STRESSES AT THE JUNCTIONS OF AXISYMMETRIC SHELLS UNDER AXIALLY VARYING LOAD.

BY

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ABSTRACT

Distribution of stresses in the neighbourhood of junctions of axisymmetric shells of different geometries with different edge restraints under axially varying internal pressure has been investigated in this theis. The shells considered are thin in which large deformations take place under load. Extensive numerical results on the axisymmetric shells have been obtained for better designs of these shells.

The method of investigation involves solution of a set of six first order nonlinear differential equations considering the large axisymmetric deformations of these shells under axially varying pressure as derived by Reissner(36). The governing nonlinear differnetial equaitons seek for that state of deformation of the shell at which, for a given pressure, the potential energy in the deformed shell is a relative minimum. The basic concept of multisegment integration as developed by Kalnins and Lestingi(24) has been utilized for obtaining the solutions of the governing equations. A computer program has been developed

incorporating the algorithm of finding the stresses and displacements of the axisymmetric shells. The information necessary for specifying a particular shell and its edge conditons and the base load are used by the program as input data.

For a given low pressure, specified in the input data, the program first finds the linear solution in terms of deformations and stresses in the shell which is followed by nonlinear solutions corresponding to the same pressure. Then pressure is increased in steps by an amount specified in the input data and nonlinear solutions are obtained and printed out for each loading step till the pressure reaches a maximum specified value.

The soundness of the method and the correctness of programming are verified by comparing the results of axisymmetric shells with that of the corresponding analytical results available in the literature. Curves are plotted based on both the linear and nonlinear solutions for depicting the stress modes at different values of the shell parameters and also for finding the locations at which stresses are maximum.

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CONTENTS

NOTATIONS

b_1 , b_{M+1} = (m, 1) matrices, contain prescribed variables at the boundary.

C = Eh, extensional rigidity

$\bar{C}$ = $(1 - \nu^2)$ s_e/R

D = $Eh^3/12\ (1-\nu^2))$, bending rigidity

$\bar{D}$ = $1/(12(1-\nu^2))\ \bar{P}_o\ \bar{T}^2\ \ \bar{R})$

E = Young's modulus of elasticity

H = horizontal stress resultant

$\bar{H}$ = H/P_oR, nondimensional horizontal stress resultant

h = Shell thickness

I = (6,6) unit matrix

K_θ , K_s = Changes of curvature of the middle surface of the shell.

$\bar{K}_\theta$ = $K_\theta\ S_e$, nondimensional value of K_θ .

$\bar{K}_s$ = K_sS_e, nondimensional value of K_s

$\bar{L}$ = $\bar{R}/\bar{P}_o\ \bar{T}$.

M = numebr of segments

M_s = meridional couple resultant

M_θ = Circumferential couple resultant

$\overline{M}_s$ = M_s/P_oRh, nondimensional value of M_s

$\overline{M}_\theta$ = $M_\theta/P_o Rh$, non-dimensional value of M_θ

N_s = meridional stress resultant

N_θ = Circumferential stress resultant

$\overline{N}_s$ = N_s/P_oR, nondimensioal value of N_s.

$\overline{N}_\theta$ = N /P_oR, nondimensional value of N_θ

P_o = outward normal pressure at the base of the shell (its positive value indicate internal pressure)

P = internal normal pressure at any point on the meridian.

$\overline{P}_o$ = P_o/E, nondimensional value of P_o

$\overline{P}$ = P/E, nondimensional vlaue of P

P_H = horizontal Component of surface load

P_V = Vertical component of surface load

Q = transverse shear stress resultant

R = radius of base circle

$\overline{R}$ = s_e/R

R_s, R_θ = Principal radii of curvature of the middle surface of the shell

r_o = distance of a point on undeformed middle surface of the shell

r = $r_o + u$, distance of a point on deformed middle surface from axis of symmetry

$\bar{r}_o$ = r_o/s_e, nondimensional value of r_o

s = distnace measured from the apex along the meridian

$\bar{s}$ = s/s_e, nondimensional value of s

s_e = total length of the shell meridian

S_i = ith segment of the shell meridian

T_1, T_{M+1} = (6,6) matrices, given by the bondary conditions.

$\bar{T}$ = R/h

u = radial displacement(normal to the axis of symmetry)

$\bar{u}$ = uEh/P_oR^2, nondimensional value of radial dispacement

V = vertical stress resultant

$\bar{V}$ = V/P_oR, nondimensinal value of vertical stress resultant

w = axial displacement

$\bar{w}$ = wEh/P_oR^2, nondimensional value of axial displacement

x = independent variable assumed in the method of solution.

x_i = vlaue of x at the ith nodal point of the segment

$y(x)$ = (6,1) matrix, contains 6 fundamental vriables

z_o = axial distance of a point on undeformed middle surface of shell from its plane

z = $z_o + w$, axial distance of a point on deformed middle surface

α = parameter of meridian of deformed shell, defined in Equation (2.4)

α_o = value of corresponding to undeformed shell

$\bar{\beta}$ = β

β = angle of rotation of normal to the middle surface of the shell

$\epsilon_\theta, \epsilon_s$ = middle surface strains.

$\bar{\epsilon}_\theta$ = $\epsilon_\theta E h s_e / P_o R^2$, nondimensional value of ϵ_θ

$\bar{\epsilon}_s$ = $\epsilon_s E h s_e / P_o R^2$, nondimensional value of ϵ_s

ϕ_o = angle between normal and axis of symmetry before deformation (meridional angle)

ν = Poisson's ratio of shell material

σ_{ai} = $N_s/h + 6M_s/h^2$, meridional stress at the extreme inner fiber

σ_{ao} = $N_s/h - 6M_s/h^2$, meridional stress at the extreme outer fiber

σ_{ci} = $N_\theta/h + 6M_\theta/h^2$, circumferential stress at the extreme inner fiber.

σ_{co} = $N_\theta/h - 6M_\theta/h^2$, circumferential stress at the extreme outer fiber

$\bar{\sigma}_{ai}$ = σ_{ai}/E, nondimensional value of σ_{ai}

$\bar{\sigma}_{ao}$ = σ_{ao}/E, nondimensional value of σ_{ao}

$\bar{\sigma}_{ci}$ = σ_{ci}/E, nondimensional value of σ_{ci}

$\bar{\sigma}_{co}$ = σ_{co}/E, nondimensinal value of σ_{co}

$(----)'$ = derivative with respect to s or $\bar{s}$

CHAPTER I

INTRODUCTION

1.1 PRELIMINARY

With the passage of time, shell structures are being utilized more and more. In many instances, axially varying load is the primary consideration in the design of various structural configurations. Shells are used as load - carring element in some part of virtually every item of modern industrial equipment. This is specially true of the marine, petrochemical industries, nuclear and aerospace where dramatic and sophisticated uses of shells are currently being made in space vehicles and missiles, submarines, nuclear reactor vessels, refinary equipments and the like. As interest in shell structures increased, more sophisticated mathematical analysis of shells were sought. Nonlinear shell analysis, which takes into account of finite shell deformation under loading as well as non linear stress-strain relations, is currently in its infancy. This type of problem requires the integration of a rather complicated system of simultaneous nonlinear differential equations or solutions of highly ill conditioned simultaneous algebraic equations.

Consequently, with the advent of large high speed computers, the authors of numerous recent papers have focussed their attention on the methods of numerical integration of thin shell equations.

Shell structures are characteristically different from others in the sense that large deformation takes place in many shells under internal or external loading. This sometimes necessitates consideration of large deformation in the formulation of the problems to obtain reasonable information of the structure. Analysis of composite shells which invariably has to account for the large deformations that take place at the junctions of shells of different geometrics, is fundamentally a subtopic of nonlinear rather than linear mechanics. The nonlinearity is introduced in the governing equations of elasticity in three ways :

a. through the strain-displacement relations.
b. through the equaitons of equilibrium of a volume element of the body, and
c. through the stress-strain relations.

In (a) and (b) retention of the nonlinear terms is conditioned by geometric considerations, that is, the necessity of taking into account the angles of rotation in determining the changes of dimension in the line elements and in formulating the conditions of equilibrium of a volume element. On the other hand, nonlinear terms appear in the third set of equations (c) if the material does not behave in a linearly elastic fashion. Hence there are two types of nonlinearity :

(i) geometric

(ii) physical

In the problems of shell structure, the angle of rotation can be large, but the strain can remain within elastic limit. The bending of a thin steel strip can be considered. Strips of good steel can straighten out without traces of residual deformation after having their ends brought together. This bears witness to the fact that, in these strips, even for large displacement and angle of rotation, the stresses do not exceed the yield point. Thus, many shell structures belong to a class of problem which are physically linear but geometrically nonlinear.

1.2 RESUME OF NONLINEAR SHELL ANALYSIS

That linear shell analysis fails to give proper information about the shell stresses and deformations in many problems can be seen in recent papers on the nonlinear shell analysis (4,5,7,9,10,11,22,24,34,36,38,41,43-53). For this reason the use of nonlinear theory has become rather widely accepted as a plausible basis for predictions of elastic strengths of thin shells of various geometries.

The basic concept of finite deflection analysis, due to Donnell(9), has been employed by numerious investigators to establish collapse loads of cylindrical shells subjected to various loadings. Finite deflection analysis has also been successful in offering reasonable predictions of the elastic buckling loads of shallow spherical caps subjected to uniformly distributed external pressure. Kaplan and Fung (24) have presented a perturbation solution to the nonlinear equations that agrees quite well with results of their experiments for very shallow clamped edge shells. Archer (1) extended these results to a greater range of shells. As can be seen from recent papers, very extensive work has been done in this field (12,15,18,22,24,26,43). Ball (2) has considered the problems of arbitrarily loaded shells of revolution and obtained solution for a clamped shallow spherical shell uniformly loaded over one half of its surface. Finite deflection studies are available for cylindrical, spherical as well as other types of shells subjected to variety of loadings and boundary conditons. In all cases the predictions of these theories are in better agreement with expremental evidence than those of the classical investigations based upon infinitesimal deformations.

Uddin (46) has found extensive numerical results on perfect spherical, ellipsoidal, conical and composite shells based on both the linear and nonlinear theories and has obtained critical pressures of different types of spherical shells. He has also obtained the solutions for spherical, ellipsoidal, conical and plate end pressure vessels (47,48,49,52) based on both the linear and nonlinear theories. For composite shells with geometrical discontinuity, he has found numerical results of stresses in the neighbourhood of junctions under uniform internal pressure.

Bushnell (6) has developed a computer software package, known as BOSOR5, for analyzing the nonlinear stress field of axisymmetric shell systems based on thin shell theory and for determining the bifurcation buckling pressures of ellipsoidal and torispherical heads joined to cylinder and subjected to internal pressure. This software is capable of taking into account of various meridional geometry and practical boundary conditons.

Haque (16) has investigated buckling of perfect ellipsoidal shells of revolution and has obtained respective critical pressures for various shell parameters. Rahman (38) has analysed the stability of imperfect ellipsoidal shells of revolution under external pressure. Extensive investigations had been carried out for imperfections of various shells and structures (19,20,21, 23, 27,30,42).

But the stresses under axially varying load of axisymmetric shells with discontinuities in slope of the meridian, taking large deformation into consideration, has not yet been studied.

1.3. OBJECTIVES OF THIS INVESTIGATION

The objectives of the present investigation are stated below :

1. The purpose of this investigation is to determine stresses at the junctions of axisymmetric shells of different geometries under axially varying load. This investigation is thus to provide some insight into the nonlinear analysis of shells of revolution under axially varying internal pressure with discontinuities in slope and curvature of the meridian.

2. The study includes only those shells which are considered to be thin and in which large deformations take place under load.

3. Distribution of stresses in the neighbourhood of junctions of axisymmetric shells of different geometries as found here are expressed in graphical forms plotted against distance along the meridian.

4. The present investigation is confined to the large deformations and thus the maximum stress in the shell is determined in order to ascertain that it is within the yield strength of the shell material, that is, it is checked whether withdrawal of internal pressure would allow the retention of original shape of the shell.

The computer program developed for the analysis may be used for various boundary conditions like completely fixed or roller supported or hinged edges.

In order to achieve these objectives, a system of six first order nonlinear ordinary differential equations with geometrical discontinuity had to be integrated as a boundary value problem. The method of Multisagment Integration had been used for solving this boundary value problem of shells of revolution undergoing axisymmetric deformation. Usually, the method of Multisegment Integration is used to solve those boundary value problems of ordinary differential equations which can not be solved by direct integration; because, direct integration losses all of its accuracy in the process of subtraction of almost equal numbers in evaluating the unknown boundary values. The method of Multisegment Integration, as used in this analysis, was first developed by Kalnins and Lestingi (24) and later applied by Uddin (46) for sloving the nonlinear problem of axisymmetric deformation of shell of revolution. The computer program used in this analysis is adopted from that of Uddin with necessary modifications to suit the requirement of solving problems of general case of shells under axially varying axisymmetric loading.

1.4. METHOD OF SOLVING NONLINEAR DIFFERENTIAL EQUATIONS

A system of nonlinear ordinary differential equaitons with geometrical discontinuities is required in solving the present problem. Unfortunately, the development of modern mathematics has provided the applied scientists hardly with any general method for solving nonlinear ordinary and nonlinear partial differential equations. The situation has been brightened considerably, however, with the development of modern digital computers and with the simultaneous revitalization and growth of the study of numerical methods.

Though there are quite a number of approximate methods available for solving nonlinear differential equaitons, there is hardly any method proved to be unique or advantageous over the other method, leaving aside its applicability to a specific problem. The methods most frequently used in solving nonlinear differential equations are :

(1) Asymptotic integration (31)

(2) Direct numerical integration (13)

(3) Finite - difference method

(4) Perturbation technique

(5) Newton's method

(6) Method of multisegment integration

(1) Asymptotic integration : It is not a general method and its scope of application is very limited. In the application of this method the solution is expressed in the form of a series where the terms of the series are the inverse powers of the largest parameter in the differential equations (31). It is very difficult to find out the terms of the series and most of the time the solution contains only the first term approximation. Considering the complexity of the shell equations and remembering that there are geometrical discontinuities at intermediate points, the posibility of obtaining a reasonably good solution by any approximate analytical method is highly unlikely.

(2) Direct Numerical Integration : The direct integration approach has certain advantages but it also has a serious disadvantage i.e. when the length of the shell is large, a loss of accuracy invariably results. This phenomenon is clearly pointed out in Ref (13). The loss of accuracy does not result from the cumulative error in integration, but it is caused by the subtruction of almost equal numbers in the process of determining unknown boundary values. It follows that for every set of geometric and material parameters of the shell there is a critical length beyond which the solution losses all its accuracy.

(3) Finite - difference method : This method is the most widely used technique for solving nonlinear differential equations. The advantage of this method over direct integration is that it can

avoid the above mentioned loss of accuracy. Here the analysis involves the solution of a large number of nonlinear algebraic equaitons which would probably have a number of solutions. Most of the time the solutions of nonlinear equaitons are obtained as the solutions of a sequence of linear equaitons. It is often difficult to distinguish between instability in the sequence of numerical calculations and the point of instability of the differential equaitons which correspond to the classical buckling pressure. It is usually the case that the finite difference method is not suitable for application to problems which contain discontinuities or rapidly varying parameters at a point.

4. Perturbation Technique : The perturbation technique is also a frequently used analytical method for solving nonlinear differential equaitons. In this technique the functions to be obtained are expressed in the form of power series in terms of a perturbation parameter and the solutions are obtained as solutions of a sequence of linear differential equations. The solutions of the linear equaitons are the terms of the series. But there must be a natural, an artificially created perturbation parameter which contributes to the nonlinearity of the problem and this parameter must be small enough so that the series is convergent.

Particularly this method is appropriate for nonlinear dynamic problem of rigid bodies (14) where a natural perturbation parameter exists and the solutions are periodic. In nonlinear shell analysis this technique is used by Archer (1) to clamped spherical shell under uniform pressure where the nondimensionalized radial displacement at the point of maximum diflection has been used as a perturbation parameter. From this solution it is seen that the computational work involved in obtaining numerical values is so extensive that it would be desirable to apply some numerical technique from the beginning. The result of this solution is compared with experimental and other results by Reiss (37) where it is shown that the perturbation solution is in serious disagreement with the rest of the results. In this problem it is required to solve a number of sets of differential equaitons where no suitable perturbation parameter is obvious which is applicable to all the sets. The convergence of the series under the present circumstances can only be established by comparing with known results, but there exist no such results.

(5) Newton's method : Newton's method for solving nonlinear differnetial equations is the extension of Newton's method for calculating roots of algebraic equations. The approach is to express the solution as the sum of two parts; the first part is a known functin and the second part is a correction to the known function. A governing equation for the correction is obtained by substituting the assumed function into the nonlinear equations and neglecting the term which are nonlinear (17). This method does not require the perturbation parameter to be small, as is necessary in the perturbation technique, but involves the solution of a sequence of linear differential equations. These linear equaions have variable coefficients and generally can not

be solved in closed form. It is paradoxical that the greatest obstacle in solving nonlinear problems is the inability to solve linear differential equations in closed form.

(6) <u>Method of multisegment integration</u> : It is the most recent method developed and used by Kalnins and Lestingi (24) to solve nonlinear differential equations. This method involves :

(a) division of the total interval into a number of segments;
(b) initial-value integration of a system of first order differential equaitons over each segment;
(c) solution of a system of matrix equaitons which ensures the continuity of the variables at the ends of the segments ;
(d) repetition of (b) and (c) till convergence is achieved;
(e) integration of an intial value problem to obtain the values of the dependent variables at any desired point within each segment.

The main advantage of this method over finite - difference method is that the solution is obtained everywhere with uniform accuracy and the iteration process with respect to the mesh size, as required in finite difference approach' is eliminated. But the feature which makes this method most attractive for this problem is that any discontinuity, either in geometry or in loading, can be easily handled by requiring that the nodal points of the segment coincide with the location of discontinuities. This method is the most accurate of all the numerical methods because the problem is solved in the form of a system of first order differential equaitons in which no derivatives of geometrical or elastic properties appear and because no further numerical derivatives have to be evaluated for obtaining the desired results in the process of computations.

CHAPTER 2

THEORY OF SHELL

2.1 INTRODUCTION

The literature on shell theories is not devoid of papers in which some of the aspects of finite displacements on the deformation of this shell are accounted for. The work of a completely general nature appears to be the papers by Chang and Chen (8) followed by a series of papers by Chen. The theory of shells developed by Chang and Chen avoids the use of displacement as unknowns in the equations. The theory is deduced from the three-dimensional theory of elasticity and then, by means of series expansion in powers of small thickness parameter, approximate theories of thin shells are derived. Other developments which also employ linear constitutive relations are founded upon the Kirchhoff hypothesis and often contain other approximations.Among these are Reissner's (36) formulation of axisymmetric deformation of shells of revolution and the more general works of Sanders (39) and Leonard. Beginning with the three dimensional field equations Naghdi and Nordgren deduced an exact, complete, and

fully general nonlinear theory of elastic shells founded upon the Kirchhoff hypothesis.

Several nonlinear theories for thin shells have been derived in increasing stages of approximation. In most cases the theories are first approximative theories in the sense that transverse shear and normal strains are neglected. Here the author has used the theory of axisymmetric deformation of shells of revolution as presented by Reissner (36), because of the fact that Reissner's derivations have extremely simple structure and that this theory differs from others in using radial and axial components of displacements and stress resultants instead of the customary practice of using normal and tangential components of displacements and stress resultants. The modified definition of displacements and stress resultants is very well suited for managing the axially varying load of composite shell problems.

2.2 REISSNER'S THEORY OF AXISYMMETRIC DEFORMATION OF SHELLS OF REVOLUTION.

The basic equations of Reissner's theory of finite axisymmetric deformations of shells of revolution are presented here for ready reference.

The equation of the meridian of the shell is written in the parametric form (Fig. 3) as,

$$r = r(s), \quad z = z(s) \qquad (2.1)$$

so that s together with polar angle θ in the x-y plane are the coordinates on the middle surface. The sloping angle ϕ of the tangents to a meridian curve is given by

$$\tan\phi = dz/dr \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.2)$$

From equation (2.2) it follows that

$$\cos\phi = r'/\alpha \quad , \sin\phi = z'/\alpha \quad \ldots\ldots\ldots \quad (2.3)$$

where the primes denote differentiation with respect to s and α is given by

$$\alpha = [(r')^2 + (z')^2]^{1/2} \ldots. \quad (2.4)$$

The principal radii of curvature of the middle surface of the shell are given by

$$R_s = \alpha/\phi' \quad , \quad R_\theta = r/\sin\phi \ldots. \quad (2.5)$$

With reference to Fig. (4a) the equation of deformed middle surface is written as

$$r = r_o + u, \quad z = z_o + w \quad \ldots\ldots\ldots \quad (2.6)$$

where the subscript $_o$ refers to the undeformed middle surface and the quantities u and w are, respectively' the radial and the axial components of displacement.

The angle enclosed by the tangents to the deformed and to the underformed shell meridian, at the same material point, is given by

$$\beta = \phi_o - \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.7)$$

With the above definition of displacements, the strain components and the curvature changes of the middle surface are given by the following equaitons :

$$\varepsilon_s = (\alpha - \alpha_o)/\alpha_o = (\cos\phi_o/\cos\phi)(1 + u'/r'_o) - 1 \quad \ldots(2.8)$$

$$\varepsilon_\theta = u/r_o \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.9)$$

$$K_s = -(\phi' - \phi'_o)/\alpha_o = \beta'/\alpha_o \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.10)$$

$$K_\theta = -(\sin\phi - \sin\phi_o)/r_o \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.11)$$

The equation containing the axial displacement w is introduced as

$$w' = \alpha \sin\phi - z_o \quad \ldots\ldots\ldots\ldots (2.12)$$

With the definition of stress resultants and couples as shown in Fig(4a) and Fig(4b) the equations are written as :

From the condition of equilibrium of forces in axial direction

$$(rV)' + r\alpha P_v = 0 \quad (2.13)$$

From the condition of equilibrium of forces in radial direction,

$$(rH)' - \alpha N_\theta + r\alpha P_H = 0 \quad (2.14)$$

From the conditon of equilibrium of moments about circumferential tangent,

$$(rM_s)' - \alpha\cos\phi\ M_\theta + r\alpha(H\sin\phi - V\cos\phi) = 0 \ldots\ldots(2.15)$$

With the assumption that the behaviour is elastic, the relations between strains and stress resultants are given by

$$C\epsilon_s = N_s - \nu N_\theta, \quad C\epsilon_\theta = N_\theta - \nu N_s \ldots\ldots\ldots\ldots (2.16)$$

$$M_s = D(K_s + \nu K_\theta), \quad M_\theta = D(K_\theta + \nu K_s) \ldots\ldots\ldots\ldots (2.17)$$

Where C = Eh, D= $Eh^3/(12(1-\nu^2))$, and h is the thickness of the shell. The radial stress resultant H and axial stress resultant V are related to N_s and transverse shear Q as follows :

$$N_s = H\cos\phi + V\sin\phi, \quad Q = -H\sin\phi + V\cos\phi \ldots\ldots \quad (2.18)$$

2.3. DERIVATION OF THE FIELD EQUATIONS

The order of the system of euquations (2.6 - 2.18) is six with respect to s, and consequently it is possible to reduce Eqns (2.6-2.18) to six first order differential equations which involves six unknowns. In the following derivation, the six fundamental variables are taken as u, β , w, V, H, M_s and the differential equations are expressed in terms of these variables. The independent variable s is taken as the distance measured from the apex along the meridian of the shell so that the differential equations can be used for all possible geometrical shapes of the meridian. With is definition of s, Eqn. (2.4) gives

$$\alpha_o = [(r'_o)^2 + (z')^2]^{1/2} = 1$$

From the geometry of the meridian, which is yet to be specified, it is known that

$$r_o = r_o(s) \qquad \text{.............(2.19)}$$

$$\phi_o = \phi_o(s) \qquad \text{.............(2.20)}$$

The following equations are rewritten from the previous section in such an order that, when evaluated serially, they are in terms of the fundamental variables.

This is done in order to keep the fundamental set of differential equations as simple as possible. Rewritting of Equns. (2.9), (2.6), (2.7), (2.11), (2.18), (2.17) yeilds.

$$\epsilon_\theta = u/r_o \qquad \text{............................(2.21)}$$

$$r = r_o + u \qquad \text{.........................(2.22)}$$

$$\phi = \phi_o - \beta \qquad \text{........................(2.23)}$$

$$K_\theta = (\sin\phi_o - \sin\phi)/r_o \qquad \text{............(2.24)}$$

$$N_s = H\cos\phi + V\sin\phi \qquad \text{..........(2.25)}$$

$$K_s = M_s/D - \nu K_\theta \qquad \text{..................(2.26)}$$

$$M_\theta = D\,(K_\theta + \nu K_s) \qquad \text{..................(2.27)}$$

Eliminating N_θ from Eqns (2.16), it is found that

$$\epsilon_s = \left(\frac{1-\nu^2}{C}\right)N_s - \nu\epsilon_\theta \qquad \text{........ (2.28)}$$

similarly, elimination of Ns from Eqns. (2.16) yields

$$N_\theta = \left(\frac{C}{1-\nu^2}\right)\left(\epsilon_\theta + \nu\epsilon_s\right) \quad \text{.......... (2.29)}$$

Rearrangement of Eqn (2.8) and substitution of α_o =1 gives

$$\alpha = 1 + \epsilon_s \quad \text{............. (2.30)}$$

Elimination of z_o from Eqn(2.12) by means of Eqn(2.3) gives

$$dw/ds = \alpha \sin\phi - \alpha_o \sin\phi = \alpha \sin\phi - \sin\phi_o \quad \text{... (2.31)}$$

Substitution of the values of ϵ_s from Eqn (2.30) and r_o from Eqn (2.3) in Eqn (2.8) gives

$$du/ds = \alpha \cos\phi - \cos\phi_o \quad \text{...........(2.32)}$$

From eqn (2.10), the expression for β' is fond to be

$$d\beta/ds = K_s \quad \text{........... (2.33)}$$

Expansion of the three equations of equilibrium and elimination of P_v, P_H and r' from these equations result in the following expressions for V', H' and M'_s :

$$dv/ds = -\alpha\,((V \cos\phi)/r - P \cos\phi) \quad \text{....... (2.34)}$$

$$dH/ds = -\alpha\,((H \cos\phi - N_\theta)/r + P \sin\phi) \quad \text{... (2.35)}$$

$$dM_s/ds = \alpha \cos\phi\, (M_\theta - M_s)/r - \alpha(H \sin\phi - V \cos\phi) \ldots \quad (2.36)$$

where P is the axially varying internal pressure, that is, P is the function of s. Eqns (2.19 - 2.36) are the nonlinear governing equations of the axisymmtric deformations of shells of revolution expressed in terms of the fundamental variables. It should be noted that this fundamental set of differential and algebraic equations are expressed in such a manner that all the quantities of physical importance are evaluated during the process of solution of these equations.

The expressions of variable internal pressure P for various kinds of shell elements are given below -

Expression for line-element :

Let the shell contain a conical frustum and is filled with a liquid of specific weight γ (Fig. (a)). Assuming that the total depth of the liquid is d from a certain point z on the axis corresponding to point s on the meridian of the shell where the gauge pressure is denoted by P_o. It is required to calculate the pressure P normal to the meridian at some other point on the shell.

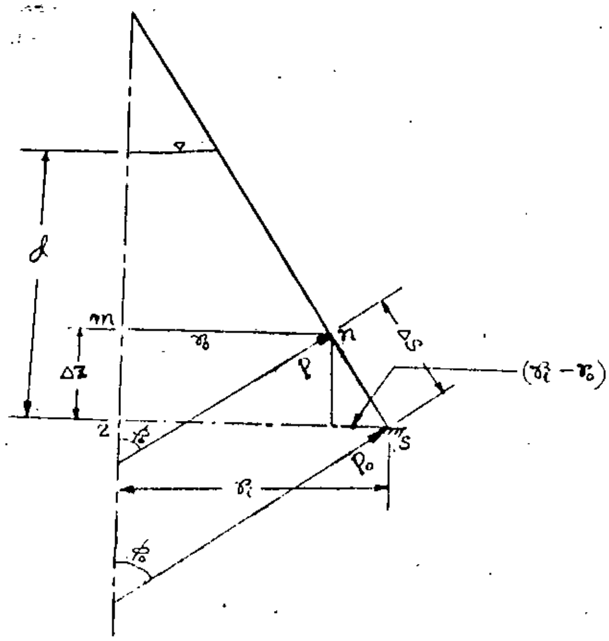

Figure -(a)

From the geometry of the shell it is seen that -

$$\Delta Z/\Delta S = \sin\phi_o,\ (r_i - r_o)/\Delta s = \cos\phi_o$$

$$P_o = \gamma d + P_a$$

where P_a is the gage pressure above the liquid surface. Now, pressure at any parallel circle mn is -

$$P = P_o - \gamma \Delta Z$$

Where P_o is the maximum pressure of the base, defined as

$$P_o = \gamma d + P_a$$

$$\text{or,}\quad P = P_o \left[1 - \frac{\Delta Z}{d + d'} \right] \qquad \text{where, } P_a = \gamma d'$$

$$\text{or,}\quad P = P_o = \left[1 - \frac{\Delta S \sin\phi}{d + d'} \right]$$

$$\text{or,}\quad P = P_o \left[1 - \frac{r_i - r_o}{\cos\phi_o} \frac{\sin\phi_o}{d + d'} \right]$$

$$\text{or,} \quad P = P_o \left[1 - \frac{r_i - r_o}{d + d'} \tan \phi_o \right]$$

$$\text{or,} \quad \frac{P}{E} = \frac{P_o}{E} \left[1 - \frac{r_i/s_e - r_o/s_e}{d/s_e + d'/s_e} \tan \phi_o \right]$$

$$\text{or} \quad \bar{P} = \bar{P}_o \left[1 - \frac{\bar{r}_i - \bar{r}_o}{\bar{d} + \bar{d}'} \tan \quad \right]$$

Expression for Circular elements :

Here, the expression of variable internal pressure P as a function of s is derived in the same manner as for line element.

