A NEW APPROACH TO SOLUTION OF MIXED BOUNDARY VALUE ELASTIC PROBLEMS

by

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ABSTRACT

This thesis deals with the analytical solutions for the stress and the displacement components in plane stress and plane strain elastic bodies under different mixed boundary conditions. Two dimensional elastic problems are either formulated to solve for two displacement components from two simultaneous second order partial differential equations or for a single stress function, known as Airy stress function, from the biharmonic partial differential equation. In the first case, solving for two functions simultaneously is extremely difficult and the method thus has hardly been used in the stress analysis of elastic bodies. Although the second formulation provides a better opportunity in seeking solutions of elastic problems, but it can be used only when the boundary conditions are known in terms of loadings. Unfortunately, all problems of elasticity are of mixed boundary conditions involving known restraints and loading at the boundary and hence the method of stress function does not provide explicit information about the stresses near the restrained boundaries.

In this thesis a new formulation for the two dimensional elastic problems is introduced. The new formulation involves seeking for a displacement function which satisfies the same biharmonic equation as that of the stress function but can provide solutions under all kinds of mixed boundary conditions.

With the help of this new formulation, a number of mixed boundary-value elastic problems are solved analytically and the solutions are presented in the form of infinite series forms. The solutions are also evaluated numerically and the results are presented in the forms of graphs. The results show that the new approach is sound and also has opened up a new horizon for seeking solution of mixed boundary-value elastic problems.

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CONTENTS

TERMINOLOGY

NOTATIONS	CONCEPT
x, y, z	Rectangular co-ordinates
Ψ	Potential function defined in terms of displacement components
φ	Airy's stress function
σ	Stress
$\sigma_x, \sigma_y, \sigma_z$	Normal stresses in x, y and z directions respectively
$\sigma_{xy}, \sigma_{yz}, \sigma_{zx}$	Shearing stresses in xy, yz, and zx planes respectively
E	Elastic modulus of the material
ν	Poisson's ratio
u, v, w	Displacement components in the x, y and z directions respectively
a, b	Dimensions of the plate
$\epsilon_x, \epsilon_y, \epsilon_z$	Strain components in x, y and z directions
$\epsilon_{xy}, \epsilon_{yz}, \epsilon_{zx}$	Shearing strain components in xy, yz, zx planes
P	A loading parameter

CHAPTER I

INTRODUCTION

1.1 Preliminary

The theory of elasticity developed into an important branch of mathematical physics which has found considerable application in the solution of engineering problems.

In treating statics in theoretical mechanics, it is stated that the conditions of equilibrium of a body or a system of bodies do not involve internal forces since they are pairwise mutually equilibrated on the basis of Newton's third law of equality of action and counteraction. The theory of elasticity sets forth the problem of determining internal forces in a solid body. These forces represent interaction between molecules; they ensure the existence of a solid body as such, its strength. They also act when no external forces are applied to the body; these forces are not in themselves the object of study in the theory of elasticity; under the action of external forces the body deforms, the mutual position of molecules changes and so do the distances between them; the action of external forces that produce deformation gives rise to additional internal forces.

In many cases, the elementary theories of strength of materials are not sufficient to describe the stress distribution in engineering structures. The elementary theory is inadequate to give information regarding local stress near the loads and near the supports of beams. It fails also in the cases when the stress distribution in bodies, all the dimensions of which are of the same order, has to be investigated.

This led to the emergence of a special trend in physics, called mathematical physics. Among the great number of problems conforming this new branch of science it is necessary to mention the need for a profound investigation of the properties of elastic materials and for the construction of a mathematical theory which would permit studying as completely as possible the internal forces occuring in an elastic body under the action of external forces, as well as the deformation of a body i.e. the change of its shape.

1.2 Literature Review

Elementary methods of strength of materials were the primary tools of the practicing engineers for handling the problems of engineering structures. However these methods are often found inadequate to furnish satisfactory information regarding local stresses near the loads and near the supports of the structures. The elementary theory gives no means of investigating stresses in regions of sharp variation in cross section of beams or shafts. Stresses in screw threads, around various shapes of holes in structures, near contact points on gear teeth, rollers and balls of bearings, have all remained beyond the scope of elementary theories. It is thus obvious that, for the designers of modern machines, recourse to the more powerful methods of the Theory of Elasticity is an absolute necessity.

Considerable progress has been made in recent year in solving important practical problems of stress analysis using the methods of Theory of Elasticity. In cases where a rigorous solution could not be obtained, approximate methods have been developed. In other cases, where even approximate methods could not be developed, solutions have been obtained by using experimental methods. Photoelastic methods, soap-film methods, application of strain gages,Moire Fringe are some of these experimental methods applied in the study of stress concentration at points of sharp variation of cross-sectional dimensions and at sharp fillets of reentrant corners. These results have considerably influenced the modern design of machine parts and helped in many cases to improve the construction by eliminating weak spots from which crack may start.

The field of elasticity is mainly concerned with the solutions of two dimensional problems as most of the three dimensional problems may be resolved into a two dimensional one or remain beyond the scope of analytical studies and have to be tackled experimentally.

Although the theories of elasticity had long been established, the solutions of practical problems started mainly after the introduction of a stress function by G.B. Airy[2]. The Airy stress function is governed by a fourth order partial differential equation and the stress components are related to it through its various second order derivatives. Solutions were initially sought through various polynomial expressions of the stress function[8,11], but the success of this approach was very limited. Using these polynomial expressions, an elementary derivation of the effect of the shearing force on the curvature of the deflection curve of beams were made by Rankine[6] in England and by Grashof[3] in Germany. The problem of stress in masonry dams is of great practical interest and has been attempted by various authors[7,14] using polynomial expressions for the stress functions. But it should be noted that the solutions thus obtained do not satisfy the conditions at the bottom of the dam where it is connected with the foundation and would predict reasonable values of stress in the region far away from the foundation on account of Saint-Venant's Principle[1].

The first application of trigonometric series in the solution of elastic problems using stress function method was given by M.C. Ribiere in his thesis[4]. Further progress in the application of these solutions was made by L.N.G. Filon[10]. Several particular examples were worked by F. Bleich[12]. Using Fourier Series, Beyer[18] solved the problem of a continuous beam on equidistant supports under gravity loading. Stress function technique has also been used by Ribiere[9] for analyzing the stress around a circular hole in a plate, Sadowsky[16] for stresses around a slender hole, Flamant[5] for stresses around a concentrated load on a straight boundary and Stokes[19] for stresses around a concentrated load on a beam. For better understanding of this approach, two examples are cited in the Appendix A.1 and A.2. The first example is the analysis of deep beams and the second one is a rectangular plate loaded with parabolic forces.

For complex shapes of boundary and also for restrained boundary, the difficulties of obtaining analytical solutions become formidable. These difficulties were partially avoided by resorting to experimental methods, such as the measurement of extensometers and strain gages or the photoelastic method. Using photoelasticity, Hetenyi[21] investigated the stresses in the threads of a bolt and nut fastening. Most of the experimental investigations of elastic problems are reported in the "Handbook of Experimental Stress Analysis", 1950[26] and by Frocht[24] in "Photoelasticity".

Numerical methods could not gain much popularity in the field of elasticity. This is mainly because, as pointed out by Uddin[28], the boundary conditions in terms of restraints can not be discretized in term of the stress function. Of course when two dimensional elastic problems are formulated in terms of displacement components u and v, the problem of discretization of the conditions at the restrained boundaries are removed, but the resulting algebraic equations become ill-conditioned and often diverge in the iterative method of solutions. Uddin[28] numerically solved two of the mixed boundary-value problems using the displacement formulation and found that the algebraic equations resulting from discretization can only be solved if the restraint at the boundary is specified on more than 25% of the boundary-perimeter. It should also be pointed out that numerical method is unattractive when the effect of discontinuity is of major interest in the investigation which is always the case of mixed boundary-value problems.

1.3 Objective of the Thesis

a. Finding a new approach for solving mixed boundary-value plane stress and plane strain elastic problems through the introduction of a displacement potential function.

b. Establishing the reliability and suitability of this new approach through its analytical application.

c. Solutions of specific problems of mixed boundary conditions for which no exact solution is available at present in the literature.

1.4 Practical Applications

Although the analytical method developed here is applicable for the solution of stresses in a plate loaded in its own plane with all possible boundary conditions, its range of application is fairly wide.

Though a body has always three dimensions and stresses develop in three perpendicular planes, two different simplifications may be made for a wide range of problems. In one class, the body is loaded by forces applied at the boundary, parallel to the plane of the body and distributed uniformly over the thickness and the dimension of the body perpendicular to the plane of loading is small enough to permit the free expansion of the elements of the body in that direction. This class of problem is known as plane stress problem. In other class, a similar simplification is possible at the other extreme when the dimension of the body in the z-direction is very large or when the two opposite ends of the body are confined between fixed smooth rigid planes, so that displacement in the axial direction is prevented. Under these circumstances the problems may be solved for stress in a plane region and they are called plane-strain problems.

There are many important problems of this kind. For example, a retaining wall with lateral pressure, a culvert or tunnel, a cylindrical tube with internal pressure, a cylindrical roller compressed by forces in a diametral plane as in a roller bearing. In each case, of course, the loading must not vary along the length. For better understanding of two types of problems, let us consider an example as shown in fig.1. In this figure, if the thickness b is small enough to

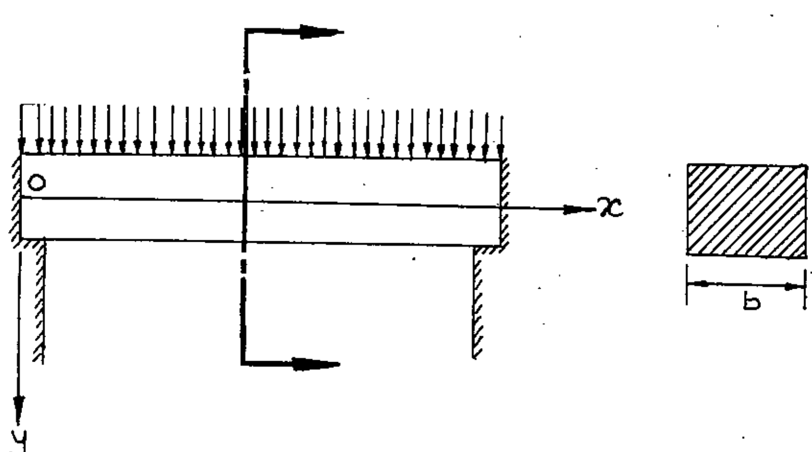


FIG. 1

permit free lateral expansion, then it constitutes a plane stress problem. On the other hand, if the thickness b is very large and the loading does not vary appreciably along the thickness, then the expansion in the z-direction is completely prevented and the problem becomes a plane strain problem.

So, we can resolve many problems of practical importance under any one of the above mentioned classes. Although, only rectangular boundary is considered but still it covers many important problems of practical interest.

CHAPTER - II

FORMULATION OF THE PROBLEMS

2.1 Introduction

The solution of a problem in elasticity is usually to find the stress distribution in an elastic body and, in some cases, to find the strain at any point due to given body forces and given conditions at the boundary of the body. To determine the stress at a point, we must find the six stress components. These six components satisfy the three equations of equilibrium. Since three equations are not sufficient, we have to take the six relations defining the strain components in terms of the three displacement components and the six stress strain relations. Thus we have altogether 15 unknowns and 15 equations. This system of equations is generally sufficient for the solution of an elasticity problem.

Let us take an infinitesimal cubic element from an elastic body with sides parallel to the coordinate axes. To ensure the equilibrium of the element, six forces will act on the six different faces of the element. The forces acting on a face may be resolved into two components-one perpendicular to the plane of the face and the other parallel to the face. The stress component acting perpendicular to the face is called the normal stress and usually denoted by "σ" with a subscript to indicate its direction of action. In the same way, the two stress components acting parallel to the face are known as shearing stresses and indicated by the same notation with double subscripts- the first indicating the direction of the normal to the face and the second indicating the direction of the component of the stress. The notations and positive directions of the stress components are illustrated in fig. (2-1). According to general conventions, the normal stress is taken positive when producing tension and negative when producing compression. On any side the direction of the positive shearing stress coincides with the positive direction of the axis if the outward normal on this side has the positive direction of the corresponding axis. If the outward normal has a direction opposite to positive axis the positive shearing stress will also have the opposite direction of the corresponding axis.

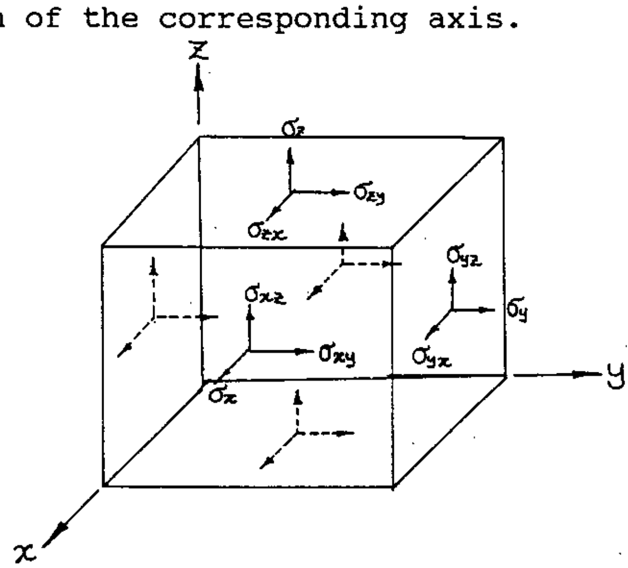


FIG. (2.1)

To describe the stresses acting on the six sides of the element three symbols σ_x, σ_y, σ_z are necessary for normal stress, and six symbols σ_{xy}, σ_{yx}, σ_{zx}, σ_{xz}, σ_{yz}, σ_{zy} for shearing stress. A consideration of the equilibrium of the cubic element shows that, for two perpendicular sides of the cubic element, the components of shearing stress perpendicular to the line of intersection of these sides are equal. Mathematically stated, these conditions are: $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$ and $\sigma_{yz} = \sigma_{zy}$. Thus the nine components of stress are reduced to six.

The six quantities σ_x, σ_y, σ_z, σ_{xy}, σ_{yz}, σ_{zx} are therefore sufficient to describe the stresses acting on the coordinate planes through a point; these will be called the components of stress at the point.

In discussing the deformation of an elastic body it is assumed that there are enough constraints to prevent the body from moving as a rigid body so that no displacements of particles of the body are possible without a deformation of it. These deformations of the body can be uniquely specified by assigning three elongations in three perpendicular directions and three shear strains related to the same directions. These directions are taken as the direction of the co-ordinate axes and the letter ϵ is used to denote the strain components with the same subscripts to this letter as for the stress components. If the components of a particle in the body are specified by u, v and w parallel to the coordinate axes x, y and z, respectively, then the relations between the components of strain and the components of displacement are given by

$$\epsilon_x = \frac{\partial u}{\partial x}$$

$$\epsilon_y = \frac{\partial v}{\partial y}$$

$$\epsilon_z = \frac{\partial w}{\partial z}$$

$$\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad [2.1]$$

$$\epsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\epsilon_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

Linear relations between the components of stress and the components of strain are known generally as **Hooke's law.** By the application of Hooke's law and the principle of superposition, the relations between the components of stress and the components of strain are given by,

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu\ (\sigma_x + \sigma_y)]$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu\ (\sigma_x + \sigma_y)] \qquad [2.2]$$

$$\epsilon_{xy} = \frac{2(1+\nu)}{E}\ \sigma_{xy}$$

$$\epsilon_{yz} = \frac{2(1+\nu)}{E}\ \sigma_{yz}\ , \text{ and}$$

$$\epsilon_{zx} = \frac{2(1+\nu)}{E}\sigma_{zx}$$

where E is the modulus of elasticity and ν the Poission's ratio.

In case of plane stress problems, σ_z, σ_{xz}, σ_{yz} are zero on both the surfaces of a thin plate and it can further be assumed that they are zero throughout the thickness of the plate. It can also be assumed that the other components σ_x, σ_y and σ_{xy} are independent of z and, hence, are functions of x and y only.

When the body is prevented from elongating or contracting in the z-direction, then the components u and v of the displacement are functions of x and y but are independent of the coordinate z. Since the longitudinal displacement w is zero, Eqs.(2.1) give,

$$\epsilon_z = \frac{\partial w}{\partial z} = 0$$

$$\epsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$$

$$\epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$

The longitudinal normal stress, σ_z can be found in terms of σ_x and σ_y by means of Hooke's law, Eqs.(2.2). Since $\epsilon_z = 0$, we find,

$$\sigma_z - \nu\ (\sigma_x + \sigma_y) = 0$$

or, $\sigma_z = \nu\ (\sigma_x + \sigma_y)$

These normal stresses act over the cross sections, including the ends, where they represent forces required to maintain the plane strain provided by the fixed smooth rigid planes. Thus the plane strain problem, like plane stress problem, reduces to the determination of σ_x, σ_y and σ_{xy} as functions of x and y only.

2.2 Differential Equations of Equilibrium

Let us consider the equilibrium of a small rectangular block of sides h, k and thickness unity as shown in fig. (2.2).

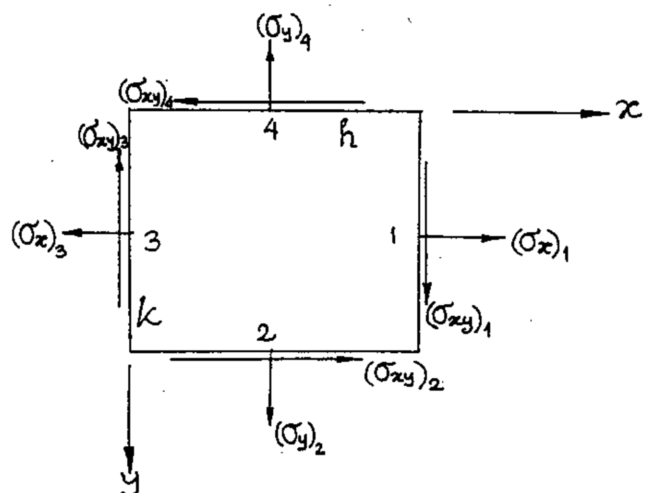


FIG. (2.2)

The stress acting on the faces 1,2,3,4 and their positive directions are indicated in the figure. The symbols σ_x, σ_y, σ_{xy} refer to the midpoint x, y of the rectangle in the figure 2.2. If X, Y denote the components of body force per unit volume, the equation of equilibrium for forces in the x and y directions are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + X = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + Y = 0 \qquad [2.3]$$

If in these equations of equilibrium of plane stress and plane strain the body forces are assumed, as they will be throughout this work, to be absent, then the equations (2.3) become,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xv}}{\partial y} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \qquad [2.4]$$

In a two dimensional problem it is necessary to solve the differential equations of equilibrium (2.4). These two equations containing three stress components σ_x, σ_y, σ_{xy} are not sufficient for the determination of these components. The problem is a statically indeterminate one and, in order to obtain the solution, the elastic deformation of the body must also be considered. For two dimensional problems, three strain components are considered. These three strain components are expressed by two functions u and v and there exists a certain relation between the strain components that can easily be obtained from equations (2.1). This relationship may be obtained by eliminating u and v from the equations (2.1) which give,

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} \qquad [2.5]$$

Thus, to ensure the existence of continuous functions u and v, the strain components have to satisfy the equation (2.5) known as the condition of compatibility. To express the equation of compatibility in terms of the stress components the strain components present in equation (2.5) have to be eliminated by their relations with the stress components. These relations can be obtained from the first, second and fourth of (2.2) by replacing σ_z with zero in case of plane stress and with $\nu(\sigma_x + \sigma_y)$ in case of plane strain. The relations are,

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y]$$

$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x] \qquad [2.6]$$

$$\epsilon_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy}$$

in case of plane stress and

$$\epsilon_x = \frac{1}{E}[(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y]$$

$$\epsilon_y = \frac{1}{E}[(1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x] \qquad [2.7]$$

$$\epsilon_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy}$$

in case of plane strain.

Thus, using the relations of (2.6) for the elimination of strain components from

equation (2.5), the compatibility condition becomes,

$$\frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) + \frac{\partial^2}{\partial^2 y}(\sigma_x - \nu\sigma_y) = 2(1+\nu)\frac{\partial^2\sigma_{xy}}{\partial x\partial y}$$

Further elimination of the term $\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$ with the help of equilibrium equations transforms the conditions of compatibility into its final form

$$\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right)(\sigma_x + \sigma_y) = 0 \qquad [2.8]$$

The equations (2.4) and (2.8) associated with the appropriate boundary conditions should theoretically be sufficient to determine the three functions in a plane region. But practically it is extremely difficult to solve for the three variables simultaneously even under the simplest boundary conditions. The problem becomes far more complex when the boundary conditions are mixed. This fact lead us to seek the existence of another function with which the desired variables are related in natural way or can be defined in terms of the required variables so that the determination of that function will uniquely determine the function sought for. The necessity of this approach will become evident when the boundary conditions are discussed in detail in the section dealing with the feasibility of the numerical evaluation of the functions in the light of the boundary conditions.

2.3 General Analytical Approach

In the analytical approach the usual practice is to introduce a new function $\varphi(x,y)$, known as the stress function or Airy's Stress function and defined as,

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} \qquad [2.9]$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

The function $\varphi(x,y)$ defined by the equations (2.9) satisfies the equilibrium equations (2.4) and must satisfy the compatibility equation. When the stress components are eliminated from equation (2.8) by substituting from equations (2.9), the condition of compatibility in terms of φ becomes,

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \qquad [2.10]$$

Thus the solution of the plane problem in terms of stress is reduced to the integration of one partial differential equation (2.10) of the fourth order; if the function $\varphi(x,y)$ is determined from this equation, then stresses are easily found at any point of a body by formulas (2.9). Equation (2.10) is the most promising one for the solution of the stress either analytically or numerically, provided the boundary conditions are also known in terms of stress. But the prospects are dimmed both for numerical and analytical solution when the boundary conditions are mixed. This fact compels us to look for other suitable avenues. In this respect, the most natural step would be to investigate the possibility of solving the problem in terms of the displacement components. This approach appears favourable for a number of reasons:

1. The stress components can easily be determined if the displacement components are known, while the reverse is not true.

2. In some problems the direct determination of displacements might be the more required one.

3. The problem is solved in terms of physical quantities which provide a ready check on the solution.

According to the above mentioned approach, the problem is to be formulated in terms of displacement components. And to accomplish that, the stress components should first be expressed as functions of strain components. Solving the equations (2.6) for the stress components, we get

$$\sigma_x = \left(\frac{E}{1-\nu^2}\right)(\epsilon_x+\nu\epsilon_y)$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}(\epsilon_{xy}) \qquad [2.11]$$

$$\sigma_y = \left(\frac{E}{1-\nu^2}\right)(\epsilon_y+\nu\epsilon_x)$$

Eliminating the strain components from equations (2.11) by substituting from equations (2.1), the expressions for the stress components in terms of the displacement components are obtained as,

$$\sigma_x = \left(\frac{E}{1-\nu^2}\right)\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right)$$

$$\sigma_y = \left(\frac{E}{1-\nu^2}\right)\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right) \qquad [2.12]$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Substituting the stress components in the equations (2.4) from the expressions (2.12) and simplifying these equations, we obtain the equilibrium equations in the following forms;

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 u}{\partial y^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 v}{\partial x \partial y} = 0$$

$$\frac{\partial^2 v}{\partial y^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 v}{\partial x^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 u}{\partial x \partial y} = 0 \qquad [2.13]$$

These two homogeneous elliptic partial differential equations with the appropriate boundary conditions should be sufficient for the evaluation of the two functions u and v, and the knowledge of these functions over the region concerned will uniquely determine the stress components which can be evaluated with the help of the system of equations (2.12). Therefore the problem is simultaneously solved for all the required quantities.

2.4 New Analytical Approach

Although the two homogeneous elliptic partial differential equations in terms of u and v (2.13) are sufficient to solve mixed boundary value elastic problems but in reality it is difficult to solve for two functions simultaneously. So, to overcome this difficulty, investigations are necessary to convert the above two equations into a single equation of a single function. If that function is defined in terms of the displacement functions u and v, then the determination of that function uniquely determines the stress functions sought for.

In this approach, the possibility of the existence of a potential function defined in terms of the displacement components, is investigated. By this approach, as in the case of Airy stress function, the problem is reduced to the determination of one function only. If a new function $\Psi(x,y)$ is defined as,

$$u = \frac{\partial^2 \psi}{\partial x \partial y}$$

$$v = -\left[\left(\frac{1-\nu}{1+\nu}\right)\frac{\partial^2 \psi}{\partial y^2} + \left(\frac{2}{1+\nu}\right)\frac{\partial^2 \psi}{\partial x^2}\right] \qquad [2.14]$$

and the displacement components in the equations (2.13) are replaced by the equations (2.14), then the first equation is,

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 u}{\partial y^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 v}{\partial x \partial y}$$

$$= \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \psi}{\partial x \partial y}\right) + \left(\frac{1-\nu}{2}\right)\left(\frac{\partial^4 \psi}{\partial x \partial y^3}\right) +$$

$$\left(\frac{1+\nu}{2}\right)\frac{\partial^2}{\partial x \partial y}(\{-\left(\frac{1-\nu}{1+\nu}\right)\frac{\partial^2 \psi}{\partial y^2} + \left(\frac{2}{1+\nu}\right)\frac{\partial^2 \psi}{\partial x^2}$$

$$= \frac{\partial^4 \psi}{\partial x^3 \partial y} + \left(\frac{1-\nu}{2}\right)\frac{\partial^4 \psi}{\partial y^3 \partial x} - \left(\frac{1-\nu}{2}\right)\frac{\partial^4 \psi}{\partial x \partial y^3} - \frac{\partial^4 \psi}{\partial x^3 \partial y} \equiv 0$$

Therefore, Ψ has only to satisfy the second equation. Expressing this equation in terms of Ψ, the condition that Ψ has to satisfy is

$$\frac{\partial^2 v}{\partial y^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 v}{\partial x^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 u}{\partial x \partial y} = 0.$$

or,

$$\frac{\partial^2}{\partial y^2}\{-\left(\frac{1-\nu}{1+\nu}\right)\frac{\partial^2 \psi}{\partial y^2} - \left(\frac{2}{1+\nu}\right)\frac{\partial^2 \psi}{\partial x^2}\} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2}{\partial x^2}\{-\left(\frac{1-\nu}{1+\nu}\right)\frac{\partial^2 \psi}{\partial v^2}$$
$$- \left(\frac{2}{1+\nu}\right)\frac{\partial^2 \psi}{\partial x^2}\} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 \psi}{\partial x \partial y}\right) = 0.$$

or,

$$-\frac{1-\nu}{1+\nu}\frac{\partial^4 \psi}{\partial y^4} - \left(\frac{2}{1+\nu}\right)\frac{\partial^4 \psi}{\partial x^2 \partial y^2} - \frac{(1-\nu)^2}{2(1+\nu)}\frac{\partial^4 \psi}{\partial x^2 \partial y^2}$$
$$- \left(\frac{1-\nu}{1+\nu}\right)\frac{\partial^4 \psi}{\partial x^4} + \left(\frac{1+\nu}{2}\right)\frac{\partial^4 \psi}{\partial x^2 \partial y^2} = 0.$$

or,

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0.$$

Therefore, the problem is again reduced to the evaluation of only one function Ψ of x and y from the biharmonic partial differential equation. If this approach is feasible then the amount of computational time might be expected to be less than the other approach of solving for two functions as in (2.13).

CHAPTER – III

BOUNDARY CONDITIONS

3.1 General Consideration of the Boundary Conditions:

Equations (2.3) must be satisfied at all points throughout the volume of the body. The stress components vary over the volume of the plates; and at the boundary they must be such as to be in equilibrium with the external forces on the boundary of the plate, so that external forces may be regarded as a continuation of the internal stress distribution.

In practical cases, along the edge of a plate, two things may be known:

a. displacements
b. loading or stress

Both the displacements and stress are defined by their respective components. These components are:

1. normal displacement
2. tangential displacement
3. normal stress
4. tangential stress

At any point on the boundary, out of these four quantities two are known at a time. Therefore, four quantities taken two at a time will create six different boundary conditions. These six boundary conditions are given by

i. normal displacement
tangential displacement

ii. normal displacement
tangential stress

iii. tangential displacement
normal stress

iv. normal stress
tangential stress

v. normal displacement
normal stress

vi. tangential stress
tangential displacement

Out of these six possible combinations the last two combinations, namely (v) and (vi), do not generally exist in physical problems. Therefore, at any point on the boundary, the first four possible boundary conditions are concerned with. As the shape of the boundary to be considered is rectangular, the plate may be oriented so that its edges are parallel to the co-ordinate axes. In that case, the normal and the tangential stress and displacements at the boundary are the co-ordinate components of stress and displacement inside the plate. Therefore, when the four boundary conditions are stated mathematically and in terms of the functions to be determined, then,

(1) $u = f_1 (x,y)$
$v = f_2 (x,y)$

(2) $u = f_1 (x,y)$
$\sigma_{xy} = f_3 (x,y)$ for x,y on the boundary [3.1]

(3) $v = f_2 (x,y)$
$\sigma_x = f_4 (x,y)$

(4) $\sigma_x = f_4 (x,y)$
$\sigma_{xy} = f_3 (x,y)$

on an edge parallel to y-axis and

(1) $u = f_5 (x,y)$
$v = f_6 (x,y)$

(2) $u = f_6 (x,y)$
$\sigma_{xy} = f_7 (x,y)$ for x,y on the boundary [3.1]

(3) $v = f_5 (x,y)$
$\sigma_y = f_8 (x,y)$

(4) $\sigma_y = f_8 (x,y)$
$\sigma_{xy} = f_7 (x,y)$

on an edge parallel to x-axis.

