dP}{dt} = \frac{dT}{dt} + \frac{dB}{dt} + \frac{dZ}{dt} + \frac{dS}{dt} + \frac{dC}{dt} = q + \pi_0 \delta S - \alpha_0 T - \beta_1 TB + \theta_1 \beta_1 TB - \alpha_1 B - \beta_2 BZ + \theta_2 \beta_2 BZ$$
$$- \alpha_2 Z + \pi_1 \alpha_1 B + \pi_2 \alpha_2 Z - \delta S + q_c - \alpha_3 C - \eta B - \delta_1 S$$

$$= q + \pi_0 \delta S - \alpha_0 T - TB\left(\beta_1 - \theta_1 \beta_1\right) - \alpha_1 B - BZ\left(\beta_2 - \theta_2 \beta_2\right)$$
$$- \alpha_2 Z + \pi_1 \alpha_1 B + \pi_2 \alpha_2 Z - \delta S + q_c - \alpha_3 C - \eta B - \delta_1 S$$

$$\Rightarrow \frac{dP}{dt} \le q + \pi_0 \delta S - \alpha_0 T - \alpha_2 Z + \pi_1 \alpha_1 B + q_c - \alpha_3 C$$

$$\Rightarrow \frac{dP}{dt} \le q + q_c - \alpha_0 T + \pi_1 \alpha_1 B - \alpha_2 Z - \alpha_3 C + \pi_0 \delta S$$

$$\Rightarrow \frac{dP}{dt} + \mu P \le q + q_c - \alpha_0 T + \pi_1 \alpha_1 B - \alpha_2 Z - \alpha_3 C + \pi_0 \delta S + \mu T + \mu B + \mu Z + \mu S + \mu C$$

$$\Rightarrow \frac{dP}{dt} + \mu P \le q + q_c + T\left(\mu - \alpha_0\right) + B\left(\mu + \pi_1 \alpha_1\right) + Z\left(\mu - \alpha_2\right) + S\left(\mu - \alpha_3\right) + C\left(\mu + \pi_0 \delta\right)\pi_0 \delta S = Q$$

Applying the theory of differential inequality

$$\Rightarrow P \le \frac{Q}{\mu} + C_0 e^{-\mu t}$$

For $t \to \infty$ we have $0 \le P \le \dfrac{Q}{\mu}$

Therefore, all the solutions of the system are bounded.

3.4.2 Positivity analysis

Now we will prove all the variables in the system model equations are positive. If $T(0) \geq 0,\ \mathrm{B}(0) \geq 0,\ \mathrm{Z}(0) \geq 0, \mathrm{S}(0) \geq 0,\ C(0) \geq 0$ then the solutions of the model are positive.

3.4.2.1 Positivity analysis for the concentration of nutrients

The first differential equation of the model describes the change of nutrient which is given below:

$$\frac{dT}{dt} = q + \pi_0 \delta S - \alpha_0 T - \beta_1 TB$$

Taking only linear parts of the above equation, we get

$$\frac{dT}{dt} \geq -\alpha_0 T$$

$$\Rightarrow \frac{dT}{dt} + \alpha_0 T \geq 0$$

We now find the integrating factor of the above inequality,

$$I.F. = e^{\int \alpha_0 dt} = e^{\alpha_0 t}$$

Multiplying by the integrating factor $e^{\alpha_0 t}$, we get

$$e^{\alpha_0 t} \frac{dT}{dt} + e^{\alpha_0 t} \alpha_0 T \geq 0$$

$$\Rightarrow \frac{d}{dt}(Te^{\alpha_0 t}) \geq 0$$

Integrating both sides with respect to t, we get

$$Te^{\alpha_0 t} \geq l_1;\ \left[l_1 \text{ is an integrating factor}\right]$$

$$\Rightarrow T \geq l_1 e^{-\alpha_0 t}$$

For the initial condition $T(0) = T_0$, we obtain the following condition

$$T_0 \geq l_1$$

From these two we get,

$$T(t) \geq T(0) e^{-\alpha_0 t}$$

$$\forall t \geq 0 \text{ and } \alpha_0 \in R$$

$$e^{-\alpha_0 t} > 0$$

$$\therefore T(t) > 0$$

The concentration of nutrients is always non negative for all time $t \geq 0$

3.4.2.2 Positivity analysis of the density of algae

The second differential equation of the model describes the change of algae which is given below:

$$\frac{dB}{dt} = \theta_1 \beta_1 TB - \alpha_1 B - \beta_2 BZ$$

Taking only linear parts of the above equation, we get

$$\frac{dB}{dt} \geq -\alpha_1 B$$

$$\Rightarrow \frac{dB}{dt} + \alpha_1 B \geq 0$$

We now find the integrating factor of the above inequality,

$$I.F. = e^{\int \alpha_1 dt} = e^{\alpha_1 t}$$

Multiplying by the integrating factor $e^{\alpha_1 t}$, we get

$$e^{\alpha_1 t} \frac{dB}{dt} + e^{\alpha_1 t} \alpha_1 B \geq 0$$

$$\Rightarrow \frac{d}{dt}(Be^{\alpha_1 t}) \geq 0$$

Integrating both sides with respect to t, we get

$$Be^{\alpha_1 t} \geq l_2$$

$$\Rightarrow B \geq l_2 e^{-\alpha_1 t}$$

For the initial condition $B(0) = B_0$, we obtain the following condition

$$B_0 \geq l_2$$

From these two we get,

$$B(t) \geq B(0) e^{-\alpha_1 t}$$

$$\forall t \geq 0 \text{ and } \alpha_1 \in R$$

$$e^{-\alpha_1 t} > 0$$

$$\therefore B(t) > 0$$

The density of algae is always non negative for all time $t \geq 0$.

3.4.2.3 Positivity analysis of the density of zooplankton

The fourth differential equation of the model describes the density of detritus which is given below:

$$\frac{dZ}{dt} = \theta_2 \beta_2 BZ - \alpha_2 Z$$

Taking only linear parts of the above equation, we get

$$\frac{dZ}{dt} \geq -\alpha_2 Z$$

$$\Rightarrow \frac{dZ}{dt} + \alpha_2 Z \geq 0$$

We now find the integrating factor of the above inequality,

$$I.F. = e^{\int \alpha_2 dt} = e^{\alpha_2 t}$$

Multiplying by the integrating factor $e^{\delta t}$, we get

$$e^{\alpha_2 t} \frac{dZ}{dt} + e^{\alpha_2 t} \alpha_2 Z \geq 0$$

$$\Rightarrow \frac{d}{dt}(Ze^{\alpha_2 t}) \geq 0$$

Integrating both sides with respect to t, we get

$$\Rightarrow Ze^{\alpha_2 t} \geq l_3$$

$$\Rightarrow Z \geq l_3 e^{-\alpha_2 t}$$

For the initial condition $Z(0) = Z_0$, we obtain the following condition

$$Z_0 \geq l_3$$

From these two we get,

$$Z(t) \geq Z(0) e^{-\alpha_2 t}$$

$$\forall t \geq 0 \text{ and } \alpha_2 \in R$$

$$e^{-\alpha_2 t} > 0$$

$$\therefore Z(t) > 0$$

The density of zooplankton is always non negative for all time $t \geq 0$.

3.4.2.4 Positivity analysis of the density of detritus

The fourth differential equation of the model describes the density of detritus which is given below:

$$\frac{dS}{dt}=\pi_1\alpha_1 B+\pi_2\alpha_2 Z-\delta S$$

Taking only linear parts of the above equation, we get

$$\frac{dS}{dt}\geq -\delta S$$

$$\Rightarrow \frac{dS}{dt}+\delta S\geq 0$$

We now find the integrating factor of the above inequality,

$$I.F.=e^{\int \delta dt}=e^{\delta t}$$

Multiplying by the integrating factor $e^{\delta t}$, we get

$$e^{\delta t}\frac{dS}{dt}+e^{\delta t}\delta S\geq 0$$

$$\Rightarrow \frac{d}{dt}(Se^{\delta t})\geq 0$$

Integrating both sides with respect to t, we get

$$Se^{\delta t}\geq l_4$$

$$\Rightarrow S\geq l_4 e^{-\delta t}$$

For the initial condition $S\left(0\right)=S_0$, we obtain the following condition

$$S_0\geq l_4$$

From these two we get,

$$S\left(t\right)\geq S\left(0\right)e^{-\delta t}$$

$$\forall t\geq 0 \text{ and } \delta\in R$$

$$e^{-\delta t}>0$$

$$\therefore S\left(t\right)>0$$

The density of detritus is always non negative for all time $t\geq 0$.

3.4.2.5 Positivity analysis of the concentration of the dissolved oxygen

The fifth differential equation of the model describes the change of algae which is given below:

$$\frac{dC}{dt} = q_c - \alpha_3 C + \eta B - \delta_1 S$$

Taking only linear parts of the above equation, we get

$$\frac{dC}{dt} \geq -\alpha_3 C$$

$$\Rightarrow \frac{dC}{dt} + \alpha_3 C \geq 0$$

We now find the integrating factor of the above inequality,

$$I.F. = e^{\int \alpha_3 dt} = e^{\alpha_3 t}$$

Multiplying by the integrating factor $e^{\alpha_3 t}$, we get

$$e^{\alpha_3 t}\frac{dC}{dt} + e^{\alpha_3 t}\alpha_3 C \geq 0$$

$$\Rightarrow \frac{dC}{dt}(Ce^{\alpha_3 t}) \geq 0$$

Integrating both sides with respect to t, we get

$$Ce^{\alpha_3 t} \geq l_5$$

$$\Rightarrow C \geq l_5 e^{-\alpha_3 t}$$

For the initial condition $C(0) = C_0$, we obtain the following condition

$$C_0 \geq l_5$$

From these two we get,

$$C(t) \geq C(0)e^{-\alpha_3 t}$$

$$\forall t \geq 0 \text{ and } \alpha_3 \in R$$

$$e^{-\alpha_3 t} > 0$$

$$\therefore C(t) > 0$$

The concentration of dissolved oxygen is always non negative for all time $t \geq 0$

3.4.3 Equilibrium analysis of the model

The mathematical model is governed by the differential equations (3.1) - (3.5).

Now, for equilibrium point of the model,

$$\left(\frac{dT}{dt}\right)_{(T^*,B^*,Z^*,S^*,C^*)} = q + \pi_0 \delta S^* - \alpha_0 T^* - \beta_1 T^* B^* = 0 \tag{3.11}$$

$$\left(\frac{dB}{dt}\right)_{(T^*,B^*,Z^*,S^*,C^*)} = \theta_1 \beta_1 T^* B^* - \alpha_1 B^* - \beta_2 B^* Z^* = 0 \tag{3.12}$$

$$\left(\frac{dZ}{dt}\right)_{(T^*,B^*,Z^*,S^*,C^*)} = \theta_2 \beta_2 B^* Z^* - \alpha_2 Z^* = 0 \tag{3.13}$$

$$\left(\frac{dS}{dt}\right)_{(T^*,B^*,Z^*,S^*,C^*)} = \pi_1 \alpha_1 B^* + \pi_2 \alpha_2 Z^* - \delta S^* = 0 \tag{3.14}$$

$$\left(\frac{dC}{dt}\right)_{(T^*,B^*,Z^*,S^*,C^*)} = q_c - \alpha_3 C^* - \eta B^* - \delta_1 S^* = 0 \tag{3.15}$$

From equation (3.13), we have,

$$\theta_2 \beta_2 B Z^* - \alpha_2 Z^* = 0$$

$$\Rightarrow Z^* \left(\theta_2 \beta_2 B^* - \alpha_2\right) = 0$$

$$\therefore\ Z^* = 0 \text{ Or, } \theta_2 \beta_2 B^* - \alpha_2 = 0$$

$$\Rightarrow B^* = \frac{\alpha_2}{\theta_2 \beta_2} \tag{3.16}$$

Putting $Z^* = 0$ in equation (3.12), we have,

$$\theta_1 \beta_1 T^* B^* - \alpha_1 B^* - 0 = 0$$

$$\Rightarrow B^* \left(\theta_1 \beta_1 T^* - \alpha_1\right) = 0$$

$$\text{So, } B^* = 0 \text{ Or, } \theta_1 \beta_1 T^* - \alpha_1 = 0$$

$$\Rightarrow T^* = \frac{\alpha_1}{\theta_1 \beta_1} \tag{3.17}$$

Putting the values of $Z^* = 0$ and $B^* = 0$ in equation (3.14), we have,

$$S^* = 0$$

Putting the values of $B^* = Z^* = S^* = 0$ in equation (3.15), we have,

$$q_c - \alpha_3 C^* = 0$$

$$\therefore C^* = \frac{q_c}{\alpha_3}$$

Putting $B=S=0$ in equation (3.11),

$$\alpha_0 T^* = q$$

$$\Rightarrow T^* = \frac{q}{\alpha_0}$$

$\therefore$ The 1st equilibrium point is $E_1\left(T^*,B^*,Z^*,S^*,C^*\right)=\left(\frac{q}{\alpha_0},0,0,0,\frac{q_c}{\alpha_3}\right)$.