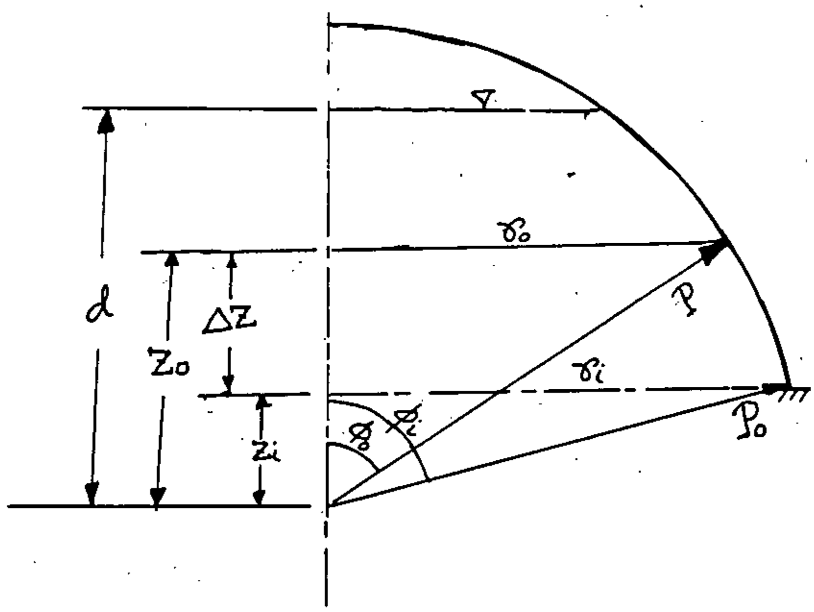

Figure -(b)

From the geometry of the shell-

$$Z_i = \frac{r_i}{\tan \phi_i}$$

$$Z_o = \frac{r_o}{\tan\phi_o}$$

Therefore, $\Delta Z = Z_o - Z_i = \frac{r_o}{\tan\phi_o} - \frac{r_i}{\tan\phi_i}$

Pressure P at any point on the circular meridian is

$$P = P_o - \gamma\Delta Z$$

$$\text{or,} \quad P = P_o \left[1 - \frac{\Delta Z}{d + d'}\right]$$

$$\text{or,} \quad P = P_o \left[1 - \frac{1}{d+d'}\left(\frac{r_o}{\tan\phi_o} - \frac{r_i}{\tan\phi_i}\right)\right]$$

Nondimensionalization of P yields

$$\bar{P} = \bar{P}_o \left[1 - \frac{1}{\bar{d}+\bar{d}'}\left(\frac{\bar{r}_o}{\tan\phi_o} - \frac{\bar{r}_i}{\tan\phi_i}\right)\right]$$

3.4. EQUATIONS FOR THE APEX

The fundamental set of equations derived in the previous sections is singular at the pole (Fig.1). In order to remove this sigularity, the conditon that all the physical quantities must be regular at the pole should be imposed. From the symmetry at the pole it is found that

$$u = \quad = 0$$

and as there is no concentrated load at the pole, it follows that

$$V = 0$$

In the following derivation it is assumed that s is measured from the pole of the axisymmetric shell.

Since ϵ_θ and ϵ_θ' must be regular at s = 0 Eqn.(2.21) gives

$$\lim_{s \to 0} \epsilon_\theta = u'/r_o' \quad \text{(By L' Hospitals' principle)}$$

and

$$\lim_{s \to 0} \epsilon_\theta' = \frac{u'' r_o' - u' r_o''}{2(r_o')^2}$$

From eqn (2.3), it is found that $r_o' = \cos\phi$

and therefore, $r_o'' = -\sin\phi_o . \phi_o'$.

Substitution of the values of r_o' and r_o'' into the expression of ϵ_o and ϵ_o' yeilds

$$\lim_{s \to 0} \epsilon_\theta = u'/\cos\phi_o \text{.........................} \quad (2.37)$$

$$\lim_{s \to 0} \epsilon_\theta' = \frac{u'' \cos\phi_o + u'\phi_o' \sin\phi_o}{2\cos^2\phi_o} \quad \text{.............} (2.38)$$

Similarly, the following equations can be deduced from eqns (2.19) - 2.36) by taking the limit as $s \to 0$

$$\lim_{s \to 0} \phi = \phi_o \quad \text{.................} \quad (2.39)$$

$$\lim_{s \to 0} \phi' = \phi_o' - \beta' \quad \text{.................} \quad (2.40)$$

$$\lim_{s \to 0} K_\theta = \beta' \quad \text{.................} \quad (2.41)$$

$$\lim_{s \to 0} K_\theta' = \frac{1}{2}(\beta'' - \phi'\beta' \tan\phi_o) \text{..........} \quad (2.42)$$

$$\lim_{s \to 0} N_s = H \cos\phi_o \quad \text{.........} \quad (2.43)$$

$$\lim_{s\to o} N_s' = H'\cos\phi_o - H\phi'\sin\phi_o + V'\sin\phi_o \quad \ldots(2.44)$$

$$\lim_{s\to o} M_\theta' = \lim_{s\to o}(D(1-\nu^2)K_\theta' + \nu M_s') \quad \ldots\ldots\ldots (2.45)$$

$$\lim_{s\to o} N_\theta' = \lim_{s\to o}(C\,\epsilon_\theta' + \nu N_s') \quad \ldots\ldots\ldots (2.46)$$

$$\lim_{s\to o} \alpha = (1+\frac{1-\nu^2}{C} H\cos\phi_o - \frac{u'/\nu}{\cos\phi_o} \ldots (2.47)$$

$$\lim_{s\to o} \alpha' = \lim_{s\to o}\left(\frac{1-\nu^2}{C} N_s' - \nu\epsilon_\theta'\right) \quad \ldots .(2.48)$$

$$\lim_{s\to o} u' = ((1-\nu)/C)\, H\cos^2\phi_o \quad \ldots\ldots (2.49)$$

$$\lim_{s\to o} \beta' = M_s/(D(1+\nu)) \quad \ldots\ldots\ldots (2.50)$$

$$\lim_{s\to o} w' = \frac{1-\nu}{C} H\sin\phi_o\cos\phi_o \quad \ldots . (2.51)$$

Substitution of Eqn (2.49) in Eqn (2.47) gives

$$\lim_{s\to o} \alpha = 1+\frac{1-\nu}{C} H\cos\phi_o \ldots\ldots (2.52)$$

Now $$\lim_{s\to o} \frac{V}{r} = \frac{V'}{\alpha\cos\phi_o} \quad \ldots\ldots\ldots (2.53)$$

Substituting Eqn. (2.53) in Eqn. (2.34) and solving for V/ at the apex, it is found that

$$\lim_{s\to o} V' = \frac{1}{2}\alpha P\cos\phi_o \quad \ldots\ldots\ldots (2.54)$$

Differentiating Eqn. (2.32) and taking the limit as $s\to o$, the expression for u'' at the pole can be derived as

$$\lim_{s \to 0} u'' = \left(\frac{2}{2+\nu}\right)\left(\frac{1-\nu^2}{C}\right) N_s' \left(\cos\phi_o + \alpha\beta'\sin\phi_o\right) - u'\phi_o'\tan\phi_o$$

hence from equation (2.46)

$$\lim_{S \to 0} N_\theta' = \frac{1}{2+\nu}\left((1+2\nu) N_s' + C\alpha\beta' \tan\phi_o\right).$$

Taking the limit of Equation (2.35) and eliminating N_θ',

$$\lim_{s\ o} H' = \frac{1}{3}\left((1-\nu)\phi' H + \frac{\alpha C \beta'}{\cos\phi_o}\right)\tan\phi_o - \frac{\alpha P}{2}\sin\phi_o \quad \ldots(2.55)$$

In order to evaluate M_s' at the pole, the expression of M_θ' in terms of M_s' has to be derived first, Differentiating Eqn (2.35) and taking the limit as $s \to o$,

$$\lim_{s \to o} \beta'' = \frac{2}{2+\nu}\left(M_s'/D + \frac{\nu\beta'\phi'}{2}\tan\phi_o\right)$$

which, when substituted in Eqn.(2.45), gives

$$\lim_{s \to 0} M_\theta' = \left(\frac{1+2\nu}{2+\nu}\right)M_s' - \left(\frac{1-\nu^2}{2+\nu}\right)\phi'\beta'\tan\phi_o$$

Taking the limit of Eqn. (2.36) and eleminating M_θ', the expression for M_s' is found to be

$$\lim_{s \to o} M_s' = -\frac{1}{3}\left(\alpha(2+\nu) H \sin\phi_o + D(1-\nu^2)\beta'\phi'\tan\phi_o\right) \quad \ldots(2.56)$$

Thus Eqns (2.49), (2.50), (2.51), (2.54), (2.56) form the fundamental set of differential equations applicable only at the pole, where α and ϕ' appearing in these equations are given by

Eqns. (2.52), and (2.40) respectively. These equations can further be simplified if it is assumed that the curvature of the underformed shell is continuous at the pole. In this case,

ϕ= o and, Thus fundamental set becomes -

$$u' = (1-\nu)\ H/C \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.57)$$

$$\beta' = M_s/(D(1+\nu)) \quad \ldots\ldots\ldots (2.58)$$

$$w' = 0 \quad \ldots\ldots\ldots\ldots (2.59)$$

$$\alpha = 1 + (1-\nu)\ H/C \quad \ldots\ldots\ldots\ldots (2.60)$$

$$V' = \alpha P/2 \quad \ldots\ldots\ldots\ldots (2.61)$$

$$H' = 0 \quad \ldots\ldots (2.62)$$

$$M_s' = 0 \quad \ldots\ldots\ldots\ldots (2.63)$$

2.5. LINEARIZED EQUATIONS OF AXISYMMETRIC SHELLS

Highly nonlinear equations are derived in sections 2.3 and 2.4. These nonlinear equations are always solved by the method of iteration in which arbitrary initial values have to be assigned to the fundamental dependent variables. Unless the initial values assigned to the dependent variables are a good approximation to their actual values, the iteration process fails to converge. For achieving convergence in the iteration process of solving nonliear equations, it is usually necessary to solve first the linearized version of the given nonlinear equaitons. The results of the linear solutions are then assigned as the initial values to the dependent variables of the nonlinear equaitons. The linear governing equations of axisymmetric deformation of shells of revolution are thus derived in this section.

The equations of small deflection theory follow from the forgoing Eqns. (2.19 - 2.36) together with (2.25) to the undeformed shell and by omitting all nonlinear terms in the remaining equations of the fundamental sets (2.19 - 2.36). The resulting equations are recorded below for ready reference :

$$\epsilon_\theta = u/r_o \quad . \tag{2.64}$$

$$K_\theta = \beta \cos \phi_o / r_o \tag{2.65}$$

$$N_s = H \cos\phi_o + V \sin\phi_o \tag{2.66}$$

$$\epsilon_s = (1 - \nu^2) N_s/C - \nu\epsilon_\theta \tag{2.67}$$

$$K_s = M_s/D - \nu K_\theta \tag{2.68}$$

$$N_\theta = \left(\frac{C}{1-\nu^2}\right) (\epsilon_\theta + \nu \epsilon_s) \tag{2.69}$$

$$M_\theta = D (K_\theta + \nu K_s) \tag{2.70}$$

$$w' = \epsilon_s \sin\phi_o - \beta\cos\phi_o \quad (2.71)$$

$$u' = \epsilon_s \cos\phi_o + \beta\sin\phi_o \quad (2.72)$$

$$\beta' = K_s \quad (2.73)$$

$$V' = -((V/r_o) \cos\phi_o - P \cos\phi_o) \quad (2.74)$$

$$H' = -((H\cos\phi_o - N_\theta)/r_o + P \sin\phi_o) \quad (2.75)$$

$$M_s' = -((M_s-M_\theta)\cos\phi_o)/r_o-(H \sin\phi_o - V \cos\phi_o) \quad (2.76)$$

The corresponding linearized equaitons at the pole are obtained in the same manner as Eqns. (2.64-2.76), Expressios for u', β' and w' remain the same, whereas, the three equations for equilibrium reduce to

$$V' = (P \cos\phi_o) /2 \quad (2.77)$$

$$H' = \frac{1}{3} ((1-\nu) \phi_o' H + \frac{C\beta'}{\cos\phi_o}) \tan\phi_o - \frac{P \sin\phi_o}{2} \quad (2.78)$$

$$M_s' = -\frac{1}{3}((2+\nu)H \sin\phi_o + D(1-\nu^2)\beta'\phi_o' \tan\phi_o) \quad (2.79)$$

In the case of continuous curvature of the meridian at the apex the linearized equaitons applicable at the pole remain the same as the Eqns (2.57-2.63) except that the value of α is to be replaced by unity in Eqn(2.61).

2.6. BOUNDARY CONDITIONS FOR AXISYMMETRIC SHELLS

The general bondary conditons of a shell at an edge, s_1 = constant, are to prescribe, in Sanders (39) notations,

$$
\left.
\begin{array}{ll}
N_{11} & \text{or } u_1, \\
N_{12} \quad \frac{1}{2}(3R_2^{-1} - R_1^{-1})M_{12} + \frac{1}{2}(N_{11} + N_{22}) & \text{or } u_2, \\
Q_1 + \alpha_2^{-1}\frac{\partial M_{12}}{\partial S_2} - \phi_1 N_{11} - \phi_2 N_{12} & \text{or } w, \\
\text{and} \quad M_{11} & \text{or } \phi_1,
\end{array}
\right\} \quad \ldots(2.80)
$$

where S_1 and S_2 are the shell coordinates along the principal lines of curvature, N and M are the stress and couple resultants; ϕ's are the rotations about respective axis; u and w are tangential and normal displacement components. When the quantities in Eqns (2.80) are specialized for axisymmetric deformations of shells of revolution, they reduce to prescribing

$$
\begin{array}{ll}
N_{11} & \text{or } u_1, \\
Q_1 - \phi_1 N_{11} & \text{or } w, \\
\text{and } M_{11} & \text{or } \phi_1
\end{array}
\quad \ldots\ldots\ldots\ldots(2.81)
$$

at an edge, s_1 = constant, From (3.81), it is seen that the boundary conditions consist of the specification of rotational, tangential and normal restraints at the edge. But in most of the practical cases of shell problems, the conditions of the horizontal and vertical restraints are known rather than those of the normal and tangential restraints, so it is concluded that it will be preferable to specify the bondary conditons in terms of the horizontal and vertical restraints from the point of view of practical applications. When this is done the boundary conditions in terms of the notations used in the body of this thesis will be to prescribe

$$\begin{aligned} &H \text{ or } u \\ &M_s \text{ or } \beta \qquad \qquad \ldots\ldots\ldots\ldots(2.82) \\ \text{and } &V \text{ or } W \end{aligned}$$

at the edge , s = constant.

2.7 NONDIMENSIONALIZATION OF THE EQUATIONS

It is always desirable to solve any engineering problem in terms of nondimensional quantities in order to decrease the number of input of physical parameters as well as to increase applicability of the solution. With this in mind and also to make the variables more or less of the same order of magnitude, the desplacement components and stress resultants are expressed as ratios of their actual values to those of the cercumferential desplacement and stress resultant of an unrestrained thin cylindrical shell. The independent variable s is normalized in such a manner that s_*, the total length of the shell meridian corresponds to unity (Fig.1). The normalized quantities are defined mathematically by the following equations;

$$\bar{s} = s/s_e, \quad \bar{u} = \frac{uEh}{P_o R^2}, \quad \bar{H} = \frac{H}{P_oR}, \quad \bar{V} = \frac{V}{P_oR}, \quad \bar{\beta} = \beta$$

$$\bar{M}_s = \frac{M_s}{P_oRh}, \quad \bar{M}_\theta = \frac{M_\theta}{P_oRh}, \quad \bar{N}_s = \frac{N_s}{P_oR}, \quad \bar{N}_\theta = \frac{N_\theta}{P_oR}$$

$$\bar{\epsilon}_\theta = \epsilon_\theta Eh\, s_e/(P_oR)^2, \quad \bar{\epsilon}_s = \epsilon_s Ehs_e/(P_oR)^2, \quad \bar{K}_\theta = K_\theta s_e \quad \ldots(2.83)$$

$$\bar{K}_s = K_s.s_e, \quad \bar{w} = \frac{wEh}{P_oR^2}, \quad \bar{C} = (1-\nu^2)s_e/R, \quad \bar{P}_o = B/E,$$

$$\bar{T} = R/h, \quad \bar{R} = s_e/R, \quad \bar{D} = 1/[12(1-\nu^2)\,\bar{P}_o\,\bar{T}^2\,\bar{R}], \quad \bar{P} = P/E,$$

$$\bar{L} = \bar{R}/(\bar{P}_o\bar{T}), \quad \bar{r}_o = r_o/s_e,$$

Where R is is the radius of the cylindrical part in case of pressure vessel problems or in general R= R_o at s_e. With the help of normalized qualities defined in Eqn (2.83), the fundamental set of Eqns (2.64-2.79) (linear theory) becomes

$$\bar{\epsilon}_\theta = \bar{u}/\bar{r}_o \quad \ldots\ldots\ldots\ldots\ldots(2.84)$$

$$\bar{K}_\theta = \bar{\beta}\cos\phi_o/\bar{r}_o \quad (2.85)$$

$$\bar{N}_s = \bar{H}\cos\phi_o + \bar{V}\sin\phi_o \quad (2.86)$$

$$\bar{\epsilon}_s = \bar{C}\bar{N}_s - \nu\,\bar{\epsilon}_\theta \quad (2.87)$$

$$\bar{K}_s = \bar{M}_s/\bar{D} - \nu\,\bar{K}_\theta \quad (2.88)$$

$$\bar{N}_\theta = (\bar{\epsilon}_\theta + \nu\,\bar{\epsilon}_s)\,/\,\bar{C} \quad (2.89)$$

$$\bar{M}_\theta = \bar{D}\,(\bar{K}_\theta + \nu\,\bar{K}_s) \quad (2.90)$$

$$\bar{w}' = \bar{\epsilon}_s\sin\phi_o - \bar{\beta}\cos\phi_o\,.\,\bar{L} \quad (2.91)$$

$$\bar{u}' = \bar{\epsilon}_s\cos\phi_o + \bar{\beta}\sin\phi_o.\,\bar{L} \quad (2.92)$$

$$\bar{\beta}' = \bar{K}_s \quad (2.93)$$

$$\bar{V}' = -(\bar{V}\,\cos\phi_o/\bar{r}_o - \bar{R}\,\bar{f}(s)\cos\phi_o \quad (2.94)$$

$$\bar{H}' = -((\bar{H}\cos\phi_o - \bar{N}_\theta)/\bar{r}_o + \bar{R}\,\bar{f}(s)\sin\phi_o \quad (2.95)$$

$$\bar{M}_s' = -\cos\phi_o(\bar{M}_s - \bar{M}_\theta)/\bar{r}_o - \bar{R}.\bar{T}.(\bar{H}\sin\phi_o - \bar{V}\cos\phi_o) \quad ..(2.96)$$

The corresponding nonlinear equations of the fundamental set in nondimensional form are as follows :

$$\bar{\epsilon}_\theta = \bar{u}/\bar{r}_o \quad ....(2.97)$$

$$\phi = \phi_o - \bar{\beta} \quad ....(2.98)$$

$$\bar{K}_\theta = (\sin\phi_o - \sin\phi)/\bar{r}_o \quad ....(2.99)$$

$$\bar{N}_s = \bar{H}\cos\phi + \bar{V}\sin\phi \quad ....(2.100)$$

$$\bar{\epsilon}_s = \bar{C}\bar{N}_s - \nu\bar{\epsilon}_\theta \quad ....(2.101)$$

$$\bar{K}_s = \bar{M}_s/\bar{D} - \nu\bar{K}_\theta \quad ....(2.102)$$

$$\bar{N}_\theta = (\bar{\epsilon}_\theta + \nu\bar{\epsilon}_s)/\bar{C} \quad ....(2.103)$$

$$\bar{M}_\theta = \bar{D}(\bar{K}_\theta + \nu\bar{K}_s) \quad ....(2.104)$$

$$\bar{\alpha} = \bar{L} + \bar{\epsilon}_s \quad ....(2.105)$$

$$\bar{r} = \bar{L}.\bar{r}_o + \bar{u} \quad ....(2.106)$$

$$\bar{w}' = \bar{\alpha}\sin\phi - \bar{L}\sin\phi_o \quad ....(2.107)$$

$$\bar{u}' = \bar{\alpha}\cos\phi - \bar{L}\cos\phi_o \quad .... (2.108)$$

$$\bar{\beta}' = \bar{K}_s \quad .... (2.109)$$

$$\bar{V}' = -\bar{\alpha}\cos\phi\,(\bar{V}/\bar{r} - \bar{P}\bar{T}) \quad .... (2.110)$$

$$\bar{H}' = -\bar{\alpha}((\bar{H}\cos\phi - \bar{N}_\theta)/\bar{r} + \bar{P}\bar{T}\sin\phi) \quad .. (2.111)$$

$$\bar{M}_s' = \bar{\alpha}\cos\phi(\bar{M}_\theta - \bar{M}_s)\bar{r} - \bar{\alpha}\bar{P}\bar{T}^2(\bar{H}\sin\phi - \bar{V}\cos\phi) \quad (2.112)$$

The equations at the pole corresponding to the nonlinear set take the folowing form after normalization :

$$\bar{u}' = (1-\nu)\,\bar{R}\,\bar{H}\cos^2\phi_o \quad ... (2.113)$$

$$\bar{w}' = (1-\nu)\ \bar{R}\ \bar{H}\ \cos\phi_o\ \sin\phi_o \quad \ldots\ (2.114)$$

$$\bar{\beta}' = \bar{M}_s\ /\ ((1-\nu)\ \bar{D}) \quad \ldots\ (2.115)$$

$$\bar{\alpha} = \bar{L} + (1-\nu)\ \bar{R}\ \bar{H}\ \cos\phi_o \quad \ldots\ (2.116)$$

$$\bar{V}' = \frac{1}{2}\bar{\alpha}\bar{P}\ \bar{T}\ \cos\phi_o \quad \ldots\ (2.117)$$

$$\bar{H}' = \frac{1}{3}\left((1-\nu)\phi'\bar{H} + \bar{\alpha}\ \bar{\beta}'\ (\bar{R}\cos\phi_o)\right)\tan\phi_o - \frac{1}{2}\bar{\alpha}\ \bar{P}\ \bar{T}\ \sin\phi_o \quad \ldots\ (2.118)$$

$$\bar{M}_s' = \frac{1}{3}(\bar{\alpha}\bar{P}\ \bar{T}^2\ \bar{H}\ \sin\phi_o + \beta'\phi'\tan\phi_o/(12\ \bar{P}\ \bar{R}\ \bar{T}^2)) \quad \ldots\ (2.119)$$

Eqns (2.113-2.119) may be simplified in case of continuous meridian at the pole as :

$$\bar{u}' = \bar{C}\bar{H}/(1+\nu) \quad \ldots\ (2.120)$$

$$\bar{w}' = 0 \quad \ldots\ (2.121)$$

$$\bar{\beta}' = \bar{M}_s/((1+\nu)\ \bar{D}) \quad \ldots\ (2.122)$$

$$\bar{V}' = \bar{\alpha}\ \bar{P}/2 \quad \ldots\ (2.123)$$

$$\bar{H}' = 0 \quad \ldots\ (2.124)$$

$$\bar{M}_s' = 0 \quad \ldots\ (2.125)$$

Eqns. (2.113 - 2.125) may be linearized as before to obtain the corresponding equaitons at the pole for the linear theory. The nondimensionalized form employed here will make the linear solutions independent of the loading parameter.

It should be noted that some of the nondimensional shell parameters in Eqns.(2.83) are defined in terms of s_e which will depend on the geometry of the meridian and thus should be derived for each individual case. In some cases there is no closed form expression for s_e and, therefore, s_e has to be evaluated either from a series expression or by numerical integration. The smae is true for the expressions of r_o and ϕ_o in terms of s. There may not be any closed form expressions for r_o and ϕ_o and thus numerical integration has to be used. The evaluation of shell parameters and the expressions of r_o and ϕ_o in terms of $\bar{s}$ for general case of composite shells of revolution are given below

<u>General Case of shells of revolution</u>

For the general composite shell whose meridian is composed of cylindrical, spherical and conical elements (Fig.1), the total length s_e of the shell meridian has to be determined for each individual case. The constant $\bar{R}$, defined as s_e/R (R being the radius of the shell at the base), is then directly read in by the program. In addition the value of ϕ_o for each element at its starting point along with its type (that is, cylindrical or pherical or conical element) is required.

<u>Line element</u> : If a segment s_i of the meridian is a line element, the meridional angle ϕ_o remains constant over the segment s_i and its value is

$$\phi_o = (\phi_o)_i \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.126)$$

Where subscript i refers to the starting point of the element. The expression for r_o becomes

$$\bar{r}_o = (\bar{r}_o)_i - ((\bar{s})_i - \bar{s}) \cos(\phi_o)_i \ \ldots\ldots \ (2.127)$$

Circular element:

If any segment s_i of the meridian is a circular element, the quantities $\bar{r}_o$ and over this segment s_i are given by

$$\phi_o = (\phi_o)_i - \frac{((\bar{s})_i - \bar{s}) \sin(\phi_o)_i}{(\bar{r}_o)_i} \quad \text{....(2.128)}$$

$$\bar{r}_o = \frac{(\bar{r}_o)_i \sin\phi_0}{\sin(\emptyset_o)_i} \quad \text{.......(2.129)}$$

Elliptic element :

If a segment s_i of the meridian is a portion of an ellipse, the quantities $\emptyset_o$ and r_o at any point over this segment have to be evaluated from the numerical integration of eqn (2.128) for which the values of $(\emptyset_o)_i$ and Z are necessary.

CHAPTER 3

METHOD OF SOLUTION

3.1 INTRODUCTION TO MULTISEGMENT INTEGRATION

The fundamental set of linear differential equaitons (2.84 - 2.96) and nonlinear differential equations (2.97-2.112) along with their corresponding forms at the apex and the boundary conditons (2.82) have to be integrated over a finite range of the independent variable s. But the numerical integration of these equaitons is not possible beyond a very limited range of s due to the loss of accuracy in solving for the unknown boundary values, as pointed out by Kalnins (22), That is why , the multisegment method of integration developed by Kalnins and Lestingi (24) has been used in this analysis.

It is supposed that a set of 6 first order nonlinear differential equations are given to be :

$$\left.\begin{aligned} \frac{dy_1(x)}{dx} &= f_1\,(x, y_1(x), y_2(x) \ldots\ldots\ldots\ldots\ldots y_6(x)) \\ \frac{dy_2(x)}{dx} &= f_2\,(x, y_1(x), y_2(x) \ldots\ldots\ldots\ldots\ldots y_6(x)) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{dy_6(x)}{dx} &= f_6\,(x, y_1(x), y_2(x) \ldots\ldots\ldots\ldots\ldots y_6(x)) \end{aligned}\right\} \quad (3.1)$$

where, ($y_k(x)$, K = 1,6) are dependent foundamental variables, and x is the independent variable.

The above equations can be written in the form

$$\frac{dy(x)}{dx} = F\,(x, y_1(x),\ y_2\,(x)\ \ldots\ldots\ldots\ldots,\ y_6(x)) \qquad \text{------(3.2)}$$

where $y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \text{------} \\ \text{------} \\ y_6(x) \end{bmatrix}$, (6,1) fundamental variable matrix,

and $F = \begin{bmatrix} f_1 \\ f_2 \\ \text{--} \\ \text{--} \\ f_6 \end{bmatrix}$, (6,1) matrix of nonlinear functions of fundamental variables

It is assumed here for convenience that the first 3 eleterms of $y(x_1)$ and last 3 elements of $y\,(x_{M+1})$ are prescribed by the boundary conditions, where x_1 is the starting boundary and x_{M+1} is the finishing boundary (Fig. 5).

If at the initial point x_1 of the segment s_1 (Fig - 5), a set of values $y\,(x_1)$ is prescribed for the variables of Eqns. (3.2), then the variables at any x within s_1 can be expressed as

$$y(x) = f(y_1(x_1),\ y_2(x_1), \ldots\ldots\ldots\ldots,\ y_6(x_1)) \qquad \ldots..(3.3)$$

where the function f is uniquely dependent on x and the system of equaitons (3.2).

From the set of equations (3.3), the expression for small change in the values of the independent variables can be written as

$$\left.\begin{aligned}
\partial y_1(x) &= \frac{\partial y_1(x)}{\partial y_1(x_i)}\partial y_1(x_i) + \frac{\partial y_1(x)}{\partial y_2(x_i)}\partial y_2(x_i) + \cdots + \frac{\partial y_1(x)}{\partial y_6(x_i)}\partial y_6(x_i) \\
\partial y_2(x) &= \frac{\partial y_2(x)}{\partial y_1(x_i)}\partial y_1(x_i) + \frac{\partial y_2(x)}{\partial y_2(x_i)}\partial y_2(x_i) + \cdots + \frac{\partial y_2(x)}{\partial y_6(x_i)}\partial y_6(x_i) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\partial y_6(x) &= \frac{\partial y_6(x)}{\partial y_1(x_i)}\partial y_1(x_i) + \frac{\partial y_6(x)}{\partial y_2(x_i)}\partial y_2(x_i) + \cdots \frac{\partial y_6(x)}{\partial y_6(x_i)}\partial y_6(x_i)
\end{aligned}\right\} \quad (3.4)$$

Eqns. (3.4) can be written in matrix form as

$$\begin{bmatrix} \partial y_1 \\ \partial y_2 \\ \cdots \\ \cdots \\ \partial y_6 \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1(x)}{\partial y_1(x_i)} & \frac{\partial y_1(x)}{\partial y_2(x_i)} & \cdots\cdots & \frac{\partial y_1(x)}{\partial y_6(x_i)} \\ \frac{\partial y_2(x)}{\partial y_1(x_i)} & \frac{\partial y_2(x)}{\partial y_2(x_i)} & \cdots\cdots & \frac{\partial y_2(x)}{\partial y_6(x_i)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_6(x)}{\partial y_1(x_i)} & \frac{\partial y_6(x)}{\partial y_2(x_i)} & \cdots\cdots & \frac{\partial y_6(x)}{\partial y_6(x_i)} \end{bmatrix} \begin{bmatrix} \partial y_1(x_i) \\ \partial y_2(x_i) \\ \cdots \\ \cdots \\ \partial y_6(x_i) \end{bmatrix} \quad (3.5)$$

or $\partial y(x) = Y_i(x)\,\partial y(x_i)$ (3.5a)

where

$$Y_i(x) = \begin{bmatrix} \frac{\partial y_1(x)}{\partial y_1(x_i)} & \frac{\partial y_1(x)}{\partial y_2(x_i)} & \cdots\cdots & \frac{\partial y_1(x)}{\partial y_6(x_i)} \\ \frac{\partial y_2(x)}{\partial y_1(x_i)} & \frac{\partial y_2(x)}{\partial y_2(x_i)} & \cdots\cdots & \frac{\partial y_2(x)}{\partial y_6(x_i)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_6(x)}{\partial y_1(x_i)} & \frac{\partial y_6(x)}{\partial y_2(x_i)} & \cdots\cdots & \frac{\partial y_6(x)}{\partial y_6(x_i)} \end{bmatrix}$$

$$\partial y(x) = \begin{bmatrix} \partial y_1(x) \\ \partial y_2(x) \\ \cdots \\ \cdots \\ \partial y_6(x) \end{bmatrix} \quad (3.5b) \quad \text{and} \quad y(x_i) = \begin{bmatrix} \partial y_1(x_i) \\ \partial y_2(x_i) \\ \cdots \\ \cdots \\ \partial y_6(x_i) \end{bmatrix} \quad (3.5c)$$

Equations (3.5) are expressed in finite difference form as

$$(y(x)-y^t(x)) = Y_i(x)\,(y(x_i) - y^t(x_i) - y^t(x_i)) \quad ---(3.6)$$

where y(x) denotes an iterated solution state based on the condition of continuity of the variables at the nodal points and $y^t(x)$ denotes a trial solution state. Evaluating Equns. (3.6) at $x = x_i$, it is found that

$$(y(x_i) - y^t(x_i)) = Y_i(x_i)(y(x_i)-y^t(x_i)) \quad --------(3.7)$$

Therefore, $Y_i(x_i) = I$

where I denotes (6,6) unit matrix. Evaluating Eqns, (3.6) at $x = x_{i+1}$, it is found that

$$(y(x_{i+1}) - y^t(x_{i+1})) = Y_i(x_{i+1})(y(x_i)-y^t(x_i)) \quad .... \quad (3.8)$$

Equns (3.8) can be rearranged as

$$Y_i(x_{i+1})\; y(x_i) - y(x_{i+1}) = -Z_i(x_{i+1}) \quad ....(3.9)$$

where, $Z_i(x_{i+1}) = y^t(x_{i+1}) - Y_i(x_{i+1})\, y^t(x_i)$.

