INVESTIGATION AND ANALYSIS OF AN ARBITRARILY SHAPED DIELECTRIC OBSTACLE IN A RECTANGULAR WAVEGUIDE.

A Thesis

Presented to the Department of Electrical Engineering, Bangladesh University of Engineering and Technology, Dacca in partial fulfilment of the requirements for the Degree of Master of Science in Electrical Engineering.

by

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1979.

CERTIFICATE

This is to certify that this work has been done by me and it has not been submitted elsewhere for the award of any degree or diploma or for publication.

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ABSTRACT

The development of the dominant mode equivalent circuit parameters for a three-dimensional dielectric obstacle inside a rectangular waveguide requires solution of an intergral equation containing a suitable dyadic Green's function. The derivation of the integral equation involves conversion of a two-region scattering problem into two one-region scattering problems by utilizing the linearity property of Maxwell's curl equations. The point matching moment method with a pulse function type subsectional basis vectors has been used for finding the secondary volume current density at the centres of the small rectangularly shaped cells formed by subdividing the dielectric obstacle. Subsequently values of these secondary current density are used in the mathematical formulations for obtaining the dominant mode equivalent circuit parameters. An illustrative example has been solved using a Fortran IV language program. Some modified methods regarding the solution of the problem have been presented.

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Contd....

CHAPTER-I

INTRODUCTION

On the analysis of an arbitrarily shaped dielectric obstacle in a rectangular waveguide.

The effects of obstacles and discontinuities on electromagnetic fields in a waveguide have been outstanding problems for a long time. Many of them, essentially two-dimensional, have been solved and were summarized by Marcuvitz (1). The general three-dimensional obstacle-discontinuity problem has remained virtually unsolved inspite of the advent of high speed digital computers and the method of moment, introduced in E.M.Field theory by Harrington (2), which permits dealing with problems not solvable by exact method.

The lack of published research activities on three-dimensional waveguide discontinuities has been mainly due to non-availability of Green's Functions in the waveguide region. A dyadic Green's function for rectangular waveguides based on the use of eigenvector functions M and N was presented by Tai(3) and it was later revised by the same author (4). Collin (5) has discussed the question of incompleteness of the E and H modes in the source region of a waveguide and has shown that an additional term must be added to the classical representation of the E.M.field in order to derive a complete solution that is valid both in the source and source free region. Sami (6) has recently presented a method of deriving the Dyadic Green's Function for rectangular waveguides and cavities using the theory of distribution and has shown that if one carefully defines the derivatives in the distribution sense and applies the correct completeness property of the modes, it is then possible to construct the complete solution of the entire structure just by employing the scalar eigenfunctions of Helmholtz equation and has pointed out that this procedure may also be used to determine the complete form of the Dyadic Green's Functions for non-rectangular waveguides and cavities.

Recently Wang (7) has presented a method of analyzing a three-dimensional dielectric obstacle in a rectangular waveguide by applying moment method to an integral equation involving Dyadic Green's function and has pointed out that this technique could be extended to ferromagnetic obstacles and for highly conductive obstacles a surface type Green's Function may be more desirable.

The purpose of this Thesis work is to develop a method for finding the dominant mode equivalent circuit parameters for a dielectric obstacle discontinuity inside a rectangular waveguide. The method involves solution of an integral equation involving a dyadic Green's function by moment method for obtaining the secondary current induced in the dielectric obstacle and subsequent use of these induced secondary current values to find the dominant mode equivalent circuit parameters.

The method can be extended for finding the multimode equivalent circuit parameter matrix. However, it should be noted that waveguides operate usually in a narrow band near the dominant mode and dominant mode equivalent circuit is the most usual representation for a discontinuity in a waveguide.

In course of this investigation a modified method of deriving the dyadic Green's function for a dielectric obstacle in a rectangular waveguide has been developed. A modified method of summing the double infinite series for evaluating the elements of the matrix resulting from the matrization of the relevant dyadic integral equation by moment method has also been suggested. A modified method of moment for solving the problem has also been presented.

CHAPTER-II.

FORMULATION OF THE PROBLEM.

Art-1: The Integral Equation involving the Dyadic Green's Function for a 3-dimensional arbitrarily shaped dielectric obstacle in a rectangular waveguide.

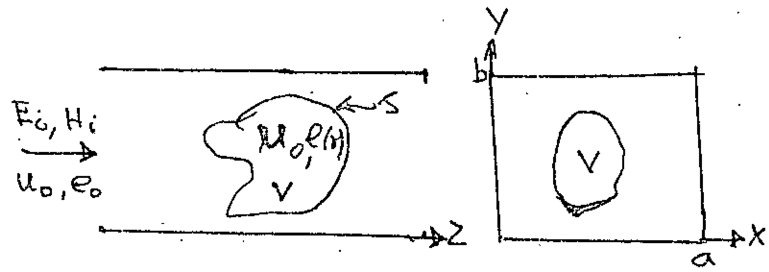

Fig-1. A 3-dimensional arbitrarily shaped dielectric body illuminated with incident field E_i,H_i(with propagation factor e^{jwt} implicit) travelling toward +z direction.

The problem to be considered is depicted in Fig. 1. A three-dimensional arbitrarily shaped heterogeneous body of volume V enclosed by surface S is illuminated with electromagnetic field E_i,H_i incident towards +z direction. At any position vector r the obstacle body has permittivity e(r) and permeability u_o. Outside V the medium is homogeneous with permittivity e_o and permeability u_o.

The object occupying the volume V may be regarded as disturbing or scattering the field that would exist if the object were not present, that is, if all the space had the properties:admittivity Y_o= jwe_o, and impedivity Z_o = jwu_o.

With the scattering object present the total fields E & H satisfy Maxwell's equations in the form:

Outside V

$$\nabla X H = Y_o E + J_o \qquad \nabla X E = -Z_o H \qquad (1)$$

Inside V

$$\nabla X H = Y_1 E = Y_o E + J \qquad \nabla X E = Z_o H = -Z_o H \qquad (2)$$

where J_o is the impresed source outside V, and $J = (Y_1 - Y_o)E$, wherein $Y_1 = jwe(r)$, is the induced source throughout V. $Y_1 = jwe(r)$ and $Z_o = jwu_o$ are the admittivity & impedivity, respectively, of the dielectric in V. Since Equations(1) & (2) are of the same form they may be written together, and applied to all space both inside and outside V, as follows:-

$$\nabla X H = Y_o E + J_o + J \qquad \nabla X E = -Z_o H \qquad (3)$$

The linearity of Maxwall's equations can be utilized in solving the scattering problem under consideration. The consequance of adopting Equation(1),(2)& (3) is that the scattering problem has been replaced with a problem involving sources in homogeneous media. The total field quantities E and H may be broken into components whose equations are:-

In a medium with u_o, e_o,

$$\nabla X H_i = Y_o E_i + J_o - J \qquad \nabla X E_i = -Z_o H_i \qquad (4)$$

In a medium with $u_o, e(r)$

$$\nabla X H_i = Y_1 E_i - J \qquad \nabla X E_i = -Z_o H_i \qquad (6)$$

In a medium with u_o, e_o		In a medium with $u_o, e(r)$	
$\nabla X H_s = Y_o E_s + J$	(5a)	$\nabla X H_s = Y_1 E_s + J$	(7a)
$\nabla X E_s = -Z_o H_s$	(5b)	$\nabla X E_s = -Z_o H_s$	(7b)

Combining Eqns(4) & (5),

$$\left.\begin{aligned} \nabla X(H_i+H_s) &= Y_o(E_i+E_s) + J_o \\ \nabla X(E_i+E_s) &= -Z_o(H_i+H_s) \end{aligned}\right\} \quad (8)$$

Combining Eqns(6) & (7),

$$\left.\begin{aligned} \nabla X(H_i+H_s) &= Y_1(E_i+E_s) \\ \nabla X(E_i+E_s) &= -Z_o(H_i+H_s) \end{aligned}\right\} \quad (9)$$

Comparing Equation(1) with(8) and Equation(2) with (9)

We find:-

$E = (E_i+E_s)$, $H = (H_i+H_s)$ relations are applicable both inside and outside the dielectric obstacle in V. The advantage of decomposing E and H in this way is that the two-region problem is now tackled by two homogeneous-medium problems with E_s and H_s obtainable by solving similar equations whose Green's functions take into account guide boundary and J distribution conditions.

Taking curl of Eqn(5b) and then using Eqn(5a), we have,

$$\nabla X \nabla X E_s - K_o^2 E_s = -jwu_o J \qquad (10)$$

where $K_o^2 = w^2 u_o e_o$

Using electric type of Dyadic Green's function(to be derived)

$$\bar{\bar{G}}e(r,r') = \bar{\bar{G}}e_o(r,r') - \frac{\hat{z}\hat{z}d(r-r')}{K_o^2} \quad .. \quad .. \quad (11)$$

where $d(r-r')$ is the Dirac delta function, the Solution of Equation(10) is given by using Green's theorem or superposition principle:-

$$E_s(r) = -jwu_o \int_V \bar{\bar{G}}e\,(r,r').\,J(r')dv' \quad .. \quad .. \quad (12).$$

where r is the position vector, r' and dv' refer to the region in V.

Using Eqns(11) and (12) in the relation $E_i+E_s = E$ and using the relation $J=jw\,[e(r)-e_o]\,E$ we have,

$$-jwu_o\int_V \bar{\bar{G}}e_o(r,r')\cdot J(r')dv' - \frac{J(r)}{jw(e(r)-e_o)} - \frac{J_z(r)\bar{z}}{jw\,\rho_o} = -E_i(r)\quad ..(13).$$

where $J_z = J.\bar{z}$

Taking curl of Equation (7b) and then using (7a) we have,

$$\nabla X \nabla X E_s - K^2(r)\; E_s = -jwu_oJ \quad .. \quad .. \quad (14).$$

where $K^2(r) = w^2u_oe(r)$

In an analogous manner the Dyadic Green's function

$$\bar{\bar{G}}e'(r,r') = \bar{\bar{G}}e_o(r,r') - \frac{\bar{z}\,\bar{z}\; d\;(r-r')}{K^2(r)} \quad .. \quad .. \quad (15).$$

may be derived.

The solution to Equation (14) is given by using Green's Theorem or Superposition principle :-

$$E_s(r) = -jwu_o\int_V \bar{\bar{G}}e'(r,r')\cdot J(r')\;dv' \quad .. \quad .. \quad (16).$$

where r is the position vector of $E_s(r)$ & $J(r)$ in V, r' and dv' refer to $J(r')$ in dv' region in V.

Using Equation(15) and (16) in the relation $E_i+E_s=E$ and using the relation $J = jw\,[e(r)-e_o]E$ we have,

$$-jwu_o\int_V \bar{\bar{G}}e_o\;(r,r')\cdot J(r')dv' - \frac{J(r)}{jw(e(r)-e_o)} - \frac{J_z(r)\bar{z}}{jwe(r)} = -E_i(r)\quad ..\;(17).$$

where $J_z = J.\bar{z}$

Equation (17) is the integral Equation to be solved for J(r) by moment method for given $E_i(r)$.

Equation (17) is the integral Equation to be solved for J(r) by moment method for given $-E_i(r)$.

In the next article $\bar{\bar{G}}e(r,r')$ for a rectangular waveguide filled homogeneously with medium (u_o, e_o) and with current density distribution $J = jw(e(r)-e_o)E$ in volume V is derived.

Art-2: Derivation of Electric Type Dyadic Green's Function $\bar{\bar{G}}e(r,r')$:

In deriving an expression for $\bar{\bar{G}}e(r,r')$ we follow the procedure of Sami (6). The geometry of the problem is as shown in Fig.1. The waveguide with its dimensions a and b along x and y axes respectively and aligned along z-axis, is assumed to be of a perfect conductor and is excited by an electric current density distribution J contained in finite volume V.

The Maxwall's curl equations for the scattered field and the relevant boundary conditions are :-

$$\nabla X H_s = J + jwe_o E_s \qquad \text{in the waveguide.} \qquad \text{.. (1a)}$$
$$\nabla X E_s = -jwu_o H_s$$

$$\bar{\bar{n}} X E_s = 0 \qquad \text{On the wall} \qquad \text{.. (1b)}$$

where $\bar{\bar{n}}$ is an unit vector normal to the guide walls.

As already outlined this is equivalent to assuming the waveguide filled with a homogeneous isotropic medium with parameters e_o, u_o and representing the scattering effect of the dielectric with $e(r), u_o$ in V by a current density distribution $J = jw(e(r)-e_o)E$ wherein $E = E_i + E_s$.

In order to obtain the unique solution, the field components must also satisfy the Sommerfeld radiation condition along the z-axis:-

$$\lim_{Z\to\infty} z\left(\nabla X\left(\begin{matrix}E\\H\end{matrix}\right) \pm jk_o\bar{\bar{z}}\left(\begin{matrix}E\\H\end{matrix}\right)\right) = 0. \text{ (For a matched condition).}$$

for either E(with H absent) or H(with E absent) and $K_o^2 = w^2 u_o e_o$.

Using this and the Green's vector theorem identity we may have, $E_s=-jwu_o\int_V \bar{\bar{G}}e(r,r').\ J(r')dv'$ .. .. (2).

The Maxwell dot equation yields $\nabla.H=0$ in the waveguide (3a)

The relevant boundary condition is $\bar{n}.H=0$ on the wall (3b)

From the Curl equations:- $\nabla X\nabla XE_s-K_o^2E_s=-jwu_oJ$ (4a)

The relevant boundary condition is $\bar{n}XE_s=0$ (4b)

Also from the curl equation:- $\nabla X\nabla XH_s-K_o^2H_s=\nabla XJ$ (5a)

The relevant boundary conditions are $\bar{n}.H_s=0$, $\bar{n}X\nabla XH_s=0$ on the wall (5b)

In order to solve Equation (4) we introduce the electric type dyadic Green's function $\bar{\bar{G}}e(r,r')$ defined by

$$\nabla X\nabla X\bar{\bar{G}}e-K_o^2\bar{\bar{G}}e=\bar{\bar{I}}\ d(r-r') \text{ in the guide} \quad .. \quad (5c)$$

$$\bar{n}X\bar{\bar{G}}e = 0 \text{ on the wall.} \quad (5d)$$

where unit dyadic function $\bar{\bar{I}} = \bar{x}\bar{x} + \bar{y}\bar{y} + \bar{z}\bar{z}$.

$\bar{\bar{G}}e(r,r')$ is the response at r due to a concentrated current density $\bar{\bar{I}}d(r-r')$ at r'.

The magnetic dyadic Green's function $\bar{\bar{G}}_m(r,r')$ corresponding to Eqns (5a) and (5b) is defined by :-

$$\nabla X\nabla X\bar{\bar{G}}_m-K_o^2\bar{\bar{G}}_m=\nabla X\bar{\bar{I}}\ d\ (r-r') \text{ in the guide} \quad .. \quad (6a)$$

$$\left.\begin{array}{l}\bar{n}.\bar{\bar{G}}_m = 0\\ \bar{n}X\nabla X\bar{\bar{G}}_m = 0\end{array}\right| \text{ on the wall} \quad .. \quad (6b)$$

$\bar{\bar{G}}_m(r,r')$ is the response at r due to a concentrated current density $\bar{\bar{I}}d(r-r')$ at r'.

Eqn(6) may be rewritten as follows:-(Using $\nabla X\nabla=\nabla\nabla.-\nabla^2$)

$$(\nabla^2+K_o^2)\bar{\bar{G}}_m=-\nabla X\bar{\bar{I}}\ d(r-r') \quad .. \quad .. \quad (7a)$$

$$\left.\begin{array}{l}\bar{n}.\bar{\bar{G}}_m = 0\\ \bar{n}X\nabla X\bar{\bar{G}}_m = 0\end{array}\right| \text{ on the wall} \quad .. \quad (7b)$$

To facilitate solving Eqn(7) another Green's function

$\bar{\bar{g}}_m(r,r'')$ defined as follows is introduced:

$$(\nabla^2+k_o^2)\bar{\bar{g}}_m = -\bar{\bar{I}}\ d(r-r'') \quad .. \quad .. \quad (8a)$$

$$\left.\begin{array}{l} \bar{n}\cdot\bar{\bar{g}}_m = 0 \\ \bar{n}X\nabla X\bar{\bar{g}}_m = 0 \end{array}\right\} \text{ on the wall} \quad .. \quad (8b)$$

Applying Green's vector theorem identity we have,

$$\bar{\bar{G}}_m(r,r') = \int \bar{\bar{g}}_m(r,r'')\cdot \nabla''X\bar{\bar{I}}\ d(r'-r'')dv'' \quad .. \quad .. \quad (9)$$

where ∇'' implies curl operation with respect to r''. [Ref. Page 33]

It is to be noted that from Eqn(1a) we may obtain the following equation relating $\bar{\bar{G}}e$ and $\bar{\bar{G}}_m$

$$k_o^2\bar{\bar{G}}e = \nabla X\bar{\bar{G}}_m - \bar{\bar{I}}\ d(r-r') \quad .. \quad .. \quad (10)$$

Eqn(10) is arrived at as follows:-

From (4a) when $J \Rightarrow \bar{\bar{I}}\ d(r-r')$, $\dfrac{E_s}{jwu_o} \Rightarrow \bar{\bar{G}}_e$

From (5a) when $J \Rightarrow \bar{\bar{I}}\ d\ (r-r')\ H_s \Rightarrow Gm$

Thus putting $J \Rightarrow \bar{\bar{I}}\ d(r-r')$, $E_s \Rightarrow -\ jwu_o\ \bar{\bar{G}}_e$ and $H_s \Rightarrow \bar{\bar{G}}m$

in Eqn (1a) we have,

$$\nabla X\bar{\bar{G}}_m = \bar{\bar{I}}\ d\ (r-r') + k_o^2\ \bar{\bar{G}}e$$

where the symbol $\Rightarrow$ implies "Corresponds to".

Eqn.(8a) may be written as :-

$$(\nabla^2+k_o^2)(g_m^{xx}\bar{x}\bar{x}+g_m^{yy}\ \bar{y}\bar{y} + g_m^{zz}\ \bar{z}\bar{z}) = -(\bar{x}\bar{x}+\bar{y}\bar{y}+\bar{z}\bar{z})d\ (r-r'')$$

Hence Eqn(7) may be written componentwise as :-

X-Component (With upper sign for $z > z''$ and Lower sign for $z < z''$)

$$(\nabla^2+K_o^2)\, g_m^{xx}\, \bar{\bar{x}}\bar{x} = -d(r-r'')\bar{x}\bar{x} \quad .. \quad ..(11a)$$

$$\left.\begin{array}{l} \bar{n}.g_m^{xx}\,\bar{x}\bar{x} = 0 \\ \bar{n}x\left(\dfrac{\partial g_m^{xx}}{\partial z}\bar{y} - \dfrac{\partial g_m^{xx}}{\partial y}\bar{z}\right)\bar{x} = 0 \end{array}\right| \text{ on the wall .. } (11a')$$

The corresponding solution is :-

$$g_m^{xx} = \sum_{n=0}^{\alpha} \sum_{m=0}^{\alpha} A_{nm}^{\pm} \operatorname{Sin}\frac{n\Pi x}{a} \operatorname{Cos}\frac{m\Pi y}{b} \exp(\mp G_{nm} z) \quad (11d)$$

[illegible]

where $G_{nm}^2 = \left(\left(\frac{n\Pi}{a}\right)^2 + \left(\frac{m\Pi}{b}\right)^2 - K_o^2\right)$ and $\bar{n}=\bar{x}$ &/or $\bar{y}$.

Y-Component

$$(\nabla^2+K_o^2)\, g_m^{yy}\, \bar{y}\bar{y} = -d(r-r')\bar{y}\bar{y} \quad .. \quad .. \quad (11b)$$

$$\left.\begin{array}{l} \bar{n}.g_m^{yy}\,\bar{y}\bar{y} = 0 \\ \bar{n}x\left(\dfrac{\partial g_m^{yy}}{\partial x}\bar{z} - \dfrac{\partial g_m^{yy}}{\partial z}\bar{x}\right)\bar{y} = 0 \end{array}\right| \text{ on the wall .. } (11b')$$

The corresponding solution is :-

$$g_m^{yy} = \sum_{n=0}^{\alpha} \sum_{m=0}^{\alpha} B_{nm}^{\pm} \operatorname{Cos}\frac{n\Pi x}{a} \operatorname{Sin}\frac{m\Pi y}{b} \exp(\mp G_{nm} z) \quad .. \quad (11e)$$

Z-Component

$$(\nabla^2+K_o^2)\, g_m^{zz}\, \bar{z}\bar{z} = -d(r-r')\bar{z}\bar{z} \quad .. \quad .. \quad (11c)$$

$$\left.\begin{array}{l} \bar{n}.g_m^{zz}\,\bar{z}\bar{z} = 0 \\ \bar{n}x\left(\dfrac{\partial g_m^{zz}}{\partial y}\bar{x} - \dfrac{\partial g_m^{zz}}{\partial x}\bar{y}\right)\bar{z} = 0 \end{array}\right| \text{ On the wall. } \quad (11c')$$

The corresponding solution is :-

$$g_m^{zz} = \sum_{n=0}^{\alpha} \sum_{m=0}^{\alpha} C_{nm}^{\pm} \operatorname{Cos}\frac{n\Pi x}{a} \operatorname{Cos}\frac{m\Pi y}{b} \exp(\mp G_{nm} z) \quad .. \quad (11f)$$

Evaluation of $A^{\pm}_{nm}$: [8] Pp 199-200.

If Equation (11a) is multiplied by $\sin\left(\frac{n\Pi x}{a}\right)\cos\left(\frac{m\Pi y}{b}\right)$ and integrated over the guide cross-section we obtain,

$$\left(\frac{\partial^2}{\partial z^2} - G^2_{nm}\right) g^{xx}_{nm}(z) = -\sin\left(\frac{n\Pi x''}{a}\right)\cos\left(\frac{m\Pi y''}{b}\right) \quad \ldots\ldots(11g)$$

where $g^{xx}_{nm}(z) = \int_0^a\int_0^b g^{xx}_m(r, r'') \sin\frac{n\Pi x}{a}\cos\frac{m\Pi y}{b}\,dxdy$

Equation (11g) is arrived at as follows: -

For $\bar{n} = \bar{x}$ at $x = 0$ and $x = a$

1st B.C. (boundary condition) yields $g^{xx}_m \bar{x} = 0$ } i.e. $g^{xx}_m \Rightarrow \sin\frac{n\Pi x}{a}$

2nd B.C. yields $0 = 0$

For $\bar{n} = \bar{y}$ at $y = 0$ & $y = b$

1st B.C. yields $0 = 0$

2nd B.C. yields $-\frac{\partial g^{xx}_m}{\partial y}\bar{\bar{x}}\bar{\bar{x}} = 0$ } i.e. $g^{xx}_m \Rightarrow \cos\frac{m\Pi y}{b}$

Combining these results for all possible modes:

$$g^{xx}_m = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A^{\pm}_{nm}\sin\frac{n\Pi x}{a}\cos\frac{m\Pi y}{b}\exp(\pm G_{nm}z) \quad ..(11d)$$

where the upper sign is for $z > z''$ and the lower sign if for $z < z''$

Substituting (11d) in (11a) we have,

$$\left(\frac{\partial^2}{\partial z^2} - \left(\frac{n\Pi}{a}\right)^2 - \left(\frac{m\Pi}{b}\right)^2 + K_o^2\right) g^{xx}_m(r, r'') = -d(r-r'') = -d(x-x'')d(y-y'')d(z-z'')$$

Multiplying this equation by $\sin\frac{n\Pi x}{a}\cos\frac{m\Pi y}{b}$ and integrating over guide cross-section we have:-

$$\int_0^a\int_0^b \left(\frac{\partial^2}{\partial z^2} - G^2_{nm}\right) g^{xx}_m \sin\frac{n\Pi x}{a}\cos\frac{m\Pi y}{b}\,dxdy$$

$$= -\int_0^a\int_0^b d(r-r'')\sin\frac{n\Pi x}{a}\cos\frac{m\Pi y}{b}\,dxdy$$

or $$\left(\frac{\partial^2}{\partial z^2} - G^2_{nm}\right)\int_0^a\int_0^b g^{xx}_m \sin\frac{n\Pi x}{a}\cos\frac{m\Pi y}{b}\,dxdy$$

$$= -\sin\frac{n\Pi x''}{a}\cos\frac{m\Pi y''}{b}\,d(z-z'')$$

or $$\left(\frac{\partial^2}{\partial z^2} - \mathcal{G}_{nm}^2 \right) g_{nm}^{xx}(z) = - \sin \frac{n\Pi x''}{a} \cos \frac{m\Pi y''}{b} \delta(z-z'') \qquad (11g)$$

d(z-z'') .. .. .. (119).

where $$g_{nm}^{xx}(z) = \int_0^a\int_0^b g_m^{xx} \sin \frac{n\Pi x}{a} \cos \frac{m\Pi y}{b}\, dxdy =$$

$$= \begin{cases} A_{nm}^{-} \dfrac{ab}{e_{on} e'_{om}} e^{\mathcal{G}_{nm} z} \quad .. & \text{for } z < z'' \\ A_{nm}^{+} \dfrac{ab}{e_{on} e'_{om}} e^{-\mathcal{G}_{nm} z} \quad .. & \text{for } z > z'' \end{cases}$$

N.B. $e_{on} = \begin{cases} 1 \text{ for } n = 0 \\ 2 \text{ otherwise.} \end{cases}$ & $e'_{om} = \begin{cases} 1 \text{ for } m = 0 \\ 2 \text{ otherwise.} \end{cases}$

At $Z = Z''$, $g_{nm}^{xx}(z)$ is continuous.

Hence $$A_{nm}^{-} \frac{ab}{e_{on} e_{om}} e^{\mathcal{G}_{nm} z''} = A_{nm}^{+} \frac{ab}{e_{on} e_{om}} e^{-\mathcal{G}_{nm} Z''} \quad ...(11h)$$

Also since $g_{nm}^{xx}(z)$ is continuous at $Z = Z''$, integrating (11g) from $Z''-\Delta$ to $Z''+\Delta$ over Z'' and letting Δ approach zero, we have,

$$\left. \frac{\partial g_{nm}^{xx}(z)}{\partial z} \right]_{Z''_-}^{Z''_+} = - \sin \frac{n\Pi x''}{a} \cos \frac{m\Pi y''}{b}$$

This implies that $\dfrac{\partial g_{nm}^{xx}(z)}{\partial z}$ is discontinuous at $Z = Z''$.

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Thus, $\mathcal{G}_{nm} A^{+}_{nm} \frac{ab}{e_{on} e'_{om}} e^{-\mathcal{G}_{nm} z''} - \mathcal{G}_{mn} A^{-}_{nm} \frac{ab}{e_{on} e'_{om}} e^{\mathcal{G}_{nm} z''}$

$$= - \sin \frac{n\pi x''}{a} \cos \frac{m\pi y''}{b} \quad \ldots \quad \ldots \quad (11i)$$

Solving (11h) and (11i) $A^{+}_{nm} e^{-\mathcal{G}_{nm} z''} = A^{-}_{nm} e^{\mathcal{G}_{nm} z''}$

$$= \frac{e_{on} \; e'_{om}}{\mathcal{G}_{nm} 2ab} \sin \frac{n\pi x''}{a} \cos \frac{m\pi y''}{b}$$

Thus $g^{xx}_{m} (r,r'') = \left[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{e_{on} e'_{om}}{2ab \mathcal{G}_{nm}} , e^{-\mathcal{G}_{nm} |z-z''|} \right.$

$$\left. \sin \frac{n\pi x}{a} \sin \frac{n\pi x''}{a} \cos \frac{m\pi y}{b} \cos \frac{m\pi y''}{b} \right]$$

Similarly we can evaluate the co-efficients $B^{\pm}_{nm}$ and $C^{\pm}_{nm}$

and obtain expressions for $g^{yy}_{m} (r,r'')$ and $g^{zz}_{m} (r,r')$

Combining $g^{xx}_{m} (r,r'')$, $g^{yy}_{m}(r,r'')$ and $g^{zz}_{m}(r,r'')$ we have

$$\overline{\overline{g}}_{m} (r,r'') = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{e_{on} e'_{om}}{2ab \mathcal{G}_{nm}} , e^{-\mathcal{G}_{nm} |z-z''|}$$

$$\cdot \left[\begin{array}{l} \bar{x}\bar{x} \left(\sin \frac{n\pi x}{a} \sin \frac{n\pi x''}{a} \cos \frac{m\pi y}{b} \cos \frac{m\pi y''}{b} \right) \\ + \bar{y}\bar{y} \left(\cos \frac{n\pi x}{a} \cos \frac{n\pi x''}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y''}{b} \right) \\ + \bar{z}\bar{z} \left(\sin \frac{n\pi x}{a} \sin \frac{n\pi x''}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y''}{b} \right) \end{array} \right] \quad \ldots (12)$$

Substituting Eqn(12) in Eqn(9) we now perform the integration to solve for $\bar{\bar{G}}_m$. The integration can be performed in a simple way by incroporating some elementary theorems of the distribution theory. To this end the following relations of the are used to simplify the process of integrating Eqn(9): $\nabla'' X \bar{\bar{I}}\ d\ (r'-r'')$

$$= (\bar{x}\ \frac{\partial}{\partial x''} + \bar{y}\ \frac{\partial}{\partial y''} + \bar{z}\ \frac{\partial}{\partial z''})\ X\ (\bar{x}\bar{x}+\bar{y}\bar{y}+\bar{z}\bar{z})d(r'-r'')$$

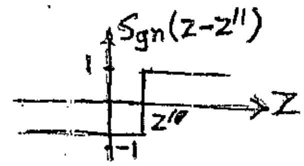
$$= (\frac{\partial d}{\partial z''}\ \bar{y} - \frac{\partial d}{\partial y''}\ \bar{z})\ \bar{x} + (\frac{\partial d}{\partial x''}\ \bar{z} - \frac{\partial d}{\partial z''}\bar{x})\bar{y} + (\frac{\partial d}{\partial y''}\ \bar{x} - \frac{\partial d}{\partial x''}\bar{y})\bar{z} \quad (13)$$

(Since $\bar{x}x\bar{x} = \bar{y}x\bar{y} = \bar{z}x\bar{z} = 0,\ \bar{x}x\bar{y} = \bar{z},\ \bar{y}x\bar{z} = \bar{x},\ \bar{z}x\bar{x} = \bar{y}$)

$$\frac{\partial}{\partial z''}\ e^{-G_{nm}|z-z''|} = G_{nm}\ \mathrm{Sgn}\ (z-z'').\ e^{-G_{nm}|z-z''|} \quad \left[\begin{matrix}\text{Ref.(3)}\\ \text{P.48.}\end{matrix}\right] \quad (14)$$

Sgn(z-z")... ... (14)Ref.9).