3.2 Boundary Conditions for the System of Equations in Terms of the Stress Components:

Considering the system of equations in terms of the stress components, given by the equations (2.4) and (2.8), in conjunction with boundary conditions of (3.1) it is seen that those boundary conditions known in terms of displacement have to be expressed in terms of the unknown functions. In this respect, the necessary equations are given by (2.12). In these equations the individual stress components are expressed as functions of the normal and tangential derivatives of the displacement components. To evaluate these functions the knowledge of the displacement components only on the boundary is not sufficient. More precisely the normal derivative present here can only be evaluated if the displacement component concerned is also known within the boundary. The only imaginable way to overcome this difficulty is to evaluate the displacement components simultaneously with the stress components. But the complexity involved in that approach is so enormous that it would be far better to look for an alternate method.

3.3 Boundary Conditions for the Stress Function φ:

The relation between stress components and the unknown function φ are given by,

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = \frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right)$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = \frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right) \qquad [3.2]$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

From these equations it is found that, if the boundary conditions are given in terms of the stress components, the derivatives of the unknown function φ present in (3.2) can be evaluated on the boundary. But the evaluation of these derivatives of φ is not possible if the boundary conditions are given in any one of the forms (1), (2) and (3) of (3.1). The impossibility is, as before, due to the presence of normal derivatives of the displacements in the equations (3.2). Therefore the function φ can be solved only if the boundary conditions are known in the form (4) of (3.1).

3.4 Boundary Conditions for the System of Equations in Terms of Displacement Components:

For the system of equations (2.13) the boundary conditions have to be entirely expressed in terms of the displacement components. An examination of the boundary conditions (3.1) shows that the boundary conditions are available either directly in terms of the unknown functions concerned or indirectly, in terms of the stress components. In the latter case the necessary relations between the stress components and the unknown functions are given by (3.2). From these equations it is seen that these functions u and v can be determined without the knowledge of the stress components anywhere except on the boundary - which is exactly the given boundary conditions. Therefore, the boundary conditions do not present any serious difficulty in this case.

3.5 Boundary Conditions for the Function Ψ:

In order to solve the problem by solving the function Ψ of the biharmonic equation, defined by the relations (2.14), the boundary conditions (3.1) should be expressed in Ψ. These conditions are given by

$$u = \frac{\partial^2 \psi}{\partial x \partial y}$$

$$v = -\left[\left(\frac{1-\nu}{1+\nu}\right)\frac{\partial^2 \psi}{\partial y^2} + \frac{2}{1+\nu}\frac{\partial^2 \psi}{\partial x^2}\right]$$

$$\sigma_x = \left(\frac{E}{1-\nu^2}\right)\left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y}\right)$$

$$= \left(\frac{E}{1-\nu^2}\right)\left(\frac{\partial^3\psi}{\partial x^2\partial y} - \nu\frac{\partial^3\psi}{\partial y^3}\right)$$

$$\sigma_y = \left(\frac{E}{1-\nu^2}\right)\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right)$$

$$= -\left[\frac{E}{(1+\nu)^2}\right]\left[\frac{\partial^3\psi}{\partial y^3} + (\nu+2)\frac{\partial^3\psi}{\partial x^2\partial y}\right]$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$= \frac{E}{(1+\nu)^2}\left[\nu\frac{\partial^3\psi}{\partial x\partial y^2} - \frac{\partial^3\psi}{\partial x^3}\right]$$

From these expressions it is found that, as far as boundary conditions are concerned, there is no technical difficulty in this approach. No knowledge of the displacement and the stress components away from the boundary is needed to evaluate the function Ψ at the boundary. Moreover, when compared with the approach of solving the problem in terms of displacement components, it has the advantage that, the evaluation of only one function at a time, are concerned. Whereas, in the other case, two functions have to be evaluated simultaneously.

CHAPTER — IV

SOLUTIONS OF PROBLEMS

4.1 General Considerations

The concept of an engineering differential system consists of four essential and equally important parts:

1. Proper understanding of physical phenomena and fundamental principles.

2. Derivation of governing equation (mathematical models) for given physical situations.

3. Development of routines, both classical and numerical, for solution of governing equations.

4. Proper application of boundary and initial conditions and evaluation and interpretation of solutions.

Until all four parts are knit together through a rational engineering mathematical analysis, including a "statical" check of the tentative solution and the appropriate boundary and initial conditions, we do not have a "complete" solution to a given engineering differential system.

Attention devoted at times to the understanding of phenomena; at other times to the formulation of models and to the establishment of solutions to equations pertaining to certain boundary value problems. In obtaining a solution to an engineering differential system, a thorough understanding of the underlying phenomena of the problem as well as a complete engineering mathematical analysis, must be found out. The mathematical model is then extremely useful for gaining an insight and understanding of the real physical world. The procedure is, in short, the uniting of pure physics and mathematics with a view to possible usefulness.

The equilibrium problem is essentially one of describing the steady-state configuration of a physical system. This can usually be achieved by specifying the magnitudes of state variables like stresses, displacements, pressures, velocities, temperatures, etc. at a finite number of points. In numerical methods this is accomplished by transforming the differential system into a set of simultaneous algebraic equations by the usual finite difference techniques and then solving the algebraic set. The solution can not be found at one point until it is known at all points.

In this thesis work, equilibrium problems are dealt with, which are also boundary-value problems. The boundary conditions are prescribed around an entire closed boundary, in contrast to the propagation problem, where all or most of the condition may be prescribed at one portion of an open boundary. In equilibrium problems the differential operators are of the elliptic type. The operators of the biharmonic equation are of the elliptic type, which occur most frequently in engineering differential systems.

4.2 Solution Procedure

A very powerful method of solving boundary value problems is the so-called trial function or trial solution method. Attempt made here to solve the fourth order homogeneous partial differential equation

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0$$

for Ψ with different trial functions.

This differential equation is solved by assuming a solution of a particular form involving a number of arbitrary constants and functions and then by determining the nature of these constants and functions by proper application of the boundary conditions.

It is seen that various combinations of trigonometric and hyperbolic functions offer suitable choices for analytic functions and if these functions can be expressed as an infinite series, then construction of solutions of differential equations becomes more accurate. In the light of the ubiquitous problems which display aspects of a periodic and a discontinuous nature, those infinite series known as Fourier series attain a place of special importance.

The Fourier series is probably the most commonly used of all the series for the solution of physical problems. It is a trigonometric series which can be used for the expansion of an arbitrary function. A much greater degree of generality is attained by taking the function as Fourier series. The usefulness of the Fourier series is due, in part, to the fact that certain functions which can not be expanded in power series form can still be represented by Fourier series. The reason for this is that the coefficients of the power series contain derivatives of the function; hence these derivatives must exist uniquely in order to obtain the power series expansion. Many functions which are not differentiable, including certain types of discontinuous functions, can be expanded in Fourier series.

4.3 Solutions of Specific Problems

PROBLEM-1: A problem of rectangular plates is considered where the two opposing edges of the plate are roller supported while the boundary conditions at the other two edges are unspecified at the moment.

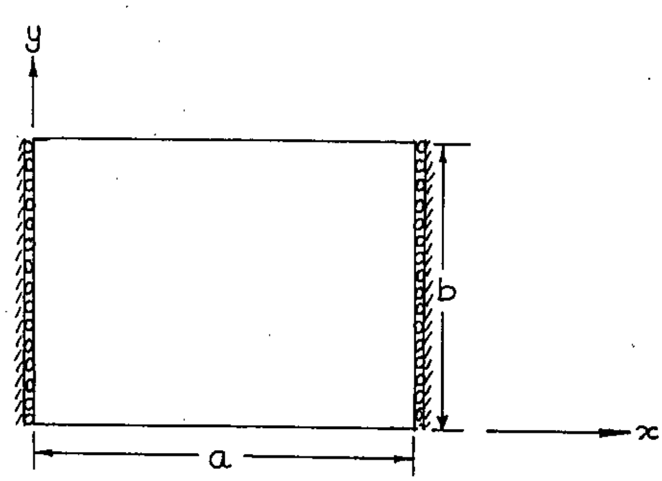


FIG. (4.1)

The plate is considered to be of unit thickness and its configuration with respect to co-ordinate axes is shown in fig. (4.1). In this case if the function Ψ is assumed to be,

$$\psi = \sum_{m=1}^{\infty} Y_m \cos\frac{m\pi x}{a} + Ay^3$$

Where Y_m is a function of y only, then Y_m has to satisfy the ordinary differential equation.

$$Y_m^{////} - 2\frac{m^2\pi^2}{a^2} Y_m^{//} + \frac{m^4\pi^4}{a^4} Y_m = 0$$

The general solution of this differential equation is given by,
$Y_m = A_m\cosh\alpha y + B_m\alpha y\sinh\alpha y + C_m\sinh\alpha y + D_m\alpha y\cosh\alpha y$
where A_m, B_m, C_m and D_m are constants and $\alpha = m\pi/a$.

The stress and displacement components are given by,

$$u = -\sum_{m=1}^{\infty} Y_m^{/}\alpha Sin\alpha x.$$

$$v = -\sum_{m=1}^{\infty}\{(\frac{1-\nu}{1+\nu})Y_m^{//} - (\frac{2\alpha^2}{1+\nu})Y_m\}\cos\alpha x - \frac{6(1-\nu)}{(1+\nu)} Ay$$

$$\sigma_x = -\frac{E}{(1+\nu)^2}[\sum_{m=1}^{\infty}\{\alpha^2 Y_m^{/} + \nu Y_m^{///}\}\cos\alpha x + 6\nu A]$$

$$\sigma_y = -\frac{E}{(1+\nu)^2}[\sum_{m=1}^{\infty}\{Y_m^{///} - (2+\nu)\alpha^2 Y_m^{/}\} \cos\alpha + 6A]$$

$$\sigma_{xy} = -\frac{E}{(1+\nu)^2}[\sum_{m=1}^{\infty}\{\nu\alpha Y_m^{//} + \alpha^3 Y_m\} \sin\alpha x]$$

where the prime ($^{/}$) indicates differentiation with respect to y.

It is seen that the boundary conditions,

$u = 0$ at $x = 0$ and $x = a$

$\sigma_{xy} = 0$ at $x = 0$ and $x = a$

are satisfied automatically.

Substituting the different derivatives of Y_m in the expressions of the stress and displacement components we get,

$$\sigma_y = \frac{E}{(1+\nu)}[\sum_{m=1}^{\infty} \alpha^3 \sinh\alpha y(A_m - t\,B_m) + \alpha^3 \cosh\alpha y(C_m - t\,D_m) + \alpha^4 y\cosh\alpha y\,B_m + D_m\,\alpha^4 y\sinh\alpha y\}\cos\alpha) - \frac{6EA}{(1+\nu)^2}. \quad where, \quad t = \frac{1-\nu}{1+\nu}$$

$$\sigma_{xy} = -\frac{E}{(1+\nu)^2}[\sum_{m=1}^{\infty} \alpha^3\{\cosh\alpha y\{A_m(1+\nu)+2\nu B_m\}+\sinh\alpha y\{C_m(1+\nu)+2\nu D_m\} + \alpha y\sinh\alpha y B_m(1+\nu) + D_m(1+\nu)\alpha y\cosh\alpha y\}\sin\alpha x]$$

$$u = -\sum_{m=1}^{\infty} \alpha^2\{(A_m + B_m)\sinh\alpha y + (C_m + D_m)\cosh\alpha y + B_m\alpha y\cosh\alpha y + D_m\alpha y\sinh\alpha y\}\cos\alpha x$$

$$v = \sum_{m=1}^{\infty} \alpha^2\{(A_m - 2tB_m)\cosh\alpha y + (C_m - 2tD_m)\sinh\alpha y + B_m\alpha y\sinh\alpha y + D_m\alpha y\cosh\alpha y\}\alpha\cos\alpha x - 6\frac{(1-\nu)}{(1+\nu)}Ay$$

Now, we are in a position to apply any feasible boundary conditions on the edges $y = 0$ and $y = b$ and thereby determine the values of the constants A_m, B_m, C_m and D_m. For example, if we consider that the edge $y = 0$ is on roller support, then the boundary conditions at this edge become,

$$v]_{y=0} = 0$$

and, $\sigma_{xy}]_{y=0} = 0$

For the boundary y = b, let it be assumed that it is loaded with an arbitrary normal load given by $\sigma_y]_{y=b} = f(x)$

Then the boundary conditions at y = b become,

$$\sigma_{xy}]_{y=b} = 0$$

$$\sigma_y]_{y=b} = f(x) = E_o + \sum_{m=1}^{\infty} E_m \cos \frac{m\pi x}{a}$$

Applying the boundary conditions at y = 0, then,

as cosh(0) = 1 and sinh(0) = 0,

$$v]_{y=0} = \sum \alpha^2 (A_m - 2tB_m) \cosh\alpha x = 0$$

As this condition should be satisfied for all values of x, therefore,

$A_m - 2tB_m = 0$ .... [1]

Also,

$$\sigma_{xy}]_{y=0} = - \frac{E}{(1 + \nu)^2} [\sum_{m=1}^{\infty} \alpha^3 \{A_m (1 + \nu) + 2\nu B_m\}] \sin\alpha x = 0$$

Therefore,

$A_m(1 + \nu) + 2\nu B_m = 0$ .... [2]

From (1) and (2),

$A_m = 0$
$B_m = 0$

Therefore,

$$\sigma_y = \frac{E}{1+\nu}\left[\sum_{m=1}^{\infty}\{\alpha^3(C_m - t\ D_m)\cosh\alpha y + D_m\alpha y\sinh\alpha y\}\cos\alpha x\right] - \frac{6EA}{(1+\nu)^2}$$

$$\sigma_{xy} = -\frac{E}{(1+\nu)^2}\left[\sum_{m=1}^{\infty}\alpha^3\{(C_m(1+\nu) + 2\nu\ D_m)\sinh\alpha y + D_m(1+\nu)\alpha y\cosh\alpha y\}\sin\alpha x\right]$$

From the boundary conditions at y = b,

$$\sigma_y]_{y=b} = \frac{E}{1+\nu}\left[\sum_{m=1}^{\infty}\alpha^3\{(C_m - t\ D_m)\cosh\alpha b + D_m\alpha b\sinh\alpha b\}\cos\alpha x - \frac{6EA}{(1+\nu)^2} = E_o + \sum_{m=1}^{\infty}E_m\cos\frac{m\pi x}{a}\right.$$

Equating co-efficients,

$$-\frac{6EA}{(1+\nu)^2} = E_o \quad or,\ A = -\frac{E_o(1+\nu)^2}{6E}$$

and,

$$\frac{E}{(1+\nu)}\alpha^3[(C_m - tD_m)\cosh\alpha b + D_m\alpha b\sinh\alpha b] = E_m \qquad ....[3]$$

Also from,

$$\sigma_{xy}]_{y=b} = \frac{-E}{(1+\nu)^2}\left[\sum_{m=1}^{\infty}\alpha^3\{(C_m(1+\nu) + 2\nu D_m)\sinh\alpha b + D_m(1+\nu)\alpha b\cosh\alpha b\right]\sin\alpha x = 0$$

it is found that

$$\{C_m(1 + \nu) + 2\nu D_m\}\sinh\alpha b + D_m(1 + \nu)\alpha b\cosh\alpha b = 0 \qquad \ldots.[4]$$

From equations [3] and [4]

$$D_m = \frac{E_m(1+\nu)}{E\alpha^3\cosh\alpha b N_m}$$

and

$$C_m = \frac{E_m(1+\nu)}{E\alpha^3\cosh\alpha b N_m}[N_m + t - \alpha b\tanh\alpha b]$$

Where, $N_m = \alpha b\ (\tanh\alpha b - \coth\alpha b) - 1$

The solutions are,

$$\sigma_x = -\left[\sum_{m=1}^{\infty}\frac{E_m}{N_m\cosh\alpha b}\{(N_m + t - \alpha b\tanh\alpha b + t')\cosh\alpha y\right.$$

$$\left. + \alpha y\sinh\alpha y\}\right]\cos\alpha x - E_o\nu \qquad \textit{where } t' = \frac{1 + 3\nu}{1 + \nu}$$

$$\sigma_y = \sum_{m=1}^{\infty}\frac{E_m}{N_m\cosh\alpha b}\{(N_m - \alpha b\tanh\alpha b)\cosh\alpha y + \alpha y\sinh\alpha y\}\cos\alpha x + E_o$$

$$\sigma_{xy} = -\sum_{m=1}^{\infty}\frac{E_m}{N_m\cosh\alpha b}\{(N_m - \alpha b\tanh\alpha b + 1)\sinh\alpha y + \alpha y\cosh\alpha y\}\sin\alpha x$$

$$u = -\sum_{m=1}^{\infty}\frac{E_m(1+\nu)}{\alpha E N_m\cosh\alpha b}\{(t - \alpha b\coth\alpha b)\cosh\alpha y + \alpha y\sinh\alpha y\}\sin\alpha x$$

$$v = \sum_{m=1}^{\infty}\frac{E_m}{E\alpha N_m\cosh\alpha b}\{-(t + 1 + \alpha b\coth\alpha b)\sinh\alpha y$$

$$+ \alpha y\cosh\alpha y\}\cos\alpha x + \frac{E_o(1 - \nu^2)y}{6E}$$

In the above expressions E_o and E_m are the coefficients of Fourier series representation of the loading. To take a specific example, the problem shown in fig. (4.2), is considered,

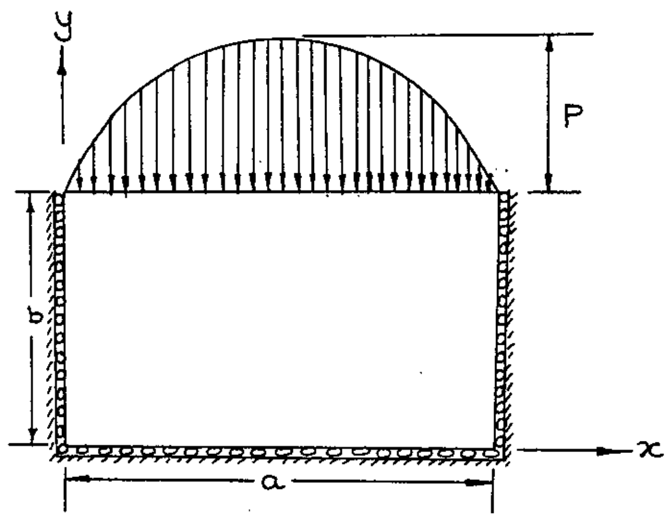


FIG. (4.2)

Let the loading be given by $\sigma_y]_{y=b} = \frac{4P}{a^2}(x^2-ax)$, as shown in figure (4.2).

Then, $E_o = \frac{1}{a}\int_o^a \frac{4P}{a^2}(x^2-ax)\,dx$

$$= \frac{1}{a}[\frac{4P}{a^2}][\frac{a^3}{3} - \frac{a^3}{2}] = -\frac{4P}{6} = -\frac{2P}{3}$$

$$E_m = \frac{2}{a}\int_o^a [\frac{4P}{a^2}](x^2-ax)\cos\alpha x\,dx,$$

$$= \frac{8P}{a^3}\int_o^a (x^2-ax)\cos\alpha x dx$$

$$= \frac{8P}{a^3} \cdot \frac{a}{\alpha^2} (\cos\alpha a + 1)$$

$$= \frac{8P}{a^2\alpha^2} \ (\cos\alpha a + 1) \ = \frac{8P}{m^2\pi^2}\{\cos(m\pi) + 1\}$$

$$= \frac{16P}{a^2m^2\pi^2}, \ \textit{for} \ m = 2,4,6,.......$$

$$= 0, \quad \textit{for} \quad m = 1,3,5,.......$$

The solution for this particular problem is,

$$\sigma_y = \frac{16P}{\pi^2} \sum_{m=2,4,6,....}^{\infty} \left\{\frac{1}{m^2E_m\cosh\alpha \frac{m\pi b}{a}}\right\}\left\{(N_m - \frac{m\pi b}{a}\tanh\frac{m\pi h}{a})\right.$$

$$\left.\cos\frac{m\pi y}{a} + \frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right\}\cos\frac{m\pi x}{a} - \frac{2P}{3}$$

$$\sigma_{xy} = -\frac{16P}{\pi^2} \sum_{m=2,4,6,...}^{\infty} \left\{\frac{1}{m^2N_m\cosh\frac{m\pi b}{a}}\right\} \left\{(N_m - \frac{m\pi b}{a}\tanh\frac{m\pi b}{a} + 1)\sinh\frac{m\pi y}{a}\right.$$

$$\left. + \frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right\}\sin\frac{m\pi x}{a}$$

$$\sigma_x = -\frac{16P}{\pi^2} \sum_{m=2,4,6,...}^{\infty} \left\{\frac{1}{m^2N_m\cosh\frac{m\pi h}{a}}\right\} \left\{(N_m + t - \alpha b\tanh\frac{m\pi b}{a} + t')\cosh\frac{m\pi}{a}\right.$$

$$\left. + \frac{m\pi}{a}y\sinh\frac{m\pi y}{a}\right\}\cos\frac{m\pi x}{a} - \frac{2P\nu}{3}$$

$$u = -\frac{16Pa}{\pi^3E} \sum_{m=2,4,6,...}^{\infty} \frac{(1+\nu)}{m^3N_m\cosh\frac{m\pi b}{a}}\left\{(t - \frac{m\pi b}{a}\coth\frac{m\pi b}{a})\cosh\frac{m\pi y}{a}\right.$$

$$\left. + \frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right\}\sin\frac{m\pi x}{a}$$

$$v = \frac{16Pa}{\pi^3E} \sum_{m=2,4,6}^{\infty} \frac{(1+\nu)}{m^3N_m\cosh\frac{m\pi b}{a}}\left\{-(1 + t + \frac{m\pi b}{a}\coth\frac{m\pi b}{a})\sinh\frac{m\pi}{a}y\right.$$

$$\left. + \frac{m\pi y}{a}\cosh\frac{m\pi h}{a}\right\}\cos\alpha x - \frac{2P(1-\nu^2)y}{18E}$$

PROBLEM-2:

For the problems where the edge y = 0 is fixed as shown in fig. (4.3), the boundary conditions are,

$$u\,]_{y=o} = 0 \quad and \quad v\,]_{y=o} = 0$$

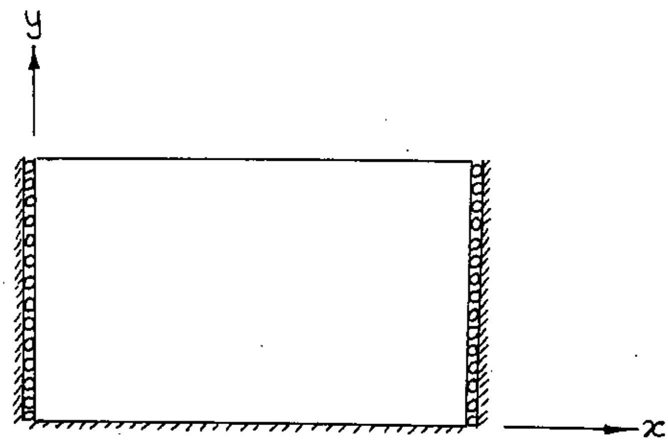


FIG. (4.3)

Therefore, from the expressions of u and v as derived in problem-1, we have,

$$C_m + D_m = 0 \qquad \text{or,} \quad C_m = -D_m$$

$$A_m - 2tB_m = 0 \qquad \text{or,} \quad A_m = 2tB_m$$

The expressions for stresses and displacements become:

$$\sigma_y = \frac{E}{(1+\nu)}\Big[\sum_{m=1,2,...}^{\infty} \{\alpha^3 tB_m \sinh\alpha y - \alpha^3(1+t)D_m \cosh\alpha y$$

$$+ B_m\alpha^4 y\cosh\alpha y + D_m\alpha^4 y\sinh\alpha y\}\cos\alpha x\Big] - \frac{6EA}{(1+\nu)^2}$$

$$= \left(\frac{E}{1+\nu}\right) \left[\sum_{m=1}^{\infty} \{\alpha^3 B_m (t\sinh\alpha y + \alpha y\cosh\alpha y) + \alpha^3 D_m(\alpha y\sinh\alpha y - (1+t)\cosh\alpha y)\}\cos\alpha x\right] - \frac{6EA}{(1+\nu)^2}$$

$$\sigma_x = -\frac{E}{1+\nu}\sum_{m=1}^{\infty} \alpha^3 [\{2tB_m + t'B_m\}\sinh\alpha y + (-D_m + t'D_m)\cosh\alpha y + B_m\alpha\cosh\alpha y + D_m\alpha y\sinh\alpha y]\cos\alpha x - \frac{6E\nu A}{(1+\nu)^2}$$

$$= -\frac{E}{(1+\nu)}\sum_{m=1}^{\infty} \alpha^3 [\{(2t+t')\sinh\alpha y + \alpha y\cosh\alpha y\}B_m + \{(t'-1)\cosh\alpha y + \alpha y\sinh\alpha y\}D_m]\cos\alpha x - \frac{6E\nu A}{(1+\nu)^2}$$

$$\sigma_{xy} = -\frac{E}{(1+\nu)^2}\left[\sum_{m=1}^{\infty} \alpha^3\{B_m(2\cosh\alpha y + (1+\nu)\alpha y\sinh\alpha y) + D_m\{(\nu-1)\sinh\alpha y + (1+\nu)\alpha y\cosh\alpha y\}\right]\sin\alpha x$$

$$u = -\left[\sum_{m=1}^{\infty} \alpha^2\{B_m\{(1+2t)\sinh alpha y + \alpha y\cosh\alpha y\} + D_m\alpha y\sinh\alpha y\right]\sin\alpha x$$

$$v = \sum_{m=1}^{\infty} \alpha^2\{D_m(\alpha y\cosh\alpha y - (1+2t)\sinh\alpha y) + B_m\alpha y\sinh\alpha y]\cos\alpha x - 6\left(\frac{1-\nu}{1+\nu}\right) Ay$$

Now, suppose the boundary conditions at y = b are,

$$\sigma_y]_{y=b} = f(x) = E_o + \sum_{m=1}^{\infty} E_m\cos\alpha x$$

$$\sigma_{xy}]_{y=b} = 0.$$

From the above conditions,

$$-\frac{6EA}{(1+\nu)} = E_o$$

and,

$$\frac{E}{1+\nu}\{\alpha^3 B_m(t\sinh\alpha b + \alpha b\cosh\alpha b) + \alpha^3 D_m(\alpha b\sinh\alpha b - (1+t)\cosh\alpha b$$

$$= E_m \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots[5]$$

$$\frac{E\alpha^3}{(1+\nu)^2}[B_m(2\cosh\alpha b + (1+\nu)\alpha b\sinh\alpha b) + D_m(\nu-1)\sinh\alpha b$$

$$+ (1+\nu)\alpha b\cosh\alpha b] = 0 \ \ldots\ldots[6]$$

From equation (6)

$$B_m = -D_m\frac{(\nu-1)\sinh\alpha b + (1+\nu)\alpha b\cosh\alpha b}{2\cosh\alpha b + (1+\nu)\alpha b\sinh\alpha b} = -D_m\frac{\alpha b - t\tanh\alpha b}{\alpha b\tanh\alpha + \frac{2}{1+\nu}}$$

Substituting in [5], we get,

$$-D_m\{\frac{\alpha b - t\tanh\alpha b}{\alpha b\tanh\alpha b + \frac{2}{1+\nu}}\}\{t\sinh\alpha b + \alpha b\cosh\alpha b\}$$

$$+ D_m[\alpha b\sinh\alpha b - (1+t)\cosh\alpha b\} = \frac{E_m(1+\nu)}{E\alpha^3}$$

$$B_m = \frac{E_m(1+\nu)}{E\alpha^3}\{\frac{\alpha b\cosh\alpha b - t\sinh\alpha b}{1+\alpha^2b^2+(1+2t)\cosh^2\alpha b}\}$$

$$D_m = -\frac{E_m(1+\nu)}{E_\alpha^3}\{\frac{\alpha b\sinh\alpha b + (1+t)\cosh\alpha b}{1+\alpha^2b^2+(1+2t)\cosh^2\alpha b}\}$$

The solution is,

$$\sigma_x = -\sum_{m=1}^{\infty}\frac{E_m}{1+\alpha^2b^2+(1+2t)\cosh^2\alpha b}[\{(2t+t')\sinh\alpha y$$

$$+ \alpha y\cosh\alpha y\}\{\alpha b\cosh\alpha b - t\sinh\alpha b\}$$

$$-\{(t'-1)\cosh\alpha y + \alpha y\sinh\alpha y\}\{\alpha b\sinh\alpha b$$

$$+ (1+t)\cosh\alpha b\}]\cos\alpha x - E_o\nu.$$

$$\sigma_y = \sum_{m=1}^{\infty}\frac{E_m}{1+\alpha^2b^2+(1+2t)\cosh^2\alpha b}[(\alpha b\cosh\alpha b - t\sinh\alpha b)$$

$$(t\sinh\alpha y + \alpha y\cosh\alpha y) - \{\alpha b\sinh\alpha b + (1+t)\cosh\alpha b\}\{\alpha y\sinh\alpha y$$

$$-(1+t)\cosh\alpha y\}]\cos\alpha x + E_o.$$

$$\sigma_{xy} = -\sum_{m=1}^{\infty} \frac{E_m}{1 + \alpha^2 b^2 + (1 + 2t)\cosh^2\alpha b} [\{\alpha b\cosh\alpha b - t\sinh\alpha b)$$

$$\{(1+t)\cosh\alpha y + \alpha y\sinh\alpha y\} - \{\alpha b\sinh\alpha b + (1+t)\cosh\alpha b\}$$

$$\{\alpha y\cosh\alpha y - t\sinh\alpha y\}]\sin\alpha x$$

$$u = -\frac{(1 + \nu)}{E} \sum_{m=1}^{\infty} \frac{E_m/\alpha}{1 + \alpha^2 b^2 + (1 + 2t)\cosh^2\alpha b}$$

$$[(\alpha b\cosh\alpha b - t\sinh\alpha b)\{\alpha y\cosh\alpha y + (1+2t)\sinh\alpha y\}$$

$$- \{\alpha b\sinh\alpha b + (1+t)\cosh\alpha b\}\alpha y\sinh\alpha y]\sin\alpha x$$

$$v = \frac{1 + \nu}{E} \sum_{m=1}^{\infty} \frac{E_m/\alpha}{1 + \alpha^2 b^2 + (1 + 2t)\cosh^2\alpha b} [(\alpha b\cosh\alpha b - t\sinh\alpha b)$$

$$\alpha y\sinh\alpha y - \{\alpha b\sinh\alpha b + (1 + t)\cosh\alpha b\}\{\alpha y\cosh\alpha y$$

$$- (1 + 2t)\sinh\alpha y\}]\cos\alpha + \frac{(1 - \nu^2)E_o y}{E}$$

Therefore, the problem is solved with arbitrary loading at y = b and the other boundaries are as shown in fig. [4.4]

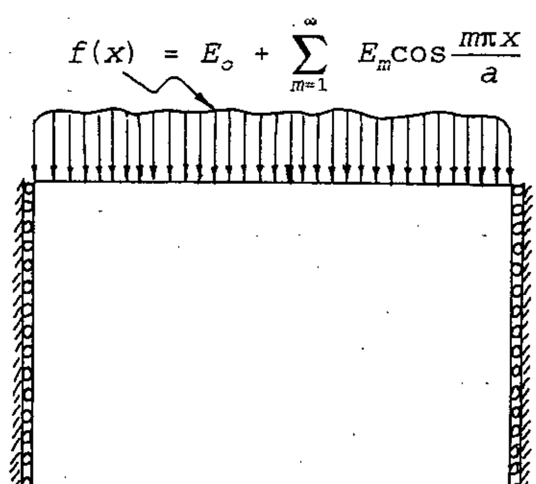


FIG. (4.4)

As an example, the solution for the problem of Fig. [4.4] with the loading of Fig. [4.5] is considered.