In Eqns (3.9), $y(x_i)$, $y(x_{i+1})$ and $Y_i(x_{i+1})$ are unknown. In order to determine the elements of $Y_i(x)$, the th column of $Y_i(x)$ can be regarded as a set of new variables, which is a solution of an initial value problem governed within each segment by a linear system of first order differential equations, obtained from Equns (3.2) by differentiating with respect to y_j in (x_i) in the form

$$\frac{d}{dy_j(x_i)} \frac{dy}{dx} = \frac{d}{dy_j(x_i)} (F(x, y_1, y_2 \ldots y_6))$$

which gives,

$$\frac{d}{dx}\left(\frac{dy}{dy_j(x_i)}\right) = \frac{dF}{dy_j(x_i)} \qquad \ldots\ldots\ldots\ldots \quad (3.10)$$

Thus the columns of the matrix $Y_i(x)$ are defined as the solutions of 6 initial value problems governed by (3.9) in s_i (with j = 1,2.......6) having initial values specified by Eqns (3.7). It should be noted that the initial value integration is possible only when the original equations of y are already integrated with the initial value of y^t x_i). Now to obtain the iterated solution, Eqns (3.9) are written as a partitioned matrix product of the form

$$\begin{bmatrix} y^1(x_{i+1}) \\ \hline y^2(x_{i+1}) \end{bmatrix} = \left[\begin{array}{c|c} Y_{1i}(x_{i+1}) & Y_{2i}(x_{i+1}) \\ \hline Y_{3i}(x_{i+1}) & Y_{4i}(x_{i+1}) \end{array}\right] \begin{bmatrix} y^1(x_i) \\ \hline y^2(x_i) \end{bmatrix} + \begin{bmatrix} Z_{1i}(x_{i+1}) \\ \hline Z_{2i}(x_{i+1}) \end{bmatrix} \ldots\ldots (3.11)$$

where $y^1(x_{i+1}) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $y^2(x_{i+1}) = \begin{bmatrix} y_4 \\ y_5 \\ y_6 \end{bmatrix}$

So, a pair of equations can be written from Eqn. (3.11) to replace each of eqns (3.9) as

$$\left.\begin{aligned} ((Y_{1i}(x_{i+1}))(y^1(x_i)) + (Y_{2i}(x_{i+1}))(y^2(x_i)) - (y^1(x_{i+1})) &= -Z_{1i}(x_{i+1}), \\ (Y_{3i}(x_{i+1}))(y^1(x_i)) + (Y_{4i}(x_{i+1}))(y^2(x_i)) - (y^2(x_{i+1})) &= -Z_{2i}(x_{i+1}). \end{aligned}\right\} (3.12)$$

Replacement of Eqns (3.9) is done to seperate known bounday conditions from the unknowns. Thus from Eqns (3.12), a simultaneous systems of 2M linear metrix equations is obtained in which the known cefficients (Y_{ji} (x_{i+1})) and ($Z_{ij}(x_{i+1})$) are (3,3) and 3,1) matrices respectively, and the unknows ($y^j(x_i)$) are (3,1) matrices. Since ($y^1(x_1)$) and (y^2 (x_{M+1}) are known from the boundary conditions, there are exactly 2M unknowns : ($y^1(x_{i+1})$with i = 2,3....., M+1, and ($y^2(x_i)$) with i = 1,2,3 .....,M.

The problem is, therefore, well set in order to obtain the solution of the linear equations (3.12), Gaussian elimination method is used. Gaussian elimination method leads to a triangularized set of linear equaitons which for the specific case of Equns. (3.12), takes the following form :

$$\begin{bmatrix} E_1 & -I & 0 & 0\ldots 0 & 0 \\ 0 & C_1 & -I & 0\ldots 0 & 0 \\ 0 & 0 & E_2 & -I\ldots 0 & 0 \\ \cdot\cdot & \cdots & & & \\ \cdot\cdot & \cdots & & & \\ 0 & 0 & 0 & 0\ldots E_M & -I \\ 0 & 0 & 0 & 0\ldots 0 & C_M \end{bmatrix} \begin{bmatrix} y^2(x_1) \\ y^1(x_2) \\ y^2(x_2) \\ \cdots \\ \cdots \\ y^2(x_M) \\ y^1(x_{M+1}) \end{bmatrix} = \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ \cdots \\ \cdots \\ A_M \\ B_M \end{bmatrix}$$

$$(E_i)\ (y^2(x_i)) - (y^1(x_{i+1})) = (A_i)$$

or ......(3.13)

$$(C_i)\ (y^1(x_{i+1})) - (y^2(x_i + 1)) = (B_i)$$

for, i = 1,2,3.......M. Using the rotations(Z_{JI}) and (Y_{ji}) in place of the symbols, $(Z_{ji}(x_{i+1})$ and $(Y_{ji}\ (x_{i+1}))$, the (3,3) matrices (E_i) and(C_i) in the Eqns. (3.13) are defined by

$$(E_1) = (Y_{21}),\ (C_1) = Y_{41})$$

and $(E_i) = (Y_{2i}) + (Y_{1i})\ (C_{i-1})^{-1}$

$(C_i) = ((Y_{4i}) + (Y_{3i})\ (C_{i-1})^{-1}(E_i)^{-1}$

for i = 2,3,........., M.

The (3,1) matrices (A_i) and (B_i) are given by

$$(A_1) = -(Z_{11}) - (Y_{11})\ (y^1\ (x_1))$$

$$(B_1) = -(Z_{21}) - (Y_{31})\ (y^1\ (x_1) - (Y_{41})(E_1)^{-1}\ (A_1)$$

and $(A_i) = -(Z_{1i}) - (Y_{1i})\ (C_{i-1})^{-1}\ (B_{i-1}),$

$(B_i) = -(Z_{2i}) - (Y_{3i})\ (C_{i-1})^{-1}\ (B_{i-1}) - ((Y_{4i}) + (Y_{3i})\ (C_{i-1})^{-1})\ (E_i)^{-1}\ (A_i)$

for i = 2,3,........,M-1.

and $(A_M) = -(Z_{1M}) - (Y_{1M})\ (C_{M-1})^{-1}(B_{M-1})$

$(B_M) = (y^2\ (x_{M+1})) - (Z_{2M}) - (Y_{3M})\ (C_{M-1})^{-1}(B_{M-1}) - ((Y_{4M}) + (Y_{3M})\ (C_{M-1})^{-1})\ (E_M)\ (A_M)$

The unknowns of (3.13) are obtained by

$(y^1\ (x_{M+1})) = (C_M)^{-1}\ (B_M)$

$(y^2(x_M)) = (E_M)^{-1}((\ y^1\ (x_{M+1})) + (A_M)),$

and $(y^1(x_{M-i+1})) = (C_{M-i})^{-1}((y^2(x_{M-i+1})) + (B_{M-i}))$

$(y^2(x_{M-i})) = (E_{M-i})^{-1}((y^1(x_{M-i+1})) + (A_{M-i}))$.

for $i = 1,2,3..........., M-1$.

Assuming $y(x_i)$ as the next trial solution, $y^t(x_i)$, the process is repeated until the integration results of Eqns. (3.1) at x_{i+1}, as obtained from the integrations in segment S_i with the initial values $y(x_i)$, match with the elements of $y(x_{i+1})$ as obtained from (3.9) and also with the bondary conditions at x_{M+1}. This completes the formal solution of the problem. Therefore, the method of multisegment integration involves the following steps :

(i) Initial-value integrations of Eqns. (3.1) in each of M segments. To start,the initial values $y_j(x_i)$ for the integration over any segment are arbitrary.

(ii) Initial value integration for the six additional sets of variables of matrix (3.5a) over each of M segments.

(iii) Solution of M matrix equaitons which ensures the continuity of variables of Eqns (3.2) at the nodal points of the segments including the given boundary conditions at the two end nodal points.

(iv) Repetition of steps (i) to (iii) with initial values $y_j(x_i)$ of steps (i) replaced each time by their improved values obtained in step (iii) from the solution of continuity equaiton. The process is continued until the values of the variables of Eqns (3.2) at the end point of any segment as obtained from the initial value integration in step (i) match with their initial values in the next segment obtained from the solutions of the continuity equatios in step (iii).

3.2 DERIVATION OF ADDITIONAL EQUAITONS

In the multisegment integration technique for a set of ordinary differential equations it has already been noted that in addition to the integration of the given equations, it is required to integrate another 6 set of equations represented by (3.10). Thus in order to apply the method of multisegment integrtion, differential equations corresponding to Eqns. (3.10) for the 36 additional variables as represented in (3.59) have to be derived. These differential equations can be obtained by differentiating Eqns. (2.84-2.96) for the linear solution and Eqns. (2.97-2.112) for nonlinear solution with respect to each fundamental variable. As the variables in any column of (3.5a) have the same form, it is required to derive here the system of equaitons (3.10) for the variables of any column of (3.59) where the new variables are identified from the fundamental variables by the subscript a.

From the nonlinear equations (2.97 - 2.112), differentiation in succession gives

$$\bar{\epsilon}_{\theta a} = \bar{u}_a/\bar{r}_o \quad \ldots\ldots\ldots\ldots \quad (3.14)$$

$$\phi_a = -\bar{\beta}_a \quad \ldots\ldots\ldots\ldots \quad (3.15)$$

$$\bar{K}_{\theta a} = \bar{\beta}_a \cos\phi / \bar{r}_o \quad \ldots\ldots\ldots\ldots \quad (3.16)$$

$$\bar{N}_{sa} = (\bar{H}_a - \bar{V}\bar{\beta}_a)\cos\phi + (\bar{H}\bar{\beta}_a + \bar{V}_a)\sin\phi \ldots\ldots (3.17)$$

$$\bar{\epsilon}_{sa} = \bar{C}\bar{N}_{sa} - \nu\bar{\epsilon}_{\theta a} \quad \ldots\ldots\ldots (3.18)$$

$$\bar{K}_{sa} = \bar{M}_{sa}/\bar{D} - \nu\bar{K}_{\theta a} \quad \ldots\ldots\ldots (3.19)$$

$$\bar{N}_{\theta a} = (\bar{\epsilon}_{\theta a} + \nu\bar{\epsilon}_{sa})/\bar{C} \quad \ldots\ldots\ldots (3.20)$$

$$\bar{M}_{\theta a} = \bar{D}(\bar{K}_{\theta a} + \nu\bar{K}_{sa}) \quad \ldots\ldots\ldots (3.21)$$

$$\bar{\alpha}_a = \bar{\epsilon}_{sa} \quad \ldots\ldots\ldots (3.22)$$

$$\bar{r}_a = \bar{u}_a \quad \ldots\ldots\ldots (3.23)$$

$$\bar{u}'_a = \bar{\alpha}_a\cos\phi + \bar{\beta}_a\bar{\alpha}\sin\phi \quad \ldots\ldots\ldots (3.24)$$

$$\bar{w}'_a = \bar{\alpha}_a\sin\phi - \bar{\alpha}\bar{\beta}_a\cos\phi \quad \ldots\ldots\ldots (3.25)$$

$$\bar{\beta}'_a = \bar{K}_{sa} \quad \ldots\ldots\ldots (3.26)$$

$$\bar{V}'_a = -(\bar{\alpha}_a\cos\phi + \bar{\alpha}\bar{\beta}_a\sin\phi)(\bar{V}/\bar{r} - \bar{P}\bar{T}) - \bar{\alpha}\cos\phi(\bar{V}_a/\bar{r} - \bar{V}\bar{r}_a/\bar{r}^2) . (3.27)$$

$$\bar{H}'_a = -\bar{\alpha}_a((\bar{H}\cos\phi - \bar{N}_\theta)/\bar{r} + \bar{P}\bar{T}\sin\phi) - \bar{\alpha}((\bar{H}_a\cos\phi + \bar{\beta}_a\bar{H}\sin\phi - \bar{N}_{\theta a} - \bar{u}_a(\bar{H}\cos\phi - \bar{N}_\theta)/\bar{r})/\bar{r} - \bar{P}\bar{T}\bar{\beta}_a\cos\phi) . (3.28)$$

$$\bar{M}'_{sa} = (\bar{\alpha}_a\cos\phi + \bar{\beta}_a\bar{\alpha}\sin\phi)((\bar{M}_\theta - \bar{M}_s)/\bar{r} + \bar{P}\bar{T}^2\bar{V}) + \bar{\alpha}(\cos\phi(\bar{P}\bar{T}^2\bar{V}_a + (\bar{M}_{\theta a} - \bar{M}_{sa} - \bar{u}_a(\bar{M}_\theta - \bar{M}_s)/\bar{r}) - \bar{P}\bar{T}^2\bar{H}_a\sin\phi) - \bar{P}\bar{T}^2\bar{H}(\bar{\alpha}_a\sin\phi - \bar{\alpha}\bar{\beta}_a\cos\phi) . \ldots\ldots\ldots (3.29)$$

At the pole, the corresponding equations are obtained from (2.113-2.119) as

$$\bar{u}'_a = (1-\nu)\,\bar{R}\,\bar{H}_a\cos^2\phi_o \quad \ldots\ldots\ldots \quad (3.30)$$

$$\bar{w}'_a = (1-\nu)\,\bar{R}\,\bar{H}_a\cos\phi_o\sin\phi_o \quad \ldots\ldots\ldots \quad (3.31)$$

$$\bar{\beta}'_a = \bar{M}_{sa}/((1-\nu)\,\bar{D}) \quad \ldots\ldots\ldots \quad (3.32)$$

$$\bar{\alpha}'_a = (1-\nu)\,\bar{R}\,\bar{H}_a\cos\phi_o \quad \ldots\ldots\ldots \quad (3.33)$$

$$\bar{V}'_a = \frac{1}{2}\,\bar{P}\,\bar{T}\cos\phi_o\cdot\bar{\alpha}_a \quad \ldots\ldots\ldots \quad (3.34)$$

$$\bar{H}'_a = \frac{1}{3}\left((1-\nu)(\phi'\bar{H}_a-\bar{\beta}'_a\bar{H}) + (\bar{\alpha}_a\bar{\beta}' + \bar{\alpha}\bar{\beta}_a)/(\bar{R}\cos\phi_o)\right)\tan\phi_o - \frac{1}{2}\bar{\alpha}_a\,\bar{P}\,\bar{T}\sin\phi_o \quad \ldots\ldots (3.35)$$

$$\bar{M}'_{sa} = \frac{1}{3}\,(\bar{P}\,\bar{T}^2\sin\phi_o\,(\bar{\alpha}_a\bar{H} + \bar{\alpha}\,\bar{H}_a + (\bar{\beta}'_a\phi' - \bar{\beta}'\phi'_a)\tan\phi_o/(12\bar{P}\,\bar{R}\,\bar{T}^2)) \quad \ldots\ldots\ldots (3.36)$$

Eqns. (3.14-3.29) which takes the form (3.30-3.36) at s = o, have to be integrated as initial value problem 6 times in each segment with the initial values given by (3.7). It should be noted that the equaitons (3.14-3.36) contain not only the variables of (3.5a) but also the variables of the fundamental set. Thus eqns. (3.14-3.36) cannot be integrated unless the fundamental variables are stored for use in Eqns (3.14-3.36). It should be further pointed out that one point integration formula can not be used for the integrtion of Equns (3.14-3.36) since this formula needs evaluation of derivatives at intermediate points where the variables are never evaluated.

The corresponding equaitons for the linear theory are given by the homogeneous form of Eqns. (2.84-2.96) and thus readily obtainable by dropping the load terms in Eqns.(2.84-2.96).

3.3 TREATMENT OF BONDARY CONDITIONS

In the introduction of the method of multisegment integration, it was assumed that the first 3 elements of y (x) at x_1 and last 3 elements of y (x) at x_{M+1} were prescribed as the bondary conditions. But, in general, the boundary conditions are given as

$$T_1 y(x_1) = b_1 \text{ at } x_1, \text{ and} \qquad \ldots\ldots\ldots\ldots\ldots (3.37)$$

$$T_{M+1}\, y(x_{M+1}) = b_{M+1} \text{ at } x_{M+1}$$

in which any 3 elements of b_1 and any 3 elements of b_{M+1} are specified as boundary conditons. The sysmbols T_1 and T_{M+1} represent nonsingular (6,6) matrices which are known from the specification of the boundary conditons at the ends of the interval.

By rearranging the rows of T_1 and T_{M+1} in a special order, Eqns. (3.37) can always be stated in a manner such that the prescribed elements of b_1 and b_{M+1} become respectively the first 3 and last 3 elements of b_1 and b_{M+1} when this is achieved, evaluation of (3.9) at i = 1 and i =M, and then elimination of $y(x_1)$ and $y(x_{M+1})$ by means of (3.37) yields.

$y(x_1)$ and $y(x_{M+1})$ by means of (3.37) yields.

$$Y_1(x_2)\,T_1^{-1}\,b_1 - y(x_2) = -Z_1(x_2) \quad \text{.........(3.38)}$$

$$T_{M+1}\,Y_M(x_{M+1})\,y(x_M) - b_{M+1} = -T_{M+1}\,Z_M(x_{M+1}) \quad \text{....(3.39)}$$

The form and notation of (3.9) can be retained if it is regarded that the coefficient matrices $Y_1(x_2)$, $Y_M(x_{M+1})$, $Z_M(x_{M+1})$ occurring in (3.9) represent $Y_1(x_2)\,T_1$, $T_{M+1}\,Y_M(x_{M+1})$ and $T_{M+1}\,Z_M(x_{M+1})$ respectively.

In doing so, the solution of (3.9) will not yield $y(x_1)$ and $y(x_{M+1})$ but rather the transformed variables b_1 and b_{M+1}. When $y(x_1)$ and $y(x_{M+1})$ are derived they can be obtained by the inversion of the matrix equations (3.37).

It should be noted here that with reference to the boundary conditions (2.82) stated in terms of the fundamental variables, it is obvious that the matrices T_1 and T_{M+1} are both unit matrices of order 6. The construction of T_1 and T_{M+1}, in accordance with any possible statement of (2.82), so that the Eqns (3.37) are in order, is treated in Appendix A.

CHAPTER 4

RESULTS AND DISCUSSION

4.1. Reliability and Validity of the Analysis :

It is always desirable that the solutions obtained by any new technique should be compared with the available results in the literature in order to determine the reliability and validity of the method employed. In other words it should be ascertained that no error due to logic is committed in formulating the problem, in method of solution and, in this particular case, no mistake is made in the computer programming. Keeping all these in mind, a number of standard problems are solved with the present method of solution and later the results are compared with the corresponding analytical solution or solution by some other method as available in the literature. On the basis of this comparsion, reliability and validity of the method employed here are determined.

The multisegment method of integration and the governing equations of shells as used in the present analysis, had been used by a number of authors earlier. Uddin (46)used this method in finding the solution for pressurized composite shell with clamped edge made-up of an inverted conical frustum, a cylindrical part, and a spherical part. He also found the variation of meridional stress and circumferential stress along the meridian of an ellipsoidal-head pressure vessel based on both the linear and nonlinear theories by multisegment integration which had earlier been worked out by Kraus et al (28) and it was found that there was hardly any difference between these two results. Haque (16) took the full advantage of the fact that a hemispherical shell with radius A and a semiellipsoidal shell with the ratio of major to minor exes, B/A = 1, are identical and found that the solution for ellipsoidal shells with B/A = 1 differed from that for hemispherical shells available in the literature (3) after six digits. Rahman (38) obtained the solutions of imperfect semi-ellipsoidal shells with rigidly fixed edges in which different values of parameters, degree of imperfaction and position of imperfect segment were used. Rahman observed that his results of imperfect ellipsoidal shells converged to those of Haque when imperfections were gradually reduced.

The above developments prove that the multisegment method of integration and the linear and nonlinear governing equations of shells as employed in this analysis is highly accurate.Actually, in an indirect way, the accuracy of the method of multisegment integration is self ascertaining. Once the values of the fundamental variables at the nodal points are known from the multisegment method of integration, the fundamental set of the governing differential equations can be integrated over each segment of the meridian as an initial value integration of the fundamental set of differential equations. If the values of the fundamental variables at the end of each segement s_i, as obtained from the initial value integration, match upto six or seven digits with their respective initial values for the respective subsequent segments s_{i+1} for i = 1,2,3..... M and also with the boundary conditions at the edges, then it can be concluded that the results are correct upto six or seven digits of their numerical figures.

Further, for establishing the reliability and validity of the method, a cylindrical shell containing a fluid of density γ, fixed at the base and free at the upper end, was considered. This particular problem was solved by the present method of solution because an approximate analytical solution, based on the general theory of cylindrical shells is available in the literature in closed form (45). Here, for solving the Cylindrical shell problem, axially varying internal pressure on the shell surface was assumed to be applied by a liquid column of specific weight γ.

The shell meridian was divided into ten seg-ments of equal lengths. The shell and its parameters are presented in Fig. (6). Using the computer programme of the present analysis the result of this cylindrical shell is obtained based on both the linear and nonlinear theories under axially varying load. These results compare quite well with the analytical solution of linear theory (45), as observed in Table - 1. The tabular results show that the computer results are slightly different from the analytical solutions at the upper portion of the cylinder. These differences may be attributed to the fact that the boundary conditions at the ends of the shell meridian and the differential equaitons of Ref (45) can not be considered very appropriate for this problem. The analytical solution of Ref (45) is for an inner liquid column of height equal to that of the cylinder itself whereas the computer results are for a liquid column of hight less than the hight of the cylinder. It should furhter be pointed out that the linear theory employed in Ref (45) is entirely different and very approximate in comparison to the linear theory of Reissner, the theory employed in the present analysis. Also, it should be noted that the objective of Ref (45) was to obtain only the maximum values of u, M_s and M_θ at the fixed edge of the shell which is hardly influenced by the boundary conditions at the upper edge whereas in the present analysis exact boundary conditions at both the ends of the shell meridian were employed in this computations. The graphical representation of the analytical and the present linear and nonlinear solutions of this cylindrical shell are shown in Figs. 6 and 7. Analytical solution for N, based on membrane theory of Ref. (45), for this cylindrical shell is also plotted in figure 7. Other results of the presnt analysis of cylindrical shell, of figure 6, are presented in figures 8 to 11. Pertinent results of the membrane theory are also shown in figures 9 and 10. As observed here, the results of linear theory are highly conservative in comparison to that of nonlinear theory, specifically in the region of edge fixity and junction. The results of membrane theory, whenever pertinent, are obsrbed to be much closer to nonlinear results and thus superior to the linear results. Looking at the stresses, if can be concluded that the membrane theory predicts quite acceptable values of stresses except at the end fixity.

From this comparisons it can be conclude that the governing equations, the method of solution and the algorithms incorporated in the computer program are sound and free from both the conceptual and accidential errors.

4.2. Results and Discussion :

The method of investigation employed here is quite versatile to handle any problem of the general case of composite shells under axially varying load. Here, axially variable internal or external pressure load on the shell surface is considered to be applied by a liquid column of a certain specific weight γ.

The input variables of the composite shells as required in the present method of solution are edge conditions, total number of segments of the shell meridian, base-radius to thickness ratio and Poisson's ratio of shell material. Here each segment of the composite shells is considered to be of uniform thickness but different segment may have different thicknesses. Meridional length of the composite shell may be divided into any number of segements, equal or unequal in length. The results of this study as presented here is confined to only one kind of end fixity as, otherwise, the results would be too volumenous and the time required would be very long.

It happens that the composite shells as studied here are commonly used as water towers, ships, under water crafts, pressure vessels, etc., with ring stiffened edges which very nearly approximate the boundary conditions of rigidly fixed edge. Thus the results presented here are of major practical importance.

The computer program which obtains the solution in the present method of analysis first finds the solution in terms of stresses and displacements based on the linear theory for an initial value of the axially varying pressure as assigned by the investigator. Then the solution based on the nonlinear theory is obtained for the same loading through iterations; from here on, the loading parameter is increased in small steps to find solution for the new loading, taking solution of previous loading as initial values for the variables. In this investigation the following input variables are required to be prescribed.

EM1 = Increasing step of base pressure

SO1 = Number of desired loaidng steps

M = Number of segments.

IZ = Indicator of type of Problem.

IG(I) = Indicator of type of a segment.

APH(I) = Meridional angle at the starting point of each segment.

RC = S_e/R , Normalised base radius.

EMO = P_o/E , Normalised base pressure.

Tk(I) = R/h , Thickness ratio for each segment.

AN = ν , Poisson's ratio

X(1,I), I = 1 to M +1 , meridional distance from the opex.

X(J,I), J = 2, 7 and I=1, M+1; initial values of six fundamental variables

H, β, w, u, βV, Boundary Conditions at starting and finishing boundary.

IS1,IS2,IS3 , Indicators of boundary conditons at base.

IF1,IF2,IF3, Indicators of boundary conditons at upper end.

All the results obtained in this investigation are based on the nonlinear theory, because nonlinear theory gives much better prediction than linear theory at higher loadings. But the results of linear theory are also presented here in order to point out its short-comings at higher loading.The solution for each shell studied is also presented in the tabular form so that the exact magnitude of moments and stresses can easily be checked.

The results of individual shell of different parametric values are presented seperately and their individual trends are also discussed separately.

(a) Types of the Composite Shells Investigated :

Solutions were obtained for Composite shells made-up of a cylindrical part, a circular part and a conincal frustum (Figs. 1 and 2).

Shell - I :

This composite shell consists of a cylindrical part at the lower end and closed at the top with a spherical part as shown in Figure 1. For this shell, the thickness ratio, R/h = 200, for all the segments, Poisson's ratio, = 0.3 and the base pressure, P_o/E = 0.256×10^{-5}. For fixed lower edge the boundary value of the fundamental variables are : $\bar{H} = 0.0$, $\bar{\beta} = 0.0$ and $\bar{w}$ = 0.0 and for closed top the three boundary conditions are :
$\bar{u} = 0$, $\bar{\beta} = 0$, and $\bar{V} = 0$. The numerical values of various moments and displacements at 10 equidistant locations along the meridian are presented in Table 2.

The present investigation is based on the Reissner's theory of axisymmetric deformation of shells of revolution which is founded on the assumption that the stress in the shell material is always with in the elastic limit. That is, if for a particular material, the stress level in the shell at a particular loading exceeds the yield strength, the results are not valid for that material. For this reason it has to be checked that the stresses found for any load do not exceed the corresponding yield strength of the material. From the detail results of this shell, it is found that the nondimentional meridional stress σ_{ao} occuring at the base ($\bar{s}$ = 1.0), has a maximum values of 0.66881×10^{-3}. Considering the shell material to be steel, the numerical value of this stress is σ_{ao}= 138 Mpa. Since high strength steels have yield strength as high as 1890 Mpa, the maximum stress in the shell is much below the yield strength of the shell material and thus the shell deformation is within the elastic limit.

Results of this composite shell are shown in Figs. 12 to 20. Figure 12 shows the deformed and undeformed shape of the shell under axially varying load. It is observd that the deformed shell is wavy in the region between $\bar{s}$ = 1.0 and $\bar{s}$ = 0.6 and it is of particular interest that the region between $\bar{s}$ =0.2 and $\bar{s}$ = 0.0 bends inward under internal liquid pressure whereas the remaining portion bends outward. It is to be noticed here that the shell is filled up with a liquid of specific weight γ up to $\bar{s}$ = 0.2. The linear, nonlinear and analytical membrane solutions of the various quantities are plotted against meridional distance in Figs. 15 to 20. The plotting of axial and circumferential stresses for this shell are shwon in Figs. 15 to 18. Fig. 15 shows the distribution of axial stress at the inner fiber in shell No.1. Normally, had there been no edge restrain and no junctions in the shell, the development of axial stress in the shell could hardly be justified. Only tensile circumferential stress could have been explained. A rough estimate of the maximum value of this circumferential stress by simple thin shell formula gives it a numerical value of 0.51200 x 10^{-3} whereas the maximum value of the axial stress here is 1.01 x 10^{-3} according to linear theory and according to nonlinear theory the corresponding axial stress value is 0.67 x 10^{-3}.

The existence of axial stress is entirely due to bending at the junctions and at the edge restraint which is not accounted for in the simple membrane theory of shell. Normally a shell has the tendency of straightening-up at the junctions under load. The distribution of axial stress in figure 15 is fully in conformity with this general tendency of shell. However, a few interesting points should be noted here. First, the junctions in a shell plays a havocal role in inducing stress which has no bearing with the concept of membrane theory of thin shell. Second, the prediction of linear theory is highly inadequate in this shell. It predicts a highly exaggerated value in comparison to nonlinear theory. The difference between the predictions of the two theories can easily be explained. The linear theory assumes that shell retains the original geometry and as a result has to exert a higher moment to straighten the shell at the junctions. But the nonlinear theory take the shape of the shell under load as its true form. The shell under load is already straightened up to a large extent and it has to exert a far lesser moment for further straightening up.

Fig.15 indicates that the junctions are under high tensions. Maximum tension is at the junction, $\bar{s}$ = 0.7, as expected in case of a shell containing liquid inside. But junctions are under high compression as indicated by the outer axial stresses, which is shown in Fig. 16. High tension and compression occured at the junctions for inner and outer fibers of the shell respectively because of bendings and discontinuties of radius of curvature. Figs. 17 and 18 show that the distribution of the inner and outer circumferential stresses are of approximately the same qualitative nature as the inner and outer axial stresses, respectively. But contribution of maximum axial stresses are about 3 times the contribution of circumferential stresses.