Where sgn (z-z") is representable as :-



$$\int_V \left(\frac{\partial d}{dz''}(z'-z'')\right) f\ dv'' = - \left.\frac{df}{dz''}\right]_{z''=z'} \quad (15)\ (\text{Ref.9}).$$

Applying Eqns (13), (14) & (15) in eqn (9) and performing the differentiation - integration the following result is obtained:-

$$\bar{\bar{G}}_m(r,r') = \int_V \bar{\bar{g}}_m\ (r,r''),\ \nabla'' x\ \bar{\bar{I}}\ d(r'-r'')dv''$$

[illegible] $\frac{dg_m}{dz''}$ [illegible]

$$= \int_V \Big[-g_m^{xx}\, \bar{\bar{xx}} \cdot \frac{\partial d}{\partial z''}\, \bar{\bar{xy}} + g_m^{xx}\, \bar{\bar{xx}} \cdot \frac{dd}{dy''}\, \bar{\bar{xz}} + g_m^{yy}\, \bar{\bar{yy}} \cdot \frac{\partial d}{\partial z''}\, \bar{\bar{yx}} - g_m^{yy}\, \bar{\bar{yy}} \cdot \frac{\partial d}{\partial x''}\, \bar{\bar{yz}} - g_m^{zz}\, \bar{\bar{zz}} \cdot \frac{\partial d}{\partial y''}\, \bar{\bar{zx}} + g_m^{zz}\, \bar{\bar{zz}} \cdot \frac{\partial d}{\partial x''}\, \bar{\bar{zy}} \Big] dv''$$

$$= \Big\{ \frac{\partial g_m^{xx}}{\partial z''}\, \bar{\bar{xy}} - \frac{\partial g_m^{xx}}{\partial y''}\, \bar{\bar{xz}} - \frac{dg_m^{yy}}{dz''}\, \bar{\bar{yx}} + \frac{dg_m^{yy}}{dx''}\, \bar{\bar{yz}} + \frac{dg_m^{zz}}{dy''}\, \bar{\bar{zx}} - \frac{dg_m^{22}}{dx''}\, \bar{\bar{zy}} \Big\}_{\substack{x''=x' \\ y''=y' \\ z''=z'}}$$

$$= \Big[G_m^{xy}\, \bar{\bar{xy}} + G_m^{xz}\, \bar{\bar{xz}} + G_m^{yx}\, \bar{\bar{yx}} + G_m^{yz}\, \bar{\bar{yz}} + G_m^{zy}\, \bar{\bar{zx}} + G_m^{zy}\, \bar{\bar{zy}} \Big]$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{e_{on}\, e'_{om}}{2ab\,\mathcal{G}_{nm}} \Big\{ \bar{\bar{xy}}\, \mathcal{G}_{nm}\, \mathrm{Sgn}(z-z')\,\cdot$$

$$\mathrm{Sin}\frac{n\Pi x}{a}\, \mathrm{Sin}\frac{n\Pi x'}{a}\, \mathrm{Cos}\frac{m\Pi y}{b}\, \mathrm{Cos}\frac{m\Pi y'}{b}$$

$$+ \bar{\bar{xz}}\left(\frac{m\Pi}{b}\right) \mathrm{Sin}\frac{n\Pi x}{a}\, \mathrm{Sin}\frac{n\Pi x'}{a}\, \mathrm{Cos}\frac{m\Pi y}{b}\, \mathrm{Sin}\frac{m\Pi y'}{b} - \bar{\bar{yx}}\, \mathcal{G}_{nm}\, \mathrm{Sgn}(z-z')\,\cdot$$

$$\mathrm{Cos}\frac{n\Pi x}{a}\, \mathrm{Cos}\frac{n\Pi x'}{a}\, \mathrm{Sin}\frac{m\Pi y}{b}\, \mathrm{Sin}\frac{m\Pi y'}{b} - \bar{\bar{yz}}\left(\frac{n\Pi}{a}\right) \mathrm{Cos}\frac{n\Pi x}{a}\,\cdot$$

$$\mathrm{Sin}\frac{n\Pi x'}{a}\, \mathrm{Sin}\frac{n\Pi x'}{a}\, \mathrm{Sin}\frac{m\Pi y}{b}\, \mathrm{Sin}\frac{m\Pi y'}{b} - \bar{\bar{zx}}\left(\frac{m\Pi}{b}\right) \mathrm{Cos}\frac{n\Pi x}{a},$$

$\cos\frac{n\Pi x'}{a}\cos\frac{m\Pi y}{b}\sin\frac{m\Pi y'}{b} + \bar{z}\bar{y}\left(\frac{n\Pi}{a}\right)\cos\frac{n\Pi x}{a}$

$\sin\frac{n\Pi x'}{a}\cos\frac{m\Pi y}{b}\cos\frac{m\Pi y'}{b}\Big]$

Substituting this value of $\bar{\bar{G}}_m$ in Eqn(10) we obtain $\bar{\bar{G}}_e$ as follows:

$$\bar{\bar{G}}_e = \frac{1}{K_o^2}\nabla\times\bar{\bar{G}}_m - \bar{\bar{I}}d(r-r') = \Big[\nabla\times\left(G_m^{xx}\bar{x} + G_m^{yx}\bar{y} + G_m^{zx}\bar{z}\right)\bar{x}$$

$$+ \nabla\times\left(G_m^{xy}\bar{x} + G_m^{yy}\bar{y} + G_m^{zy}\bar{z}\right)\bar{y} + \nabla\times\left(G_m^{xz}\bar{x} + G_m^{yz}\bar{y}\right.$$

$$\left.+ G_m^{zz}\bar{z}\right)\bar{z}\Big] - \Big[\bar{x}\bar{x}\,d(r-r') + \bar{y}\bar{y}\,d(r-r') + \bar{z}\bar{z}\,d(r-r')\Big]$$

$$= \Big[\left(\frac{\partial G_m^{yx}}{\partial x}\bar{z}\bar{x} - \frac{\partial G_m^{yx}}{\partial z}\bar{x}\bar{x}\right) + \left(\frac{\partial G_m^{zx}}{\partial y}\bar{x}\bar{x} - \frac{\partial G_m^{zx}}{\partial x}\bar{y}\bar{x}\right)$$

$$+ \left(\frac{\partial G_m^{xy}}{\partial z}\bar{y}\bar{y} - \frac{\partial G_m^{xy}}{\partial y}\bar{z}\bar{y}\right) + \left(\frac{\partial G_m^{zy}}{\partial y}\bar{x}\bar{y} - \frac{\partial G_m^{zy}}{\partial x}\bar{y}\bar{y}\right)$$

$$+ \left(\frac{\partial G_m^{xz}}{\partial z}\bar{y}\bar{z} - \frac{\partial G_m^{xz}}{\partial y}\bar{z}\bar{z}\right) + \left(\frac{\partial G_m^{yz}}{\partial x}\bar{z}\bar{z} - \frac{\partial G_m^{yz}}{\partial z}\bar{x}\bar{z}\right)\Big]$$

$$-\bar{\bar{I}}\,d(r-r') = \Big[\bar{x}\bar{x}\left(-\frac{\partial G_m^{yx}}{\partial z} + \frac{\partial G_m^{zx}}{\partial y} - \frac{d(r-r')}{K_o^2}\right) + \bar{y}\bar{y}$$

$$\left(\frac{\partial G_m^{xy}}{\partial z} - \frac{\partial G_m^{zy}}{\partial x} - \frac{d(r-r')}{K_o^2}\right) + \bar{z}\bar{z}\left(-\frac{\partial G_m^{xz}}{\partial y} + \frac{\partial G_m^{yz}}{\partial x}\right)$$

$$+ \bar{x}\bar{y}\left(\frac{\partial G_m^{zy}}{\partial y}\right) + \bar{y}\bar{x}\left(-\frac{\partial G_m^{zx}}{\partial x}\right) + \bar{y}\bar{z}\left(\frac{\partial G_m^{xz}}{\partial z}\right)$$

$$+ \bar{z}\bar{y}\left(-\frac{\partial G_m^{xy}}{\partial y}\right) + \bar{z}\bar{x}\left(\frac{\partial G_m^{yz}}{\partial x}\right) + \bar{x}\bar{z}\left(-\frac{\partial G_m^{yz}}{\partial z}\right)\Big]$$

$$-\bar{z}\bar{z}\,\frac{1}{K_o^2}\,d(r-r')$$

For simplifiying we use the following :-

$$\frac{d}{dz}\left[e^{-G_{nm}|z-z'|}\right] = -G_{nm}\, Sgn(z-z')\, e^{-G_{nm}|z-z'|}$$

$$-\frac{d}{dz}\left(Sgn(z-z')\, e^{-G_{nm}|z-z'|} \right) = \left(-G_{nm} + 2d(z-z')\right) e^{-G_{nm}|z-z'|} \quad \ldots (17)$$

and the completeness relations:- $d(x-x')\, d(y-y')$

$$= \left[\sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{e_{on}\, e'_{om}}{ab} \operatorname{Sin}\frac{n\Pi x}{a} \operatorname{Sin}\frac{n\Pi x'}{a} \operatorname{Cos}\frac{m\Pi y}{b} \operatorname{Cos}\frac{m\Pi y'}{b} \right]$$

$$= \left[\sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{e_{on} e'_{om}}{ab} \operatorname{Cos}\frac{n\Pi x}{a} \operatorname{Cos}\frac{m\Pi x'}{a} \operatorname{Sin}\frac{m\Pi y}{b} \operatorname{Sin}\frac{m\Pi y'}{b} \right] \quad \ldots (18)$$

Equation (18) is the mode completness relation for cylindrical Waveguides with normal mode functions separable into a transverse part and an axial part & constant coordinate curves coincide with boundary of the Waveguides(8). Using these relations we obtain the following

result:-

$$\bar{\bar{G}}_e = \frac{1}{K_o^2} \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{e_{on}\, e'_{om}}{2ab\, G_{nm}} e^{-G_{nm}|z-z'|} \left[\bar{\bar{xx}} \left(\left(\frac{m\Pi}{b}\right)^2 - G_{nm}^2 \right) \left(\operatorname{Cos}\frac{n\Pi x}{a} \operatorname{Cos}\frac{n\Pi x'}{a} \operatorname{Sin}\frac{m\Pi y}{b} \operatorname{Sin}\frac{m\Pi y'}{b} \right) + \bar{\bar{yy}} \left(\left(\frac{n\Pi}{a}\right)^2 - G_{nm}^2 \right) \left(\operatorname{Sin}\frac{n\Pi x}{a} \operatorname{Sin}\frac{n\Pi x'}{a} \operatorname{Cos}\frac{m\Pi y}{b} \operatorname{Cos}\frac{m\Pi Y'}{b} \right) + \bar{\bar{zz}} \left(\left(\frac{m\Pi}{b}\right)^2 + \left(\frac{n\Pi}{a}\right)^2 \right) \right.$$

$$+\left(\frac{m\Pi}{a}\right)^2\Big)\left(\sin\frac{n\Pi x}{a}\sin\frac{n\Pi x'}{a}\sin\frac{m\Pi y}{b}\sin\frac{m\Pi y'}{b}\right)+\bar{\bar{xy}}\left(-\left(\frac{m\Pi}{b}\right)\left(\frac{n\Pi}{a}\right)\right).$$

$$\left(\frac{n\Pi}{a}\right)\Big)\Big)\left(\cos\frac{n\Pi x}{a}\sin\frac{n\Pi x'}{a}\sin\frac{m\Pi y}{b}\cos\frac{m\Pi y'}{b}\right)+\bar{\bar{yx}}$$

$$+\bar{\bar{yx}}\left(-\left(\frac{m\Pi}{b}\right)\left(\frac{n\Pi}{a}\right)\right).\left(\sin\frac{n\Pi x}{a}\cos\frac{n\Pi x'}{a}\cos\frac{m\Pi y}{b}\sin\frac{m\Pi y'}{b}\right)$$

$$\sin\frac{m\Pi y'}{b}\Big)+\bar{\bar{yz}}\left(-\left(\frac{m\Pi}{b}\right)\Gamma_{nm}\,\mathrm{Sgn}\,(z-z')\right).\left(\sin\frac{n\Pi x}{a}\sin\frac{n\Pi x'}{a}\right.$$

$$\left.\cos\frac{m\Pi y}{b}\cos\frac{m\Pi y'}{b}\right)+\bar{\bar{zy}}\left(+\left(\frac{m\Pi}{b}\right)\Gamma_{mn}\,\mathrm{Sgn}(z-z')\right)\left(\cos\frac{n\Pi x}{a}\sin\frac{n\Pi x'}{a}\right.$$

$$\left.\sin\frac{n\Pi x'}{a}\sin\frac{m\Pi y}{b}\cos\frac{m\Pi y'}{b}\right)+\bar{\bar{zx}}\left(+\left(\frac{n\Pi}{a}\right)\Gamma_{nm}\,\mathrm{Sgn}\,(z-z')\right).$$

$$\left(\sin\frac{n\Pi x}{a}\cos\frac{n\Pi x'}{a}\sin\frac{m\Pi y}{b}\sin\frac{m\Pi y'}{b}\right)+\bar{\bar{xz}}\left(-\left(\frac{n\Pi}{a}\right).\right.$$

$$\left.\Gamma_{nm}\,\mathrm{Sgn}\,(z-z')\right).\left(\cos\frac{n\Pi x}{a}\sin\frac{n\Pi x'}{a}\sin\frac{m\Pi y}{b}\sin\frac{m\Pi y'}{b}\right)\Big]$$

$$-\frac{1}{K_o^2}\,\bar{\bar{zz}}\,d(r-r')=\bar{\bar{G}}_{eo}\,(r,r')-\frac{1}{K_o^2}\,\bar{\bar{zz}}\,d(r-r')\quad\ldots\quad\ldots(19)$$

Putting $\Gamma_{nm} = jK_{nm}$ where $-K_{nm}^2 = \Gamma_{nm}^2 = K_c^2 - k_o^2$

and $K_c^2 = \left[\left(\frac{n\Pi}{a}\right)^2 + \left(\frac{m\Pi}{b}\right)^2\right]$ & $jK_{nm} = \Gamma_{nm} = \sqrt{(K_c^2 - k_o^2)}$ ½

We have the following form for $\bar{\bar{G}}_{eo}$

$$\bar{\bar{G}}_{eo}(r,r') = -\frac{j}{2abk_o^2}\sum_{n=o}^{\infty}\sum_{m=o}^{\infty}\frac{e_{on}e'_{om}}{k_{nm}}\,e^{-jK_{nm}|z-z'|}\,.$$

$$\left[\bar{\bar{xx}}\left(K_o^2-\left(\frac{n\Pi}{a}\right)^2\right)\cos\frac{n\Pi x}{a}\cos\frac{n\Pi x'}{a}\sin\frac{m\Pi y}{b}\sin\frac{m\Pi y'}{b}\right.$$

$$+\bar{y}\bar{y}\left(K_o^2 - (\frac{m\pi}{b})^2\right) \sin\frac{n\pi x}{a} \sin\frac{n\pi x'}{a} \cos\frac{m\pi y}{b} \cos\frac{m\pi y'}{b}$$

$$+\bar{z}\bar{z}\left((\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2\right) \sin\frac{n\pi x}{a} \sin\frac{n\pi x'}{a} \sin\frac{m\pi y}{b} \sin\frac{m\pi y'}{b}$$

$$+\bar{x}\bar{y}\left(\ominus(\frac{n\pi}{a}) (\frac{m\pi}{b})\right) (\cos\frac{n\pi x}{a} \sin\frac{n\pi x'}{a} \sin\frac{m\pi y}{b} \cos\frac{m\pi y'}{b}$$

$$+\bar{y}\bar{x}\left(-(\frac{n\pi}{a}) (\frac{m\pi}{b})\right) \sin\frac{n\pi x}{a} \cos\frac{n\pi x'}{a} \cos\frac{m\pi y}{b} \sin\frac{m\pi y'}{b}$$

$$+\bar{y}\bar{z}\left(\mp jK_{nm} (\frac{m\pi}{b})\right) \sin\frac{n\pi x}{a} \sin\frac{n\pi x'}{a} \cos\frac{m\pi y}{b} \sin\frac{m\pi y'}{b}$$

$$+\bar{z}\bar{y}\left(\pm jK_{nm} (\frac{m\pi}{b})\right) \sin\frac{n\pi x}{a} \sin\frac{n\pi x'}{a} \sin\frac{m\pi y}{b} \cos\frac{m\pi y'}{b}$$

$$+\bar{z}\bar{x}\left(\pm jK_{nm} (\frac{n\pi}{a})\right) \sin\frac{n\pi x}{a} \cos\frac{n\pi x'}{a} \sin\frac{m\pi y}{b} \sin\frac{m\pi y'}{b})$$

$$+\bar{x}\bar{z}\left(\mp jK_{nm} (\frac{n\pi}{a})\right) \cos\frac{n\pi x}{a} \sin\frac{n\pi x'}{a} \sin\frac{m\pi y}{b} \sin\frac{m\pi y'}{b} \Big]$$

for $z \gtrless z'$ .. .. (20).

Art-3:-

A modified method of deriving $\bar{\bar{G}}_e$:

Following the suggestion of Collin ((8) pp 222-223) and a procedure very similar to that of Art-2, a mofified procedure of deriving the dlectric type dyadic Green's function $\bar{\bar{G}}_e$ is presented as follows:-

Defining vector potential A_s by $u_o H_s = \nabla X A_s$ and defining scalar potential ψ_s by $E_s = -jwA_s - \nabla \psi_s$ and using them in Maxwell's Curl equation $\nabla X H_s = jwe_o E_s + J$ and using the Lorentz condition $\nabla\cdot A_s = -jwu_o e_o \psi_s$ we have

$$\nabla X \nabla X A_s = u_o J + k_o^2 A_s - jwu_o e_o \nabla \psi_s$$

or $$-\nabla^2 A_s + \nabla\nabla\cdot A_s = u_o J + k_o^2 A_s - jwu_o e_o \nabla \psi_s$$

$$\therefore \nabla^2\left(\frac{A_s}{u_o}\right) + k_o^2\left(\frac{A_s}{u_o}\right) = -J$$

and $$\frac{E_s}{-jwu_o} = \left(1 + \frac{1}{k^2}\nabla\nabla\cdot\right)\left(\frac{A_s}{u_o}\right)$$ (Using Lorentz condition in ψ_s defining equation)

For $J \Rightarrow \bar{\bar{I}}d(r-r')$, $\frac{A_s}{u_o} \Rightarrow \bar{\bar{G}}_a$ and $\frac{E_s}{-jwu_o} \Rightarrow \bar{\bar{G}}_e$

we have

$$\nabla^2 \bar{\bar{G}}_a + k_o^2 \bar{\bar{G}}_a = -\bar{\bar{I}}\, d(r-r') \text{ and } \bar{\bar{G}}_e = \left(1 + \frac{1}{k_o^2}\nabla\nabla\cdot\right)\bar{\bar{G}}_a$$

Thus due to a current distribution $J(r,r')$, throughout V, in dv' element

$$E_s(r) = -jwu_o \int_V \bar{\bar{G}}_e(r,r') J(r,r')\, dv'$$ (Eqn (12) of Art -2.)

To Solve for $\bar{\bar{G}}_a$ we set

$$\nabla^2\bar{\bar{G}}_a + k_0^2\bar{\bar{G}}_a = -\bar{\bar{I}}\, d(r-r') \text{ in the waveguide}$$

$$\nabla\cdot\bar{\bar{G}}_a = 0 \quad \left[\text{from } \nabla\cdot E_s = -\frac{\rho_s}{\epsilon} = \nabla\cdot\left(-jwu_0\left(1+\frac{1}{k_0^2}\nabla\nabla\cdot\right)A\right)\right] \quad \text{on the wall.}$$

$$\bar{n}\cdot\nabla\times\bar{\bar{G}}_a = 0 \left[\text{ from } \bar{n}\cdot H_s = \bar{n}\cdot\nabla\times A_s = 0.\right]$$

Component wise:-

$$(\nabla^2+k_0^2)\, G_a^{xx}\ \bar{x}\bar{x} = -d(r-r')\ \bar{x}\bar{x}$$

$$\frac{\partial}{\partial x} G_a^{xx}\ \bar{x} = 0$$

$$\bar{n}\cdot\left(\frac{\partial G_a^{xx}}{\partial z}\bar{y} - \frac{\partial G_a^{xx}}{\partial y}\bar{z}\right)\bar{x} = 0 \quad \text{on the wall.}$$

$$(\nabla^2+k_0^2)\, G_a^{yy}\ \bar{y}\bar{y} = -d(r-r')\ \bar{y}\bar{y}$$

$$\frac{\partial G_a^{yy}}{\partial y}\bar{y} = 0$$

$$\bar{n}\cdot\left(\frac{\partial G_a^{yy}}{\partial x}\bar{z} - \frac{\partial G_a^{yy}}{\partial z}\bar{x}\right)\bar{y} = 0 \quad \text{on the wall.}$$

$$(\nabla^2+k_0^2)\, G_a^{zz}\ \bar{z}\bar{z} = -d(r-r')\ \bar{z}\bar{z}$$

$$\frac{\partial G_a^{zz}}{\partial z}\bar{z} = 0$$

$$\bar{n}\cdot\left(\frac{\partial G_a^{zz}}{\partial y}\bar{x} - \frac{\partial G_a^{zz}}{\partial x}\bar{y}\right)\bar{z} = 0 \quad \text{on the wall.}$$

$$\therefore \quad G_a^{xx} = \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} A_{nm}^{\pm} \operatorname{Cos} \frac{n\Pi x}{a} \operatorname{Sin} \frac{m\Pi y}{b} e^{\mp \Gamma_{nm} z}$$

$$G_a^{yy} = \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} B_{nm}^{\pm} \operatorname{Sin} \frac{n\Pi x}{a} \operatorname{Cos} \frac{m\Pi y}{b} e^{\mp \Gamma_{nm} z}$$

$$G_a^{zz} = \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} C_{nm}^{\pm} \operatorname{Sin} \frac{n\Pi x}{a} \operatorname{Sin} \frac{n\Pi y}{b} e^{\mp \Gamma_{nm} z}$$

These solutions are obtained by the method of separation of variables with assumed propagations factor $e^{\mp \Gamma_{nm} z}$.

where $\Gamma_{nm}^2 = \left[\left(\frac{n\Pi}{a} \right)^2 + \left(\frac{m\Pi}{b} \right)^2 - K_o^2 \right]$ & using boundary conditions.

In G_a^{xx}, G_a^{yy}, G_a^{zz} expressions, the uppper sign is for $z > z'$ and the lower sign is for $z < z'$ ..

Proceeding as in Art:-2 the coefficients $A_{nm}^{\pm}$, ~~andx~~ $B_{nm}^{\pm}$ and $C_{nm}^{\pm}$ may be evaluated. When so evaluated we obtains $\bar{\bar{G}}_a$ as follows :-

$$\bar{\bar{G}}_a = G_a^{xx} \,\overline{\overline{xx}} + G_a^{yy} \,\overline{\overline{yy}} + G_a^{zz} \,\overline{\overline{zz}} = \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} e^{-\Gamma_{nm}|z-z'|} \cdot \frac{e_{on} e'_{om}}{2ab\Gamma_{nm}^2},$$

$$\Big[\overline{\overline{xx}} \operatorname{Cos} \frac{n\Pi x}{a} \operatorname{Cos} \frac{n\Pi x'}{a} \operatorname{Sin} \frac{m\Pi y}{b} \operatorname{Sin} \frac{m\Pi y'}{b}$$

$$+ \overline{\overline{yy}} \operatorname{Sin} \frac{n\Pi x}{a} \operatorname{Sin} \frac{n\Pi x'}{a} \operatorname{Cos} \frac{m\Pi y}{b} \operatorname{Cos} \frac{m\Pi y'}{b}$$

$$+ \bar{z}\bar{z} \, Sin \frac{n\Pi x}{a} \, Sin \frac{n\Pi x'}{a} \, Sin \frac{m\Pi y}{b} \, Sin \frac{m\Pi y'}{b} \Big]$$

Using this expression of $\bar{\bar{G}}_a$, $\bar{\bar{G}}_e$ is obtained as follows:-

$$\bar{\bar{G}}_e = \left(1 + \frac{1}{k_o^2} \nabla\nabla \cdot\right) \bar{\bar{G}}_a = \left[\bar{\bar{G}}_a + \left\{\frac{1}{k_o^2} \nabla\left(\bar{x} \frac{\partial}{\partial x} + \bar{y} \frac{\partial}{\partial y} + \bar{z} \frac{\partial}{\partial z}\right) \cdot \right.\right.$$

$$\left.\left.(\bar{x}\bar{x} G_a^{xx} + \bar{y}\bar{y} G_a^{yy} + \bar{z}\bar{z} G_a^{zz})\right\}\right]$$

$$= \left[\bar{\bar{G}}_a + \frac{1}{k_o^2}\left(\bar{x} \frac{\partial}{\partial x} + \bar{y} \frac{\partial}{\partial y} + \bar{z} \frac{\partial}{\partial z}\right)\left(\bar{x} \frac{\partial G_a^{xx}}{\partial x} + \bar{y} \frac{\partial G_a^{yy}}{\partial y} + \bar{z} \frac{\partial G_a^{zz}}{\partial z}\right)\right]$$

$$= (\bar{x}\bar{x} G_a^{xx} + \bar{y}\bar{y} G_a^{yy} + \bar{z}\bar{z} G_a^{zz})$$

$$+ \frac{1}{k_o^2}\left[\begin{array}{l} \bar{x}\bar{x} \frac{\partial^2 G_a^{xx}}{\partial x^2} + \bar{x}\bar{y} \frac{\partial^2 G_a^{yy}}{\partial x \partial y} + \bar{x}\bar{z} \frac{\partial^2 G_a^{zz}}{\partial x \partial z} \\ + \bar{y}\bar{x} \frac{\partial^2 G_a^{xx}}{\partial y \partial x} + \bar{y}\bar{y} \frac{\partial^2 G_a^{yy}}{\partial y^2} + \bar{y}\bar{z} \frac{\partial^2 G_a^{zz}}{\partial y \partial z} \\ + \bar{z}\bar{x} \frac{\partial^2 G_a^{xx}}{\partial z \partial x} + \bar{z}\bar{y} \frac{\partial^2 G_a^{yy}}{\partial z \partial y} + \bar{z}\bar{z} \frac{\partial^2 G_a^{zz}}{\partial z^2} \end{array}\right]$$

$$= \left[\bar{x}\bar{x}\left(G_a^{xx} + \frac{\partial^2 G_a^{xx}}{k_o^2 \partial x^2}\right) + \bar{y}\bar{y}\left(G_a^{yy} + \frac{\partial^2 G_a^{yy}}{k_o^2 \partial y^2}\right) + \bar{z}\bar{z}\left(G_a^{zz} + \frac{\partial^2 G_a^{zz}}{k_o^2 \partial z^2} + \right.\right.$$

$$\left.\frac{1}{k_o^2} \delta(r-r')\right) + \bar{x}\bar{y}\left(\frac{\partial^2 G_a^{yy}}{k_o^2 \partial x \partial y}\right) + \bar{y}\bar{x}\left(\frac{\partial^2 G_a^{xx}}{k_o^2 \partial y \partial x}\right) + \bar{y}\bar{z}\left(\frac{\partial^2 G_a^{zz}}{k_o^2 \partial y \partial z}\right)$$

$$\left. + \bar{z}\bar{y}\left(\frac{\partial^2 G_a^{yy}}{k_o^2 \partial z \partial y}\right) + \bar{z}\bar{x}\left(\frac{\partial^2 G_a^{xx}}{k_o^2 \partial z \partial x}\right) + \bar{x}\bar{z}\left(\frac{\partial^2 G_a^{zz}}{k_o^2 \partial x \partial z}\right)\right] - \bar{z}\bar{z}\frac{1}{k_o^2} \delta(r-r')$$

Using the relation $\frac{\partial}{\partial z} e^{-\Gamma_{nm}|z-z'|} = -\Gamma_{mn} \operatorname{Sgn}(z-z') e^{-\Gamma_{nm}|z-z'|}$

$$\frac{\partial}{\partial z} \left(\operatorname{Sgn}(z-z') e^{-\Gamma_{nm}|z-z'|} \right) = \left(-\Gamma_{nm} + 2d(z-z') \right) e^{-\Gamma_{nm}|z-z'|}$$

and the mode completness relation:- $d(x-x')d(y-y') = \left[\sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \right.$

$$\left. \frac{e_{on}\, e_{om}}{ab} \operatorname{Sin} \frac{n\pi x}{a} \operatorname{Sin} \frac{n\pi x'}{a} \operatorname{Sin} \frac{m\pi y}{b} \operatorname{Sin} \frac{m\pi y'}{b} \right]$$

We obtain the following expression for $\bar{\bar{G}}_e$:-

$$\bar{\bar{G}}_e = \frac{1}{k_o^2} \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{e_{on}\, e_{om}}{2ab\Gamma_{nm}}\, e^{-\Gamma_{nm}|z-z'|} \left[\bar{x}\bar{x} \left(\left(\frac{m\pi}{b}\right)^2 - \Gamma_{nm}^2 \right) \right.$$

$$\operatorname{Cos} \frac{n\pi x}{a} \operatorname{Cos} \frac{n\pi x'}{a} \operatorname{Sin} \frac{m\pi y}{b} \operatorname{Sin} \frac{m\pi y'}{b}$$

$$+\bar{y}\bar{y} \left(\left(\frac{n\pi}{a}\right)^2 - \Gamma_{nm}^2 \right) \operatorname{Sin} \frac{n\pi x}{a} \operatorname{Sin} \frac{n\pi x'}{a} \operatorname{Cos} \frac{m\pi y}{b} \operatorname{Cos} \frac{m\pi y'}{b}$$

$$+\bar{z}\bar{z} \left(\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2 \right) \operatorname{Sin} \frac{n\pi x}{a} \operatorname{Sin} \frac{n\pi x'}{a} \operatorname{Sin} \frac{m\pi y}{b} \operatorname{Sin} \frac{m\pi y'}{b}$$

$$+\bar{x}\bar{y} \left(-\left(\frac{m\pi}{b}\right)\left(\frac{n\pi}{a}\right) \right) \operatorname{Cos} \frac{n\pi x}{a} \operatorname{Sin} \frac{n\pi x'}{a} \operatorname{Sin} \frac{m\pi y}{b} \operatorname{Cos} \frac{m\pi y'}{b}$$

$$+\bar{y}\bar{x} \left(-\left(\frac{m\pi}{b}\right)\left(\frac{n\pi}{a}\right) \right) \operatorname{Sin} \frac{n\pi x}{a} \operatorname{Cos} \frac{n\pi x'}{a} \operatorname{Cos} \frac{m\pi y}{b} \operatorname{Sin} \frac{m\pi y'}{b}$$

$$+\bar{y}\bar{z} \left(-\left(\frac{m\pi}{b}\right) \Gamma_{nm} \operatorname{Sgn}(z-z') \right) \operatorname{Sin} \frac{n\pi x}{a} \operatorname{Sin} \frac{n\pi x'}{a} \operatorname{Cos} \frac{m\pi y}{b} \operatorname{Sin} \frac{m\pi y'}{b}$$

$$+\bar{z}\bar{y} \left(+\left(\frac{m\pi}{b}\right) \Gamma_{mn} \operatorname{Sgn}(z-z') \right) \operatorname{Sin} \frac{n\pi x}{a} \operatorname{Sin} \frac{n\pi x'}{a} \operatorname{Sin} \frac{m\pi y}{b} \operatorname{Cos} \frac{m\pi y'}{b}$$

$$+\bar{z}\bar{x} \left(+\left(\frac{n\pi}{a}\right) \Gamma_{nm} \operatorname{Sgn}(z-z') \right) \operatorname{Sin} \frac{n\pi x}{a} \operatorname{Cos} \frac{n\pi x'}{a} \operatorname{Sin} \frac{m\pi y}{b} \operatorname{Sin} \frac{m\pi y'}{b}$$

$$\left. +\bar{x}\bar{z} \left(-\left(\frac{n\pi}{a}\right) \Gamma_{nm} \operatorname{Sgn}(z-z') \right) \operatorname{Cos} \frac{n\pi x}{a} \operatorname{Sin} \frac{n\pi x'}{a} \operatorname{Sin} \frac{m\pi y}{b} \operatorname{Sin} \frac{m\pi y'}{b} \right]$$

$$-\bar{z}\bar{z} \frac{1}{k_o^2} d(r-r').$$

Appendix-I: ON DYADS and DYADIC Green's Function(4) and Radiation Condition [4]

A dyad or a dyadic function $\bar{\bar{D}}$ is defined by :-

$\bar{\bar{D}} = \bar{A}\,\bar{B}$, where the vector function $\bar{A}$ is the anterior element and the vector function $\bar{B}$ is the posterior element.

A dyad plays the role very similar to that of a matrix.

Scalar Product between $\bar{\bar{D}}$ and $\bar{C}$:

Anterior scalar product : $\bar{C}.\bar{\bar{D}} = (\bar{C}.\bar{A})\bar{B} = \bar{B}(\bar{C}.\bar{A}) = \bar{B}\,(\bar{A}.\bar{C})$

Posterior scalar product: $\bar{\bar{D}}.\bar{C} = \bar{A}\,(\bar{B}.\bar{C}) = (\bar{B}.\bar{C})\bar{A} = (\bar{C}.\bar{B}.)\bar{A}$

The above two identities suggests $\bar{\bar{D}}^t = \bar{B}\,\bar{A}$

where superiscript t inplies "transpose".

Thus $\bar{\bar{D}}.\bar{C} = \bar{C}.\bar{\bar{D}}^t$

The resultant of the scalar product between $\bar{\bar{D}}$ and $\bar{C}$ is a vector function.

Vector Product between $\bar{\bar{D}}$ and $\bar{C}$:-

Anterior vector product $\bar{C} \times \bar{\bar{D}} = (\bar{C} \times \bar{A})\,\bar{B}$

Posterior vector product $\bar{\bar{D}} \times \bar{C} = \bar{A}\,(\bar{B} \times \bar{C})$

The resultant of the vector product between $\bar{\bar{D}}$ and $\bar{C}$ is a dyadic function.

e.g. In rectanguar coordinate system

$\bar{\bar{D}} = \bar{D}^{(x)}\,\bar{x} + \bar{D}^{(y)}\,\bar{y} + \bar{D}^{(z)}\,\bar{z}$

$$= \begin{bmatrix} A_x B_x \bar{x}\bar{x} + A_x B_y \bar{x}\bar{y} + A_x B_z \bar{x}\bar{z} \\ + A_y B_x \bar{y}\bar{x} + A_y B_y \bar{y}\bar{y} + A_y B_z \bar{y}\bar{z} \\ + A_z B_x \bar{z}\bar{x} + A_z B_y \bar{x}\bar{y} + A_z B_z \bar{z}\bar{z} \end{bmatrix}$$

where $\bar{D}^{(x)} = B_x \bar{A} = A_x B_x \bar{x} + A_y B_x \bar{y} + A_z B_x \bar{z}$

$\bar{D}^{(y)} = B_y \bar{A} = A_x B_y \bar{x} + A_y B_y \bar{y} + A_z B_y \bar{z}$

$\bar{D}^{(z)} = B_z \bar{A} = A_x B_z \bar{x} + A_y B_z \bar{y} + A_z B_z \bar{z}$

The above form of writing $\bar{\bar{D}}$ is most usual. An alternative form (not used usually) is :-

$\bar{\bar{D}} = \bar{x}\,{}^{(x)}\bar{D} + \bar{y}\,{}^{(y)}\bar{D} + \bar{z}\,{}^{(z)}\bar{D}$ where ${}^{(x)}\bar{D} = \bar{x}.\bar{\bar{D}}$, ${}^{(y)}\bar{D} = \bar{y}.\bar{\bar{D}}$, ${}^{(z)}\bar{D} = \bar{z}.\bar{\bar{D}}$

Such expanded representation of $\bar{\bar{D}}$ illustrates the concept that a dyadic function is a composite of three vectors functions. Divergence of $\bar{\bar{D}}$: $\nabla.\bar{\bar{D}} = (\nabla.\bar{D}^{(x)})\bar{x} + (\nabla.\bar{D}^{(y)})\bar{y} + (\nabla.\bar{D}^{(z)})\bar{z}$. Thus $\nabla.\bar{\bar{D}}$ is a vector function.

Curl of $\bar{\bar{D}}$: $\nabla \times \bar{\bar{D}} = (\nabla \times \bar{D}^{(x)})\bar{x} + (\nabla \times \bar{D}^{(y)})\bar{y} + (\nabla \times \bar{D}^{(z)})\bar{z}$

$\nabla \times \bar{\bar{D}}$ is a dyadic function.

Unit Dyad $\bar{\bar{I}}$: $\bar{\bar{I}} = \bar{x}\bar{x} + \bar{y}\bar{y} + \bar{z}\bar{z}$

The following identities manifest the properties of $\bar{\bar{I}}$.

$\bar{A}.\bar{\bar{I}} = \bar{\bar{I}}.\bar{A} = \bar{A}$ & $\nabla.(\bar{\bar{I}}\psi) = \nabla\psi$ where ψ is a scalar function.

Green's vector Theorem and its application:-

With $\bar{A} = \bar{Q} \times \nabla \times \bar{P} - \bar{P} \times \nabla \times \bar{Q}$

Using the vector identity $\nabla.(\bar{C} \times \bar{D}) = \bar{D}.\nabla \times \bar{C} - \bar{C}.\nabla \times \bar{D}$, by assuming $\bar{C} = \bar{Q}$, $\bar{D} = \nabla \times \bar{P}$ in 1st term of $\bar{A}$ and then by assuming $\bar{C}' = \bar{P}$, $\bar{D}' = \nabla\times\bar{Q}$ in 2nd term of $\bar{A}$, we have

$$\nabla\cdot\bar{A} = \bar{P}\cdot\nabla\times\nabla\times\bar{Q} - \bar{Q}\cdot\nabla\times\nabla\times\bar{P}$$

Applying Gauss theorem to this equation,

$$\int_V \nabla\cdot\bar{A}\, dv = \oint_S \bar{A}\cdot\bar{ds}$$

or
$$\int_V \left(\bar{P}\cdot\nabla\times\nabla\times\bar{Q} - \bar{Q}\cdot\nabla\times\nabla\times\bar{P}\right) dv = \oint_S (\bar{Q}\times\nabla\times\bar{P} - \bar{P}\times\nabla\times\bar{Q})\bar{ds}$$

This is Green's vecter theorem:

Now we let $\bar{P} = \bar{E}_s(r)$, $\bar{Q} = \left[\bar{\bar{G}}_e(r,r')\cdot \bar{a}\right]$ where $\bar{a}$ denotes a constant, arbitrary vector.

Putting these values in Green's vector therorem, we have

$$\int_V \left[\bar{E}_s(r)\cdot\left(\nabla\times\nabla\times\bar{\bar{G}}_e^{*}(r,r')\cdot\bar{a}\right) - \left(\bar{\bar{G}}_e(r,r')\cdot\bar{a}\right)\cdot\left(\nabla\times\nabla\times E_s(r)\right)\right] dv$$

$$= \oint_S \left[\left(\bar{\bar{G}}_e(r,r')\cdot\bar{a}\right)\times\left(\nabla\times E_s(r)\right) - \bar{E}_s(r)\times\left(\nabla\times\bar{\bar{G}}_e(r,r')\cdot\bar{a}\right)\right]\bar{ds}$$

$$= -\oint_S \left[\left(\nabla\times\bar{E}_s(r)\right)\times\left(\bar{\bar{G}}_e(r,r')\cdot\bar{a}\right) + \left(\bar{E}_s(r)\times\left(\nabla\times\bar{\bar{G}}_e(r,r')\cdot a\right)\right)\right]\cdot\bar{ds}$$

$$= -\oint_S \left[\left(\bar{n}\times\nabla\times\bar{E}_s(r)\right)\cdot\left(\bar{\bar{G}}_e(r,r')\cdot\bar{a}\right) + \left(\bar{n}\times\bar{E}_s(r)\right)\cdot\left(\nabla\times\bar{\bar{G}}_e(r,r')\cdot\bar{a}\right)\right] ds$$

where we have used $\bar{ds} = \bar{n}ds$

and the relation $(\bar{a})\times(\bar{b})\cdot\bar{C} = (\bar{C}\times\bar{a})\cdot\bar{b}$

[N.B. $\bar{n}$ denotes an outward normal]

In the volume integral part of this equation we put,

$$\nabla\times\nabla\times\bar{\bar{G}}_e(r,r') = k_o^2\bar{\bar{G}}_e(r,r') + \bar{\bar{I}}\, d(r-r')$$ [from eqn. 5(G) Ch. II.]

$$\nabla\times\nabla\times\bar{E}_s(r) = K_o^2\,\bar{E}_s - jwu_oJ(r)$$ [from eqn 4(a) Ch. II.]

and use the relation $\int_V \bar{E}_s(r)\cdot\bar{\bar{I}}\, d(r-r')dv = E_s(r')$

and rearranging terms we have [with $k_o^2\ \bar{E}_s(r)\cdot\bar{\bar{G}}_e(r,r')$ term cancelled].

$$\bar{E}_s(r')\cdot\bar{a} = -jwu_o\int_V \bar{J}(r)\cdot\bar{\bar{G}}_e(r,r')\cdot\bar{a}\,dv$$

$$-\oint\left[\left(\bar{n} \times \nabla \times \bar{E}_s(r)\right)\cdot\bar{\bar{G}}e(r,r')\cdot\bar{a} + \left(\bar{n}\times E_s(r)\right)\cdot\nabla\times\bar{\bar{G}}e(r,r')\cdot\bar{a}\right]ds$$

Replacing $\nabla\times\bar{E}_s$ by $-jwu_0\bar{H}s$, interchanging the primed and the unprimed variables and deleting $(\cdot\bar{a})$ from the above equation, since $\bar{a}$ is a constant vecor, we have

$$\bar{E}_s(r)=-jwu_0\int_V \bar{J}(r')\cdot\bar{\bar{G}}e(r',r)dv'$$

$$-\oint_S\left[\left(-jwu_0\left(\bar{n}\times\bar{H}s(r')\right)\bar{\bar{G}}e(r',r)\right)+\left(\bar{n}\times\bar{E}s(r')\right)\cdot\nabla'\times\bar{\bar{G}}e(r',r)\right]ds'$$

where $(\nabla'\times)$ means that the curl operation has to be performed in the primed coordinate system.