For second equilibrium:

Again Applying $Z=0$ and $T^*=\frac{\alpha_1}{\theta_1\beta_1}$ in equation (3.12) we get

$$B^* = \frac{-(\alpha\alpha_1 - q\theta_1\beta_1)}{\alpha_1\beta_1 - \alpha_1\pi_0\theta_1\beta_1\pi_1}$$

Now applying $Z=0$, T^* B^* in equation (3.11) then,

$$q+\pi_0\delta S - \alpha_0 T^* - \beta_1 T^* B^* = 0$$

$$\Rightarrow \pi_0\delta S^* = \alpha_0 T^* + \beta_1 T^* B^* - q$$

$$\therefore S^* = \frac{\alpha_0 T^* + \beta_1 T^* B^* - q}{\pi_0\delta}$$

Applying $Z^*=0$, T^* B^* and S^* in (3.15)

$$q_c - \alpha_3 C^* - \eta B^* - \delta_1 S^* = 0$$

$$\Rightarrow \alpha_3 C^* = q_c - \eta B^* - \delta_1 S^*$$

$$\Rightarrow C^* = \frac{q_c - \eta B^* - \delta_1 S^*}{\alpha_3}$$

The 2nd equilibrium point is

$$E_2\left(T^*,B^*,Z^*,S^*,C^*\right)=\left(\begin{array}{l}\frac{\alpha_1}{\theta_1\beta_1},\frac{\alpha_0\alpha_1 - q\theta_1\beta_1}{\alpha_1\beta_1(\pi_0\pi_1\theta_1 - 1)},0,\frac{\alpha_0 T^* + \beta_1 T^* B^* - q}{\pi_0\delta},\\ \frac{q_c - \eta B^* - \delta_1 S^*}{\alpha_3}\end{array}\right)$$

$\therefore$ The second equilibrium point is $E_2\left(T^*,\mathrm{B}^*,0,\mathrm{S}^* \, C^*\right)$

Where,

$$T^* = \frac{\alpha_1}{\theta_1\beta_1} \tag{3.18}$$

$$B^* = \frac{\left(q\theta_1\beta_1 - \alpha_0\alpha_1\right)}{\alpha_1\beta_1 - \alpha_1\pi_0\theta_1\beta_1\pi_1} \tag{3.19}$$

$$S^* = \frac{\alpha_0 T^* + \beta_1 T^* B^* - q}{\pi_0\delta} \tag{3.20}$$

$$C^* = \frac{q_c - \eta B^* - \delta_1 S^*}{\alpha_3} \tag{3.21}$$

For third equilibrium point,

We have,

$$B^* = \frac{\alpha_2}{\theta_2\beta_2}$$

From (3.14), we get

$$\pi_1\alpha_1 B^* + \pi_2\alpha_2 Z^* - \delta S^* = 0$$

$$\Rightarrow \delta S^* = \pi_1\alpha_1 B^* + \pi_2\alpha_2 Z^*$$

$$\Rightarrow S^* = \frac{\pi_1\alpha_1 B^* + \pi_2\alpha_2 Z^*}{\delta}$$

Again from (3.12)

$$\theta_1\beta_1 T B^* - \alpha_1 B^* - \beta_2 B^* Z^* = 0$$

$$\Rightarrow \theta_1\beta_1 T^* B^* = \alpha_1 B^* + \beta_2 B^* Z^*$$

$$\Rightarrow T^* = \frac{\alpha_1 B^* + \beta_2 B^* Z}{\theta_1\beta_1 B^*}$$

Now putting values of T^* and S^* in equation (3.11)

$$q + \pi_0\delta\frac{\left(\pi_1\alpha_1 B^* + \pi_2\alpha_2 Z^*\right)}{\delta} - \alpha_0\left(\frac{\alpha_1 B^* + \beta_2 B^* Z^*}{\theta_1\beta_1 B^*}\right) - \beta_1 B^*\left(\frac{\alpha_1 B^* + \beta_2 B^* Z^*}{\theta_1\beta_1 B^*}\right) = 0$$

$$\Rightarrow q + \pi_0\pi_1\alpha_1 B^* + \pi_0\pi_2\alpha_2 Z^* - \frac{\alpha_0\alpha_1}{\theta_1\beta_1} - \frac{\alpha_0\beta_2}{\theta_1\beta_1}Z^* - \frac{\alpha_1 B^*}{\theta_1} - \frac{\beta_2 B^* Z^*}{\theta_1} = 0$$

$$\Rightarrow \pi_0\pi_2\alpha_2 Z^* - \frac{\alpha_0\beta_2}{\theta_1\beta_1}Z^* - \frac{\beta_2 B^* Z^*}{\theta_1} = \frac{\alpha_0\alpha_1}{\theta_1\beta_1} + \frac{\alpha_1 B^*}{\theta_1} - q - \pi_0\pi_1\alpha_1 B^*$$

$$\Rightarrow Z^* = \frac{\dfrac{\alpha_0\alpha_1}{\theta_1\beta_1} + \dfrac{\alpha_1 B^*}{\theta_1} - q - \pi_0\pi_1\alpha_1 B^*}{\pi_0\pi_2\alpha_2 - \dfrac{\alpha_0\beta_2}{\theta_1\beta_1} - \dfrac{\beta_2 B^*}{\theta_1}}$$

Putting Z^* and B^* in equation (3.10)

$$\pi_1\alpha_1 B^* + \pi_2\alpha_2 Z^* - \delta S^* = 0$$

$$\Rightarrow S^* = \frac{\pi_1\alpha_1 B^* + \pi_2\alpha_2 Z^*}{\delta}$$

Putting B^*, Z^*, S^* in (3.11)

$$q_c - \alpha_3 C^* - \eta B^* - \delta_1 S^* = 0$$

$$\Rightarrow C^* = \frac{q_c - \eta B^* - \delta_1 S^*}{\alpha_3}$$

Putting S^*, B^* in equation (3.12)

$$\alpha_0 T^* - \beta_1 T^* B^* = q + \pi_0 \delta S^*$$

$$\Rightarrow T^* = \frac{q + \pi_0\delta S^*}{\alpha_0 - \beta_1 B^*}$$

$\therefore$ The 3rd equilibrium point is, $E_3\left(T^*, \mathrm{B}^*, \mathrm{Z}^*, \mathrm{S}^*, C^*\right)$

Where,

$$B^* = \frac{\alpha_2}{\theta_2\beta_2} \tag{3.22}$$

$$Z^* = \frac{\frac{\alpha_0\alpha_1}{\theta_1\beta_1} + \frac{\alpha_1 B^*}{\theta_1} - q - \pi_0\pi_1\alpha_1 B^*}{\pi_0\pi_2\alpha_2 - \frac{\alpha_0\beta_2}{\theta_1\beta_1} - \frac{\beta_2 B^*}{\theta_1}} \tag{3.23}$$

$$S^* = \frac{\pi_1\alpha_1 B^* + \pi_2\alpha_2 Z}{\delta} \tag{3.24}$$

$$C^* = \frac{q_c - \eta B^* - \delta_1 S^*}{\alpha_3} \tag{3.25}$$

$$T^* = \frac{q + \pi_0\delta S^*}{\alpha_0 - \beta_1 B^*} \tag{3.26}$$

Therefore, we can write,

The 1st equilibrium point is, $E_1\left(T^*, B^*, Z^*, S^*, C^*\right) = \left(\frac{q}{\alpha_0}, 0, 0, 0, \frac{q_c}{\alpha_3}\right)$

The 2nd equilibrium point is

$$E_2\left(T^*,B^*,Z^*,S^*,C^*\right)=\left(\frac{\alpha_1}{\theta_1\beta_1},\frac{\left(q\theta_1\beta_1-\alpha_0\alpha_1\right)}{\alpha_1\beta_1-\alpha_1\pi_0\theta_1\beta_1\pi_1},0,\frac{\alpha_0T^*+\beta_1T^*B^*-q}{\pi_0\delta},\frac{q_c-\eta B^*-\delta_1S^*}{\alpha_3}\right)$$

And the 3rd equilibrium point is

$$E_3\left(T^*,B^*,Z^*,S^*,C^*\right)=\left(\begin{array}{l}\dfrac{q+\pi_0\delta S^*}{\alpha_0-\beta_1B^*},\dfrac{\alpha_2}{\theta_2\beta_2},\dfrac{\alpha_0\alpha_1+\alpha_1\beta_1B^*-q\theta_1\beta_1-\pi_0\pi_2\alpha_1\theta_1\beta_1B^*}{\pi_0\pi_2\alpha_1\theta_1\beta_1-\alpha_0\beta_2-\beta_1\beta_2B^*},\\ \dfrac{\pi_1\alpha_1B^*+\pi_2\alpha_2Z^*}{\delta},\dfrac{q_c-\eta B^*-\delta_2S^*}{\alpha_3}\end{array}\right)$$

<u>Case-I:</u> $E_1\left(T^*,B^*,Z^*,S^*,C^*\right)=\left(\frac{q}{\alpha_0},0,0,0,\frac{q_c}{\alpha_3}\right)$ always exists.

This equilibrium of model explains that if the density of algae and density of zooplankton are not participating in the system then the equilibrium level of nutrients will reach to the value $\frac{q}{\alpha_0}$ and the equilibrium concentration of dissolved oxygen will reach to the value $\frac{q_c}{\alpha_3}$. Here we also note that since detritus is formed due to death of algae and zooplankton, both are not participating in the system. hence the equilibrium density of detritus will be zero.

<u>Case-II:</u>

$$E_2\left(T^*,B^*,Z^*,S^*,C^*\right)=\left(\frac{\alpha_1}{\theta_1\beta_1},\frac{q\theta_1\beta_1-\alpha_0\alpha_1}{\alpha_1\beta_1\left(\pi_0\pi_1\theta_1-1\right)},0,\frac{\alpha_0T^*+\beta_1T^*B^*-q}{\pi_0\delta},\frac{q_c-\eta B^*-\delta_1S^*}{\alpha_3}\right)$$

always exists, provided the following conditions are satisfied:

$$q\theta_1\beta_1-\alpha_0\alpha_1>0\ ,$$

$$q_c-\eta B^*-\delta_1S^*>0$$

The second equilibria E_2 of model is obtained when algae participating in the system whereas zooplankton population is not participating in this equilibrium point. In this case the equilibrium level of concentration of organic pollutant, density of algae, density of detritus and concentration of dissolved oxygen will reach to the values $T^*,B^*,S^*and\,C^*$ respectively. These values are explicitly given by equations (3.18), (3.19), (3.20) and (3.21) respectively.