Figures 15 to 18 also show that the analytical membrane results are much closer to nonlinear results. So, it is noted that membrane theory predicts better results than the linear theory and those results are quite acceptable except at the end fixity and shell junctions.

Figures 13 and 14 show the distribution of meridional and circumferential bending moments along the meridian. In these figures it is noted that the meridional bending moment is the dominating contributor to stresses in the shell. Considerable amount of bending moments are developed at the junctions which gradually decrease with the decrease in loading along the meridian. The difference between the results of linear and nonlinear theories are shown in the figures. The maximum stress in this shell is the meridional stress at the inner surface of the junctions. Although the meridional bending stress at the junction as predicted by the linear theory is much higher than the actual stress as indicated by the nonlinear results, it still remains to be the maximum of all the stresses. The most interesting observation in Fig. 13 is that the amount of bending moment developed in the spherical tip of this shell is practically zero. Had there been no spherical top the bending moment at the apex of the shell would definitely have been much greater. This is a clear indication of the fact that the best possible way of avoiding the stress concentration at the junction is to use a spherical ring there.

Figure 14 shows that the distribution of the circumferential bending moment is approximately of the same qualitative nature as the meridional bending moment.

Figs. 19 and 20 show the membrane state of axial and circumferential stress resultants, $\overline{N}_s$ and $\overline{N}_\theta$. Fig. 19 shows that the maximum positive value of $\overline{N}_s$ occurs at the base ($\bar{s}$ = 1.0) of the shell and gradually decreases with the decrease in internal pressure. At locations, $\bar{s}$ = 0.0 , 0.10, 0.20, and 0.30 the compressive values of $\overline{N}_s$ indicate that the shell is under compression meridionally under liquid pressure.

Fig. 20 shows that the maximum circumferential stress resultant occurs near the base of the shell meridian. Compressive value of $\overline{N}_\theta$ is obtained at the junction $\bar{s}$ = 0.7. It should be noted here that the circumferential stress resultant is of much greater magnitude in comparison to that of the axial stress resultant. Analytical results of $\overline{N}_\theta$ based on membrane theory are also presented in Figure 20. It should be noted here that the analytical results are very close to nonlinear results except at the base of the shell.

In the absence of edge restraint, $\bar{N}_s$ would be zero along the edge. Thus $\bar{N}_s$ is induced in the shell because of the restraint at the edge.

Shell - 2 :

This shell is exactly of the same geometry and boundary conditons as shell - 1 except that the thickness ratio, R/h = 300 and pressure at the base $P_o/E = 0.356 \times 10^{-5}$. The numerical values of different quantities for axially variable loadings, specially the components of displacement and moment at 10 equidistant locaitons on the meridian are presented in Table - 3.

In order to ascertain that Reissner's theory of axisymmetric deformations holds good in the analysis of this shell, it is required to show that the deformations are elastic. Thus the values of the maximum stresses at the junctions would have to be less than the yield strength of the shell material. From the detail results of this shell, nondimensional value of maximum meridional stress at the junction ($\bar{s} = 0.7$), $\bar{\sigma}_{s1} = 2.517 \times 10^{-3}$ according to linear theory and 0.445×10^{-3} according to nonlinear theory. Considering shell material to be steel, corresponding numerical value of maximum meridional stress is found as $\bar{\sigma}_{ao}$ = 503.4 Mpa at the junction ($\bar{s}$ = 0.7), which is much below the yield strength of high strength steels.So the deformations of this shell are elastic.At the apex $\bar{\sigma}_{ai}$ = -0.16831 x 10^{-8}. For the same material, its numerical vlaue is very small than that of the maximum value. The linear and nonlinear solutions for stresses and moments are plotted against meridional distance in figures 21 to 27. Analytical results based on membrane theory are also plotted in Figures (21 - 23,26,27). These results show that the membrane theory can predict the state of stress in these thin shells more accurately than the linear bending theory.

Here also, the stresses conform to the general expectation. Fig. 23 shows the distribution of the inner circumferential stress which is maximum in the line element near the junction ($\bar{s}$=1.0 to $\bar{s}$= 0.8) according to linear theory and its numerical value of 1.0057 x 10^{-3} whereas the maximum vlaue is this circumferential stress by simple thin shell formula is 1.068 x 10^{-3}. The distribution of circumferential and meridional bending moments for this shell are shown in figures 24 and 25. Figure 25 indicates that the meridional bending moment is maximum at the junctions ($\bar{s}$ = 0.7 and $\bar{s}$ = 0.5) and at the base ($\bar{s}$ = 1.0) due to bending at the junctions and at the edge restraint. The numerical value of maximum nondimensional meridional bending moment is 3.3741 x 10^{-1} at the junction $\bar{S}$ = 0.7 according to linear theory and the corresponding nonlinear value is 0.541×10^{-1}. Between the junctions the curve of $\overline{M}_s$ takes a wavy form. The value of $\overline{M}_s$ gradually decreases with the decrease in loadings and becomes very small above the liquid surface.

Fig. 24 shows that the distribution of the circumferential bending moment has approximately the same qualitative nature as the meridional bending moment. But it is seen that contribution of maximum circumferential moment to the stress is about ν times the contribution of the maximum meridional moment. It shows further that the distribution given by the nonlinear solution differs substantially from that of the linear solution which is already discussed with reference to shell - 1.

Figures 26 and 27 show the distribution of the nondimentional meridional and circumferential stress resultants, respectively, against the meridional distance of the shell. The linear solution of $\overline{N}_s$ is maximum at the base ($\overline{s}$= 1.0) and it remains high up to $\overline{s}$ = 0.7 due to uniform slope of the cylindrical part. From the location, $\overline{s}$ = 0.7, the value of $\overline{N}_s$ decreases gradually along the meridian because of low loadings and reduction in the circumferential of radius of curvature.

Figures 26 and 27 also show that the results of membrane theory are almost identical to nonlinear results. Thus membrane theory predicts quite acceptable valus of stress resultants except at the end fixity.

Fig. 27 indicates that the magnitude of $\bar{N}_\theta$ gradually decreases towards the junctions. Specifically, it has ebcome compressive at the junction, $\bar{s}$ = 0.7 due to the general tendency of shell and it is maximum in between the base ($\bar{s}$ = 1.0) and the junction $\bar{s}$ = 0.7. After the location $\bar{s}$ = 0.7, the value of $\bar{N}_\theta$ decreases and it is nearly zero at the apex ($\bar{s}$ = 0.0). Due to the edge - restraint the circumferential stress resultant at the base is approximately zero. Figs. 26 and 27 also indicate that $\bar{N}_s$ is very small in comparison to $\bar{N}_\theta$ because internal load is mainly resisted by the circumferential straining of the shell.

It is noted here that the stresses increase with the increase in loadings and also with the increase in R/h ratio.

Shell - 3 :

This is another Composite shell consisting of a cylinderical part, a circular part and a conical frustum. The base of the shell is a cylindrical part and the top is closed with a spherical part, like shell - 1 and shell - 2. But the locations of various elements, meridional angle $(\phi_o)_i$ for each segment at the lower end and the thickness ratio for each segment are different from that of shell - 1 and shell 2 . Here the junctions are located at the points $\bar{s}$ = 0.7, $\bar{s}$ = 0.5 and $\bar{s}$ = 0.3 from the apex. The meridional angle $(\phi_o)_i$ at the lower end of each of the segments are : $(\phi_o)_1 = 90°$, $(\phi_o)_2 = 90°$, $(\phi_o)_3 = 90°$, $(\phi_o)_4 = 78°$, $(\phi_o)_5 = 66.96°$, $(\phi_o)_6 = 45°$, $(\phi_o)_7 = 45°$, $(\phi_o)_8 = 35°$, $(\phi_o)_9 = 23.27°$ and $(\phi_o)_{10} = 13°$.

Initially the shell is considered to be filled with a liquid of specific weight γ up to the segment S_8.This particular shell is shown in Fig. 2. The numerical values of moments and displacements at ten equidistant locations on the meridion are presented in Table - 4. The nondimentional inner meridional stress $\bar{\sigma}_{ai}$ at the base is maximum where its numerical value is 0.2844 x 10^{-2} and at the apex $\bar{\sigma}_{ai}$ = 0.26156 x 10^{-4}. The maximum stress at the base becomes 568.8 Mpa and at the apex 5.23 Mpa, if the material is steel. So the deformation of the shell meridian is within the elastic limit. The base and the junctions of the shell meridion are under high tension axially at the inner surface.

The linear and nonlinear solutions of bending moments along the shell meridian are presented in graphical forms in Figs. 28 and 29. It should be mentioned here that the maximum values of $\bar{M}_s$ and $\bar{M}_\theta$ have occured at the base in this shell whereas the respective values are maximum at the junction $\bar{s}$ = 0.7 in case of shell - 1 and shell - 2 . Fig. 28 shows that the maximum value of $\bar{M}_s$ is 0.263 at the base and 0.224 at the junction, $\bar{s}$ = 0.7 according to linear theory. For different geometry the slopes at the junctions and the radius of curvatures of this shell are lesser than that of shell - 1 and shell - 2. So, in relation to original geometry this shell is more straightened at the junctions than the shell - 1 and shell - 2. That is why the maximum moment and stress are developed at the base rather than at the junctions of this shell.

Figs. 30 and 31 show the distribution of nondimensional axial stresses at the inner and outer surfaces of the shell.Fig.31 shows that the base and the junctions are under high compression axially at the outer surface, while the neighbourhood of the junctions and middle portions of the cylindrical, spherical and conical parts are under tensions. The maximum stress is obtained at the base ($\bar{s}$ = 1.0) due to end restraint. Fig. 32 shows that the maximum inner circumferential stresses are developed in the middle portions of the respective parts of the shell. The maximum numerical vlaue of circumferential stress is 1.416×10^{-3} whereas the rough estimate of the maximum value of this circumferential stress by simple thin shell formula gives it a numerical value of 1.500×10^{-3}. The same qualitative nature is obtained for the distributions of circumferential stress resultants which is shown in Fig. 33. Above the liquid surface a little compressive stress is developed due to discontinuties of loadings. Fig. 34 shows that the distribution of $\bar{N}_s$ given by the nonlinear solution differs substantially from that of the linear solution.

Analytical results based on membrane theory are also presented in Figures 30 to 34

(b) Built-in Edge Hemispherical shell :

For this shell both the linear and nonlinear solutions are obtained and presented in graphical forms so that the difference between these two results can be readily checked . It should be noted here that in all the graphs presented, the linear solution may be considered as equivalent to the nonlinear solution at zero loading.

In Figs. 35 and 36 the nondimensional values of M_θ and M_s for hemispherical shell are plotted, respectively, against the meridional length of the shell for R/h equal to 200. The peak values of the meridional bending moment based on both the linear and nonlinear theories have almost the same magnitude and are identical in distribution in the hemispherical and in the cylindrical shell for the smae loadings and for the same R/h ratios. The maximum bending moment is obtained at the base ($\bar{s}$ = 1.0) of the shell meridian, where the shell edge is assumed to be restrained against rotation.The same magnitude of the edge bending moment for the spherical and the cylinderical edge segment shows that bending moment due to edge restraint is independent of shell geometry.

It should be mentioned here that the circumferential bending moment is approximately ν times the meridional bending moment as dictated by the governing equation and verified here in figures 35 and 36.

Fig. 37 presents the distribution of the circumferential stress resultant $\bar{N}_\theta$ for both the linear and nonlinear solutions. The values of $\bar{N}_\theta$ obtained from analytical membrane solution are also presented in figure 37. It shows that the distribution given by the nonlinear and membrane solutions differ substantially from that of the linear solution. In the absence of edge restraint, a roughly estimated maximum value of $\bar{N}_\theta$ is 0.5. As seen in figure 37 $\bar{N}_\theta$ has exceeded this value because of edge restraint. The zero value of $\bar{N}_\theta$ at the edge is easily explained. Because of edge fixity the shell could not expand circumferentially. Hence, no circumferential stress could be induced in the shell at the edge. The wavy nature in the distribution of $\bar{N}_\theta$ is quite in conformity with the distribution of circumferential moment distribution.

Fig. 38 shows the distribution of $\bar{N}_s$, which decreases with decrease in loading along the meridian. In the absence of edge restraint, $\bar{N}_s$ would be zero along the edge. Thus, $\bar{N}_s$ is induced in the shell because of restraint at the edge.

Figs. 39 and 40 show the distribution of the nondimensional circumferential stresses at the inner and outer fibers of the shell. It is observed that the circumferential stress has almost the same magnitude at the inner and outer fibre. This shows that circumferential stress is mainly induced by the internal liquid pressure. Analytical membrane results of circumferential stressea are also presented in figures 39 and 40. The results based on membrane theory are observed to be much closer to nonlinear results and thus superior to the linear results.

The distribution of the meridional stress at the inner and outer fibers in the hemispherical shell is shown in Figs. 41 and 42. The distribution of stresses and their peak values for both the hemispherical and one end fixed cylindrical shells are almost identical. This shows that meridianal stress in both these shells is entirely due to edge restraint. The maximum value occurs at the inner fiber at the base in both the cylindrical and hemispherical shells. The maximum value of $\bar{\sigma}_{ai}$ is equal to 0.62549×10^{-3}. This stress becomes 18764.7 psi Considering the material of the shell to be steel. So the deformation is within the elastic zone. The difference between the solutions of the two theories increases with the increase in load. The numerical values of various displacements and moments are presented in Table -5.

CHAPTER 5

CONCLUSIONS

The stress problems of axisymmetric shells under axially varying internal pressure has been investigated in this thesis. The axisymmetric shells under investigation may be composed of spherical, conical and cylindrical segments and the two edge of the shell, top and bottom, may have any kind of edge-fixity including the provision of completely closed top. The axially varying load may be considered as that exerted by a liquid column contained either inside or outside the shell. Solution is obtained for varying height of the liquid column subjected to any pressure on its top surface. Analysis of axisymmetric shells based on both the linear and nonlinear theories have been achieved here. The nonlinear theory of axisymmetric shells as developed by Reissner (36) has been used in this analysis. The basic concept of multisegment integration developed by Kalnins and Lestingi (24) has been employed to obtain the solutions of the nonlinear equations of shells. The soundness of the theory, the method of solution, the criterion of finding the internal pressure along the meridian and the computer program used for numerical results are all checked by comparing the solutions of a one end fixed cylindrical shell of uniform thickness ratio with those of an analytical solutino of the same shell under the same conditions.

The comparison shows that the method of solution, the governing equations and the computer programme are all free from any error and based on sound hypothesis.

Based upon the results of various problems presented here, the following conclusions are made :

(1) The linear theory of shells is, in general, very conservative in predicting the state of stresses and deformations in the axisymmetric shells.

(2) Any discontinuity in geometry of the meridian induces bending stresses in the shell. If the change in geometry is also associated with the discontinuity of slope, then the maximum values of bending moments occur at the junction. Under this circumstance the inner fiber meridional stresses become usually the maximum of all the stresses of the shell under internal pressure except those produced by the end fixity.

(3) If the included angle of a junction is less than 180 degrees then a circumferentially compressive zone is developed there under load.

(4) The magnitude of the bending moment developed at the junction is observed to increase with the decrease of the included angle at the junction.

(5) In designing axisymmetric shells with discontinuity of slope of the meridian care has to be taken of the extreme stress concentration at the junction.

(6) The best possible way of avoiding the stress concentration at the junction is to use a spherical ring matching in slope with the two neighbouring segments.

(7) In this shells the membrane theory is observed to be superior to linear bending theory in predicting the actual state of stress even if the shells have geometrical discontinuity. It can thus be concluded that the linear bending theory should not be used in analyzing stresses in shells except perhaps in finding the effect of edge fixity in absence of a nonlinear theory. The prediction of stresses at the restrained edge, by the linear bending theory is always found to be highly conservative.

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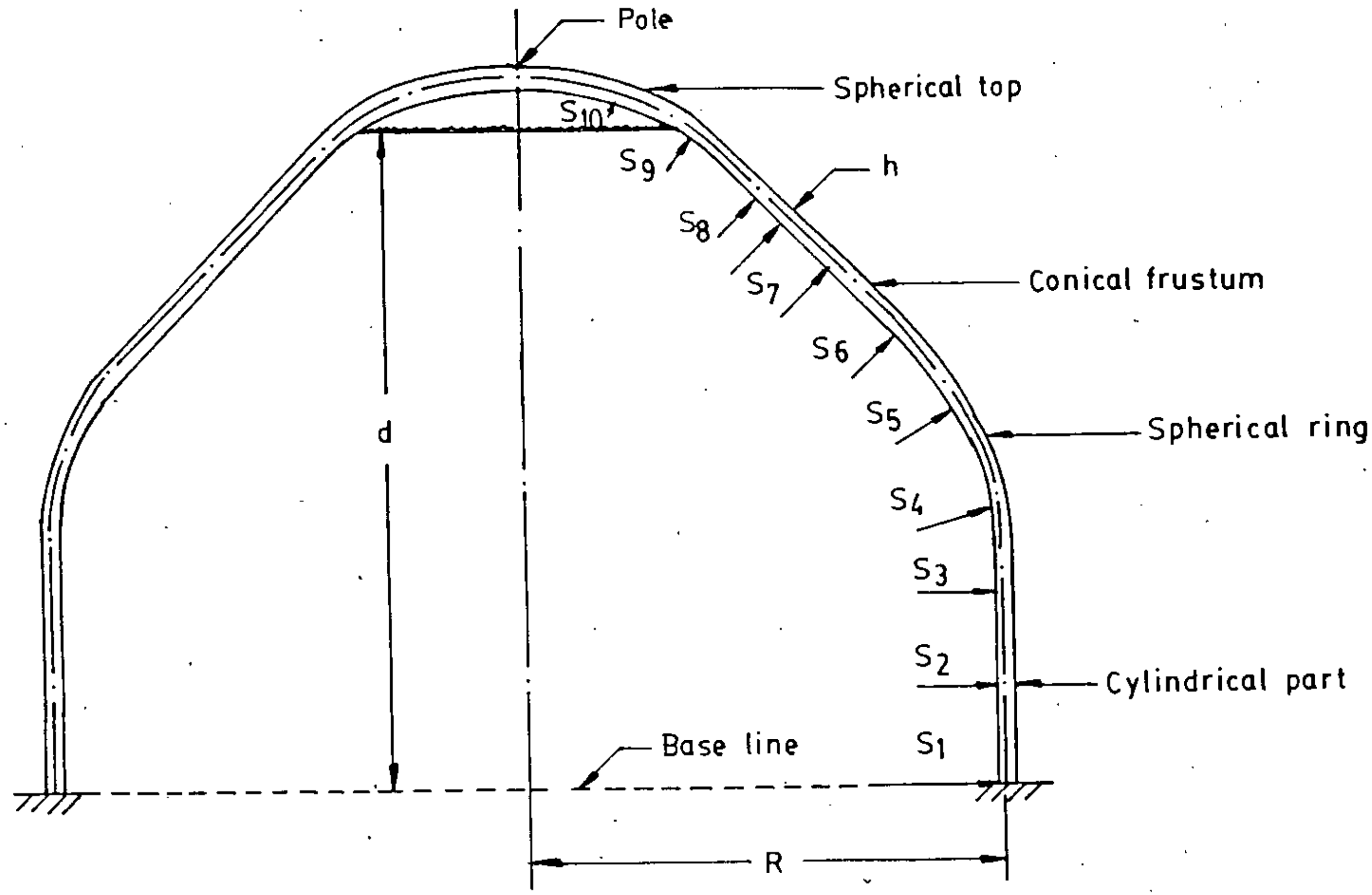


Fig. 1 A composite shell consisting of a cylindrical part at the bottom edge followed by a spherical ring, A conical frustum and a spherical top, R is the radius at the bottom edge, Se is the total meridional distance from apex to the base circle. d is the total depth of liquid. This shell is reffered as shell no. 1 and shell no. 2

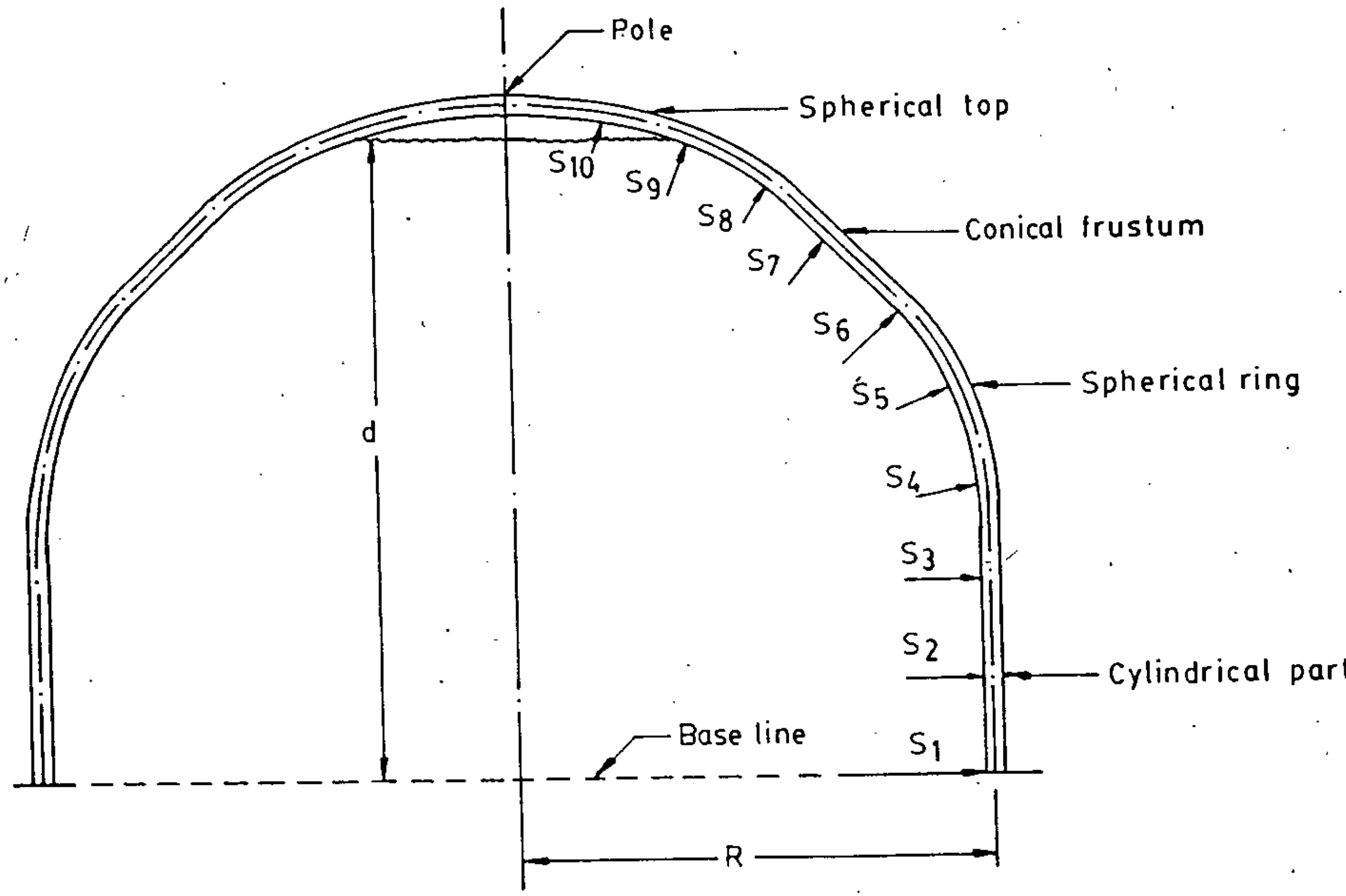


Fig. 2 A composite shell consisting of a cylindrical part, Spherical ring a conical frustum and a spherical top. R is the radius at the base. Se is the total meridional distance from the apex to the base circle, d is the total depth of liquid. This shell is reffered as shell no. 3