With $jwu_0 = jk_0\eta_0$ and $\bar{H}s = \frac{1}{\eta_0}\bar{r}\times\bar{E}s$ (As far field radiation approximation and considering the waveguide to be of infinite volue i.e. input and output matched, surface S recedes to infinity then as a result of radiation condition the surface integral of the above equation becomes zero i.e.

$$\oint_S\left(\bar{n}\times\bar{E}s(r')\right)\cdot\left(\nabla'\times\bar{\bar{G}}e(r',r)-jk_0\bar{n}\times\bar{\bar{G}}e(r',r)\right)ds'=0$$

N.B. Radiation condition :- $\lim_{z\to\infty}\left[\nabla'\times\bar{\bar{G}}_0(r,r')-jk_0\bar{z}\times\bar{\bar{G}}_0(r,r')\right]=0$

Hence $\bar{E}_s(r) = -jwu_o \int_V \bar{J}(r')\cdot\bar{\bar{G}}_e(r',r)\,dv'$

$$= -jwu_o \int_V \bar{\bar{G}}_e^t(r',r)\cdot\bar{J}(r')\,dv' = -jwu_o \int_V \bar{\bar{G}}_e(r,r')\cdot\bar{J}(r')\,dv'$$

The last experession results from the identity that

$$\bar{\bar{G}}_e^t(r',r) = \bar{\bar{G}}_e(r,r')$$

This is the effect either of the radiation condition or of the specific boundary condition. To clarify further this point we define two vector functions $\bar{P}$ and $\bar{Q}$ using a Green's function $\bar{\bar{G}}$ of the same boundary conditions but with two diferent source locations r_a and r_b such that

$\bar{P} = \bar{\bar{G}}(r,r_a)\cdot\bar{a}$ and $\bar{Q} = \bar{\bar{G}}(r,r_b)\cdot\bar{b}$

where $\bar{a}$ and $\bar{b}$ denote two different constant arbitrary vectors. Applying the vector Green's theorem to these two functions we obtain,

$$\bar{a}\cdot\bar{\bar{G}}(r_a,r_b)\cdot\bar{b} = \bar{b}\cdot\bar{\bar{G}}(r_b,r_a)\cdot\bar{a}$$

The surface integral vanishes either becuase of the radiation condition or becuase of the specific boundary conditions which $\bar{\bar{G}}$ must satisfy. Thus this equation implies $\bar{\bar{G}}^t(r_a,r_b) = \bar{\bar{G}}(r_b,r_a)$. Thus Eqn.(12) of Ch.II is proved. This equation may also be written as:- $\bar{E}_s(r) = -jwu_o \int_V \bar{\bar{G}}_e(r,r'')\cdot\bar{J}(r'')\,dv''$ (*)

With this Eqn(9) of Ch.II may be proved. In an analogous way, with ⇒ implying "componentwise correspondence" & ⇛ implying "correspondence as a whole", we can prove Eqn (9) of Ch.II with $\left[\frac{E_s(r)}{-jwu_o}\right] \Rightarrow \bar{\bar{G}}_m(r,r')$, $J(r'') \Rightarrow \nabla''\times\bar{\bar{I}}\delta(r'-r'')$ and $\bar{\bar{G}}_e(r,r'') ⇛ \bar{\bar{g}}_m(r,r'')$ in Eqn (*).

It is to be noted that radiation condition stated above is the modified waveguide version of the free space radiation condition and rests on the condition that at the large distance from the source $\nabla'\times\bar{\bar{G}}_e(r',r) \approx jk_o\bar{z}\times\bar{\bar{G}}_e(r',r)$

N.B. In above, correspondence relations $E_s(r)$, $\bar{\bar{G}}_e(r,r'')$, $J(r'')$ are with respect to $r = r'$ as coordinate origin and $\bar{\bar{G}}_m(r,r')$, $\bar{\bar{g}}_m(r,r'')$, $\nabla''\times\bar{\bar{I}}\delta(r'-r'')$ are with respect to $r = o$ as origin of the relevant coordinate system.

Appendix-II.

On the mode completeness relation (18) of Ch.-II. [8]
With normal mode function separable into a transverse part e_n and an axial part e_{zn} & constant coordinate surfaces coinciding with waveguide boundary,
Let E_t be an arbitrary transverse electric field in a given guide. The normal mode functions e_n form a complete set, provided the integrated mean square error in representing or approximating E_t by a finite series of N normal mode functions tends to zero as N tends to infinity. If we approximate E_t by $E_a = \sum_{n=1}^{N} C_n e_n$ where C_n is given by

$c_n = \langle E_t, e_n \rangle = \int_S E_t \cdot e_n \, ds$, we have

$$\langle E_t - E_a, E_t - E_a \rangle = \int_S (E_t - E_a) \cdot (E_t - E_a) \, ds \geqslant 0$$

$$\text{or} \langle E_t, E_t \rangle - 2 \langle E_a, E_t \rangle + \langle E_a, E_a \rangle \geqslant 0$$

Assuming that the functions e_n form an orthonormal set such that $\langle e_n, e_m \rangle = \int_S e_n \cdot e_m ds = \delta_{mn} = \begin{cases} 0 \text{ for } n \neq m \\ 1 \text{ for } n = m \end{cases}$

$$\langle E_t, E_t \rangle - 2 \left\langle \sum_{n=1}^{N} c_n e_n, E_t \right\rangle + \left\langle \sum_{n=1}^{N} c_n e_n, \sum_{n=1}^{N} c_n e_n \right\rangle \geqslant 0$$

$$\text{or} \langle E_t, E_t \rangle - 2 \sum_{n=1}^{N} c_n \langle e_n, E_t \rangle + \sum_{n=1}^{N} c_n^2 \geqslant 0$$

$$\text{or} \langle E_t, E_t \rangle - 2 \sum_{n=1}^{N} c_n^2 + \sum_{n=1}^{N} C_n^2 \geqslant 0$$

$$\text{Or} \langle E_t, E_t \rangle - \sum_{n=1}^{N} c_n^2 \geqslant 0$$

If, when N tends to infinity

$$\langle E_t, E_t \rangle - \sum_{n=1}^{N\to\infty} c_n^2 = 0$$

then we say that the mode set e_n is complete. This is also called the closure property of the set e_n. E_t must be at least piecewise continuous, if the above eqality equation is to be valid. Physical fields are, of course, sufficiently well behaved to satisfy these requirements.

An equivalent statement of the completeness property is the following relation:-

$$\sum_n e_n(x', y')\, e_n(x,y) = (\bar{\bar{x}}\hat{x}+\bar{y}\bar{y})\, d(x-x')\, d(y-y') = \bar{\bar{I}}_2\, d(x-x')\, d(y-y')$$

~~d(x-x') d(y-y')~~ where $\bar{\bar{I}}_2$ is the 2-dimensional unit dyadic.

This equation is seen to be the expansion of the unit source of this ... on the the right hand side. The validity of this relation involving delta function may be checked by scalar-post-multiplying both sides by $e_m(x,y)$ and integrating over the guide cross-section.

Thus LHS= $\sum_n e_n(x',y') \int_S e_n(x,y) \cdot e_m(x,y)\,ds = \sum_n e_n(x',y')\,\delta_{mn}$

$= e_m(x',y')$ [By orthogonal property of e_n] and

RHS $= \int_S \bar{\bar{I}}_2 \cdot e_m(x,y)\,d(x-x')\,d(y-y')\,ds = e_m(x',y')$

[By the property of delta function]

Using the above mode completeness relation the expansion for an arbitrary field E_t follows by superposition i.e.

$$E_t(x,y) = \int_S E_t(x',y') \cdot \bar{\bar{I}}_2 \, d(x-x')\, d(y-y')\, dx'dy'$$

$$= \sum_n \int_S E_t(x',y') \cdot e_n(x',y')\, e_n(x,y)\, dx'dy'$$

$$= \sum_n \int_S E_t(x',y') \cdot e_n(x',y')\, e_n(x,y)\, dx'dy'$$

$$= \sum_n C_n e_n(x,y)$$

It is to be noted that apart from its use in the representation of an arbitrary field E_t relation like the completeness relation having an infinite sine and cosine function series on one side & scalar d(x-x') d(y-y') function on the other may be considered to be the purely mathematical expansion function representing d(x-x') d(y-y') whose validity is proved by multiplying by a testing function on both sides and integrated over an infinite xy-plane. In waveguide cases, the testing function reduces to an eigenfunction, and the infinite x-y plane reduces to the waveguide cross-section.

This set of equations can be written in matrix form as

$$(l_{mn}) (\mathcal{L}_n) = (g_m) \qquad \ldots \qquad (5).$$

where

$$(l_{mn}) = \begin{pmatrix} \langle W_1 \, Lf_1 \rangle & \langle W_1 \, Lf_2 \rangle & \ldots & \ldots \\ \langle W_2 \, Lf_1 \rangle & \langle W_2 \, Lf_2 \rangle & \ldots & \ldots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix};$$

$$(\mathcal{L}_n) = \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \vdots \end{pmatrix}; \quad (g_m) = \begin{pmatrix} \langle w_1, g \rangle \\ \langle w_2, g \rangle \\ \vdots \end{pmatrix}$$

If the matrix (l_{mn}) is nonsingular its inverse $(l_{mn})^{-1}$ exists.

In that case

$$(\mathcal{L}_n) = (l_{mn})^{-1} (g_m) \qquad \ldots \qquad (6)$$

and

$$f = (f_n) (\mathcal{L}_n) = (f_n)(l_{mn})^{-1} (g_m) \qquad \ldots \qquad (7).$$

where matrix $(f_n) = (f_1, f_2, f_3 \ldots \quad \ldots)$

The solution may be exact or approximate, depending on the choice of f_n and W_m. If the matrix (l_{mn}) is of infinite order, it can be inverted only in special cases, for example, if it is diagonal. The classical eigenfunction method leads to a diagonal matrix and can be thought of as a special case of method of moments. If the sets f_n and W_m are finite, the matrix is of finite order, and can be inverted by known methods. The main task in the matrization of the functional equation by the moment method is the choice of f_n and W_m. The f_n should be linearly independent and chosen so that some sorts of superposition like that in Eqn(2) can approximate f reasonably well. The W_m should also be linearly independent and chosen so that the scalar producets $\langle W_m, g \rangle$ depend on relatively independent properties of g. Some additional factors which affect the choice

Thus for N=1, $l_{11} = \frac{1}{3} g_1 = \frac{11}{30}$ and $\mathcal{L}_1 = \frac{11}{30}$ from Eqn(5)

For N =2,
$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{pmatrix} = \begin{pmatrix} \frac{11}{30} \\ \frac{7}{12} \end{pmatrix} \therefore \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{10} \\ \frac{2}{3} \end{pmatrix}$$

For N=3,
$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{3}{5} \\ \frac{1}{2} & \frac{4}{3} & 1 \\ \frac{3}{5} & 1 & \frac{9}{7} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \end{pmatrix} = \begin{pmatrix} 11/30 \\ 7/12 \\ 51/70 \end{pmatrix} \therefore \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{3} \end{pmatrix}$$

For N=3 we obtain the exact solution $f = \sum_{n}^{N=3} \mathcal{L}_n f_n$

For N $\geq$ 3 we obtain the same exact solution.

Solution by Point matching method with entire domain basis

With $f_n = x - x^{n+1}$ Let $x_m = \frac{m}{N+1}$, m = 1, 2,3, ...N be the equipaced points in the interval $0 \leq x \leq 1$ at which the equation is to be satisfied exactly. For this we let $W_m = d\ (x - x_m)$ and we have $l_{mn} = n(n+1) \left(\frac{m}{N+1}\right)^{n-1}$,

$$g_m = 1 + 4 \left(\frac{m}{N+1}\right)^2$$

Thus for N=1, N=2 we again have an approximate. Solution and for N$\geq$3, we have the exact solution.

Solution by Subsectional bases:

With $x_m = \frac{m}{N+1}$, m = 1,2,3, ...N as the equispaced points in the interval $0 \leq x \leq 1$, let $f_n = P(x - x_m)$ or

$f_n = T\ (x - x_n)$

where pulse function $P(x) = \begin{cases} 1 & |x| < \frac{1}{2(N+1)} \\ 0 & |x| > \frac{1}{2(N+1)} \end{cases}$ and triangle

function $T(x) = 1 - |x| (N+1)$

for $|x| \leq \frac{1}{N+1}$

$T(x) = 0$ for $|x| > \frac{1}{N+1}$

Since L [P(x)] functions are not in the range of L. We take $f_n = T(x-x_n)$, $LT(x-x_n) = (N+1) \left[-d(x-x_{n-1}) + 2d(x-x_n) - d(x-x_{n+1}) \right]$

Choosing $W_m = P(x-x_m)$, $l_{mn} = \begin{cases} 2(N+1) & m = n \\ -(N+1) & |m-n| = 1 \\ 0 & |m-n| > 1 \end{cases}$;

& $g_m = \frac{1}{N+1} \left[1 + \frac{4m^2 + \frac{1}{3}}{(N+1)^2} \right]$; N = 5 yields discrete values of f, coinciding with exact f(x).

Art-2: Solution of the obstacle problem by the point-matching method of moment:

Although there exists numerical methods by which the integral operator obstacle equation could be solved numerically, the complexity of three-dimensional and the dyadic green's function involved indicates that further computational complication in the scalar multiplication process of matrizing the functional equation must be avoided as far as possible. This naturally implies that point-matching method of moment may be adopted for the obstacle problem. Even in the relatively simple case of plane wave incidence on obstacles in un-bounded free space, only point-matching method together with rectangularly sided cells has been attempted for the volume type of integral equations(10). Fortunately, this process has been found to be capable of producing good numerical results(7). Thus point-matching method with rectangularly sided cells is employed in the present anglysis.

The volume V occupied by the dielectric is first divided into L equal rectangular-sided cells ΔV_l, $l=1$, 2,......L, each of which has constant dimensions $\Delta X, \Delta y$ and ΔZ. The incident electric field, assumed to be uniform inside the lth cell is designated $E_l^i(r_l)$, where r_l represents the centre of the lth cell.

The corresponding current in the dielectric may be expressed as

$$J(r) = \sum_{l=1}^{L} \sum_{k=1}^{3} \bar{u}_k J_l^k \, P^l(r) \quad \ldots (1)$$

where $\bar{u}_k$ denotes a unit vector and $P^l(r) = \begin{cases} 1 & \text{for } r \text{ in } \Delta V_l \\ 0 & \text{otherwise} \end{cases}$

and $\bar{U}_1 = \bar{x}$, $\bar{U}_2 = \bar{y}$, $\bar{U}_3 = \bar{z}$,

The equation to be solved is :- [Eqn(17)Ch-II Art-I].

$$-jwu_o \int_V \bar{\bar{G}}_{eo}(r,r').J(r')dv' - \frac{J(r)}{jw[e(r)-e_o]} - \frac{(\bar{z}.J(r)\bar{z}}{jwe_o(r)} = -E_i(r) \quad (2)$$

In the point matching method we generate a set of linear equations by first substituting Eqn(1) into Eqn(2) and then performing a scalar product on the resulting equation with the testing function $W_q^p(r) = d(r-r_q)\,\bar{u}_p$ for p=1,2,3 and q= 1,2, ... L.

where $\bar{u}_p$ is a unit vector.

The scalar product between vectors f and g is defined as

$$\langle f, g\rangle = \int_V f.g \, dv$$

Thus $\langle W_q^p(r), -E_i(r)\rangle = -\int_V d(r-r_q)\bar{U}p.E_i(r)dv'$

$= -\int_V d(r-r_q)\, E_i^p(r)\, dv' = -E_i^p(r_q) = -C_q^{pq}$ for $p = 1, 2, 3$

$q = 1, 2, \ldots L$.

$$\left\langle W_q^p(r), \frac{[\bar{z}.J(r)]\bar{z}}{jwe(r)}\right\rangle = \left\langle W_q^p(r), \frac{\left[\bar{u}_3 . \sum_{l=1}^{L}\sum_{k=1}^{3} \bar{u}_k J_k^k P^l(r)\right]\bar{u}_3}{jw\, e(r)}\right\rangle$$

$$= \int_V \frac{d(r-r_q)\bar{U}p}{jwe(r)} . \left[\bar{u}_3 . \sum_{l=1}^{L}\sum_{k=1}^{3} \bar{u}_k J_k^k P^l(r)\right]\bar{u}_3 dv'$$

$$= \int_V \sum_{k=1}^{3}\sum_{l=1}^{L} \frac{J_k^k [\delta_3^p . \delta_3^k]}{jwe(r)} P^l(r) d(r'-r_q) dv'$$

$$= \sum_{k=1}^{3}\sum_{l=1}^{L} \frac{J_k^k d_q^l [\delta_3^p . \delta_3^k]}{jwe_l(r_q)}$$ for $p=1,2\ 3$ and $q=1,2,..L$.

$$\left\langle W_q^p(r), \frac{J(r)}{jw[e(r)-e_o]}\right\rangle = \int_V d(r-r_q)\bar{u}_p . \frac{\sum_{k-1}^{3}\sum_{l=1}^{L} \bar{U}_k J_k^k P^l(r)}{jw[e(r)-e_o]} dv'$$

$$= \int_V \sum_{k=1}^{3}\sum_{l=1}^{L} \frac{J_k^k \delta_k^p P^l(r)}{jw[e(r)-e_o]} d(r-r_q) dv' = \sum_{K=1}^{3}\sum_{l=1}^{L}$$

$$= \sum_{K=1}^{3}\sum_{l=1}^{L} J_k^l \frac{\delta_q^l\, \delta_k^i}{jw[e(r_q)-e_o]}$$ for $p=1,2,3$ & $q = 1,2, \ldots L$.

[N.B. $\int_V p^l(r) d(r-r_q)\, dv' = P^l(r_q) = \delta_q^l$]

Rewriting $\int_V \bar{G}_{eo}(r,r').J(r')dv' = \int_V \left[\sum_{k=1}^{3}\bar{u}_p \sum_{K=1}^{3}\bar{u}_k\, G_{eo}^{pk}(r,r')\right]$.

$$\sum_{k=1}^{3} \bar{u}_k \sum_{l=1}^{L} J_1^k P^1(r')\Big] dv' \Big\} = \Big\{ \sum_{p=1}^{3} \bar{u}_p \sum_{K=1}^{3} \int_V G_{eo}^{pk} (r,r')$$

$$\times \sum_{l=1}^{L} J_{\ell}^k P^1(r') dv' \Big\} = \Big\{ \sum_{p=1}^{3} \bar{u}_p \sum_{k=1}^{3} \sum_{l=1}^{L} J_{\ell}^k \int_{\Delta V_\ell} G_{eo}^{pk} (r,r') dv' \Big\}$$

For a fixed p and q in the scalar multiplication $\Big[\Big\langle W_q^p(r),$

$$j\omega u_0 \int_V \bar{\bar{G}}_{eo} (r,r') \cdot J(r') dv' \Big\rangle \Big] = \Big\langle W_q^p(r),\ j w u_o \int_V \bar{\bar{G}}_{eo} (r,r') \cdot J(r') dv' \Big\rangle$$

$$= j\omega u_0 \int_V \Big[d(r-r_q)\ \bar{u}_p \Big] \cdot \Big[\bar{u}_p \sum_{K=1}^{3} \sum_{l=1}^{3} J_{\ell}^k \int_{\Delta V_\ell} G_{eo}^{pk} (r,r')\ dv' \Big] dv'$$

$$= j w u_o \sum_{K=1}^{3} \sum_{l=1}^{L} J_{\ell}^k \int_{\Delta V_\ell} G_{eo}^{pk} (r_q,r') dv' \quad \text{for } p=1,2,3 \text{ and}$$

$q=1,2, \ldots L.$

Hence this point-matching moment method yields the following set of linear equations:-

$$\sum_{K=1}^{3} \sum_{l=1}^{L} J_1^k A_{1q}^{pk} = -C_q^p \quad (3) \quad \text{where} \quad C_q^p = +E_i^p (r_q) \ldots (3a)$$

$$\text{and } A_{\ell q}^{pk} = -j w u_o \cdot Q_{1q}^{pk} - \frac{\delta_q^1}{j w} \left[\frac{\delta_k^p}{[e(r_q) - e_o]} + \frac{\delta_3^k \cdot \delta_3^p}{e(\gamma_q)} \right] \ldots (3b).$$

$$\text{wherein } Q_{1q}^{pk} = \int_{\Delta V_\ell} G_{eo}^{pk} (r_q, r')\ dv'$$

When expanded and put in matrix form Eqn 3(a) appears as:-

$$
-j\omega\epsilon_0\mu
\left[\begin{array}{ccc|ccc|ccc}
Q'^{11}_{11} & Q^{12}_{11} & Q^{13}_{11} & Q^{11}_{21} & Q^{12}_{21} & Q^{13}_{21} & Q^{11}_{L1} & Q^{12}_{L1} & Q^{13}_{L1} \\
Q^{21}_{11} & Q'^{22}_{11} & Q^{23}_{11} & Q^{21}_{21} & Q^{22}_{21} & Q^{23}_{21} & Q^{21}_{L1} & Q^{22}_{L1} & Q^{23}_{L1} \\
Q^{31}_{11} & Q^{32}_{11} & Q'^{33}_{11} & Q^{31}_{21} & Q^{32}_{21} & Q^{32}_{21} & Q^{31}_{L1} & Q^{32}_{L1} & Q^{33}_{L1} \\
\hline
Q^{11}_{12} & Q^{12}_{12} & Q^{13}_{12} & Q'^{11}_{22} & Q^{12}_{22} & Q^{13}_{22} & Q^{11}_{L2} & Q^{12}_{L2} & Q^{13}_{L2} \\
Q^{21}_{12} & Q^{22}_{12} & Q^{23}_{12} & Q^{21}_{22} & Q'^{22}_{22} & Q^{23}_{23} & Q^{21}_{L2} & Q^{22}_{L2} & Q^{23}_{L2} \\
Q^{31}_{12} & Q^{32}_{12} & Q^{33}_{12} & Q^{31}_{22} & Q^{32}_{22} & Q'^{33}_{22} & Q^{31}_{L2} & Q^{32}_{L2} & Q^{33}_{L2} \\
\hline
Q^{11}_{1L} & Q^{12}_{1L} & Q^{13}_{1L} & Q^{11}_{2L} & Q^{12}_{2L} & Q^{13}_{2L} & Q'^{11}_{LL} & Q^{12}_{LL} & Q^{13}_{LL} \\
Q^{21}_{1L} & Q^{22}_{1L} & Q^{23}_{1L} & Q^{21}_{2L} & Q^{22}_{2L} & Q^{23}_{2L} & Q^{21}_{LL} & Q'^{22}_{LL} & Q^{23}_{LL} \\
Q^{31}_{1L} & Q^{32}_{1L} & Q^{33}_{1L} & Q^{31}_{2L} & Q^{32}_{2L} & Q^{33}_{2L} & Q^{31}_{LL} & Q^{32}_{LL} & Q'^{33}_{LL}
\end{array}\right]
\left[\begin{array}{c}
J^1_1 \\ J^2_1 \\ J^3_1 \\ J^1_2 \\ J^2_2 \\ J^3_2 \\ J^1_L \\ J^2_L \\ J^3_L
\end{array}\right]
= -
\left[\begin{array}{c}
C^1_1 \\ C^2_1 \\ C^3_1 \\ C^1_2 \\ C^2_2 \\ C^3_2 \\ C^1_L \\ C^2_L \\ C^3_L
\end{array}\right]
\quad ..(4)
$$

where the primed diagonal elements corresponding to Eqn (17) of Ch.II for μ_o,e(r) medium are :-

$$
Q'^{11}_{11} = Q^{11}_{11} - \left[\frac{1}{k_1^2(r_1) - k_o^2}\right], \qquad Q'^{22}_{11} = Q^{22}_{11} - \left[\frac{1}{k_1^2(r_1) - k_o^2}\right],
$$

$$
Q'^{33}_{11} = Q^{33}_{11} - \left[\frac{1}{K_1^2(r_1) - k_o^2} + \frac{1}{k_1^2(r_1)}\right] \quad .. (4
$$

$$Q'^{11}_{22} = Q^{11}_{22} - \left[\frac{1}{k_1^2(r_2)-k_o^2}\right], \quad Q'^{22}_{22} - \left[\frac{1}{k_1^2(r_2)-k_o^2}\right],$$

$$Q'^{33}_{22} = Q^{33}_{22} - \left[\frac{1}{k_1^2(r_2)-k_o^2} + \frac{1}{K_1^2(r_2)}\right]$$

- - - - - - - - - - - - - - -

$$Q'^{11}_{LL} = Q^{11}_{LL} - \left[\frac{1}{k_1^2(r_L)-k_o^2}\right], \quad Q'^{22}_{LL} = Q^{22}_{LL} - \left[\frac{1}{k_1^2(r_L)-k_o^2}\right],$$

$$Q'^{33}_{LL} = Q^{33}_{LL} - \left[\frac{1}{k_1^2(r_L)-k_o^2} + \frac{1}{k_1^2(r_L)}\right]$$

and where the primed diagonal elements corresponding to Eqn (13) of Ch.II. for U_o, Θ_o medium are :-

$$Q'^{11}_{11} = Q^{11}_{11} - \left[\frac{1}{k_1^2(r_1)-k_o^2}\right], \quad Q'^{22}_{11} = Q^{22}_{11} - \left[\frac{1}{k_1^2(r_1)-k_o^2}\right],$$

$$Q'^{33}_{11} = Q^{33}_{11} - \left[\frac{1}{K_1^2(r_1)-k_o^2} + \frac{1}{k_o^2}\right]$$

$$Q'^{11}_{22} = Q^{11}_{22} - \left[\frac{1}{k_1^2(r_2)-k_o^2}\right], \quad Q'^{22}_{22} = Q^{22}_{22} - \left[\frac{1}{k_1^2(r_2)-k_o^2}\right],$$

$$Q'^{33}_{22} = Q^{33}_{22} - \left[\frac{1}{k_1^2(r_2)-K_o^2} + \frac{1}{k_o^2}\right]$$

- - - - - - - - - - - - - - -

$$Q'^{11}_{LL} = Q^{11}_{LL} - \left[\frac{1}{k_1^2(r\ \ )-k_o^2}\right], \quad Q'^{22}_{LL} = Q^{22}_{LL} - \left[\frac{1}{k_1^2(r_L)-k_o^2}\right],$$

$$Q'^{33}_{LL} = Q^{33}_{LL} - \left[\frac{1}{k_1^2(r_L)-k_o^2} + \frac{1}{k_o}\right]$$

It is to be noted that in Equation(4) subscript indices q &/or ℓ refer to the position vector and the superscript indices p & /or k refer to the unit vectors $\bar{x}$, $\bar{y}$, $\bar{z}$.

For finding the integral $Q^{pk}_{lq} = \int_{\Delta V_\ell} G^{pk}_{eo}(r_q, r')\, dv'$ we use the following relations:-

$$\int_{x_\ell - \Delta x_\ell/2}^{x_\ell + \Delta x_\ell/2} \sin\frac{n\pi x'}{a}\, dx' = \sin\frac{n\pi x_\ell}{a}\left[\frac{\sin(n\pi \Delta x_\ell/2a)}{(n\pi/2a)}\right]; \quad \int_{x_\ell - \Delta x_\ell/2}^{x_\ell + \Delta x_\ell/2} \cos\frac{n\pi x'}{a}\, dx'$$

$$= \cos\frac{n\pi x_\ell}{a}\left[\frac{\sin(n\pi \Delta x_\ell/2a)}{(n\pi/2a)}\right];$$

$$\int_{Y_\ell - \Delta Y_\ell/2}^{Y_\ell + \Delta Y_\ell/2} \sin\frac{m\pi Y'}{b}\, dy' = \sin\frac{m\pi Y_\ell}{b}\left[\frac{\sin(m\pi \Delta y_\ell/2b)}{(m\pi/2b)}\right]$$

$$\int_{Y_\ell - \Delta Y_\ell/2}^{Y_\ell + \Delta Y_\ell/2} \cos\frac{m\pi y}{b}\, dy' = \cos\frac{m\pi Y_\ell}{b}\left[\frac{\sin(m\pi \Delta y_\ell/2b)}{(m\pi/2b)}\right]$$

<u>For $Z_q \neq Z_1$ cases</u>

$$I = I_1 = \int_{Z_\ell - \Delta Z_\ell/2}^{Z_\ell + \Delta Z_\ell/2} e^{-jK_{nm}|Z_q - z'|}\, dz' \quad \text{for } Z_q \gtrless Z' \quad \left[Z_\ell - \frac{\Delta Z_\ell}{2} \le Z' \le Z_\ell + \frac{\Delta Z_\ell}{2}\right]$$

$$= \begin{cases} \int_{Z_\ell - \Delta Z_\ell/2}^{Z_\ell + \Delta Z_\ell/2} e^{-jK_{nm}(Z_q - Z')}\, dz' & \text{for } Z_q > Z' \approx Z_\ell \\ \int_{Z_\ell - \Delta Z_\ell/2}^{Z_\ell + \Delta Z_\ell/2} e^{jK_{nm}(Z_q - Z')}\, dz' & \text{for } Z_q < Z' \approx Z_\ell \end{cases}$$

$$= \begin{cases} \frac{2}{K_{nm}} e^{-jk_{nm}(Z_q - Z_1)} \sin K_{nm}\frac{\Delta Z_\ell}{2} & \text{for } Z_q > Z' \\ \frac{2}{K_{nm}} e^{jK_{nm}(Z_q - Z_1)} \sin K_{nm}\frac{\Delta Z_\ell}{2} & \text{for } Z_q < Z' \end{cases}$$

$$= \frac{2}{K_{nm}} e^{-jK_{nm}|Z_q - Z_1|} \sin K_{nm}\frac{\Delta Z_\ell}{2} \quad \text{for } Z_q \gtrless Z_\ell.$$

<u>For $Z_q = Z_1$ cases with pk = 11, 22, 33, 12, 21</u>

$$I = I_2 = \int_{Z_\ell - \Delta Z_\ell/2}^{Z_\ell} e^{-jKnm(Z_q - Z')} dz' \ (\text{for } Z_q > Z') + \int_{Z_\ell}^{Z_\ell + \Delta Z_\ell/2} e^{jknm(Z_q - Z')} dz \ (\text{for } Z_q < Z')$$

$$(\text{for } Z_q \quad Z) = \left\{ e^{-jKnm\,\Delta Z_\ell/2} - 1 \right\} \frac{2\cos Knm(Z_q - Z_\ell)}{jK_{nm}}$$

$$= \frac{2}{jKnm} \left(e^{-jKnm\,\Delta Z_\ell/2} - 1 \right) \text{for } Z_q = Z_1$$

<u>For $Z_q = Z_1$ cases</u> : with pk = 32,23,31, 13.

$$I = I_3 = \pm \left\{ \int_{Z_\ell - \Delta Z_\ell/2}^{Z_\ell} e^{-jKnm(Z_q - Z')} dz' (\text{for } Z_q > Z') - \int_{Z_\ell}^{Z_\ell + \Delta Z_\ell/2} e^{jK_{nm}(Z_q - Z)} dz' (\text{for } Z_q < Z') \right\} = \pm \left\{ e^{-jKnm\frac{\Delta Z_1}{2}} - 1 \right\} \frac{2\sin K_{nm}(Z_q - Z_\ell)}{K_{nm}}$$

$$= 0 \text{ for } Z_q = Z_\ell$$

With these integrations carried out Q_{1q}^{pk} takes the following form :-

$$Q_{1q}^{pk} = -\frac{j}{2abk_o^2} \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{e_{on} e'_{om}}{K_{nm}} \, IQF_{nm}^{pk} \qquad (5)$$

where

$$I = \begin{cases} I_1 & \text{for } Z_q \neq Z_1 \ \& \text{ for all pk values} \\ I_2 & \text{for } Z_q = Z_1 \ \& \text{ for pk} = 11,22,33,12,21 \\ I_3 = 0 & \text{for } Z_q = Z_1 \ \& \text{ for pk} = 32,33,\ 31,\ 13 \end{cases}$$

$$Q = \left[\frac{\sin\left(n\Pi\,\Delta x_1/2a\right)}{(n\Pi/2a)} \right] \left[\frac{\sin\left(m\Pi\,\Delta y_1/2b\right)}{(m\Pi/2b)} \right]$$

$$F_{nm}^{pk} = \left[\text{the } (n,m)\text{th term of } G_{eo}^{pk}(r_q, r_l)\right] \cdot e^{jk_{nm}|Z_q - Z_l|}$$

N.B., For pk = 23,32,31,13 F_{nm}^{pk} includes the $\pm$ sign depending on $Z_q > Z_l$ or $Z_q < Z_l$. pk=23 implies p=2 and k=3 ; pk=32 implies p=3 and k=2 & so on.

<u>Art-3:</u> (a) Evaluation Formulas for Q_{lq}^{pk} elements with $Z_q \neq Z_l$

With $Z_q \neq Z_l$, the Q_{lq}^{pk} elements have the exponential factor $e^{-jk_{nm}|Z_q - Z_l|} = e^{-\mathcal{G}_{nm}|Z_q - Z_l|}$. Except for the few terms with first few values of n and m all the remaining terms of a Q_{lq}^{pk} element are real. If $\mathcal{G}_{nm}$ is real i.e. $\left[\left(\frac{n\Pi}{a}\right)^2 + \left(\frac{m\Pi}{b}\right)^2\right] > k_o^2$ then the double infinite series terms of Q_{lq}^{pk} decrease rapidly with increasing n and m. Therefore a Q_{lq}^{pk} element terms with real $\mathcal{G}_{mn}$ can be computed with a finite series truncated according to a precision criterion established by the value of $\mathcal{G}_{nm}|Z_q - Z_l|$. Depending on the value of $|Z_q - Z_l|$ approximately 16x8 upto 21x12 terms were used for n and m in the typical examples reported in a paper by J.J.H.Wang(7)

To this value of the truncated series we add the few terms (usually 2 or so) with imaginary $\mathcal{G}_{nm}$ to obtain the value of a Q_{lq}^{pk} element.