Here, $f(x) = -\dfrac{2Px}{a}$ when $0 \le x \le \dfrac{a}{2}$

$= \dfrac{2Px}{a} - 2P$ when $\dfrac{a}{2} \le x \le a$

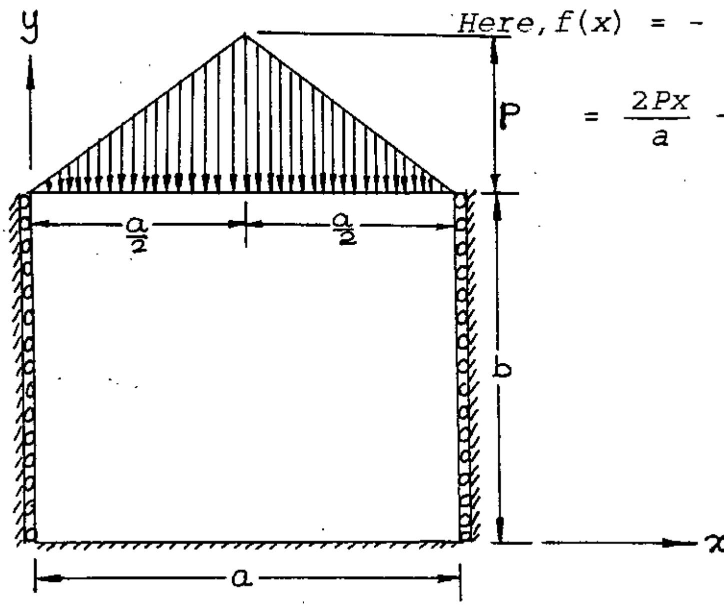


FIG. (4.5)

Here, $E_o = \frac{1}{2}\{\frac{1}{a}\int_{-a}^{a} f(x)\,dx\} = \frac{1}{a}\int_{o}^{a} f(x)\,dx = \frac{1}{a}\int_{o}^{a/2} f(x)\,dx + \frac{1}{a}\int_{a/2}^{a} f(x)\,dx$

$$= \frac{1}{a}\int_{o}^{a/2} - \frac{2Px}{a}dx + \frac{1}{a}\int_{a/2}^{a}(\frac{2Px}{a} - 2P)\,dx = -\frac{P}{2}$$

$$E_m = \frac{2}{a}\int_{o}^{a/2} f(x)\cos\alpha x dx + \frac{2}{a}\int_{a/2}^{a} f(x)\cos\alpha x dx$$

$$= \frac{2}{a}\int_{o}^{a/2} - \frac{2Px}{a}\cos\alpha x dx + \frac{2}{a}\int_{a/2}^{a}(\frac{2Px}{a} - 2P)\,dx$$

$$= -\frac{4P}{m\pi}\sin\frac{m\pi}{2} - \frac{4P}{m^2\pi^2}(\cos\frac{m\pi}{2} - 1) + \frac{4P}{m\pi}\sin\frac{m\pi}{2}$$

$$+ \frac{4P}{m^2\pi^2}(\cos m\pi - \cos\frac{m\pi}{2})$$

$$= -\frac{4P}{m^2\pi^2}(2\cos\frac{m\pi}{2} - \cos m\pi - 1) \quad for, \quad m = 2,6,10,14,\ldots..$$

PROBLEM-3:

In this class of mixed boundary-value problems, it is assumed that at the edges x = 0 and x = a, the normal stress is zero and the tangential displacement is zero. Such a problem can be visualized by assuming that these two edges are stiffened by adding additional stiffener as shown in fig. (4.6).

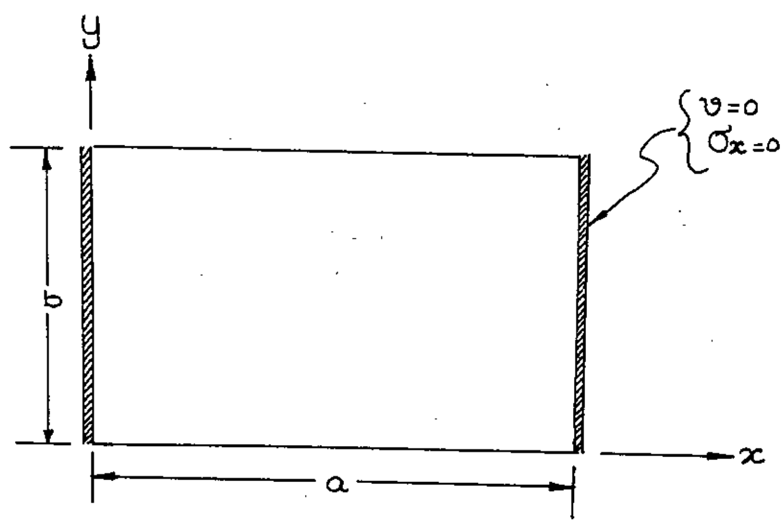


FIG. (4.6)

In this case let the solution for Ψ is assumed to be,

$$\psi = \sum_{m=1}^{\infty} Y_m \sin\frac{m\pi x}{a}$$

As before, in order to satisfy the biharmonic equations,

$$Y_m = A_m\cosh\frac{m\pi y}{a} + B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + C_m\sin\frac{m\pi y}{a} + D_m\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}$$

The corresponding expressions for the stress and displacement components are given by,

$$u = \sum_{m=1}^{\infty} \alpha^2 [(A_m + B_m)\sinh\alpha y + (C_m + D_m)\cosh\alpha y + B_m \alpha y \cosh\alpha y + D_m \alpha y \sinh\alpha y] \cos\alpha x$$

$$v = \sum_{m=1}^{\infty} \alpha^2 [(A_m - 2tB_m)\cosh\alpha y + (C_m - 2tD_m)\sinh\alpha y + B_m \alpha y \sinh\alpha y + D_m \alpha y \cosh\alpha y] \sin\alpha x$$

$$\sigma_y = \sum_{m=1}^{\infty} \frac{E\alpha^3}{(1 + \nu} [(A_m - t\, B_m)\sinh\alpha y + (C_m - tD_m)\cosh\alpha y + B_m \alpha y \cosh\alpha y + D_m \alpha y \sinh\alpha y] \sin\alpha x$$

$$\sigma_{xy} = \sum_{m=1}^{\infty} \frac{E\alpha^3}{(1 + \nu)^2} [\{A_m(1 + \nu) + 2\nu B_m\}\cosh\alpha y + \{C_m(1+\nu) + 2\nu D_m\} \sinh\alpha y + B_m(1+\nu)\alpha y \sinh\alpha y + D_m(1+\nu)\alpha y \cosh\alpha y] \cos\alpha x$$

$$\sigma_x = -\sum_{m=1}^{\infty} \frac{E\alpha^3}{(1+\nu)} [(A_m + \frac{1+3\nu}{1+\nu} B_m)\sinh\alpha y + (C_m + \frac{1+3\nu}{1+\nu} D_m)\cosh\alpha y + B_m \alpha y \cosh\alpha y + D_m \alpha y \sinh\alpha y] \sin\alpha x$$

Thus it is seen that the boundary conditions $\sigma_x = v = 0$ at both x = 0 and x = a are satisfied. The four constants in the expression of Ψ can be determined from the known boundary conditions at the other two opposite edges.

First suppose that the edge $y = o$ is on roller while the edge $y = b$ is loaded with an arbitrary normal loading as shown in fig. (4.7).

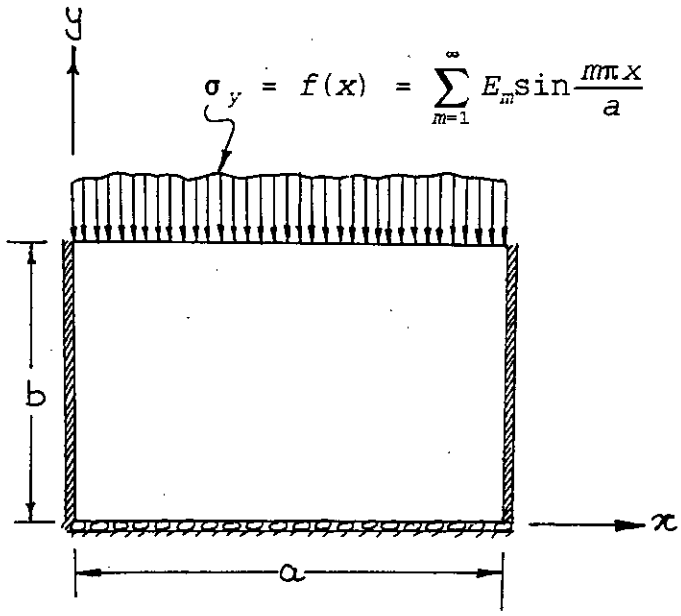


FIG. (4.7)

Then, $\sigma_{xy}]_{y=o} = 0$

To satisfy this condition for all values of x,

$A_m (1 + \nu) + 2 \nu B_m = 0.$

Also, from $v]_{y=o} = 0$, the following is found

$A_m - 2tB_m = 0$

Therefore,

$A_m = 0, \; B_m = 0$

From the condition that $\sigma_{xy}]_{y=o} = 0$, the following equation is found,

$$\{C_m(1 + \nu) + 2\nu D_m\}\sinh\alpha b + D_m(1 + \nu)\alpha b\cosh\alpha b = 0 \quad ....[7]$$

Also, from

$$\sigma_y]_{y=b} = \sum_{m=1}^{\infty} \frac{E\alpha^3}{1+\nu}\{(C_m - tD_m)\cosh a\alpha b + D_m\alpha b\sinh\alpha b\}\sin\alpha x = f(x) = \sum_{m=1}^{\infty} E_m\sin\alpha x$$

The following relation is found,

$$(C_m - tD_m)\cosh\alpha b + D_m\sinh\alpha b = \frac{E_m(1 + \nu)}{E\alpha^3} \quad ....[8]$$

From equations [7] and [8],

$$C_m = \frac{E_m(1 + \nu)}{E\alpha^3}\left\{\frac{\alpha b\cosh\alpha b + \frac{2\nu}{1+\nu}\sinh\alpha b}{\alpha b + \sinh\alpha b\cosh\alpha b}\right\}$$

$$D_m = -\frac{E_m(1+\nu)\sinh\alpha b}{E\alpha^3(\alpha b + \sinh\alpha b\cosh\alpha b)}$$

For a particular loading, say $\sigma_y]_{y=b} = -P$, ,

$$E_m = \frac{2}{a}\int_o^a (-P\sin\frac{m\pi x}{a})\, dx$$

$$= \frac{2}{a}\left[\frac{Pa}{m\pi}\cos\frac{m\pi x}{a}\right]_o^a$$

$$= \frac{2P}{m\pi}(\cos m\pi - 1)$$

$$= -\frac{4P}{m\pi}, \quad for \quad m = 1,3,..........$$

$$= 0, \quad for \quad m = 2,4,.......$$

The solution for this particular problem is,

$$\sigma_y = -\frac{4P}{\pi}\sum_{m=1,3,5,\ldots}^{\infty}\frac{1}{m(\frac{m\pi b}{a}+\cosh\frac{m\pi b}{a}\sinh\frac{m\pi b}{a})}[(\frac{m\pi b}{a}\cosh\frac{m\pi b}{a}$$

$$+\sinh\frac{m\pi b}{a})\cosh\frac{m\pi y}{a}-\sinh\frac{m\pi b}{a}(\frac{m\pi y}{a})\sinh\frac{m\pi y}{a}]\sin\frac{m\pi x}{a}$$

$$\sigma_{xy} = -\frac{4P}{\pi}\sum_{m=1,3,5,\ldots}^{\infty}\frac{1}{m(\frac{m\pi b}{a}+\sinh\frac{m\pi b}{a}\cosh\frac{m\pi b}{a})}[\frac{m\pi b}{a}\cosh\frac{m\pi b}{a}\sinh\frac{m\pi y}{a}$$

$$-\sinh\frac{m\pi b}{a}(\frac{m\pi y}{a})\cosh\frac{m\pi y}{a}]\cos\alpha x$$

$$\sigma_x = \frac{4P}{\pi}\sum_{m=1,3,5,\ldots}^{\infty}[\frac{1}{m(\frac{m\pi b}{a}+\sinh\frac{m\pi b}{a}\cosh\frac{m\pi b}{a})}]$$

$$\{(\frac{m\pi b}{a}\cosh\frac{m\pi b}{a}-\sinh\frac{m\pi b}{a})\cosh\frac{m\pi}{a}y$$

$$-\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\sinh\frac{m\pi b}{a}\}\sin\frac{m\pi xy}{a}$$

$$u = -\frac{4P}{\pi}\sum_{m=1,3,5}^{\infty}\frac{(1+\nu)}{E\alpha m(\alpha b+\sin\alpha b\cos h alpha b)}[(\alpha b - t\sinh\alpha b)\cosh\alpha b$$

$$-\sinh\alpha b(\alpha y)\sinh\alpha y]\cos\alpha x$$

$$v = -\frac{4P}{\pi}\sum_{m=1,3,5}^{\infty}\frac{(1+\nu)}{E\alpha m(\alpha b+\sinh\alpha b\cosh\alpha b)}[(\alpha b+\frac{2}{\nu}\sinh\alpha b)\sinh\alpha y$$

$$+\sinh\alpha b(\alpha y)\cosh\alpha y]\sin\alpha x.$$

PROBLEM - 4:

If it is assumed that the edge y = 0 is fixed as shown in fig. (4.8) then the boundary conditions are,

$u]_{y=0} = 0$ *and* $v]_{y=0} = 0$

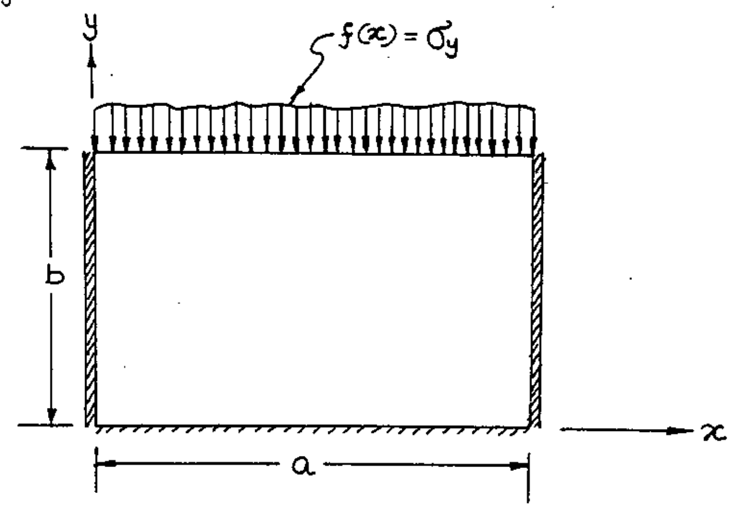


FIG. (4.8)

From $u]_{y=o} = 0, \Rightarrow C_m + D_m = 0$ *Therefore,* $C_m = - D_m$

From $v]_{y=o} = 0 \Rightarrow A_m - 2t\, B_m = 0$ *or,* $A_m = 2t\, B_m$

The expressions for σ_{xy} and σ_y become.

$$\sigma_{xy} = \sum_{m=1}^{\infty} \frac{E\alpha^3}{(1 + \nu)^2} [B_m\{2\cosh\alpha y + (1 + \nu)\alpha y\sinh\alpha y\}$$

$$+ D_m\{(\nu - 1)\sinh\alpha y + (1 + \nu)\alpha y\cosh\alpha y\}] \cos\alpha x$$

$$\sigma_y = \sum_{m=1}^{\infty} \frac{E\alpha^3}{(1 + \nu)} [B_m\{t\sinh\alpha y + \alpha y\cosh\alpha y\}$$

$$+ D_m(\alpha y\sinh\alpha y - (1 + t)\cosh\alpha y)] \sin\alpha x.$$

If it is assumed that, $\sigma_{xy}]_{y=b} = 0$ *and* $\sigma_y]_{y=b} = f(x) = \sum_{m=1}^{\infty} E_m\sin\alpha x.$

then,

$$B_m[2\cosh\alpha b + (1+\nu)\alpha b\sinh\alpha b] + D_m[(\nu-1)\sinh\alpha b + (1+\nu)\alpha b\cosh\alpha b] = 0.$$

$$B_m \left[t\sinh\alpha b + \alpha b\cosh\alpha b\right] + D_m \left[\alpha b\sinh\alpha b - (1 + t)\cosh\alpha b\right]$$

$$= \frac{E_m(1 + \nu)}{E\alpha^3}$$

Solving these equations, constants are evaluated as follows,

$$B_m = \frac{E_m(1 + \nu)}{E\alpha^3} \left\{\frac{\alpha b\cosh\alpha b - t\sinh\alpha b}{t^2 + \alpha^2 b^2 + (1 + 2t)\cosh^2\alpha b + 2\alpha bt\sinh\alpha b cpsh\alpha b}\right\}$$

$$C_m = -\frac{E_m(1 + \nu)}{E\alpha^3} \left\{\frac{\alpha b\sinh\alpha b + (1 + t)\cosh\alpha b}{t^2 + \alpha^2 b^2 + (1 + 2t)\cosh^2\alpha b + 2\alpha bt\cosh\alpha b\sinh\alpha}\right.$$

E_m is found out according to the normal loading at y = b and therefore, the problem is solved for all normal loading conditions at y = b.

For example, the solution for the problem of Fig. [4.9] is considered,

$$f(x) = -\frac{Px}{a} = \sum_{m=1}^{\infty} E_m \sin\alpha x$$

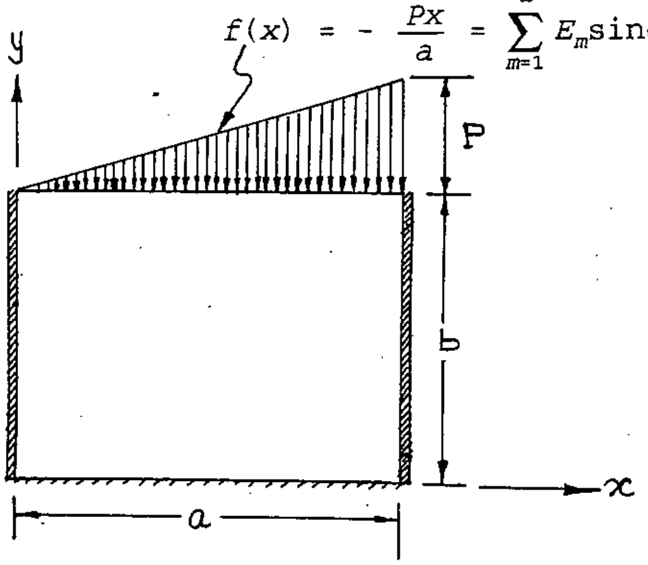


FIG. (4.9)

Here, $E_m = \frac{2}{a}\int_o^a f(x)\sin\alpha x dx = \frac{2}{a}.\frac{P}{a}\int_o^a - \frac{Px}{a}\sin\alpha x dx = -\frac{2P}{a^2}\int_o^a x\sin\alpha x dx$

$$= -\frac{2P}{a^2}[-\frac{x\cos\alpha x}{\alpha}]_o^a - \frac{2P}{a^2}\int_o^a \frac{\cos\alpha x}{\alpha}dx = \frac{2P}{a^2}[\frac{x\cos\alpha x}{\alpha}]_o^a - \frac{2P}{a\alpha}[\frac{\sin\alpha x}{\alpha}$$

$$= \frac{2P}{a^2\alpha}[\alpha\cos\alpha x]_o^a = \frac{2P}{a^2\alpha}[a\cos\alpha x] = \frac{2P}{a^2\alpha}a\cos m\pi = \frac{2P}{a\alpha}\cos m\pi$$

$$= \frac{2P}{a\alpha}(-1)^m, \quad for \quad m = 1, 2, 3, \ldots.$$

The solution for this particular problem is,

$$u=\sum_{m=1,2,3,...}^{\infty}\left[\frac{2P(-1)^{m}\ (1+\nu)a}{m^2\pi^2E\{\alpha^2b^2+(1+2t)\cosh^2\alpha b+t^2+2\alpha bt\cosh\alpha b\sinh\alpha b\}}\right.$$

$$[\alpha b\cosh\alpha b-t\sinh\alpha b)\{(1+2t)\sinh\alpha y+\alpha y\cosh\alpha y\}-$$

$$\alpha y\sinh\alpha y\{\alpha b\sinh\alpha b+(1+t)\cosh\alpha b\}]\cos\alpha x$$

$$v=\sum_{m=1,2,3,...}^{\infty}\left[\frac{2P(-1)^{m}\ (1+\nu)a}{m^2\pi^2E\{\alpha^2b^2+(1+2t)\cosh^2\alpha b+t^2+2\alpha bt\cosh\alpha b\sinh\alpha b\}}\right]$$

$$[(\alpha b\cosh\alpha b-t\sinh\alpha b)\alpha y\sinh\alpha y-$$

$$\{\alpha y\cosh\alpha y-(1+2t)\sinh\alpha yy\}(\alpha b\sinh\alpha b+(1+t)\cosh\alpha b)]\sin\alpha x$$

$$\sigma_y=\sum_{m=1,2,3,...}^{\infty}\left[\frac{2P(-1)^{m}}{m\pi\{\alpha^2b^2+(1+2t)\cosh^2\alpha b+t^2+2\alpha bt\cosh\alpha b\sinh\alpha b\}}\right]$$

$$[(\alpha b\cosh\alpha b-t\sinh\alpha b)(t\sinh\alpha y+\alpha y\cosh\alpha y)-$$

$$\{\alpha b\sinh\alpha b+(1+t)\cosh\alpha b\}\{\alpha y\sinh\alpha y-(1+t)\cosh\alpha y\}]\sin\alpha x$$

$$\sigma_{xy}=\sum_{m=1,2,3,...}^{\infty}\left[\frac{2P(-1)^{m}}{(1+\nu)m\pi\{\alpha^2b^2+(1+2t)\cosh^2\alpha b+t^2+2\alpha bt\cosh\alpha b\sinh\alpha b\}}\right]$$

$$[\{2\cosh\alpha y+(1+\nu)\alpha y\sinh\alpha y\}(\alpha b\cosh\alpha b-t\sinh\alpha b)-$$

$$\{(\nu-1)\sinh\alpha y+(1+\nu)\alpha y\cosh\alpha y\}\{\alpha b\sinh\alpha b+(1+t)\cosh\alpha b\}]\cos\alpha x$$

$$\sigma_x=\sum_{m=1,2,3,...}^{\infty}\left[\frac{2P(-1)^{m}}{m\pi\{\alpha^2b^2+(1+2t)\cosh^2\alpha b+t^2+2\alpha bt\cos b\alpha b\sinh\alpha b\}}\right]$$

$$[(\frac{3+\nu}{1+\nu}\sinh\alpha y+\alpha y\cosh\alpha y)(\alpha b\cosh\alpha b-\frac{2\nu}{1+\nu}\sinh\alpha b)-$$

$$(\alpha y\sinh\alpha y-2\nu\cosh\alpha y)\{\alpha b\sinh\alpha b+(1+t)\cosh\alpha\}]\sin\alpha x$$

PROBLEM - 5:

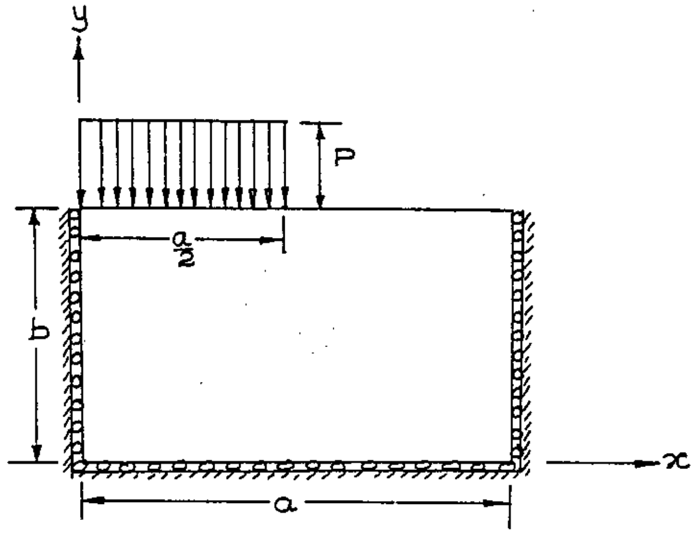


FIG. (4.10)

In this problem, the boundary conditions are the same as problem-1, only difference is the loading.

Here, $E_o = - P/2$

$$E_m = -\frac{2P}{m\pi}(-1)^{\frac{m-1}{2}}, \quad for \quad m = 1,3,5,.....$$

$$= 0, \quad for \quad m = 2,4,6,......$$

The solution for this particular problem is,

$$u = -\sum_{m=1,3,5,\ldots}^{\infty}\left[\frac{2P(-1)^{\frac{m-1}{2}}}{m\pi E\alpha\cosh\frac{m\pi b}{a}N_m}\right]$$

$$\left[\left(t-\frac{m\pi b}{a}\right)\cosh\frac{m\pi y}{a} + \frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}$$

$$v = -\sum_{m=1,3,5,\ldots}^{\infty}\left[\frac{2P(-1)^{\frac{m-1}{2}}}{m\pi E\alpha\cosh\frac{m\pi b}{a}N_m}\right]$$

$$\left[\{-(1+t)+m\pi\}\sinh\frac{m\pi y}{a}+\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\cos\frac{m\pi x}{a}$$

$$-\frac{P(1-\nu^2)y}{2E}$$

$$\sigma_{xy}=\sum_{m=1,3,5,\ldots}^{\infty}\left[\frac{-2P(-1)^{\frac{m-1}{2}}}{m\pi N_m\cosh\frac{m\pi}{a}}\right]$$

$$\left[\left(\frac{m\pi b}{a}\sinh\frac{m\pi y}{a} - \frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right)\right]\sin\frac{m\pi x}{a}$$

$$\sigma_y = \sum_{m=1,3,5,\ldots}^{\infty}\left[\frac{-2(-1)^{\frac{m-1}{2}}}{m\pi N_m\cosh\frac{m\pi b}{a}}\right]$$

$$\left[\left\{\left(-1-\frac{m\pi b}{a}\right)\cosh\frac{m\pi y}{a}+\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right\}\right]\cos\alpha x - \frac{p}{2}$$

$$\sigma_x=\sum_{m=1,3,5,\ldots}^{\infty}\frac{2P(-1)^{\frac{m-1}{2}}}{m\pi\cosh\frac{m\pi b}{a}N_m}\{(N_m+t-\alpha b\tanh\alpha b)\cosh\alpha y +$$

$$t'\cosh\alpha y+\alpha y\sinh\alpha y\}\cos\alpha x - \frac{p\nu}{2}$$

CHAPTER V
RESULTS AND DISCUSSIONS

5.1 Introduction

For studying the soundness of the present formulation for two-dimensional mixed boundary-value elastic problems, the analytical solutions obtained earlier were evaluated numerically. In obtaining numerical values, the plates were assumed to be square (a/b = 1) and made of ordinary steel ($E = 2 \times 10^{11}$ N/m^2, $\nu = 0.3$). Numerical values were obtained at a number of sections of the plates, i.e. at different constant values of y for varying x. With the numerical values so obtained, graphs are plotted at each section (at each constant value of y) for varying x for each of the five quantities of interest, namely u, v, σ_x, σ_y and σ_{xy}, of plane-stress and plane-strain problems. In order to make the results nondimensional, the displacements are expressed as the ratio of actual displacement to the size of the plate and the stresses are expressed as the ratio of the actual stress to the applied loading parameter P.