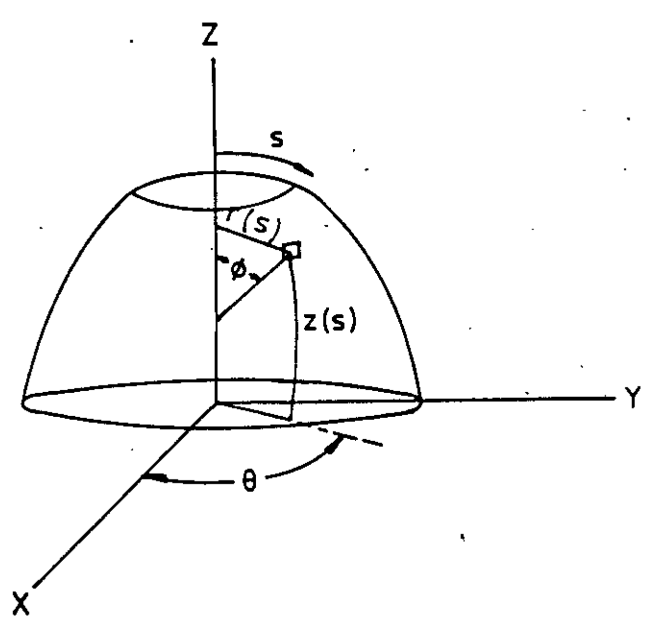


Fig. 3 Middle surface of shell

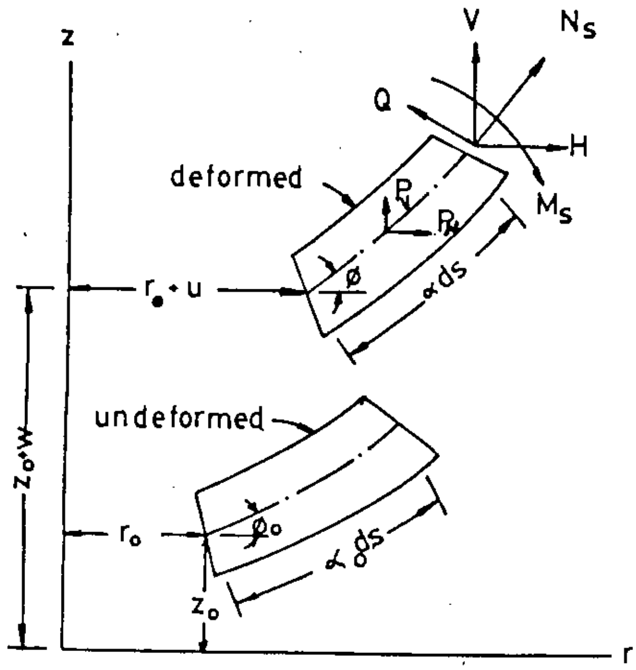


Fig. 4 (a) Side view of element of shell in undeformed and deformed states

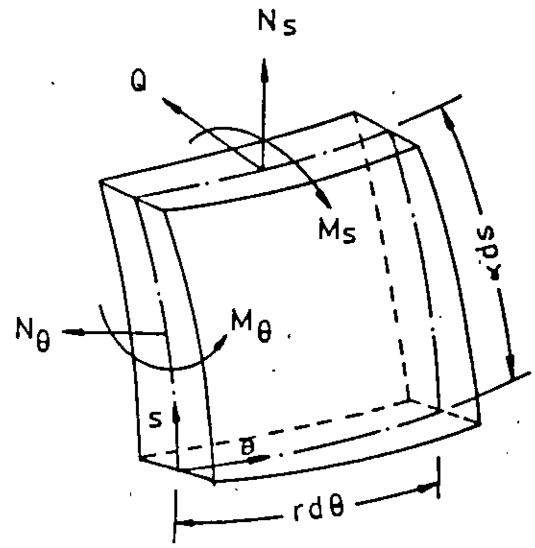


Fig. 4 (b) Element of shell showing stress resultants and couples

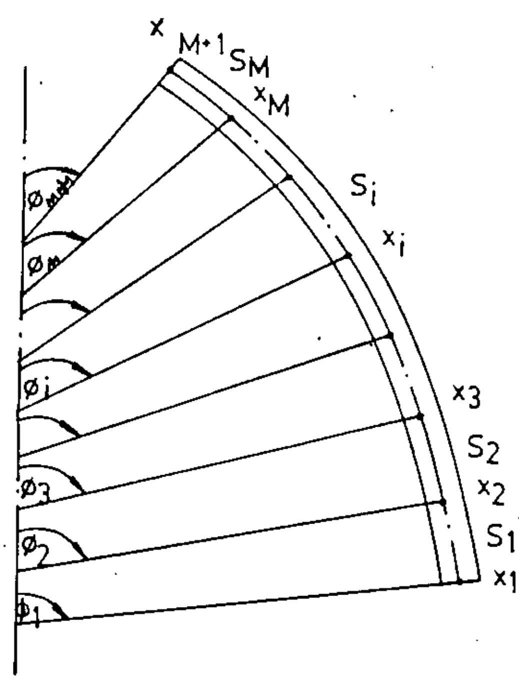


Fig. 5 Division for multisemen integration

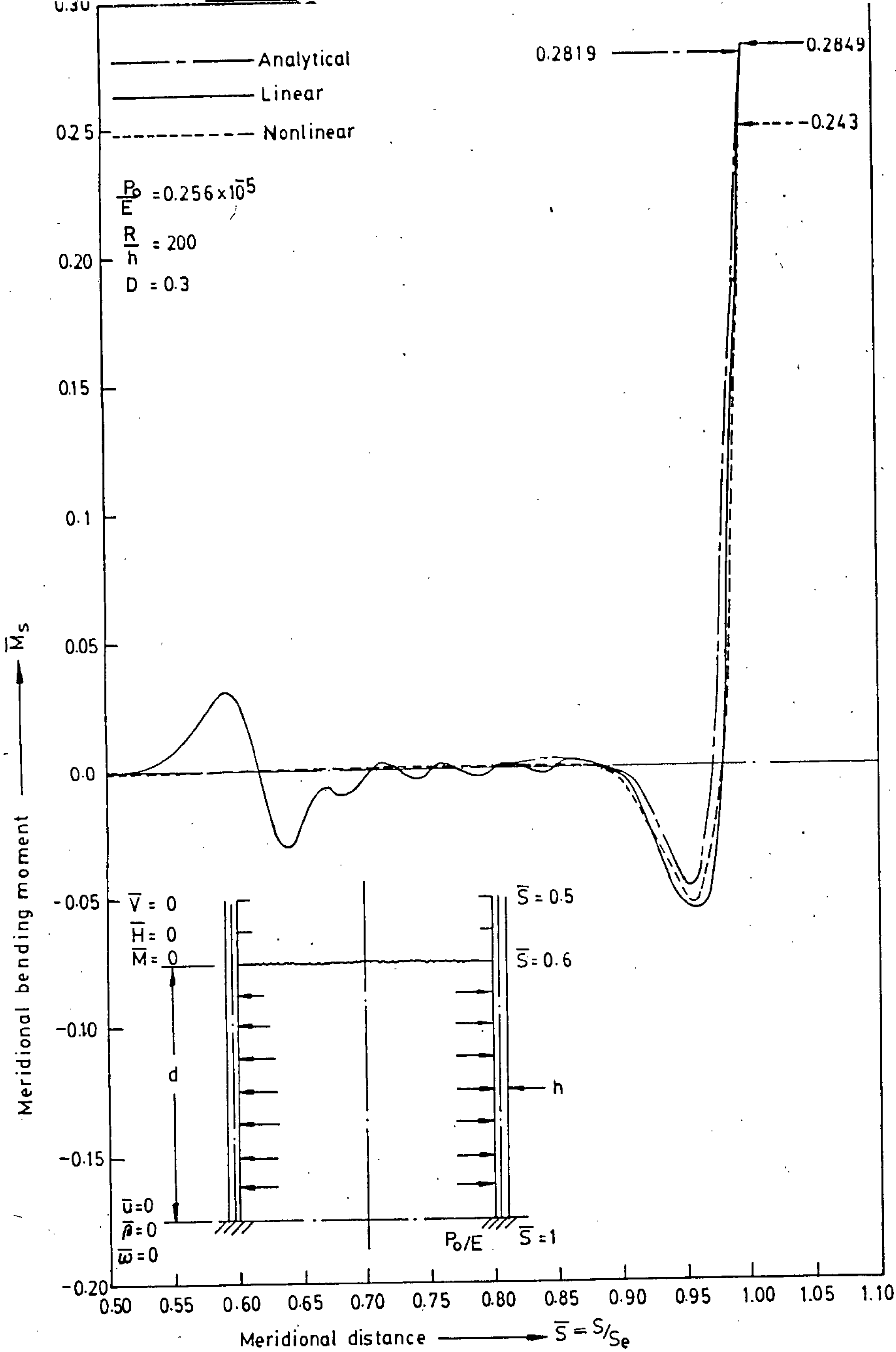


Fig. 6 Distribution of meridional bending moment in cylindrical shell

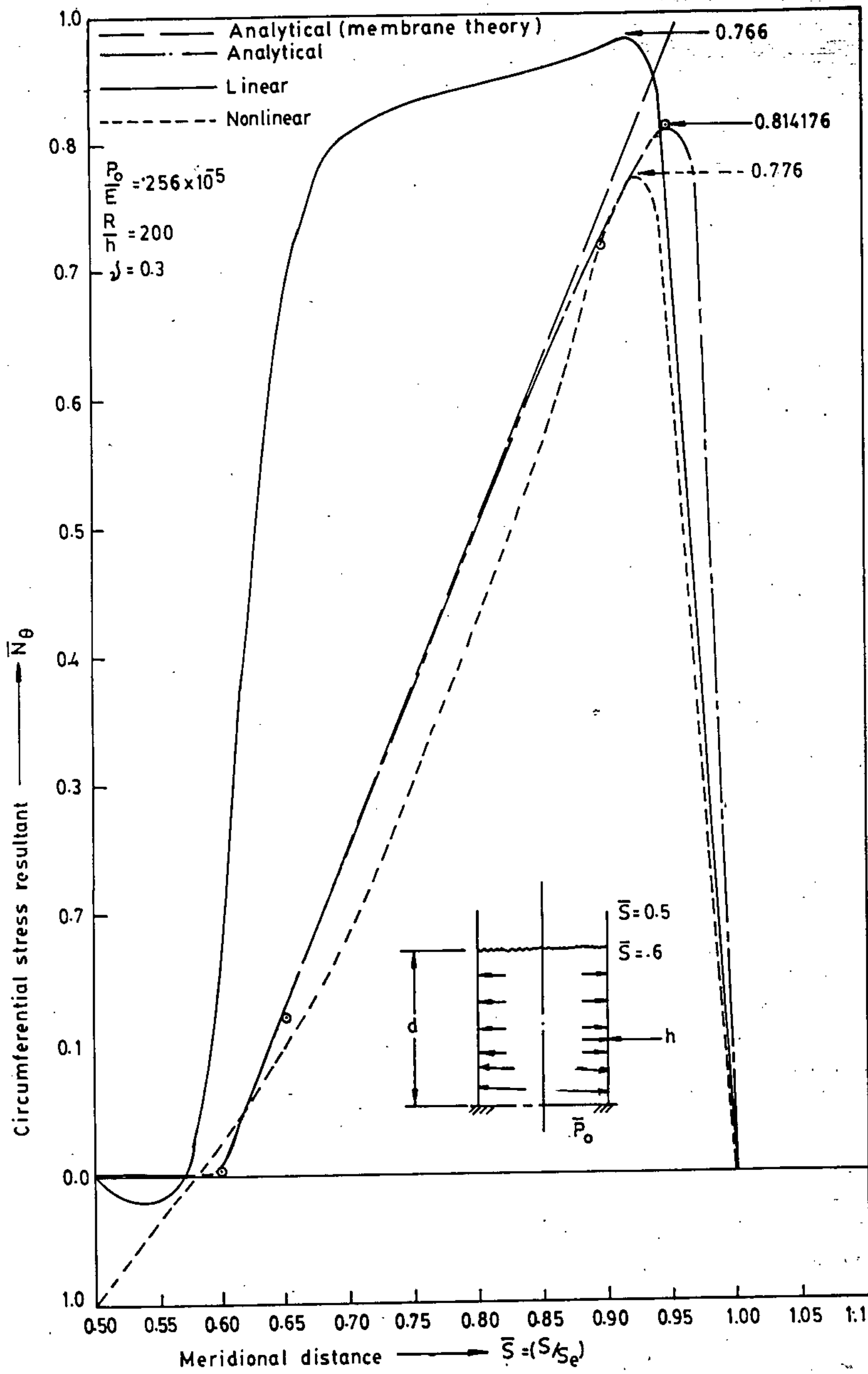


Fig. 7 Distribution of circumferential stress resultant in cylindrical shell

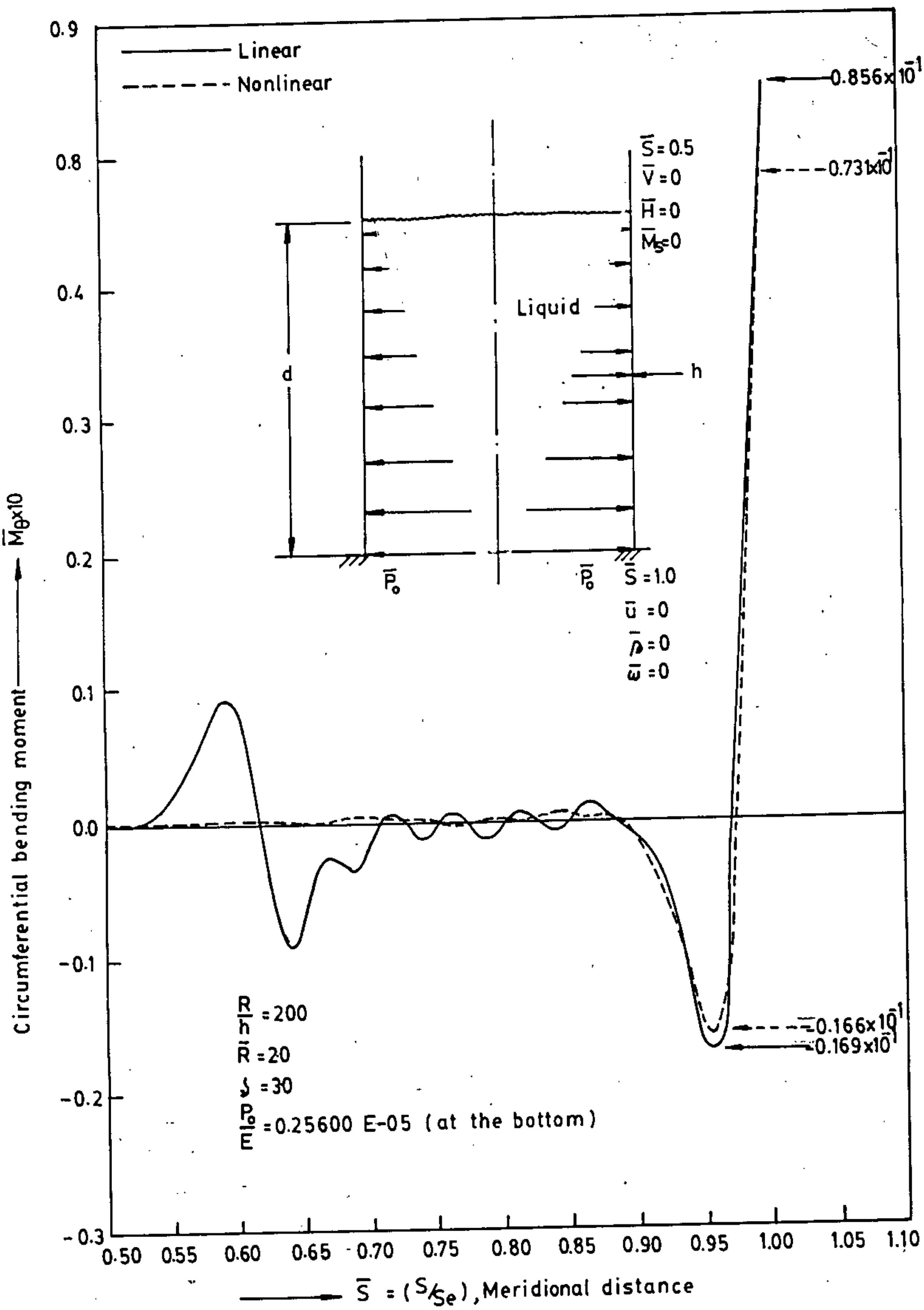


Fig. 8 Distribution of circumferential bending moment in cylindrical shell

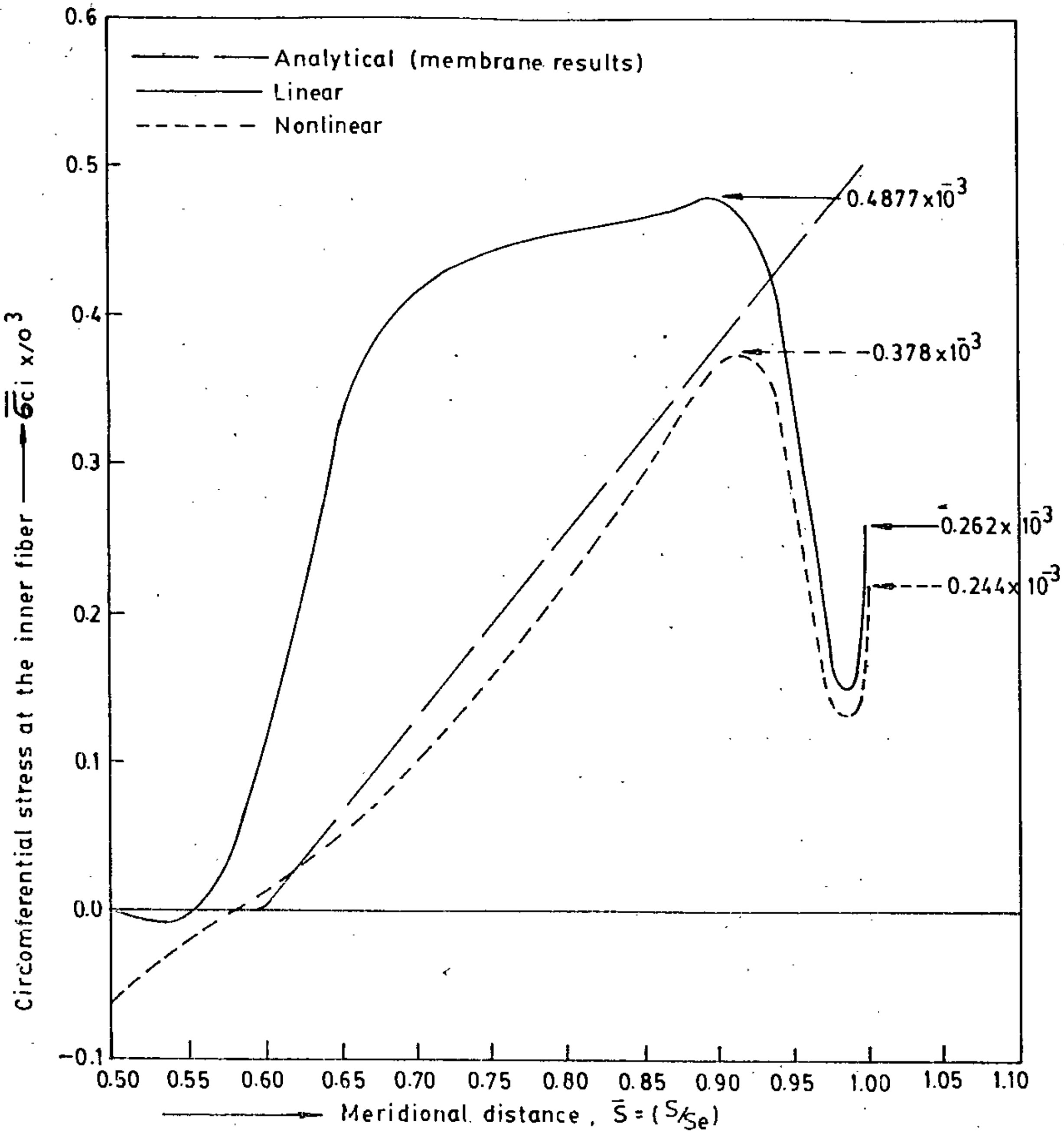
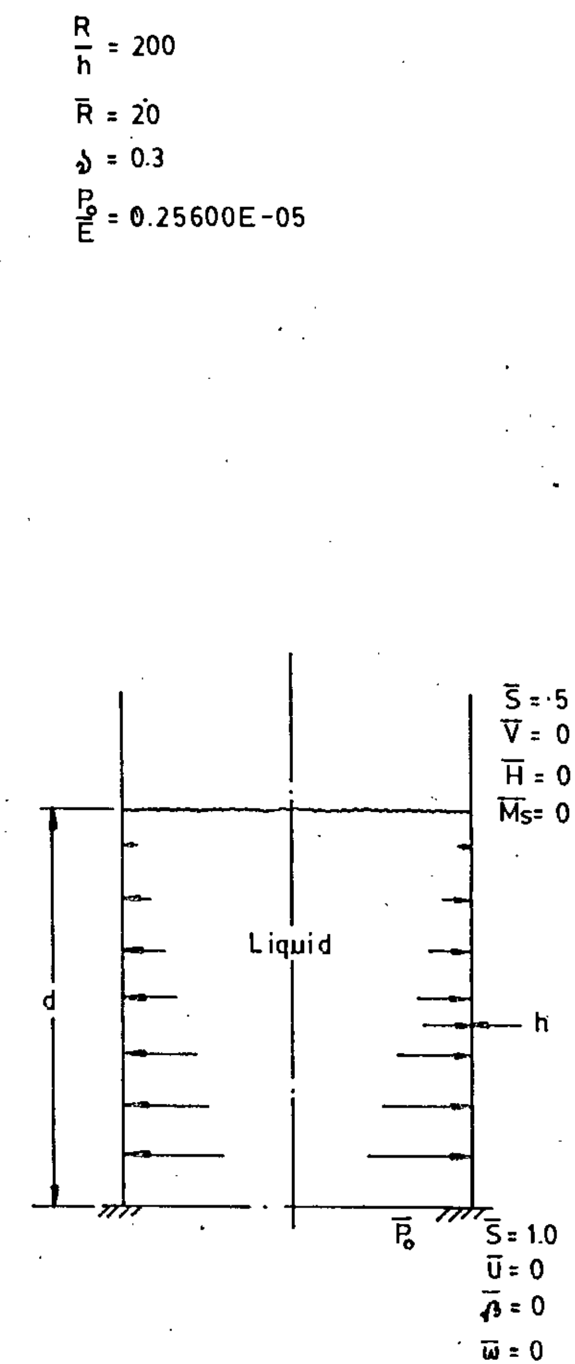