Case-III:

$$E_3\left(T^*,B^*,Z^*,S^*,C^*\right)=\begin{pmatrix} \dfrac{q+\pi_0\delta S^*}{\alpha_0+\beta_1 B^*},\dfrac{\alpha_2}{\theta_2\beta_2},\dfrac{\alpha_0\alpha_1+\alpha_1\beta_1B^*-q\theta_1\beta_1-\pi_0\pi_2\alpha_1\theta_1\beta_1B^*}{\pi_0\pi_2\alpha_1\theta_1\beta_1-\alpha_0\beta_2-\beta_2B^*}, \\ \dfrac{\pi_1\alpha_1B^*+\pi_2\alpha_2Z^*}{\delta},\dfrac{q_c-\eta B^*-\delta_2S^*}{\alpha_3} \end{pmatrix}$$

always exists, provided the following conditions are satisfied:

$$q\theta_1\beta_1-\alpha_0\alpha_1>0\ ,$$

$$\alpha_0\alpha_1+\alpha_1\beta_1B^*-q\theta_1\beta_1-\pi_0\pi_2\alpha_1\theta_1\beta_1B^*>0$$

$$q_c-\eta B^*-\delta_1S^*>0$$

The third equilibria E_3 of model is obtained when both algae and zooplankton are participating in the system. In this case the equilibrium level of concentration of organic pollutant, density of algae, density of detritus and concentration of dissolved oxygen will reach to the values $T^*, B^*, Z^*, S^* and\, C^*$ respectively. These values are explicitly given by equations (3.22) - (3.26) respectively.

3.4.4 Stability analysis of the model

We can check the stability of different equilibrium points of the model and by computing the eigen-value or applying Routh's Hurwitz criterion. Now the Jacobian matrix of the model (3.1) - (3.5) is,

$$J_{\left(T^*,B^*,Z^*,S^*,C^*\right)}=\begin{pmatrix} f_T\left(T^*,B^*,Z^*,S^*,C^*\right) & f_B\left(T^*,B^*,Z^*,S^*,C^*\right) & f_Z\left(T^*,B^*,Z^*,S^*,C^*\right) & f_S\left(T^*,B^*,Z^*,S^*,C^*\right) & f_C\left(T^*,B^*,Z^*,S^*,C^*\right) \\ g_T\left(T^*,B^*,Z^*,S^*,C^*\right) & g_B\left(T^*,B^*,Z^*,S^*,C^*\right) & g_Z\left(T^*,B^*,Z^*,S^*,C^*\right) & g_S\left(T^*,B^*,Z^*,S^*,C^*\right) & g_C\left(T^*,B^*,Z^*,S^*,C^*\right) \\ k_T\left(T^*,B^*,Z^*,S^*,C^*\right) & k_B\left(T^*,B^*,Z^*,S^*,C^*\right) & k_Z\left(T^*,B^*,Z^*,S^*,C^*\right) & k_S\left(T^*,B^*,Z^*,S^*,C^*\right) & k_C\left(T^*,B^*,Z^*,S^*,C^*\right) \\ h_T\left(T^*,B^*,Z^*,S^*,C^*\right) & h_B\left(T^*,B^*,Z^*,S^*,C^*\right) & h_Z\left(T^*,B^*,Z^*,S^*,C^*\right) & h_S\left(T^*,B^*,Z^*,S^*,C^*\right) & h_C\left(T^*,B^*,Z^*,S^*,C^*\right) \\ p_T\left(T^*,B^*,Z^*,S^*,C^*\right) & p_B\left(T^*,B^*,Z^*,S^*,C^*\right) & p_Z\left(T^*,B^*,Z^*,S^*,C^*\right) & p_S\left(T^*,B^*,Z^*,S^*,C^*\right) & p_C\left(T^*,B^*,Z^*,S^*,C^*\right) \end{pmatrix}$$

$$\Rightarrow J_{\left(T^*,B^*,Z^*,S^*,C^*\right)}=\begin{pmatrix} -\alpha_0-\beta_1B^* & -\beta_1T^* & 0 & \pi_0\delta & 0 \\ \theta_1\beta_1B^* & \theta_1\beta_1T^*-\alpha_1 & -\beta_2B^* & 0 & 0 \\ 0 & \theta_2\beta_2Z^* & \theta_2\beta_2B^*-\alpha_2 & 0 & 0 \\ 0 & \pi_1\alpha_1 & -\pi_2\alpha_2 & -\delta & 0 \\ 0 & -\eta & 0 & -\delta_1 & -\alpha_3 \end{pmatrix} \tag{3.27}$$

At equilibrium point $E_1\left(\dfrac{q}{\alpha_0},0,0,0,\dfrac{q_c}{\alpha_3}\right)$,

$$J_{E_1\left(\frac{q}{\alpha_0},0,0,0,\frac{q_c}{\alpha_3}\right)}=\begin{pmatrix} -\alpha_0 & -\frac{\beta_1 q}{\alpha_0} & 0 & \pi_0\delta & 0 \\ 0 & \frac{\theta_1\beta_1 q}{\alpha_0}-\alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \pi_1\alpha_1 & -\pi_2\alpha_2 & -\delta & 0 \\ 0 & -\eta & 0 & -\delta_1 & -\alpha_3 \end{pmatrix}$$

Now the characteristics equation of Jacobian matrix is, $|J-\lambda I|=0$

$$\therefore |J-\lambda I|=\left|\begin{pmatrix} -\alpha_0 & -\frac{\beta_1 q}{\alpha_0} & 0 & \pi_0\delta & 0 \\ 0 & \frac{\theta_1\beta_1 q}{\alpha_0}-\alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \pi_1\alpha_1 & -\pi_2\alpha_2 & -\delta & 0 \\ 0 & -\eta & 0 & -\delta_1 & -\alpha_3 \end{pmatrix}-\begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}\right|=0$$

$$\Rightarrow\begin{pmatrix} -\alpha_0-\lambda & -\frac{\beta_1 q}{\alpha_0} & 0 & \pi_0\delta & 0 \\ 0 & \frac{\theta_1\beta_1 q}{\alpha_0}-\alpha_1-\lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & \pi_1\alpha_1 & -\pi_2\alpha_2 & -\delta-\lambda & 0 \\ 0 & -\eta & 0 & -\delta_1 & -\alpha_3-\lambda \end{pmatrix}=0$$

$$\Rightarrow(-\alpha_0-\lambda)\begin{pmatrix} \frac{\theta_1\beta_1 q}{\alpha_0}-\alpha_1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ \pi_1\alpha_1 & -\pi_2\alpha_2 & -\delta-\lambda & 0 \\ -\eta & 0 & -\delta_1 & -\alpha_3-\lambda \end{pmatrix}-\left(-\frac{\beta_1 q}{\alpha_0}\right)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\pi_2\alpha_2 & -\delta-\lambda & 0 \\ 0 & 0 & -\delta_1 & -\alpha_3-\lambda \end{pmatrix}$$

$$+\pi_0\delta\begin{pmatrix} 0 & \frac{\theta_1\beta_1 q}{\alpha_0}-\alpha_1-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & \pi_1\alpha_1 & -\pi_2\alpha_2 & 0 \\ 0 & -\eta & 0 & -\alpha_3-\lambda \end{pmatrix}=0$$

$$\Rightarrow(\alpha_0+\lambda)\left(\frac{-\theta_1\beta_1 q}{\alpha_0}+\alpha_1+\lambda\right)(\delta+\lambda)(\alpha_3+\lambda)(\eta\lambda+\theta_2)=0$$

$$\therefore(\alpha_0+\lambda)=0\ \text{ or,}\left(-\frac{\theta_1\beta_1 q}{\alpha_0}+\alpha_1+\lambda\right)=0\ \text{ or,}(\delta+\lambda)=0\ \text{ or,}(\alpha_3+\lambda)=0\ \text{ or, } n\lambda+\theta_2=0$$

$$\therefore \lambda = -\alpha_0 \;, \lambda = -\delta \,, \lambda = -\alpha_3 \,, \lambda = \frac{-\theta_2}{\eta} \text{ and } \lambda = \frac{-\left(\alpha_0\alpha_1 - \theta_1\beta_1 q\right)}{\alpha_0}$$

Hence, the eigenvalues, $\lambda = -\alpha_0, -\delta, -\alpha_3, \frac{-\theta_2}{\eta}, \frac{-\left(\alpha_0\alpha_1 - \theta_1\beta_1 q\right)}{\alpha_0}$

It is known that if the eigenvalues of the characteristic equation at an equilibrium point of any system are negative then the system is considered as asymptotically stable at that point.

The eigenvalues for equilibrium point $E_1\left(\frac{q}{\alpha_0}, 0, 0, 0, \frac{q_c}{\alpha_3}\right)$ are,

$$\lambda = -\alpha_0, -\delta, -\alpha_3, \frac{-\theta_2}{\eta}, \frac{-\left(\alpha_0\alpha_1 - \theta_1\beta_1 q\right)}{\alpha_0}$$

Here, the system will be asymptotically stable if and only if $\alpha_0\alpha_1 > \theta_1\beta_1 q$.

Now, at equilibrium point $E_3\left(T^*, B^*, Z^*, S^*, C^*\right)$, the Jacobian becomes

$$\Rightarrow J_{\left(T^*,B^*,Z^*,S^*,C^*\right)} = \begin{pmatrix} -\alpha_0 - \beta_1 B^* & -\beta_1 T^* & 0 & \pi_0\delta & 0 \\ \theta_1\beta_1 B^* & \theta_1\beta_1 T^* - \alpha_1 & -\beta_2 B^* & 0 & 0 \\ 0 & \theta_2\beta_2 Z^* & \theta_2\beta_2 B^* - \alpha_2 & 0 & 0 \\ 0 & \pi_1\alpha_1 & -\pi_2\alpha_2 & -\delta & 0 \\ 0 & -\eta & 0 & -\delta_1 & -\alpha_3 \end{pmatrix}$$

Now the characteristics equation of Jacobian matrix is, $\left|J - \lambda I\right| = 0$

$$\left|J - \lambda I\right| = \left| \begin{pmatrix} -\alpha_0 - \beta_1 B^* & -\beta_1 T^* & 0 & \pi_0\delta & 0 \\ \theta_1\beta_1 B^* & \theta_1\beta_1 T^* - \alpha_1 & -\beta_2 B^* & 0 & 0 \\ 0 & \theta_2\beta_2 Z^* & \theta_2\beta_2 B^* - \alpha_2 & 0 & 0 \\ 0 & \pi_1\alpha_1 & -\pi_2\alpha_2 & -\delta & 0 \\ 0 & -\eta & 0 & -\delta_1 & -\alpha_3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{pmatrix} -\alpha_0 - \beta_1 B^* - \lambda & -\beta_1 T^* & 0 & \pi_0\delta & 0 \\ \theta_1\beta_1 B^* & \theta_1\beta_1 T^* - \alpha_1 - \lambda & -\beta_2 B^* & 0 & 0 \\ 0 & \theta_2\beta_2 Z^* & \theta_2\beta_2 B^* - \alpha_2 - \lambda & 0 & 0 \\ 0 & \pi_1\alpha_1 & -\pi_2\alpha_2 & -\delta - \lambda & 0 \\ 0 & -\eta & 0 & -\delta_1 & -\alpha_3 - \lambda \end{pmatrix} = 0$$

$$\Rightarrow \left\{-\alpha_0 - \beta_1 B^* - \lambda\right\} \begin{pmatrix} \theta_1\beta_1 T^* - \alpha_1 - \lambda & -\beta_2 B^* & 0 & 0 \\ \theta_2\beta_2 Z^* & \theta_2\beta_2 B^* - \alpha_2 - \lambda & 0 & 0 \\ \pi_1\alpha_1 & -\pi_2\alpha_2 & -\delta - \lambda & 0 \\ -\eta & 0 & -\delta_1 & -\alpha_3 - \lambda \end{pmatrix} +$$

$$\beta_1 T^* \begin{pmatrix} \theta_1\beta_1 B^* & -\beta_2 B^* & 0 & 0 \\ 0 & \theta_2\beta_2 B^* - \alpha_2 - \lambda & 0 & 0 \\ 0 & -\pi_2\alpha_2 & -\delta - \lambda & 0 \\ 0 & 0 & -\delta_1 & -\alpha_3 - \lambda \end{pmatrix} + \pi_0\delta \begin{pmatrix} \theta_1\beta_1 B^* & \theta_1\beta_1 T^* - \alpha_1 - \lambda & -\beta_2 B^* & 0 \\ 0 & \theta_2\beta_2 Z^* & \theta_2\beta_2 B^* - \alpha_2 - \lambda & 0 \\ 0 & \pi_1\alpha_1 & -\pi_2\alpha_2 & 0 \\ 0 & -\eta & 0 & -\alpha_3 - \lambda \end{pmatrix} = 0$$