In order to develop working formulas and expressions for summing the double infinite series we proceed as follows:-

Rewriting Q^{pk}_{lq} as follows

$$Q^{pk}_{lq} = C \sum_{n=o}^{\infty} e_{on} \; a^{pk}_{lq}(n) \sum_{m=o}^{\infty} e'_{om} \, b^{pk}_{lq}(n,m) \quad .$$

where $C = \dfrac{1}{2abk_o^2}$

$a^{pk}_{lq}(n)$ is the part of Q^{pk}_{lq} comprising sine and cosine functions involving n only and the factor $\frac{1}{n}$ (if any)

$b^{pk}_{lq}(n,m)$ is the part of Q^{pk}_{lq} comprising sine and cosine functions involving m only and a factor involving m, $\Gamma_{nm} = jK_{nm}$, jK_{nm}, and $e^{-\Gamma_{nm}|Z_q - Z_l|} \sinh \Gamma_{nm} \frac{\Delta Z_l}{2}$

Expanding Q^{pk}_{lq} we have :-

$$Q^{pk}_{lq} = C \left[a^{pk}_{lq}(0) \sum_{m=o}^{\infty} e'_{om} \, b^{pk}_{lq}(0,m) \right] + C \left[a^{pk}_{lq}(1) \sum_{m=0}^{\infty} e'_{om} \, b^{pk}_{lq}(1,m) \right]$$

$$+ C \left[2a^{pk}_{lq}(NT) \sum_{m=o}^{\infty} e'_{om} \, b^{pk}_{lq}(NT,m) \right] + \;,, \quad \ldots$$

Now $a^{pk}_{lq}(0) \; b^{pk}_{lq}(0,0) = 0$ for all pk

$a^{pk}_{lq}(0) = 0$ for all pk except pk = 11

$b^{pk}_{lq}(NT,0) = 0$ for all pk except pk=22, NT=1,2,3 . . [Chosen for a particular truncation]

Thus we have (expanding upto n=NT & m = MT)

$$\left.\begin{aligned} Q^{11}_{1q} &= 2C\, a^{11}_{1q}(0) \sum_{m=1}^{MT} b^{11}_{1q}(0,m) + 4C \sum_{n=1}^{NT} a^{11}_{1q}(n) \sum_{m=1}^{MT} b^{11}_{1q}(n,m) \\ Q^{22}_{1q} &= 2C \sum_{n=1}^{NT} a^{22}_{1q}(n)\; b^{22}_{1q}(n\;\; 0) \\ &\quad + 4C \sum_{n=1}^{NT} a^{22}_{1q}(n) \sum_{m=1}^{MT} b^{22}_{1q}(n,m) \\ Q^{pk}_{1q} &= 4C \sum_{n=1}^{NT} a^{pk}_{1q}(n) \sum_{m=1}^{MT} b^{pk}_{1q}(n,m) \quad \text{for other pk values} \end{aligned}\right\} \text{I}$$

Similarly we may rewrite Q^{pk}_{1q} as follows :-

$$Q^{pk}_{1q} = C \sum_{m=0}^{\infty} e'_{om}\; b^{pk}_{1q}(m) \sum_{n=0}^{\infty} e_{on}\; a^{pk}_{1q}(n,m)$$

Where $C = \dfrac{1}{2abk_o^2}$

$b^{pk}_{1q}(m)$ is the part of Q^{pk}_{1q} Comprising Sine and Cosine functions involving m only and the factor $\frac{1}{m}$ (if any)

$a^{pk}_{1q}(n,m)$ is the part of Q^{pk}_{1q} Comprising Sine and Cosine functions involving n only and a factor involving $n, m, G_{nm} = j K_{nm}$

and $e^{-G_{nm}|z_q - z_l|} \cdot \sinh G_{nm} \frac{\Delta z_l}{2}$ [illegible]

Expanding Q^{pk}_{1q} We have :-

$$Q^{pk}_{1q} = C\left[b^{pk}_{1q}(0)\sum_{n=0}^{\infty} e_{on}\, a^{pk}_{1q}(n,0)\right] + C\left[2b^{pk}_{1q}(1)\sum_{n=0}^{\infty} e_{on}\, a^{pk}_{1q}(n,1)\right]$$

$$+ \ldots + C\left[2b^{pk}_{1q}(MT)\sum_{n=0}^{\infty} e_{on}\, a^{pk}_{1q}(n, MT)\right] + \ldots$$

$$= C\left[b^{pk}_{1q}(0)\, a^{pk}_{1q}(0,0) + 2b^{pk}_{1q}(0)\sum_{n=1}^{\infty} a^{pk}_{1q}(n,0)\right] + \ldots$$

$$+ C\left[2b^{pk}_{1q}(1)\, a^{pk}_{1q}(0,1) + 4b^{pk}_{1q}(1)\sum_{n=1}^{\infty} a^{pk}_{1q}(n,1)\right] \ldots\ldots$$

$$+ C\left[2b^{pk}_{1q}(MT)\, a^{pk}_{1q}(0,MT) + 4b^{pk}_{1q}(MT)\sum_{n=1}^{\infty} a^{pk}_{1q}(n,MT)\right] + \ldots$$

Now $b^{pk}_{1q}(0)\, a^{pk}_{1q}(0,0) = 0$ for all pk

$$b^{pk}_{1q}\ (0) = 0 \quad \text{for all } pk \quad \text{except} \quad pk = 22$$

$$a^{pk}_{1q}\ (0, MT) = 0 \quad \text{for all } pk \quad \text{except} \quad pk = 11, \ MT = 1, 2, 3 \ldots\ldots$$

[chosen for a particular truncation]

Thus we have (expanding up to n = NT & m = MT)

$$Q^{11}_{1q} = 2C \sum_{m=1}^{MT} b^{11}_{1q}(m)\ a^{11}_{1q}(0, m) + 4C \sum_{m=1}^{MT} b^{11}_{1q}(m) \sum_{n=1}^{NT} a^{pk}_{1q}(n, m)$$

$$\sum_{n=1} a^{11}_{1q}(n, m)$$

$$Q^{22}_{1q} = 2C\ b^{22}_{1q}(0) \sum_{n=1}^{NT} a^{22}_{1q}(n, 0) + \ldots\ldots$$

$$+ 4C \sum_{m=1}^{MT} b^{22}_{1q}(m) \sum_{n=1}^{NT} a^{22}_{1q}(n, m) \qquad \text{II}$$

$$Q^{pk}_{1q} = 4C \sum_{m=1}^{MT} b^{pk}_{1q}(m) \sum_{n=1}^{NT} a^{pk}_{1q}(n, m) \quad \text{for other } pk \text{ values}$$

Either of the Equation I or Equation II may be used for evaluation of the double series of Q^{pk}_{1q} elements according to convenience.

The explicit expressions for Q^{pk}_{1q} elements for various pk are as follows :- [for a specific problem the few imaginary G_{nm} terms of a Q^{pk}_{1q} element can easily be picked up]

Using Equation I

$$Q_{1q}^{11} = \Bigg[\Big\{2C\left(\frac{4b}{\pi}\Delta x_l\right)\sum_{m=1}^{MT}\frac{k_o^2}{m G_{om}^2} e^{-G_{om}|z_q - z_l|} \cdot \sinh G_{om}\frac{\Delta z_l}{2} \sin\frac{m\pi y_q}{b} \cdot \sin\frac{m\pi y_l}{b}\sin\frac{m\pi \Delta y_l}{2b}\Big\}$$

$$+\Big\{4C\left(\frac{8ab}{\pi^2}\right)\sum_{n=1}^{NT}\frac{1}{n}\cos\frac{n\pi x_q}{a}\cos\frac{n\pi x_l}{a}\sin\frac{n\pi\Delta x_l}{2a}\sum_{m=1}^{MT}\frac{\left[k_o^2-\left(\frac{n\pi}{a}\right)^2\right]}{m G_{nm}^2} \cdot e^{-G_{nm}|z_q - z_l|} \cdot \sinh G_{nm}\frac{\Delta z_l}{2} \cdot \sin\frac{m\pi y_q}{b} \sin\frac{m\pi y_l}{b} \cdot \sin\frac{m\pi\Delta y_l}{2b}\Big\}\Bigg] \quad (21.1)$$

Using Equation II

$$Q_{1q}^{22} = \Bigg[\Big\{2C\left(\frac{4a}{\pi}\Delta y_l\right)\sum_{n=1}^{NT}\frac{k_o^2 e^{-G_{no}|z_q - z_l|}}{n G_{no}^2} \cdot \sinh G_{no}\frac{\Delta z_l}{2} \cdot \sin\frac{n\pi x_q}{a} \cdot \sin\frac{n\pi x_l}{a}\sin\frac{n\pi\Delta x_l}{2a}\Big\} + \Big\{4C\left(\frac{8ab}{\pi^2}\right)\sum_{m=1}^{MT}\frac{1}{m}\cos\frac{m\pi y_q}{b} \cdot \cos\frac{m\pi y_l}{b} \cdot$$

$$\sin\frac{m\pi\Delta y_l}{2b} \cdot \sum_{n=1}^{NT}\frac{\left[k_o^2-\left(\frac{m\pi}{b}\right)^2\right]}{n G_{nm}^2} \cdot e^{-G_{nm}|z_q - z_l|} \cdot \sinh G_{nm}\frac{\Delta z_l}{2} \cdot \sin\frac{n\pi x_q}{a}\sin\frac{n\pi x_l}{a} \cdot \sin\frac{n\pi\Delta x_l}{2a}\Big\}\Bigg] \quad (21.2)$$

Using Equation I

$$Q_{1q}^{33} = \Bigg[\Big\{4C\left(\frac{8ab}{\pi^2}\right)\sum_{n=1}^{NT}\frac{1}{n}\sin\frac{n\pi x_q}{a}\sin\frac{n\pi x_l}{a}\sin\frac{n\pi\Delta x_l}{2a} \cdot \ldots$$

$$\cdot \sum_{m=1}^{MT} \frac{\left[\left(\frac{n}{a}\right)^2 + \left(\frac{mT}{b}\right)^2\right]}{m\, G_{nm}^2} \cdot e^{-Gnm \left| zq - z_l \right|} \cdot \mathrm{Sin\,h}\; Gnm \frac{\Delta z_l}{2} \cdot \mathrm{Sin} \frac{m \pi y_q}{b} \cdot$$

$$\mathrm{Sin} \frac{m\pi y_l}{b} \cdot \mathrm{Sin} \frac{m\pi \Delta y_l}{2b} \Bigg\} \quad (21.3)$$

Using Equation I

$$Q_{1q}^{12} = \Big[4C(-8) \sum_{n=1}^{NT} \mathrm{Cos} \frac{n\pi x_q}{a} \; \mathrm{Sin} \frac{n\pi x_l}{a} \; \mathrm{Sin} \frac{n\pi \Delta x_l}{2a}$$

$$\cdot \sum_{m=1}^{MT} \frac{e^{-Gnm \left| zq - z_l \right|}}{G_{nm}^2} \cdot \mathrm{Sin\,h}\; G_{nm} \frac{\Delta z_l}{2} \; \mathrm{Sin} \frac{m\pi y_q}{b} \; \mathrm{Cos} \frac{m\pi y_l}{b} \; \mathrm{Sin} \frac{m\pi \Delta y_l}{2b} \Big] \quad (21.4)$$

Using Equation I

$$Q_{1q}^{21} = \Big[4C(-8) \sum_{n=1}^{NT} \mathrm{Sin} \frac{n\pi x_q}{a} \; \mathrm{Cos} \frac{n\pi x_l}{a} \; \mathrm{Sin} \frac{n\pi \Delta x_l}{2a}$$

$$\cdot \sum_{m=1}^{MT} \frac{e^{-Gnm \left| zq - z_l \right|}}{G_{nm}^2} \cdot \mathrm{Sin\,h}\; G_{nm} \frac{\Delta z_l}{2} \cdot \mathrm{Cos} \frac{m\pi y_q}{b} \; \mathrm{Sin} \frac{m\pi y_l}{b} \; \mathrm{Sin} \frac{m\pi \Delta y_l}{2b} \Big] \quad (21.5)$$

Using Equation I

$$Q_{1q}^{23} = \Big[4C\left(\frac{+8a}{-\pi}\right) \sum_{n=1}^{NT} \frac{1}{n} \mathrm{Sin} \frac{n\pi x_l}{a} \; \mathrm{Sin} \frac{n\pi x_l}{a} \; \mathrm{Sin} \frac{n\pi \Delta x_l}{2a}$$

$$\cdot \sum_{m=1}^{MT} \frac{e^{-Gnm \left| zq - z_l \right|}}{G_{nm}} \cdot \mathrm{Sin\,h}\; G_{nm} \frac{\Delta z_l}{2} \; \mathrm{Cos} \frac{m\pi y_q}{b} \; \mathrm{Sin} \frac{m\pi y_l}{b} \; \mathrm{Sin} \frac{m\pi \Delta y_l}{2b} \Big] \quad (21.6)$$

Using Equation I

$$Q_{1q}^{32} = \Bigg[4C \left(\frac{-8a}{+\pi} \right) \sum_{n=1}^{NT} \frac{1}{n} \operatorname{Sin} \frac{n\pi x_q}{a} \operatorname{Sin} \frac{n\pi x_l}{a} \operatorname{Sin} \frac{n\pi \Delta x_l}{2a}$$

$$\cdot \sum_{m=1}^{MT} \frac{e^{-G_{nm} \left| z_q - z_l \right|}}{G_{nm}} \operatorname{Sin h} G_{nm} \frac{\Delta z_l}{2} \operatorname{Sin} \frac{m\pi y_q}{b} \operatorname{Cos} \frac{m\pi y_l}{b} \operatorname{Sin} \frac{m\pi \Delta y_l}{2b} \Bigg] \quad (21.7)$$

Using Equation II

$$Q_{1q}^{31} = \Bigg[4C \left(\frac{-8b}{+\pi} \right) \sum_{m=1}^{MT} \frac{1}{m} \operatorname{Sin} \frac{m\pi y_q}{b} \operatorname{Sin} \frac{m\pi y_l}{b} \operatorname{Sin} \frac{m\pi \Delta y_l}{2b}$$

$$\cdot \sum_{n=1}^{NT} \frac{e^{-G_{nm} \left| z_q - z_l \right|}}{G_{nm}} \operatorname{Sin h} G_{nm} \frac{\Delta z_l}{2} \operatorname{Sin} \frac{n\pi x_q}{a} \operatorname{Cos} \frac{n\pi x_l}{a} \operatorname{Sin} \frac{n\pi \Delta x_l}{2a} \Bigg] \quad (21.8)$$

$$Q_{1q}^{13} = \Bigg[4C \left(\frac{+8b}{-\pi} \right) \sum_{m=1}^{MT} \frac{1}{m} \operatorname{Sin} \frac{m\pi y_q}{b} \operatorname{Sin} \frac{m\pi y_l}{b} \operatorname{Sin} \frac{m\pi \Delta y_l}{2b}$$

$$\cdot \sum_{n=1}^{NT} \frac{e^{-G_{nm} \left| z_q - z_l \right|}}{G_{nm}} \operatorname{Sin h} G_{nm} \frac{\Delta z_l}{2} \operatorname{Cos} \frac{n\pi x_q}{a} \operatorname{Sin} \frac{n\pi x_l}{a} \operatorname{Sin} \frac{n\pi \Delta x_l}{2a} \Bigg] \quad (21.9)$$

In the last four expressions the upper sign is for $z_q < z_l$, the lower Sign is for $z_q > z_l$

Art . 3(b) Evaluation Formula for Q_{1q}^{pk} elements with $z_q = z_l$

For pk = 11 From Equation (21.1)

$$Q_{1q}^{11} = j \left[Q_{1q} \text{ with } \left(e^{-G_{nm}|z_q - z_1|} . \sinh G_{nm} \frac{\Delta z_1}{2} \right) \text{ replaced by } 1 \right]$$

$$- j \left[Q_{1q}^{11} \text{ with } \left(e^{-G_{nm}|z_q - z_1|} . \sinh G_{nm} \frac{\Delta z_1}{2} \right) \text{ replaced by } e^{-G_{nm} \frac{\Delta z_1}{2}} \right]$$

The second part of Q_{1q}^{11} can be dealt with in the manner of dealing with Q_{1q}^{11} with $z_q \neq z_1$ outlined in Art 3(a)

Ist part of $Q_{1q}^{11} = \Bigg\{ 2C \left(\frac{j4b\,\Delta x_1}{4\pi} \right) \sum_{m=1}^{MT} \frac{a_o^2}{m(m^2 - a_o^2)} \Bigg[\sin \frac{m\pi}{b}\left(y_q + y_1 - \frac{\Delta y_1}{2}\right) - \sin \frac{m\pi}{b}\left(y_q + y_1 + \frac{\Delta y_1}{2}\right)$

$$+ \sin \frac{m\pi}{b}\left(y_q - y_1 + \frac{\Delta y_1}{2}\right) - \sin \frac{m\pi}{b}\left(y_q - y_1 - \frac{\Delta y_1}{2}\right) \Bigg] \Bigg\} + \Bigg\{ 4C \left(-\frac{j8ab}{4\pi^2} \right) \sum_{n=1}^{NT} \frac{1}{n} \cos \frac{n\pi x_q}{a}$$

$$\cos \frac{n\pi x_1}{a} \sin \frac{n\pi \Delta x_1}{2a} \sum_{m=1}^{MT} \frac{a_n^2}{m(m^2 + a_n^2)} \Bigg[\sin \frac{m\pi}{b}\left(y_q + y_1 - \frac{\Delta y_1}{2}\right) - \sin \frac{m\pi}{b}\left(y_q + y_1 + \frac{\Delta y_1}{2}\right)$$

$$+ \sin \frac{m\pi}{b}\left(y_q - y_1 + \frac{\Delta y_1}{2}\right) - \sin \frac{m\pi}{b}\left(y_q - y_1 - \frac{\Delta y_1}{2}\right) \Bigg] \Bigg\}$$

Where $a_o^2 = \frac{b^2 k_o^2}{\pi^2}$, $a_n^2 = \frac{b^2}{\pi^2}\left[\left(\frac{n\pi}{a}\right)^2 - k_o^2 \right]$ and the factor $\left[\sin \frac{m\pi}{b} y_q \; \sin \frac{m\pi}{b} y_1 \; \sin \frac{m\pi \Delta y_1}{2b} \right]$ is rewritten as above.

Writing $\frac{a_o^2}{m(m^2 - a_o^2)} = \left[\frac{m}{m^2 - a_o^2} - \frac{1}{m}\right]$ and $\frac{a_n^2}{m(m^2+a_n^2)}$

$= \left[\frac{1}{m} - \frac{m}{m^2 + a_n^2}\right]$ and using the following formulas [7]

$$\sum_{m=1}^{\infty} \frac{\sin mx}{m} = \frac{\pi - x}{2} = f_1(x) \qquad 0 < x < 2\pi$$

$$\sum_{m=1}^{\infty} \frac{m \sin mx}{m^2 + a_n^2} = \frac{\pi \sinh a_n (\pi - x)}{2 \sinh a_n \pi} = f_2(n, x) \left[\begin{array}{l}\text{N.B.}\\ a_n^2 > 0,\\ 0 < x < 2\pi\end{array}\right]$$

0 x 2

$$\sum_{m=1}^{\infty} \frac{m \sin mx}{m^2 - a_o^2} = \frac{\sin a_o (\pi - x)}{2 \sin a_o \pi} = g_1(0, x)$$

[N.B. a_o is noninteger, $0 < x < 2\pi$

N.B. This formula is to be used when $a_n^2 < 0$]

With $\theta_1^{11} = \frac{\pi}{b}\left(y_q + y_1 - \frac{\Delta y_1}{2}\right)$, $\theta_2^{11} = \frac{\pi}{b}\left(y_q - y_1 + \frac{\Delta y_1}{2}\right)$

$\theta_3^{11} = \frac{\pi}{b}\left(y_q + y_1 + \frac{\Delta y_1}{2}\right)$, $\theta_4^{11} = \frac{\pi}{b}\left(y_q - y_1 - \frac{\Delta y_1}{2}\right)$

and θ_1^{11}, θ_2^{11}, θ_3^{11} ; θ_4^{11} reduced to within 0 to 2π range by

adding $\pm 2\pi$ and with $G_{1q}^{11}(0) = \left\{\sum_{i=1}^{2} \left[g_1(0, \theta_i^{11}) - f_1(\theta_i^{11})\right]\right.$

$$\left. - \sum_{i=3}^{4} \left[g_1(0, \theta_i^{11}) - f_1(\theta_i^{11})\right]\right\}$$

and $F_{1q}^{11}(n) = \left\{ \sum_{j=1}^{2} \left[f_1(\theta_j^{11}) - f_2(n, \theta_j^{11}) \right] - \right.$ ....

$\left. - \sum_{j=3}^{4} \left[f_1(\theta_j^{11}) - f_2(n, \theta_j^{11}) \right] \right\}$. ... Part of Q_{1q}^{11} .....

1st part of $Q_{\ell q}^{11}$

$$= \left[\left\{ 2C \left(\frac{j4b\Delta x_1}{4\pi} \right) G_{1q}^{11}(0) \right\} + \left\{ 4C \left(\frac{-j8ab}{4\pi^2} \right) \sum_{n=1}^{NT} \frac{1}{n} \cos \frac{n\pi x_q}{a} \cdot \cos \frac{n\pi x_\ell}{a} \sin \frac{n\pi \Delta x_1}{2a} \cdot F_{1q}^{11}(n) \right\} \right]$$

[N.B. This is with NT → ∞]

Wang (7) reports that some 20 x 20 terms are needed to compute Q_{1q}^{11} to a high degree of accuracy in typical cases. This is a marked improvement since by direct truncation procedure summation of even 140 x 140 terms does not produce convergence.

For Pk = 22 From Eqn. (21.2)

$$Q_{1q}^{22} = \left\{ j \left[Q_{1q}^{22} \text{ with } \left(e^{-G_{nm} |z_q - z_1|} \cdot \sinh G_{nm} \frac{\Delta z_1}{2} \right) \ldots \text{ replaced by } 1 \right] - j \left[Q_{1q}^{22} \text{ with } \left(e^{-G_{nm} |z_q - z_1|} \cdot \sinh \frac{\Delta z_1}{2} \right) \text{ replaced by } e^{-G_{nm} \frac{\Delta z_1}{2}} \right] \right\}$$

The second part of Q_{1q}^{22} can be dealt with in the manner of dealing with Q_{1q}^{22} with $z_q \neq z_1$ outlined in Art. 3 (a)

The first part of Q_{1q}^{22} can be dealt with in the manner the Ist part of Q_{1q}^{11} was dealt with.

Thus with $\theta_1^{22} = \frac{\pi}{a}\left(x_q + x_1 - \frac{\Delta x_1}{2}\right)$, $\theta_2^{22} = \frac{\pi}{a}\left(x_q - x_1 + \frac{\Delta x_1}{2}\right)$,

$\theta_3^{22} = \frac{\pi}{a}\left(x_q + x_1 + \frac{\Delta x_1}{2}\right)$, $\theta_4^{22} = \frac{\pi}{a}\left(x_q - x_1 - \frac{\Delta x_1}{2}\right)$

reduced within 0 to 2π range,

$$G_{1q}^{22}(0) = \left\{\sum_{j=1}^{4}\left[G_1(0, \theta_j^{22}) - F_1(\theta_j^{22})\right]\right\}, F_{1q}^{22}(m) = \ldots\ldots$$

$$\left\{\sum_{j=1}^{4}\left[F_1(\theta_j^{22}) - F_2(m, \theta_j^{22})\right]\right\}.$$ Ist part of $Q_{1q}^{22} = \left\{\left[2C\left(\frac{j\,4a\Delta y_1}{4\pi}\right)\right.\right.$

$$\left. .G_1^{22}(0)\right] + \left[4C\left(\frac{-j\,8ab}{4\pi^2}\right)\sum_{m=1}^{NT}\frac{1}{m}\cos\frac{m\pi y_q}{b}\cos\frac{m\pi y_1}{b}\sin\frac{m\pi\Delta y_1}{2b}\right.$$

$$\left.\left. .F_{1q}^{22}(m)\right]\right\}$$

(with $NT \to \infty$)

Where
$$G_{1q}^{22}(0) = \left\{\begin{array}{l}\sum_{j=1}^{2}\left[G_1(0, \theta_j^{22}) - F_1(\theta_j^{22})\right] \\ -\sum_{j=3}^{4}\left[G_1(0, \theta_j^{22}) - F_1(\theta_j^{22})\right]\end{array}\right\}$$

$$F_{1q}^{22}(m) = \left\{\begin{array}{l}\sum_{j=1}^{2}\left[F_1(\theta_j^{22}) - F_2(m, \theta_j^{22})\right] \\ -\sum_{j=3}^{4}\left[F_1(\theta_j^{22}) - F_2(m, \theta_j^{22})\right]\end{array}\right\}$$

and
$$F_1(y) = \sum_{n=1}^{\infty}\frac{\sin ny}{n} = \frac{\pi - y}{2},$$

$$F_2 \;(m_1\, y) = \sum_{n=1}^{\infty} \frac{n \;\mathrm{Sin}\; ny}{n^2 + b_m^2} = \frac{\pi \;\mathrm{Sinh}\, b_m\, (\pi - y)}{2\;\mathrm{Sinh}\, b_m} \quad \ldots\ldots$$

$$\text{where} \quad b_m^2 = \frac{a^2}{\pi^2} \left[\left(\frac{m\pi}{b} \right)^2 - k_o^2 \right]$$

$$G_1\;(o_1\, y) = \sum_{n=1}^{\infty} \frac{n \;\mathrm{Sin}\; ny}{n^2 - b_o^2} = \frac{\pi \;\mathrm{Sin}\; b_o\;(\pi - y)}{2\;\mathrm{Sin}\; b_o \pi} \quad \text{where} \quad \ldots\ldots$$

$$b_o^2 = \frac{a^2}{\pi^2}\; k_o^2 \quad (\text{a noninteger value})$$

<u>For k = 33</u> From Eqn. (21, 3)

$$Q_{1q}^{33} = \left\{ j \left[Q_{1q}^{33} \text{ with } \left(e^{-Gnm\,|z_q - z_1|} \cdot \mathrm{Sinh}\; Gnm \frac{\Delta z_1}{2} \right) \text{ replaced by } 1 \right] \right.$$

$$\left. - j \left[Q_{1q}^{33} \text{ with } \left(e^{-Gnm\,|z_q - z_1|} \cdot \mathrm{Sinh}\; Gnm \frac{\Delta z_1}{2} \right) \text{ replaced by} \ldots e^{-Gnm \frac{\Delta z_1}{2}} \right] \right\}$$

The Second part of Q_{1q}^{33} can be dealt with in the manner of dealing with Q_{1q}^{33} with $z_q \neq z_1$ outlined in Art. 3 (a)

$$\text{Ist part of } Q_{1q}^{33} = \left\{ 4C \left(\frac{+\, j8ab}{4\,\pi^2} \right) \sum_{n=1}^{NT} \frac{1}{n} \;\mathrm{Sin}\; \frac{n\pi x_q}{a} \;\mathrm{Sin}\; \frac{n\pi x_1}{a} \;\cdot \right.$$

$$\cdot\; \mathrm{Sin}\; \frac{n\pi\Delta x_1}{2a} \sum_{m=1}^{MT\to\infty} \left[\frac{1}{m} - \frac{a_o^2}{a_n^2\, m} + \frac{a_o^2}{a_m^2\,(m^2 + \alpha_n^2)} \right] \ldots$$

$$\left. \cdot \left[\mathrm{Sin}\; m\theta_1^{33} + \mathrm{Sin}\; m\theta_2^{33} - \mathrm{Sin}\; m\theta_3^{33} - \mathrm{Sin}\; m\theta_4^{33} \right] \right\}$$

Where $\theta_1^{33} = \theta_1^{11}$, $\theta_2^{33} = \theta_2^{11}$, $\theta_3^{33} = \theta_3^{11}$, $\theta_4^{33} = \theta_4^{11}$

$$\text{With } F_{1q}^{33}\;(n) = \left\{ \sum_{j=1}^{2} \left[f_1\,(\theta_j^{33}) - \frac{a_o^2}{a_n^2}\, f_1\,(\theta_j^{33}) + \frac{a_o^2}{a_n^2}\, f_2(n, \theta_j^{33}) \right] \right.$$

$$- \sum_{j=3}^{4} \left[f_1 (\theta_j^{33}) - \frac{a_o^2}{a_n^2} f_2 (n, \theta_j^{33}) + \frac{a_o^2}{a_n^2} f_2 (n, \theta_j^{33}) \right] \Bigg\}$$

and $f_1(x)$, $f_2(n,x)$ are as defined on Page 58.

Ist part of $Q_{1q}^{33} = \Bigg\{ 4C \left(\frac{\pm j8ab}{4\Pi^2} \right) \sum_{n=1}^{NT} \frac{1}{n} \operatorname{Sin} \frac{n\Pi x_q}{a}$

$\cdot \operatorname{Sin} \frac{n\Pi x_1}{a} \operatorname{Sin} \frac{n\Pi \Delta x_1}{2a} \cdot F_{1q}^{33} (n) \Bigg\}$

For $k = 12$ From Eqn. (21.4)

$$Q_{1q}^{12} = \Bigg\{ j \left[Q_{1q}^{12} \text{ with } \left(e^{-G_{nm} |z_q - z_1|} \cdot \operatorname{Sinh} G_{nm} \frac{\Delta z_1}{2} \right) \text{replaced by} \right.$$

$$-j \left[Q_{1q}^{12} \text{ with } \left(e^{-G_{nm} |z_q - z_1|} \cdot \operatorname{Sin h} G_{nm} \frac{\Delta z_1}{2} \right) \text{replaced by} \right.$$

$$\left. e^{-G_{nm} \frac{\Delta z_1}{2}} \right] \Bigg\}$$

The second part of Q_{1q}^{12} can be dealt with in the manner of dealing with

Q_{1q}^{12} with $z_q \neq z_1$ outlined in Art 3 (a)

Ist part of $Q_{1q}^{12} = \Bigg\{ 4C \left(\frac{-j8}{4} \right) \sum_{n=1}^{NT} \operatorname{Cos} \frac{n\Pi x_q}{a} \operatorname{Sin} \frac{n\Pi x_1}{a}$

$$\cdot \operatorname{Sin} \frac{n\Pi \Delta x_1}{2a} \sum_{m=1}^{MT\to\infty} \frac{\frac{b^2}{\Pi^2}}{\left(m^2 + a_n^2\right)} \left[\begin{array}{l} - \operatorname{Cos} m\theta_1^{12} + \operatorname{Cos} m\theta_2^{12} \\ + \operatorname{Cos} m\theta_3^{12} - \operatorname{Cos} m\theta_4^{12} \end{array} \right] \Bigg\}$$

Where $\theta_1^{12} = \theta_1^{11}$, $\theta_2^{12} = \theta_2^{11}$, $\theta_3^{12} = \theta_3^{11}$, $\theta_4^{12} = \theta_1^{11}$

Using the relations [7] :- $\displaystyle\sum_{m=1}^{\infty} \frac{\operatorname{Cos} mx}{m^2 + a_n^2} = \frac{\Pi}{2a_n} \frac{\operatorname{Cos h} (\Pi - x)}{\operatorname{Sin h} a_n\Pi} - \frac{1}{2a_n^2}$

$= f_3 (n, x)$

and $F_{1q}^{12}(n) = \left\{ \frac{b^2}{\pi^2} \left[-f_3(n, \theta_1^{12}) + f_3(n, \theta_2^{12}) + f_3(n, \theta_3^{12}) \ldots - f_3(n, \theta_4^{12}) \right] \right\}$

Ist part of $Q_{1q}^{121} = \left\{ 4C \left(\frac{-j8}{4} \right) \sum_{n=1}^{NT} \cos \frac{n\pi x_q}{a} \sin \frac{n\pi x_1}{a} \right.$

$\left. \cdot \sin \frac{n\pi \Delta x_1}{2a} \cdot F_{1q}^{121}(n) \right\}$

N.B.- $\sum_{m=1} \frac{\cos mx}{m^2 - a^2} = \frac{1}{2a^2} - \frac{\pi}{2} \frac{\cos a(\pi - x)}{a \sin a\pi} \quad 0 \leq x \leq 2\pi$

For K = 21 From Equation (21.5)

$$Q_{1q}^{21} = \left\{ j \left[Q_{1q}^{21} \text{ with } e^{-G_{nm}|z_q - z_1|} \cdot \sinh G_{nm} \frac{\Delta z_1}{2} \text{ replaced by } 1 \right] - j \left[Q_{1q}^{21} \text{ with } e^{-G_{nm} z_q - z_1} \cdot \sin G_{nm} \frac{\Delta z_1}{2} \text{ replaced by } e^{-G_{nm}\frac{\Delta z_1}{2}} \right] \right\}$$

The second part of Q_{1q}^{21} can be dealt with in the manner of dealing with Q_{1q}^{21} with $z_q \neq z_1$ outlined in Art. 3 (a)

Ist part of $Q_{1q}^{21} = \left\{ 4C \left(\frac{-j8}{4} \right) \sum_{n=1}^{NT} \sin \frac{n\pi x_q}{a} \cos \frac{n\pi x_1}{a} \ldots \right.$

$\left. \sin \frac{n\pi \Delta x_1}{2a} \sum_{m=1}^{MT \to \infty} \left[\cos m\theta_1^{21} + \cos m\theta_2^{21} - \cos m\theta_3^{21} - \cos m\theta_4^{21} \right] \right\}$ Where $\theta_1^{21} = \theta_1^{11}$, ...

$\theta_2^{21} = \theta_2^{11}$, $\theta_3^{21} = \theta_3^{11}$, $\theta_4^{21} = \theta_4^{11}$

Using the relations [7] :- $\sum_{m=1} \frac{\cos mx}{m^2 + a_n^2} = \frac{\pi}{2a_n} \frac{\cosh (\pi - x)}{\sinh a_n \pi} - \frac{1}{2a_n^2}$

$= f_3(n, x)$

and $F_{1q}^{21}(n) = \left\{ \frac{b^2}{\pi^2} \left[f_3(n_1 \theta_1^{21}) \right.\right.$

$$\left.\left. + f_3(n, \theta_2^{21}) - f_3(n, \theta_3^{21}) - f_3(n, \theta_4^{21}) \right] \right\}$$

Ist part of $Q_{1q}^{21} = \left\{ 4C \left(\frac{-j8}{4} \right) \sum_{n=1}^{NT} \mathrm{Sin} \frac{n\pi x_q}{a} \mathrm{Cos} \frac{n\pi x_1}{a} \right.$

$$\left. . \mathrm{Sin} \frac{n\pi \Delta x_1}{2a} \cdot F_{1q}^{21}(n) \right\}$$

For pk = 23, 32, 31, 13

The Q_{1q}^{pk} elements are zero because of the zero value of the factor I in these cases for $z_q = z_1$ in Eqn (5) of Art 2 of Ch. III.

Art. 4(a) A suggested Modified Method of Computing Q_{1q}^{pk} elements with $z_q = z_1$

As found in Art. 3 (b) the 2nd part of Q_{1q}^{pk} elements with $z_q = z_1$ has the factor $e^{-G_{nm}\frac{\Delta z_1}{2}}$. The rapidity of the convergence of the double infinite summation of the 2nd part of Q_{1q}^{pk} elements with $z_q = z_1$ depends largely on the cell size paraments $\frac{\Delta z_1}{2}$. With the number of cells increased for greater details and accuracy of the obstacle di-electric field quantities and with the cell size decreased for finding the field quantities in a heteregeneous obstacle di-electric the number of Q_{1q}^{pk} elements with $z_q = z_1$ will be increased and the rapidity of Q^{pk}_{1q} element computation (for $Z_q = Z_1$) will be much decreased respectively.