Fig. 9 Distribution of circumferential stress at the inner fiber



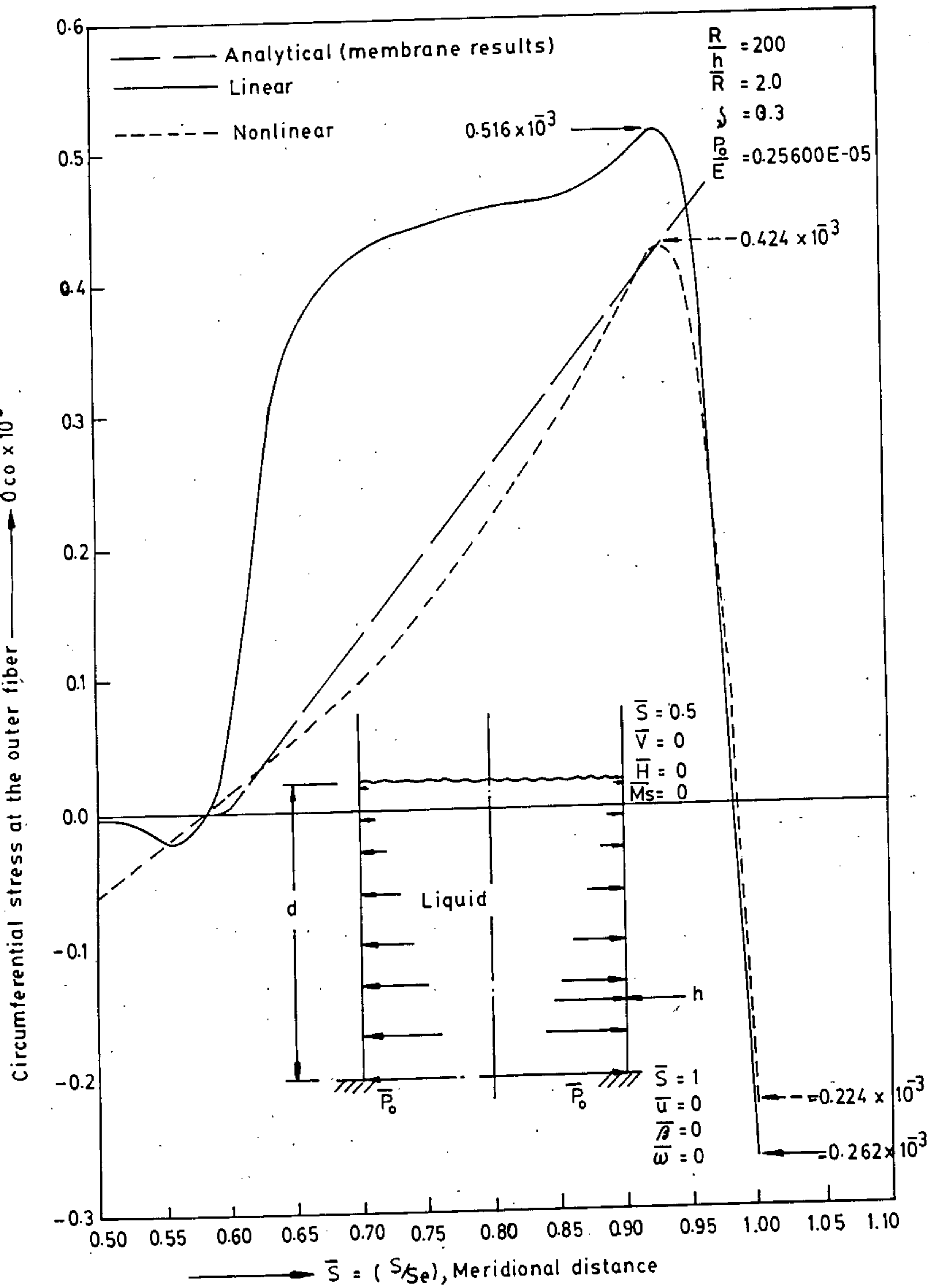


Fig. 10 Distribution of circumferential stress at the outer fiber

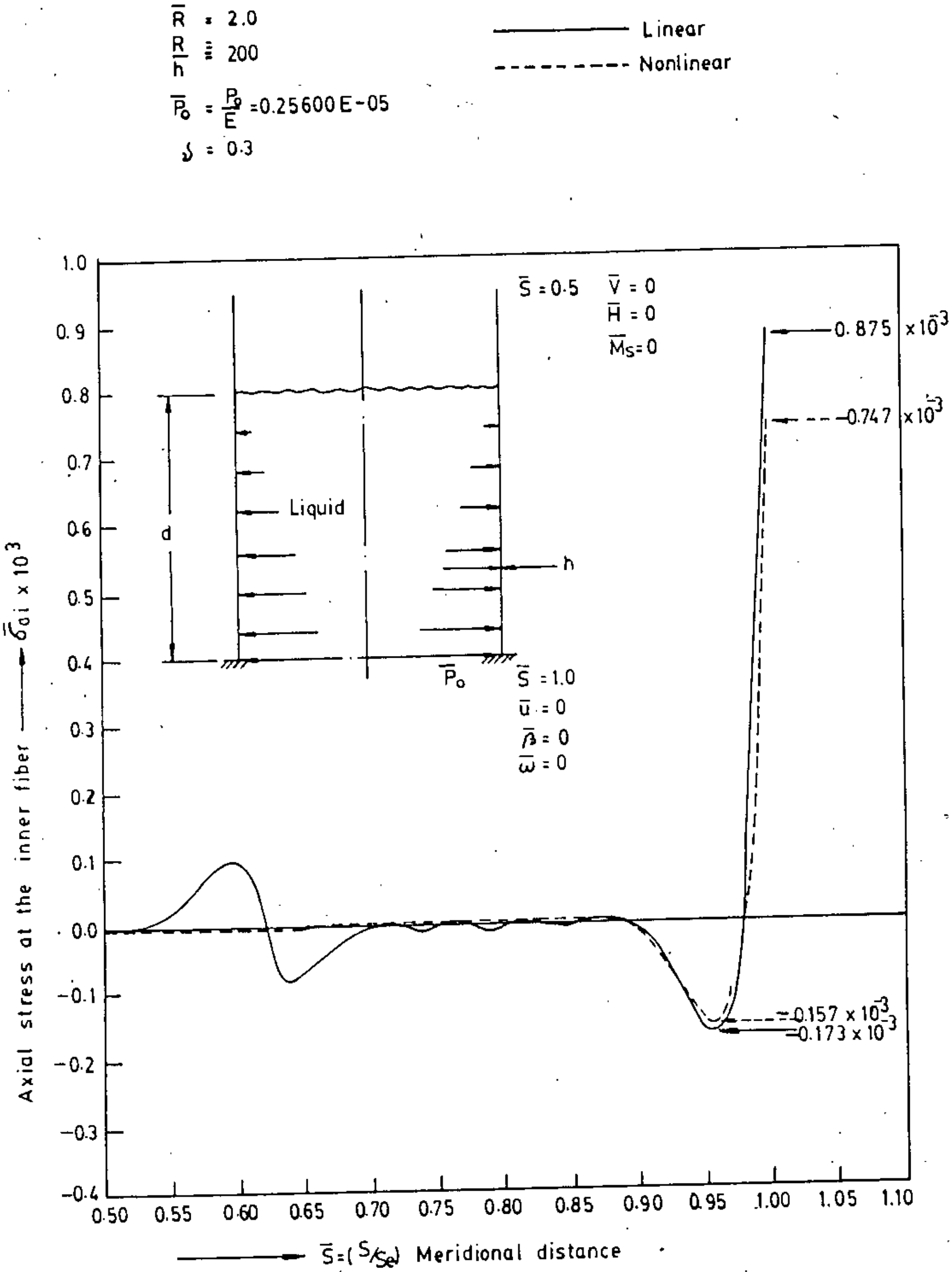


Fig. 11 Distribution of axial stress at the inner fiber

Shell no. 1

$\overline{P}_o = 0.25600E-05$

$\frac{R}{h} = 200$

$\nu = 0.3$

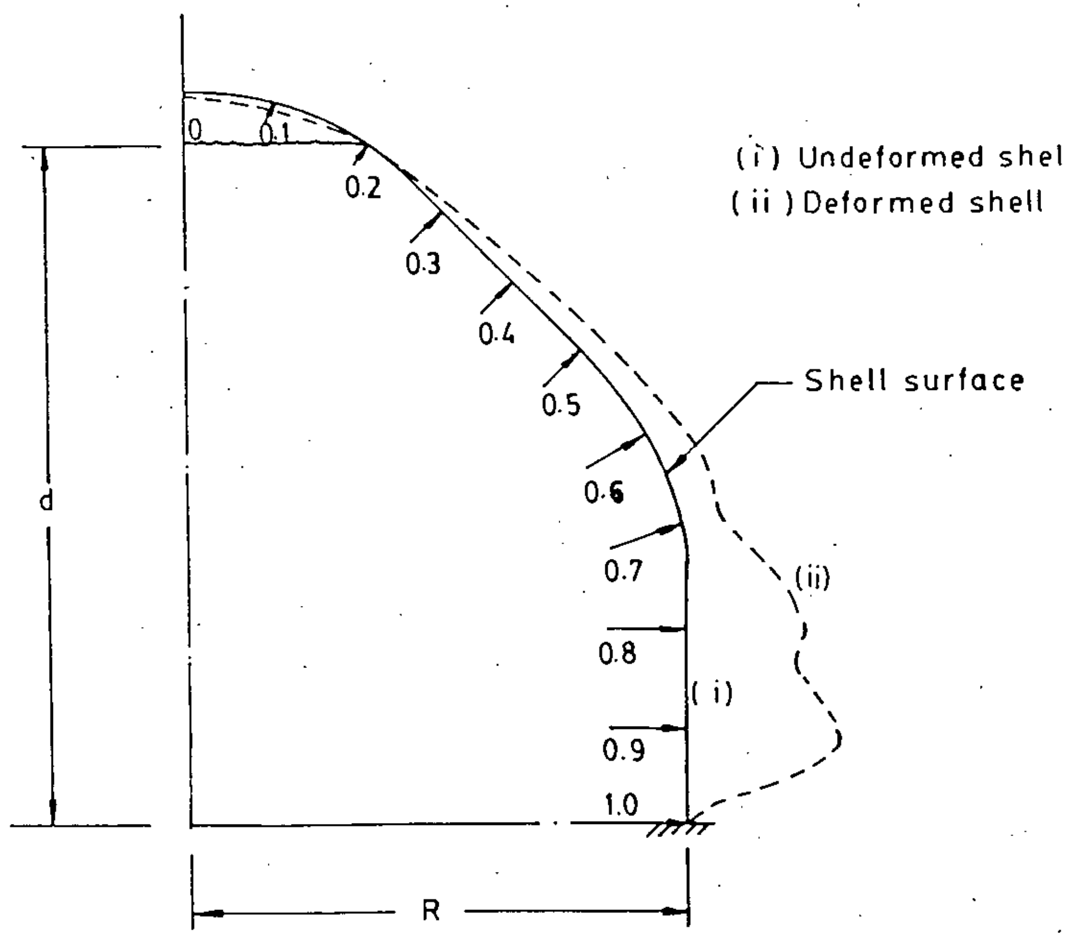


Fig. 12 Undeformed and deformed shape of the shell

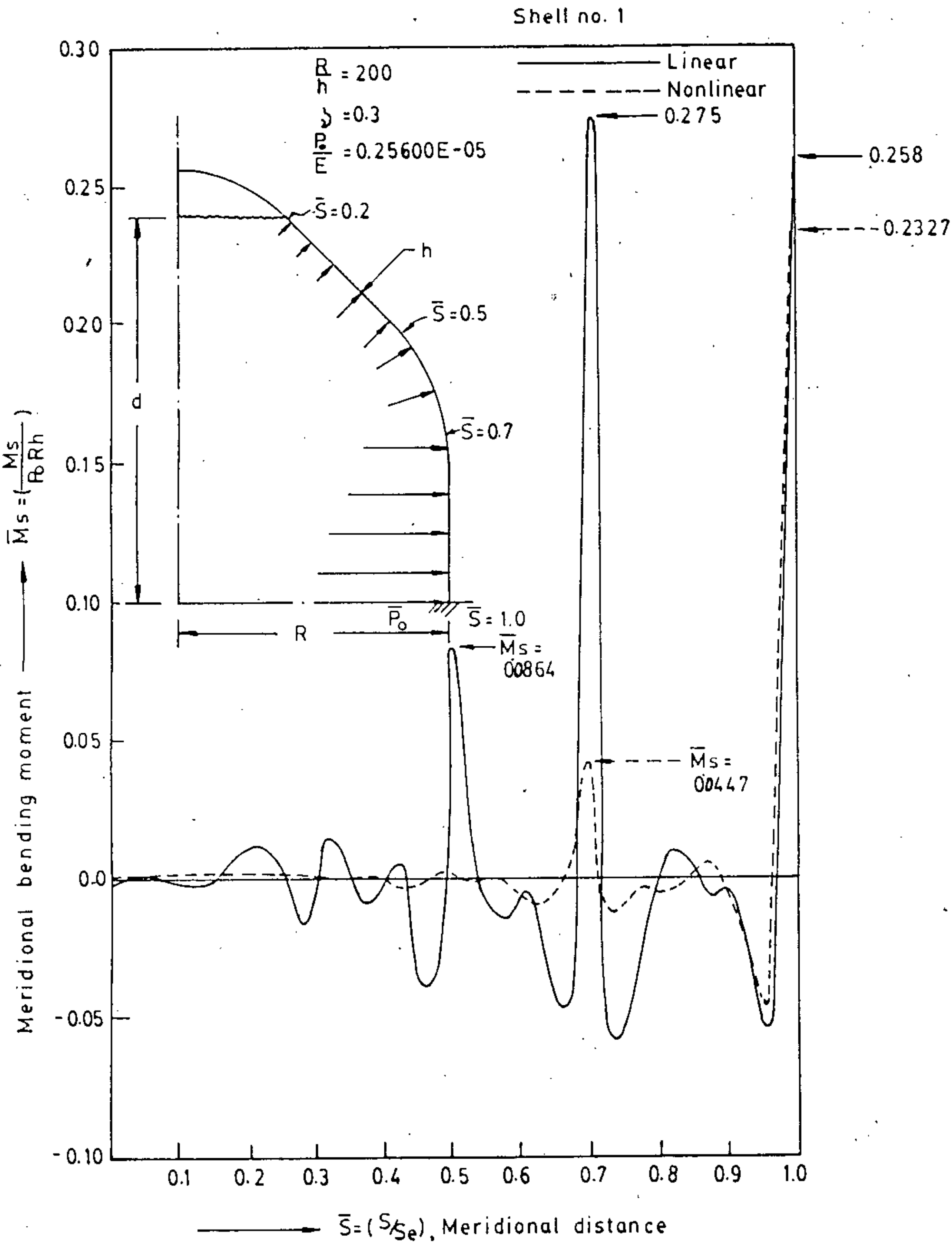


Fig. 13 Distribution of meridional bending moment shell no. 1

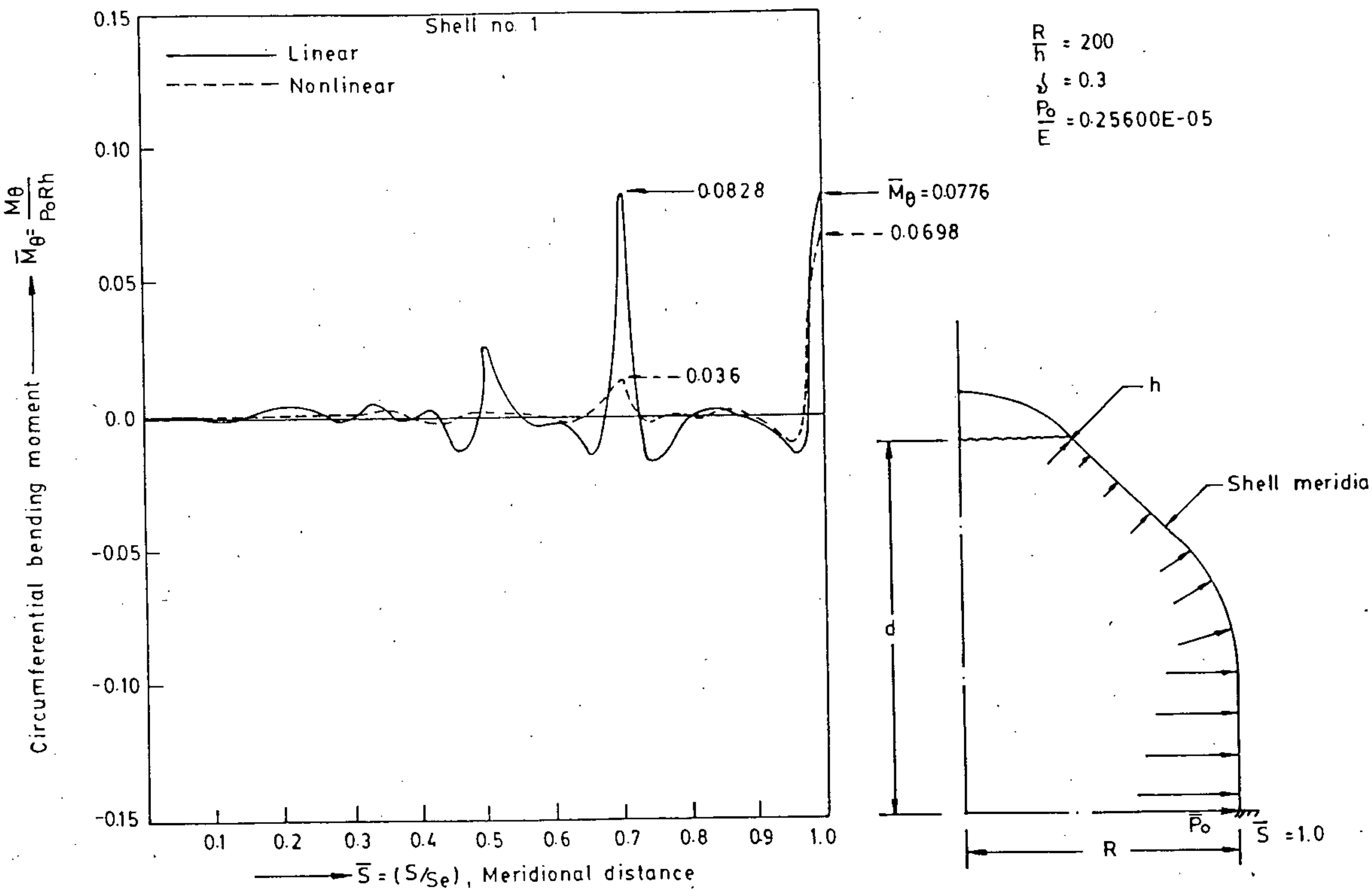


Fig. 14 Distribution of circumferential bending moment in shell 1

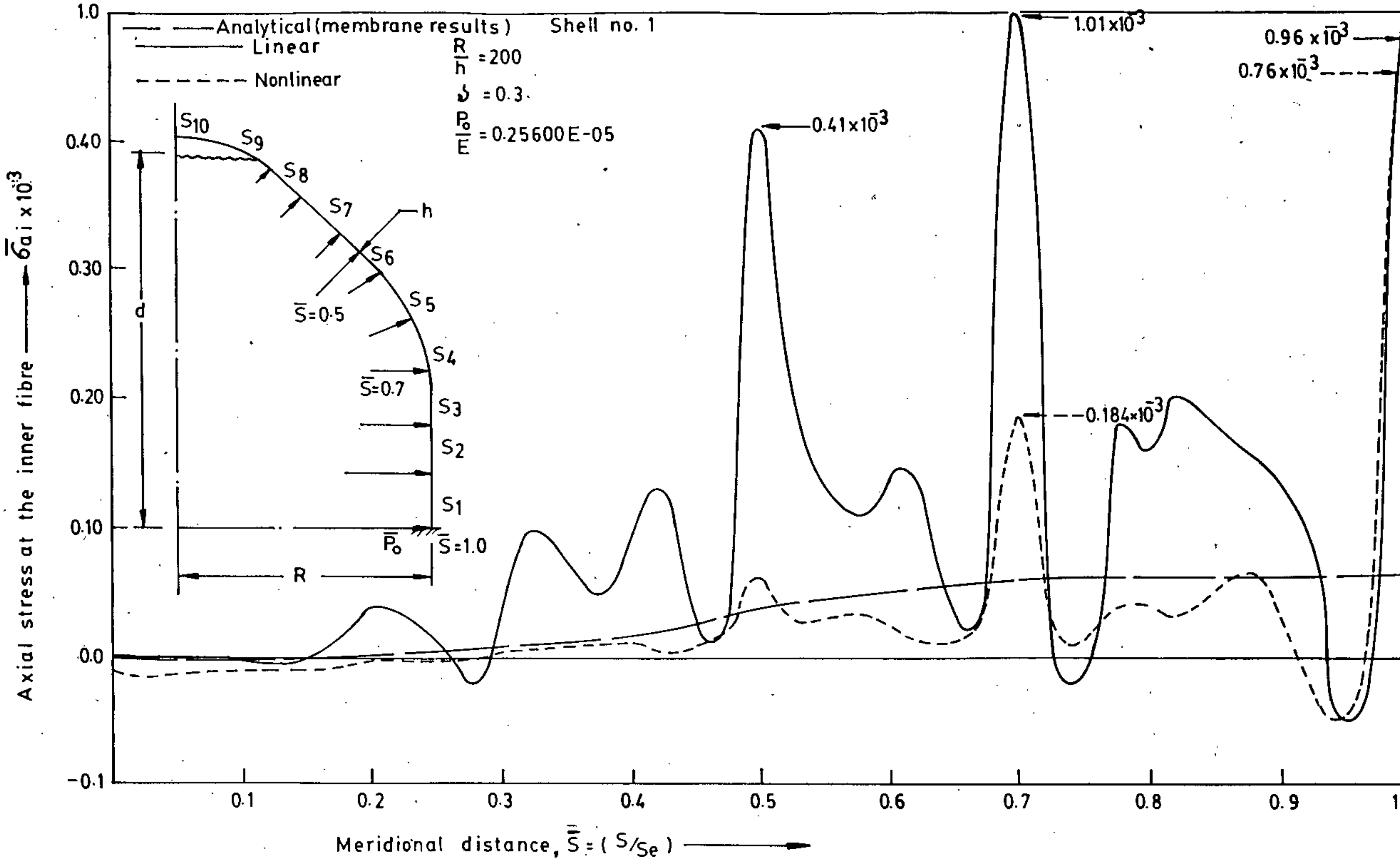


Fig. 15 Distribution of axial stress at the inner surface in shell no. 1

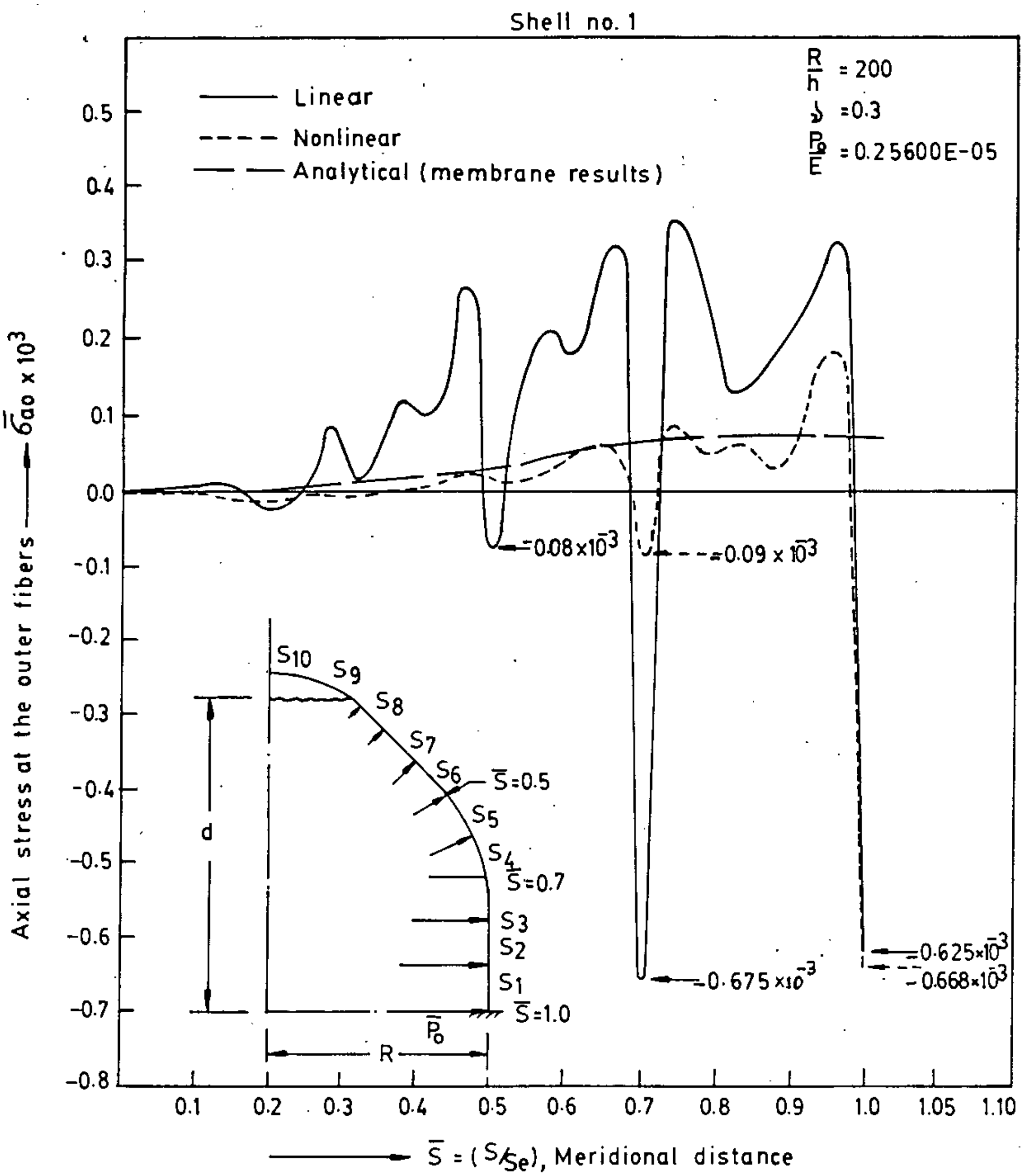


Fig. 16 Axial stress at outer surface in shell no. 1

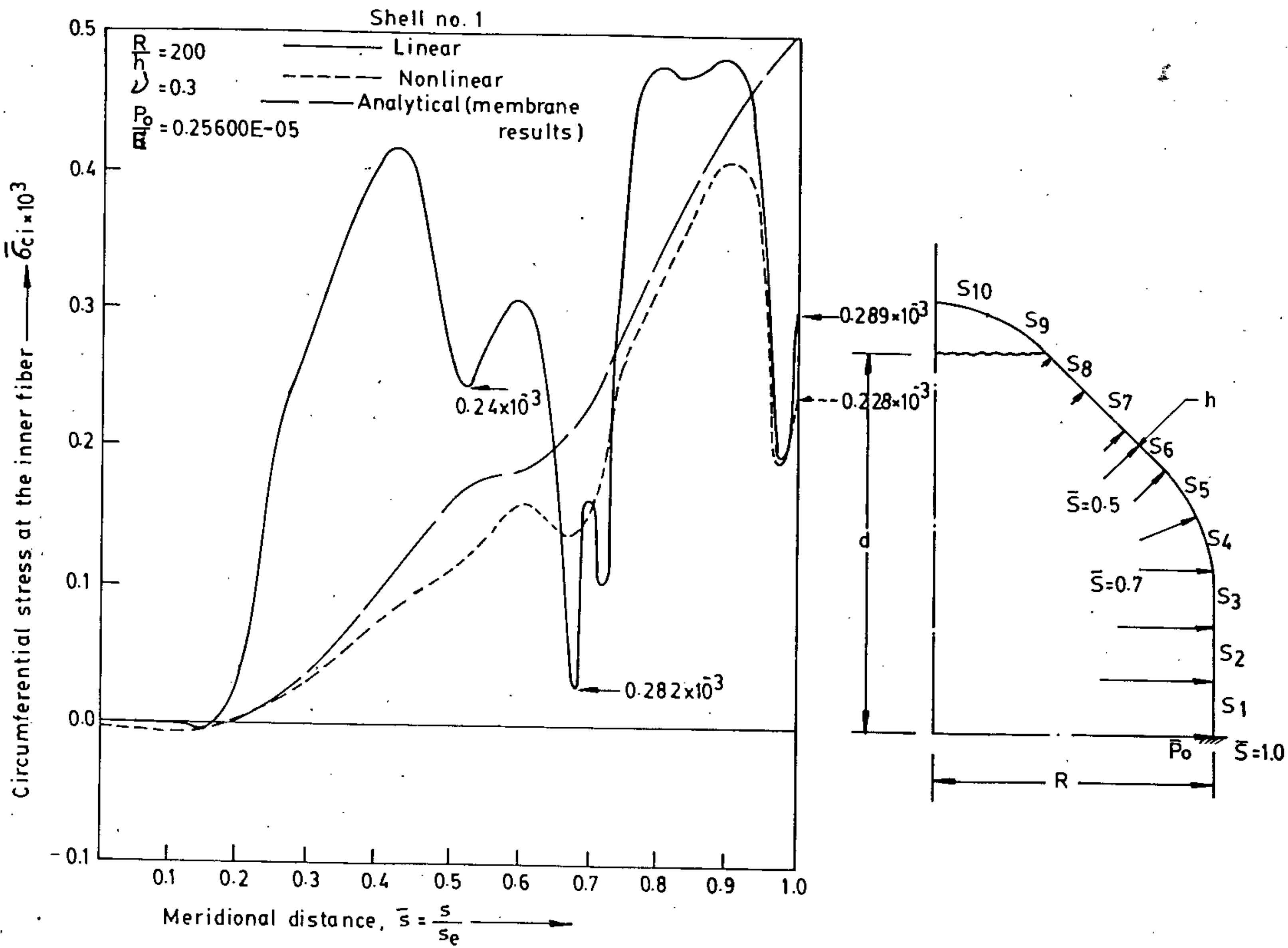


Fig. 17 Distribution of circumferencial stress at inner surface in shell no. 1

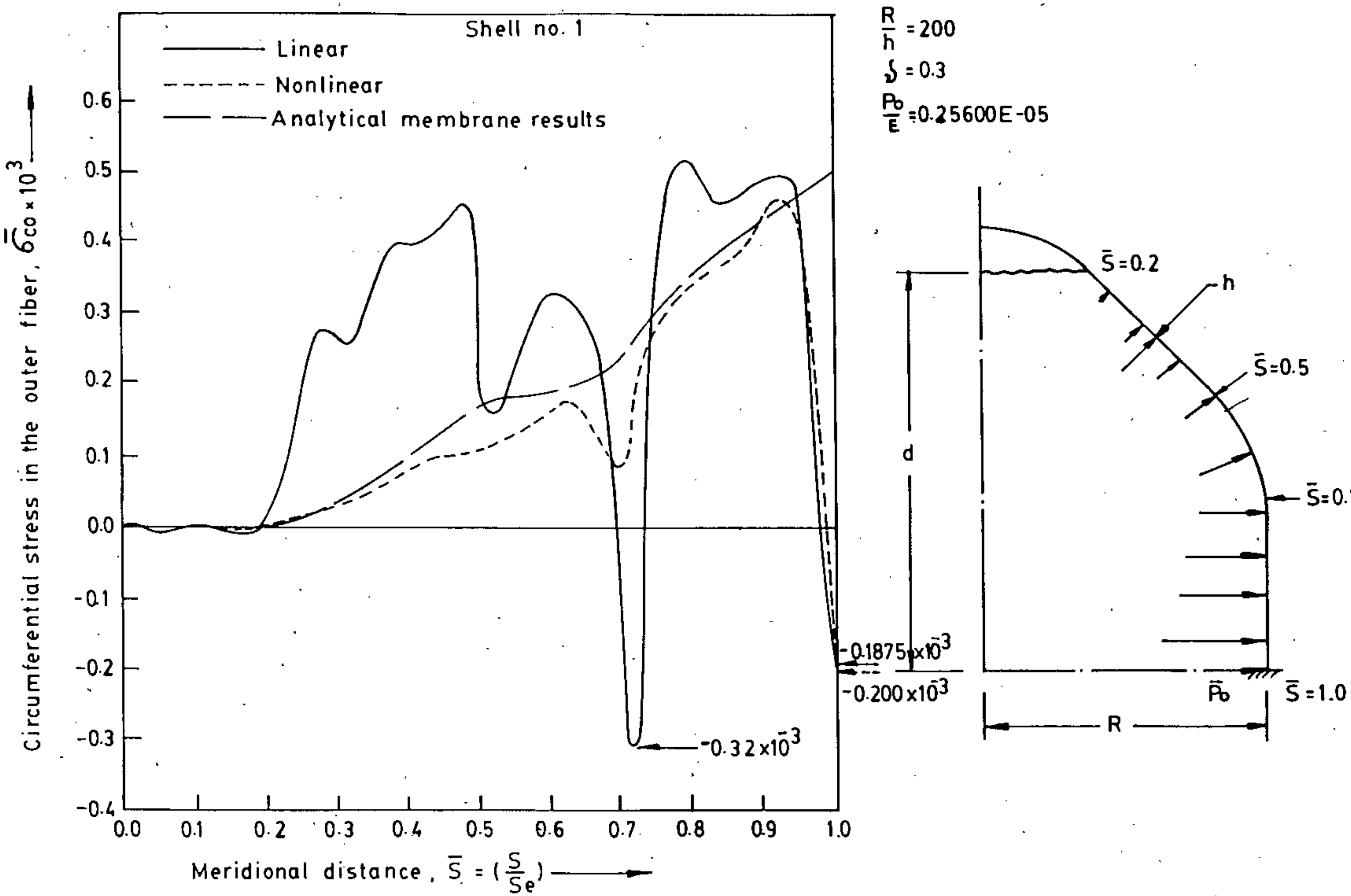


Fig. 18 Distribution of circumferential stress at the outer fiber in shell no. 1

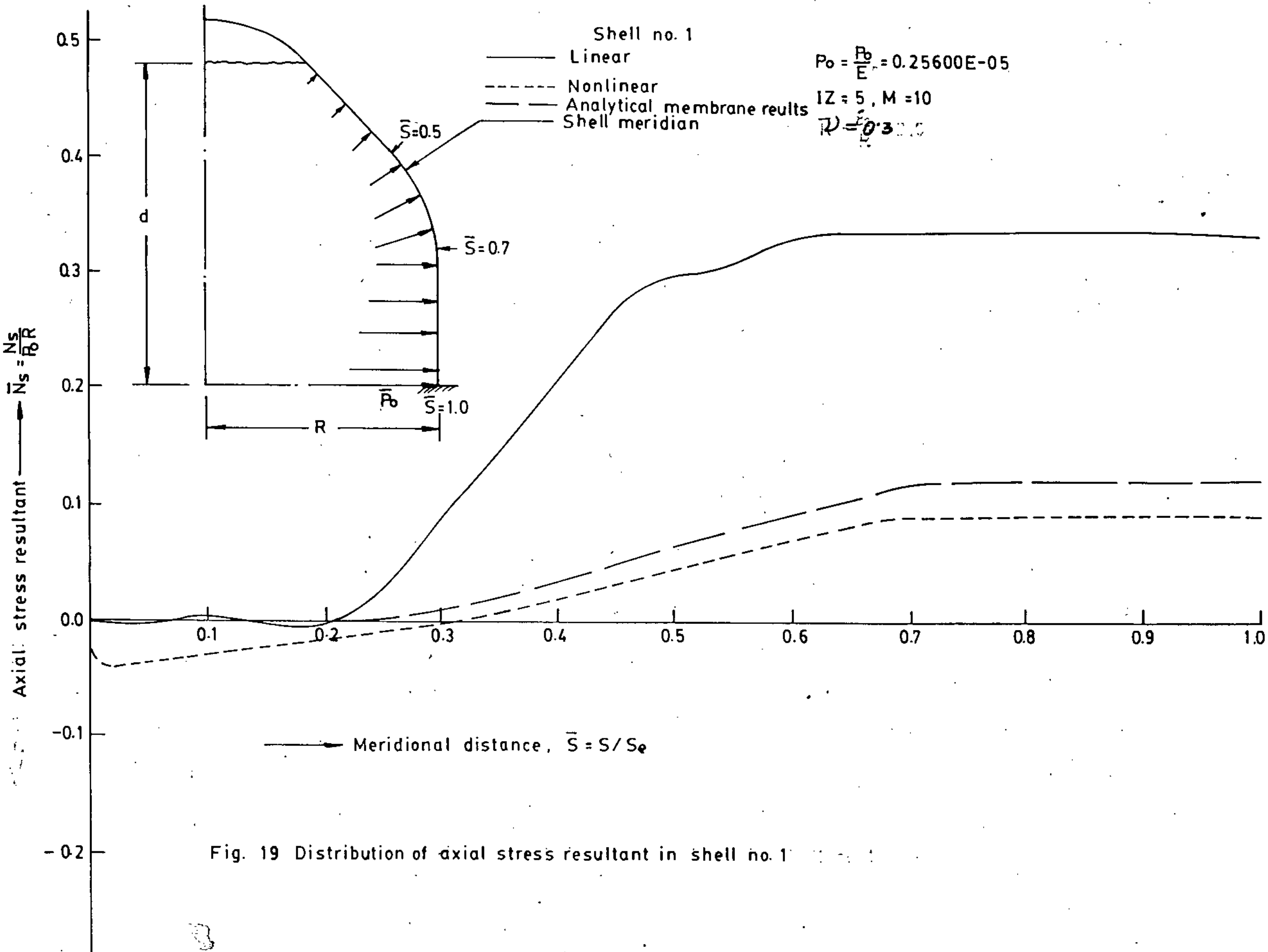
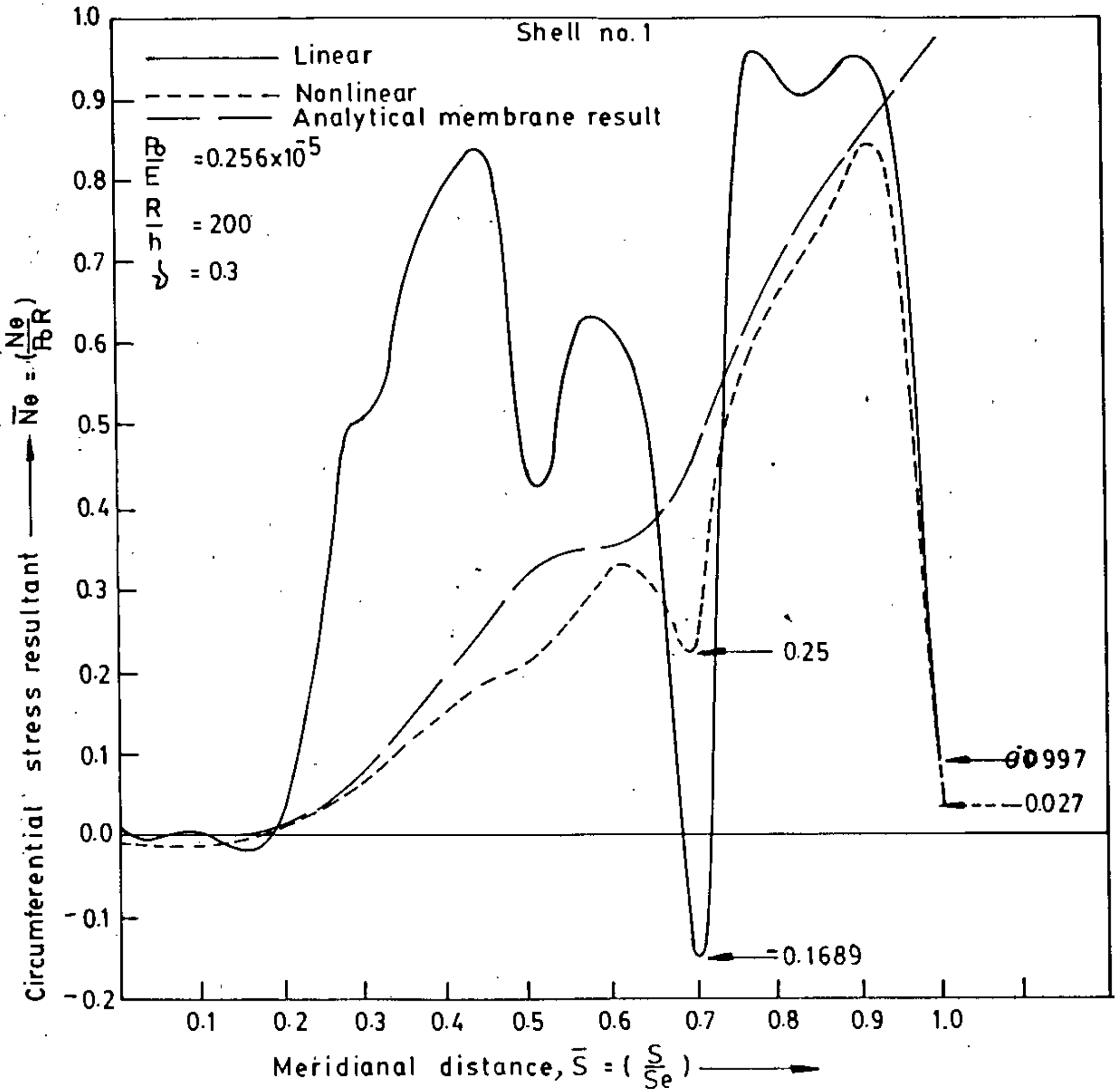