$$\Rightarrow\left[\left(-\alpha_0-\beta_1B^*-\lambda\right)\left\{\left(\theta_1\beta_1T^*-\alpha_1-\lambda\right)\begin{pmatrix}\theta_2\beta_2B^*-\alpha_2-\lambda & 0 & 0\\ -\pi_2\alpha_2 & -\delta-\lambda & 0\\ 0 & -\delta_1 & -\alpha_3-\lambda\end{pmatrix}+\beta_2B^*\begin{pmatrix}\theta_2\beta_2Z^* & 0 & 0\\ \pi_1\alpha_1 & -\delta-\lambda & 0\\ -\eta & -\delta_1 & -\alpha_3-\lambda\end{pmatrix}\right\}\right]$$
$$+\beta_1T^*\left(\theta_1\beta_1B^*\right)\left(\theta_2\beta_2B^*-\alpha_2-\lambda\right)\left(\theta_2\beta_2B^*-\alpha_2-\lambda\right)\left(\delta+\lambda\right)\left(\alpha_3+\lambda\right)+$$
$$+\pi_0\delta\left(\theta_1\beta_1B^*\right)\left(\theta_2\beta_2Z^*\right)\left(\alpha_3+\lambda\right)\pi_2\alpha_2=0$$
$$\Rightarrow\left[\left(-\alpha_0-\beta_1B^*-\lambda\right)\left\{\left(\theta_1\beta_1T^*-\alpha_1-\lambda\right)\left(\theta_2\beta_2B^*-\alpha_2-\lambda\right)\left(\delta+\lambda\right)\left(\alpha_3+\lambda\right)+\beta_2B^*\left(\theta_2\beta_2Z^*\right)\left(\delta+\lambda\right)\left(\alpha_3+\lambda\right)\right\}\right]$$
$$+\beta_1T^*\left(\theta_1\beta_1B^*\right)\left(\theta_2\beta_2B^*-\alpha_2-\lambda\right)\left(\theta_2\beta_2B^*-\alpha_2-\lambda\right)\left(\delta+\lambda\right)\left(\alpha_3+\lambda\right)+$$
$$+\pi_0\delta\left(\theta_1\beta_1B^*\right)\left(\theta_2\beta_2Z^*\right)\left(\alpha_3+\lambda\right)\pi_2\alpha_2=0$$
$$\Rightarrow\left\{\left(-\alpha_0-\beta_1B^*-\lambda\right)\left(\theta_1\beta_1T^*-\alpha_1-\lambda\right)\left(\theta_2\beta_2B^*-\alpha_2-\lambda\right)\left(-\delta-\lambda\right)\left(-\alpha_3-\lambda\right)+\beta_2B^*\left(-\alpha_0-\beta_1B^*-\lambda\right)\left(\theta_2\beta_2Z^*\right)\left(\delta+\lambda\right)\left(\alpha_3+\lambda\right)\right\}$$
$$+\beta_1T^*\left(\theta_1\beta_1B^*\right)\left(\theta_2\beta_2B^*-\alpha_2-\lambda\right)\left(\delta+\lambda\right)\left(\alpha_3+\lambda\right)+$$
$$+\pi_0\delta\left(\theta_1\beta_1B^*\right)\left(\theta_2\beta_2Z^*\right)\left(\alpha_3+\lambda\right)\pi_2\alpha_2=0$$
$$\Rightarrow\lambda^5-\left(A_1+A_2+A_3+A_4+A_5\right)\lambda^4+$$
$$\left(A_6+A_7+A_1A_2+A_1A_3+A_1A_4+A_2A_3+A_1A_5+A_2A_4+A_2A_5+A_3A_4+A_3A_5+A_4A_5\right)\lambda^3$$
$$-\begin{pmatrix}A_1A_6+A_3A_7+A_4A_6+A_4A_7+A_5A_6+A_5A_7+A_1A_2A_3+A_1A_2A_4+A_1A_2A_5+A_1A_3A_4+A_1A_3A_5+A_2A_3A_4\\ +A_1A_4A_5+A_2A_3A_5+A_2A_4A_5+A_3A_4A_5\end{pmatrix}\lambda^2$$
$$+\begin{pmatrix}A_8+A_1A_4A_6+A_1A_5A_6+A_3A_4A_7+A_3A_5A_7+A_4A_5A_6+A_4A_5A_7\\ +A_1A_2A_3A_4+A_1A_2A_3A_5+A_1A_2A_4A_5+A_1A_3A_4A_5+A_2A_3A_4A_5\end{pmatrix}\lambda$$
$$-\left(A_5A_8+A_1A_4A_5A_6+A_1A_2A_3A_4+A_1A_2A_3A_4A_5\right)=0$$

Therefore,

$$\lambda^5+a_1\lambda^4+a_2\lambda^3+a_3\lambda^2+a_4\lambda+a_5=0$$

Where,

$$a_1=-\left(A_1+A_2+A_3+A_4+A_5\right)$$
$$a_2=\left(A_6+A_7+A_1A_2+A_1A_3+A_1A_4+A_2A_3+A_1A_5+A_2A_4+A_2A_5+A_3A_4+A_3A_5+A_4A_5\right)$$
$$a_3=-\begin{pmatrix}A_1A_6+A_3A_7+A_4A_6+A_4A_7+A_5A_6+A_5A_7+A_1A_2A_3+A_1A_2A_4+A_1A_2A_5+A_1A_3A_4+A_1A_3A_5+A_2A_3A_4\\ +A_1A_4A_5+A_2A_3A_5+A_2A_4A_5+A_3A_4A_5\end{pmatrix}$$
$$a_4=\begin{pmatrix}A_8+A_1A_4A_6+A_1A_5A_6+A_3A_4A_7+A_3A_5A_7+A_4A_5A_6+A_4A_5A_7\\ +A_1A_2A_3A_4+A_1A_2A_3A_5+A_1A_2A_4A_5+A_1A_3A_4A_5+A_2A_3A_4A_5\end{pmatrix}$$
$$a_5=-\left(A_5A_8+A_1A_4A_5A_6+A_1A_2A_3A_4+A_1A_2A_3A_4A_5\right)$$

And

$$A_1=-\alpha_0-\beta_1B^*$$

$A_2 = \theta_1\beta_1 T^* - \alpha_1$

$A_3 = \theta_2\beta_2 B^* - \alpha_2$

$A_4 = -\delta$

$A_5 = -\alpha_3$

$A_6 = \beta_2 B^* \theta_2 \beta_2 Z^*$

$A_7 = \beta_1 T^* \theta_1 \beta_1 B^*$

$A_8 = \pi_0 \delta \theta_1 \beta_1 B^* \theta_2 \beta_2 Z^*$

According to the Routh's Hurwitz criterion, the necessary conditions are found with the help of table array-

$$b_1 = \frac{a_1 a_2 - a_3}{a_1}$$

$$c_1 = \frac{(a_1 a_2 - a_3)a_3 - (a_1 a_4 - a_5)a_1}{a_1 a_2 - a_3}$$

$$d_1 = \frac{(a_1 a_2 - a_3)(a_1 a_4 - a_5)a_3 - a_1(a_1 a_4 - a_5)^2 - a_5(a_1 a_2 - a_3)}{a_1 a_3(a_1 a_2 - a_3) - a_1^2(a_1 a_4 - a_5)}$$

$$e_1 = a_5$$

It is stable when $b_1 > 0$, $c_1 > 0$, $d_1 > 0$ and $e_1 > 0$

The equilibrium point $E_3(T^*, B^*, Z^*, S^*, C^*)$ will be asymptotically stable if and only if

$a_1 a_2 - a_3 > 0$, $(a_1 a_2 - a_3)a_3 - (a_1 a_4 - a_5)a_1 > 0$,
$(a_1 a_2 - a_3)(a_1 a_4 - a_5)a_3 - a_1(a_1 a_4 - a_5)^2 - a_5(a_1 a_2 - a_3) > 0$ and $a_5 > 0$.

Theorem: The equilibrium $E_i (i = 1, 2)$ is unstable whenever E_{i+1} exists. The equilibrium point E_1 is locally stable if $\dfrac{\pi_0^2}{\alpha_0 + \beta_1 B^*} < \dfrac{T^*}{2\theta_1} \min\left[\dfrac{1}{\pi_1^2 \alpha_1^2}, \dfrac{2}{\pi_2^2 \theta_2 \alpha_2^2}\right]$.

Proof: Linearizing system by using the transformations

$T = T^* + T_1$, $B = B^* + B_1$, $Z = Z^* + Z_1$, $S = S^* + S_1$ and $C = C^* + C_1$

And the positive definite function

$$V = \frac{1}{2}\left(T_1^2 + m_1 B_1^2 + m_2 Z_1^2 + m_3 S_1^2 + m_4 C_1^2\right)$$

Where m_1, m_2, m_3 and m_4 are some positive constants to be chosen appropriately.

We obtain, along the solution as follows

$$\frac{dV}{dt} = -\left(\alpha_0 + \beta_1 B^*\right)T_1^2 - \frac{T^*}{\theta_1}B_1^2 - \frac{T^*}{\theta_1\theta_2}Z_1^2 - m_3\delta S_1^2 - m_4\alpha_3 C_1^2 + T_1S_1\left(\pi_0\delta\right) +$$
$$B_1S_1\left(m_3\pi_1\alpha_1\right) + Z_1S_1\left(m_3\pi_2\alpha_2\right) + B_1C_1\left(-m_4\eta\right) + S_1C_1\left(-m_4\delta_1\right)$$

Where $m_1 = \dfrac{T^*}{\theta_1 B^*}$ and $m_2 = \dfrac{T^*}{\theta_1\theta_2 Z^*}$. Here we note that $\dfrac{dV}{dt}$ will be the negative definite if the following conditions are satisfied

$$m_3 > \frac{\pi_0^2\delta}{\alpha_0 + \beta_1 B^*}$$

$$m_3 < \frac{T^*\delta}{2\pi_1^2\theta_2\alpha_1^2}$$

$$m_3 < \frac{T^*\delta}{\pi_2^2\theta_1\theta_2\alpha_2^2}$$

$$m_4 < \frac{T^*\alpha_3}{\theta_1\eta^2}$$

$$m_4 < m_3\frac{\alpha_3\delta}{2\delta_1^2}$$

We can choose a positive m_3 and m_4 hence if the following condition is satisfied:

$$\frac{\pi_0^2}{\alpha_0 + \beta_1 B^*} < \frac{T^*}{2\theta_1}\min\left[\frac{1}{\pi_1^2\alpha_1^2}, \frac{2}{\pi_2^2\theta_2\alpha_2^2}\right] \qquad \text{(Proved)}$$

3.4.5 Characteristics of state variables

i. **Characteristics of the concentration of dissolved oxygen $\left(C^*\right)$ and the density of algae $\left(B^*\right)$ with respect to q (cumulative rate of discharge of nutrients)**

Substituting the values of T^* , Z^* and S^* in the model, we have,

$$j\left(C^*, B^*, q\right) = q_c - \alpha_3 C^* - \eta B^* - \frac{\delta_1\pi_1\alpha_1 B^*}{\delta}$$

$$k\left(C^*,B^*,q\right)=\frac{\beta_1\theta_1 q\beta_2\theta_2B^*}{\alpha_0\beta_2\theta_1-\beta_1\alpha_2}+\frac{\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1B^*}{\alpha_0\beta_2\theta_1-\beta_1\alpha_2}-\alpha_1B^*$$

$$\therefore \frac{dC^*}{dq}=\frac{\begin{vmatrix}\frac{\partial j}{\partial B^*} & \frac{\partial j}{\partial q}\\ \frac{\partial k}{\partial B^*} & \frac{\partial k}{\partial q}\end{vmatrix}}{\begin{vmatrix}\frac{\partial j}{\partial C^*} & \frac{\partial j}{\partial B^*}\\ \frac{\partial k}{\partial C^*} & \frac{\partial k}{\partial B^*}\end{vmatrix}}=\frac{\frac{\partial j}{\partial B^*}\cdot\frac{\partial k}{\partial q}-\frac{\partial j}{\partial q}\cdot\frac{\partial k}{\partial B^*}}{\frac{\partial j}{\partial C^*}\cdot\frac{\partial k}{\partial B^*}-\frac{\partial j}{\partial B^*}\cdot\frac{\partial k}{\partial C^*}}$$