The soundness and accuracy of the formulation and the method of solutions are judged against the following cirteria:

i. The values of different parameters in a model of structure in terms of its nature of variation and magnitude for any loading can be predicted intuitively based on experience.

ii. the physical symmetry or antisymmetry of the model and that of loading are always reflected in symmetric or antisymmetric distribution of the parameters in the body of the structure.

iii. The famous Saint Venant's Principle[20] must be found true in the distribution of values of every parameter within the body of the structure. This principle states that any sharp variation or abrupt change or concentration in the values of a parameter on the boundary of a body must gradually be distributed or smoothened up within the body with increasing distances from the relevant point of concentration or sharp-change on the boundary.

It should be pointed out here that in cases where the elastic problems are solved in terms of the stress function φ, the solutions are generally available in terms of stress components σ_x, σ_y and σ_{xy}. This is because of the fact that the deformations are to be obtained by simultaneous integration of φ with respect to x and y which bring in additional unkonwn function of x and y in the expressions of the deformation parameters u and v and require lengthy and complicated evaluation process. Further, the stresses obtained from stress function φ are never valid at the restrained boundaries.

In the present formulation of the problem in terms of the displacement function Ψ, all the parameters of interest in the solution of elastic problems, namely u, v, σ_x, σ_y and σ_{xy}, are readily obtained as soon as Ψ is known as all of them are expressed as summation of different derivations of Ψ. Moreover, the solutions are always exact anywhere within the body as the function Ψ is obtained by satisfying the governing equations as well as all the boundary conditions, whether they are specified in terms of loading or restraints at the boundary.

5.2 Discussion of the Results

Problem No. 1:

In this problem, the variation of displacement component u with respect to x (Figure 6.1) is sinusoidal. At the top edge of the plate, where the load is applied, its value increases first then becomes zero, and again it reaches a maximum negative value then becomes zero at the end. Moving from top edge, the magnitude of variation decreases and at y = 0.927b & 0.926b, the values are negative for a small length of the plate, then it follows the same variation but again before becoming zero at the end edge, it becomes positive for a small portion of plate. For the rest of the sections beginning from y = 0.8b, this phenomena is completely reversed. At the other sections also it becomes negative first then becomes positive with a zero value at the middle as before. It should be noted that the displacement at the ends and the mid section are zero.

The distribution of displacement component v (Fig. 6.2) is observed to be in good agreement with the physical model of the plate. As the loading is given on the top edge; so the displacement will be maximum there and it will gradually decrease as we move towards the bottom edge. This phenomena is readily seen from the graph. Another thing is to be noted here that at the top edge the distribution is not linear rather it is curved and maximum at the middle of the plate. This non-linearity decreases gradually towards the bottom and becomes almost linear at the bottom edge. So, we see that according to Saint Venant's principle, the non-linearity distribution at the top edge vanishes completely at the bottom edge. It is always negative as displacement here is opposite to positive direction of y.

The distribution of σ_x (Fig.6.3) is also non-linear at every section and maximum at the top edge. We know that $\sigma_x = \frac{E}{(1+\nu)^2}(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y})$ i.e. it is the summation of slope of u and ν times the slope of v. Therefore, from the distribution of u at top edge, the slope is negative at x = 0.5a and the slope of v is also negative at that position. As a result of which σ_x is maximum at x = 0.5a on the top edge. And for any other section these two slopes are not of the same sign. So the value of σ_x decreases from maximum value. From the distribution of σ_x at top edge, we see that the plate is in compression for certain portion then it is in tension for rest of the length of the plate. Here, also the Saint Venant's principle is applicable, as we see that at bottom edge the distribution has slight non-linear characteristic.

The distribution of σ_y (Fig. 6.4) is also in good agreement with physical characteristic of the plate i.e. maximum at the top edge and minimum at the bottom. Distributions are also not linear and the non linearity decreases toward the bottom and slightly curved at the bottom edge. This also conforms to the Saint Venant's principle. From the distribution, it can be concluded that the plate is always in compression in vertical direction as the distribution is always negative throughout the length of the plate.

From the distribution of σ_{xy} (Fig. 6.5), it is seen that its value is maximum at y = 0.8b and zero at the top and bottom, which is quite obvious as the plate is on roller support. Here, also the distribution is sinusoidal. For the locations below top edge, the magnitude of the shear stress increases gradually upto y = 0.8b then the value again decreases gradually and becomes zero at the bottom edge, as the plate is on roller support.

Problem No. 2:

Here the distribution of u (Fig. 6.6) is almost the same as of prob.1.

Displacement component v (Fig. 6.7) varies slightly in non-linear way at the top edge, and at y = 0.8b, the non-linearity is maximum but in the opposite direction and for other sections non-linear phenomena disappears and becomes almost flat as in the problem-1.

Stress component σ_x (Fig. 6.8) varies from positive value to negative value, becomes maximum at the half length of the plate. And for rest of the sections, although they remain within the negative region, but still they vary from a maximum negative value to a minimum negative value. But one thing is common, that is, either maximum or minimum, all these occur at the half length. The variation of σ_y (Fig. 6.9) is the same as problem-1. The distribution of σ_{xy} (Fig. 6.10) is the same as problem-1.

Problem No. 3:

The curves, showing the displacement component u (Fig. 6.11), are all cosine curves. They vary from positive value to negative value with zero value at the half length of the plate. The variation of v (Fig. 6.12) is sinusoidal. It is maximum at the top edge where the loading is given and minimum at the bottom. From the curves of σ_x (Fig. 6.13) it is seen that negative maximum occurs at the top edge. And for other sections its value is completely positive and becomes maximum for $y = 0.25b$. For $y = 0.25b$, $y = 0.1b$ and $y = 0.0$, the variation of σ_x is almost same.

The curves for stress component σ_y (Fig. 6.14) are all sine curves. And this nature is less evident at the top edge where a uniform loading P is given. So, at that edge the ratio, σ_y/P should be 1 for the whole length of the plate. From the curve it is also seen that the ratio is almost 1 with slight variation and at two opposite corners at top edge, which are also the point of singularity, there this ratio is zero. The shear stress σ_{xy} (Fig. 6.15) is maximum at y = 0.8b and minimum i.e. of zero magnitude at y = b & y = 0.

Problem No. 4:

The distribution of displacement component u (Fig. 6.16) is positive maximum at x = 0, then decreases gradually, becomes zero at x = 0.7a and reaches a maximum negative value at the other end of the plate. But the case is reversed for other sections. And also the transition from positive to negative does not occur at a fixed point like problems-1,2,3, where the loading is symmetric.

The magnitude of v (Fig. 6.17) is maximum at the top edge and decreases to a minimum value of zero, at the bottom. But here the maximum value does not occur at the half length i.e. at x = 0.5a of the plate; rather it is now shifted rightward. This phenomena is quite logical as the distribution of loading is triangular and maximum at right end of the plate and also due to the fact that v = 0 at the right end because of additional stiffener.

The variation of σ_x (Fig. 6.18) remains positive throughout the whole length of the plate and reaches a maximum value after x = 0.5a and it is zero at both ends of the plate. The magnitude of σ_x at the bottom edge is negligible as compared to other sections. The distribution of σ_y (Fig. 6.19) is similar to other problems i.e. maximum at top edge and minimum at the bottom but only difference is that the maximum value does not occur at half length of the plate. It is now slightly shifted to the right as the load concentration is maximum at right end of the plate.

The shear stress σ_{xy} (Fig. 6.20) is maximum at y = 0.8b and minimum at the top edge. And at the bottom edge, which is fixed the magnitude of σ_{xy} is of negative value. Also the point of transition is not located at the half length of the plate. The maximum and minimum values of σ_{xy} at any particular section occur at the two opposite edges which are stiffened by stiffener.

Problem No. 5:

In this problem, although the loading is not symmetric but still the variation of displacement component u (Fig. 6.21) is symmetric around the vertical mid section of the plate. At the top edge, the variation starts from zero, reaches a maximum value and then becomes zero at the other end. The maximum value occurs at x = 0.5a. At the mid section of the plate i.e. at y = 0.5b, the maximum negative value occurs.

As the loading is uniform for half of the length of the plate and it is zero for rest half, the displacement component v (Fig. 6.22) is maximum at the left top corner point and its magnitude decreases gradually towards the other end. This is in good agreement with the physical characteristics of the plate. The influence of loading becomes zero at the bottom edge and which is evident from the curve of zero value at the bottom. And also the decreasing rate decreases from top edge to bottom edge.

The stress component σ_x (Fig. 6.23), at the top edge varies from a negative maximum value at the left end and reaches a maximum value at the other end with zero value at x = 0.8a. And for rest of the sections, variation remains within the negative region, never reaches a positive value. The variation is minimum at mid section i.e. at y = 0.5b. The value of σ_y (Fig. 6.24)is negative maximum at the top left end as before, then decreases gradually and gains a positive value at the right end. This characteristic can also be attributed from the curve for y = 0.8b. and for rest of the sections, σ_y remains with the negative region.

The case of σ_{xy} (Fig. 6.25) is the same as before i.e. of zero value at top & bottom edge and maximum at y = 0.8b. The maximum value occurs at the location x = 0.5a.

CHAPTER VI
CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

Included in this thesis is a new approach to the solution of mixed boundary-value elastic problems. In this new approach the elastic problem is formulated in terms of a displacement function, Ψ. This displacement function Ψ may be considered as parallel to Airy's Stress function φ since both of them have to satisfy the same biharmonic partial differential equation and may be considered as potential functions- φ being a stress potential while Ψ a displacement potential in the two dimensional stress analysis.

The major difference in the two approaches lie in the fact that while φ formulations can be used in solving problem in which boundary conditions are specified in terms of loading only, whereas Ψ-formulations can be used for all boundary conditions-either in terms of loading or restraints or any combination of them.

It should be pointed out here that the φ-formulation has been used in solving mixed boundary problems of two-dimensional elasticity but the boundary conditions specified as restraints were satisfied approximately in an over-all nature and the solutions thus obtained were not satisfactory in predicting stresses in the neighbourhood of the restrained boundary. However, the Ψ-formulation does not suffer from this shortcoming and obtains exact solutions satisfying all kinds of boundary conditions and valid for the entire region of interest.

Earlier, two-dimensional mixed boundary-value stress problems were solved in terms of the two displacement functions, u and v, but obtaining two functions simultaneously, satisfying two simultaneous partial differential equations and the mixed boundary conditions, is extremely difficult and hence hardly any solution is obtained for mixed boundary-value problems this way.

Using the present Ψ-formulation, a number of mixed boundary-value elastic problems are solved analytically and the solutions are presented in this thesis. All these analytical solutions of different problems are evaluated numerically and then presented graphically. As the solutions are functions of two independent variables, the numerical values are obtained at different sections of the structural body, keeping one of the independent variable constant. The graphs obtained from this scheme, showing variation of different solution parameters like various displacement and stress components with varying x, provide a better comprehension of the nature of solutions.

The study of the graphical results of different problems for the important parameters namely the relevant stress and displacement components, establish the soundness of the formulation as well as the appropriateness of this new approach.

As established by obtaining solutions of various mixed boundary-value problems in this thesis, the new approach has provided a very bright prospect of investigating stresses in the regions of boundary restraints. It is thus expected that, with time, solutions would be obtained for various practical problems in order to provide better insight and further understanding of the stress distribution in the critical regions of the restrained boundaries of structural problems.

6.2 Recommendations for Further Works

The new approach will provide a great scope for the investigations of mixed boundary-value elastic problems which hitherto remained beyond appropriate analysis. In this connection, the following works are recommended for further investigations.

1. Although the solutions presented in this thesis for various mixed boundary-value problems of rectangular plates are valid for all sizes of the plates, the numerical results are obtained only for square plates. The effect of b/a ratio on the deformation and the stress distribution should be studied. It is expected that very interesting results will be revealed in extreme cases of b/a ratios.

2. Numerical values are obtained only at sections normal to y-axis. Although this gives a general idea about the stress distribution in the plate, it would be interesting to study the stress distribution over sections normal to x-axis.

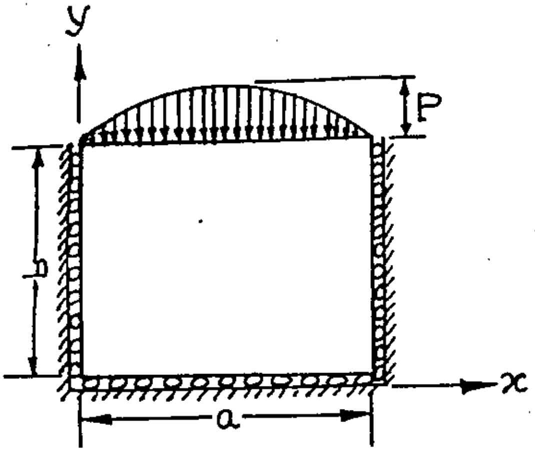
3. The solutions presented in this thesis are restricted to particular mixed boundary conditions at $x = 0$ and $x = a$, but not restricted to any particular type at $y = 0$ and $y = b$. It is thus imperative that solutions should be obtained for various boundary conditions of interest at $y = 0$ and $y = b$.

4. Solutions presented here are only for particular types of boundary conditions at $x = 0$ and $x = a$ of a rectangular plate. Attempt should be made for obtaining analytical solutions for other types of boundary conditions at these two opposing boundaries.

5. Solutions obtained here are all confined to rectangular plates. No effort was made to obtain solutions of plates of arbitrary shapes of boundary. It is very unlikely that analytical solutions can be found for arbitrary shapes of boundary. But problems of arbitrary shapes can be solved by numerical methods using the present formulation of the elastic problems. It is thus suggested that a computer program should be developed based on finite-difference scheme to solve problems of arbitrary boundary shape and arbitrary mixed boundary conditions.

6. There are various approximate analytical methods like that of Raleigh-Ritz. Although these approximate methods do not provide exact solutions but the accuracy provided is good enough for engineering purpose. Attempt should thus be made to solve the mixed boundary-value problems by these approximate method in terms of the function Ψ.

7. The present formulation of the problem is presented in rectangular co-ordinates. It may be transformed into polar coordinates. This would thus provide scope for solving problems of circular plates subjected to mixed boundary conditions.



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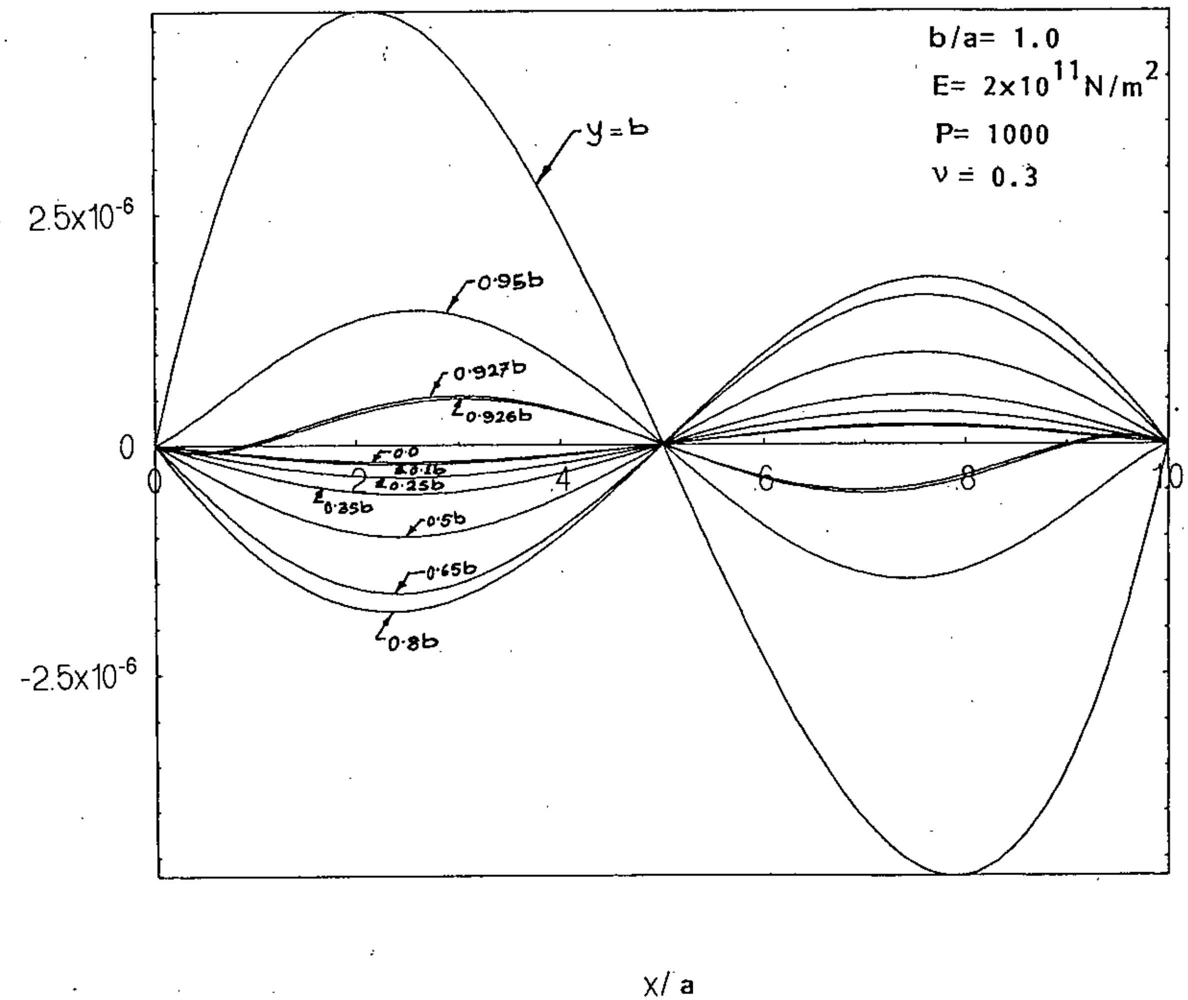
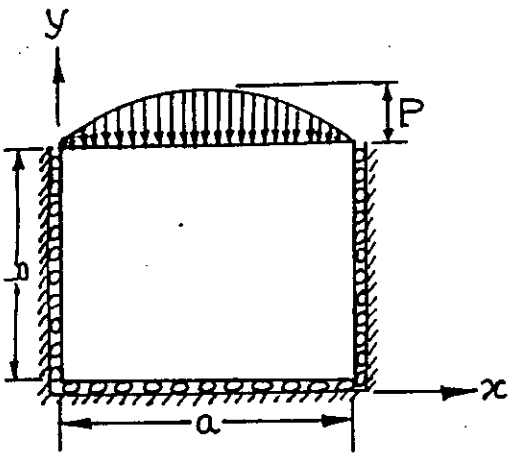


Fig 6.1: Distribution of displacement component u at different sections of the plate



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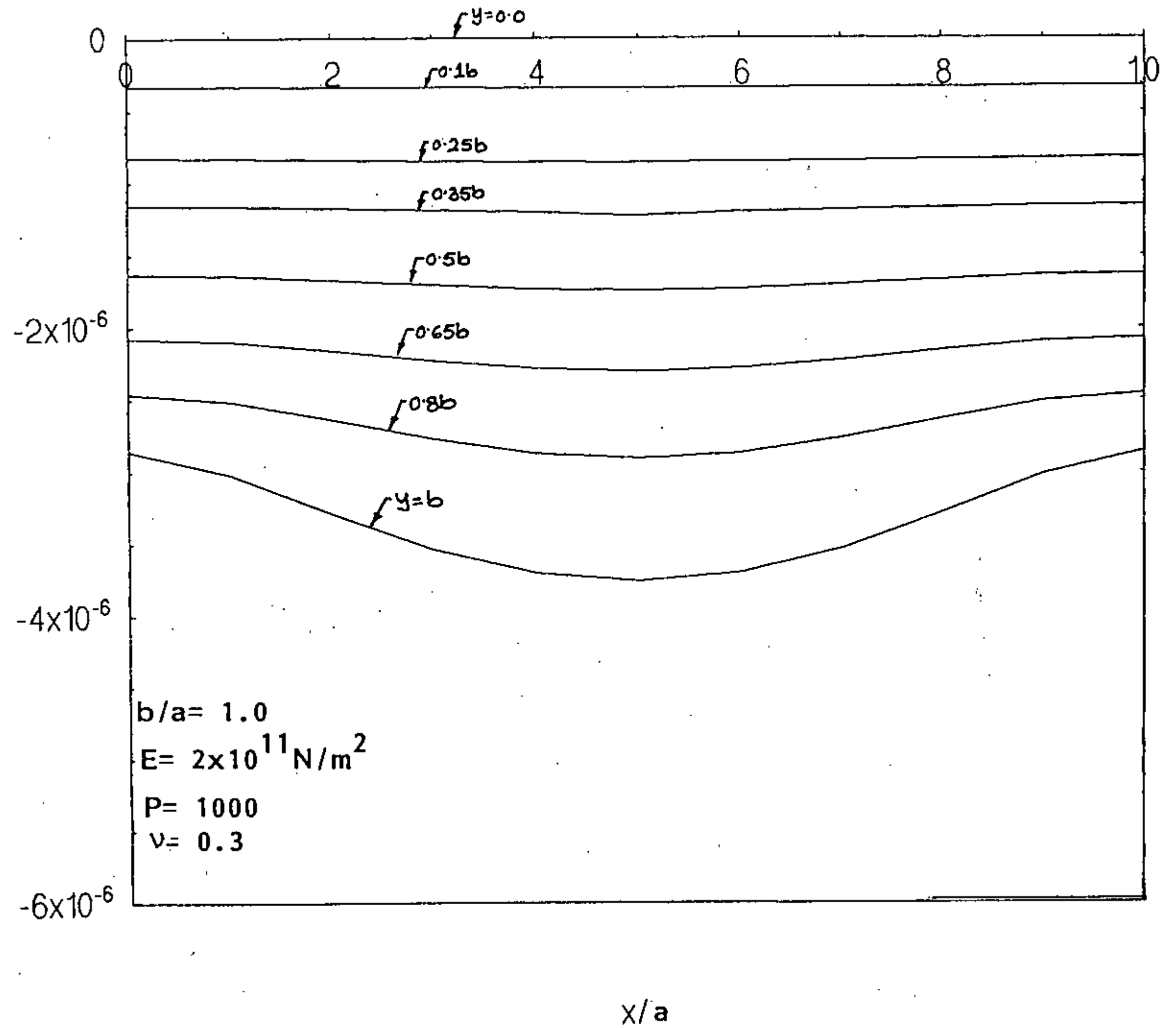

Fig 6. 2: Distribution of displacement component v at different sections of the plate

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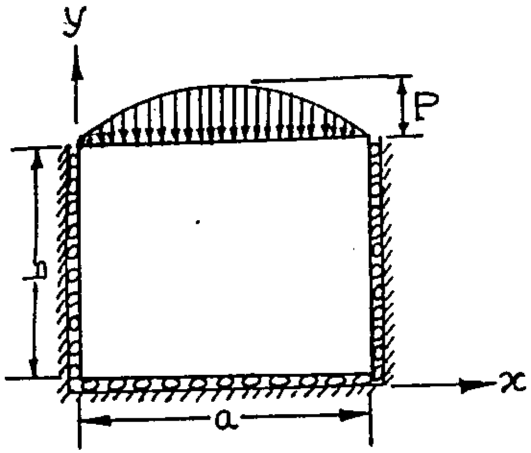


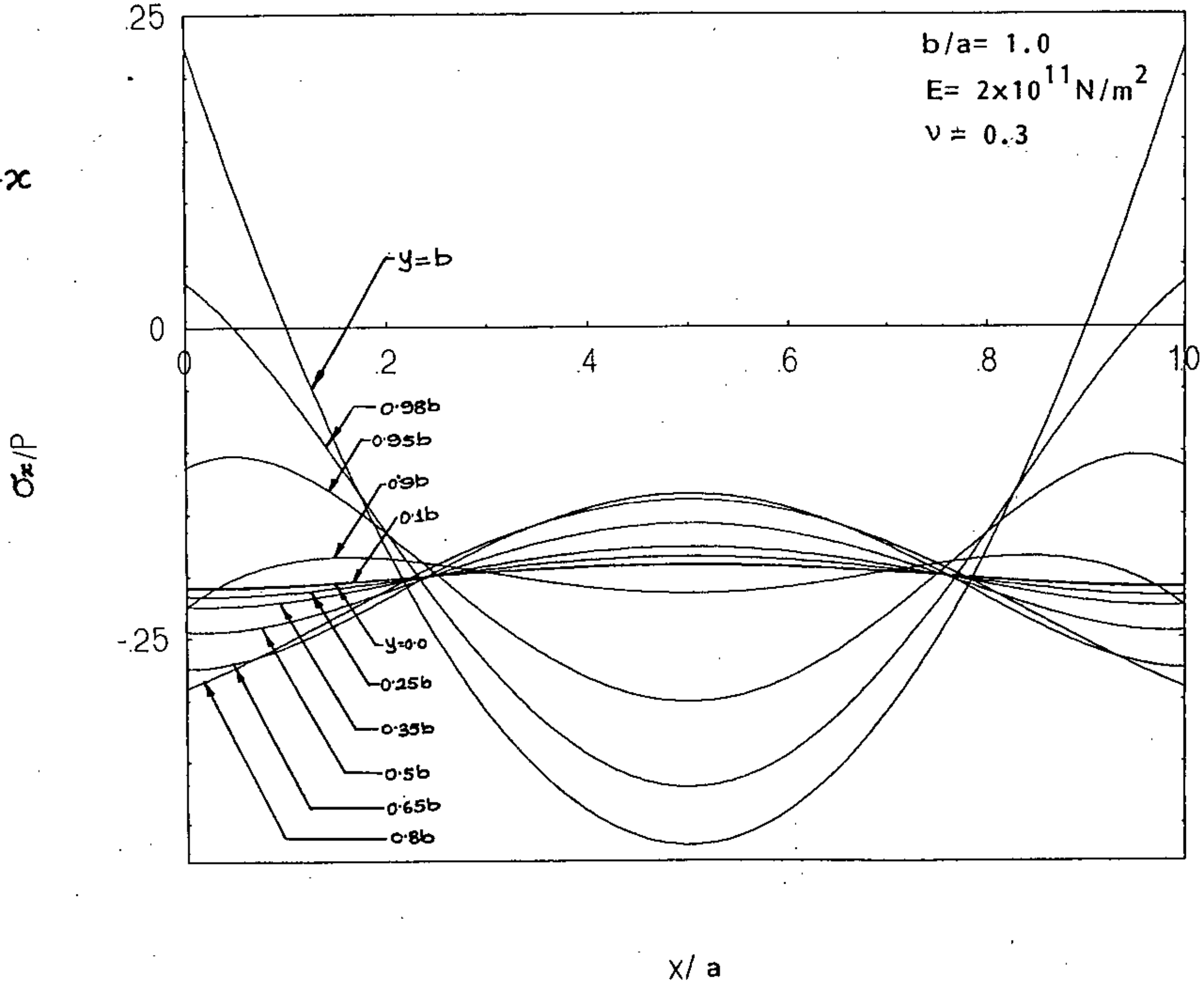
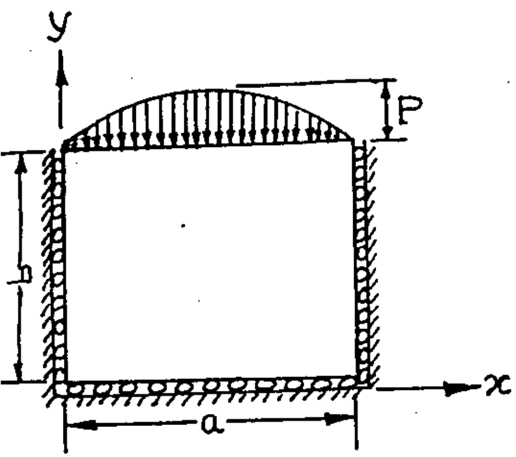