Integral trasforms have been used in a variety of ways to sum certain types of series in closed form. They are found useful in many cases in converting relatively complicated series into simpler ones which are more easily summed or in converting relatively slowly converging series into much more rapidly converging ones. A.D. Wheelon (11) is one of the several authors who investigated the application of laplace transforms to the summation of infinite series and here we follow the method suggested by Wheelon (11)

The laplace transform of a function f (u) is

$$F(p) = \int_0^\infty du\, e^{-up} f(u)$$

If we have a series for which the function F(n) represents the general term or summand then we can identify the transform variable p with the dummy index of summation n and sum both sides of this with respect to n over 0 to ∞.

$$\text{Thus} \quad \sum_{n=0}^{\infty} F(n) = \sum_{n=0}^{\infty} \int_0^\infty du\, e^{-un} f(u) = \int_0^\infty du\, f(u) \sum_{n=0}^{\infty} (e^{-u})^n$$

The second form follows from the interchange of the summation and the integration process based on the theory of convergence.

Now, with $F(u) = \int_0^{\infty} du\, f(u)\, (e^{-u})^n$

if we multiply both sides of this equation by $F_{1q}^{pk}(n)$ which does not spoil the convergence of the ensuing summation we obtain a more general series form :-

$$\sum_{n=0}^{\infty} F(n)\, F_{1q}^{pk}(n) = \int_0^{\infty} du\, f(u) \sum_{n=0}^{\infty} F_{1q}^{pk}(n)\, (e^{-u})^n \qquad \text{III}$$

Similarly with the dummy index of summation m over 0 to ∞ we have

$$\sum_{m=0}^{\infty} F(m)\, F_{1q}^{pk}(m) = \int_0^{\infty} du\, f(u) \sum_{m=0}^{\infty} F_{1q}^{pk}(m)\, (e^{-u})^m \qquad \text{IV}$$

Such conversions provide rapidly convergent integro - summation for a slowly convergent series.

From Equation (21), making $NT \to \infty$ and noting $b_{1q}^{pk}(n, 0) = 0$ for pk = 11, 33, 12, 21, we obtain the following relations :-

$$\left.\begin{aligned} &\text{2nd part of } Q_{1q}^{11} = \Big\{ 2C\,(C_1^{11}) \sum_{m=0}^{\infty} b_{1q}^{11}(0,m) + 4C\,(C_2^{11}) \,. \\ &\quad . \sum_{n=1}^{NT} a_{1q}^{11}(n) \sum_{m=0}^{\infty} b_{1q}^{11}(n,m) \Big\} \\ &\text{2nd part of } Q_{1q}^{pk} = \Big\{ 4C(C_3^{pk}) \sum_{n=1}^{NT} a_{1q}^{pk}(n) \sum_{m=0}^{\infty} b_{1q}^{pk}(n,m) \Big\} \\ &\quad \text{for } pk = 33, 12, 21, \text{ and } C_1^{11}, C_2^{11}, C_3^{pk} \text{ are constants.} \end{aligned}\right\} \text{V}$$

C_2^{11}, C_3^{11}, C_3^{pk} are constants

Similarly from Equation(21) making $NT \to \infty$ and noting $a_{1q}^{pk}(0,n)=0$ for $pk = 22, 33, 12, 21,$ we obtain

$$\left.\begin{aligned}
&\text{2nd part of } Q_{1q}^{22} = \left\{ 2C\,(C_1^{22}) \sum_{n=0}^{\infty} a_{1q}^{22}(n,0) + 4C\,(C_2^{22}) \sum_{m=1}^{MT} b_{1q}^{22}(m) \cdot \sum_{n=0}^{\infty} a_{1q}^{22}(n,m) \right\} \\
&\text{2nd part of } Q_{1q}^{pk} = \left\{ 4C(C_4^{pk}) \sum_{m=1}^{MT} b_{1q}^{pk}(m) \sum_{n=0}^{\infty} a_{1q}^{22}(n,m) \right\},
\end{aligned}\right\} \quad \text{VI}$$

for $pk = 33, 12, 21$ and C_1^{22}, C_2^{22}, C_4^{pk} are constants

Applying Equation IV to infinite series part of Q_{1q}^{pk} for $pk = 11, 33, 12, 21,$ we have, (for a fixed n)

$$S_{1q}^{pk}(n) = \sum_{m=0}^{\infty} b_{1q}^{pk}(n,m) = \sum_{m=0}^{\infty} F(m)\, F_{1q}^{pk}(m)$$

$$= \int_0^{\infty} du\, f(u) \sum_{m=0}^{\infty} F_{1q}^{pk}(m)\, e^{-um} = \int_0^{\infty} du\, f(u)\, S^{pk}(u) \qquad \text{VII}$$

Where $F(m) = \dfrac{1}{m^2 + a_n^2}$ and $f(u) = \dfrac{\sin(a_n u)}{a_n}$ for

$a_n = \dfrac{b}{\pi}\sqrt{\left(\dfrac{n\pi}{a}\right)^2 - k_0^2}$ and $F(m) = \dfrac{1}{m^2 - a_n^2}$ and $f(u) =$

$\dfrac{\sinh(a_n u)}{a_n}$ for $j a_n = j\dfrac{b}{\pi}\sqrt{k_0^2 - \left(\dfrac{n\pi}{a}\right)^2}$

and $S^{pk}(u) = \displaystyle\sum_{m=0}^{\infty} F_{1q}^{pk}(m)\, e^{-um} \approx \sum_{m=0}^{MT} F_{1q}^{pk}(m)\, e^{-um}$

for $pk = 11, 33, 12, 21$

For FK = 11 ,

$$\text{With} \quad F_{1q}^{11}(m) = \left\{ \frac{\frac{b^2}{\Pi^2}\left[K_0^2 - \left(\frac{n\Pi}{a}\right)^2\right]}{m} \cdot \left(e^{-G_{nm}\frac{\Delta z_1}{2}} \right) \cdot \sin\frac{m\Pi y_q}{b} \sin\frac{m\Pi y_1}{b} \sin\frac{m\Pi \Delta y_1}{2b} \right\}; \quad S^{11}(u) = \sum_{m=o}^{MT} F_{1q}^{11}(n)\, e^{-um};$$

$$S_{1q}^{11}(n) = \int_0^{\infty} du\, f(u)\, S^{11}(u) \approx \int_0^{UT} du\, f(u)\, S^{11}(u)$$

Initially applying truncated trapezoridal rule , we have,

$$S_{1q}^{11}(n) = \frac{\Delta u}{2}\Big[f(0)\, S^{11}(0) + 2f(1)\, S^{11}(1) + \ldots\ldots + 2f(2k)\, S^{11}(2k) + f(2k+1)\, S^{11}(2k+1) \Big]$$

$$\approx \Delta u \sum_{\frac{u}{\Delta u}=1}^{2k} f(u)\, S^{11}(u) \quad \text{for selecting suitable } \Delta u \text{ and } K.$$

Finally applying Simpson's rule , we have

$$S_{1q}^{11}(n) = \left[\left\{ \frac{2\Delta u}{3} \sum_{\frac{u}{\Delta u}=0,2,\ldots}^{2k} f(u)\, S^{11}(u) + \frac{2\Delta u}{3} \cdot \sum_{\frac{u}{\Delta u}=1,3,5,\ldots}^{2K-1} f(u)\, S^{11}(u) \right\} + \left\{ \frac{\Delta u}{3} \begin{pmatrix} f(0)\, S^{11}(0) \\ + f[(2k+1)\Delta u] \cdot S^{11}[(2k+1)\Delta u] \end{pmatrix} \right\}\right]$$

$$\text{2nd part of} \quad Q_{1q}^{11} = \left[\left\{ 2C\left(-j\frac{4b}{\Pi}\Delta x_1\right) S_{1q}^{11}(0) \right\} + \left\{ 4C\left(-j\frac{8ab}{\Pi^2}\right) \cdot \sum_{n=1}^{NT} \frac{1}{n} \cos\frac{n\Pi x_q}{a} \cos\frac{n\Pi x_1}{a} \sin\frac{n\Pi\Delta x_1}{2a} \cdot S_{1q}^{11}(n) \right\}\right]$$

For $k = 22$ Applying Equation III to infinite series part of Q_{1q}^{22},

We have ; (for a fixed m)

$$S_{1q}^{22}(m) = \sum_{n=0}^{\infty} a_{1q}^{22}(n,m) = \sum_{n=0}^{\infty} F(n)\, F_{1q}^{22}(n) =$$

$$= \int_0^{\infty} du\, f(u) \sum_{n=0}^{\infty} F_{1q}^{22}(n)\, e^{-un} = \int_0^{\infty} du\, f(u)\, S^{22}(u)$$

where $F(n) = \frac{1}{n^2 + b_m^2}$ and $f(u) = \frac{\operatorname{Sin}(b_m u)}{m}$ for $b_m = \frac{a}{\pi}\sqrt{\left(\frac{m\pi}{b}\right)^2 - k_o^2}$

and $F(n) = \frac{1}{n^2 - b_m^2}$ and $f(u) = \frac{\operatorname{Sin} h\,(b_m u)}{m}$ for $j\, b_m = j\,\frac{a}{\pi}\sqrt{k_o^2 - \left(\frac{m\pi}{b}\right)^2}$

and $S^{22}(u) = \sum_{n=0}^{\infty} F_{1q}^{22}(n)\, e^{-un} \approx \sum_{n=0}^{NT} F_{1q}^{22}(n)\, e^{-un}$

With $F_{1q}^{22}(n) = \left[\frac{\frac{a^2}{\pi^2}\left[k_o^2 - \left(\frac{m\pi}{b}\right)^2\right]}{n} \left(e^{-\zeta_{nm}\frac{\Delta Z_1}{2}} \right) \operatorname{Sin}\frac{n\pi x_q}{a} \cdot \operatorname{Sin}\frac{n\pi x_1}{a}\, \operatorname{Sin}\frac{n\pi \Delta x_1}{2a} \right]$

$$S_{1q}^{22}(m) = \int_0^{\infty} du\, f(u)\, S^{22}(u) \approx \int_0^{UT} du\, f(u)\, S^{11}(u)$$

Evaluating $S_{1q}^{22}(m)$ in the manner $S_{1q}^{11}(n)$ is evaluated, we obtain,

2nd part of $Q_{1q}^{22} = \left[\left\{ 2C\left(-j\,\frac{4a}{\pi}\,\Delta y_1\right) S_{1q}^{22}(0) \right\} + \left\{ 4C\left(-j\,\frac{8ab}{\pi^2}\right) \cdot \sum_{m=1}^{MT} \frac{1}{m} \operatorname{Cos}\frac{m\pi y_q}{b} \operatorname{Cos}\frac{m\pi y_1}{b} \operatorname{Sin}\frac{m\pi \Delta y_1}{2b} \cdot S_{1q}^{22}(m) \right\} \right]$

For PK = 33

With $F_{1q}^{33}(m) = \left\{ \frac{-\frac{b^2}{\pi^2}\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]}{m} \cdot \left(e^{-\Gamma_{nm}\frac{\Delta z_1}{2}}\right) \cdot \sin\frac{m\pi y_q}{b} \cdot \right.$

$\left. \cdot \sin\frac{m\pi y_1}{b} \sin\frac{m\pi\Delta y_1}{2b} \right\}$, From Equation VII $S_{1q}^{33}(n) = \int_0^{\alpha} du\, f(u)\, S^{33}(u)$

$\approx \int_0^{UT} du\, f(u)\, S^{33}(\mu)$, Evaluating $S_{1q}^{33}(n)$ in the manner $S_{1q}^{11}(n)$

is evaluated, we have,

2nd part of $Q_{1q}^{33} = \left[4C\left(-j\frac{8ab}{\pi^2}\right) \sum_{n=1}^{NT} \frac{1}{n} \sin\frac{n\pi x_q}{a} \sin\frac{n\pi x_1}{a} \cdot \right.$

$\left. \cdot \sin\frac{n\pi\Delta x_1}{2a} \cdot S_{1q}^{33}(n) \right]$

For PK = 12

with $F_{1q}^{12}(m) = \left[\frac{b^2}{\pi^2} \left(e^{-\Gamma_{nm}\frac{\Delta z_1}{2}}\right) \cdot \sin\frac{m\pi y_q}{b} \cos\frac{m\pi y_1}{b} \sin\frac{m\pi\Delta y_1}{2b} \right]$

From Equation VII

$$S_{1q}^{12}(n) = \int_0^{\alpha} du\, f(u)\, S^{12}(\mu) \approx \int_0^{UT} du\, f(\mu)\, S^{12}(n)$$

Evaluating $S_{1q}^{12}(n)$ in the manner $S_{1q}^{11}(n)$ is evaluated, we have,

2nd part of $Q_{1q}^{12} = \left[4C(+j8) \sum_{n=1}^{NT} \cos\frac{n\pi x_q}{a} \sin\frac{n\pi x_1}{a} \sin\frac{n\pi\Delta x_1}{2a} S_{1q}^{12}(n) \right]$

For PK = 21

With $F_{1q}^{21} = \left[\frac{b^2}{\pi^2} \left(e^{-\Gamma_{nm}\frac{\Delta z_1}{2}}\right) \cos\frac{m\pi y_q}{b} \sin\frac{n\pi y_1}{b} \sin\frac{m\pi\Delta y_1}{2b} \right]$

From Equation VII

$$S_{1q}^{21}(n) = \int_0^{\alpha} du\, f(u)\, S^{21}(u) \approx \int_0^{UT} du\, f(u)\, S^{21}(u)$$

Evaluating S_{1q}^{21} (n) in the manner S_{1q}^{11} (n) is evaluated, we have,

$$\text{2nd part of } Q_{1q}^{21} = \left[4C\ (+j8\ )\ \sum_{n=1}^{NT} \sin \frac{n\pi x_q}{a} \cos \frac{n\pi x_1}{a} \sin \frac{n\pi \Delta x_1}{2a} \cdot S_{lq}^{21}(n) \right]$$

Finding the 2nd part of Q_{1q}^{pk} elements as shown above we may add to it the respective value of the Ist part of Q_{1q}^{pk} elements with $z_q = z_1$ computed by the method outlined in Art. 3 (b) and get the whole value of Q_{1q}^{pk} elements with $z_q = z_1$. It is to be noted that for pk = 23,32, 31,13 the Q_{1q}^{pk} elemants with $z_q = z_1$ are each equal to zero. This method may be found useful for Q_{1q}^{12}, Q_{1q}^{21} with $|z_q - z_1|$ small for $z_q \neq z_1$ cases as well.

Art. 4(b). A suggested Modified method of computing Q_{1q}^{pk} with $z_q \neq z_1$ and pk = 23,32,31,13.

With $\frac{1}{G_{nm}} \cdot e^{-G_{nm} |z_q - z_1|} \cdot \sinh G_{nm} \frac{\Delta z_1}{2}$ factor in Q_{1q}^{pk} element with $|z_q - z_1|$ small & [for pk = 23,32,31,13] involving double infinite series we may apply Poisson's summation Formula (8), for evaluation of the Q_{1q}^{pk} elements with rapid convergence, utilizing Fourier transform. This may also be useful when there a large number of such elements to be computed.

According to the Poisoon's summation formula (Appendis III) the summation of an infinite series is given by

$$\sum_{n=-\infty}^{\infty} g\left(\frac{2\pi n}{d}\right) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{d}\right) = \frac{d}{2\pi}\sum_{n=-\infty}^{\infty} f(dn)$$

Where the fourier integral relation

$$f(x) = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} dw\, e^{jwx}\int_{-\infty}^{\infty} f(x)\, e^{-jwx}dx\right] \text{ defines}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dw\, e^{jwx} F(w) = \int_{-\infty}^{\infty} dw\, e^{jwx} g(w)$$

$$F(w) = \int_{-\infty}^{\infty} dx\, e^{-jwx} f(x) \text{ and } g(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-jwx} f(x)$$

and the above summation formula results after performing the summation and integration in the RHS of the following relation.

$$\sum_{n=-\infty}^{n=\infty} f(dn) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\sum_{n=-\infty}^{n=\infty} e^{jwdn}\, dw \quad \text{where } x = dn \text{ and}$$

d is such that the series converges.

Putting $d = 2a$

$$\sum_{n=-\infty}^{n=\infty} g\left(\frac{n\pi}{a}\right) = \frac{a}{\pi}\sum_{n=-\infty}^{n=\infty} f(2na) \qquad \text{IX}$$

Now $$f(z) = \int_{-\infty}^{\infty} dw\, e^{jwz}\, g(w) = \int_{-\infty}^{\infty}\frac{e^{jwz}\, e^{jwz' - d\sqrt{w^2+\beta^2}}}{\sqrt{w^2+\beta^2}}\, dw$$

$$= 2K_o\left\{\beta\sqrt{d^2 + (z+z')^2}\right\}$$ where $K_o\{x\}$ is the modified Bessel function of the second kind [8, Pp 267 – 269] of x

Putting $W = \frac{n\pi}{a}$; $z' = r\cos\theta_0$, $d = r\sin\theta$, $\beta^2 = k_m^2 = (\frac{m\pi}{b})^2 - k_o^2$

relation IX yields

$$\sum_{n=-\infty}^{\infty} \frac{\exp\left[j\frac{n\pi}{a} r \cos\theta - r\sin\theta\sqrt{(\frac{n\pi}{a})^2 + k_m^2}\right]}{\sqrt{(\frac{n\pi}{a})^2 + k_m^2}}$$

$$= \frac{a}{\pi} \sum_{n=-\infty}^{\infty} 2K_o\left\{ k_m\sqrt{r^2 + (2na)^2 + 4na\,r\cos\theta}\right\}.$$

Where LHS $= \sum_{n=-\infty}^{\infty} g\left(\frac{n\pi}{a}\right)$ and RHS $= \sum_{n=-\infty}^{\infty} f(2na)$

Hence $\sum_{n=-\infty}^{n=\infty} e^{j\frac{n\pi}{a} r\cos\theta} \cdot \frac{1}{G_{nm}} \cdot e^{-G_{nm} r \sin\theta}$ ......

$$= \frac{a}{\pi} \sum_{n=-\infty}^{n=\infty} 2K_o\left\{ k_m\sqrt{r^2 + (2na)^2 + 4na\,r\cos\theta}\right\} \qquad \text{X}$$

Analogously $\sum_{m=-\infty}^{m=\infty} e^{j\frac{m\pi}{b} r\cos\theta} \cdot \frac{1}{G_{nm}} \cdot e^{-G_{nm} r \sin\theta}$ ....

$$= \frac{b}{\pi} \sum_{m=-\infty}^{m=\infty} 2K_o\left\{ k_n\ r^2 + (2mb)^2 + 4mb\,r\cos\theta\right\} \qquad \text{XI}$$

where $k_n = \sqrt{(\frac{n\pi}{a})^2 - k_o^2}$

For bk=23 From Equation (21.6)

$$Q_{1q}^{23} = \Bigg\{ 4C\left(\mp\frac{a}{\pi}\right) \sum_{n=1} \frac{1}{n} \sin\frac{n\pi x_q}{a} \sin\frac{n\pi x_1}{a} \sin\frac{n\pi\Delta x_1}{2a} .$$

$$\cdot \sum \epsilon'_{om} \left[\frac{e^{-G_{nm}\left(|z_q - z_1| - \frac{\Delta z_1}{2}\right)}}{G_{nm}} - \frac{e^{-G_{nm}\left(|z_q - z_1| + \frac{\Delta z_1}{2}\right)}}{G_{nm}}\right].$$

$$\times \left[\begin{array}{l} \cos\frac{m\pi}{b}\left(y_q + y_1 - \frac{\Delta y_1}{2}\right) + \cos\frac{m\pi}{b}\left(y_q - y_1 + \frac{\Delta y_1}{2}\right) \\ - \cos\frac{m\pi}{b}\left(y_q + y_1 + \frac{\Delta y_1}{2}\right) - \cos\frac{m\pi}{b}\left(y_q - y_1 - \frac{\Delta y_1}{2}\right) \end{array}\right]\Bigg\}$$

Let us consider the Ist term to be summed over m from 0 to ∞.

1st term

$$= \left[\sum_{m=0}^{\infty} \epsilon'_{om} \cdot \frac{e^{-G_{nm}\left(|z_q - z_1| - \frac{z_1}{2} \right)}}{G_{nm}} \cdot \cos \frac{m\pi}{b} \left(y_q + y_1 - \frac{\Delta y_1}{2} \right) \right]$$

$$= \left[\sum_{m=0}^{\infty} \epsilon'_{om} \cdot \frac{e^{-G_{nm} \cdot (\gamma_1 \sin \theta_1)}}{G_{nm}} \cdot \cos \left(\frac{m\pi}{b} \gamma_1 \cos \theta_1 \right) \right]$$

[N.B. By putting $\left(|z_q - z_1| - \frac{\Delta z_1}{2} \right) = \gamma_1 \sin \theta_1$

$\left(y_q - y_1 + \frac{\Delta y_1}{2} \right) = \gamma_1 \cos \theta_1$]

$$= \left[\sum_{m=-\infty}^{m=\infty} e^{\, j \frac{m\pi}{b} \gamma_1 \cos \theta_1} \cdot \frac{1}{G_{nm}} \cdot e^{-G_{nm} \gamma_1 \sin \theta_1} \right]$$

[N.B. Since $\cos \frac{m\pi}{b} \gamma_1 \cos \theta_1)$ is an even function over m and $\epsilon'_{om} = \begin{cases} 1 \text{ for } m = 0 \\ 2 \text{ otherwise} \end{cases}$]

$$= \left[\frac{b}{\pi} \sum_{m=-\infty}^{n=\infty} 2 K_0 \left\{ k_n \sqrt{\gamma_1^2 + (2mb)^2 + 4mb \, \gamma_1 \cos \theta_1} \right\} \right] = K_1^{23} (n) \quad \text{(by Eqn. XI)}$$

Similarly the other seven series terms to be summed over m from 0 to ∞ may be dealt with. The sum of these eight terms yields the expression denoted by K_{1q}^{23} (n)

Hence $$Q_{1q}^{23} = \left[4C \left(\frac{-}{+} \frac{a}{\pi} \right) \sum_{n=1}^{NT} \frac{1}{n} \sin \frac{n\pi x_q}{a} \sin \frac{n\pi x_1}{a} \sin \frac{n\pi \Delta x_1}{2a} \cdot K_{1q}^{23} (n) \right]$$

For $k = 32$ From Eqn. (21.7)

$$Q_{1q}^{32} = \Bigg\{ 4C \left(\frac{-a}{+\pi} \right) \sum_{n=1}^{NT} \frac{1}{n} \sin \frac{n\pi x_q}{a} \sin \frac{n\pi x_1}{a} \sin \frac{n\pi \Delta x_1}{2a} \ldots$$

$$\sum_{m=0}^{\infty} e'_{om} \cdot \left[\frac{e^{-G_{nm}\left(|z_q - z_1| - \frac{\Delta z_1}{2} \right)}}{G_{nm}} - \frac{e^{-G_{nm}\left(|z_q - z_1| + \frac{\Delta z_1}{2} \right)}}{G_{nm}} \right].$$

$$\times \left[\begin{array}{l} \cos \frac{m\pi}{b}\left(y_q - y_1 - \frac{\Delta y_1}{2} \right) - \cos \frac{m\pi}{b}\left(y_q + y_1 - \frac{\Delta y_1}{2} \right) \\ + \cos \frac{m\pi}{b}\left(y_q - y_1 + \frac{\Delta y_1}{2} \right) - \cos \frac{m\pi}{b}\left(y_q + y_1 - \frac{\Delta y_1}{2} \right) \end{array} \right] \Bigg\}$$

Thus proceeding in the manner outlined for the case of Q_{1q}^{23}

$$Q_{1q}^{32} = \Bigg\{ 4C \left(\frac{-a}{+\pi} \right) \sum_{n=1}^{NT} \frac{1}{n} \sin \frac{n\pi x_q}{a} \sin \frac{n\pi x_1}{a} \sin \frac{n\pi \Delta x_1}{2a} K_{1q}^{32}(n) \Bigg\}$$

For $k = 31$ From Eqn. (21,8)

$$Q_{1q}^{31} = \Bigg\{ 4C \left(\frac{-b}{+\pi} \right) \sum_{m=1}^{MT} \frac{1}{m} \sin \frac{m\pi y_q}{b} \sin \frac{m\pi y_1}{b} \sin \frac{m\pi \Delta y_1}{2b} \ldots$$

$$\cdot \sum_{n=0}^{\infty} e_{on} \left[\frac{e^{-G_{nm}\left(|z_q - z_1| - \frac{\Delta z_1}{2} \right)}}{G_{nm}} - \frac{e^{-G_{nm}\left(|z_q - z_1| + \frac{\Delta z_1}{2} \right)}}{G_{nm}} \right]$$

$$\times \left[\begin{array}{l} \cos \frac{n\pi}{a}\left(x_q - x_1 - \frac{\Delta x_1}{2} \right) - \cos \frac{n\pi}{a}\left(x_q + x_1 + \frac{\Delta x_1}{2} \right) \\ + \cos \frac{n\pi}{a}\left(x_q + x_1 + \frac{\Delta x_1}{2} \right) - \cos \frac{n\pi}{a}\left(x_q + x_1 - \frac{\Delta x_1}{2} \right) \end{array} \right] \Bigg\}$$

Proceeding in the manner outlined for the case of Q_{1q}^{23} except that Eqn. X is used in stead of Eqn. XI

$$Q_{1q}^{31} = \left\{ 4C \left(\mp \frac{b}{\pi} \right) \sum_{m=1}^{MT} \frac{1}{m} \sin \frac{m\pi y_q}{b} \sin \frac{m\pi y_1}{b} \sin \frac{m\pi \Delta y_1}{2b} K_{1q}^{31}(m) \right\}$$

<u>For PK = 13</u> From Eqn. (21, 9)

$$Q_{1q}^{13} = \left\{ 4C \left(\pm \frac{b}{\pi} \right) \sum_{m=1}^{MT} \frac{1}{m} \sin \frac{m\pi y_q}{b} \sin \frac{m\pi y_1}{b} \sin \frac{m\pi \Delta y_1}{2b} \cdots \right.$$

$$\cdot \sum_{n=0} e_{on} \left[\frac{e^{-G_{nm}\left(|z_q - z_1| - \frac{\Delta z_1}{2}\right)}}{G_{nm}} - \frac{e^{-G_{nm}\left(|z_q - z_1| + \frac{\Delta z_1}{2}\right)}}{G_{nm}} \right] \cdot$$

$$\left. \cdot \left[\begin{array}{l} \cos \frac{n\pi}{a} \left(x_q + x_1 - \frac{\Delta x_1}{2}\right) + \cos \frac{n\pi}{a} \left(x_q - x_1 + \frac{\Delta x_1}{2}\right) \\ - \cos \frac{n\pi}{a} \left(x_q + x_1 + \frac{\Delta x_1}{2}\right) - \cos \frac{n\pi}{a} \left(x_q - x_1 - \frac{\Delta x_1}{2}\right) \end{array} \right] \right\}$$

Proceeding in the manner outlined for the case of Q_{1q}^{23} except that Equation X is used in stead of Equation XI

$$Q_{1q}^{33} = \left\{ 4C \left(\pm \frac{b}{\pi} \right) \sum_{m=1}^{MT} \frac{1}{m} \sin \frac{m\pi y_q}{b} \sin \frac{m\pi y_1}{b} \sin \frac{m\pi \Delta y_1}{2b} \cdot K_{1q}^{13}(m) \right\}$$

<u>N.B.</u> In above expressions the upper sign is for $z_q < z_1$ and the lower sign is for $z_q > z_1$

Appendix III :- On the Poisson's summation Formula (Ref. 8)

Let us consider the following general series, $\sum_{n=0} f(\alpha n)$, where $f(x)$ is of such a form that its Laplace transform $F(P)$ exists.

We have by definition

$$F(P) = \int_0^{\infty} f(x) \cdot e^{-px} \, dx$$

This integral determines $F(p)$ as an analytic function of the complex variable $p = u + j\omega$ whose singularities all lie to the left of some value of $u = C$ in the p plane. The inversion integral yields

$$f(x) = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} F(P) \, e^{px} \, dp$$

If we replace x by αn and sum over n, we get,

$$\sum_{n=0}^{\infty} f(\alpha n) = \frac{1}{2\pi j} \sum_{n=0}^{\infty} \int_{C-j\infty}^{C+j\infty} e^{p\alpha n} F(p) \, dp$$

[N.B. α is such that the series & the integral converges]

This is permissible since the inversion integral holds identically for all values of x with the exception of certain values of x for which $f(x)$ may be discontinuous. In our case it is assumed that $f(x)$ is a continuous function of x. If now, the integral involving $F(P)$ is uniformly convergent, we may integchange the order of integration and summation to get

$$\sum_{n=0}^{\infty} f(\alpha n) = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} F(P) \sum_{n=0}^{\infty} e^{p\alpha n} \, dp \quad \ldots\ldots$$

$$= \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} \frac{F(p)}{1-e^{p\alpha}}\, dp \qquad \left[\text{N.B. using } \sum_{n=0}^{\infty} e^{p\alpha n} = \frac{1}{1-e^{p\alpha}} \right]$$

provided $f(x)$ is of such form that we may take $C<0$ in order that $\sum_{n=0}^{\infty} e^{p\alpha n}$ should represent a convergent series. This implies that $f(x)$ is asymptotic to $e^{-\epsilon n}$, $\epsilon > 0$, as x approaches infinity. In general, this condition is not satisfied, but in practice, we can multiply $f(\alpha n)$ by $e^{-\epsilon n}$, sum the resulting series and then take the limit as ϵ approaches zero. If we can evaluate the resulting integral, we have the sum of the series in closed form. Alternatively, we may expand the integral by the residue theorem. We choose the contour, for the relevant contour integral in this case, the line $p = C$, $C < 0$, and the semicicle, in the right half plane. Since $F(P)$ is analytic for all $u > C$, the only poles of the integrand occur at $e^{\alpha p} = 1$ or $p = \left(j\frac{2\pi n}{\alpha} \right)$, $n = 0, \pm 1, \pm 2 \ldots$. The residues at these poles of $\left[1 - e^{p\alpha}\right]^{-1}$ are $-\frac{1}{\alpha}$ and the residues at these poles of the integrand are $-\frac{1}{\alpha} F\left(j\frac{2\pi n}{\alpha}\right)$. Hence, provided the integral around the semicircle vanishes (for a number of standard types of functions most commonly used it does vanish) we get

$$\sum_{n=0}^{\infty} f(\alpha n) = \frac{1}{\alpha} \sum_{n=0}^{\infty} F\left(j\frac{2\pi n}{\alpha}\right) \text{ for } n = 0, \pm 1, \pm 2 \ldots\ldots$$

$$= \frac{1}{\alpha} \sum F\left(j\frac{2\pi n}{\alpha}\right) \text{ for } n = 0, 1\ 2 \ldots$$

The change in Sign is due to the fact that the contour is traversed in a clockwise serise . Sometimes it is possible to close the contour in the left half plane and the sum of the series is given in terms of the residues at the poles enclosed.

We may extend the definition of the Laplace transform so that it is valid for negative values of x as follows :-

$$L\ f(x) = \int_{-\infty}^{0} f(x)\ e^{-px}\ dx \qquad x < 0$$

The corresponding inversion formula yields

$$f\ (x) = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} e^{px}\ F\ (p)\ dp$$

where C now lies to the left of all singularities of F(P). For positive values of x , the contour may be closed in the left half plane and since no singularities are enclosed the intergral vanishes and $f(x) = 0$. For negative values of x the contour may be closed in the right half plane and the original function f(x) is recovered.

Let us now consider a series such as $\sum_{n=-\infty}^{\infty} f\ (\alpha n\ )$.

Imposing the condition that f(x) is integrable square over $-\infty < x < \infty$ i.e. $\int_{-\infty}^{\infty} |f(x)|^2\ dx$ exists, and this in turn inplying that f(x) has no poles on the real axis, we break up f(x) into two parts as follows :-

$$f_+(x) = \begin{cases} f(x) & x \geqslant 0 \\ 0 & x < 0 \end{cases} \qquad f_-(x) = \begin{cases} 0 & x > 0 \\ f(x) & x < 0 \end{cases}$$

The corresponding Laplace transforms are

$$F_+(p) = \int_0^{\infty} e^{-px} f_+(x)\, dx \qquad F_-(p) = \int_{-\infty}^{0} e^{-px} f_-(x)\, dx$$

The corresponding inverse transforms are

$$f_+(x) = \frac{1}{2\pi j} \int_{C_+ - j\infty}^{C_+ + j\infty} e^{px} F_+(p)\, dp \qquad f_-(x) = \frac{1}{2\pi j} \int_{C_- - j\infty}^{C_- + j\infty} e^{px} F_-(p)\, dp$$

From Parseval's theorem we find that $\int_{-\infty}^{\infty} |F(p)|^2\, dp$ exists if $f(x)$ is of integrable square type. Consequently, $F(p)$ has no poles on the imaginary axis and we may, therefore, take $C_+ < 0$ and $C_- > 0$ in the above inversion integrals, Under these conditions, we get

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2\pi j} \int_{C_+ - j\infty}^{C_+ + j\infty} \frac{F_+(p)}{1 - e^{\lambda p}}\, dp = \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} F_+\left(\frac{j\,2\pi n}{\lambda}\right)$$

on closing the contour in the right half plane,

Similarly, $$\sum_{n=-1}^{-\infty} f(\lambda n) = \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} F_-\left(\frac{j2\pi n}{\lambda}\right)$$, on closing the contour in the lefthalf plane.

Combining these results, we finally have,

$$\sum_{n=-\infty}^{\infty} f(\lambda n) = \frac{1}{\lambda} \sum_{-\infty}^{\infty} F\left(\frac{j2\pi n}{\lambda}\right)$$

If we make the following change in notation, $p = jw$, $dp = j\,dw$, then our bilateral laplace transforms become Fourier transforms and inverse bilaterial laplace transforms become inverse Fourier transforms,

$$\text{Thus } f(x) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} F(w)\, e^{jwx}\, dw = \int_{-\alpha}^{\alpha} g(w)\, e^{jwx}\, dw$$

$$F(w) = \int_{-\alpha}^{\alpha} f(x)\, e^{-jwx}\, dx \qquad g(w) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} f(x)\, e^{-jwx}\, dx$$

Since the poles at which the residues are taken into account in the summation formula are at $p = jw = j\,\frac{2\pi n}{d}$ with $x = dn$, We get,

$$\sum_{n=-\alpha}^{\alpha} f(dn) = \frac{1}{d}\sum_{n=-\alpha}^{\alpha} F\left(\frac{2\pi n}{d}\right) = \frac{2\pi}{d}\sum_{n=-\alpha}^{\alpha} g\left(\frac{2\pi n}{d}\right)$$

$$\text{or } \sum_{n=-\alpha}^{\alpha} g\left(\frac{2\pi n}{d}\right) = \frac{1}{2\pi}\sum_{n=-\alpha}^{\alpha} F\left(\frac{2\pi n}{d}\right) = \frac{d}{2\pi}\sum_{n=-\alpha}^{\alpha} f(dn)$$

This is the Poisson's Summation Formula.