Here,

$$\frac{\partial j}{\partial B^*}=-\eta-\frac{\delta_1\pi_1\alpha_1}{\delta},\ \frac{\partial j}{\partial C^*}=-\alpha_3,\ \frac{\partial k}{\partial C^*}=0,\ \frac{\partial j}{\partial q}=0,\ \frac{\partial k}{\partial q}=\frac{\beta_1\theta_1\beta_2\theta_2B^*}{\alpha_0\beta_2\theta_1-\beta_1\alpha_2},$$

$$\frac{\partial k}{\partial B^*}=\frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\beta_1\beta_2\theta_1\theta_2q-\beta_1\beta_2\theta_1\theta_2qB^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}+\frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1-\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1B^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}-\alpha_1$$

Now,

$$\frac{\partial j}{\partial B^*}\cdot\frac{\partial k}{\partial q}-\frac{\partial j}{\partial q}\cdot\frac{\partial k}{\partial B^*}=\left(-\eta-\frac{\delta_1\pi_1\alpha_1}{\delta}\right)\left(\frac{\beta_1\theta_1\beta_2\theta_2B^*}{\alpha_0\beta_2\theta_1-\beta_1\alpha_2}\right)$$

$$\frac{\partial j}{\partial C^*}\cdot\frac{\partial k}{\partial B^*}-\frac{\partial j}{\partial B^*}\cdot\frac{\partial k}{\partial C^*}=-\alpha_3\left\{\begin{aligned}&\frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\beta_1\beta_2\theta_1\theta_2q-\beta_1\beta_2\theta_1\theta_2qB^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}+\\&\frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1-\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1B^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}-\alpha_1\end{aligned}\right\}$$

$$\therefore \frac{dc^*}{dq}=\frac{\left(-\eta-\frac{\delta_1\pi_1\alpha_1}{\delta}\right)\left(\frac{\beta_1\theta_1\beta_2\theta_2B^*}{\alpha_0\beta_2\theta_1-\beta_1\alpha_2}\right)}{-\alpha_3\left\{\begin{aligned}&\frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\beta_1\beta_2\theta_1\theta_2q-\beta_1\beta_2\theta_1\theta_2qB^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}+\\&\frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1-\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1B^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}-\alpha_1\end{aligned}\right\}}$$

For $\delta_1\pi_1\alpha_1<\delta\eta$, $\frac{dc^*}{dq}<0$

Here the change is negative because oxygen is depleted through different ways that has been discussed before. If we use algae, oxygen would rise a little because algae produce oxygen but we didn't assume algae here.

Therefore, when the rate of discharge of nutrients from outside (q) increases, the concentration of dissolved oxygen also decreases.

Again,

$$\frac{dB^*}{dq}=\frac{\begin{vmatrix}\frac{\partial j}{\partial q} & \frac{\partial j}{\partial C^*}\\ \frac{\partial k}{\partial q} & \frac{\partial k}{\partial C^*}\end{vmatrix}}{\begin{vmatrix}\frac{\partial j}{\partial C^*} & \frac{\partial j}{\partial B^*}\\ \frac{\partial k}{\partial C^*} & \frac{\partial k}{\partial B^*}\end{vmatrix}}=\frac{\frac{\partial j}{\partial C^*}\cdot\frac{\partial k}{\partial q}-\frac{\partial j}{\partial q}\cdot\frac{\partial k}{\partial C^*}}{\frac{\partial j}{\partial C^*}\cdot\frac{\partial k}{\partial B^*}-\frac{\partial j}{\partial B^*}\cdot\frac{\partial k}{\partial C^*}}$$

$$\frac{\partial j}{\partial C^*}\cdot\frac{\partial k}{\partial q}-\frac{\partial j}{\partial q}\cdot\frac{\partial k}{\partial C^*}=-\alpha_3\frac{\beta_1\theta_1\beta_2\theta_2 B^*}{\alpha_0\beta_2\theta_1-\beta_1\alpha_2}$$

$$\frac{\partial j}{\partial C^*}\cdot\frac{\partial k}{\partial B^*}-\frac{\partial j}{\partial B^*}\cdot\frac{\partial k}{\partial C^*}=-\alpha_3\left\{\begin{aligned}&\frac{(\alpha_0\beta_2\theta_2-\beta_1\alpha_2)\beta_1\beta_2\theta_1\theta_2 q-\beta_1\beta_2\theta_1\theta_2 qB^*}{(\alpha_0\beta_2\theta_2-\beta_1\alpha_2)^2}+\\&\frac{(\alpha_0\beta_2\theta_2-\beta_1\alpha_2)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1-\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{(\alpha_0\beta_2\theta_2-\beta_1\alpha_2)^2}-\alpha_1\end{aligned}\right\}$$

$$\therefore\frac{dB^*}{dq}=\frac{-\alpha_3\frac{\beta_1\theta_1\beta_2\theta_2 B^*}{\alpha_0\beta_2\theta_1-\beta_1\alpha_2}}{-\alpha_3\left\{\begin{aligned}&\frac{(\alpha_0\beta_2\theta_2-\beta_1\alpha_2)\beta_1\beta_2\theta_1\theta_2 q-\beta_1\beta_2\theta_1\theta_2 qB^*}{(\alpha_0\beta_2\theta_2-\beta_1\alpha_2)^2}+\\&\frac{(\alpha_0\beta_2\theta_2-\beta_1\alpha_2)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1-\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{(\alpha_0\beta_2\theta_2-\beta_1\alpha_2)^2}-\alpha_1\end{aligned}\right\}}$$

$$\Rightarrow\frac{dB^*}{dq}=\frac{\frac{\beta_1\theta_1\beta_2\theta_2 B^*}{\alpha_0\beta_2\theta_1-\beta_1\alpha_2}}{\left\{\begin{aligned}&\frac{(\alpha_0\beta_2\theta_2-\beta_1\alpha_2)\beta_1\beta_2\theta_1\theta_2 q-\beta_1\beta_2\theta_1\theta_2 qB^*}{(\alpha_0\beta_2\theta_2-\beta_1\alpha_2)^2}+\\&\frac{(\alpha_0\beta_2\theta_2-\beta_1\alpha_2)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1-\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{(\alpha_0\beta_2\theta_2-\beta_1\alpha_2)^2}-\alpha_1\end{aligned}\right\}}$$

$$\therefore\frac{dB^*}{dq}>0$$

Here, the change is positive which shows a proportional relation.

Therefore, the rate of discharge of nutrients from outside (q) increases, the density of algae also increases. But after a certain period of time, it starts to decreasing because zooplankton consumes algae.

Now, using equation (3.13) we get,

$$\theta_2\beta_2 B^* \frac{dZ^*}{dq} + \theta_2\beta_2 Z^* \frac{dB^*}{dq} - \alpha_2 \frac{dZ^*}{dq} = 0$$

$$\frac{dZ^*}{dq} = \frac{\dfrac{\dfrac{\beta_1\theta_1\beta_2\theta_2 B^*}{\alpha_0\beta_2\theta_1 - \beta_1\alpha_2}}{\left\{\begin{array}{l} \dfrac{(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2)\beta_1\beta_2\theta_1\theta_2 q - \beta_1\beta_2\theta_1\theta_2 q B^*}{(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2)^2} + \\ \dfrac{(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 - \pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2)^2} - \alpha_1 \end{array}\right\}}}{\left(\beta_2\theta_2 B^* - \alpha_2\right)}$$

$$\frac{dZ^*}{dq} > 0$$

The change is positive here.

Therefore, when the rate of discharge of nutrients from outside (q) increases, there will be an increasing behavior in zooplankton.

Again, using equation (3.14), we get,

$$\pi_1\alpha_1 B^* + \pi_2\alpha_2 Z^* = \delta S^*$$

$$\Rightarrow \pi_1\alpha_1 \frac{\dfrac{\beta_1\theta_1\beta_2\theta_2 B^*}{\alpha_0\beta_2\theta_1 - \beta_1\alpha_2}}{\left\{\begin{array}{l} \dfrac{(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2)\beta_1\beta_2\theta_1\theta_2 q - \beta_1\beta_2\theta_1\theta_2 q B^*}{(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2)^2} + \\ \dfrac{(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 - \pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2)^2} - \alpha_1 \end{array}\right\}}$$

$$+\pi_2\alpha_2 \frac{\dfrac{\beta_1\theta_1\beta_2\theta_2 B^*}{\alpha_0\beta_2\theta_1 - \beta_1\alpha_2}}{\left(\beta_2\theta_2 B^* - \alpha_2\right)\left\{\begin{array}{l} \dfrac{(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2)\beta_1\beta_2\theta_1\theta_2 q - \beta_1\beta_2\theta_1\theta_2 q B^*}{(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2)^2} + \\ \dfrac{(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 - \pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2)^2} - \alpha_1 \end{array}\right\}} = \delta \frac{dS^*}{d\eta}$$

$$\Rightarrow \frac{dS^*}{dq} = \frac{\left(\beta_2\theta_2 B^* - \alpha_2\right)\frac{\pi_1\alpha_1\beta_1\theta_1\beta_2\theta_2 B^*}{\alpha_0\beta_2\theta_1 - \beta_1\alpha_2} + \frac{\pi_2\alpha_2\beta_1\theta_1\beta_2\theta_2 B^*}{\alpha_0\beta_2\theta_1 - \beta_1\alpha_2}}{\delta\left(\beta_2\theta_2 B^* - \alpha_2\right)\left\{\begin{array}{l}\frac{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)\beta_1\beta_2\theta_1\theta_2 q - \beta_1\beta_2\theta_1\theta_2 q B^*}{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)^2} + \\ \frac{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 - \pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)^2} - \alpha_1\end{array}\right\}}$$

$$\Rightarrow \frac{dS^*}{dq} > 0$$

Therefore, when the rate of discharge of nutrients from outside (q) increases, detritus amount increases.

From equation (3.11),

$$q + \pi_0\delta S^* = \alpha_0 T^* + \beta_1 T^* B^*$$

$$\Rightarrow \pi_0\delta\frac{dS^*}{dq} = \alpha_0\frac{dT^*}{dq} + \beta_1 T^*\frac{dB^*}{dq} + \beta_1 B^*\frac{dT^*}{dq}$$

$$\Rightarrow \frac{dT^*}{dq} = \frac{\frac{\pi_0\pi_1\alpha_1\beta_1\theta_1\beta_2\theta_2 B^*}{\alpha_0\beta_2\theta_1 - \beta_1\alpha_2} + \frac{\pi_0\pi_2\alpha_2\beta_1\theta_1\beta_2\theta_2 B^*}{\left(\beta_2\theta_2 B^* - \alpha_2\right)\alpha_0\beta_2\theta_1 - \beta_1\alpha_2} + \delta\frac{\beta_1 T^*\beta_1\theta_1\beta_2\theta_2 B^*}{\alpha_0\beta_2\theta_1 - \beta_1\alpha_2}}{\delta\left(\alpha_0 + \beta_1 B^*\right)\left\{\begin{array}{l}\frac{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)\beta_1\beta_2\theta_1\theta_2 q - \beta_1\beta_2\theta_1\theta_2 q B^*}{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)^2} + \\ \frac{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 - \pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)^2} - \alpha_1\end{array}\right\}}$$

$$\therefore \frac{dT^*}{dq} > 0$$

Here, the change is positive which shows a proportional relation.