Fig 6.3: Distribution of stress component σ_x at different sections of the plate



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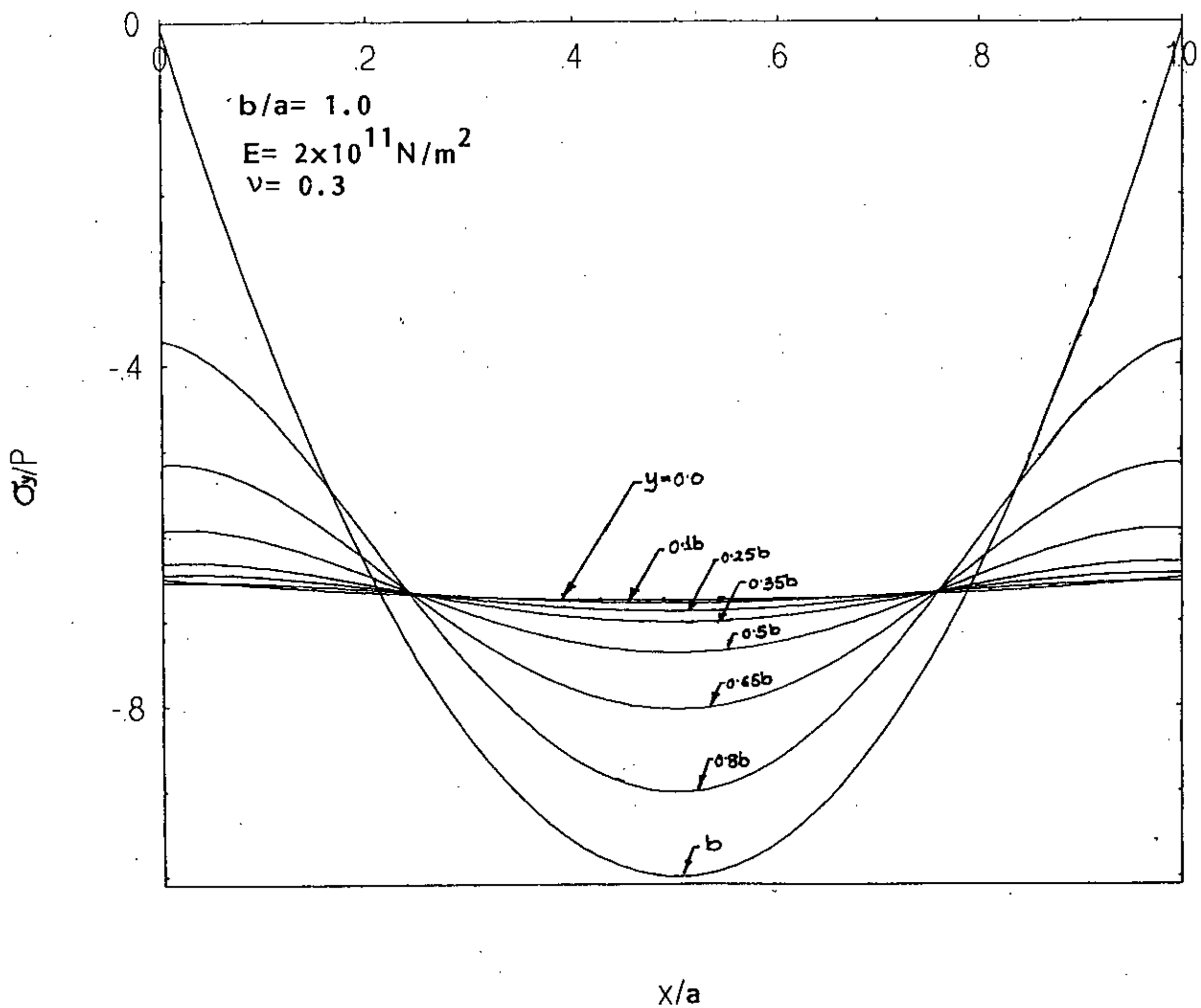

Fig 6.4: Distribution of stress component σ_y at different sections of the plate

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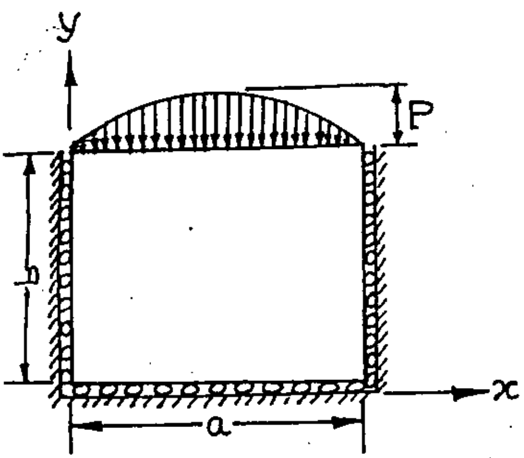


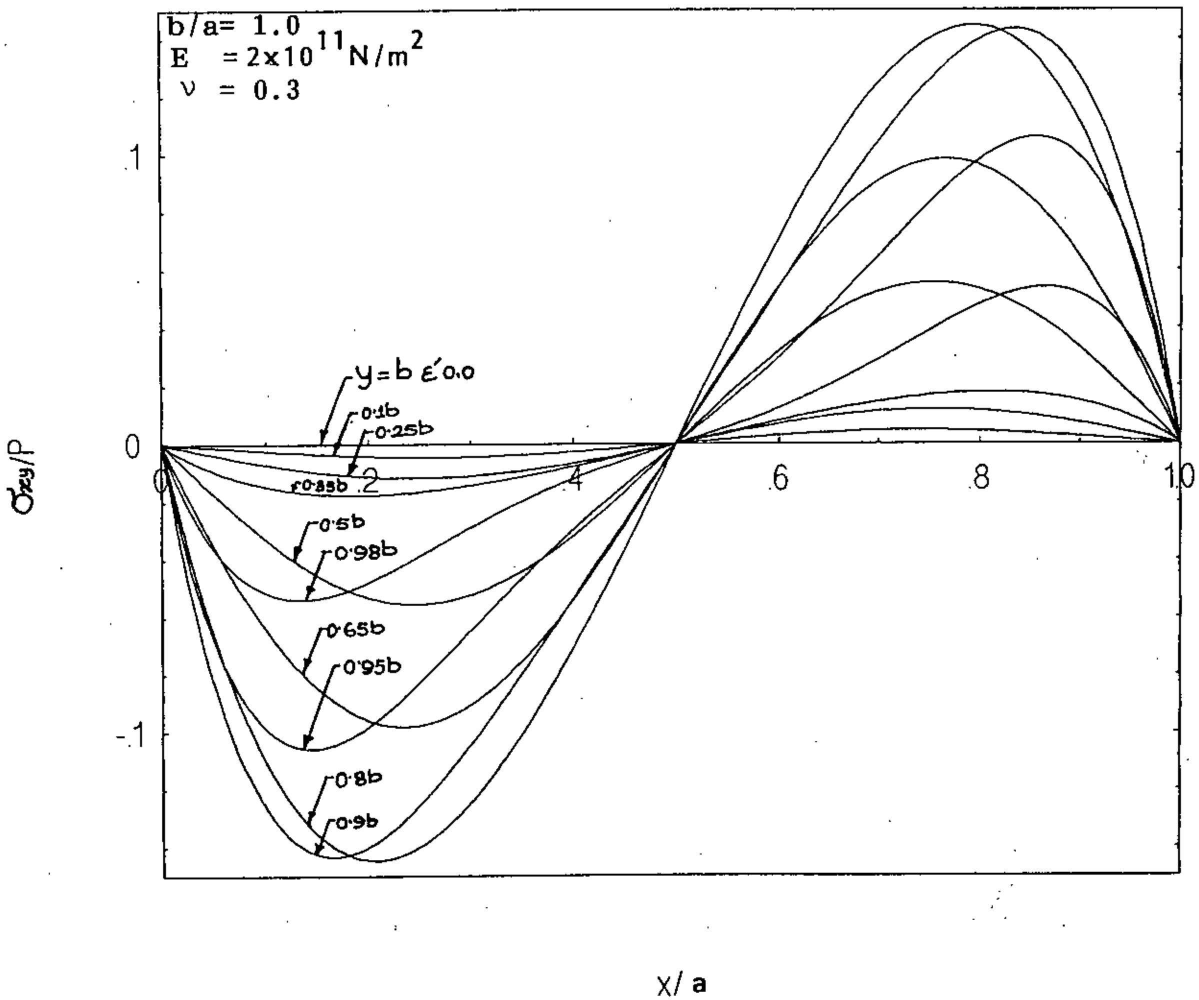
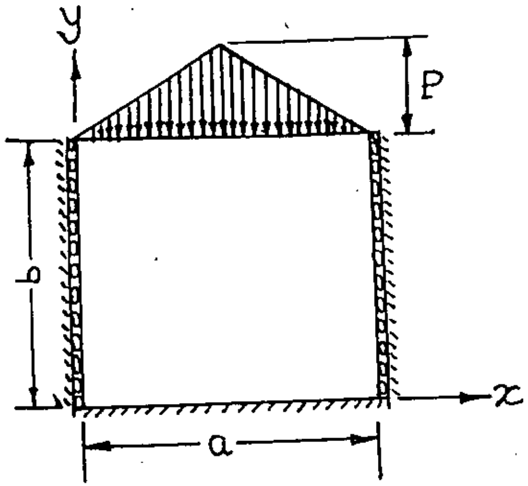


Fig 6.5: Distribution of stress component σ_{xy} at different sections of the plate



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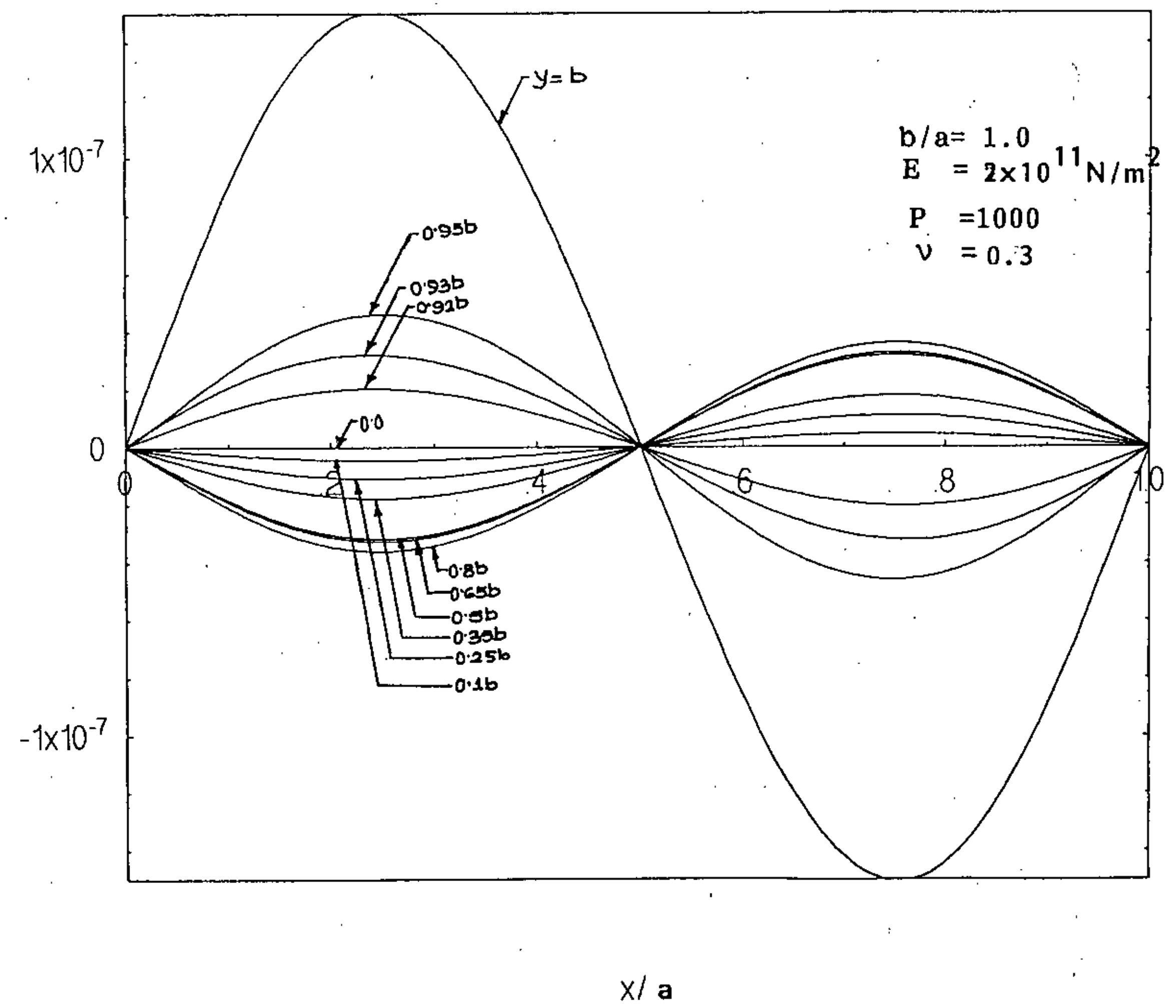
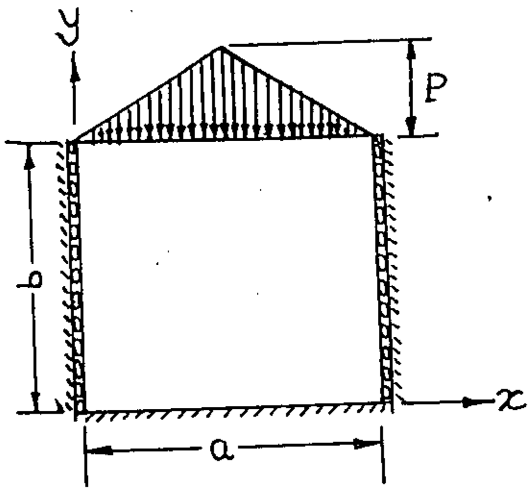


Fig 6.6: Distribution of displacement component u at different sections of the plate



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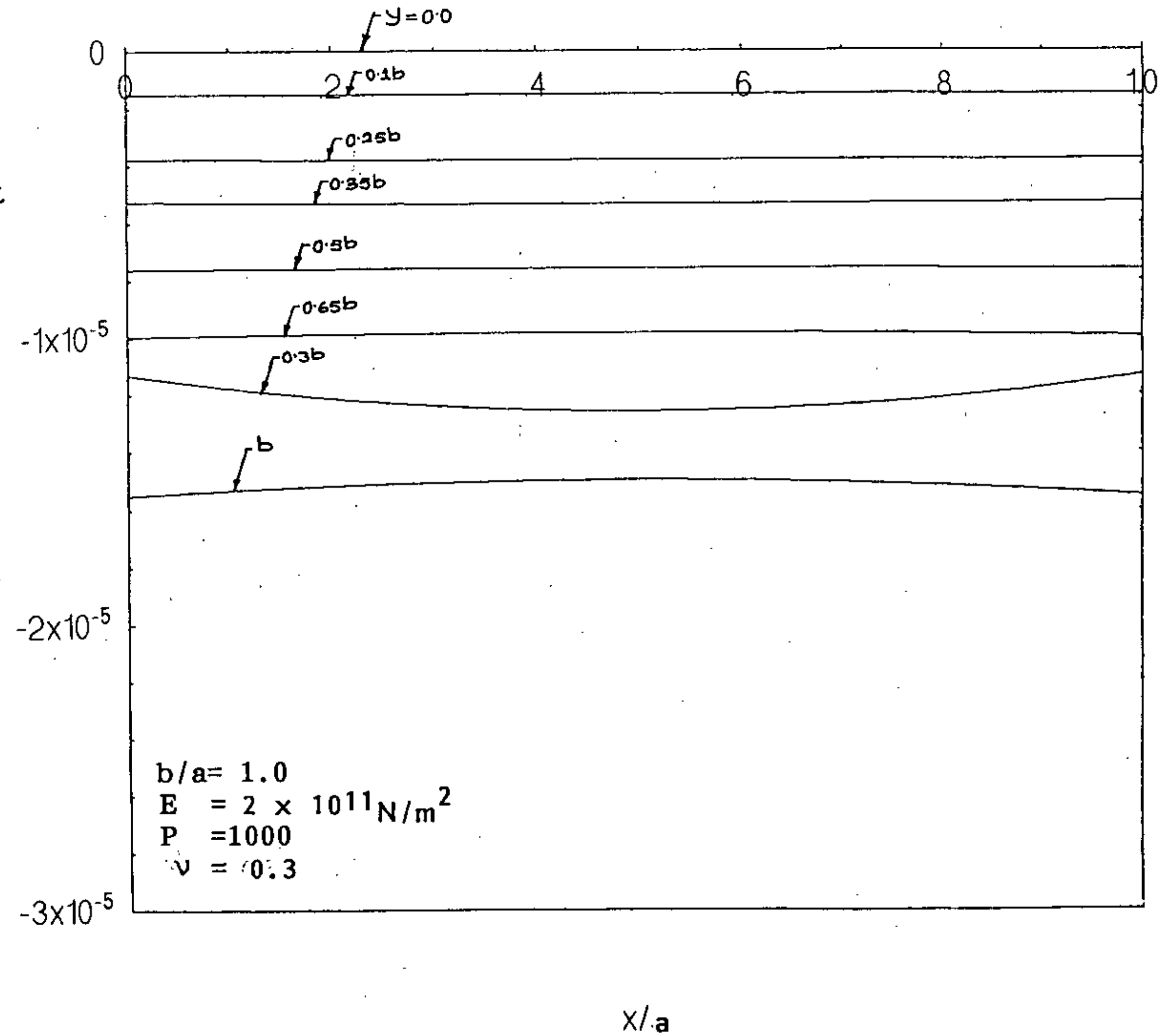
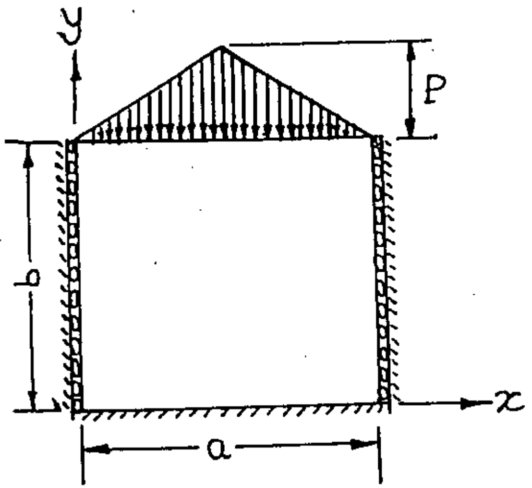


Fig. 6.7: Distribution of displacement component v at different sections of the plate



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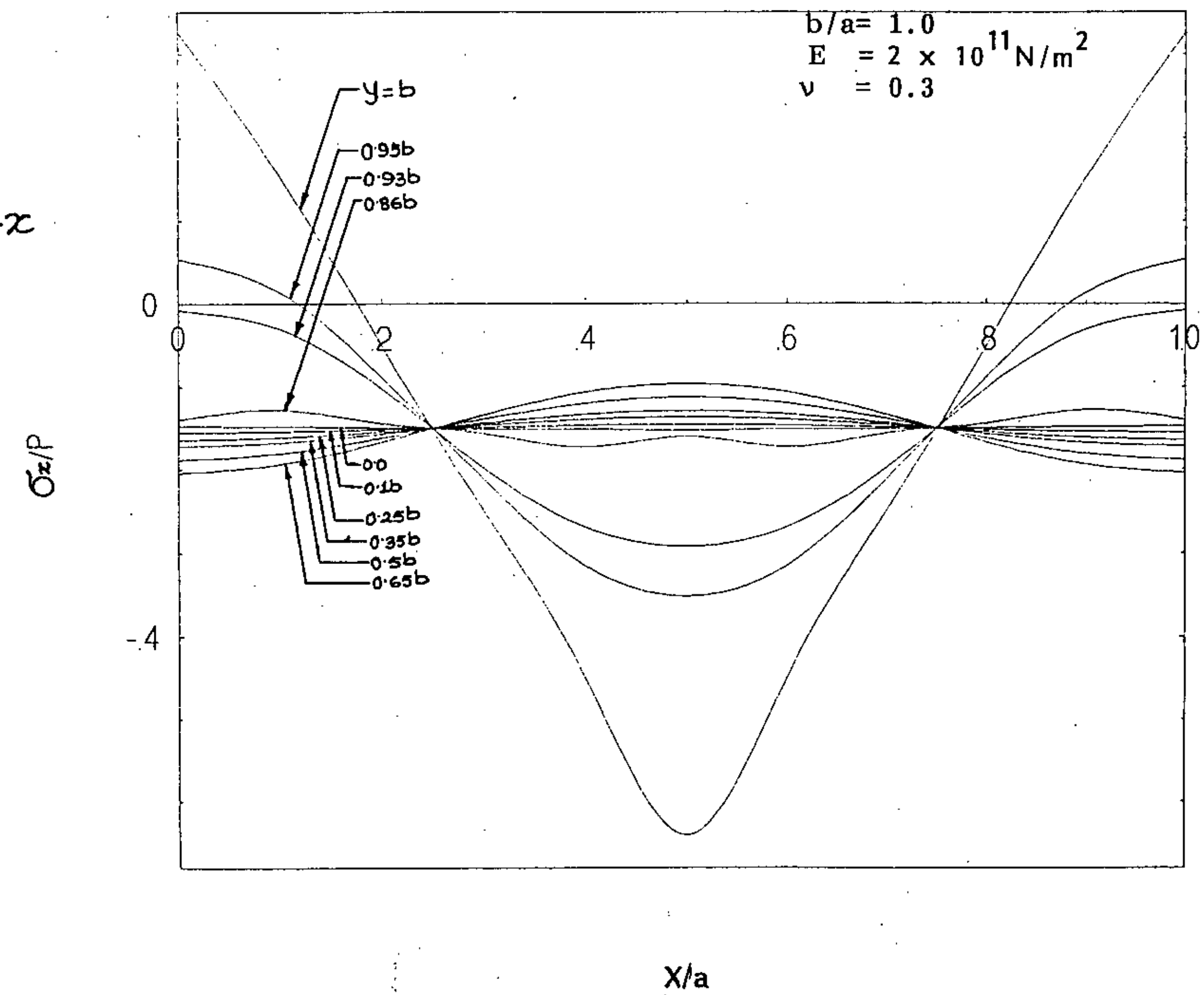
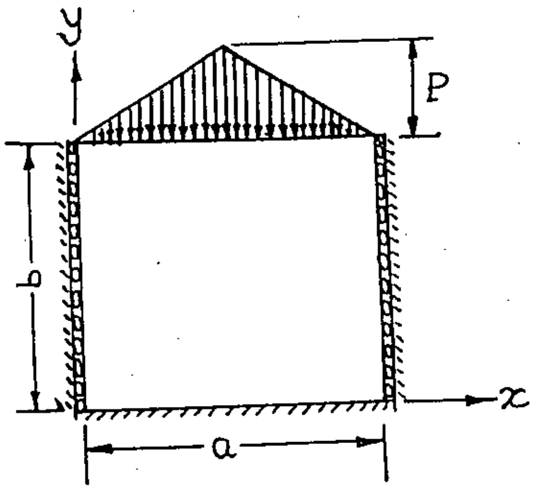


Fig. 6.8: Distribution of stress component σ_x at different sections of the plate



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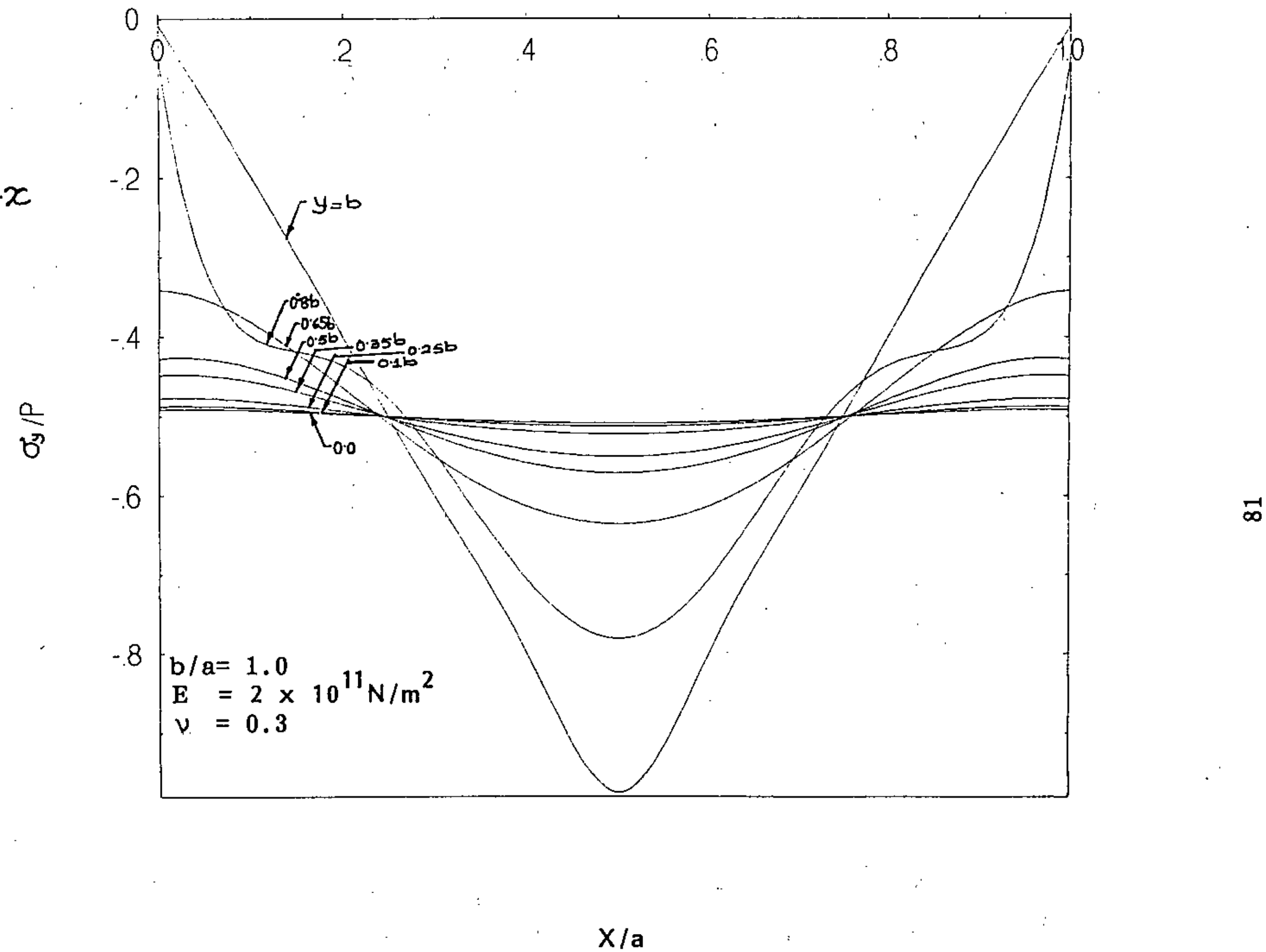
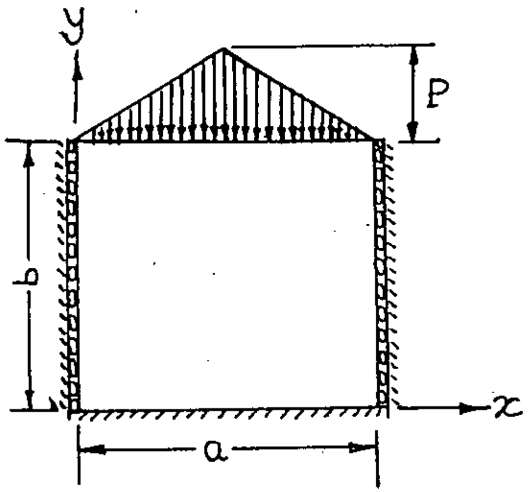


Fig. 6.9 : Distribution of stress component σ_y at different sections of the plate



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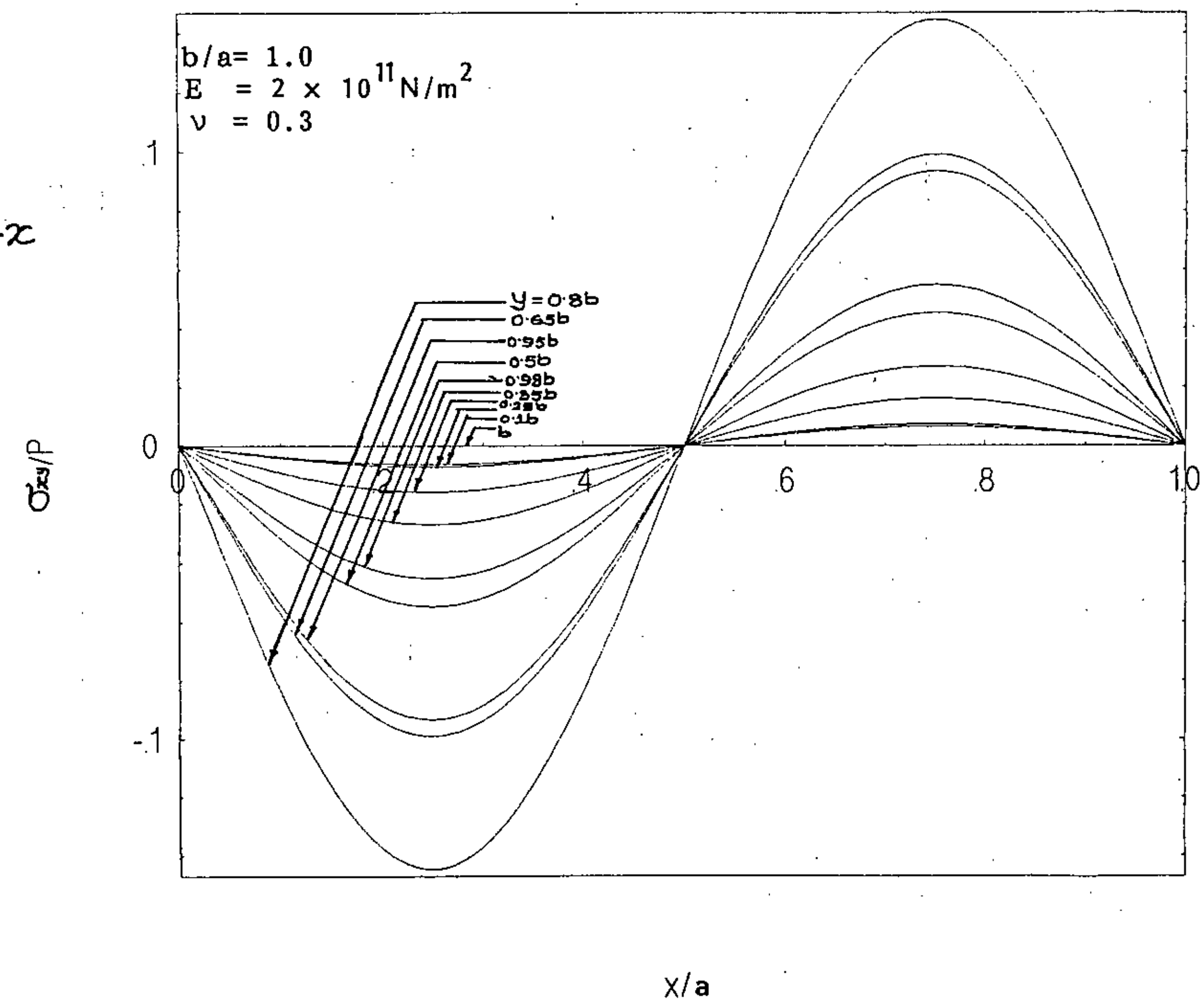