Art. 5. Solution for field quantities inside the di-electric $[\mu_0, \epsilon(\gamma)]$ in volume V in volume V. With Q_{lq}^{pk} elements computed we can solve Equation (4) along with relation 4(a) for $J_l^k(\gamma_l)$ elements with $C_q^p = E_i(\gamma_q)$ elements known on the basis of sinusoidal variation of unit amplitude and phase reference at the dielectric obstacle centre at $Z = Z_c$ for $p = \bar{x}, \bar{y}, \bar{z}$ and $q = 1, 2, \text{------} L$ and obtain the solution in the form

$$J_l^k = \frac{\chi_l^k}{-j\, w\mu_0}$$

Using the relation $E = \dfrac{J}{j\omega[\epsilon(\gamma) - e_0]}$ we obtain

$$E_1^k = E^k(\gamma_l) = \frac{J_1^k}{j\omega[e(\gamma_l)-e_0]} = \frac{jX_l^k}{k_1^2(\gamma_l) - k_0^2} \quad \text{for } k = x, y, z \ \& \ l$$

$l = 1, 2, \ldots\ldots L$

Using the relation $E_s = E - E_i$ we obtain

$$E_s^k(\gamma_l) = E^k(\gamma_l) - E_i^k(\gamma_l) = \left[\frac{X_l^k}{k_1^2(\gamma_l) - k_0^2} - C_1^k\right] \quad \text{for } k = x, y, z \text{ and } l = 1, 2, \ldots L$$

<u>Art. 6.</u> Solution for the Dominant TE_{10} Mode T - Equivalent Circuit Parameters for the di-electric obstacle discontinuity.

Since the solution for the dominant mode T - Equivalent Circuit ponaments is to be valid in the medium (u_0, e_0) we solve Equation(4) along with relation 4 (b) for $J_1^k = J^k(\gamma_1)$ elements with $C_q^p = E_i^p(\gamma_q)$ elements known on the basis of simisaidal variation of unit amptitude and phase reference in the volume V at $\gamma = \gamma_q$ for $p = x, y, z$ and $q = 1, 2, \ldots L$ and get the solution in the form

$$J_1^k = \frac{X_1^k}{-j\omega u_0}$$

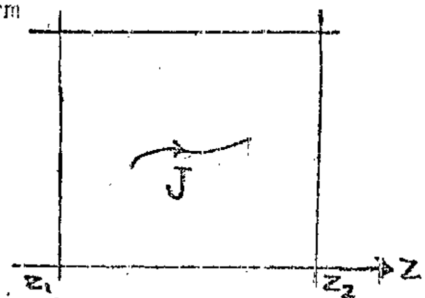


Fig.3 Terminal planes in a waveguide containing a discontinuity current J. at $z = z_1$ & $z = z_2$.

The dominant TE_{10} mode scattered field due to J in the waveguide medium (u_0, e_0) in the region between $z = z_1$ and $z = z_2$ (Fig. 3) is given by (Ref. Appendix IV) :-

$$\left.\begin{aligned} E_s^+ &= C_{10}^+\, e_{10}\, e^{-j\beta_{10}(z - z_2)} \\ H_s^+ &= C_{10}^+\, h_{10}\, e^{-j\beta_{10}(z - z_2)} \end{aligned}\right\} \text{ for } z \geqslant z_2$$

$$\left.\begin{aligned} E_s^- &= C_{10}^-\, e_{10}\, e^{j\beta_{10}(z - z_1)} \\ H_s^- &= C_{10}^-\, h_{10}\, e^{j\beta_{10}(z - z_1)} \end{aligned}\right\} \text{ for } z \leqslant z_1 \qquad \text{(A)}$$

where $e_{10} = \bar{y} \sin \frac{\pi x}{a}$, $h_{10} = - Y_w \bar{x} \sin \frac{\pi x}{a}$, $\beta_{10} = \sqrt{k_o^2 - (\frac{\pi}{a})^2}$,

Y_w = Wave impedance of the guide.

$$C_{10}^+ = \sum_{k=1}^{3} \sum_{l=1}^{L} C_{10_l}^{+k} \quad \text{and} \quad C_{10}^- = \sum_{k=1}^{3} \sum_{l=1}^{L} C_{10_l}^{-k}$$

$$C_{10_l}^{+k} = C_{10_l}^{-k} = -\frac{1}{P_{10}} \int_{\Delta V_l} e_{10} \cdot \bar{u}_k J_l^k \, dv \qquad \text{for transverse } J_l^k$$

$$C_{10_l}^{+k} = - C_{10_l}^{-k} = \frac{1}{P_{10}} \int_{\Delta V_l} e_{z10}\, \bar{u}_k J_l^k \cos\beta_{10} z\, dv \quad \text{for axial, symmatrical about } z_l,\ J_l^k$$

$$= 0 \quad (\text{Since } e_{z_{10}} = 0 \text{ for } TE_{10} \text{ mode})$$

$$P_{10} = 2 \int_0^a \int_0^b e_{10} \times h_{10} \cdot \bar{z}\, dxdy = abY_w$$

e.g.

Thus for the single cell case with $\bar{x}J_1^1, \bar{y}J_1^2, \bar{z}J_l^3$ existing

$$C_{10}^{+} = C_{10_1}^{+1} + C_{10_1}^{+2} = C + C_{10}^{+2} = -\frac{1}{P_{10}} \int_{\Delta V_\ell} \sin \frac{\pi x}{a} \, J_1^2 \, dv$$

$$= -\frac{J_1^{\,2} \Delta y_1 \Delta z_1}{P_{10}} \int_{(x_\ell - \frac{\Delta x_\ell}{2})}^{(x_1 + \frac{\Delta x_1}{2})} \sin \frac{\pi x}{a} \, dx$$

$$= \frac{J_1^2 \Delta Y_1 \Delta z_1}{a b Y_W \pi} \left[\cos \frac{\pi}{a}\left(x_1 + \frac{\Delta x_\ell}{2}\right) - \cos \frac{\pi}{a}\left(x_1 - \frac{\Delta x_1}{2}\right)\right] \ldots$$

$$= -\frac{2 J_1^{\,2} \Delta y_1 \Delta z_1}{\pi b Y_W} \sin \frac{\pi x_\ell}{a} \sin \frac{\pi \Delta x_\ell}{2a}$$

$$= -j \frac{2 x_1^{\,2} \Delta Y_1 \Delta z_1}{\pi b \beta_{10}} \sin \frac{\pi x_\ell}{a} \sin \frac{\pi \Delta x_1}{2a} \qquad (B)$$

$$\left[\text{Since } J_1^2 = \frac{x_1^2}{-j w \mu_0} = j \frac{x_1^2 Y_W}{\beta_{10}} \right]$$

Similarly we find $\beta_{10}^{-} = C_{10}^{+} = C_{10}$

It is to be noted that although there are other modes, mostly evanescent type, only the dominant mode survive attenuation in practical cases and the other modes are attenuated in a short distance from the discontinuity between terminal planes at Z_1 and Z_2 .

Considering the waveguide to be uniform and exactly of similar size and matched on the LHS and RHS of discontinuity region between Z_1 and Z_2 , [From Equation (A)]

For $Z \ll Z_1$ Total field with incident field from LHS

$$E_1 = E_i^{+} + E_s^{-} = A_{10}^{+} e_{10} e^{-j\beta_{10}(z - z_1)} + C_{10}^{-} e_{10} e^{+j\beta_{10}(z - z_1)}$$

$$H_1 = H_i^{+} + H_s^{-} = A_{10}^{+} h_{10} e^{-j\beta_{10}(z - z_1)} - C_{10}^{-} h_{10} e^{+j\beta_{10}(z - z_1)}$$

For $z \geq z_2$ total field with incident field from LHS

$$E_2 = E_i^- + E_s^+ = A_{10}^- e_{10} \, e^{j\beta_{10}(z-z_2)} + C_{10}^+ e_{10} \, e^{-j\beta_{10}(z-z_2)}$$

$$H_2 = H_i^- + E_s^+ = -A_{10} h_{10} \, e^{j\beta_{10}(z-z_2)} + C_{10}^+ h_{10} \, e^{-j\beta_{10}(z-z_2)}$$

The following equivalent transmission line voltages and currents are now introduced ;

$V_1^+ = K_1 A_{10}^+$ $\quad V_1^- = K_1 C_{10}^-$ $\quad V_2^+ = K_1 A_{10}^-$ $\quad V_2^- = K_1 C_{10}^+$

$I_1^+ = K_2 A_{10}^+$ $\quad I_1^- = K_2 C_{10}^-$ $\quad I_2^+ = -K_2 A_{10}$ $\quad I_2^- = -K_2 C_{10}^+$

Where $K_1 K_2 = \frac{ab Y_W}{2}$, $K_1/K_2 = Z_W = 1/Y_W$ [from power flow consideration] and $K_1 = \sqrt{\frac{ab}{2}}$, $K_2 = \sqrt{\frac{ab}{2}} \, Y_W$ [17]

With these the total voltages V_1 and V_2 and currents I_1 and I_2 are given by

For $z \leq z_1$

$$\left.\begin{aligned} V_1 &= V_1^+ e^{-j\beta_{10}(z-z_1)} + V_1^- e^{j\beta_{10}(z-z_1)} \\ I_1 &= I_1^+ e^{-j\beta_{10}(z-z_1)} - I_1^- e^{j\beta_{10}(z-z_1)} \end{aligned}\right\}$$

For $z \geq z_2$

$$\left.\begin{aligned} V_2 &= V_2^+ e^{j\beta_{10}(z-z_2)} + \ldots \\ &\quad + V_2^- e^{-j\beta_{10}(z-z_2)} \\ I_2 &= I_2^+ e^{j\beta_{10}(z-z_2)} - \ldots \\ &\quad - I_2^- e^{-j\beta_{10}(z-z_2)} \end{aligned}\right\}$$

For the terminal plane at $z = z_1$

$$V_1 = V_1^+ + V_1^-$$

$$I_1 = I_1^+ - I_1^-$$

For the terminal plane at $z = z_2$

$$V_2 = V_2^+ + V_2^-$$

$$I_2 = I_2^+ - I_2^-$$

The equivalent T - network representing such a reciprocal system is as in Fig. 4.

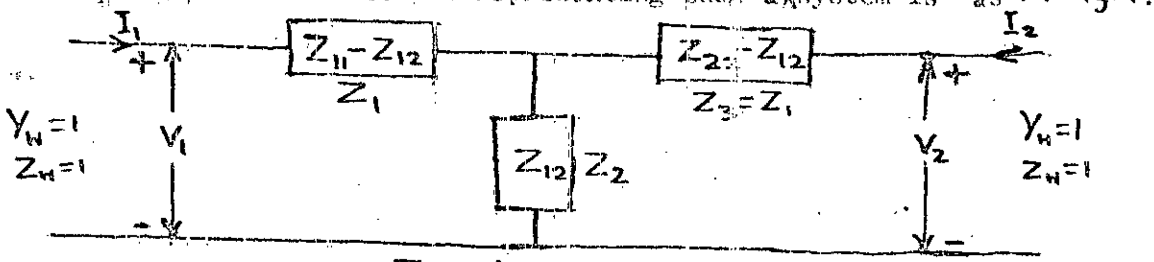


Fig. 4.

The equations relating the L H S and RHS voltages and currents are

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{12} I_1 + Z_{22} I_2$$

for a reciprocal system with $Z_{12} = Z_{21}$

For a lossless discontinuity the Z - parameters become imaginary.

For a normalized matched voltage & current system with $Y_N = 1$ (assumed), $K_1 = K_2 = K$ and for a completely symmetrical system, $A_{10}^{+} = A_{10}^{-} = A_{10}$, $C_{10}^{+} = C_{10}^{-} = C_{10}$ and under these conditions we get, [N.B. For a symmetrical system $Z_{11} = Z_{22}$]

$$V_1^{+} = V_2^{+}, \quad I_1^{+} = I_2^{+}, \quad V_1^{-} = V_2^{-}, \quad I_1^{-} = I_2, \quad V_1 = V_2, \quad I_1 = I_2$$

This implies $Z_{11} = Z_{22}$ and the above two network equations become similar i.e. $V_1 = (Z_{11} + Z_{12}) I_1$ **(C)**

Also from Fig. 4 with $Z_{22} = Z_{11}$ the input impedance $\dfrac{V_1}{I_1} = \dfrac{(Z_{11}^2 - Z_{12}^2) + Z_{11}}{1 + Z_{22}}$ **(D)**

With C_{10} known from Equation (B), $A_{10} = 1$, $K = \sqrt{\dfrac{ab}{2}}$, V_1 & I_1 are known.

Thus $\dfrac{V_1}{I_1} = Z_k$ (known). Now solving Equation (C) and (D), we get

$$Z_{11} = \frac{Z_k(1+Z_k)}{2+Z_k} \quad \text{and} \quad Z_{12} = \frac{Z_k}{2+Z_k} \quad \text{where} \quad Z_K = \frac{V_1}{I_1} = \frac{V_1^+ + V_1^-}{I_1^+ - I_1^-} = \frac{K_1 A_{10}^+ + K_1 C_{10}^-}{K_2 A_{10}^+ - K_1 C_{10}^-}$$

$$= \frac{K A_{10} + K C_{10}}{K A_{10} - K C_{10}} = \frac{A_{10} + C_{10}}{A_{10} - C_{10}} = \frac{1 + C_{10}}{1 - C_{10}}$$

Appendix IV Radiation from current source in a Waveguide (12)

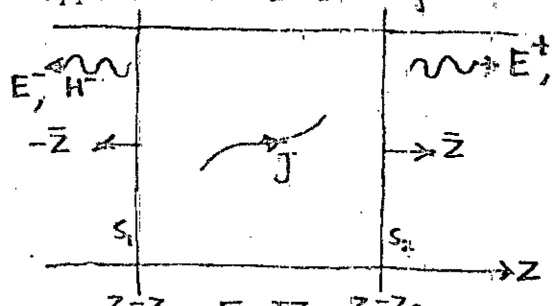


Fig IV

The adjacent Figure IV illustrates an infinitely long waveguide in which a current source J is located in the region between $z=z_1$ and $z=z_2$. The wavy arrows indicate the direction of propagation

The field radiated by this source may be expressed as a combination of allowable waveguide modes as follows :-

$$\left.\begin{aligned} E^+ &= \sum_n C_n^+ (e_n + e_{zn})\, e^{-j\beta_n (z - z_2)} \\ H^+ &= \sum_n C_n^+ (h_n + h_{zn})\, e^{-j\beta_n (z - z_2)} \end{aligned}\right\} \text{for } z \geq z_2 \qquad (Ia)$$

$$\left.\begin{aligned} E^- &= \sum_n C_n^- (e_n - e_{zn})\, e^{j\beta_n (z - z_1)} \\ H^- &= \sum_n C_n^- (-h_n + h_{zn})\, e^{j\beta_n (z - z_1)} \end{aligned}\right\} \text{for } z \leq z_1 \qquad (Ib)$$

With $e^{j\beta_n z_2}$ and $e^{-j\beta_n z_1}$ made implicit for the time being, as they are not affecting the following derivation , we have,

$$\left.\begin{aligned} E^+ &= \sum_n C_n^+ (e_n + e_{zn})\, e^{-j\beta_n z} \\ H^+ &= \sum_n C_n^+ (h_n + h_{zn})\, e^{-j\beta_n z} \end{aligned}\right\} \text{for } z \geq z_2 \qquad (IIa)$$

$$\left.\begin{aligned} E^- &= \sum_n C_n^- (e_n + e_{zn})\, e^{+j\beta_n z} \\ H^- &= \sum_n C_n^- (-h_n + h_{zn})\, e^{+j\beta_n z} \end{aligned}\right\} \text{for } z \leq z_1 \qquad (IIb)$$

Where the summation index n implies a summation over all possible TE and TM modes ; e_n and h_n are the transverse mode vectors and are functions of x and y coordinates, e_{zn} & h_{zn} are the axial mode vectors and are functions of x & y coordinates ; β_n is the mode propagation constant.

Let E_1 , H_1 represent the field yielded by the current source J and expressed by Equation II and let $E_2 = E_n^- = (e_n - e_{zn})\, e^{j\beta_n z}$ $H_2 = H_n^- = (-h_n + h_{zn})\, e^{j\beta_n z}$ be the source free field ($J_2 = 0$)within V enclosed by S i.e. the region between $z = z_1$ and $z = z_2$

By reciprocity relation $\int_S (E_1 \times H_n^- - E_n^- \times H_1) \cdot \bar{n}\, ds = \int_V E_n^- \cdot J\, dv.$

The surface integral is zero over the waveguide walls by virtue of the boundary condition $\bar{n} \times E_1 = \bar{n} \times E_n = 0$. Since the modes are orthogonal

i.e. $\int_{S_0} E_m^{\pm} \times H_n^{\pm} \cdot \bar{n}\, ds = 0 \quad n \neq m$ [Ref. (12) Pg 121-124]

all the terms except the nth in the expansion of E_1, H_1 vanish when integrated over the arbitrary waveguide cross section S_0 . Thus we have,

$$\int_{S_2} C_n^+ \left[(e_n + e_{zn}) \times (-h_n + h_{zn}) - (e_n - e_{zn}) \times (h_n + h_{zn}) \right] \cdot \bar{z}\, ds$$
$$- \int_{S_1} C_n^- \left[(e_n - e_{zn}) \times (-h_n + h_{zn}) - (e_n - e_{zn}) \times (-h_n + h_{zn}) \right] \cdot \bar{z}\, ds$$
$$= -2\, C_n^+ \int_{S_2} e_n \times h_n \cdot \bar{z}\, ds = \int_V E_n^- \cdot J\, dv \; ;$$ Since the integral over waveguide cross-section S_1 vanishes identically, Hence C_n^+ is given by

$$C_n^+ = -\frac{1}{P_n}\int_V E_n^- \cdot J\, dv = -\frac{1}{P_n}\int_V (e_n - e_{zn}) \cdot J\, e^{j\beta_n z} dv$$

If E^+, H_n^+ is chosen for the field E_2, H_2, we obtain

$$C_n^- = -\frac{1}{P_n}\int_V E_n^+ \cdot J dv = -\frac{1}{P_n}\int_V (e_n + e_{zn}) \cdot J\, e^{-j\beta_n z} dv$$

where $P_n = 2\int_{S_2} e_n \times h_n \cdot \bar{z}\, ds = 2\int_{S_0} e_n \times h_n \cdot \bar{z}\, ds$. [Since S_2 at z_2 was chosen arbitrarily.]

For transverse J at $Z = Z_l \approx 0$ i.e. Z_l is near origin, we have

$$C_n^+ = C_n^- = -\frac{1}{P_n}\int_V e_n \cdot J\, dV \quad \text{Since} \quad e_{zn} = 0$$

For axial J, at $Z = Z_l \approx 0$ i.e. Z is near origin, we have

$$C_n^+ = \frac{1}{P_n}\int_V J \cdot e_{zn} e^{j\beta_n z \to 0} dv, \quad C_n^- = -\frac{1}{P_n}\int_V J \cdot e_{zn} e^{-j\beta_n z \to 0} dv.$$

(Since $e_n \cdot J = 0$)

If J is a symmatrical function of Z, then, since e_{zn} is not a function of Z, we have

$$C_n^+ = -C_n^- = \frac{1}{P_n}\int_V J \cdot e_{zn} \cos \beta_n z\, dv$$

With C_n^+ and C_n^- thus determined we can now revoke the factors $e^{j\beta_n z_2}$ & $e^{-j\beta_n z_1}$ and have the rediation expressions given by Equation I.

CHAPTER-IV

Determination of Equivalent Circuit Parameters For A Three Dimensional Dielectric Obstacle in A Rectangular Waveguide.

With the formulation and moment method of solving of the problem involving a dielectric in an arbitrary volume V inside a rectangular waveguide discussed so far we now apply the method to an illustrative example. It is to be noted that, to represent accurately the arbitrarily shaped dilectric in the volume V the cell size of volume ΔV_1 should be made small and consequently the number of cells for the given volume should be large. This implies a Computer solution on a large scale, However to illustrats the effectiveness of the method without loosing any essential feature we undertake now the digital computer programming based on Fortran IV for a single cell problem depicted in Fig.5.

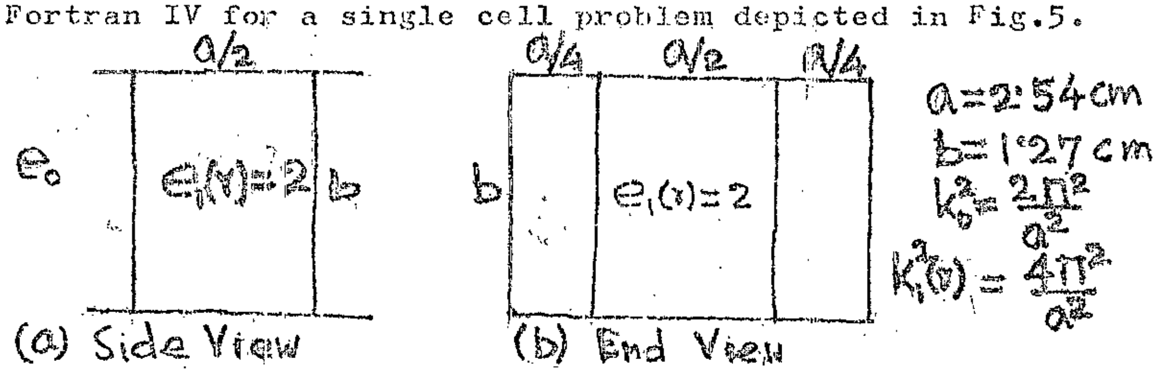


Fig.5. A rectangularly shaped dielectric in a rectangular Waveguide.

From Fig-5. $X_q = \frac{a}{2}$, $X_1 = \frac{a}{2}$; $\Delta x_1 = \frac{a}{2}$, $Y_q = \frac{a}{4}$,

$Y_1 = \frac{a}{4}$, $\Delta Y_1 = \frac{a}{2}$, $Z_q = 0$, $Z_1 = 0$, $\Delta Z_1 = \frac{a}{2}$.

With above data the relevant expressions for the nine different Q_{1q}^{kk} elements per cell of size $(\Delta X_1 \times \Delta Y_1 \times \Delta Z_1)$ are as follows :-

Q_{1q}^{11}

For $Z_q \neq Z_1$ we have from Equation (21.1)

$$Q_{1q}^{11} = \Bigg[\Bigg\{2C\left(\frac{4b\,\Delta X_1}{\pi}\right)\sum_{m=1}^{20}\frac{k_o^2\, e^{-\mathcal{G}_{om}|Z_q-Z_1|}}{m\,\mathcal{G}_{om}^2}\cdot \sinh \mathcal{G}_{om}\frac{\Delta Z_1}{2}\cdot \sin\frac{m\pi Y_q}{b}\sin\frac{m\pi Y_1}{b}\sin\frac{m\pi\Delta Y_1}{b}\Bigg\}$$

$$+\Bigg\{4C\left(\frac{8ab}{\pi^2}\right)\sum_{n=1}^{20}\frac{1}{n}\cos\frac{n\pi x_q}{a}\cos\frac{n\pi x_1}{a}\sin\frac{n\pi\Delta X_1}{2a}\cdot$$

$$\sum_{m=1}^{20}\frac{\left[k_o^2-\left(\frac{n\pi}{a}\right)^2\right]}{m\,\mathcal{G}_{nm}^2}e^{-\mathcal{G}_{nm}(Z_q-Z_1)}\cdot\sinh\mathcal{G}_{nm}\frac{\Delta Z_1}{2}\cdot\sin\frac{m\pi Y_q}{b}\sin\frac{m\pi Y_1}{b}\sin\frac{m\pi\Delta Y_1}{2b}\Bigg\}\Bigg]$$

$$= \left[\left\{\sum_{m=1}^{20}(B11)(X1)(E1LQ)\right\}+\left\{\sum_{n=1}^{20}A11N\sum_{m=1}(B11)(X11)(ELQ)\right\}\right]$$

$$= \left[\sum_{m=1}^{20} BILQM+\sum_{n=1}^{20}A11N\sum_{m=1}^{20}B11QNM\right]$$

Where

$$BII = \sin\frac{m\pi Y_q}{b}\sin\frac{m\pi Y_1}{b}\sin\frac{m\pi\Delta Y_1}{2b}$$

$$XI = 2C\left(\frac{4b\,\Delta X_1}{\pi}\right)\frac{k_o^2}{m\,G_{om}^2}$$

$$EILQ = e^{-G_{om}|Z_q-Z_1|}.\sinh G_{om}\frac{\Delta Z_1}{2}$$

$$AIIN = \frac{1}{n}\cos\frac{n\pi x_q}{a}\cos\frac{n\pi x_1}{a}\sin\frac{n\pi\Delta x_1}{2a}$$

$$XII = 4c\left(\frac{8ab}{\pi^2}\right)\frac{\left[k_o^2-\left(\frac{n\pi}{a}\right)^2\right]}{m\,G_{nm}^2}$$

$$ELQ = e^{-G_{nm}|Z_q-Z_1|}\sinh G_{nm}\frac{\Delta Z_1}{2}$$

BIILQM = (BII)(XI)(EILQ)

BIIQNM = (BII)(XII)(ELQ)

For $Z_q = Z_1$ we proceed in the manner outlined in Art 3(b) of Ch.III.

Putting $j\left(1-e^{-G_{om}\frac{\Delta Z_1}{2}}\right)$ instead of factor $\left(e^{-G_{om}|Z_q-Z_1|}.\sinh G_{om}\frac{\Delta Z_1}{2}\right)$ and putting $j\left(1-e^{-G_{nm}\frac{\Delta Z_1}{2}}\right)$ instead of factor $\left(e^{-G_{nm}|Z_q-Z_1|}.\sinh G_{nm}\frac{\Delta Z_1}{2}\right)$ in the Equation(21.1)

and rearranging terms, we have,

$$Q_{1q}^{11} = \left[\left\{j\left[2C\left(\frac{4b\,\Delta x_1}{\pi}\right)\sum_{m=1}^{20}\frac{k_o^2}{m\,G_{om}^2}\sin\frac{m\pi Y_q}{b}.\right.\right.\right.$$

$$.\sin\frac{m\pi Y_1}{b}\sin\frac{m\pi\Delta Y_1}{2b} + 4C\left(\frac{8ab}{\pi^2}\right)\sum_{n=1}^{20}\frac{1}{n}\cos\frac{n\pi X_q}{a}.$$

$$.\cos\frac{n\pi X_1}{a}\sin\frac{n\pi\Delta X_1}{2a}\sum_{m=1}^{20}\frac{\left[k_o^2-\left(\frac{n\pi}{a}\right)^2\right]}{m\,G_{nm}^2}\sin\frac{m\pi Y_q}{b}.$$

$$\left.\left.\left..\sin\frac{m\pi Y_1}{b}\sin\frac{m\pi\Delta Y_1}{2b}\right]\right\}-\right.$$

$$\sin\frac{m\pi Y_1}{b}\,\sin\frac{m\pi \Delta Y_1}{2b}\ \Big\{ j\Big[2C\left(\frac{4b\,\Delta X_1}{\pi}\right)\sum_{m=1}^{20}\frac{k_o^2}{m\Gamma_{om}^2}\,e^{-G_{om}\frac{\Delta Z_l}{2}}$$

$$e^{-k_{om}\frac{\Delta Z_l}{2}}\,\sin\frac{m\pi Y_q}{b}\,\sin\frac{m\pi Y_1}{b}\,\sin\frac{m\pi\Delta Y_1}{2b} + 4C\left(\frac{8ab}{\pi^2}\right).$$

$$\cdot\sum_{n=1}^{20}\frac{1}{n}\cos\frac{n\pi x_q}{a}\,\cos\frac{n\pi X_1}{a}\,\sin\frac{n\pi\Delta X_1}{2a}\sum_{m=1}^{20}\frac{\left[k_o^2-\left(\frac{n\pi}{a}\right)^2\right]}{m\Gamma_{nm}^2}$$

$$\cdot\, e^{-G\frac{\Delta Z_l}{2}}\,\sin\frac{m\pi Y_q}{b}\,\sin\frac{m\pi Y_1}{b}\,\sin\frac{m\pi\Delta Y_1}{2b}\Big\}\Big]$$

$$=\Big\{ j\Big[2C\left(\frac{4b\,\Delta X_1}{4\pi}\right) GIILL - 4C\left(\frac{8ab}{4\pi^2}\right)\sum_{n=2}^{20}(AIIN)(FIILLN)$$

$$-4C\left(\frac{8ab}{\pi^2}\right)(AIII)\ (FIILLI)\Big] - j\Big[\sum_{m=1}^{20} BILLM + \sum_{n=1}^{20}(AIIN)\sum_{m=1}^{20} BIILNM\Big]\Big\}$$

where as shown in Art 3(b) of Ch.III

GIILL = GII(1) + GII(2) - GII(3) - GII(4)

Wherein

$$GII(I) = -\frac{[\pi - T(I)]}{2} + \frac{\pi}{2}\,\frac{\sin a_o[\pi - T(I)]}{\sin a_o\pi} = \frac{\pi}{2}$$

$$=\frac{\pi}{2}\cdot\frac{\sin\left[\frac{bk_o}{\pi}(\pi - T(I))\right]}{\sin\left(\frac{bk_o}{\pi}\pi\right)} - \frac{[\pi - T(I)]}{2} \quad \text{for } T(1) = \frac{\pi}{b}\left(Y_q + Y_1 - \frac{\Delta Y_1}{2}\right)$$

$$T(2) = \frac{\pi}{b}\left(Y_q - Y_1 + \frac{\Delta Y_1}{2}\right),\quad T(3) = \frac{\pi}{b}\left(Y_q + Y_1 + \frac{\Delta Y_1}{2}\right),$$

$$T(4) = \frac{\pi}{b}\left(Y_q - Y_1 - \frac{\Delta Y_1}{2}\right)$$

$$\ldots = \frac{\ldots}{b} \left(Y_q - Y_1 - \frac{Y_1}{2} \right)$$

FIILLN = FII(1) + FII(2) - FII(3) - FII(4)

wherein $FII(I) = \frac{\Pi - T(I)}{2} - \frac{\Pi \operatorname{Sinh}(a_n(\Pi - T(I)))}{2 \operatorname{Sinh}(a_n \Pi)}$ with

$$a_n = \frac{b}{\Pi} \sqrt{\left(\frac{n\Pi}{a}\right)^2 - k_o^2} \qquad AIII = AIIN \Big]_{n=1}$$

FIILLI = FI (1) + FI(2) - FI(3) - FI(4)

wherein $FI(I) = \frac{\Pi - T(I)}{2} - \left(\frac{\Pi}{2}\right) \cdot \frac{\operatorname{Sin}\left[\frac{b}{a}(\Pi - T(I))\right]}{\operatorname{Sin}\left(\frac{b}{a}\Pi\right)}$ with

$$a_1 = \frac{b}{\Pi} \sqrt{\left(\frac{\Pi}{a}\right)^2 - \frac{2\Pi^2}{a^2}} = j\,\frac{b}{a}$$

BILLM = (BII) (XI) (EILL) wherein $EILL = e^{-\mathcal{G}_{om}\frac{\Delta Z_1}{2}}$

BIILNM = (BII) (XII) (ELL) wherein $ELL = e^{-\mathcal{G}_{nm}\frac{\Delta Z_1}{2}}$

Q_{1q}^{22}

For $Z_q \neq Z_1$ we have from Equation (21.2)

$$Q_{1q}^{22} = \left[\left\{ 2c \left(\frac{4a\,\Delta Y_1}{\Pi} \right) \sum_{n=1}^{20} \frac{k_o^2\, e^{-\mathcal{G}_{no}|Z_q - Z_1|}}{n\,\mathcal{G}_{no}^2} \cdot \right.\right.$$

$$\left. \cdot \operatorname{Sinh} \mathcal{G}_{no} \frac{\Delta Z_1}{2} \cdot \operatorname{Sin} \frac{n\Pi X_q}{a} \operatorname{Sin} \frac{n\Pi X_1}{q} \operatorname{Sin} \frac{n\Pi \Delta X_1}{2a} \right\}$$

$$+ \left\{ 4c \left(\frac{8ab}{\Pi^2} \right) \sum_{m=1}^{20} \frac{1}{m} \operatorname{Cos} \frac{m\Pi Y_q}{b} \operatorname{Cos} \frac{m\Pi Y_1}{b} \operatorname{Sin} \frac{m\Pi \Delta Y_1}{2b} \cdot \right.$$

$$\left.\left. \cdot \sum_{n=1}^{20} \frac{\left[k_o^2 - \left(\frac{m\Pi}{b}\right)^2 \right]}{n\,\mathcal{G}_{nm}^2} e^{-\mathcal{G}_{nm}|Z_q - Z_1|} \operatorname{Sinh} \mathcal{G}_{nm} \frac{\Delta Z_1}{2} \cdot \operatorname{Sin}\frac{n\Pi X_q}{a} \operatorname{Sin}\frac{n\Pi X_1}{a} \operatorname{Sin}\frac{n\Pi \Delta X_1}{2a} \right\} \right]$$

$$= \left[B2LQ1 + \sum_{n=2}^{20} B2LQN + \sum_{m=1}^{20} A22M \sum_{n=1}^{20} B22QNM \right]$$

where B2LQN = (B22) (X2) (E2LQ)

wherein $B22 = \sin \frac{n\pi x_q}{a} \sin \frac{n\pi x_1}{a} \sin \frac{n\pi \Delta x_1}{2a}$

$$X2 = 2C \left(\frac{4a\,\Delta y_1}{\pi} \right) \left(\frac{k_o^2}{n\gamma_{no}^2} \right)$$

$$E2LQ = e^{-\gamma_{no} |z_q - z_1|} \cdot \sinh \gamma_{no} \frac{\Delta z_1}{2}$$

$$B2LQ1 = B2LQN \Big]_{n=1} \quad \text{with } \gamma_{10} = j\frac{\pi}{a}$$

$$A22M = \frac{1}{m} \cos \frac{m\pi y_q}{b} \cos \frac{m\pi y_1}{b} \sin \frac{m\pi \Delta y_1}{2b}$$

B22QNM = (B22) (X22) (E1Q)

wherein $X22 = 4C \left(\frac{8ab}{\pi^2} \right) \frac{\left[k_o^2 - \left(\frac{m\pi}{b} \right)^2 \right]}{n\,\gamma_{nm}^2}$

For $z_q = z_1$ we proceed in the manner outlined in Art 3(b)of Ch.III. Putting $j\left(1-e^{-\gamma_{no}\frac{\Delta z_1}{2}}\right)$ instead of factor $e^{-\gamma_{no} |z_q - z_1|} \cdot \sinh \gamma_{no}\frac{\Delta z_1}{2}$ and putting $j\left(1-e^{-\gamma_{nm}\frac{\Delta z_1}{2}}\right)$ instead of factor $e^{-\gamma_{nm}|z_q - z_1|} \cdot \sinh \gamma_{nm}\frac{\Delta z_1}{2}$ in the equation (21.2) and rearranging terms, we have,

$$Q^{22}_{\ell q} = \Bigg\{ j \Bigg[2C \left(\frac{4a\,\Delta Y_1}{\Pi}\right) \sum_{n=1}^{20} \frac{k_o^2}{n G_{no}^2} \operatorname{Sin} \frac{n\Pi X_q}{a} \cdot$$

$$\cdot \operatorname{Sin} \frac{n\Pi X_1}{a} \operatorname{Sin} \frac{n\Pi \Delta X_1}{2a} + 4C \left(\frac{8ab}{\Pi^2}\right) \sum^{20} \frac{1}{m} \operatorname{Cos} \frac{m\Pi Y_q}{b} \cdot$$

$$\cdot \operatorname{Cos} \frac{m\Pi Y_1}{b} \operatorname{Sin} \frac{m\Pi \Delta Y_1}{b} \sum_{n=1}^{20} \frac{\left[k_o^2 - \left(\frac{m\Pi}{b}\right)^2\right]}{n G_{nm}^2} \operatorname{Sin} \frac{n\Pi X_q}{a}$$

$$\cdot \operatorname{Sin} \frac{n\Pi X_q}{a} \operatorname{Sin} \frac{n\,\Pi \Delta X_1}{2a} \Bigg] - j \Bigg[2C \left(\frac{4a\,\Delta Y_1}{\Pi}\right) \sum_{n-1}^{20} \frac{k_o^2}{n G_{no}^2} \cdot$$

$$\cdot e^{-G_{no}\frac{\Delta Z_\ell}{2}} \cdot \operatorname{Sin} \frac{n\Pi X_q}{a} \operatorname{Sin} \frac{n\Pi X_1}{a} \operatorname{Sin} \frac{n\Pi \Delta X_1}{2a}$$

$$+ 4C \left(\frac{8ab}{\Pi^2}\right) \sum_{m=1}^{20} \frac{1}{m} \operatorname{Cos} \frac{m\Pi Y_q}{b} \operatorname{Cos} \frac{m\Pi Y_1}{b} \operatorname{Sin} \frac{m\Pi \Delta Y_1}{2b} \cdot$$

$$\cdot \sum_{n=1}^{20} \frac{\left[k_o^2 - \left(\frac{m\Pi}{b}\right)^2\right]}{n G_{nm}^2} \operatorname{Sin} \frac{n\Pi X_q}{a} \operatorname{Sin} \frac{n\Pi X_1}{a} \operatorname{Sin} \frac{n\,\Pi \Delta X_1}{2a} \Bigg] \Bigg\}$$

$$= j \left[2C \left(\frac{4a\,\Delta Y_1}{4\Pi}\right) G22LL - 4\,C \left(\frac{8ab}{4\Pi^2}\right) \sum_{m=1}^{20} (A22M) \sum_{n=1}^{20} (F22LNM) \right]$$

$$- j \left[B2LL1 + \sum_{n=2}^{20} B2LLN + \sum_{m=1}^{20} (A22M) \sum_{n=1}^{20} B22LNM \right]$$

where as shown in Art 3(b) of Ch. III.