Therefore, when the rate of discharge of organic pollutant from outside (q) increases, it is obvious that density of organic pollutant will also increases.

ii. Characteristics of the concentration of dissolved oxygen $\left(C^*\right)$, the density of algae $\left(B^*\right)$ with respect to η (depletion rate of dissolved oxygen due to algae)

Substituting the values of T^* , Z^* and S^* in the model (3.1) – (3.5) we have

$$j\left(C^*, B^*, \eta\right) = q_c - \alpha_3 C^* - \eta B^* - \frac{\delta_1\pi_1\alpha_1 B^*}{\delta}$$

$$k\left(C^*,B^*,\eta\right)=\frac{\beta_1\theta_1 q\beta_2\theta_2 B^*}{\alpha_0\beta_2\theta_1-\beta_1\alpha_2}+\frac{\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{\alpha_0\beta_2\theta_1-\beta_1\alpha_2}-\alpha_1 B^*$$

$$\therefore\ \frac{dC^*}{d\eta}=\frac{\begin{vmatrix}\frac{\partial j}{\partial B^*} & \frac{\partial j}{\partial \eta}\\ \frac{\partial k}{\partial B^*} & \frac{\partial k}{\partial \eta}\end{vmatrix}}{\begin{vmatrix}\frac{\partial j}{\partial C^*} & \frac{\partial j}{\partial B^*}\\ \frac{\partial k}{\partial C^*} & \frac{\partial k}{\partial B^*}\end{vmatrix}}=\frac{\frac{\partial j}{\partial B^*}\cdot\frac{\partial k}{\partial \eta}-\frac{\partial j}{\partial \eta}\cdot\frac{\partial k}{\partial B^*}}{\frac{\partial j}{\partial C^*}\cdot\frac{\partial k}{\partial B^*}-\frac{\partial j}{\partial B^*}\cdot\frac{\partial k}{\partial C^*}}$$

Here,

$$\frac{\partial j}{\partial B^*}=-\eta-\frac{\delta_1\pi_1\alpha_1}{\delta},\frac{\partial j}{\partial C^*}=-\alpha_3,\frac{\partial k}{\partial C^*}=0,\frac{\partial j}{\partial \eta}=-B^*,\frac{\partial k}{\partial \eta}=0,$$

$$\frac{\partial k}{\partial B^*}=\frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\beta_1\beta_2\theta_1\theta_2 q-\beta_1\beta_2\theta_1\theta_2 qB^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}+\frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1-\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}-\alpha_1$$

Now,

$$\frac{\partial j}{\partial B^*}\cdot\frac{\partial k}{\partial \eta}-\frac{\partial j}{\partial \eta}\cdot\frac{\partial k}{\partial B^*}=B^*\left\{\begin{array}{l}\frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\beta_1\beta_2\theta_1\theta_2 q-\beta_1\beta_2\theta_1\theta_2 qB^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}+\\ \frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1-\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}-\alpha_1\end{array}\right\}$$

$$\frac{\partial j}{\partial C^*}\cdot\frac{\partial k}{\partial B^*}-\frac{\partial j}{\partial B^*}\cdot\frac{\partial k}{\partial C^*}=-\alpha_3\left\{\begin{array}{l}\frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\beta_1\beta_2\theta_1\theta_2 q-\beta_1\beta_2\theta_1\theta_2 qB^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}+\\ \frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1-\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}-\alpha_1\end{array}\right\}$$

$$\therefore\ \frac{dC^*}{d\eta}=\frac{B^*\left\{\begin{array}{l}\frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\beta_1\beta_2\theta_1\theta_2 q-\beta_1\beta_2\theta_1\theta_2 qB^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}+\\ \frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1-\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}-\alpha_1\end{array}\right\}}{-\alpha_3\left\{\begin{array}{l}\frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\beta_1\beta_2\theta_1\theta_2 q-\beta_1\beta_2\theta_1\theta_2 qB^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}+\\ \frac{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1-\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{\left(\alpha_0\beta_2\theta_2-\beta_1\alpha_2\right)^2}-\alpha_1\end{array}\right\}}$$

$$= -\frac{B^*}{\alpha_3}$$

$$\therefore \frac{dC^*}{d\eta} = -\frac{B^*}{\alpha_3} < 0$$

The change in dissolved oxygen decreases again.

Therefore, when the depletion rate of dissolved oxygen due to algae (η) increases, the concentration of dissolved oxygen also decreases.

$$\text{Again, } \frac{dB^*}{d\eta} = \frac{\begin{vmatrix} \frac{\partial j}{\partial \eta} & \frac{\partial j}{\partial C^*} \\ \frac{\partial k}{\partial \eta} & \frac{\partial k}{\partial C^*} \end{vmatrix}}{\begin{vmatrix} \frac{\partial j}{\partial C^*} & \frac{\partial j}{\partial B^*} \\ \frac{\partial k}{\partial C^*} & \frac{\partial k}{\partial B^*} \end{vmatrix}} = \frac{\frac{\partial j}{\partial C^*}\cdot\frac{\partial k}{\partial \eta} - \frac{\partial j}{\partial \eta}\cdot\frac{\partial k}{\partial C^*}}{\frac{\partial j}{\partial C^*}\cdot\frac{\partial k}{\partial B^*} - \frac{\partial j}{\partial B^*}\cdot\frac{\partial k}{\partial C^*}}$$

Now,

$$\frac{\partial j}{\partial C^*}\cdot\frac{\partial k}{\partial \eta} - \frac{\partial j}{\partial \eta}\cdot\frac{\partial k}{\partial C^*} = 0.\left(-B^*\right) + 0.\left(-\alpha_3\right) = 0$$

$$\frac{\partial j}{\partial C^*}\cdot\frac{\partial k}{\partial B^*} - \frac{\partial j}{\partial B^*}\cdot\frac{\partial k}{\partial C^*} = -\alpha_3\left\{\begin{array}{l} \frac{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)\beta_1\beta_2\theta_1\theta_2 q - \beta_1\beta_2\theta_1\theta_2 q B^*}{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)^2} + \\ \frac{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 - \pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)^2} - \alpha_1 \end{array}\right\}$$

$$\therefore \frac{dB^*}{d\eta} = \frac{0}{-\alpha_3\left\{\begin{array}{l} \frac{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)\beta_1\beta_2\theta_1\theta_2 q - \beta_1\beta_2\theta_1\theta_2 q B^*}{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)^2} + \\ \frac{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)\pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 - \pi_0\pi_1\alpha_1\alpha_2\theta_1\beta_1 B^*}{\left(\alpha_0\beta_2\theta_2 - \beta_1\alpha_2\right)^2} - \alpha_1 \end{array}\right\}}$$

$$\Rightarrow \frac{dB^*}{d\eta} = 0$$

There is no change of the density of algae.

Therefore, when the Depletion rate of dissolved oxygen due to algae η increases or decreases, there will be no change in the density of algae.

Now we get from equation (3.13),

$$\theta_2\beta_2 B^* \frac{dZ^*}{d\eta} + \theta_2\beta_2 Z^* \frac{dB^*}{d\eta} - \alpha_2 \frac{dZ^*}{d\eta} = 0$$

$$\frac{dZ^*}{d\eta} = 0 \; [\because \frac{dB^*}{d\eta} = 0]$$

There is no change of zooplankton.

Therefore, when the depletion rate of dissolved oxygen due to algae (η) increases, there would be no change in zooplankton.

Again, from equation (3.13), we get,

$$\pi_1\alpha_1 B^* + \pi_2\alpha_2 Z^* = \delta S^*$$

$$\Rightarrow \pi_1\alpha_1 \frac{dB^*}{d\eta} + \pi_2\alpha_2 \frac{dZ^*}{d\eta} = \delta \frac{dS^*}{d\eta}$$

$$\Rightarrow \frac{dS^*}{d\eta} = 0 \;\; [\because \frac{dB^*}{d\eta} = 0]$$

There is no change of detritus.

Therefore, when the depletion rate of dissolved oxygen due to algae (η) increases, there would be no change in detritus.

From equation (3.11),

$$q + \pi_0\delta S^* = \alpha_0 T^* + \beta_1 T^* B^*$$

$$\Rightarrow \pi_0\delta \frac{dS^*}{d\eta} = \alpha_0 \frac{dT^*}{d\eta} + \beta_1 T^* \frac{dB^*}{d\eta} + \beta_1 B^* \frac{dT^*}{d\eta}$$

$$\Rightarrow \frac{dT^*}{d\eta} = 0$$

There is no change of organic pollutant.

Therefore, when the depletion rate of dissolved oxygen due to algae (η) increases, there would be no change in organic pollutant.

CHAPTER FOUR: NUMERICAL ANALYSIS

4.1 Numerical Results and Discussions

We have numerically solved the model and discussed the stability of the model for different values of parameters. A logical set of parameter values is used from table 4.1 which is related to the work of Misra [7] related to our work. The initial value of cumulative concentration of nutrients (T) is 1 mg/litre, the density of algae (B) is 80 million/litre which is considered as 1 unit, the density of detritus (S) is 1 mg/litre, the density of zooplankton (Z) is 1 mg/litre and the concentration of dissolved oxygen (C) is 15 mg/litre are considered according to the work of Misra [7]. Matlab (R2021a) has been used to find the result of the model by Runge-Kutta method. The following table contains the values of the parameters.

Table 4.1: Parameter values and their descriptions

Descriptions of the parameter	**Notations**	**Values**
Cumulative rate of discharge of nutrients	q	0.5 $mgl^{-1}day^{-1}$
Natural Depletion rate of nutrients	α_0	0.005 day^{-1}
Natural depletion rate of algae	α_1	0.025 day^{-1}
Natural Depletion rate of Zooplankton	α_2	0.02 day^{-1}
Natural Depletion rate of dissolved oxygen	α_3	0.01 day^{-1}
Depletion rate of nutrients due to algae	β_1	0.025 $mg^{-1}day^{-1}$
Depletion rate of Algae due to Zooplankton	β_2	0.025 $mg^{-1}day^{-1}$
Growth rate of nutrients due to detritus	π_0	0.02
Growth rate of detritus due to algae	π_1	0.9
Growth rate of detritus due to Zooplankton	π_2	0.9
Depletion rate of detritus due to decomposing	δ	0.04 day^{-1}
Depletion rate of dissolved oxygen due to detritus	δ_1	0.06 day^{-1}

Growth rate of algae due to nutrients	θ_1	0.9
Growth rate of Zooplankton due to Algae	θ_2	0.9
Depletion rate of dissolved oxygen due to algae	η	0.02 day^{-1}
Increasing rate of dissolved oxygen by various sources	q_c	0.2 mg l^{-1}day^{-1}

While calculating the numerical result we considered the time period t from 0 to 60 days. Because we find the changes from the increasing the concentration of any individuals till the time of their depletion.

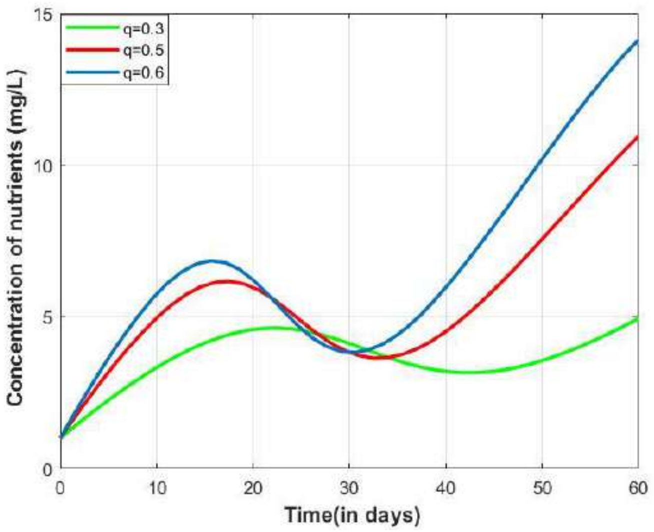

Figure 4.1: Behavior of the concentration of nutrients with time for cumulative rate of discharge of nutrients, q

The fluctuations depicted in figure 4.1 are driven by a multitude of factors intrinsic to our ecosystem model. Of all the components in this model, algae emerge as particularly significant due to the crucial role they play within the ecosystem dynamics.