Fig.6.10: Distribution of stress component σ_{xy} at different sections of the plate

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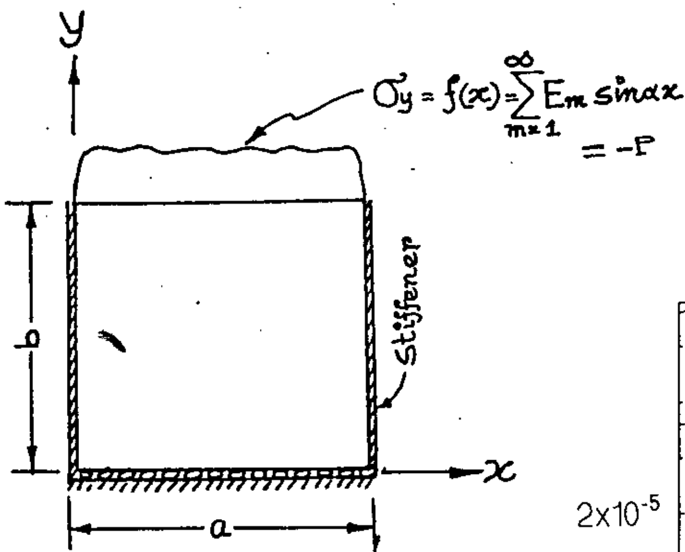

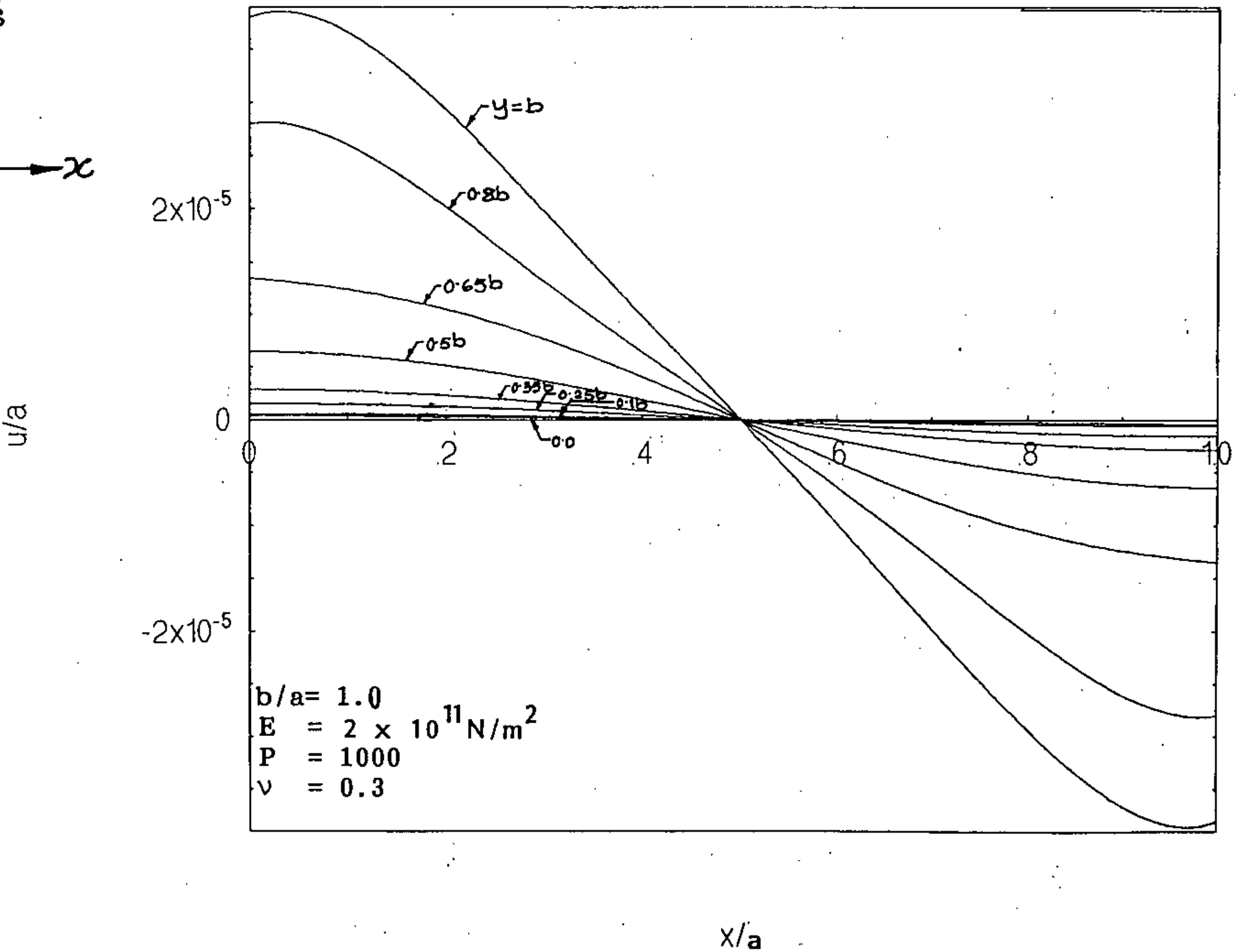

Fig. 6.11: Distribution of displacement component u at different sections of the plate

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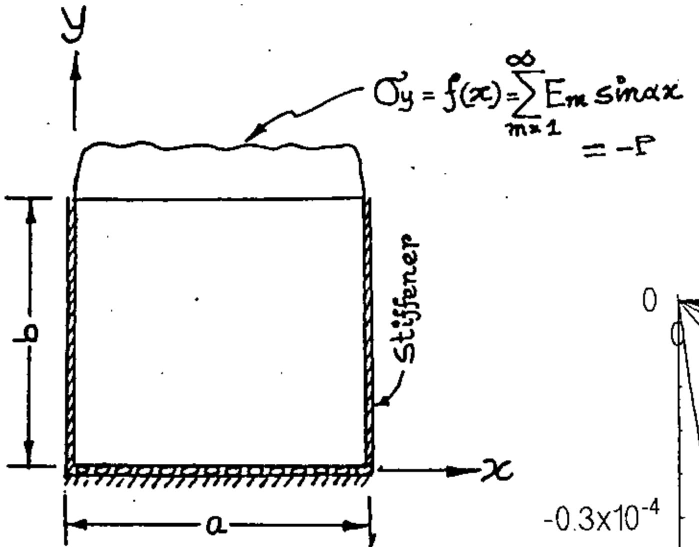


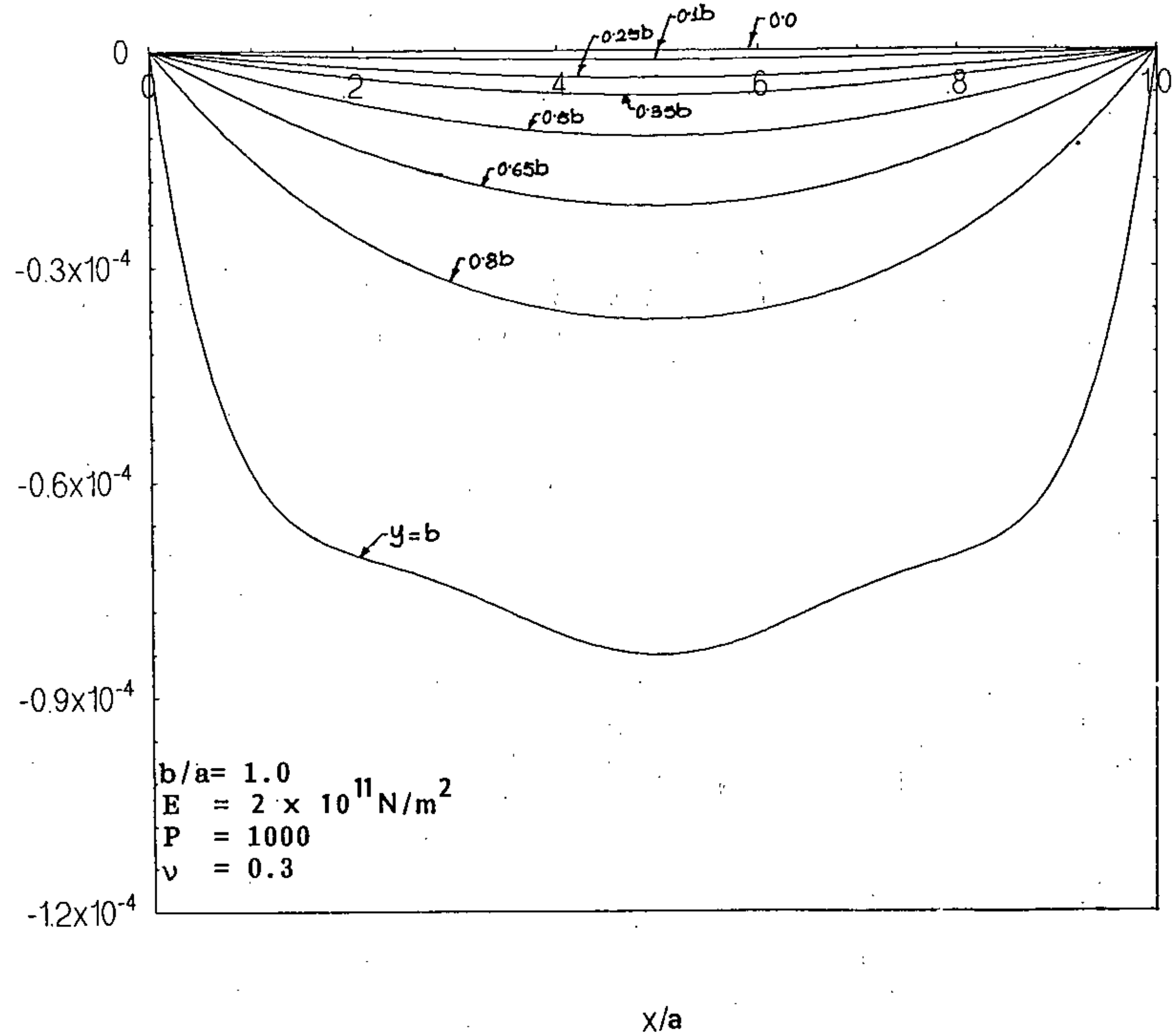


Fig.6.12: Distribution of displacement component v at different sections of the plate

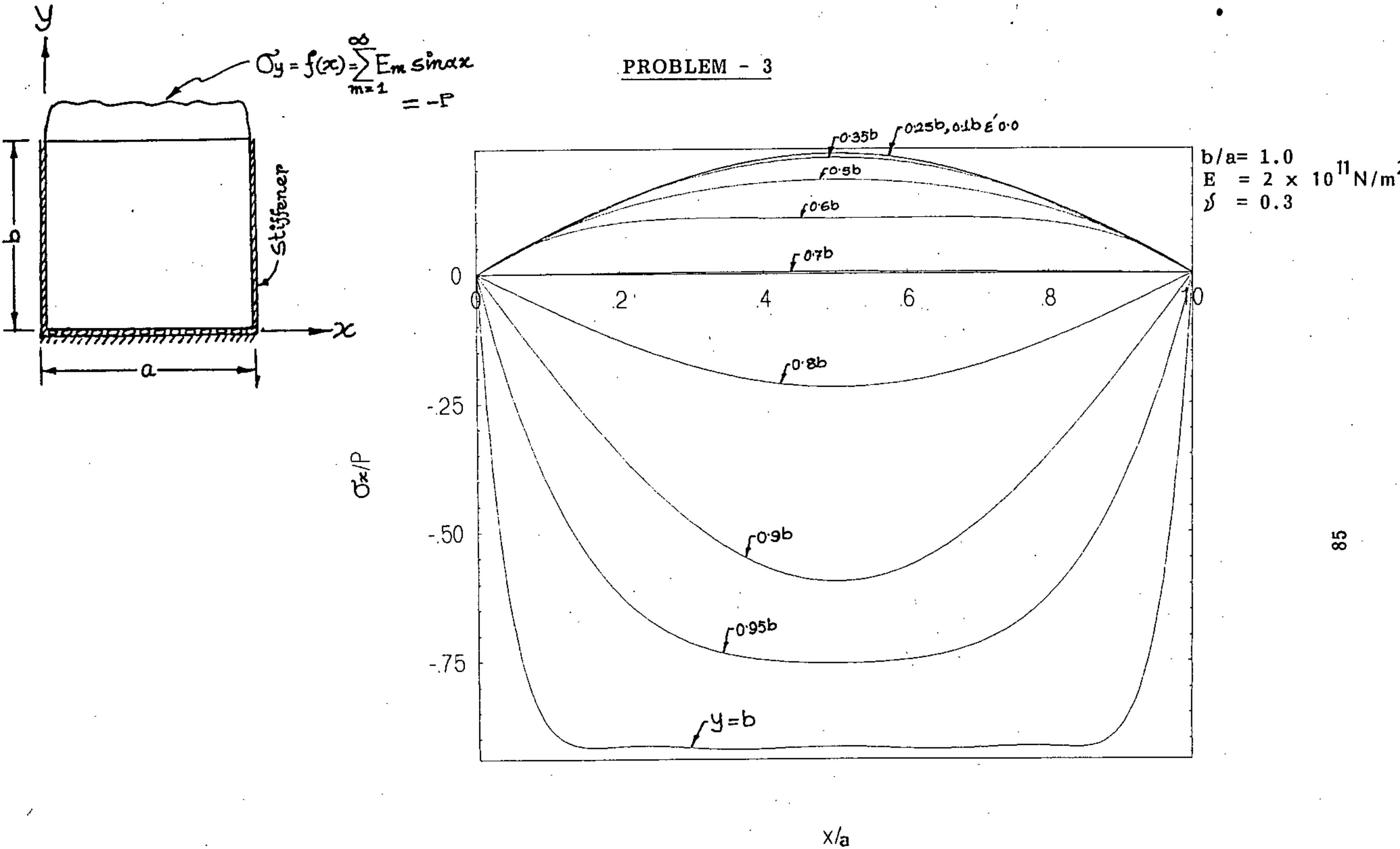
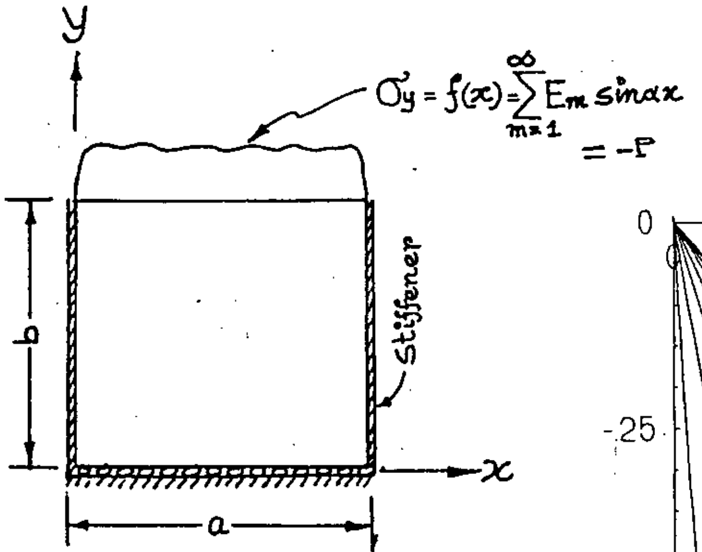


Fig.6.13: Distribution of stress component σ_x at different sections of the plate



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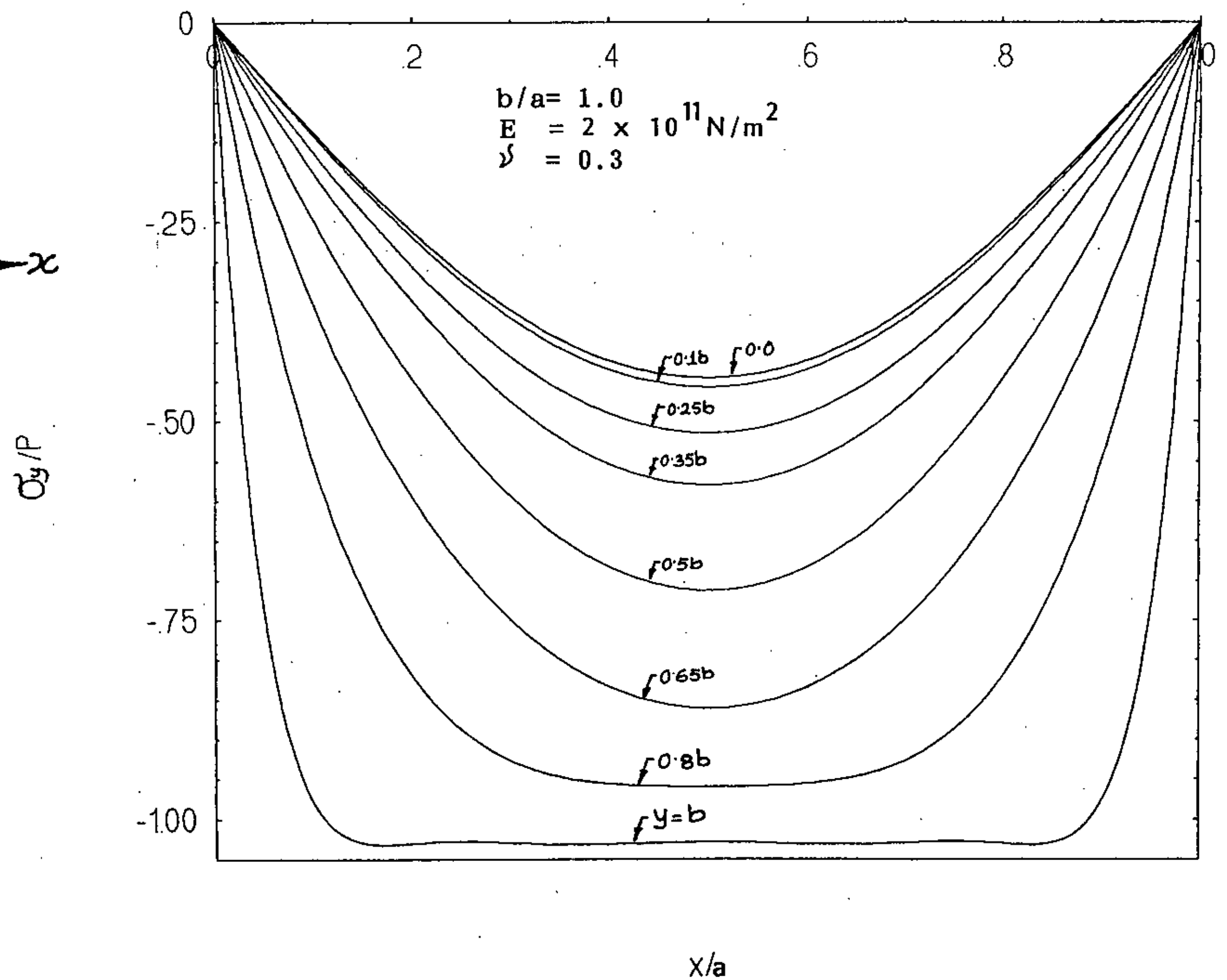

Fig.6.14: Distribution of stress component σ_y at different sections of the plate

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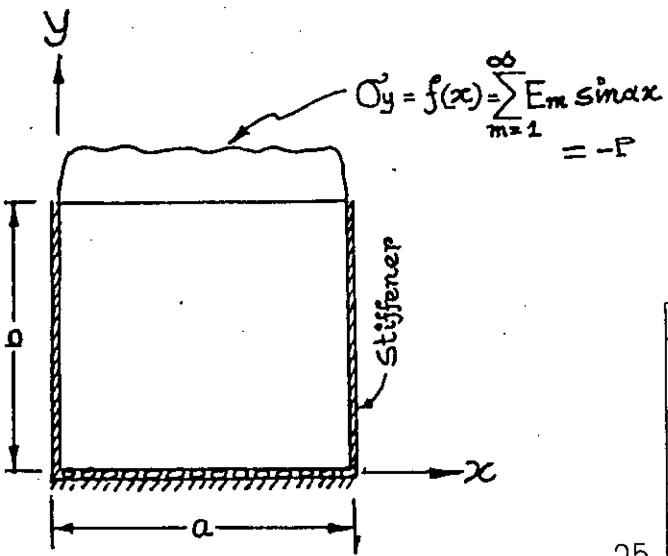

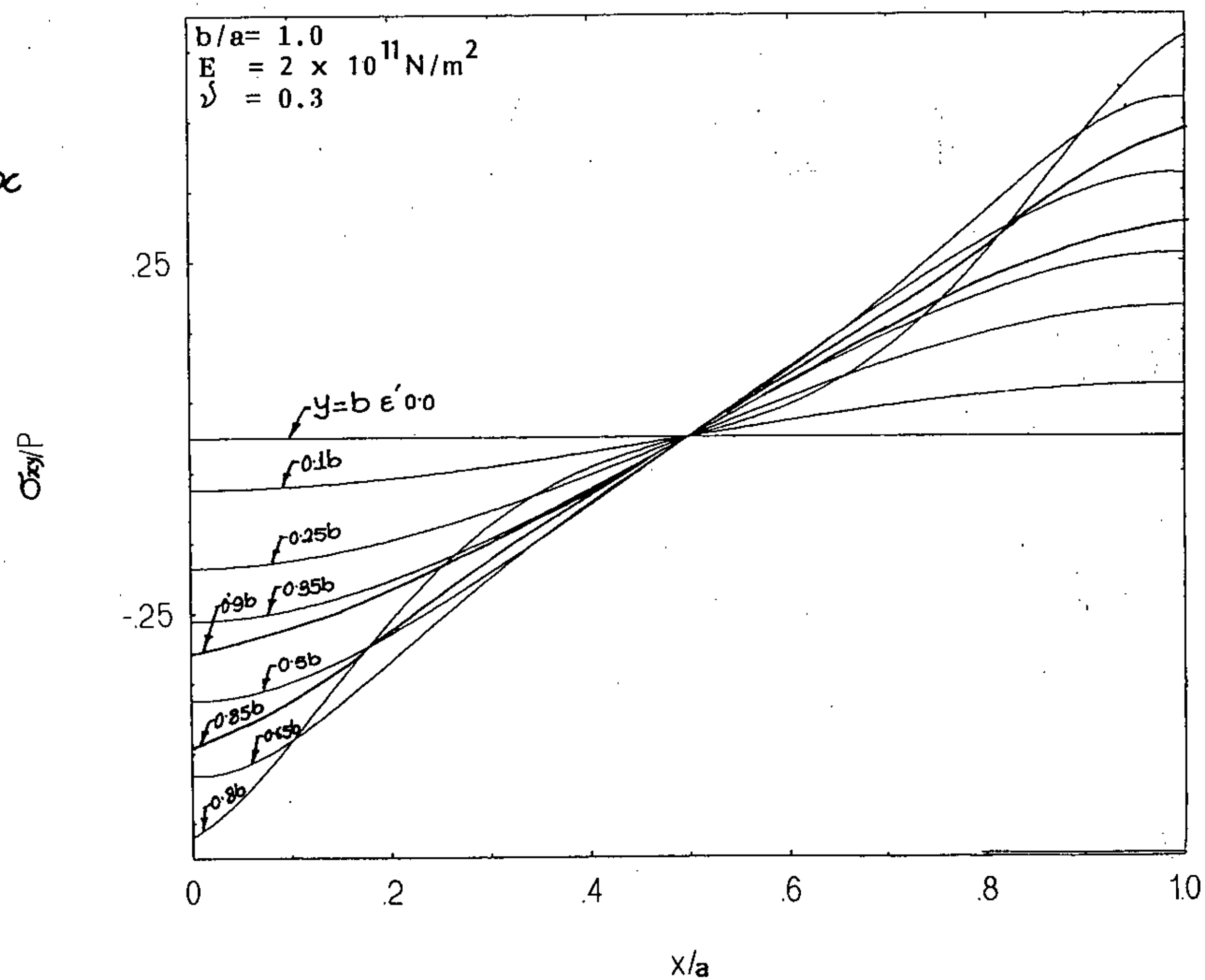

Fig.6.15: Distribution of stress component σ_{xy} at different sections of the plate

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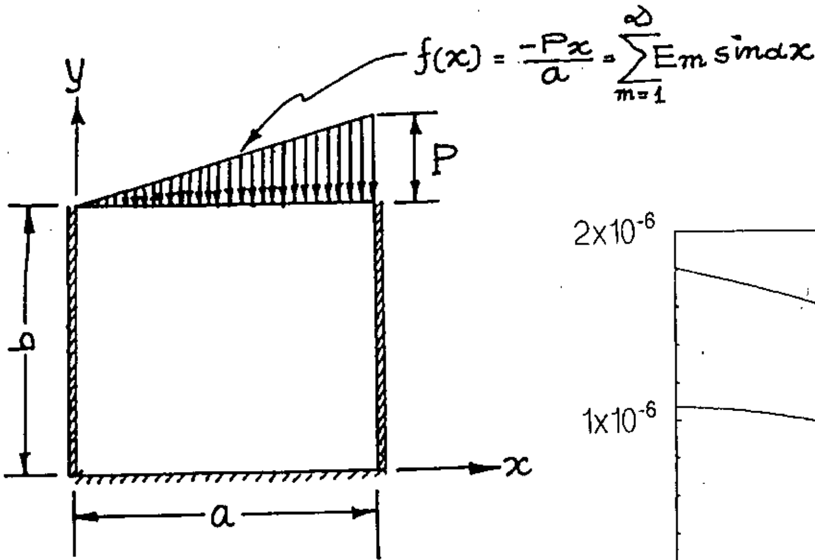


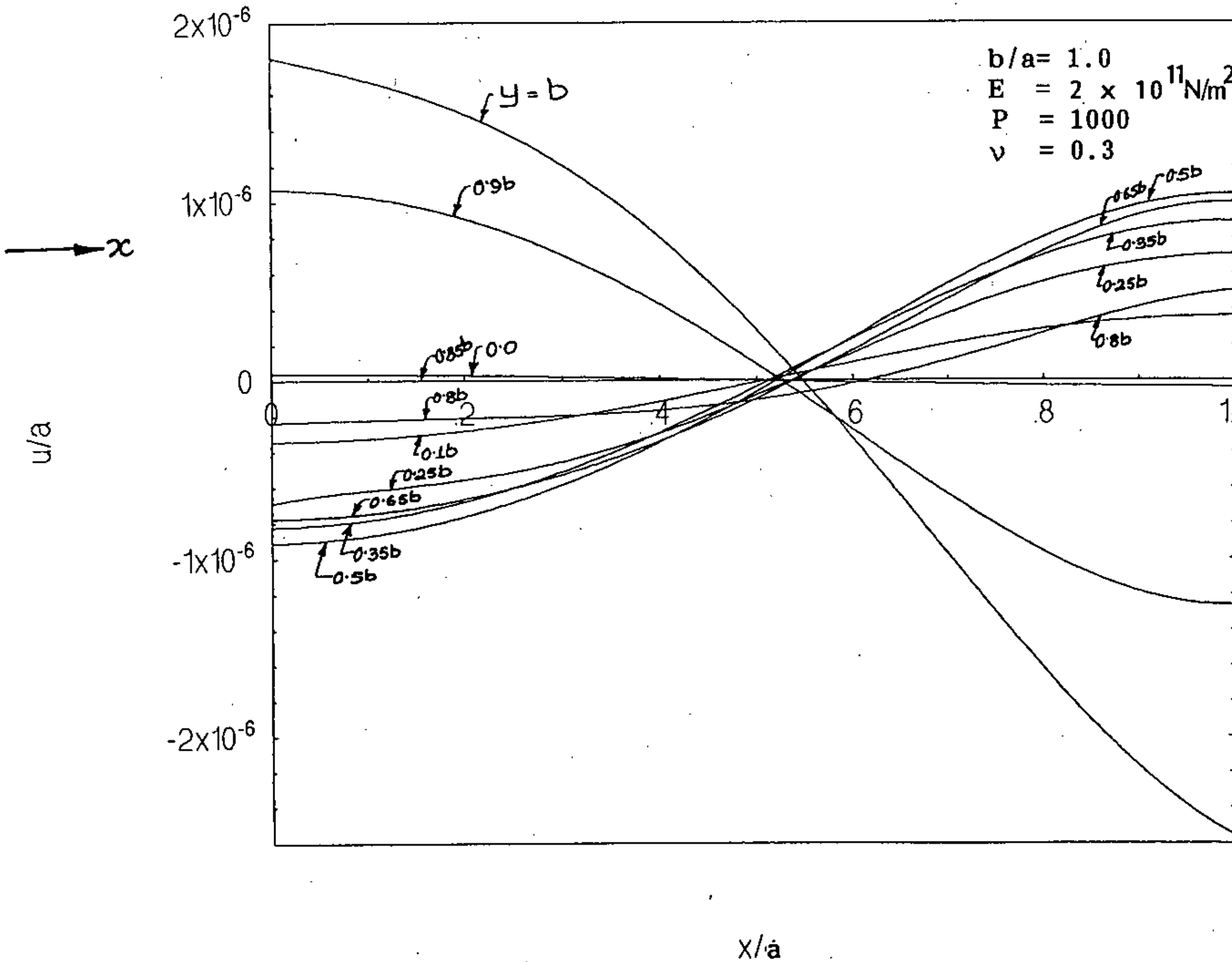
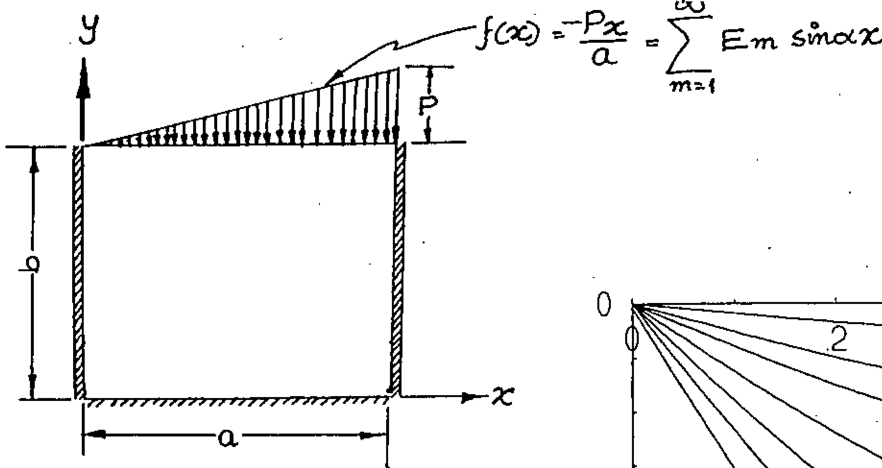