Fig. 19 Distribution of axial stress resultant in shell no. 1



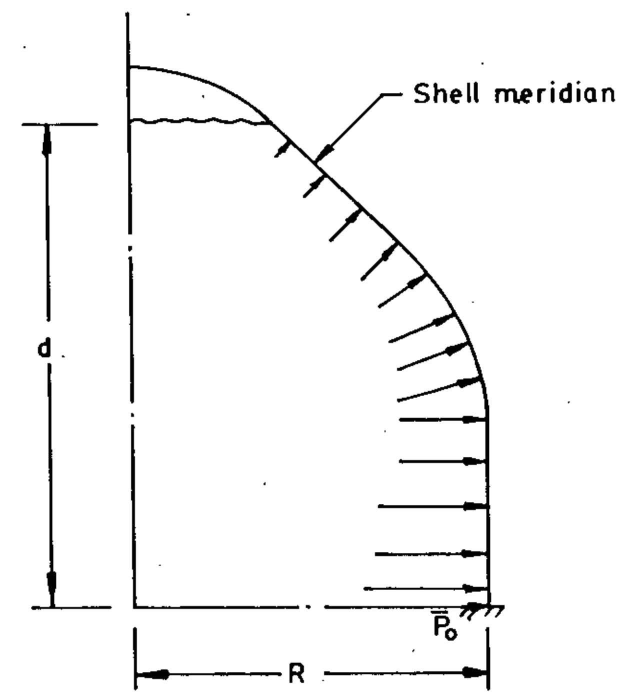


Fig. 20 Distribution of circum. stress resultant in shell no. 1

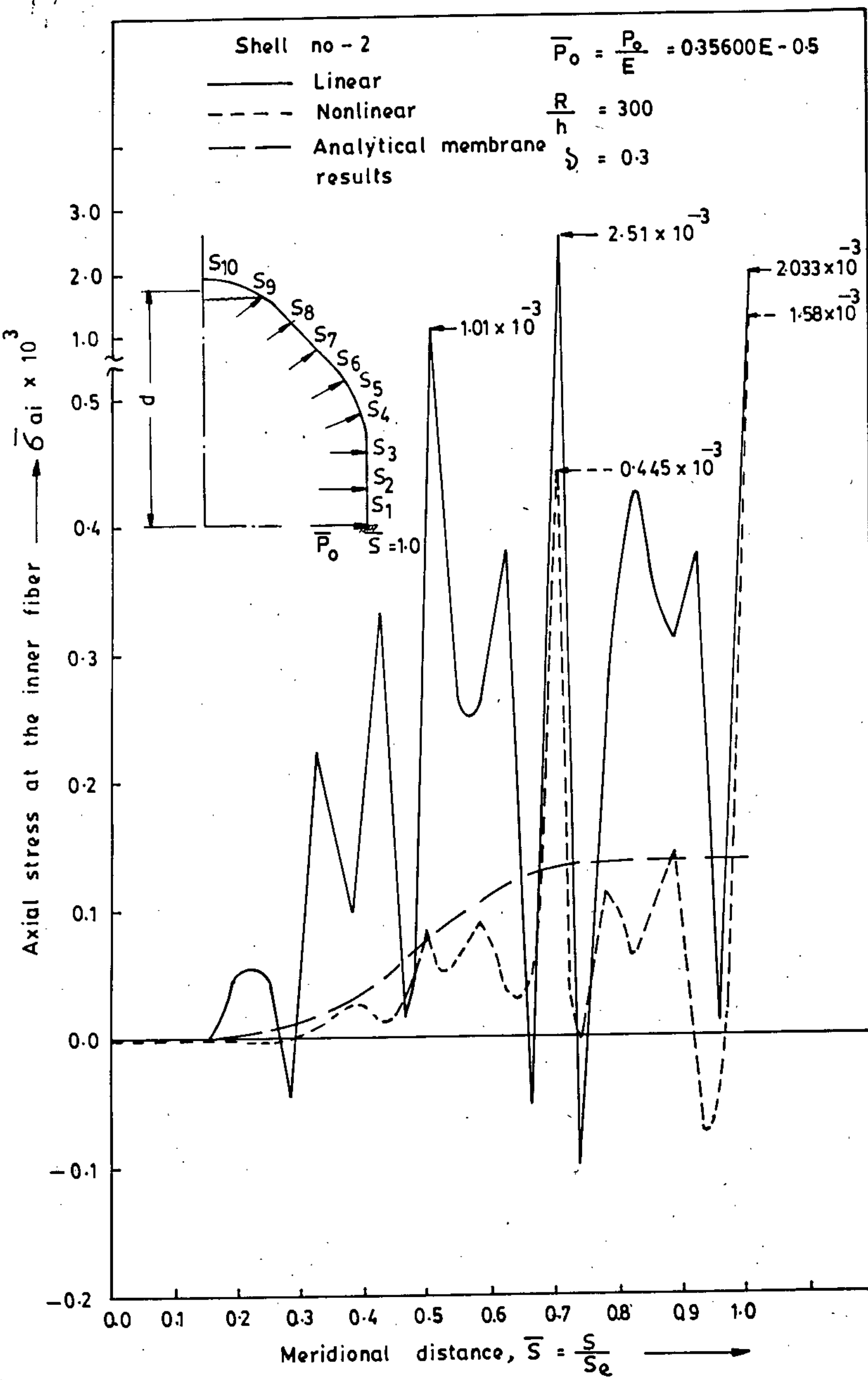


Fig. 21 Distribution of axial stress at the inner fiber in shell no - 2

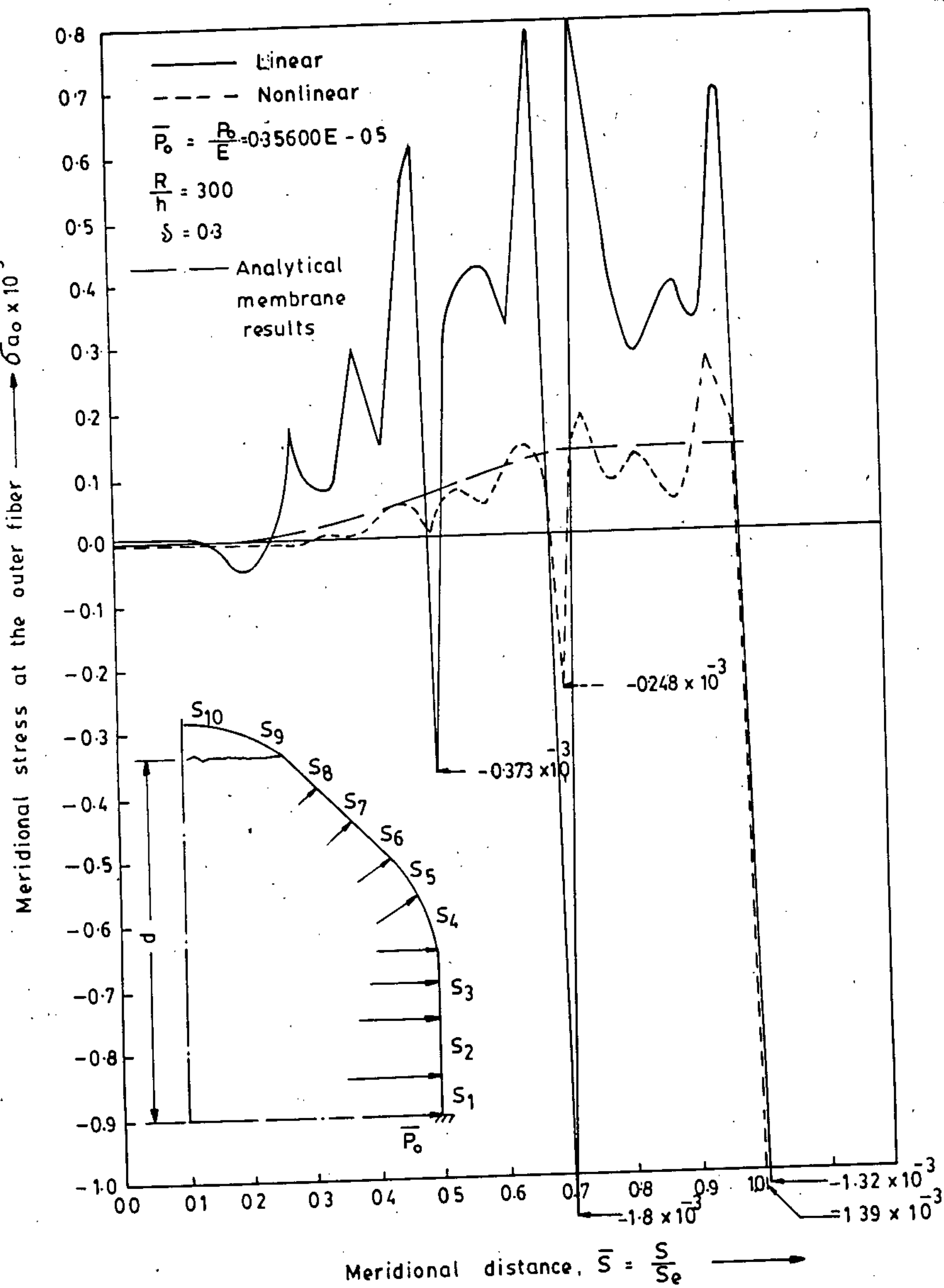


Fig.22 Distribution of meridional stress at the outer fiber in shell no -2

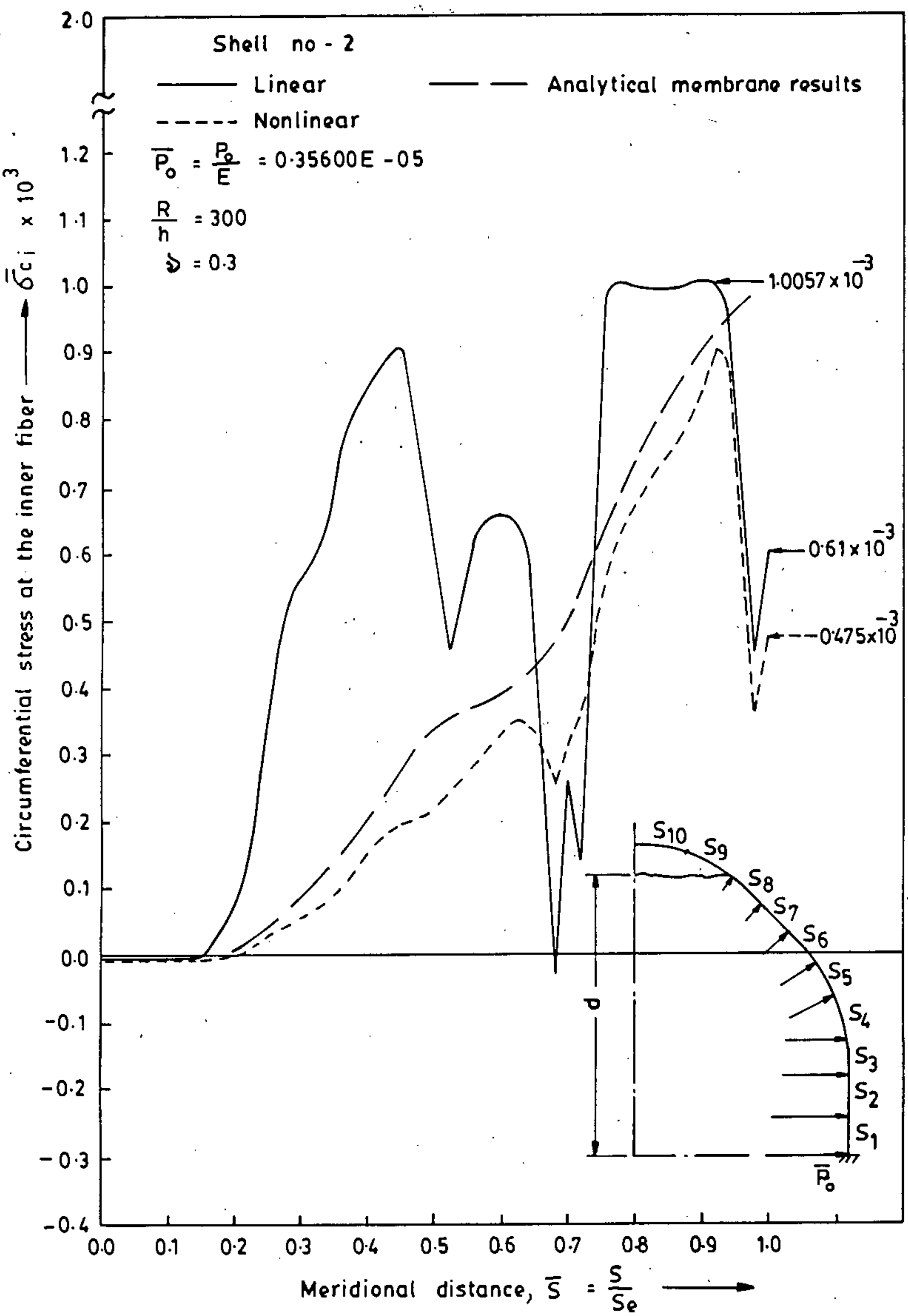


Fig- 23 Distribution of circumferential stress at the inner fiber in shell no - 2