G22LL = G22(5) + G22(6) - G22(7) - G22(8)

$$\text{wherein } G22(I) = \frac{\Pi}{2} \; \frac{\operatorname{Sin} b_o\,(\Pi - T(I))}{\operatorname{Sin} b_o\,\Pi} - \frac{[\Pi - T(I)]}{2} \quad \text{with } b_o = \frac{ak_o}{\Pi}$$

$$b_o = \frac{ak_o}{\Pi}$$

for $T(5) = \frac{\Pi}{a}\left(x_q + x_1 - \frac{\Delta x_1}{2}\right)$, $T(6) = \frac{\Pi}{a}\left(x_q - x_1 + \frac{\Delta x_1}{2}\right)$,

$T(7) = \frac{\Pi}{a}\left(x_q + x_1 + \frac{\Delta x_1}{2}\right)$, $T(8) = \frac{\Pi}{a}\left(x_q - x_1 - \frac{\Delta x_1}{2}\right)$

F22LLM = F22(5) + F22(6) − F22(7) − F22(8)

wherein $F22(I) = \frac{[\Pi - T(I)]}{2} - \left(\frac{\Pi}{2}\right) \cdot \frac{\text{Sinh}\, b_m[\Pi - T(I)]}{\text{Sinh}(b_m \Pi)(I))}$ with

$$b_m = \frac{a}{\Pi}\sqrt{\left(\frac{m\Pi}{b}\right)^2 - k_o^2}$$

B2LLN = (B22)(X2)(E2LL) wherein $E2LL = e^{-G_{no}\frac{\Delta Z_\ell}{2}}$

$B2LLI = B2LLN\Big]_{n=1}$

B22LNM = (B22)(X22) (ELL) wherein $ELL = e^{-G_{nm}\frac{\Delta Z_\ell}{2}}$

$\underline{Q^{33}_{1q}}$

For $Z_q \neq Z_\ell$ we have from Equation (21.5)

$$Q^{33}_{1q} = \left[4C\left(\frac{8ab}{\Pi^2}\right)\sum_{n=1}^{20}\frac{1}{n}\text{Sin}\frac{n\Pi x_q}{a}\ \text{Sin}\frac{n\Pi x_\ell}{a}\ \text{Sin}\frac{n\Pi\Delta x_\ell}{2a}\ .\right.$$

$$\cdot\sum_{m=1}^{20}\frac{\left[\left(\frac{n\Pi}{a}\right)^2 + \left(\frac{m\Pi}{b}\right)^2\right]}{m\ G_{nm}^2} \cdot e^{-G_{nm}|Z_q - Z_\ell|} \cdot \text{Sinh}\ G_{nm}\frac{\Delta Z_\ell}{2} \ .$$

$$\left.\cdot\ \text{Sin}\frac{m\Pi y_q}{b}\ \text{Sin}\frac{m\Pi y_1}{b}\ \text{Sin}\frac{m\Pi\Delta y_\ell}{2b}\right] = \sum_{n=1}^{20} A33N \sum_{m=1}^{20} B33QNM$$

where $A33N = \frac{1}{n}\ \text{Sin}\frac{n\Pi x_q}{a}\ \text{Sin}\frac{n\Pi x_\ell}{a}\ \text{Sin}\frac{n\Pi\Delta x_\ell}{2a}$

B33QNM = (B33) (X33) (ELQ)

Wherein $B33 = \text{Sin}\frac{m\Pi y_q}{b}\ \text{Sin}\frac{m\Pi y_1}{b}\ \text{Sin}\ m\frac{\Pi\Delta y_1}{2b}$

$$X33 = 4C\left(\frac{8ab}{\Pi^2}\right) \cdot \frac{\left[\left(\frac{n\Pi}{a}\right)^2 + \left(\frac{m\Pi}{b}\right)^2\right]}{m G_{nm}^2}$$

$$ELQ = e^{-G_{nm}|Z_q - Z_1|} \cdot \operatorname{Sinh} G_{nm}\frac{\Delta Z_1}{2}$$

For $Z_q = Z_1$ we proceed in the manner outlined in Art 3(b) of Ch.III. Putting $j\left(1-e^{-G_{nm}\frac{\Delta Z_1}{2}}\right)$ instead of factor $\left(e^{-G_{nm}|Z_q-Z_1|} \cdot \operatorname{Sinh} G_{nm}\frac{\Delta Z_1}{2}\right)$ in the Equation (21.3) and re-arranging terms, we have,

$$Q^{33}_{1q} = \Bigg\{ j\Bigg[4C\left(\frac{8ab}{\Pi^2}\right) \cdot \sum_{n=1}^{20} \frac{1}{n} \operatorname{Sin}\frac{n\Pi X_q}{a} \operatorname{Sin}\frac{n\Pi X_1}{a} \cdot \operatorname{Sin}\frac{n\Pi\Delta X_1}{2a} \cdot \sum_{m=1}^{20} \frac{\left[\left(\frac{n\Pi}{a}\right)^2 + \left(\frac{m\Pi}{b}\right)^2\right]}{m G_{nm}^2} \cdot \operatorname{Sin}\frac{m\Pi Y_q}{b} \cdot \operatorname{Sin}\frac{m\Pi Y_1}{b} \operatorname{Sin}\frac{m\Pi\Delta Y_1}{2b}\Bigg] - j\Bigg[4C\left(\frac{8ab}{\Pi^2}\right) \sum_{n=1}^{20} \frac{1}{n} \operatorname{Sin}\frac{n\Pi X_q}{a} \cdot \operatorname{Sin}\frac{n\Pi X_1}{a} \operatorname{Sin}\frac{n\Pi\Delta X_1}{2a} \cdot \sum_{m=1}^{20} \frac{\left[\left(\frac{n\Pi}{a}\right)^2 + \left(\frac{m\Pi}{b}\right)^2\right]}{m G_{nm}^2} e^{G_{nm}\frac{\Delta Z_1}{2}} \cdot \operatorname{Sin}\frac{m\Pi Y_q}{b} \operatorname{Sin}\frac{m\Pi Y_1}{b} \operatorname{Sin}\frac{m\Pi\Delta Y_1}{2b}\Bigg]\Bigg\}$$

$$= \Bigg\{ j\Bigg[4C\left(\frac{8ab}{4\Pi}\right)(A331) \cdot (F33LL1) + 4C\left(\frac{8ab}{4\Pi}\right)\sum_{n=2}^{20}(A33N)(F33LLN)\Bigg) - j\Bigg[\sum_{n=1}^{20} A33N \cdot \sum_{m=1}^{20} B33\ LNM\Bigg]\Bigg\}$$

where as shown in Art 3(b) of Ch. III

$$A33N = \frac{1}{n} \sin \frac{n\pi X_q}{a} \sin \frac{n\pi X_1}{a} \sin \frac{n\pi \Delta X_1}{2a}$$

F33LLN = F33(1) + F33(2) - F33(3) - F33(4)

$$\text{wherein } F33(I) = \left[\frac{(\pi - T(I))}{2} - \left(\frac{a_o^2}{a_n^2} \right) \cdot \frac{[\pi - T(I)]^2}{2} + \frac{\pi \sinh a_n [\pi - T(I)]}{2 \sinh(a_n \pi)} \left(\frac{a_o^2}{a_n^2} \right) \right]$$

$$\text{with } a_o = \frac{bk_o}{\pi}, \quad a_n = \frac{b}{\pi} \sqrt{\left(\frac{n\pi}{a} \right)^2 - k_o^2}$$

$$A331 = A33N \Big]_{n=1}$$

F33LL1 = F331(1) + F331(2) - F331(3) - F331(4)

$$\text{wherein } F331(I) = \left[\frac{[\pi - T(I)]}{2} - \left(a_o^2 \right) \cdot \left(- \frac{a^2}{b^2} \right) \cdot \frac{[\pi - T(I)]}{2} \right.$$

$$\left. + \left(a_o^2 \right) \cdot \left(- \frac{a^2}{b^2} \right) \cdot \left(\frac{\pi}{2} \right) \cdot \frac{\sin \frac{b}{a} [\pi - T(I)]}{\sin \left(\frac{b}{a} \pi \right)} \right]$$

B33LNM = (B33)(X33) (ELL)

$$\text{wherein } X33 = 4C \left(\frac{8ab}{\pi^2} \right) \cdot \frac{\left[\left(\frac{n\pi}{a} \right)^2 + \left(\frac{m\pi}{b} \right)^2 \right]}{m G_{nm}^2}$$

$$ELL = e^{-G_{nm} \frac{\Delta Z_1}{2}}$$

Q_{1q}^{12}

For $Z_q \neq Z_1$ we have from Equation 21.4

$$Q_{1q}^{12} = \left[4C \left(-8 \right) \sum_{n=1}^{20} \cos \frac{n\pi X_q}{a} \sin \frac{n\pi X_1}{a} \sin \frac{n\pi \Delta X_1}{2a} \right. ,$$

$$\cdot \sum_{m=1}^{20} \frac{e^{-G_{nm} |Z_q - Z_1|}}{G_{nm}^2} \sinh G_{nm} \frac{\Delta Z_1}{2} ,$$

$$\left. X \cdot \sin \frac{m\pi Y_q}{b} \cos \frac{m\pi Y_1}{b} \sin \frac{m\pi \Delta Y_1}{2b} \right]$$

$$= \sum_{n=1}^{20} A12N \sum_{m=1}^{20} B12QNM$$

where $A12N = \cos\frac{n\Pi X_q}{a} \sin\frac{n\Pi X_1}{a} \sin\frac{n\Pi \Delta X_1}{2a}$

$B12QNM = (B12)\ (X12)\ (ELQ)$

wherein $B12 = \sin\frac{m\Pi Y_q}{b} \cos\frac{m\Pi Y_1}{b} \sin\frac{m\Pi \Delta Y_1}{2b}$

$$X12 = 4C\ (-8) \cdot \left(\frac{1}{\Gamma_{nm}^2}\right)$$

$$ELQ = e^{-\Gamma_{nm}|Z_q - Z_1|} \cdot \sinh \Gamma_{nm} \frac{\Delta Z_1}{2}$$

For $Z_q = Z_1$ we proceed in the manner outlined in Art 3(b) of Ch.III.

Putting $j\ (1-e^{-\Gamma_{nm}\frac{\Delta Z_1}{2}})$ instead of factor $e^{-\Gamma_{nm}|Z_q - Z_1|} \cdot \sinh \Gamma_{nm}\frac{\Delta Z_1}{2}$ in the Equation (21.4) and re-arranging terms, we have,

$$Q^{12}_{1q} = \Big\{ j \Big[4c\ (-8) \sum_{n=1}^{20} \cos\frac{n\Pi X_q}{a} \sin\frac{n\Pi X_1}{a} \sin\frac{n\Pi \Delta X_1}{2a} \cdot \sum_{m=1}^{20} \frac{1}{\Gamma_{nm}^2} \sin\frac{m\Pi Y_q}{b} \cos\frac{m\Pi Y_1}{b} \sin\frac{m\Pi \Delta Y_1}{2b} \Big]$$

$$- j \Big[4c(-8) \sum_{n=1}^{20} \cos\frac{n\Pi X_q}{a} \sin\frac{n\Pi X_1}{a} \sin\frac{n\Pi \Delta X_1}{2a} \sum_{m=1}^{20} \frac{e^{-\Gamma_{nm}\frac{\Delta Z_1}{2}}}{\Gamma_{nm}^2} \sin\frac{m\Pi Y_q}{b} \cdot \cos\frac{m\Pi Y_1}{b} \sin\frac{m\Pi \Delta Y_1}{2b} \Big] \Big\}$$

$$= \Big\{ j \Big[4c\left(-\frac{8}{4}\right)(A121)\ (F12LL1) + 4c\left(-\frac{8}{4}\right) \sum_{n=2}^{20} (A12N)\ (F12LLN) \Big] - j \Big[\sum_{n=1}^{20} A12N \sum_{m=1}^{20} B12LNM \Big] \Big\}$$

where as shown in Art 3(b) of Ch.III.

$$A12N = \cos\frac{n\pi X_q}{a}\ \sin\frac{n\pi X_1}{a}\ \sin\frac{n\pi\Delta X_1}{2a}$$

$$F12LLN = -FLL(1) + FLL(2) + FLL(3) - FLL(4)$$

$$\text{wherein } FLL(I) = \left[\left(\frac{b^2}{\pi^2}\right)\cdot\left(\frac{\pi}{2a_n}\right)\cdot\left(\frac{\cosh\left[\pi - T(I)\right]}{\sinh\left(a_n \pi\right)}\right) - \frac{b^2}{2\pi^2 a_n^2}\right] \text{ with } a_n = \frac{b}{\pi}\sqrt{\left(\frac{n\pi}{a}\right)^2 - k_o^2} \text{ and } I = 1,2,3,4$$

$$A121 = A12N\Big]_{n=1}$$

$$F12LL1 = -FLL1(1) + FLL1(2) + FLL1(3) - FLL1(4)$$

$$\text{wherein } FLL1(I) = \left[\frac{a^2}{2\pi^2} - \left(\frac{ab}{2\pi}\right)\cdot\frac{\cos\frac{b}{a}\left[\pi - T(I)\right]}{\sin\left(\frac{b}{a}\pi\right)}\right]$$

$$B12LNM = (B12)\ (X12)\ (ELL) \text{ wherein } ELL = e^{-\varepsilon_{nm}\frac{\Delta Z_\ell}{2} \cdots \frac{Z_1}{2}}$$

Q^{21}_{1q}

For $Z_q \neq Z_1$ we have from Equation (21.5)

$$Q^{21}_{1q} = \left[4C\left(-8\right)\sum_{n=1}^{20}\sin\frac{n\pi X_q}{a}\ \cos\frac{n\pi X_1}{a}\ \sin\frac{n\pi\Delta X_1}{2a}\right.\cdot$$

$$\cdot\sum_{m=1}^{20}\frac{e^{-\varepsilon_{nm}|Z_q - Z_1|}}{\varepsilon_{nm}^2}\cdot\sinh\varepsilon_{nm}\frac{\Delta Z_1}{2}\cdot X\cos\frac{m\pi Y_q}{b}\ \sin\frac{m\pi Y_1}{b}\cdot$$

$$\left.\cdot\sin\frac{m\pi\Delta Y_1}{2b}\right] \equiv \sum_{n=1}^{20} A21N \sum_{m=1}^{20} B21QNM$$

$$\text{where } A21N = \sin\frac{n\pi X_q}{a}\ \cos\frac{n\pi X_1}{a}\ \sin\frac{n\pi\Delta X_1}{2a}$$

$$B21QNM = (B21)\ (X21)\ (ELQ)$$

wherein $B21 = \cos\frac{m\Pi Y_q}{b}\sin\frac{m\Pi Y_1}{b}\sin\frac{m\Pi\Delta Y_1}{2b}$

$X21 = X12 = 4C\left(-8\right).\left(\frac{1}{G_{nm}^2}\right)$

$ELQ = e^{-G_{nm}\left|Z_q-Z_1\right|}.\sinh G_{nm}\frac{\Delta Z_1}{2}$

For $Z_q = Z_1$ we proceed in the manner outlined in Art 3(b) of Ch.III. Putting $j\left(1-e^{-G_{nm}\frac{\Delta Z_1}{2}}\right)$ instead of factor $e^{-G_{nm}\left|Z_q-Z_1\right|}.\sinh G_{nm}\frac{\Delta Z_1}{2}$ in the Equation (21.5) and re-arranging terms, we have,

$$Q_{1q}^{21} = \left\{j\left[4C\left(-8\right)\sum_{m=1}^{20}\sin\frac{n\Pi X_q}{a}\cos\frac{n\Pi X_1}{a}.\sin\frac{n\Pi\Delta X_1}{2a}\sum_{m=1}^{20}\frac{1}{G_{nm}^2}\cos\frac{m\Pi Y_q}{b}\sin\frac{m\Pi Y_1}{b}\sin\frac{m\Pi\Delta Y_1}{2b}\right]\right.$$

$$\left.-j\left[4C\left(-8\right)\sum_{n=1}^{20}\sin\frac{n\Pi X_q}{a}\cos\frac{n\Pi X_1}{a}\sin\frac{n\Pi\Delta X_1}{2a}\sum_{m=1}^{20}\frac{e^{-G_{nm}\frac{\Delta Z_1}{2}}}{G_{nm}^2}.\cos\frac{m\Pi Y_q}{b}\sin\frac{m\Pi Y_1}{b}\sin\frac{m\Pi\Delta Y_1}{2b}\right]\right\}$$

$$= \left\{j\left[4C\left(-\frac{8}{4}\right)(A211)(F21LL1) + 4C\left(-\frac{8}{4}\right)\sum_{n=2}^{20}(A21N)(F21LLN)\right] - j\left[\sum_{n=1}^{20}A21N\sum_{m=1}^{20}B21LNM\right]\right\}$$

where F21LLN = FLL(1) + FLL(2) - FLL(3) -FLL(4)

wherein FLL(I) is already defined in the expression for F12LLN;

F12LLN $A211 = A\,21N\Big]_{n=1}$;

F21LL1 = FLL1 (1) + FLL1(2) - FLL1(3) - FLL1(4)

wherein FLL1(I) is already defined in the expression for F12LL1

B21LNM = (B21) (X21)(ELL) wherein ELL= $e^{-\epsilon_{nm}\frac{\Delta Z_1}{2}}$

In the following expressions for Q^{23}_{1q}, Q^{32}_{1q}, Q^{31}_{1q}, Q^{13}_{1q}, the upper sign is for $Z_q < Z_1$ and the lower sign is for $Z_q > Z_1$

Q^{23}_{1q}

For $Z_q \neq Z_1$ we have from Equation (21.6)

$$Q^{23}_{1q} = \left[4C\left(\pm\frac{8a}{\Pi}\right)\sum_{n=1}^{20}\frac{1}{n}\sin\frac{n\Pi X_q}{a}\sin\frac{n\Pi X_1}{a}\sin\frac{n\Pi\Delta X_1}{2a}\sum_{m=1}^{20}\frac{e^{-\epsilon_{nm}|Z_q-Z_1|}}{\epsilon_{nm}}\sinh\epsilon_{nm}\frac{\Delta Z_1}{2}\cdot\cos\frac{m\Pi Y_q}{b}\sin\frac{m\Pi Y_1}{b}\cdot\sin\frac{m\Pi\Delta Y_1}{2b}\right]$$

$$= 4C\left(\pm\frac{8a}{\Pi}\right)\sum_{n=1}^{20} A23N \sum_{m=1}^{20} B23QNM$$

Where $A23N = \frac{1}{n}\sin\frac{n\Pi X_q}{a}\sin\frac{n\Pi X_1}{a}\sin\frac{n\Pi\Delta X_1}{2a}$

B23QNM = (B23) (X23) (ELQ)

Wherein $B23 = B21 = \cos\frac{m\Pi Y_q}{b}\sin\frac{m\Pi Y_1}{b}\sin\frac{m\Pi\Delta Y_1}{2b}$

$X23 = 4C\left(\frac{8a}{\Pi}\right)\left(\frac{1}{\epsilon_{nm}}\right)$

$ELQ = e^{-\epsilon_{nm}|Z_q-Z_1|}\cdot\sinh\epsilon_{nm}\frac{\Delta Z_1}{2}$

For $Z_q = Z_1$, $R_{1q}^{23} = 0$

For $Z_q \neq Z_1$ we have from Equation (21.7)

$$Q_{1q}^{32} = \left[4C \left(\frac{-8a}{+\Pi} \right) \sum_{n=1}^{20} \frac{1}{n} \sin \frac{n\Pi X_q}{a} \sin \frac{n\Pi X_1}{a} \sin \frac{n\Pi \Delta X_1}{2a} \cdot \sum_{m=1}^{20} \frac{e^{-G_{nm}|Z_q - Z_1|}}{G_{nm}} . \sinh G_{nm} \frac{\Delta Z_1}{2} . \sin \frac{m\Pi Y_q}{b} \cos \frac{m\Pi Y_1}{b} \sin \frac{m\Pi \Delta Y_1}{2b} \right]$$

$$= \binom{\mp}{+} \sum_{n=1}^{20} A32N \sum_{m=1}^{20} B32QNM$$

where $A32N = \frac{1}{n} \sin \frac{n\Pi X_q}{a} \sin \frac{n\Pi X_1}{a} \sin \frac{n\Pi \Delta X_1}{2a}$

$B32QNM = (B32)\ (X32)\ (ELQ)$

wherein $B32 = \sin \frac{m\Pi Y_q}{b} \cos \frac{m\Pi Y_1}{b} \sin \frac{m\Pi \Delta Y_1}{2b}$

$X32 = 4C \left(\frac{8a}{\Pi} \right) \left(\frac{1}{G_{nm}} \right) = X23$

$ELQ = e^{-G_{nm}|Z_q - Z_1|} . \sinh G_{nm} \frac{\Delta Z_1}{2}$

For $Z_q = Z_1$, $Q_{1q}^{32} = 0$

Q_{1q}^{31}

For $Z_q \neq Z_1$ we have from Equation (21.8)

$$Q_{1q}^{31} = \left[4C \left(\frac{-8b}{+\Pi} \right) \sum_{m=1}^{20} \frac{1}{m} \sin \frac{m\Pi Y_q}{b} \sin \frac{m\Pi Y_1}{b} \sin \frac{m\Pi \Delta Y_1}{2b} \cdot \sum_{n=1}^{20} \frac{e^{-G_{nm}|Z_q - Z_1|}}{G_{nm}} . \sinh G_{nm} \frac{\Delta Z_1}{2} . \times \sin \frac{n\Pi X_q}{a} \cos \frac{n\Pi X_1}{a} \sin \frac{n\Pi \Delta X_1}{2a} \right]$$

$$= \binom{\mp}{+} \sum_{m=1}^{20} A31M \sum_{n=1}^{20} B31QNM$$

where $A31M = \frac{1}{m} \sin \frac{m\pi Y_q}{b} \sin \frac{m\pi Y_1}{b} \sin \frac{m\pi \Delta Y_1}{2b}$

$B31QNM = (B31)\,(X31)\,(ELQ)$

wherein $B31 = \sin \frac{n\pi X_q}{a} \cos \frac{n\pi X_1}{a} \sin \frac{n\pi \Delta X_1}{2a}$

$X31 = 4C \left(\frac{8b}{\pi} \right) \left(\frac{1}{\sigma_{nm}} \right)$

$ELQ = e^{-\sigma_{nm}(Z_q - Z_1)} \cdot \sinh \sigma_{nm} \frac{\Delta Z_1}{2}$

For $Z_q = Z_1$, $\quad Q^{31}_{1q} = 0$

Q^{13}_{1q}

For $Z_q \neq Z_1$ we have from Equation (21.9)

$$Q^{13}_{1q} = \left[4C \left(\pm \frac{8b}{\pi} \right) \sum_{m=1}^{20} \frac{1}{m} \sin \frac{m\pi Y_q}{b} \sin \frac{m\pi Y_1}{b} \sin \frac{m\pi \Delta Y_1}{2b} \cdot \sum_{n=1}^{20} \frac{e^{-\sigma_{nm}(Z_q - Z_1)}}{F_{nm}} \cdot \sinh \sigma_{nm} \frac{\Delta Z_1}{2} \cdot \cos \frac{n\pi X_q}{a} \sin \frac{n\pi X_1}{a} \sin \frac{n\pi \Delta X_1}{2a} \right]$$

$$= \left(\pm \right) \sum_{m=1}^{20} A13M \sum_{n=1}^{20} B13QNM$$

where $A13M = \left[\frac{1}{m} \sin \frac{m\pi Y_q}{b} \sin \frac{m\pi Y_1}{b} \sin \frac{m\pi \Delta Y_1}{2b} \right]$

$B13QNM = (B13)\,(X13)\,(ELQ)$

wherein $B13 = \left[\cos \frac{n\pi X_q}{a} \sin \frac{n\pi X_1}{a} \sin \frac{n\pi \Delta X_1}{2a} \right]$

$X13 = \left[4C \left(\frac{8b}{\pi} \right) \left(\frac{1}{\sigma_{nm}} \right) \right] = X31$

$ELQ = \left[e^{-\sigma_{nm}(Z_q - Z_1)} \cdot \sinh \sigma_{nm} \frac{\Delta Z_1}{2} \right]$

For $Z_q = Z_1$, $\quad Q^{13}_{1q} = 0$

With Q^{pk}_{1q} elements expressions presented as above the relevant computer program is as shown in Appendix V.

The digital Computer program consists of the main program, the sub-routine QPKLQ and the subroutine SOLVE. The main program, with the given data, utilizes the subroutine QPKLQ to find the matrix elements Q_{1q}^{pk} and then it utilizes the subroutine SOLVE to find the J_1^k elements by Gauss-Jordan elimination method[14] and, with J_1^k elements determined, it itself computes Z_{11}, Z_{12}, Z_1 and Z_2 as normalized values.

For the illustrative problem with the given dielectric volume, considered as a single cell the results obtained were as follows:-

$$Z_{11} = Z_{22} = 0{\cdot}138711 - j0{\cdot}326588$$
$$Z_{12} = Z_{21} = 0{\cdot}176307 - j0{\cdot}182039$$
$$Z_1 = Z_{11} - Z_{12} = -0{\cdot}037596 - j0{\cdot}134549$$
$$Z_2 = Z_{12} = 0{\cdot}176307 - j0{\cdot}182039$$

Subsequently the volume of the dielectric in the given illustrative problem was subdivided into three equal cells along X-direction and the results obtained thus were as follows:-

$$Z_{11} = Z_{22} = 0{\cdot}130448 - j0{\cdot}269111$$
$$Z_{12} = Z_{21} = 0{\cdot}154171 - j0{\cdot}158204$$
$$Z_1 = Z_{11} - Z_{12} = -0{\cdot}023723 - j0{\cdot}110907$$
$$Z_2 = Z_{12} = 0{\cdot}154171 - j0{\cdot}158204$$