Fig.6.16: Distribution of displacement component u at different sections of the plate



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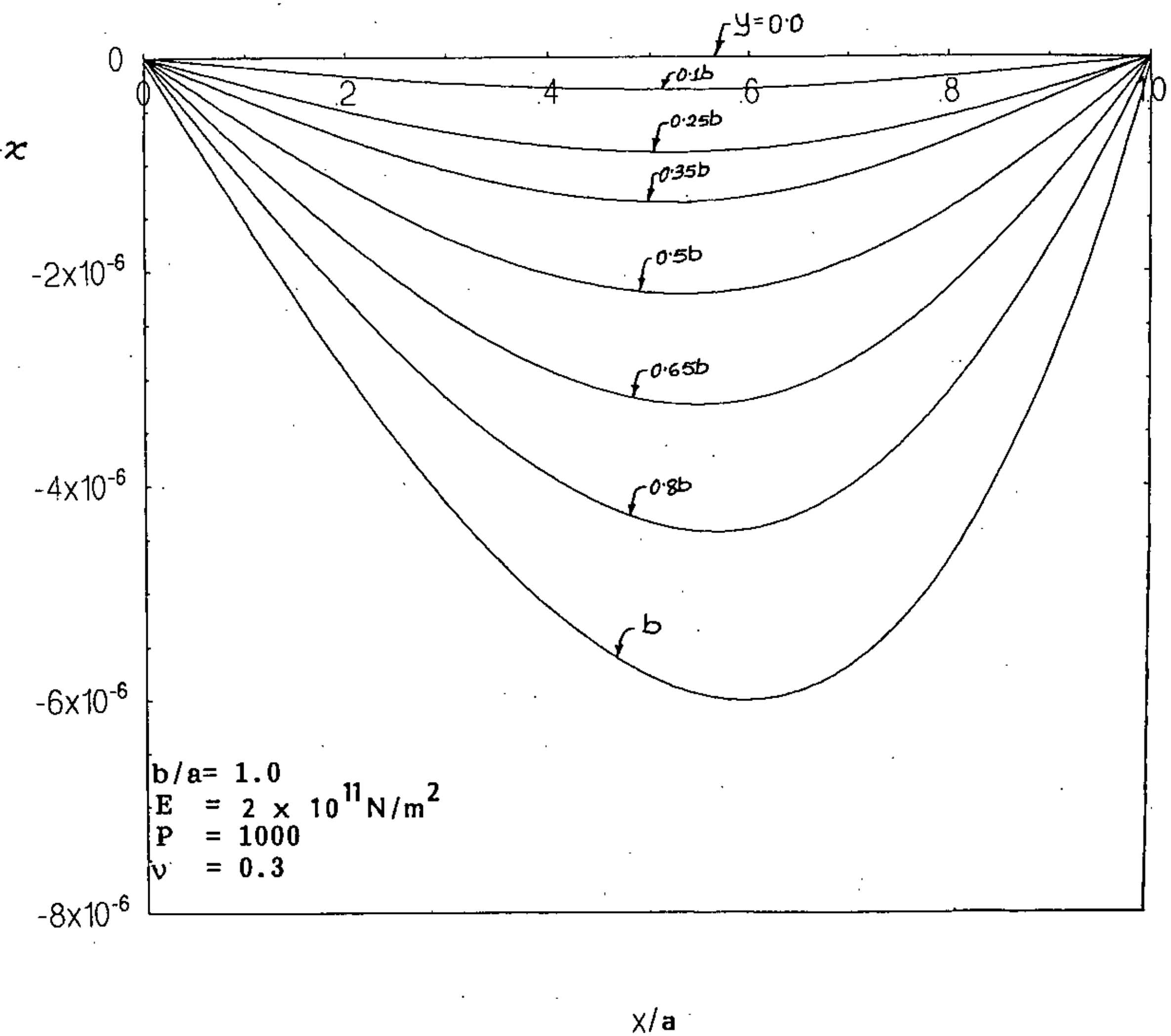
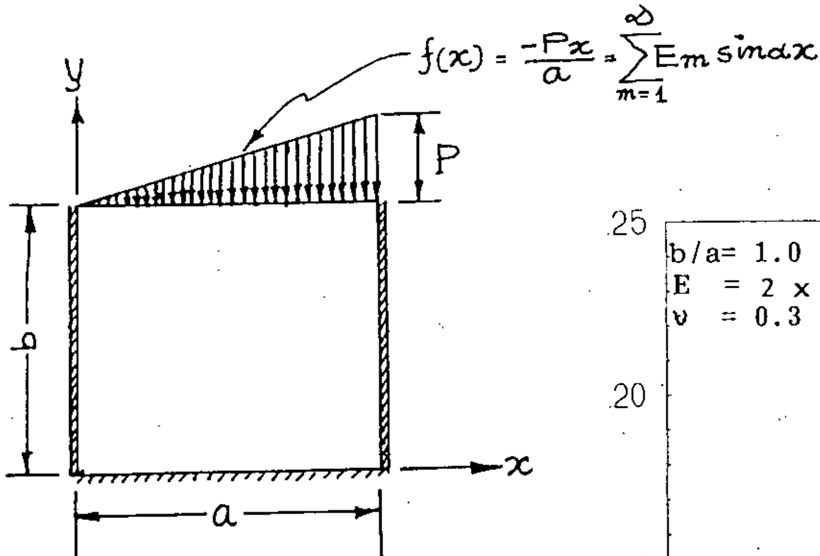


Fig. 6.17: Distribution of displacement component v at different sections of the plate



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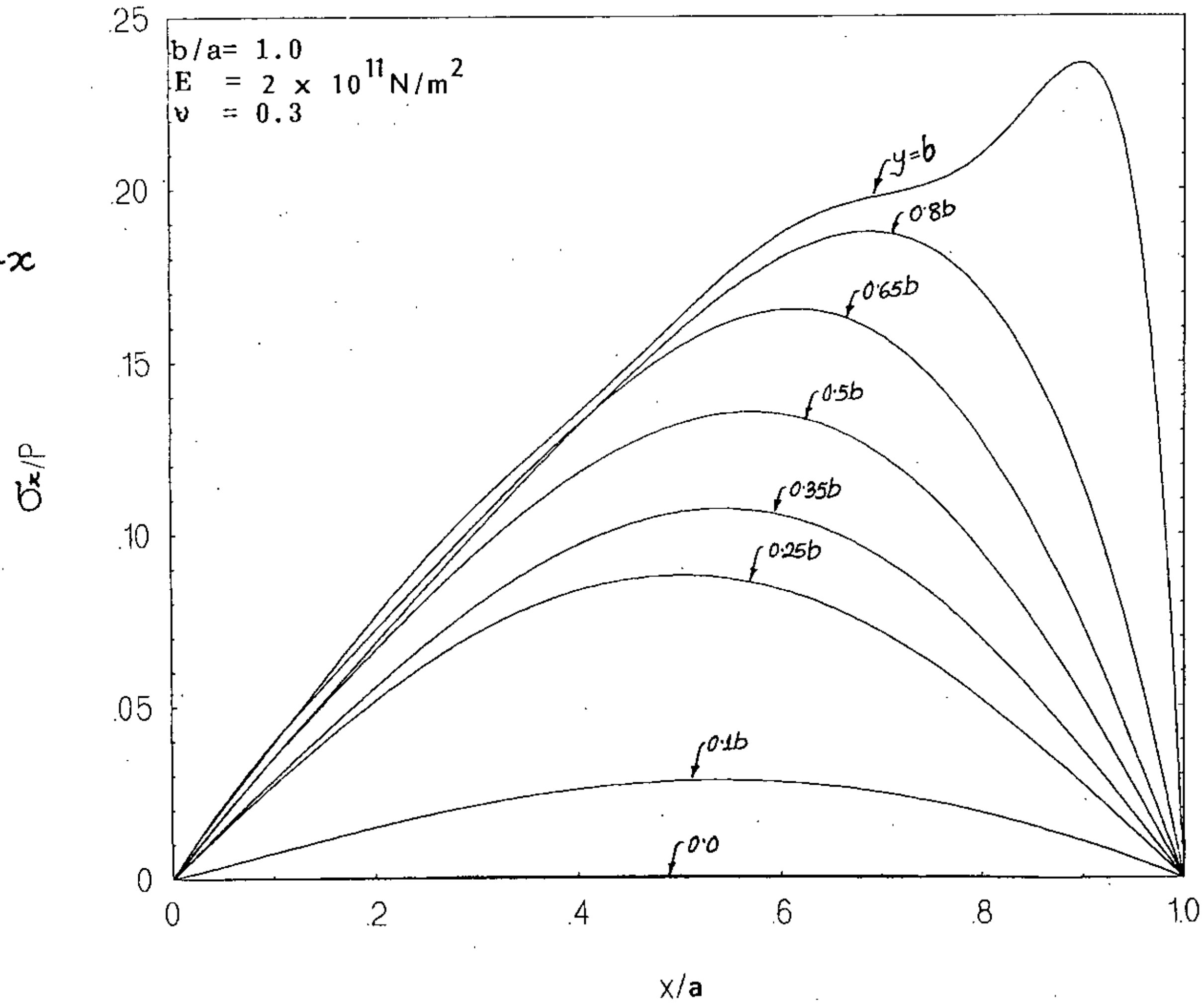
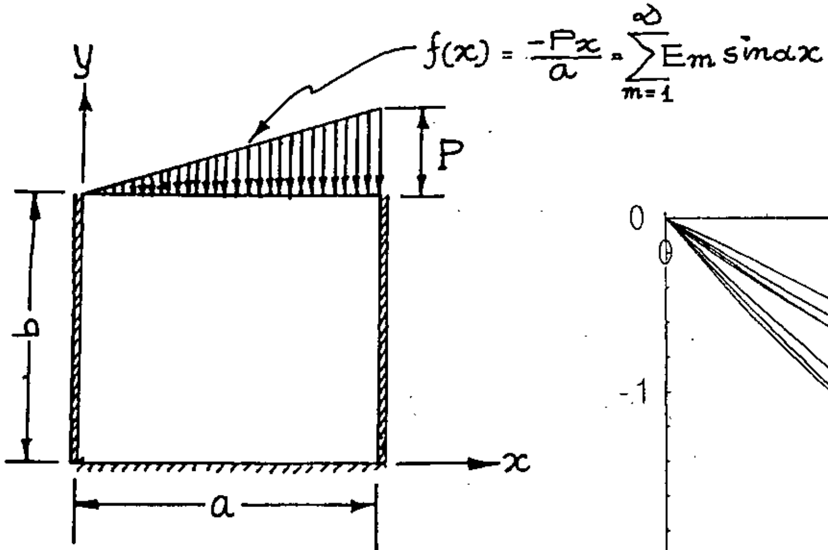


Fig.6.18: Distribution of stress component σ_x at different sections of the plate



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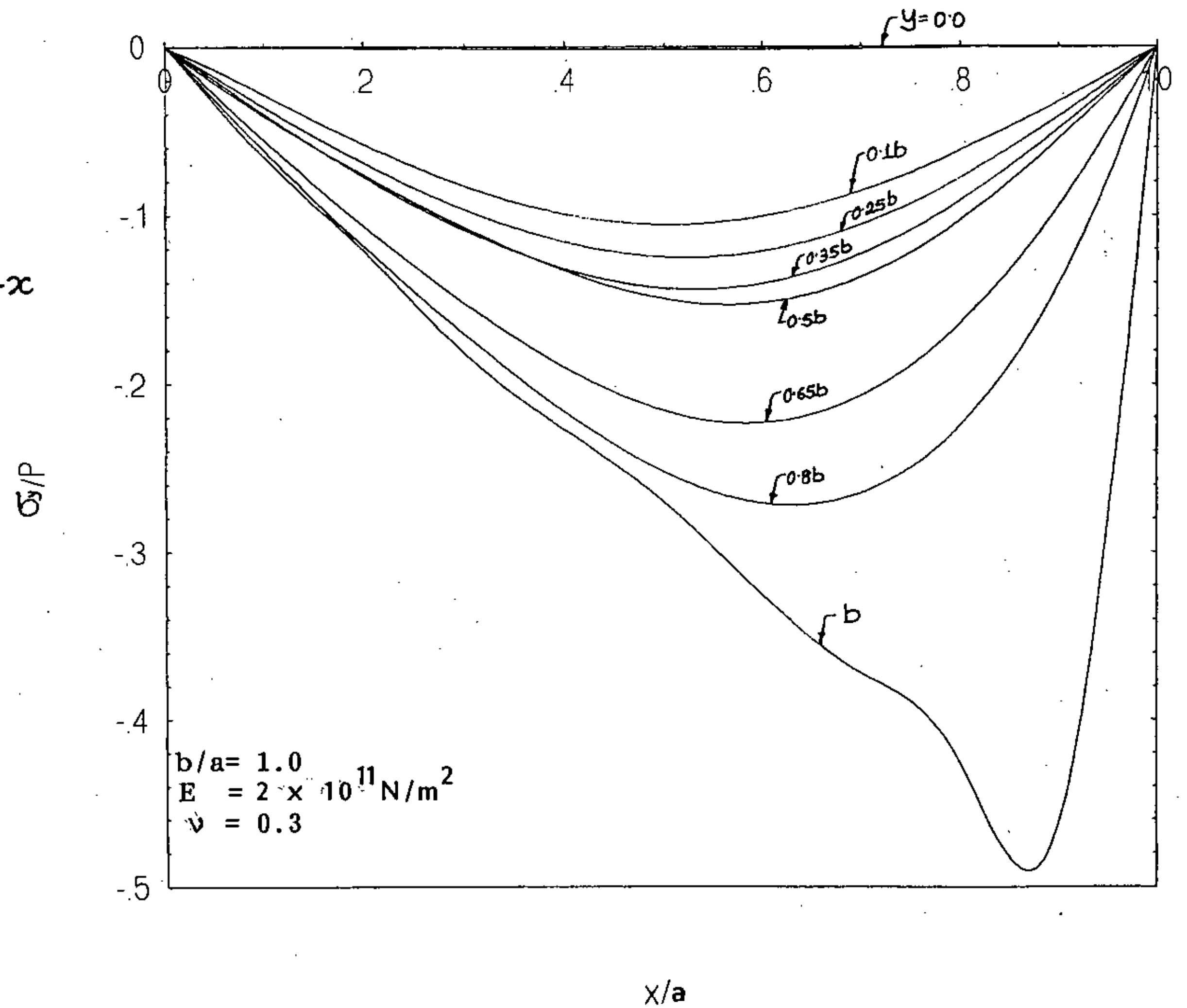
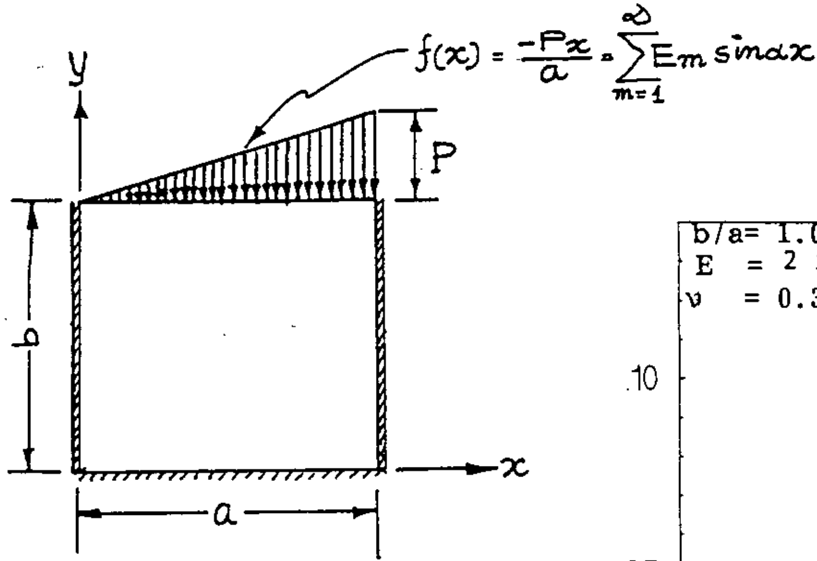


Fig.6.19: Distribution of stress component σ_y at different sections of the plate



PROBLEM - 4

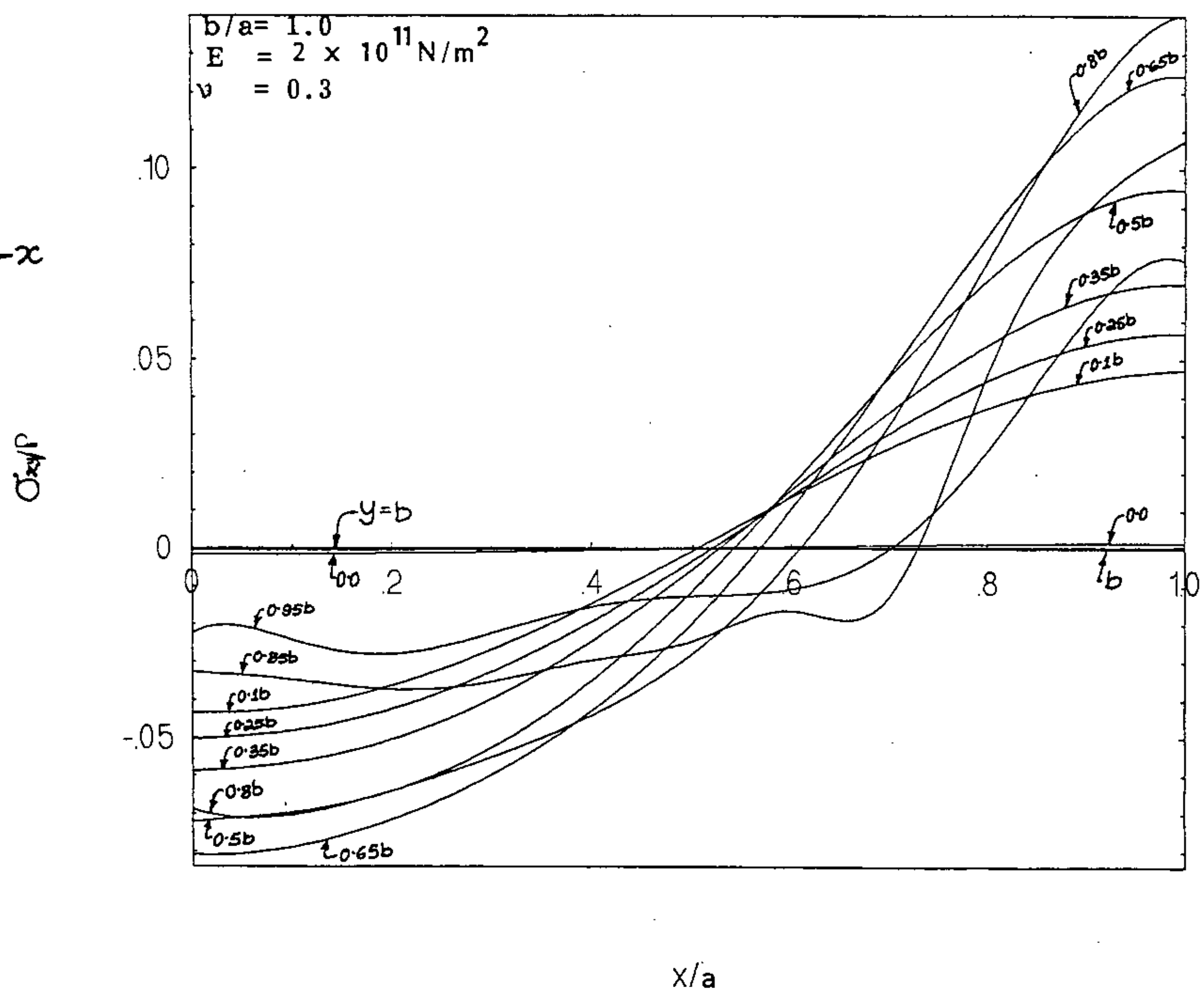
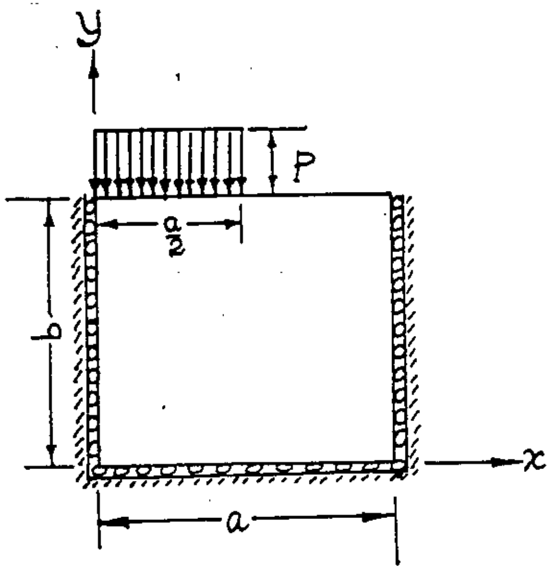


Fig.6.20: Distribution of stress component σ_{xy} at different sections of the plate

PROBLEM - 5



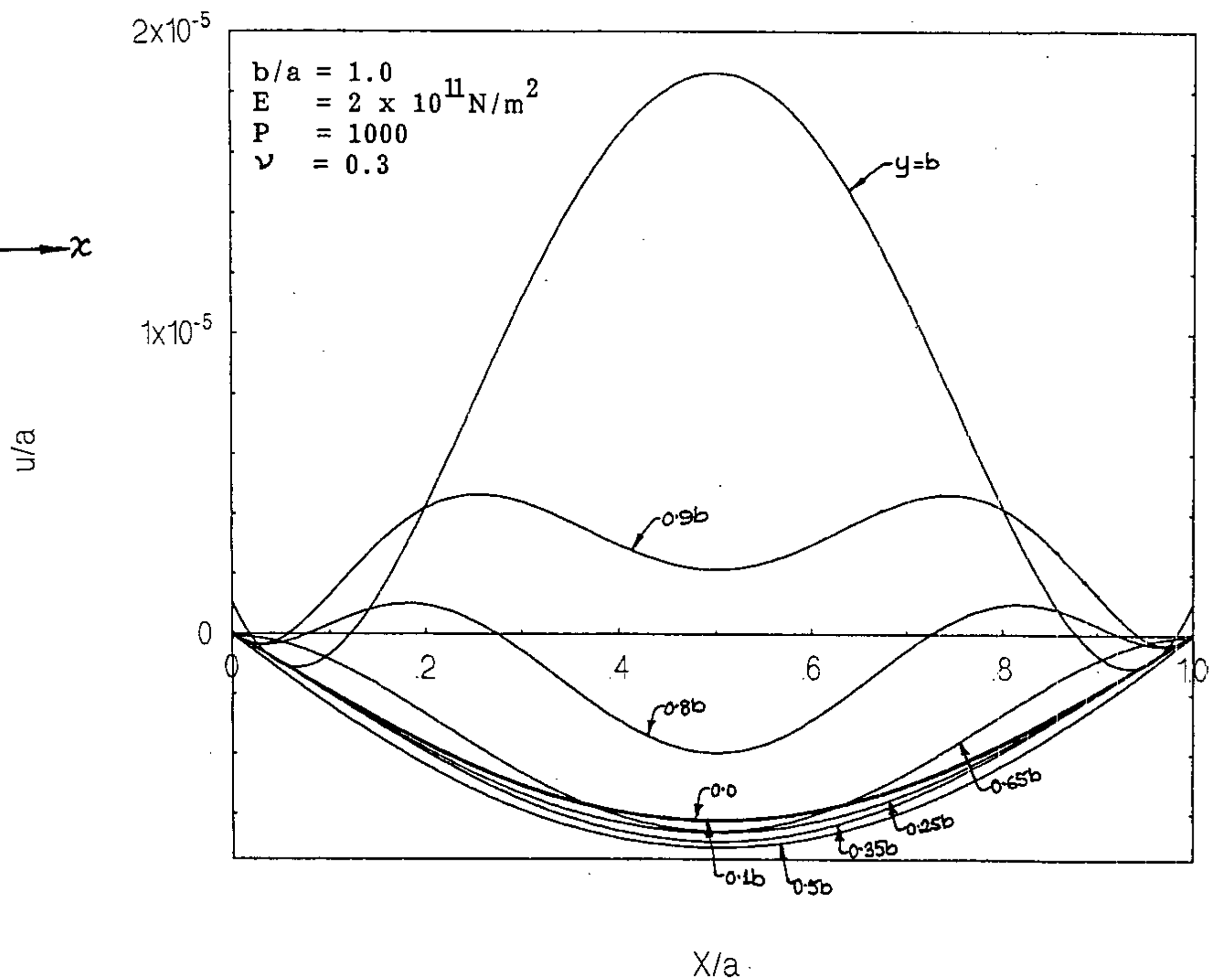
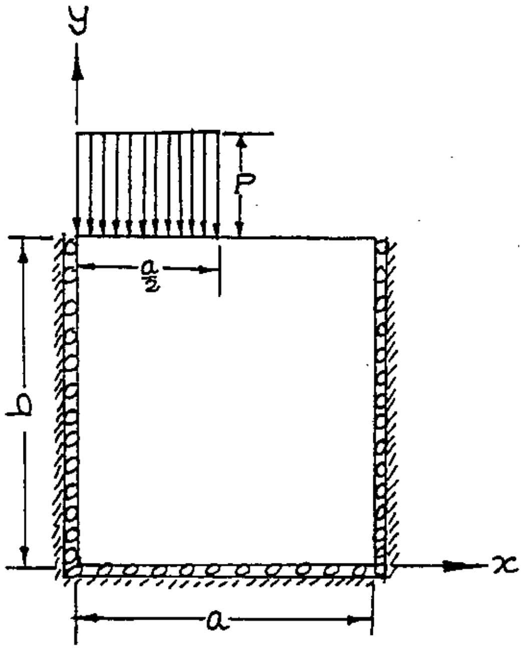


Fig. 6.21: Distribution of displacement component u at different sections of the plate.