Algae possess the unique ability to metabolize nutrients, a process which effectively reduces the concentration of these pollutants in the water. This contributes to the downward trend we see on the graph, a phenomenon observable when employing specific values for our calculations: $q = 0.3, q = 0.5$ and $q = 0.6$.

As the rate of organic pollutant discharge into the water body rises, we initially witness an increase in the overall pollutant concentration. This escalation in pollutant levels, as indicated by the figure, persists for approximately 20 days. However, beyond this period, the concentration starts to decline, a trend triggered by the action of algae.

When sufficient oxygen is present in the water, the growth rate of algae amplifies. These rapidly growing algae colonies then consume greater volumes of nutrients. In the case where the pollutant discharge rate is at q=0.3, the pollutant concentration ascends to a peak of 6 mg/L. Post this peak, the concentration commences its descent, marking the onset of the pollution reduction phase.

Similarly, at a higher discharge rate of q=0.6, the pollutant concentration soars to an even higher level of 9 mg/L. Following this surge, the graph eventually starts to display a declining trend, once again attributable to the increase in algae. This underlines the significant role of algae in regulating pollutant levels. As the algae population escalates over time, it progressively consumes more pollutants, thereby catalyzing the decline in their overall concentration.

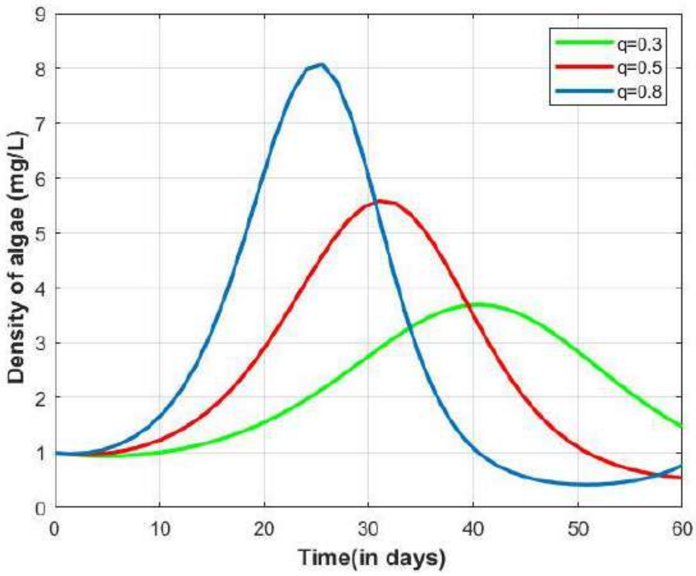

Figure 4.2: Behavior of the density of algae with time for cumulative rate of discharge of nutrients, q

The rise and fall pattern of nutrients observed in figure 4.1 is closely tied to the density of algae in the ecosystem. We have already examined how increasing amounts of algae contribute to a decrease in nutrients. However, the growth in algae also leads to changes in the population of zooplankton, another crucial component of the ecosystem.

Zooplankton play a significant role in maintaining the health of the ecosystem. They contribute to the purification of water by consuming nutrients, thereby slightly enhancing the water's clarity. In our study, we considered the values $q = 0.3, q = 0.5$ and $q = 0.8$ and figure 4.2 illustrates an initial rise in the graph (as observed).

As the cumulative rate of discharge of nutrients into the water body grows, so does the population of algae, but only up to a certain point. Following this phase of growth, the algae population begins to recede. According to the data presented in Figure 4.4, the duration of this growth phase lasts for 42 days for q=0.3, 32 days for q=0.5, and 25 days for q=0.8. Post these periods, the algae population declines, an event driven by the presence of zooplankton.

With a pollutant discharge rate of q=0.3, we witness a threefold increase in the algae population. As the value of q changes, the algae population also increases gradually. However, after a specific time interval, the graph begins to display this cyclical pattern of increase and decrease. This dynamic shift suggests a complex interplay between the discharge of nutrients, the population of algae, and the population of zooplankton within the ecosystem.

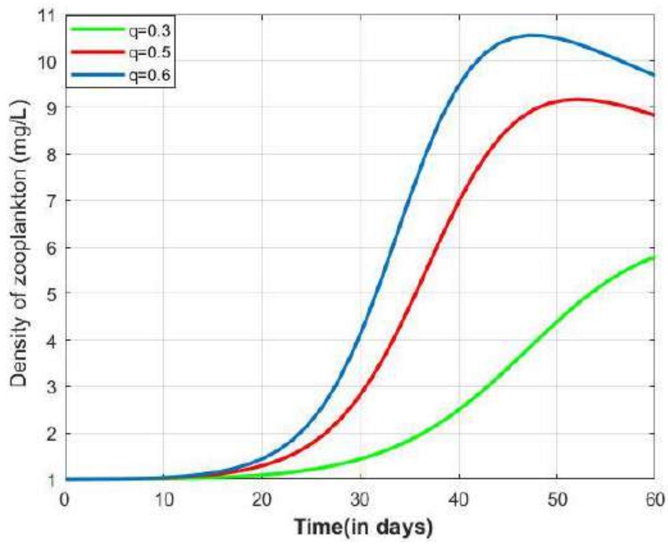

Figure 4.3: Increasing behavior of zooplankton with time for cumulative rate of discharge of nutrients, q

Figures 4.1 and 4.2, as discussed earlier, illustrate the fluctuations in nutrients and algae. However, while these changes are occurring, there's also an increase in the population of zooplankton. The behavior of zooplankton is shown in figure 4.3 and this particular pattern emerges because zooplankton consume algae.

For our calculations, we considered the values $q = 0.3, q = 0.5$ and $q = 0.6$. These values, when analyzed, cause an increase in the population of zooplankton. As the cumulative rate of discharge of nutrients into the water body escalates, so does the population of zooplankton. This relationship suggests that zooplankton growth is somehow linked to the discharge of nutrients.

The growth rate of zooplankton, however, shows two distinct phases. For the initial 30 days, the growth of zooplankton is relatively slow, suggesting a period of modest population increase. However, after the 30-day mark, we notice a significant acceleration in zooplankton growth. Despite this overall increase, there may be minor reductions in zooplankton density due to various influencing factors not explicitly accounted for in this model.

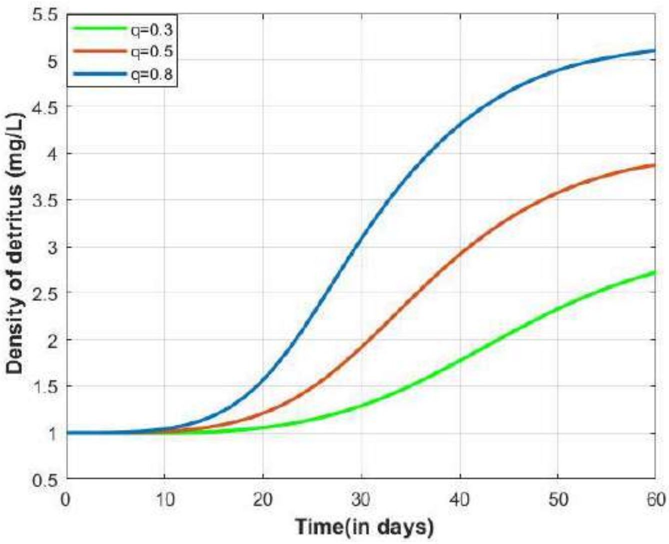

Figure 4.4: Behavior of the density of detritus with time for cumulative rate of discharge of nutrients, q

Detritus refers to dead organic material, including dead algae, zooplankton, and solid waste from nutrients. This encompasses all the dead biological material as well as waste products depicted in figures 4.1, 4.2, and 4.3.

The density of detritus escalates rapidly due to the accumulation of deceased plant matter and micro-organisms. This trend is largely driven by the consumption of algae by zooplankton, which in turn results in a larger quantity of dead organic matter.

Figure 4.4 provides a visual representation of these changes. The calculations made using the values $q = 0.3, q = 0.5$, and $q = 0.8$ are reflected in this figure. As the cumulative rate of discharge of nutrients into the water body intensifies, there's a corresponding increase in the amount of detritus.

The growth rate of detritus follows a similar trend to that observed in zooplankton. For the first 20 days, the increase in detritus is relatively modest. However, post the 20-day mark, the accumulation of detritus begins to accelerate significantly. This increase is directly related to the escalating discharge of nutrients and the subsequent increase in zooplankton and algae deaths, contributing to a larger volume of detritus in the ecosystem.

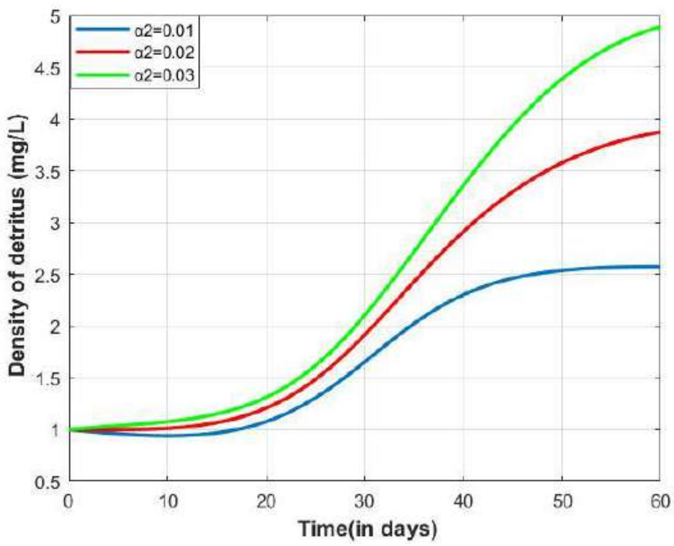

Figure 4.5: Behavior of the density of detritus with time for depletion rate of zooplankton, α_2

Figure 4.5 explains the behavior the density of detritus for different values of α_2. When the zooplankton die, they start to decompose and accumulates as detritus. If the depletion rate of zooplankton increases, the density of detritus will increase with time as well. As we took the values $\alpha_2 = 0.01$, $\alpha_2 = 0.02$ and $\alpha_2 = 0.03$, an increment is seen in the graph. In the above figure, we can see that the density of detritus started to increase after a certain period of time, and the increment is proportional to the value of α_2.

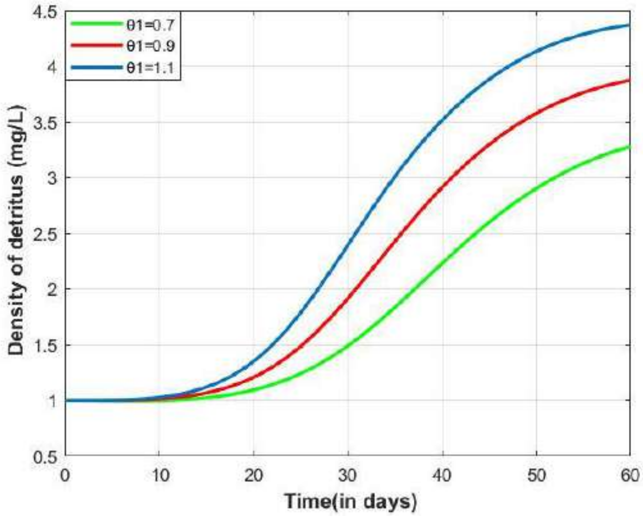

Figure 4.6: Behavior of the density of detritus with time for growth rate of algae due to nutrients, θ_1

The above figure describes the behavior of the density of detritus with respect to time for different values of the growth rate of algae due to nutrients θ_1. We took the values of θ_1 as 0.7, 0.9, and 1.1 respectively.

It is clearly visible that the density of detritus increases rapidly after a certain of time. If the algae grow faster, its population will increase rapidly. With the increasing population it is certain that the depletion rate of algae will also increase proportionally, which contributes directly to increase the density of algae.