The most outstanding feature of these results is that Z_1 parameter has a negative real part. A comment on this point has been presented in the concluding Chapter.

```
C     MAIN PROGRAM
C     NO OF CELLS=IJKLMN
      COMPLEXAA,BB,PIVOT,C10,A10,Z1,Z2,Z11,Z12,ZK,Q
     1 , S10,DET
      DIMENSION AA(36,36),BB(36,1),IPVOT(36),INDEX(36,2),PIVOT(36),Q(3
     1,3)
     1 ,X(12),Y(12),Z(12)
      COMMON D/BS/IPVOT, INDEX, PIVOT,AA,BB,MM,NN,DET/BQ/XQ,XL,DXL,
     1YQ,YL,DYL,ZQ,ZL,DZL,SKI,L,IQ,  A,B,C,SKD,Q
      IJKLMN=1
      D=.000001
      DD=0.000001
      PI=3.141592654
      READ(1,261)DXL,DYL,DZL,SKI
  261 FORMAT(3F10.8,F10.4)
      DO 262 IQ=1,IJKLMN
  263 FORMAT(3F10.8)
  262 READ(1,263)X(I),Y(I),Z(I)
      DO 290 IQ=1,IJKLMN
      DO 290 L=1,IJKLMN
      XL=X(L)
      YL=Y(L)
      ZL=Z(L)
      XQ=X(IQ)
      YQ=Y(IQ)
      ZQ=Z(IQ)
      CALL QPKLQ
      DO 202 I=1,3
      DO 202 J=1,3
  202 AA(((IQ-1)*3+I),((L-1)*3+J))=Q(I,J)
  290 CONTINUE
      NN=3*IJKLMN
      MM=1
      DET=CMPLX(0.,0.)
      DO 270 I=1,NN
      DO 270 J=1,NN
      CCCC=CABS(AA(I,J))
      IF(CCCC.LE.DD)AA(I,J)=CMPLX(0.,0.)
  270 CONTINUE
      DO 285 IQ=1,IJKLMN
      BB(((IQ-1)*3+1),1)=CMPLX(0.,0.)
      E=SIN((PI/2.54)*X(IQ))
      THETA=(PI/2.54)*Z(IQ)
      BB(((IQ-1)*3+2),1)=CMPLX((E*COS(THETA)),(-E*SIN(THETA)))
      BB(((IQ-1)*3+3),1)=CMPLX(0.,0.)
  285 CONTINUE
      WRITE(3,206)
  206 FORMAT(10X,'MATRIX BB')
      DO 230 I=1,NN
  207 FORMAT(2F10.8)
  230 WRITE(3,207)(BB(I,1))
      CALL SOLVE
      WRITE(3,208)
  208 FORMAT(10X,'SOLUTION XKL')
```

```
      DO 240 I=1,NN
  209 FORMAT(2F10.6)
  240 WRITE(3,209)(BB(I,1))
      C10=CMPLX(0.,0.)
      DO 265 L=1,IJKLMN
      CC=(-16.)*(DYL/PI)*(DZL/PI)*SIN((PI/2.54)*X(L))*SIN((PI/2.54)*DXL)
      S10=CMPLX(0.,CC)+BB(((L-1)*3+2),1)
  265 C10=C10+S10
      A10=CMPLX(1.,0.)
      ZK=(A10+C10)/(A10-C10)
      Z11=(CMPLX(1.,0.)+ZK)*(ZK/(CMPLX(2.,0.)+ZK))
      Z12=ZK/(CMPLX(2.,0.)+ZK)
      Z1=Z11-Z12
      Z2=Z12
      WRITE(3,211)
  211 FORMAT(10X,'Z11,Z12,Z1,Z2,SERIALLY')
      WRITE(3,212) Z11,Z12,Z1,Z2
  212 FORMAT(4(4X,2F10.6))
  280 CONTINUE
```

```
      SUBROUTINE QPKLQ
      COMPLEX Q    ,V
      DIMENSION Q(3,3),V(3,3),
     4               T(8),F11(4),G11(4),F22(4),G22(4),F33(4),FLL(4),
     1F1(4),S11LL5(20),S11LQ3(20),S22LL5(20),S22LQ3(20),F331(4),
     2S33LL4(20),S33LQ2(20),FLL1(4),S12LL4(20),S12LQ2(20),S21LL4(20),
     3S21LQ2(20),S23LQ2(20),S32LQ2(20),S31LQ2(20),S13LQ2(20)
      COMMON D/BQ/XQ,XL,DXL,YQ,YL,DYL,ZQ,ZL,DZL,SKI,L,IQ ,A,B,C,SKO,Q
      A=2.54
      B=1.27
      PI=3.141592654
      SKO=(2.*PI*PI)/(A*A)
      C=1/(2.*PI*PI)
      T(1)=(PI/B)*(YQ+YL-DYL)
      T(2)=(PI/B)*(YQ-YL+DYL)
      T(3)=(PI/B)*(YQ+YL+DYL)
      T(4)=(PI/B)*(YQ-YL-DYL)
      T(5)=(PI/A)*(XQ+XL-DXL)
      T(6)=(PI/A)*(XQ-XL+DXL)
      T(7)=(PI/A)*(XQ+XL+DXL)
      T(8)=(PI/A)*(XQ-XL-DXL)
      DO 10 I=1,8
    6 IF(T(I)-0.0)3,10,4
    3 T(I)=T(I)+2.*PI
      GO TO 6
    4 IF(T(I)-2*PI)10,10,5
    5 T(I)=T(I)-2*PI
      GO TO 4
   10 CONTINUE
      AO=(B/PI)*SQRT(SKO)
      BO=(A/PI)*SQRT(SKO)
      NNN=3
      MMM=3
  314 NNN1=NNN+1
      MMM1=MMM+1
      NN=NNN
      MM=MMM
      IF(ZQ-ZL)12,11,12
   11 S11LL2=0
      DO 9 I=1,4
    9 G11(I)=(PI/2.)*(SIN(AO*(PI-T(I)))/SIN(AO*PI))-(PI-T(I))*(0.5)
      G11LL=G11(1)+G11(2)-G11(3)-G11(4)
      DO 60 I=1,4
   60 F1(I)=(PI-T(I))*(0.5)-(PI/2.)*(SIN((B/A)*(PI-T(I)))/SIN(B*PI/A))
      F11LL1=F1(1)+F1(2)-F1(3)-F1(4)
      A111=COS((PI*XQ)/A)*COS((PI*XL)/A)*SIN((PI*DXL)/A)
      S11LL1=((2.*C*8.*B*DXL)/(4.*PI))*G11LL-4.*C*8.*A*B*A111*F11LL1*(1.
     1/(4.*PI*PI))
      DO 13 N=2,NN
      AN=N
      AAN=(B/PI)*SQRT((AN*PI/A)**2-SKO)
      DO 8 I=1,4
    8 F11(I)=(PI-T(I))*(0.5)-(PI/2.)*(SINH(AAN*(PI-T(I)))/SINH(AAN*PI))
      F11LLN=F11(1)+F11(2)-F11(3)-F11(4)
```

```
 13 S11LL2=S11LL2+((4.*C*8.*A*B)/(4.*PI*PI))*A11N*F11LLN
    S11LL3=0
    DO 14 M=1,MM
    BM=M
    B11=SIN(BM*PI*YQ/B)*SIN(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
    GM=SQRT((BM*PI/B)**2-SKO)
    X1=2.*C*(2.*B*DXL)*SKO*(1./(PI*BM*GM*GM))
    E1LL=EXP(-GM*DZL)
    B1LLM=B11*X1*E1LL
 14 S11LL3=S11LL3+B1LLM
    S11LL4=0
    DO 16 N=1,NN
    S11LL5(N)=0
    AN=N
    A11N=(1./AN)*COS(AN*PI*XQ/A)*COS(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
    DO 15 M=1,MM
    BM=M
    B11=SIN(BM*PI*YQ/B)*SIN(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
    GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
    X11=(4./BM)*C*(8.*A*B)*(SKO-(AN*PI/A)**2)*(1./((PI*GNM)**2))
    ELL=EXP(-GNM*DZL)
    B11LNM=B11*X11*ELL
 15 S11LL5(N)=S11LL5(N)+A11N*B11LNM
 16 S11LL4=S11LL4+S11LL5(N)
    XQ11LL=0
    YQ11LL=S11LL1-S11LL2-S11LL3-S11LL4
 21 B2=(SIN((PI/A)*XQ)*SIN((PI/A)*XL)*SIN((PI/A)*DXL))*(2*C*8.*A*DYL*
   1SKO)*(1./PI)*((-(PI*PI)/(A*A)))
    XB2LL1=B2*COS((PI/A)*DZL)
    YB2LL1=B2*(-1)*SIN((PI/A)*DZL)
    XB2LQ1=B2*COS((PI/A)*ABS(ZQ-ZL))*SIN((PI/A)*DZL)
    YB2LQ1=B2*(-1.)*SIN((PI/A)*ABS(ZQ-ZL))*SIN((PI/A)*DZL)
    DO 30 I=5,8
 30 G22(I)=(PI/2.)*(SIN(BO*(PI-T(I)))/SIN(BO*PI))-(PI-T(I))*(0.5)
    G22LL=G22(5)+G22(6)-G22(7)-G22(8)
    S22LL1=((2.*C*8.*A*DYL)/(4.*PI))*G22LL-XB2LL1
    S22LL2=0
    DO 23 M=1,MM
    BM=M
    BBM=(A/PI)*SQRT((BM*PI/B)**2-SKO)
    DO 20 I=5,8
 20 F22(I)=(PI-T(I))*(0.5)-(PI/2.)*(SINH(BBM*(PI-T(I)))/SINH(BBM*PI))
    F22LLM=F22(5)+F22(6)-F22(7)-F22(8)
    A22M=(1./BM)*COS(BM*PI*YQ/B)*COS(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
 23 S22LL2=S22LL2+((4.*C*8.*A*B)/(4.*PI*PI))*A22M*F22LLM
    S22LL3=0
    DO 24 N=2,NN
    AN=N
    GN=SQRT((AN*PI/A)**2-SKO)
    B22=SIN(AN*PI*XQ/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
    X2=2.*C*(8.*A*DYL)*SKO*(1./(PI*AN*GN*GN))
    E2LL=EXP(-GN*DZL)
    B2LLN=B22*X2*E2LL
 24 S22LL3=S22LL3+B2LLN
```

```
      S22LL4=0
      DO 26 M=1,MM
      BM=M
      S22LL5(M)=0
      A22M=(1./BM)*COS(BM*PI*YQ/B)*COS(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
      DO 25 N=1,NN
      AN=N
      GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
      B22=SIN(AN*PI*XQ/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      X22=(4./AN)*C*(8.*A*B)*(SKO-(BM*PI/B)**2)*(1./((PI*GNM)**2))
      ELL=EXP(-GNM*DZL)
      B22LNM=B22*X22*ELL
   25 S22LL5(M)=S22LL5(M)+A22M*B22LNM
   26 S22LL4=S22LL4+S22LL5(M)
      XQ22LL=Y32LL1
      YQ22LL=S22LL1-S22LL2-S22LL3-S22LL4
   31 A331=SIN((PI*XQ)/A)*SIN((PI*XL)/A)*SIN((PI*DXL)/A)
      DO 70 I=1,4
   70 F331(I)=(PI-T(I))*(0.5)-(AO*AO)*(-(A*A)/(B*B))*(PI-T(I))*(0.5)
     1+(AO*AO)*(-(A*A)/(B*B))*(PI/2.)*SIN((B/A)*(PI-T(I)))*(1./SIN((
     2B*PI)/A))
      F33LL1=F331(1)+F331(2)-F331(3)-F331(4)
      S33LL1=4.*C*(8.*A*B)*(1./(4.*PI*PI))*A331*F33LL1
      S33LL2=0
      DO 33 N=2,NN
      AN=N
      AAN=(B/PI)*SQRT((AN*PI/A)**2-SKO)
      GN=SQRT((AN*PI/A)**2-SKO)
      DO 40 I=1,4
   40 F33(I)=(PI-T(I))*(0.5)-((AO**2)/(AAN**2))*(PI-T(I))*(0.5)+((AO**2)
     1/AAN**2)*(PI/2.)*(SINH(AAN*(PI-T(I)))/SINH(AAN*PI))
      F33LLN=F33(1)+F33(2)-F33(3)-F33(4)
      B22=SIN(AN*PI*XQ/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      A33N=(1./AN)*B22
   33 S33LL2=S33LL2+4.*C*((8.*A*B)/(4.*PI*PI))*A33N*F33LLN
      S33LL3=0
      DO 35 N=1,NN
      S33LL4(N)=0
      AN=N
      B22=SIN(AN*PI*XQ/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      A33N=(1./AN)*B22
      DO 34 M=1,MM
      BM=M
      B11=SIN(BM*PI*YQ/B)*SIN(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
      B33=B11
      GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
      X33=(4./BM)*C*(8.*A*B)*((AN*PI/A)**2+(BM*PI/B)**2)*(1./((PI*GNM)
     1**2))
      ELL=EXP(-GNM*DZL)
      B33LNM=B33*X33*ELL
   34 S33LL4(N)=S33LL4(N)+A33N*B33LNM
   35 S33LL3=S33LL3+S33LL4(N)
      XQ33LL=0
      YQ33LL=S33LL1+S33LL2-S33LL3
```

```
41 A121=COS((PI*XO)/A)*SIN((PI*XL)/A)*SIN((PI*DXL)/A)
   DO 80 I=1,4
80 FLL1(I)=((A*A)/(2.*PI*PI))-((A*B)/(2.*PI))*COS((B/A)*(PI-T(I)))
  1*(1./SIN((B*PI)/A))
   F12LL1=-FLL1(1)+FLL1(2)+FLL1(3)-FLL1(4)
   S12LL1=4.*C*(-8.)*(1./4.)*A121*F12LL1
   S12LL2=0
   DO 43 N=2,NN
   AN=N
   AAN=(B/PI)*SQRT((AN*PI/A)**2-SKO)
   DO 50 I=1,4
50 FLL(I)=((B*B)/(PI*PI))*(PI/(2.*AAN*SINH(AAN*PI)))*COSH(PI-T(I))
  1-((B/(PI*AAN))**2)*0.5
   F12LLN=-FLL(1)+FLL(2)+FLL(3)-FLL(4)
   A12N=COS(AN*PI*XO/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
43 S12LL2=S12LL2+4.*C*(-8.)*(1./4.)*A12N*F12LLN
   S12LL3=0
   DO 45 N=1,NN
   S12LL4(N)=0
   AN=N
   A12N=COS(AN*PI*XO/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
   DO 44 M=1,MM
   BM=M
   GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
   B12=SIN(BM*PI*YO/B)*COS(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
   X12=4.*C*(-8.)*(1./(GNM**2))
   ELL=EXP(-GNM*DZL)
   B12LNM=B12*X12*ELL
44 S12LL4(N)=S12LL4(N)+A12N*B12LNM
45 S12LL3=S12LL3+S12LL4(N)
   XO12LL=0
   YO12LL=S12LL1+S12LL2-S12LL3
51 A211=SIN((PI*XO)/A)*COS((PI*XL)/A)*SIN((PI*DXL)/A)
   DO 380 I=1,4
380 FLL1(I)=((A*A)/(2.*PI*PI))-((A*B)/(2.*PI))*COS((B/A)*(PI-T(I)))
  1*(1./SIN((B*PI)/A))
   F21LL1=FLL1(1)+FLL1(2)-FLL1(3)-FLL1(4)
   S21LL1=4.*C*(-8.)*(1./4.)*A211*F21LL1
   S21LL2=0
   DO 53 N=2,NN
   AN=N
   AAN=(B/PI)*SQRT((AN*PI/A)**2-SKO)
   A21N=SIN(AN*PI*XO/A)*COS(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
   DO 350 I=1,4
350 FLL(I)=((B*B)/(PI*PI))*(PI/(2.*AAN*SINH(AAN*PI)))*COSH(PI-T(I))
  1-((B/(PI*AAN))**2)*0.5
   F21LLN=FLL(1)+FLL(2)-FLL(3)-FLL(4)
53 S21LL2=S21LL2+4.*C*(-8.)*(1./4.)*A21N*F21LLN
   S21LL3=0
   DO 55 N=1,NN
   S21LL4(N)=0
   AN=N
   A21N=SIN(AN*PI*XO/A)*COS(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
   DO 54 M=1,MM
```

```
      BM=M
      GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
      B21=COS(BM*PI*YQ/B)*SIN(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
      X12=4.*C*(-8.)*(1./(GNM**2))
      X21=X12
      ELL=EXP(-GNM*DZL)
      B21LNM=B21*X21*ELL
   54 S21LL4(N)=S21LL4(N)+A21N*B21LNM
   55 S21LL3=S21LL3+S21LL4(N)
      XQ21LL=0
      YQ21LL=S21LL1+S21LL2-S21LL3
   61 XQ23LL=0
      YQ23LL=0
   71 XQ32LL=0
      YQ32LL=0
   81 XQ31LL=0
      YQ31LL=0
   91 XQ13LL=0
      YQ13LL=0
   97 IF(L-IQ)90,89,90
   89 Q(1,1)=CMPLX((XQ11LL-1./(SKI-SKO)),YQ11LL)
      Q(1,2)=CMPLX(XQ12LL,YQ12LL)
      Q(1,3)=CMPLX(XQ13LL,YQ13LL)
      Q(2,1)=CMPLX(XQ21LL,YQ21LL)
      Q(2,2)=CMPLX((XQ22LL-1./(SKI-SKO)),YQ22LL)
      Q(2,3)=CMPLX(XQ23LL,YQ23LL)
      Q(3,1)=CMPLX(XQ31LL,YQ31LL)
      Q(3,2)=CMPLX(XQ32LL,YQ32LL)
      Q(3,3)=CMPLX((XQ33LL-1./(SKI-SKO)-1./SKO),YQ33LL)
      GO TO 100
   90 Q(1,1)=CMPLX(XQ11LL,YQ11LL)
      Q(1,2)=CMPLX(XQ12LL,YQ12LL)
      Q(1,3)=CMPLX(XQ13LL,YQ13LL)
      Q(2,1)=CMPLX(XQ21LL,YQ21LL)
      Q(2,2)=CMPLX(XQ22LL,YQ22LL)
      Q(2,3)=CMPLX(XQ23LL,YQ23LL)
      Q(3,1)=CMPLX(XQ31LL,YQ31LL)
      Q(3,2)=CMPLX(XQ32LL,YQ32LL)
      Q(3,3)=CMPLX(XQ33LL,YQ33LL)
      GO TO 100
   12 S11LQ1=0
      DO 17 M=1,MM
      BM=M
      B11=SIN(BM*PI*YQ/B)*SIN(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
      GM=SQRT((BM*PI/B)**2-SKO)
      X1=2.*C*(2.*B*DXL)*SKO*(1./(PI*BM*GM*GM))
      E1LQ=EXP(-GM*ABS(ZQ-ZL))*SINH(GM*DZL)
      B1LQM=B11*X1*E1LQ
   17 S11LQ1=S11LQ1+B1LQM
      S11LQ2=0
      DO 19 N=1,NN
      AN=N
      A11N=(1./AN)*COS(AN*PI*XQ/A)*COS(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      S11LQ3(N)=0
```

```
      DO 18 M=1,MM
      BM=M
      B11=SIN(BM*PI*YQ/B)*SIN(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
      GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
      X11=(4./BM)*C*(8.*A*B)*(SKO-(AN*PI/A)**2)*(1./((PI*GNM)**2))
      ELQ=EXP(-GNM*ABS(ZQ-ZL))*SINH(GNM*DZL)
      B11QNM=B11*X11*ELQ
   18 S11LQ3(N)=S11LQ3(N)+A11N*B11QNM
   19 S11LQ2=S11LQ2+S11LQ3(N)
      XQ11LQ=S11LQ1+S11LQ2
      YQ11LQ=0
   22 S22LQ1=0
      DO 27 N=2,NN
C
      AN=N
      GN=SQRT((AN*PI/A)**2-SKO)
      B22=SIN(AN*PI*XQ/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      X2=2.*C*(8.*A*DYL)*SKO*(1./(PI*AN*GN*GN))
      E2LQ=EXP(-GN*ABS(ZQ-ZL))*SINH(GN*DZL)
      B2LQN=B22*X2*E2LQ
27    S22LQ1=S22LQ1+B2LQN
      S22LQ2=0
      DO 29 M=1,MM
      S22LQ3(M)=0
      BM=M
      A22M=(1./BM)*COS(BM*PI*YQ/B)*COS(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
      DO 28 N=1,NN
      AN=N
      GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
      B22=SIN(AN*PI*XQ/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      X22=(4./AN)*C*(8.*A*B)*(SKO-(BM*PI/B)**2)*(1./((PI*GNM)**2))
      ELQ=EXP(-GNM*ABS(ZQ-ZL))*SINH(GNM*DZL)
      B22QNM=B22*X22*ELQ
   28 S22LQ3(M)=S22LQ3(M)+A22M*B22QNM
   29 S22LQ2=S22LQ2+S22LQ3(M)
      XQ22LQ=YB2LQ1
      YQ22LQ=S22LQ1+S22LQ2
   32 S33LQ1=0
      DO 37 N=1,NN
      AN=N
      B22=SIN(AN*PI*XQ/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      A33N=(1./AN)*B22
      S33LQ2(N)=0
      DO 36 M=1,MM
      BM=M
      B11=SIN(BM*PI*YQ/B)*SIN(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
      B33=B11
      GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
      X33=(4./BM)*C*(8.*A*B)*((AN*PI/A)**2+(BM*PI/B)**2)*(1./((PI*GNM)
     1**2))
      ELQ=EXP(-GNM*ABS(ZQ-ZL))*SINH(GNM*DZL)
      B33QNM=B33*X33*ELQ
   36 S33LQ2(N)=S33LQ2(N)+A33N*B33QNM
   37 S33LQ1=S33LQ1+S33LQ2(N)
```

```
      XQ33LQ=S33LQ1
      YQ33LQ=0
   42 S12LQ1=0
      DO 47 N=1,NN
      S12LQ2(N)=0
      AN=N
      A12N=COS(AN*PI*XQ/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      DO 46 M=1,MM
      BM=M
      GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
      B12=SIN(BM*PI*YQ/B)*COS(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
      X12=4.*C*(-8.)*(1./(GNM**2))
      ELQ=EXP(-GNM*ABS(ZQ-ZL))*SINH(GNM*DZL)
      B12QNM=B12*X12*ELQ
   46 S12LQ2(N)=S12LQ2(N)+A12N*B12QNM
   47 S12LQ1=S12LQ1+S12LQ2(N)
      XQ12LQ=S12LQ1
      YQ12LQ=0
   52 S21LQ1=0
      DO 57 N=1,NN
      S21LQ2(N)=0
      AN=N
      A21N=SIN(AN*PI*XQ/A)*COS(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      DO 56 M=1,MM
      BM=M
      GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
      B21=COS(BM*PI*YQ/B)*SIN(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
      X12=4.*C*(-8.)*(1./(GNM**2))
      X21=X12
      ELQ=EXP(-GNM*ABS(ZQ-ZL))*SINH(GNM*DZL)
      B21QNM=B21*X21*ELQ
   56 S21LQ2(N)=S21LQ2(N)+A21N*B21QNM
   57 S21LQ1=S21LQ1+S21LQ2(N)
      XQ21LQ=S21LQ1
      YQ21LQ=0
   62 S23LQ1=0.
      DO 64 N=1,NN
      S23LQ2(N)=0
      AN=N
      B22=SIN(AN*PI*XQ/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      A33N=(1./AN)*B22
      A23N=A33N
      DO 63 M=1,MM
      BM=M
      GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
      B21=COS(BM*PI*YQ/B)*SIN(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
      B23=B21
      X23=4.*C*(8.*A)*(1./(PI*GNM))
      ELQ=EXP(-GNM*ABS(ZQ-ZL))*SINH(GNM*DZL)
      B23QNM=B23*X23*ELQ
   63 S23LQ2(N)=S23LQ2(N)+A23N*B23QNM
   64 S23LQ1=S23LQ1+S23LQ2(N)
      IF(ZQ-ZL)65,72,66
   65 XQ23LQ=S23LQ1
```

```
      YQ23LQ=0
      GO TO 72
   66 XQ23LQ=-S23LQ1
      YQ23LQ=0
   72 S32LQ1=0
      DO 74 N=1,NN
      S32LQ2(N)=0
      AN=N
      B22=SIN(AN*PI*XQ/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      A33N=(1./AN)*B22
      A32N=A33N
      DO 73 M=1,MM
      BM=M
      GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
      B12=SIN(BM*PI*YQ/B)*COS(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
      B32=B12
      X23=4.*C*(8.*A)*(1./(PI*GNM))
      X32=-X23
      ELQ=EXP(-GNM*ABS(ZQ-ZL))*SINH(GNM*DZL)
      B32QNM=B32*X32*ELQ
   73 S32LQ2(N)=S32LQ2(N)+A32N*B32QNM
   74 S32LQ1=S32LQ1+S32LQ2(N)
      IF(ZQ-ZL)75,82,76
   75 XQ32LQ=S32LQ1
      YQ32LQ=0
      GO TO 82
   76 XQ32LQ=-S32LQ1
      YQ32LQ=0
   82 S31LQ1=0
      DO 84 M=1,MM
      S31LQ2(M)=0
      BM=M
      B11=SIN(BM*PI*YQ/B)*SIN(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
      A31M=(1./BM)*B11
      DO 83 N=1,NN
      AN=N
      GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
      B31=SIN(AN*PI*XQ/A)*COS(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      X31=4.*C*(-8.*B)*(1./(PI*GNM))
      ELQ=EXP(-GNM*ABS(ZQ-ZL))*SINH(GNM*DZL)
      B31QNM=B31*X31*ELQ
   83 S31LQ2(M)=S31LQ2(M)+A31M*B31QNM
   84 S31LQ1=S31LQ1+S31LQ2(M)
      IF(ZQ-ZL)85,92,86
   85 XQ31LQ=S31LQ1
      YQ31LQ=0
      GO TO 92
   86 XQ31LQ=-S31LQ1
      YQ31LQ=0
   92 S13LQ1=0
      DO 94 M=1,MM
      S13LQ2(M)=0
      BM=M
      B11=SIN(BM*PI*YQ/B)*SIN(BM*PI*YL/B)*SIN(BM*PI*DYL/B)
```

```
      A31M=(1./BM)*B11
      A13M=A31M
      DO 93 N=1,NN
      AN=N
      GNM=SQRT((AN*PI/A)**2+(BM*PI/B)**2-SKO)
      B13=COS(AN*PI*XQ/A)*SIN(AN*PI*XL/A)*SIN(AN*PI*DXL/A)
      X31=4.*C*(-8.*B)*(1./(PI*GNM))
      X13=-X31
      B13QNM=B13*X13*ELQ
   93 S13LQ2(M)=S13LQ2(M)+A13M*B13QNM
   94 S13LQ1=S13LQ1+S13LQ2(M)
      IF(ZQ-ZL)95,98,96
   95 XQ13LQ=S13LQ1
      YQ13LQ=0
      GO TO 98
   96 XQ13LQ=-S13LQ1
      YQ13LQ=0
   98 Q(1,1)=CMPLX(XQ11LQ,YQ11LQ)
      Q(1,2)=CMPLX(XQ12LQ,YQ12LQ)
      Q(1,3)=CMPLX(XQ13LQ,YQ13LQ)
      Q(2,1)=CMPLX(XQ21LQ,YQ21LQ)
      Q(2,2)=CMPLX(XQ22LQ,YQ22LQ)
      Q(2,3)=CMPLX(XQ23LQ,YQ23LQ)
      Q(3,1)=CMPLX(XQ31LQ,YQ31LQ)
      Q(3,2)=CMPLX(XQ32LQ,YQ32LQ)
      Q(3,3)=CMPLX(XQ33LQ,YQ33LQ)
  100  IF((NN.EQ.NNN1).AND.(MM.EQ.MMM1))GO TO 312
  315 DO 316 I=1,3
      DO 316 J=1,3
  316 V(I,J)=Q(I,J)
      WRITE(3,311)NN,MM,IQ,L
  311 FORMAT(1X,I2,5X,I2,10X,I2,5X,I2)
      DO 317 I=1,3
  321 FORMAT(3(4X,2E19.8))
  317 WRITE(3,321)(V(I,J),J=1,3)
      NN=NNN1
      MM=MMM1
      GO TO 11
  312 WRITE(3,313)NN,MM,IQ,L
  313 FORMAT(1X,I2,5X,I2,10X,I2,5X,I2)
      DO 320 I=1,3
  305 FORMAT(3(4X,2E19.8))
  320 WRITE(3,305)(Q(I,J),J=1,3)
      KKK=0
      DO 323 I=1,3
      DO 323 J=1,3
      CCC=CABS(Q(I,J)-V(I,J))
      IF(CCC.LE.D)KKK=KKK+1
  323 CONTINUE
      IF(KKK.EQ.9)GO TO 3200
      IF((NN.GT.20).OR.(MM.GT.20))GO TO 3200
      NNN=NNN+1
      MMM=MMM+1
      GO TO 314
```

```
3200 RETURN
     END
```

```
      SUBROUTINE SOLVE
      COMPLEX AA,BB,PIVOT,TT,DET
      DIMENSION AA(36,36),BB(36,1),IPVOT(36),INDEX(36,2),PIVOT(36)
       COMMON D/BS/IPVOT,INDEX, PIVOT,AA,BB,MM,NN,DET
      EQUIVALENCE(IROW,JROW),(ICOL,JCOL)
C
157   DET=CMPLX(1.,0.)
      DO 117 J=1,NN
  117 IPVOT(J)=0
      DO 135 I=1,NN
C
      TT=CMPLX(0.,0.)
      DO 108 J=1,NN
      IF(IPVOT(J).EQ.1)GO TO 108
  113 DO 123 K=1,NN
      IF(IPVOT(K)-1)143,123,181
  143 IF(CABS(TT).GE.CABS(AA(J,K))) GO TO 123
  183 IROW=J
      ICOL=K
      TT=AA(J,K)
  123 CONTINUE
  108 CONTINUE
      IPVOT(ICOL)=IPVOT(ICOL)+1
C
      IF(IROW.EQ.ICOL)GO TO 109
  173 DET=-DET
      DO 112 LL=1,NN
      TT=AA(IROW,LL)
      AA(IROW,LL)=AA(ICOL,LL)
  112 AA(ICOL,LL)=TT
      IF (MM.LE.0)GO TO 109
  133 DO 102 LL=1,MM
      TT=BB(IROW,LL)
      BB(IROW,LL)=BB(ICOL,LL)
  102 BB(ICOL,LL)=TT
  109 INDEX(I,1)=IROW
      INDEX(I,2)=ICOL
      PIVOT(I)=AA(ICOL,ICOL)
      DET=DET*PIVOT(I)
C
      AA(ICOL,ICOL)=CMPLX(1.,0.)
      DO 105 LL=1,NN
  105 AA(ICOL,LL)=AA(ICOL,LL)/PIVOT(I)
      IF(MM.LE.0)GO TO 147
  166 DO 152 LL=1,MM
  152 BB(ICOL,LL)=BB(ICOL,LL)/PIVOT(I)
C
  147 DO 135 LI=1,NN
      IF(LI.EQ.ICOL)GO TO 135
  121 TT=AA(LI,ICOL)
      AA(LI,ICOL)=CMPLX(0.,0.)
      DO 189 LL=1,NN
  189 AA(LI,LL)=AA(LI,LL)-AA(ICOL,LL)*TT
      IF(MM.LE.0)GO TO 135
```

```
  118 DO 168 LL=1,MM
  168 BB(LI,LL)=BB(LI,LL)-BB(ICOL,LL)*TT
  135 CONTINUE
C
  122 DO 103 I=1,NN
      LL=NN-I+1
      IF(INDEX(LL,1).EQ.INDEX(LL,2))GO TO 108
  119 IROW=INDEX(LL,1)
      JCOL=INDEX(LL,2)
      DO 149 K=1,NN
      TT=AA(K,JROW)
      AA(K,JROW)=AA(K,JCOL)
      AA(K,JCOL)=TT
  149 CONTINUE
  103 CONTINUE
  181 RETURN
      END
```

Chapter-V

A Suggested Modified Moment Method

Of the different types of moment methods introduced in Art-1 of Ch.III the evaluation of l_{mn} matrix elements become laborious even on a high speed computer since at least two integrations may have to be performed numerically incase if the operator L is an integral operator. In the point-matching method the use of dirac delta functions as testing functions at discrete points reduces the laborious process of integration in finding l_{mn} matrix elements. However the accuracy of the solution depends not only on the number of points of the physical problem at which solution values are desired but also on their location. In the entire domain basis method use of a basis spanning the whole domain may result in rapid convergence, but the derivation of the l_{mn} elements is usually not efficient in terms of computer time. On the other hand in the sub-sectional basis method the use of subspace basis functions usually afford efficient derivation of the l_{mn} elements but the number of matrix elements becomes large for representing the whole problem especially when the whole problem involves a considerably odd-shaped curved boundary requiring many cell segments to approximate the problem geometry. One might conclude, then, that it

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would be desirable to have an alternative approach that would have the advantages of both the entire domain and sub-domain approaches. One such method a applied to a wire antenna has been discussed by R. Mittra et al [16]. This approach may be applied to the three-dimensional dielectric obstacle problem Equation (4) in Art 2 of Ch.III written below in matrix form:

$$-j\omega\mu_0\left[Q^{pk}_{lq}\right]\left[J^k_l\right] = -\left[C^p_q\right] \qquad (1*)$$

we consider firstly the one-dimensional Case and then extend it to three-dimensional case.

From Art-1 of Ch.III we rewrite Equation (5)

$$[l_{mn}][\alpha_n] = [g_m] \quad \text{- - - - -} \quad (1) \qquad \text{wherein}$$

$$[l_{mn}] = \begin{bmatrix} l_{11} & l_{12} & \cdots \\ l_{21} & l_{22} & \cdots \\ - & - & - - \end{bmatrix}, \quad [\alpha_n] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{bmatrix}, \quad [g_m] = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \end{bmatrix}$$

Now Column vector $[\alpha_n] = \alpha_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + \cdots$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ - & - & - & \cdots \end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\vdots\end{bmatrix} = \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\vdots\end{bmatrix} \qquad (1a)$$

Thus the basis vectors are subsectional pulse functions,

$$[f_1] = \begin{bmatrix}1\\0\\0\\\vdots\end{bmatrix}, \quad [f_2] = \begin{bmatrix}0\\1\\0\\\vdots\end{bmatrix}, \quad \cdots$$

If we change to a new set of basis vector which are nonsubsectional sinusoidal function vectors

$$[f_1'] = \begin{bmatrix}\cos\frac{\pi l_1}{L}\\ \cos\frac{\pi l_2}{L}\\ -\\ -\end{bmatrix}, \quad [f_2'] = \begin{bmatrix}\cos\frac{2\pi l_1}{L}\\ \cos\frac{2\pi l_2}{L}\\ -\\ -\end{bmatrix} \quad \text{where } l_1 = 1, l_2 = 2, \cdots l_L = L.$$

then $$[\alpha_n] = \alpha_1'[f_1'] + \alpha_2'[f_2'] + \cdots = [T_{mn}][\alpha_n'] \qquad (2)$$

$$\text{where } [T_{mn}] = \begin{bmatrix} \cos\frac{\pi l_1}{L} & \cos\frac{2\pi l_1}{L} & - \\ \cos\frac{\pi l_2}{L} & \cos\frac{2\pi l_2}{L} & - \\ - & - & - \end{bmatrix} \quad (2a) \quad \text{and } [\alpha_n'] = \begin{bmatrix}\alpha_1'\\\alpha_2'\\-\\-\end{bmatrix} \quad (2b)$$

where $[T_{mn}]$ is the transformation matrix

Substituting Equation (2) in Equation (1) we have

$$\left[l'_{mn}\right]\left[\lambda'_n\right] = \left[g_m\right] \qquad (3)$$

where $\left[l'_{mn}\right] = \left[l_{mn}\right]\left[T_{mn}\right]$

Now in the expansion of Equation (2)

$$\left[\lambda_n\right] = \lambda'_1\left[f'_1\right] + \lambda'_2\left[f'_2\right] + \cdots + \lambda'_k\left[f'_k\right] + \cdots + \lambda'_L\left[f'_L\right]$$

Often one usually finds that $[\lambda_n]$ is quite well approximated by retaining only the first k terms (assuming proper ordering of the $[\lambda_n]$ matrix elements) and in fact it is not unusual to obtain sufficient accuracy with $k < \frac{L}{10}$.

When this approximation is done Equation (3) becomes

$$\left[l'_{mn}\right]_a\left[\lambda'_n\right]_a = \left[g_m\right] \qquad (4)$$

where $\left[l'_{mn}\right]_a$, $\left[\lambda'_n\right]_a$, $\left[g_m\right]$ are of orders $L\times k, k\times 1, L\times 1$ respectively. Multiplying both sides by the conjugate transpass of matrix $\left[l'_{mn}\right]_a$, $\left[l'_{mn}\right]_a^{*t}$, we get the reduced matrix Equation

$$\left[l''_{mn}\right]\left[\lambda''_n\right] = \left[g''_m\right] \qquad (5)$$

where $\left[l''_{mn}\right]$, $\left[\lambda''_n\right]$, $\left[g''_m\right]$ are of orders $k\times k$, $k\times 1$, $k\times 1$ respectively.

It is reasonable to expect that going in for a suitably changed basis vectors and solving equation(3) instead of equation (1) will improve the accuracy of the solution or computer time saving.

For extending this one dimensional approach to 3-dimensional cases we proceed as follows:-

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Instead of using a scalar element in the one-dimensional case we use a scalar submatrix. Thus instead of Equations (1), (1a) we write:

$$\left[(l_{mn}) \right] \left[(\lambda_n) \right] = \left[(g_m) \right]$$

where submatrices $(l_{mn}), (\lambda_n), (g_m)$ are of orders $3\times3, 3\times1, 3\times1$ respectively.

$$\left[(\lambda_n) \right] = \begin{bmatrix} (I) \\ (0) \\ (0) \\ - \end{bmatrix} (\lambda_1) + \begin{bmatrix} (0) \\ (I) \\ (0) \\ - \end{bmatrix} (\lambda_2) + \cdots = [f_1](\lambda_1) + [f_2](\lambda_2) + \cdots \quad (1'a)$$

wherein $(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $(\lambda_1) = \begin{pmatrix} \lambda_1^1 \\ \lambda_1^2 \\ \lambda_1^3 \end{pmatrix}$, $(\lambda_2) = \begin{pmatrix} \lambda_2^1 \\ \lambda_2^2 \\ \lambda_2^3 \end{pmatrix}$ - -

and superscripts 1, 2, 3 imply $\bar{x}, \bar{y}, \bar{z}$ direction components respectively.

Similarly Equations (2), and (2-a) may be written as:-

$$\left[(\lambda_n) \right] = [f_1'](\lambda_1') + [f_2'](\lambda_2') + \cdots$$

$$= \begin{bmatrix} \cos\frac{\pi l_1}{L}(I) \\ \cos\frac{\pi l_2}{L}(I) \\ - \end{bmatrix} (\lambda_1') + \begin{bmatrix} \cos\frac{2\pi l_1}{L}(I) \\ \cos\frac{2\pi l_2}{L}(I) \\ - \end{bmatrix} (\lambda_2') \quad (2')$$

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wherein $\cos\frac{\pi l_1}{L}(I) = \begin{pmatrix} \cos\frac{\pi l_1}{L} & 0 & 0 \\ 0 & \cos\frac{\pi l_1}{L} & 0 \\ 0 & 0 & \cos\frac{\pi l_1}{L} \end{pmatrix}$, $\cos\frac{\pi l_2}{L}(I) = \begin{pmatrix} \cos\frac{\pi l_2}{L} & 0 & 0 \\ 0 & \cos\frac{\pi l_2}{L} & 0 \\ 0 & 0 & \cos\frac{\pi l_2}{L} \end{pmatrix}$, - -

and $(\lambda_1') = \begin{pmatrix} \lambda_1^{1'} \\ \lambda_1^{2'} \\ \lambda_1^{3'} \end{pmatrix}$, $(\lambda_2') = \begin{pmatrix} \lambda_2^{1'} \\ \lambda_2^{2'} \\ \lambda_2^{3'} \end{pmatrix}$, - and $\left[(T_{mn}) \right] = \begin{bmatrix} \cos\frac{\pi l_1}{L}(I) & \cos\frac{2\pi l_1}{L}(I) & - & - \\ \cos\frac{\pi l_2}{L}(I) & \cos\frac{2\pi l_2}{L}(I) & - & - \\ - & - & & \end{bmatrix} \quad (2'a)$

where (λ'_n), (T_{mn}) are of orders 3×1, 3×3 respecively.

Thus this three-dimensional approach may be usefully applied to Equation (1*) and may result in either computer time saving or improving the accuracy of the pulse function basis solution in a manner outlined for the 1-dimensional case.

Chapter - VI.

C o n c l u s i o n

The development of the dominant mode equivalent circuit parameters for a three-dimensional dielectric obstacle inside a rectangular waveguide involves solution of an integral equation containing suitable dyadic Green's function. The point-matching moment method has been used for finding the secondary volume current density produced at the centres of small rectangularly shaped cells formed by subdividing the dielectric obstacle. Subsequently these secondary volume current density values are used to obtain the dominant mode equivalent circuit parameters.

The various steps of this investigation, the difficulties encountered and the ways they were dealt with are discussed below.

The first step is the formulation of the problem (Chap.I). The formulation was done starting from Maxwell's curl equation and utilizing dyadic Green's function developed by Y.R.Sami[6]. The main task in formulating the problem was conversion of the two-region scattering problem into two one-region scattering problems, by utilizing the linearity property of Maxwell's curl equations. Y.R.Sami's derivation of the relavant dyadic Green's function has been analyzed in details and following the suggestion of R.E.Collin [12] a modified way of deriving the relevant dyadic Green's function has been presented.

The second step is the solution of the formulated equation to find the secondary volume current density values in the dielectric obstacle(Ch.II). In this work the point-matching

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moment method. Using a set of pulse function type subsectional basis vectors spanning the domain of the integral operator over the subsections of the obstacle region has been used. These subsections were chosen to be rectangularly sided parallelopipeds and have been termed "Cells" by J.J.H.Wang [7]. This was done in order to reduce the computational difficulty especially the numerical integration process involved in the moment method. The procedure of solution as outlined by J.J.H.Wang [7] has been followed and computational formulas have been derived for the matrix elements resulting from the application of the point-matching moment method. These elements involve double infinite series. An alternative method of dealing with such double infinite series utilizing laplace transforms as suggested by A.D.Wheelon [11] and utilizing fourier transforms as suggested by R.E. Collin [8] has been presented.

The third step is the derivation of the formulas for the TE_{10} mode equivalent circuit parameters (Ch.III). Following the procedure of R.E.Collin [12 & 8] expressions for TE_{10} scattered field radiated by the secondary current in the obstacle were derived these expressions along with the indident field expression were used in deriving the two-part equivalent circuit parameter expressions.

An illustrative example showing the various steps of solution has been presented (Ch.IV). A digital Computer program in Fortran-IV Language has been developed using the techniques suggested by S.S.Kuo [14] and E.I. Organick [15]. The results of the program indicate quite rapid convergence in the double infinite series summation with some 20x20 terms per series and also

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indicate that an equivalent current parameter for the problem in this illustrative example has negative resistive part. The physical interpretation of this is that the fundamental mode contribute power to other modes. In this context the following remark made by R.E.Collin [8] in connection with explaining the thick iris discontinuity problem in a waveguide is networthy:- "A single evanescent mode cannot propagate real power but the combination of nonpropagating modes which decay in opposite directions does lead to a transfer of power. Thus the occurance of the negative resistance may be attributed to the presence of a number of evanenscent modes decaying in opposite directions out of the infinite number of evanescent modes existent with the propagating mode.

Following the approach discussed by R.Mittra et al [16] a modified moment method has been developed for the three-dimensional di-electric obstacle problem. The method involves a transformation matrix manipulation in the process of solving for the secondary current values by the application of point matching method. The method has the possibility of either reducing the computer time or improving the accuracy of the solution. Further investigation with this method is worth consideration.

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