PROBLEM - 5



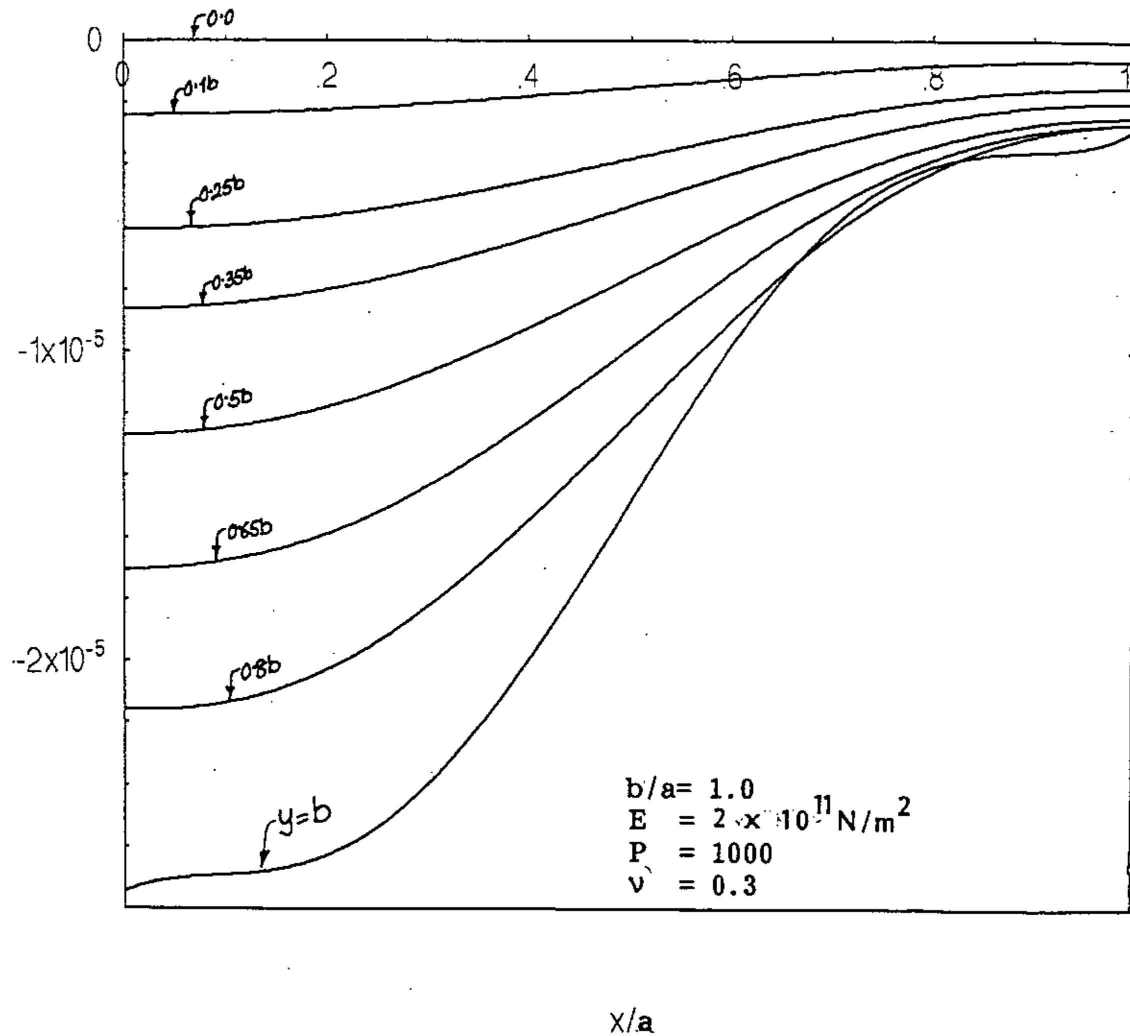
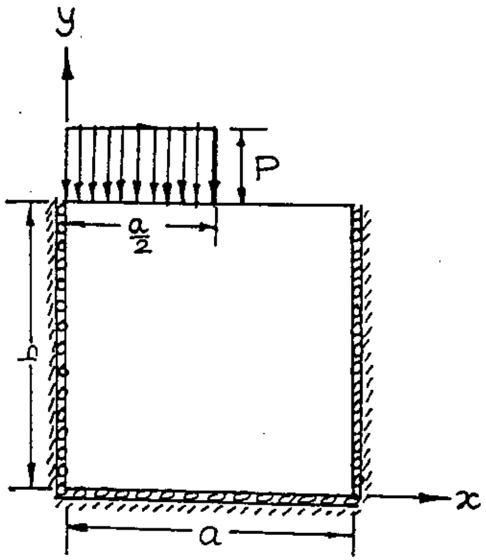
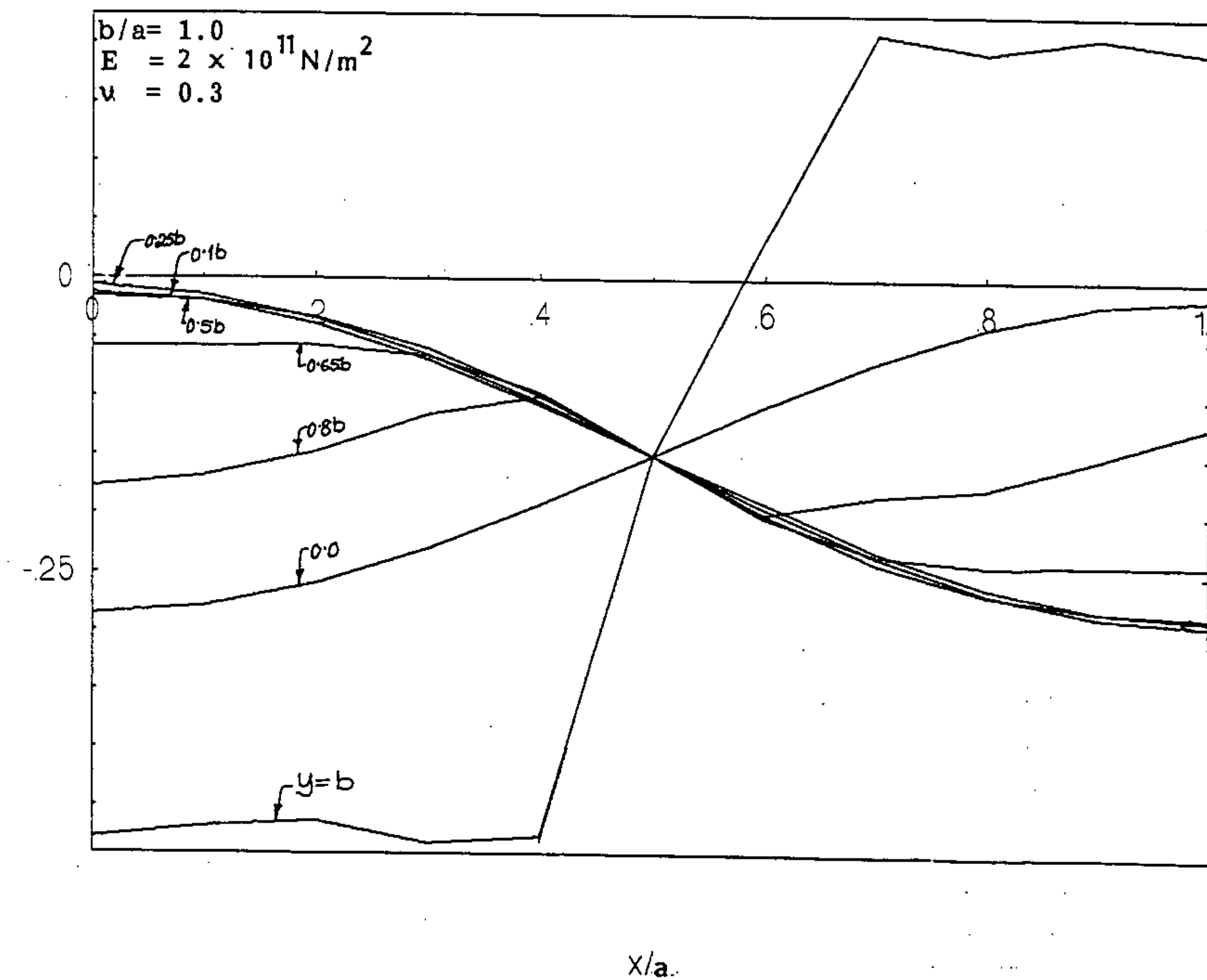


Fig 6.2[illegible]: Distri[illegible]

PROBLEM - 5



y
P
a/2
b
x
a



b/a= 1.0
E = 2 x 10^11 N/m^2
ν = 0.3
0.25b
0.1b
0.5b
0.65b
0.8b
0.0
y=b
0
-.25
0
.2
.4
.6
.8
1.0
σx/P
X/a.

PROBLEM - 5

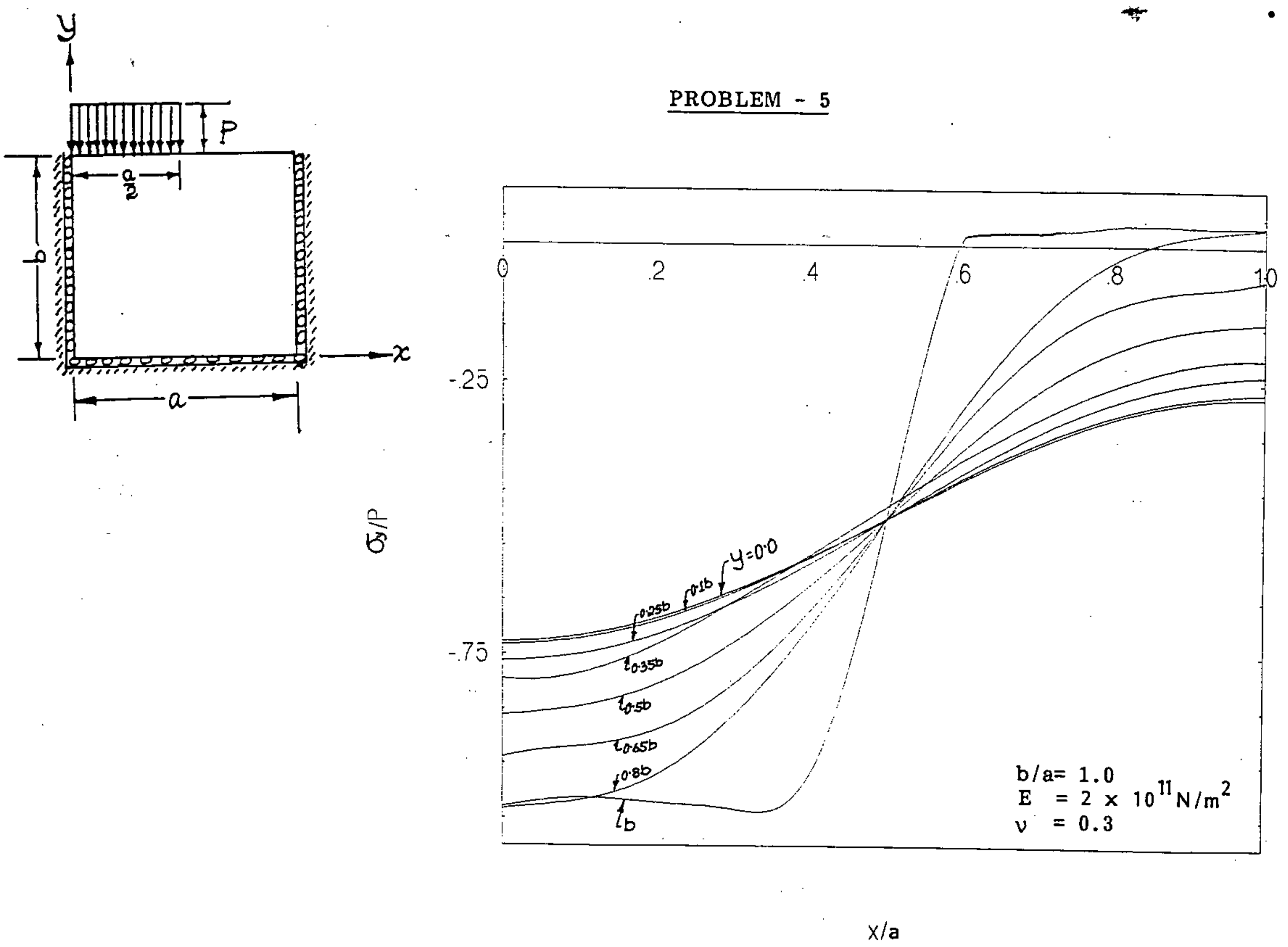
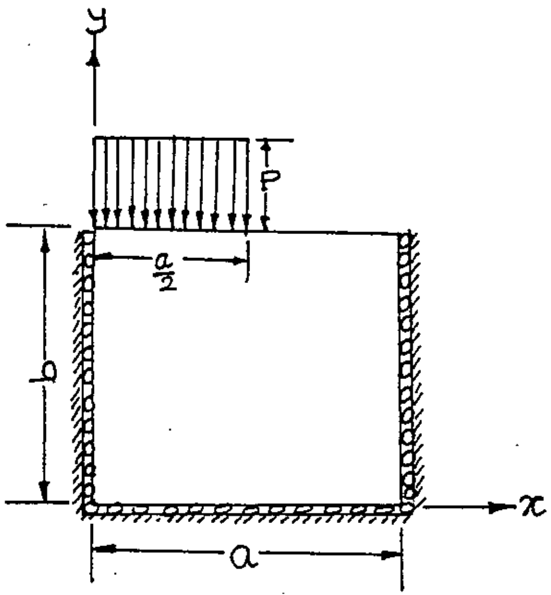


Fig.6.24: Distribution of stress component σ_y at different sections of the plate

PROBLEM - 5



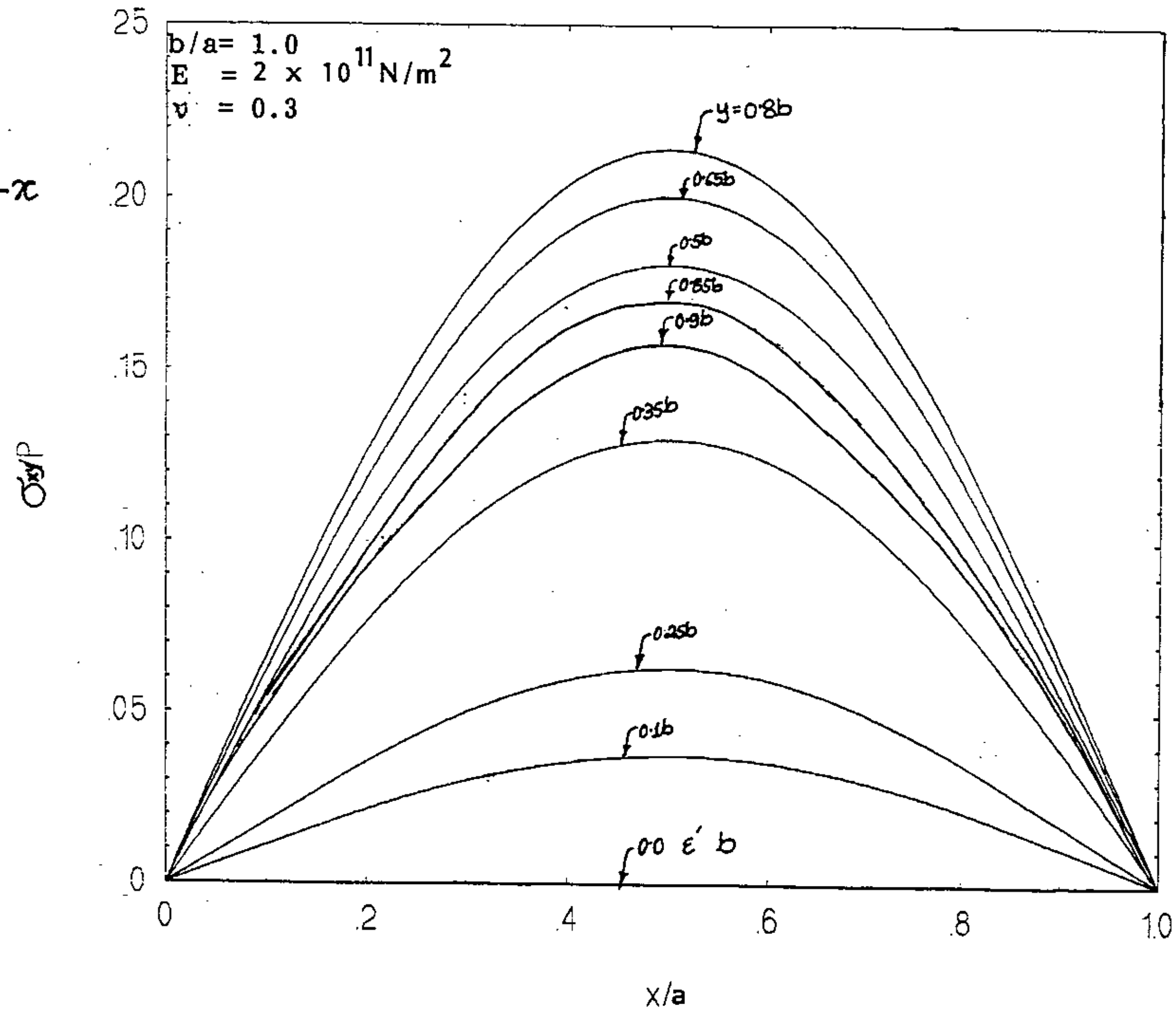


Fig [illegible]

REFERENCES

1. Mem. Savants Etrangers, vol. 14, 1855.
2. Airy, G.B., Brit. Assoc. Advanced Sci. Rept., 1862.
3. Grashof, Elastizitnt Festigkeit, 2nd ed., 1878.
4. Ribiere, M.C., Sur divers cas de la flexon des prismes rectangles, thesis, Bordeux, 1889.
5. Flamant, Compt. rend., vol. 114, p. 1465, 1892, Paris.
6. Rankine, Applied Mechanics, 14th ed., p. 344, 1895.
7. Levy, M., Compt. rend., vol. 126, p. 1235, 1898.
8. Mesnager, A., Compt. rend., vol. 132, p. 1475, 1901.
9. Ribiere, Compt. rend., vol 132, p. 315, 1901.
10. Filon, L.N.G., Phil. Trans., Series A, vol. 201, p. 63, 1903.
11. Timpe, A., Z. Math. Physik., vol. 52, p. 348, 1905.
12. Bleich, F., Bauingenieur, vol. 4, p. 255, 1923.
13. S. Timoshenko, The Approximate Solution of Two-dimensional Problems in Elasticity, Philosophical Magazine, vol. 47, p. 1095, 1924.
14. Scewald, F., Abhandl. aerodynam. Inst. Tech. Hochschule, Aachen, vol. 7, p. 11, 1927.
15. Love, A.E.H., A Treatise on the Mathematical Theory of Elasticity, 4th ed., Cambridge University Press, 1927.
16. Sadowsky, M., Z angew. Math. Mech., vol. 10, p. 77, 1930.
17. Goodier, J.N., Compression of Rectangular Blocks and the Bending of Beams by Non-linear Distribution of Bending Forces, Trans., ASME, vol 54, Paper No. APM-54-17, 1932.
18. Beyer, K., Die Statik in Eisenbetonban, 2d ed., p. 723, 1934.
19. Stokes, G.G., Mathematical and Physical Papers, vol 5, p. 238, 1935.
20. Kolosoff, G.V., Application of a Complex Variable to the Theory of Elasticity, Moscow, Objed, nauchno-tekhn, izd, 1935.

21. Hetenyi, M., J. Applied Mechanics (Trans. A.S.M.E.), vol. 10, A-93, 1943.

22. American Society of Mechanical Engineers, National Meeting of the Applied Mechanics Division, Chicago, ILL, June 16 - 17, 1944.

23. Leibenzon, L.S., Theory of Elasticity, 2nd ed., Moscow, Gostekhizdat, 1947.

24. Frocht, M.M., Photoelasticity, 2 vols., John Wiley & Sons, Inc., New York, 1948.

25. American Society of Mechanical Engineers, Annual Meeting, November 26 -December 1, 1950.

26. Wang, Chi-Teh, Applied Elasticity, MacGraw-Hill Book Company, Inc. 1953.

27. Uddin, M. Wahhaj, "Finite Difference Solution of Two-dimensional Elastic Problems with Mixed Boundary Conditions", M.Sc. Thesis, Carleton University, Canada, 1966.

28. Timoshenko, S. P. & Goodier, J. N., Theory of Elasticity, 3rd ed., New York, MacGraw-Hill Book Company, 1934.

29. "Handbook of Experimental Stress Analysis", John Wiley & Sons, Inc., New York, 1950.

APPENDIX

A.1 Analysis of Deep Beams

Conway, Chow and Morgan[25] analyzed deep Beams, shown in the following figure and its loading is also shown.

The boundary conditions are,

$|x| < c, \; y = b, \; \sigma_y = -P/2c$

$c < |x| < a, \; y = b, \; \sigma_y = 0$

$|x| < a-c, \; y = -b, \; \sigma_y = 0$

$a-c < |x| < c, \; y = -b, \; \sigma_y = -P/2c$

$y = \pm b, \quad \sigma_{xy} = 0$

$x = \pm a, \quad \sigma_{xy} = 0$

$x = \pm a, \quad \sigma_x = 0$

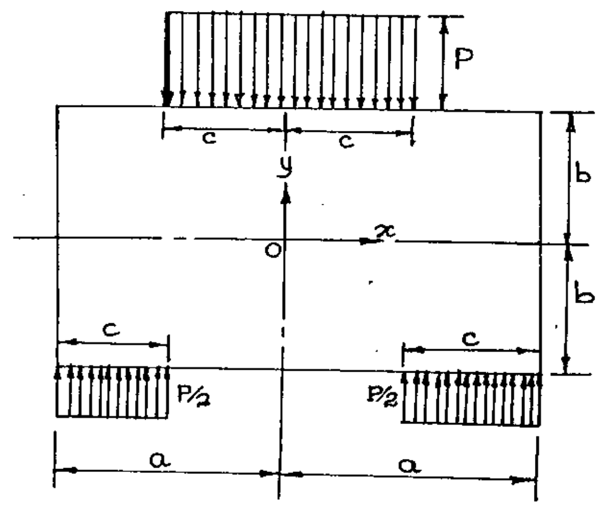


FIG. 1

It is assumed that conditions are such as to permit a two dimensional analysis i.e. the problem is considered as one of the plane stress or plane strain. This assumption justified if the thickness of the block is either very small or very large.

The solution is obtained by superimposing two stress functions. The first stress function is chosen in the form,

$$\Phi_1 = -\frac{Px^2}{4a} + \sum_{m=1}^{\infty} (A_m \cosh\alpha y + B_m y \sinh\alpha y + C_m \sinh\alpha y + D_m \cosh\alpha y) \cos\alpha x \quad \ldots [1]$$

where, A_m, B_m, C_m and D_m are constants.

Normal stress, $\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = \sum_{m=1}^{\infty} [(A_m \alpha^2 + 2B_m \alpha)\cosh \alpha y + B_m \alpha^2 y \sinh \alpha y$

$+ (C_m \alpha^2 + 2D_m \alpha)\sinh \alpha y + D_m \alpha^2 y \cosh \alpha y] \cos \alpha x$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = -\frac{P}{2a} - \sum_{m=1}^{\infty} \alpha^2 [A_m \cosh \alpha y + B_m y \sinh \alpha y + C_m \sinh \alpha y + D_m y \cosh \alpha y] \cos \alpha x \qquad [2]$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = \sum_{m=1}^{\infty} [(A_m \alpha^2 + B_m \alpha)\sinh \alpha y + B_m \alpha^2 y \cosh \alpha y + (C_m \alpha^2 + D_m \alpha)\cosh \alpha y + D_m \alpha^2 y \sinh \alpha y] \sin \alpha x$$

To satisfy the first four conditions, σ_y at the top and bottom edges are taken in the form of fourier series.

At y = b,

$$\sigma_y = -\frac{P}{2a} - \frac{P}{a}\sum_{m=1}^{\infty} \frac{\sin \alpha c}{\alpha c} \cos \alpha x$$

At y = -b,

$$\sigma_y = -\frac{P}{2a} - \frac{P}{a}\sum_{m=1}^{\infty}(-1)^m \frac{\sin\alpha c}{\alpha c}\cos\alpha x$$

Putting, y = ± b in the second of equation (2) and equating corresponding expressions for σ_y, then,

$$A_m\cosh\alpha b + B_m b\sinh\alpha b + C_m\sinh\alpha b + D_m b\cosh\alpha b = \frac{\sin\alpha c}{\alpha c}\,\frac{P}{\alpha^2 a}$$

$$A_m\cosh\alpha b + B_m b\sinh\alpha b - C_m\sinh\alpha b - D_m b\cosh\alpha b = (-1)^m\frac{\sin\alpha c}{\alpha c}\,\frac{P}{\alpha^2 a}$$

The condition of zero shearing stress on the sides y = ± b gives,

$(A_m\alpha + B_m)\sinh\alpha b + B_m\alpha b\cosh\alpha b = 0$

$(C_m\alpha + D_m)\cosh\alpha b + D_m\alpha b\sinh\alpha b = 0$

The four constants are thus determined by solving these equations,

$A_m = B_m = 0$,

$$C_m = \frac{\sin\alpha c}{\alpha c}\,\frac{P}{\alpha^2 a}\,\frac{\cosh\alpha b + \alpha b\sinh\alpha b}{\sinh\alpha b\cosh\alpha b - \alpha b}$$

$$D_m = \frac{\sin\alpha c}{\alpha c}\,\frac{P}{\alpha a}\,\frac{\cosh\alpha b}{\sinh\alpha b\cosh\alpha b - \alpha b}$$

Now, we are able to evaluate σ_x, σ_y, σ_{xy} from the equations (2)

It is evident that the condition of zero shearing stress on the sides x = ± a is automatically satisfied. Therefore, only the last one of the boundary conditions remains to be satisfied that of zero normal stress on the sides x = ± a.

In order to eliminate the normal stress on the sides $x = \pm a$ a second stress function is used.

The second stress function is determined by using strain-energy method and the function is taken in the form

$$\Phi_2 = \Phi_o + n_1\Phi_1 + n_2\Phi_2 + n_3\Phi_3 + \dots\dots\dots [5]$$

in which φ_o satisfies the boundary conditions for φ and φ_1, φ_2, φ_3, do not affect the normal and shearing stress on the sides if the boundary conditions refer to the stress. Then, n_1, n_2, n_3, ... can be found by the principle of least work.

It is sufficiently accurate to take only five term in the expression for φgiven by equation (5) and we assume a function,

$$\varphi_2 = \varphi_o + (x^2 - a^2)^2 (y^2 - a^2)^2 (n_1 y + n_2 x^2 y + n_3 y^3 + n_4 x^2 y^3)$$

In order to satisfy the boundary conditions,

$$\Phi_o = \int_o^y \int_o^y (\sigma_x)_{x=\pm a} \, d_y d_y$$

where, $(\sigma_x)_{x=\pm a}$ may be obtained from first of 3 equations of (2).

After some calculation and manipulation, we can obtain the complete solution for the second stress function φ_2, and from which we can easily calculated the values of σ_x, σ_y, σ_{xy} accordingly.

The actual stress can now be calculated by subtracting the values of σ_x, σ_y, σ_{xy} given by second stress function φ_2 from the corresponding values, given by first stress function φ_1. It is evident that the normal stress left on the vertical sides from the first stress functions are neutralized and therefore a solution satisfying all the boundary condition is obtained.

A.2 Analysis of Rectangular Plate

Gerald Pickett[22] has solved a problem with different loading as shown in the following figure.

The boundary conditions are,

$$\sigma_x = \frac{3S}{2}[1 - (\frac{y}{b})^2], \quad x = \pm a$$

$\sigma_{xy} = 0, \quad x = \pm a$

$\sigma_y = 0, \quad y = \pm b$

$\sigma_{xy} = 0, \quad y = \pm b$

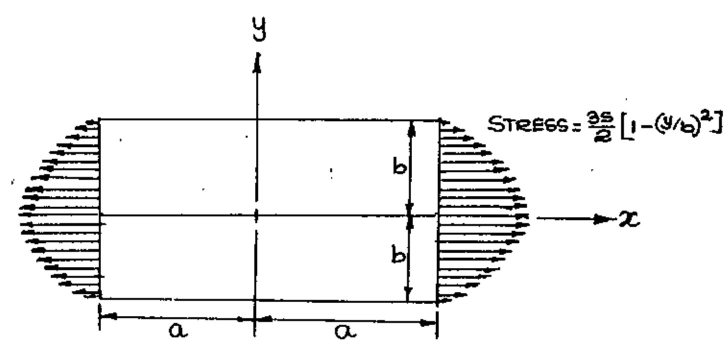


FIG. 2: Rectangular Plate Loaded with Parabolically Distributed Forces.

It may be verified by substitution that the following equations for stress satisfy the appropriate equilibrium equations for all values of the constants A_n, B_m, α_n and β_m.

$$\sigma_x = S + \sum_{n=1}^{\infty} \frac{A_n \cos\alpha_n y}{\cosh\alpha_n a} [\alpha_n x \sinh\alpha_n x - (1 + \alpha_n a \coth\alpha_n a) \cosh\alpha_n x]$$

$$- \sum \frac{B_m \cos\beta_m x}{\cosh\beta_m b} [\beta_m y \sinh\beta_m y + (1 - \beta_m \coth\beta_m) \cosh\beta_m y] \ldots . [1]$$

$$\sigma_y = - \sum_{n=1}^{\infty} \frac{A_n \cos\alpha_n y}{\cosh\alpha_n a} [\alpha_n x \sinh\alpha_n x + (1 - \alpha_n a \coth\alpha_n a) \cosh\alpha_n x]$$

$$+ \sum_{n=1}^{\infty} \frac{B_m \cos\beta_m x}{\cosh\beta_m b} [\beta_m y \sinh\beta_m y - (1 + \beta_m b \coth\beta_m) \cosh\beta_m y] \ldots . [2]$$

$$\sigma_{xy} = \sum_{n=1}^{\infty} \frac{A_n \sinh\alpha_n y}{\cosh\alpha_n a} [\alpha_n a \coth\alpha_n a \sinh\alpha_n x - \alpha_n x \cosh\alpha_n x]$$

$$+ \sum_{n=1}^{\infty} \frac{B_m \sin\beta_m x}{\cosh\beta_m b} [\beta_m b \coth\beta_m b \sinh\beta_m y - \beta_m y \coshh\beta_m y] \ldots . [3]$$

The stress σ_{xy} will be zero at the edges $x = \pm a$ if $\beta_n = m\pi/a$, and this stress will also be zero at the edges $y = \pm b$, if $\alpha_n = n\pi/b$.

The substitute of $\sigma_y = 0$ and $y = \pm b$ into equation [2] gives.

$$\sum_{n=1}^{\infty} \frac{(-1)^n A_n}{\cosh\alpha_n a} [\alpha_n x \sinh\alpha_n x + (1 - \alpha_n a \coth\alpha_n a)\cosh\alpha_n x]$$

$$= \sum_{m=1}^{\infty} B_m[\beta_m b(\tanh\beta_m b - \coth\beta_m b) - 1]\cos\beta_m x \ldots\ldots [4]$$

and the substitution of $\sigma_x = \frac{3}{2}S[1 - (y/b)^2]$ and x = ± a into

equation [1] gives,

$$\frac{3}{2}S[1-(y/b)^2] + \sum \frac{(-1)^n B_m}{\cosh\beta_m b} [\beta_m y \sinh\beta_m y + (1-\beta_m b \coth\beta_m b)\cosh\beta_m y]$$

$$= S + \sum_{n=1}^{\infty} A_n[\alpha_n a(\tanh\alpha_n a - \coth\alpha_n a) - 1]\cos\alpha_n y \ldots\ldots\ldots\ldots [5]$$

From equation [4] and [5], the values of A_n and B_m can easily be evaluated.

Thus, the stresses σ_x, σ_y, σ_{xy} can be calculated from the corresponding expressions.

Gerald[22] has also solved an another problem with different loading at the edges for x = ± a. the mathematical development is just like that previously given except that the boundary stress σ_x at $x = \pm a$ is

$$S + 2S \sum_{n=1}^{\infty} \cos \frac{n\pi y}{b}$$