That is why, with the increasing values of θ_1, the density of algae increases rapidly.

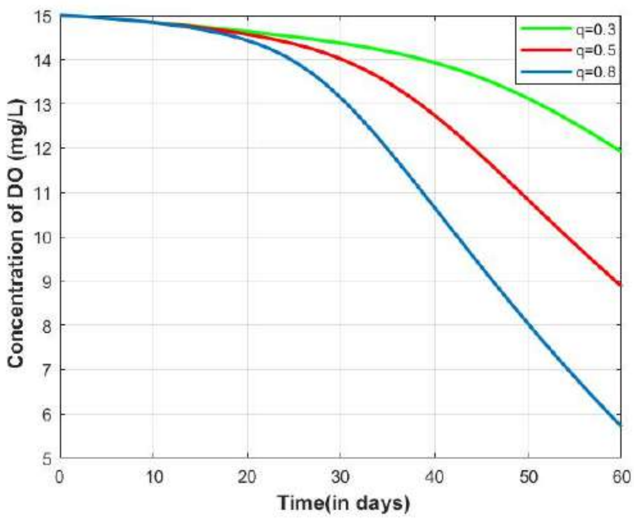

Figure 4.7: Behavior of the concentration of dissolved oxygen with time for cumulative rate of discharge of nutrients, q

In our numerical computation, we chose a time period ranging from 0 to 60 days. We focused on this span because it allowed us to observe the changes arising from increases in individual concentrations until the point of depletion. Figure 4.7 presents the fluctuations in the concentration of dissolved oxygen under various cumulative discharge rates of nutrients, mapped against time (t). We used a time interval of 60 days to evaluate the graph's behavior.

In our analysis, we used the values $q = 0.3, q = 0.5$ and $q = 0.8$. The graphical representation showed a decrease when these values were plotted. Over time, the level of dissolved oxygen showed a consistent decline. This decline was relatively subtle at first, but once the cumulative discharge rate of nutrients increased, the oxygen level decreased slightly.

However, a striking change was noted after the 20-day mark, where the decrease in the concentration of dissolved oxygen (DO) accelerated rapidly. This indicates that the increasing discharge rate of nutrients significantly impacts the level of dissolved oxygen, highlighting a rapid depletion of DO after 20 days of increased pollutant discharge.

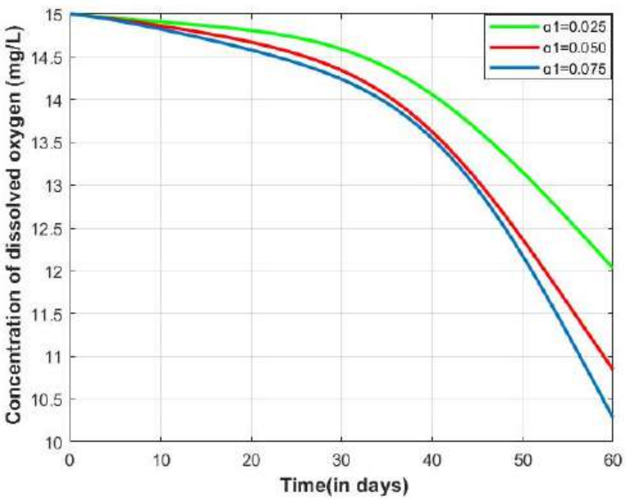

Figure 4.8: Behavior of the concentration of dissolved oxygen with time for the depletion rate of algae, α_1

Figure 4.8 provides a detailed analysis of the changes in the concentration of dissolved oxygen over time, mapped against different rates of algae depletion. The rates in question, represented by the values of α_1 are 0.025, 0.050, and 0.075 respectively.

A key finding is that the concentration of dissolved oxygen exhibits a decreasing trend with the increase in these α_1 values. This can be explained by the fact that higher rates of algae depletion result in an increased amount of detritus, and consequently, nutrients. This increase promotes an algal bloom. Thus, the underlying reason for the decline in the concentration of dissolved oxygen is the corresponding increase in the α_1 values, which enhance the nutrients leading to the algal bloom.

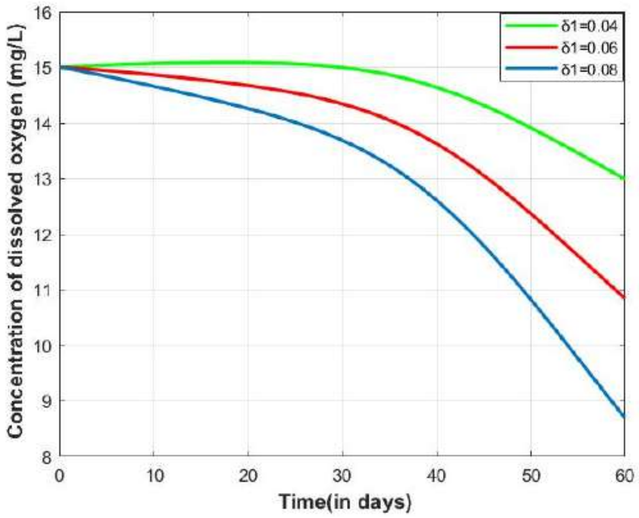

Figure 4.9: Behavior of the concentration of dissolved oxygen with time for depletion rate of dissolved oxygen due to detritus, δ_1

Figure 4.9 presents a comprehensive analysis of how the concentration of dissolved oxygen fluctuates over time based on different rates of oxygen depletion due to detritus, represented by the parameter δ_1. The values for δ_1 being investigated are 0.04, 0.06, and 0.08 respectively. An important aspect to note here is that a surge in the quantity of detritus necessitates a correspondingly larger amount of oxygen for decomposition. This process can lead a lake to become anoxic. Therefore, the more detritus present, the more oxygen will be consumed for its breakdown.

As a result, it is observed that with the escalation of δ_1 values, the concentration of dissolved oxygen experiences a significant decline. This behavior is due to the higher consumption of oxygen required for the decomposition of increasing detritus levels.

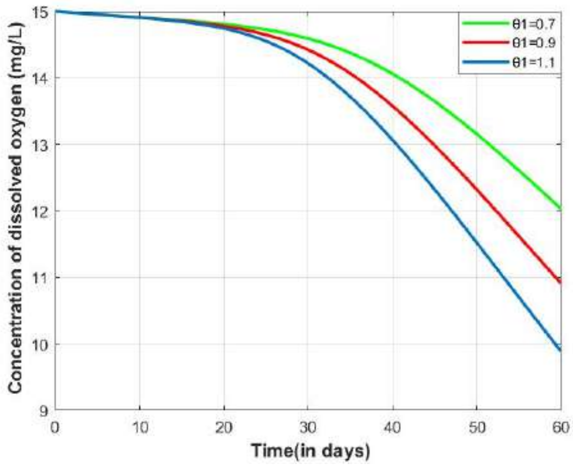

Figure 4.10: Behavior of the concentration of dissolved oxygen with time for growth rate of algae due to nutrients, θ_1

Figure 4.10 illustrates the fluctuation in the concentration of dissolved oxygen over time, corresponding to varying values of the growth rate of algae due to nutrients, denoted by θ_1. The specific θ_1 values under examination are 0.7, 0.9, and 1.1. It is observed that as the value of θ_1, indicative of the algae growth rate driven by nutrients, increases, it results in an algal bloom. This bloom subsequently causes a rapid decline in the concentration of dissolved oxygen due to the ensuing rise in detritus. This increase in detritus necessitates more oxygen for decomposition.

In conclusion, the reduction in the concentration of dissolved oxygen is directly attributed to the increased growth rate of algae due to nutrients. This is primarily because the higher volume of detritus, created by the algal bloom, consumes more oxygen for decomposition.

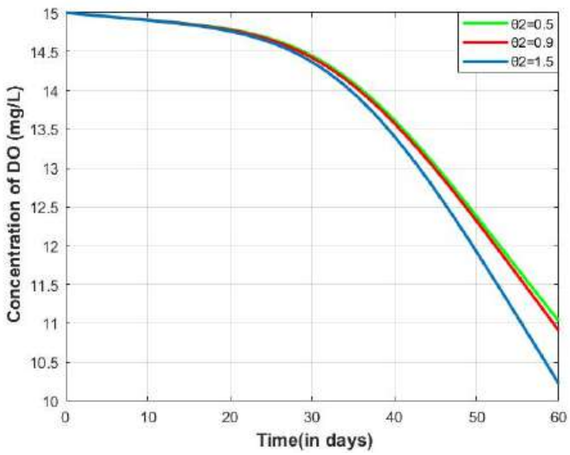

Figure 4.11: Behavior of the concentration of dissolved oxygen with time for growth rate of zooplankton due to algae, θ_2

Figure 4.11 presents an in-depth analysis of how the concentration of dissolved oxygen varies over time, with respect to the growth rate of zooplankton due to algae, represented by the parameter θ_2. The values of θ_2 considered for this study are 0.5, 0.9, and 1.5 respectively. Algae have a crucial role in promoting the population of zooplankton, mainly because zooplankton primarily feed on algae, resulting in an increased population. As the zooplankton population expands, so does their consumption of oxygen for survival. Concurrently, a larger zooplankton population also results in a higher number of deceased zooplankton. The decomposition process for these large quantities of deceased zooplankton consumes significant amounts of dissolved oxygen, leading to a drop in its overall concentration.

Hence, it is evident that the concentration of dissolved oxygen decreases in response to an increase in the growth rate of zooplankton, attributed to algae. This is mainly due to the increased oxygen demand for the decomposition of a larger number of deceased zooplankton.

CHAPTER FIVE: CONCLUSION AND FUTURE STUDIES

5.1 Conclusion

The study analyzed a nonlinear mathematical model considering the concentration of nutrients, density of algae, density of zooplankton, density of zooplankton, and concentration of dissolved oxygen as state variables. The positivity and boundedness of the state variables have been analyzed to validate the model. The equilibrium points of the model and stability of the points have been discussed elaborately as well. Finally, the behavior of the state variables with respect to different values of different parameters have been demonstrated analytically and graphically.

In conclusion, the study applying numerical methods to solve the model demonstrated the findings listed below:

- The study demonstrated significant insights into the interplay of various components within an aquatic ecosystem, including algae, zooplankton, organic pollutants, and dissolved oxygen.
- An increase in the discharge of organic pollutants into a water body over time leads to a decline in dissolved oxygen levels, due to the pollutants' effect on the ecosystem.
- Algae in the ecosystem help reduce the concentration of these pollutants, and thus are essential to preserving the health of the ecosystem. Notably, a pattern of initial growth and then decline in the algae population over time was observed. This pattern was driven by the increase in organic pollutants and the subsequent rise in zooplankton, which consume the algae.
- The study also emphasized the role of zooplankton, showing that their population grows over time as a direct response to the increase in algae, their primary food source. This growth, in turn, results in a significant increase in detritus, or dead organic matter.
- A strong correlation between the rates of algae and zooplankton depletion and the overall concentration of dissolved oxygen was noted. A higher rate of algae and zooplankton depletion resulted in more detritus, requiring more oxygen for decomposition, and thus decreasing the dissolved oxygen levels in the water body.

In summary, our study underscores the delicate balance within an aquatic ecosystem, highlighting the intertwined relationships between organic pollutants, algae, zooplankton, detritus, and dissolved oxygen. Our findings stress the importance of understanding these dynamics in order to effectively manage and preserve the health of our aquatic ecosystems.

5.2 Future Studies

Our mathematical analysis has been limited to the variables cumulative concentration of nutrients, density of algae, density of zooplankton, density of detritus, and concentration of dissolved oxygen. Bacteria is an also a vital component of the ecosystem. Bacteria decomposes dead algae and zooplankton, which requires a significant amount of dissolved oxygen. This causes a depletion in the concentration of dissolved oxygen in aquatic ecosystem.

For further research purposes the followings can be implemented:

- The density of bacteria can be considered as a state variable to better analyze the depletion of dissolved oxygen.
- A control model can be proposed implementing some treatment measures to solve the issues caused by algal bloom and to prevent this depletion of dissolved oxygen.

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