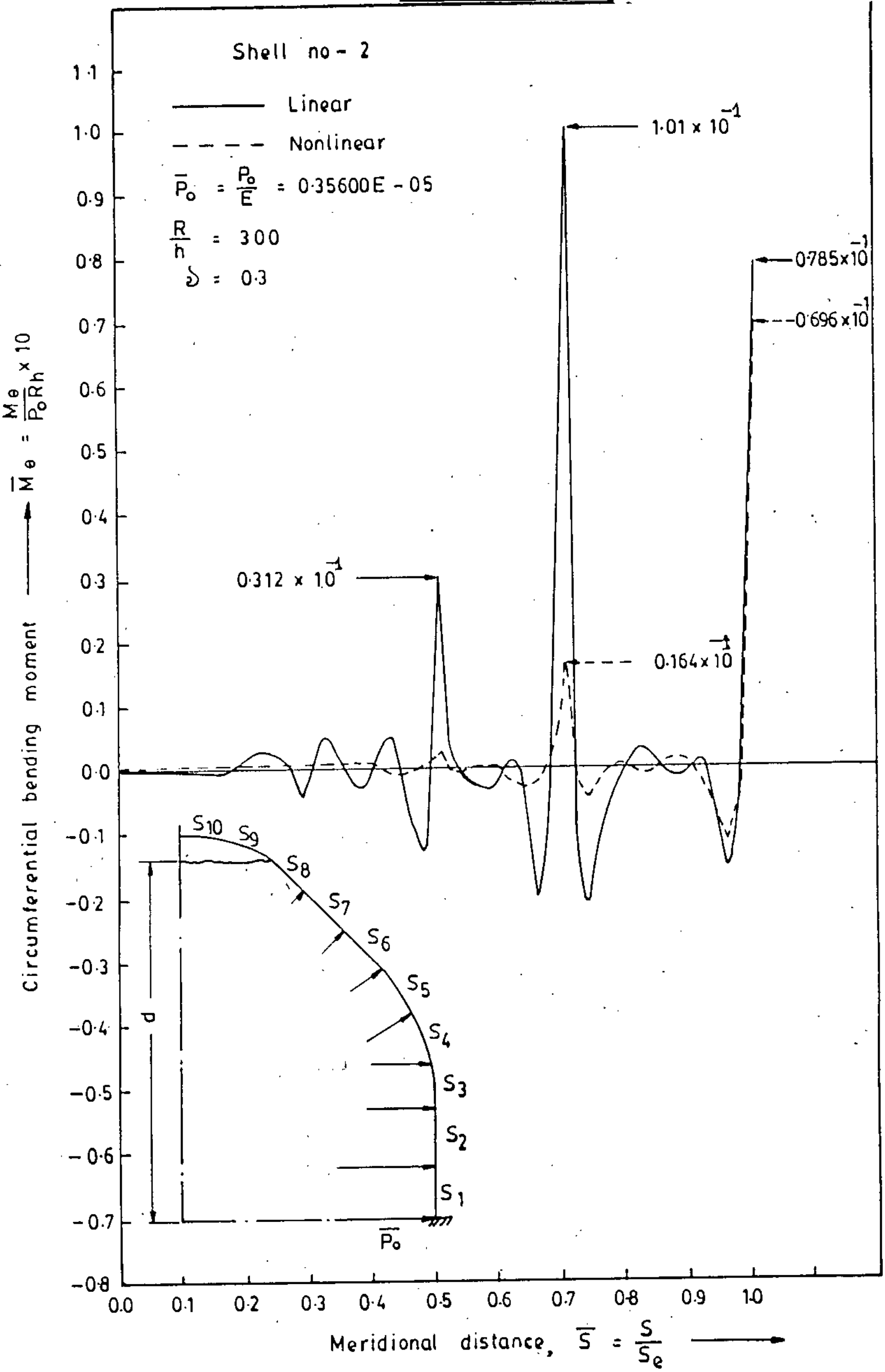


Fig. 24 Distribution of circumferential bending moment in shell no - 2

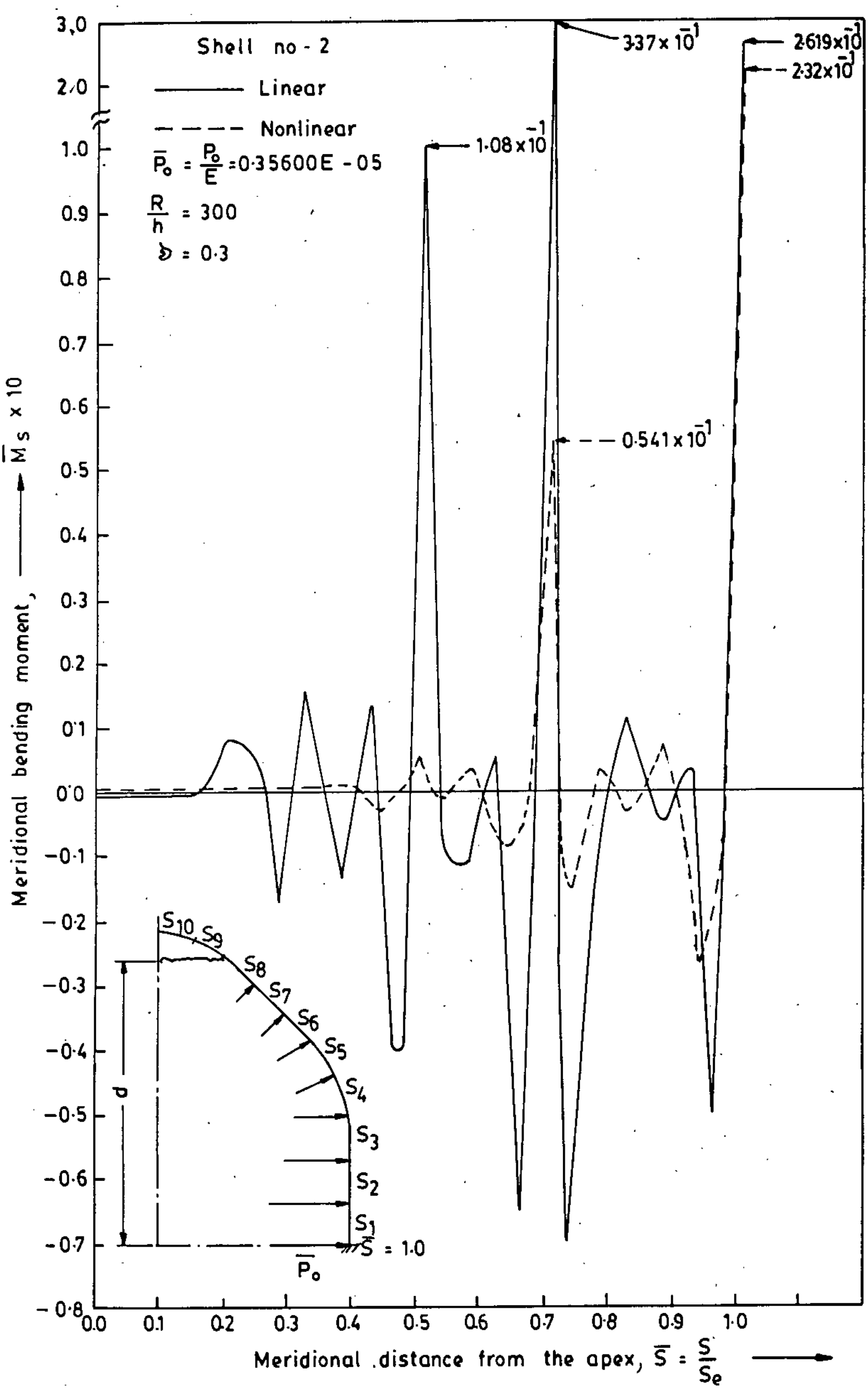


Fig.25 Distribution of meridional bending moment in shell no-2

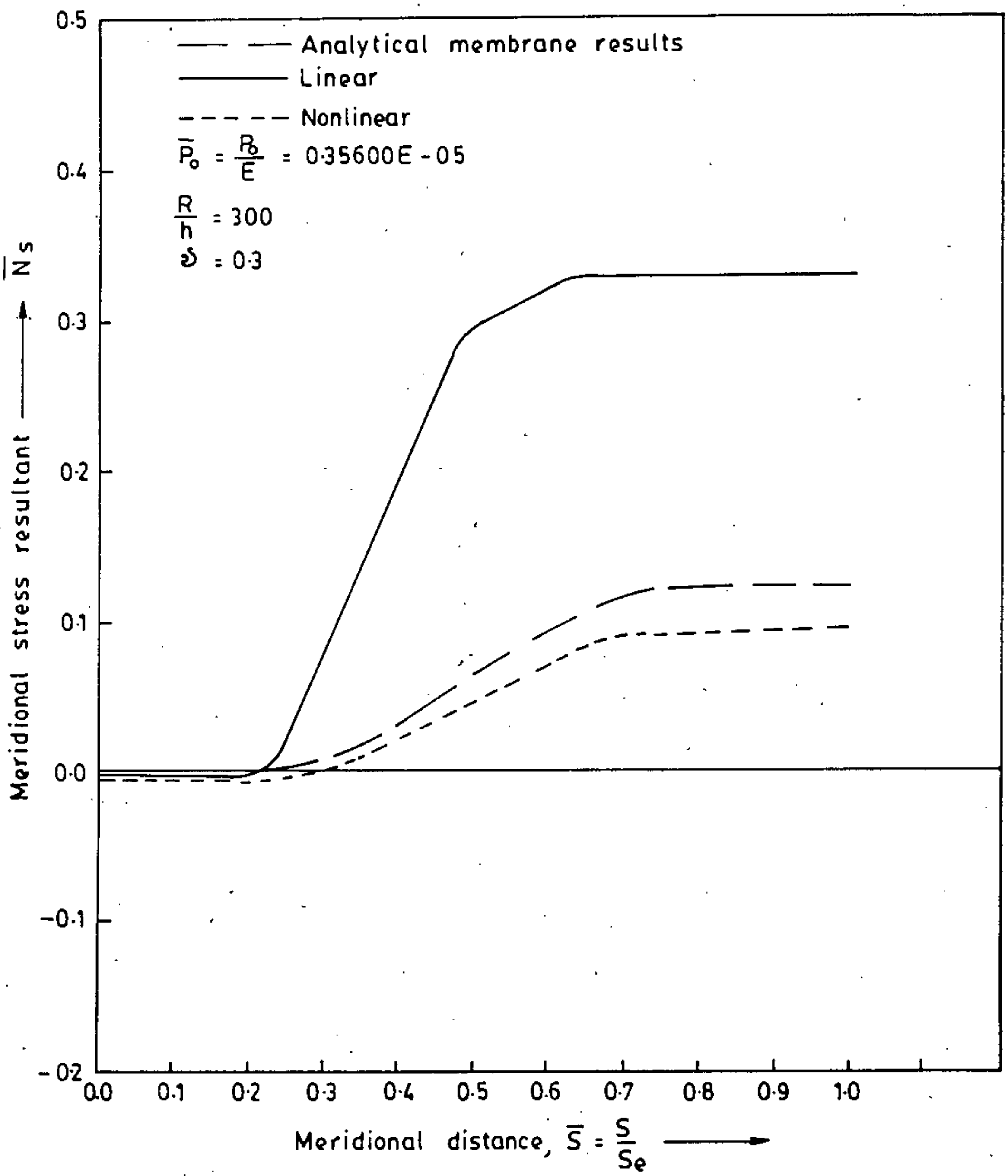


Fig.26 Distribution of meridional stress resultant in shell no-2

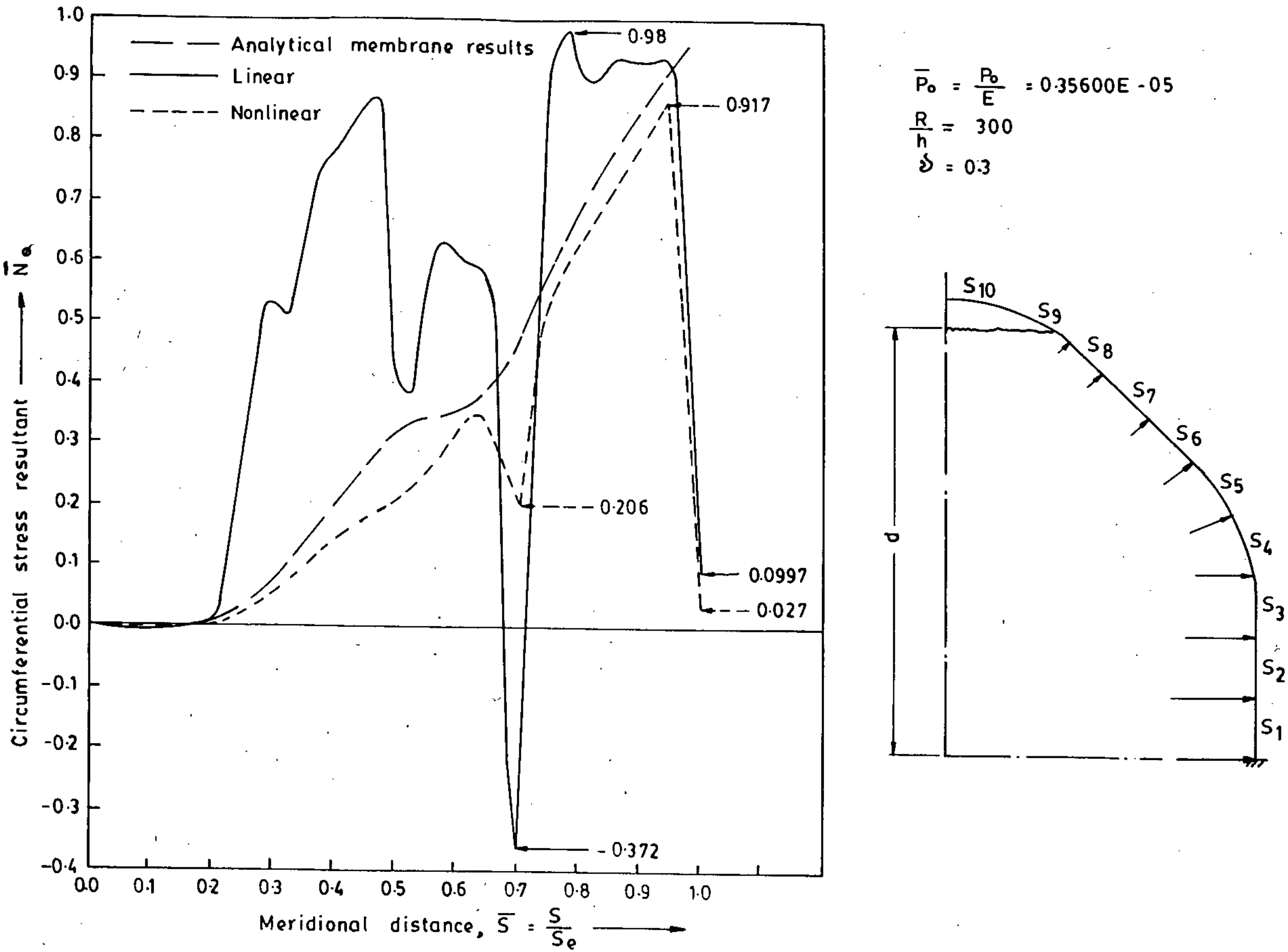


Fig. 27 Distribution of circumferential stress resultant in shell no - 2

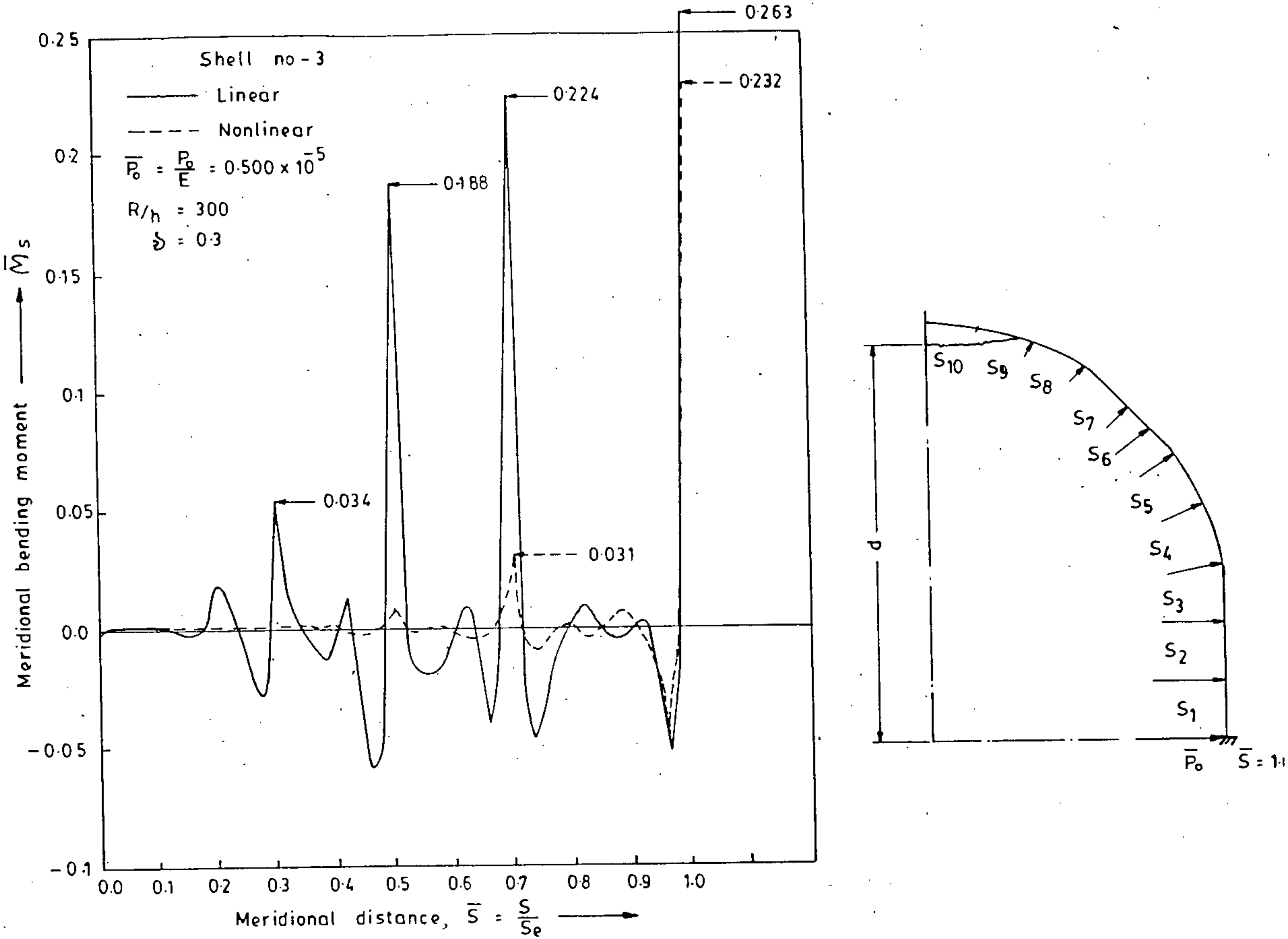


Fig. 28 Distribution of meridional bending moment in shell no - 3

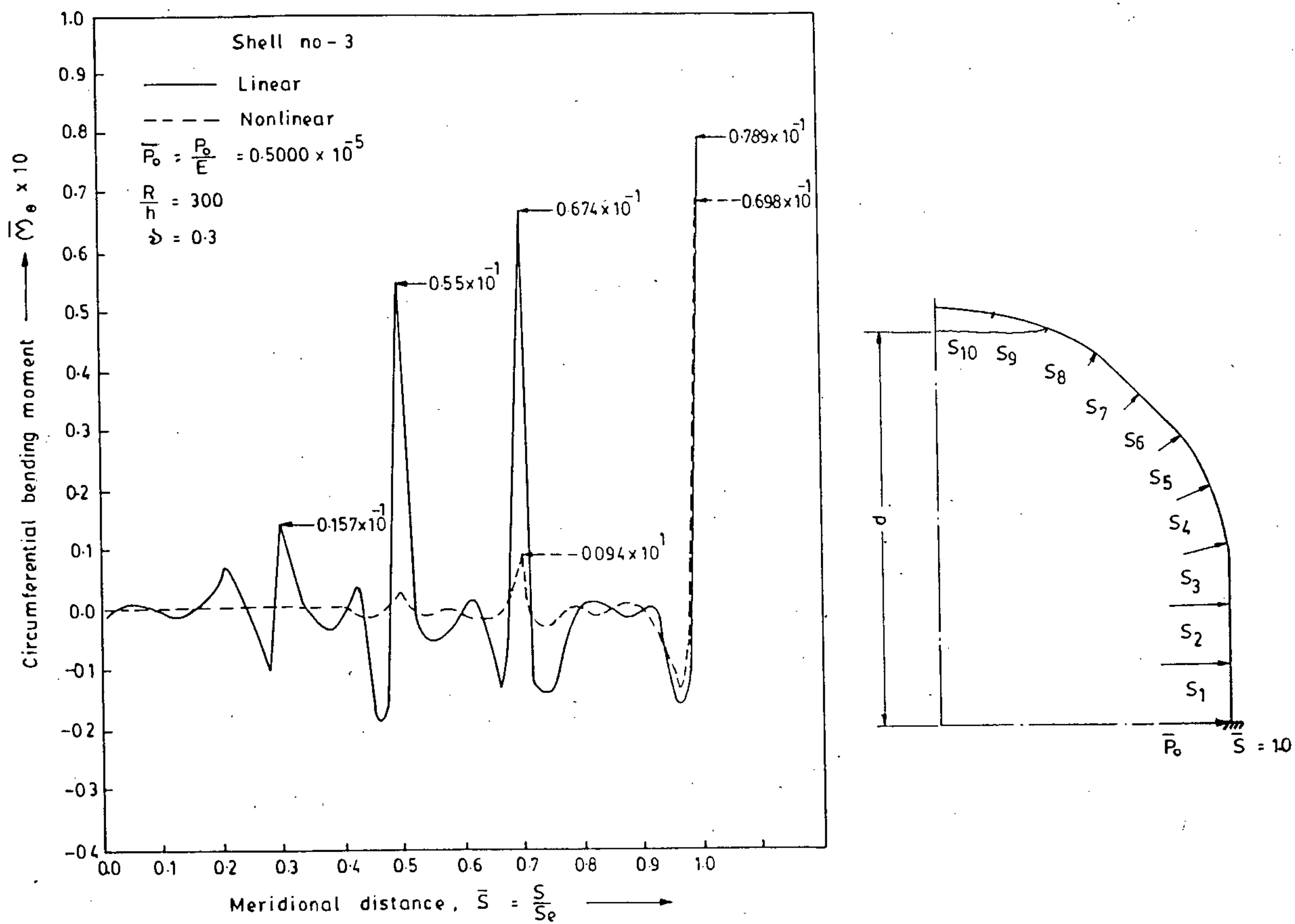


Fig. 29 Distribution of circumferential bending moment in shell no - 3

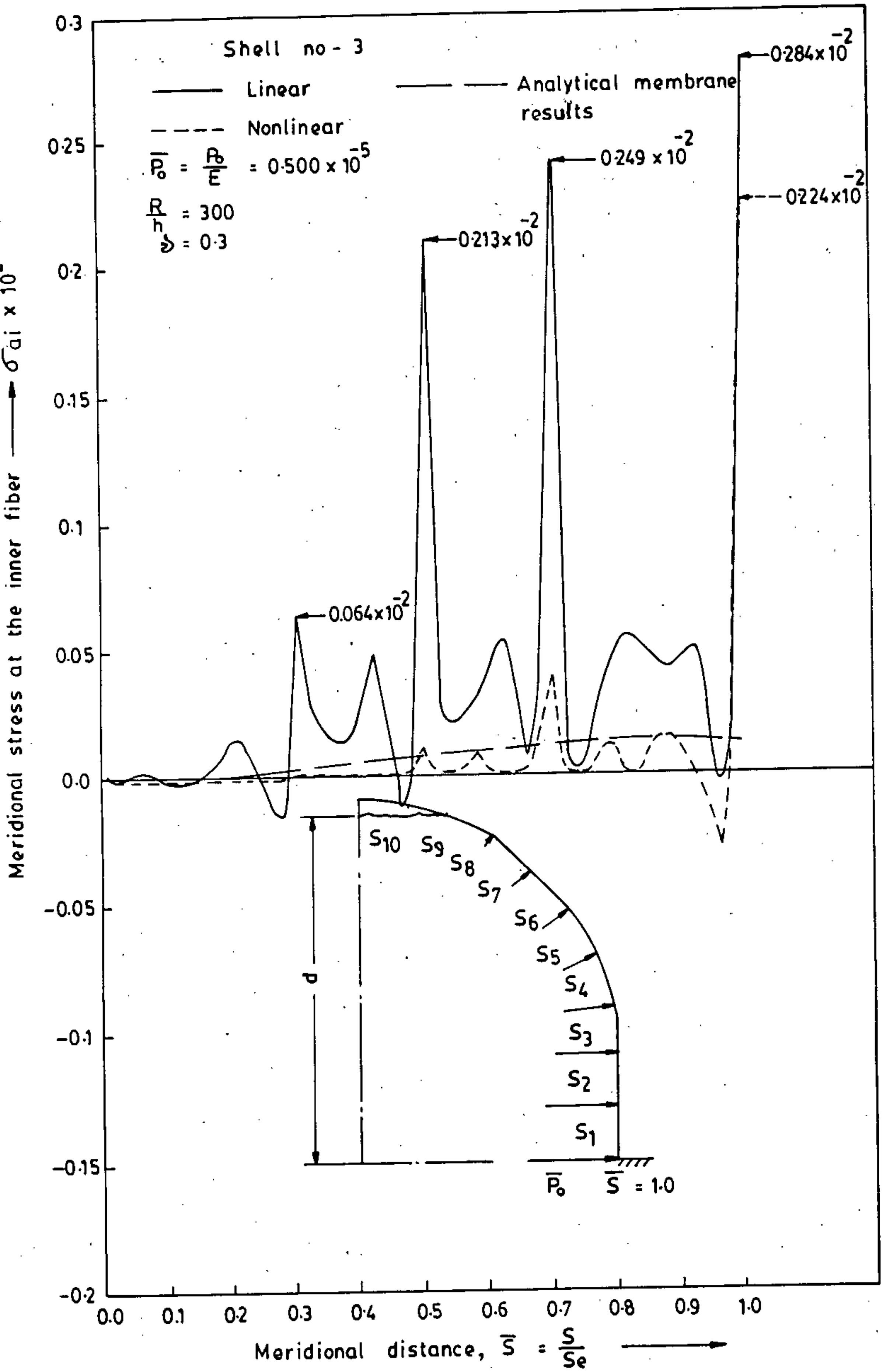


Fig. 30 Distribution of meridional stress at the inner fiber in shell no – 3

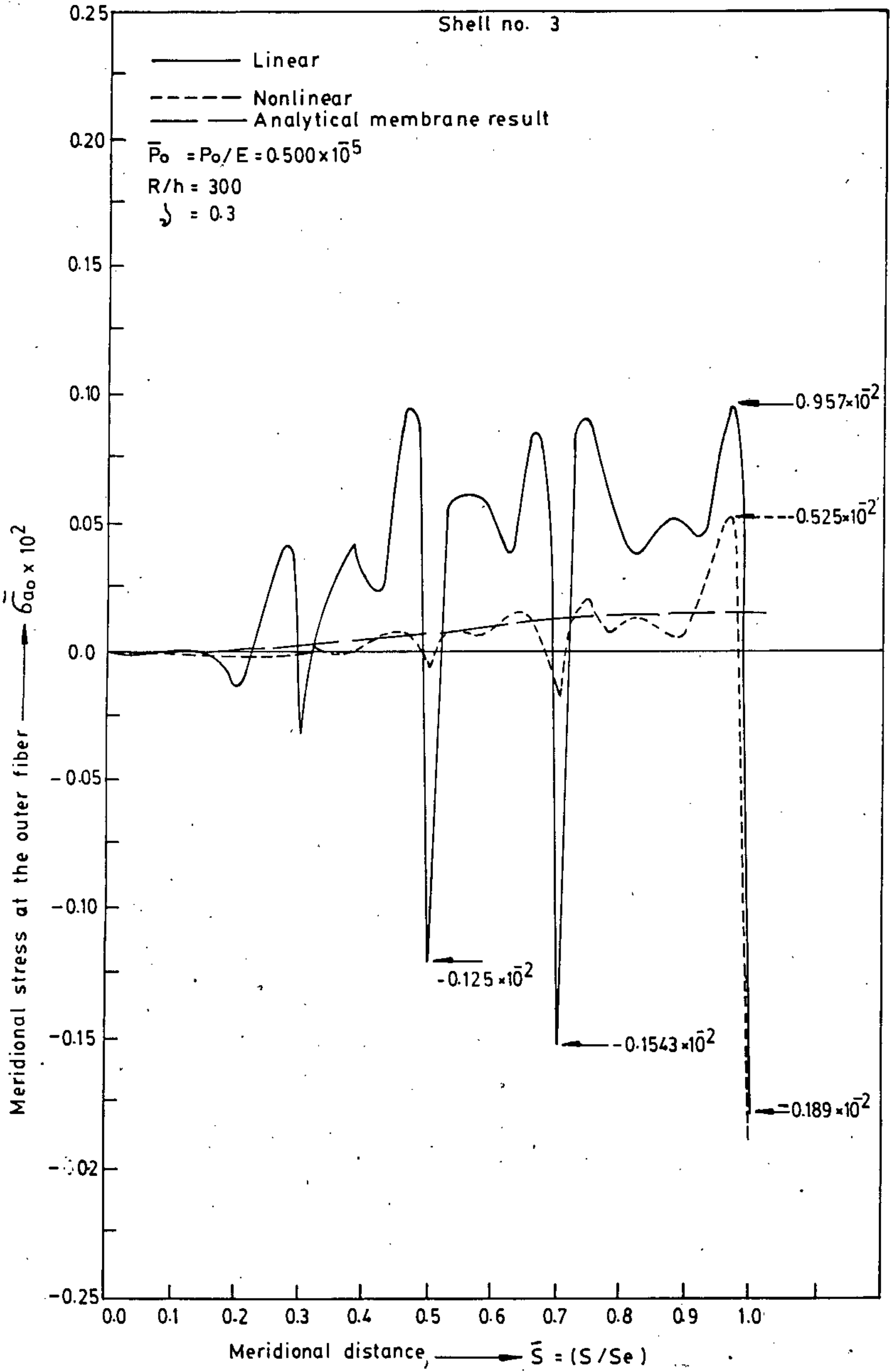
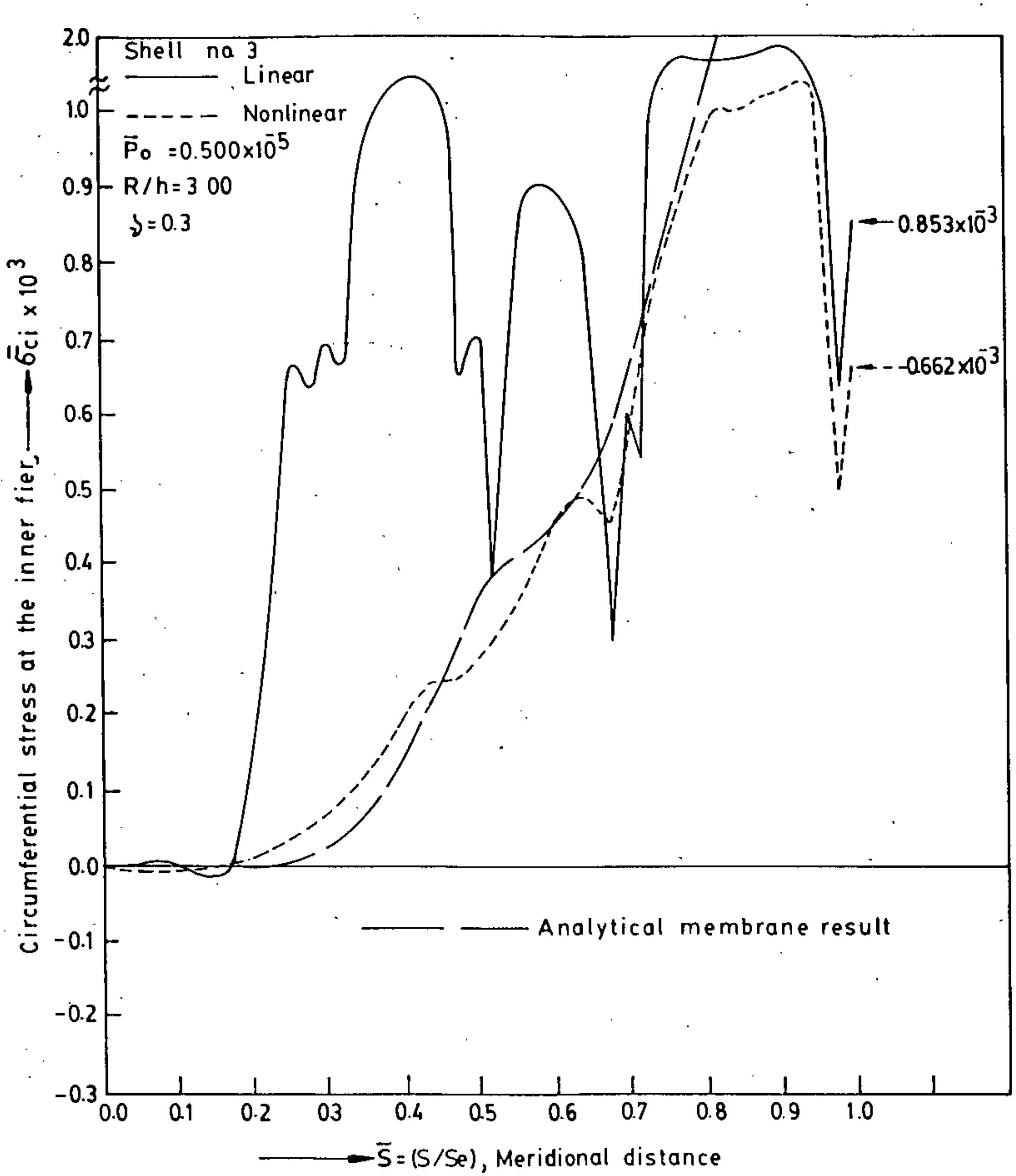


Fig. 31 Distribution of meridional stress at the outer fiber in shell no-3



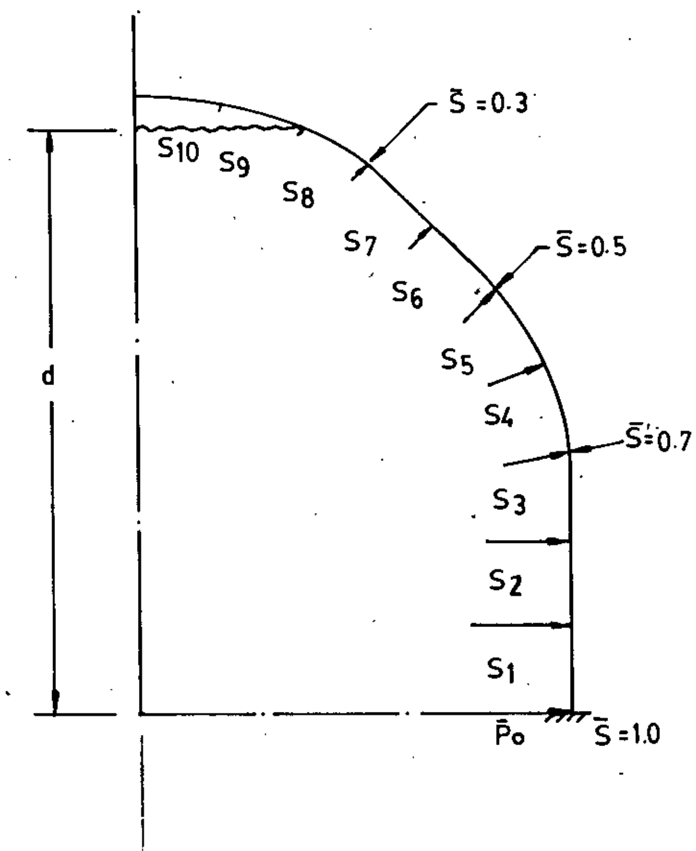


Fig. 32 Distribution of circumferential stress at the inner fiber in shell no. 3

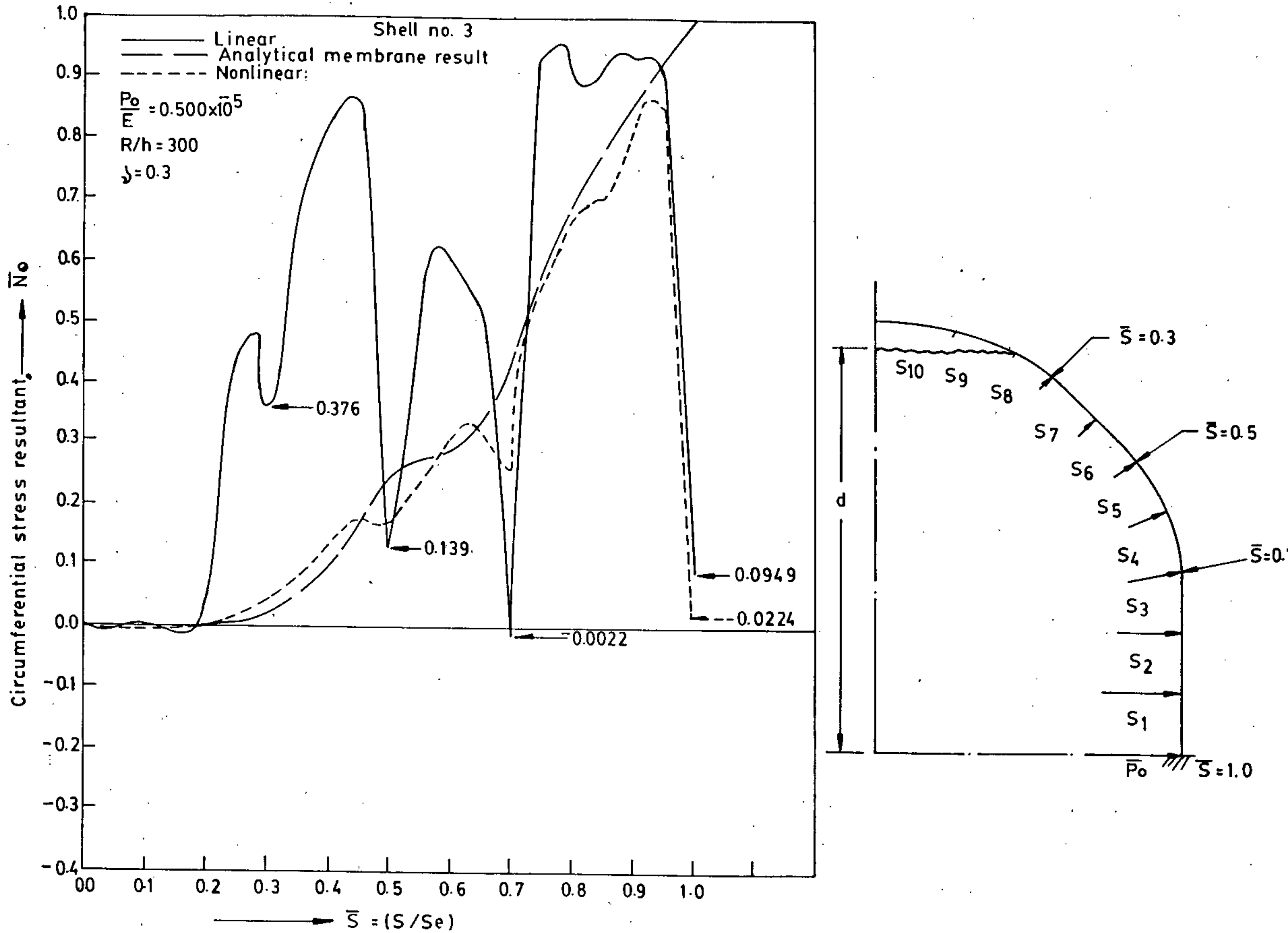


Fig. 33 Distribution of circumferential stress resultant in shell no. 3

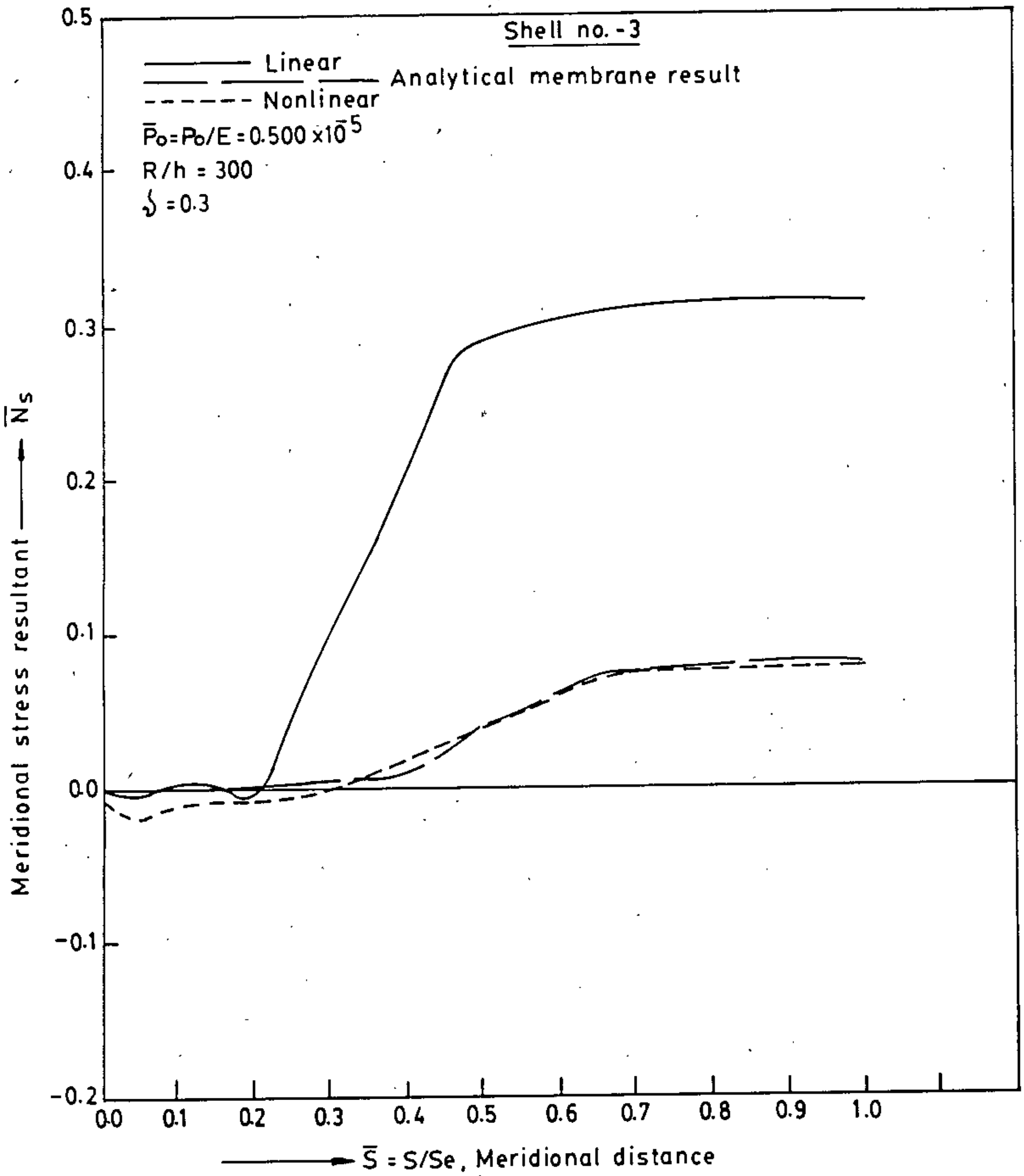


Fig. 34 Distribution of meridional stress resultant in shell no. -3

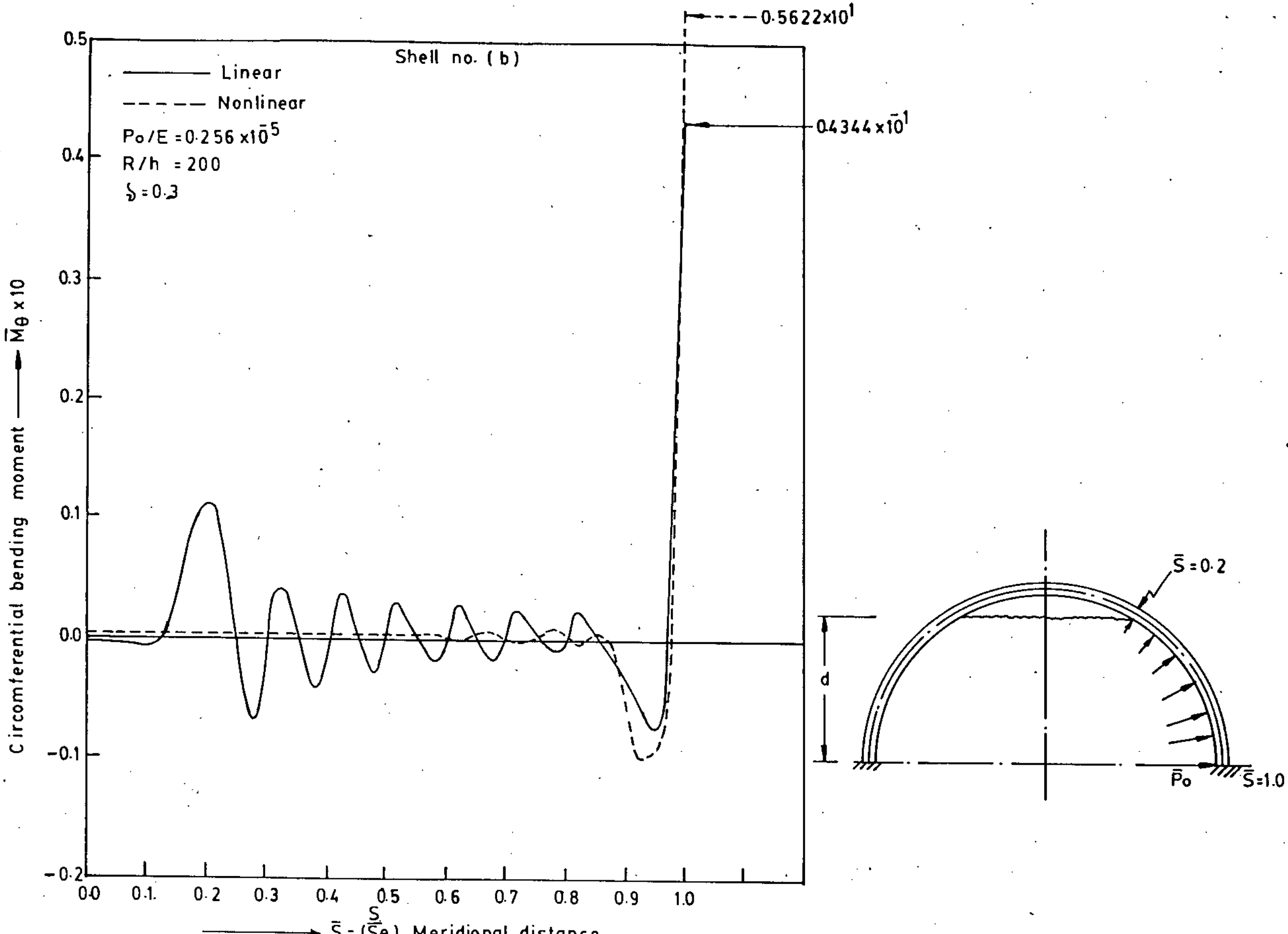


Fig. 35 Distribution of circumferential bending moment in shell no. (b)

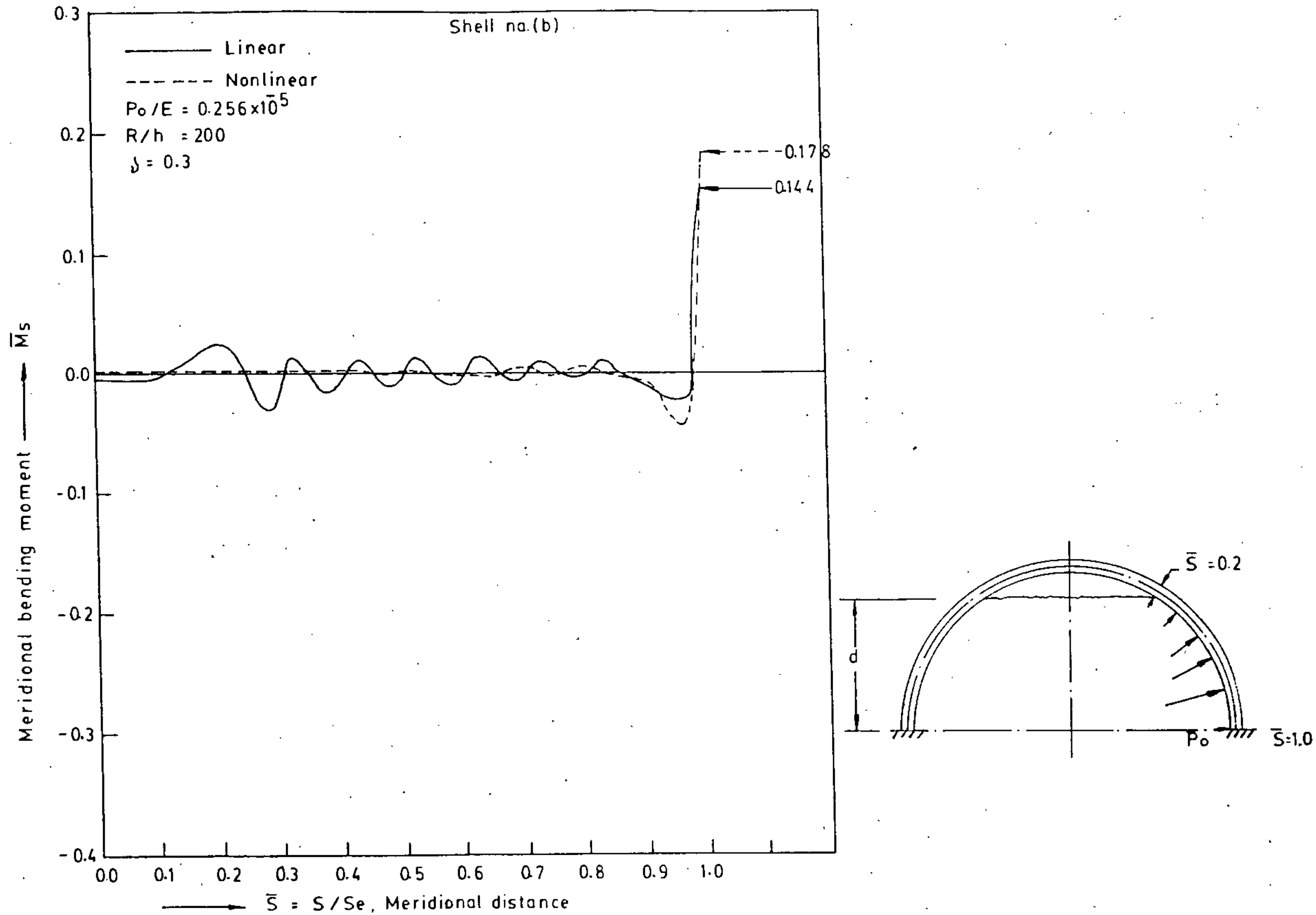


Fig. 36 Distribution of meridional bending moment in shell no. (b)

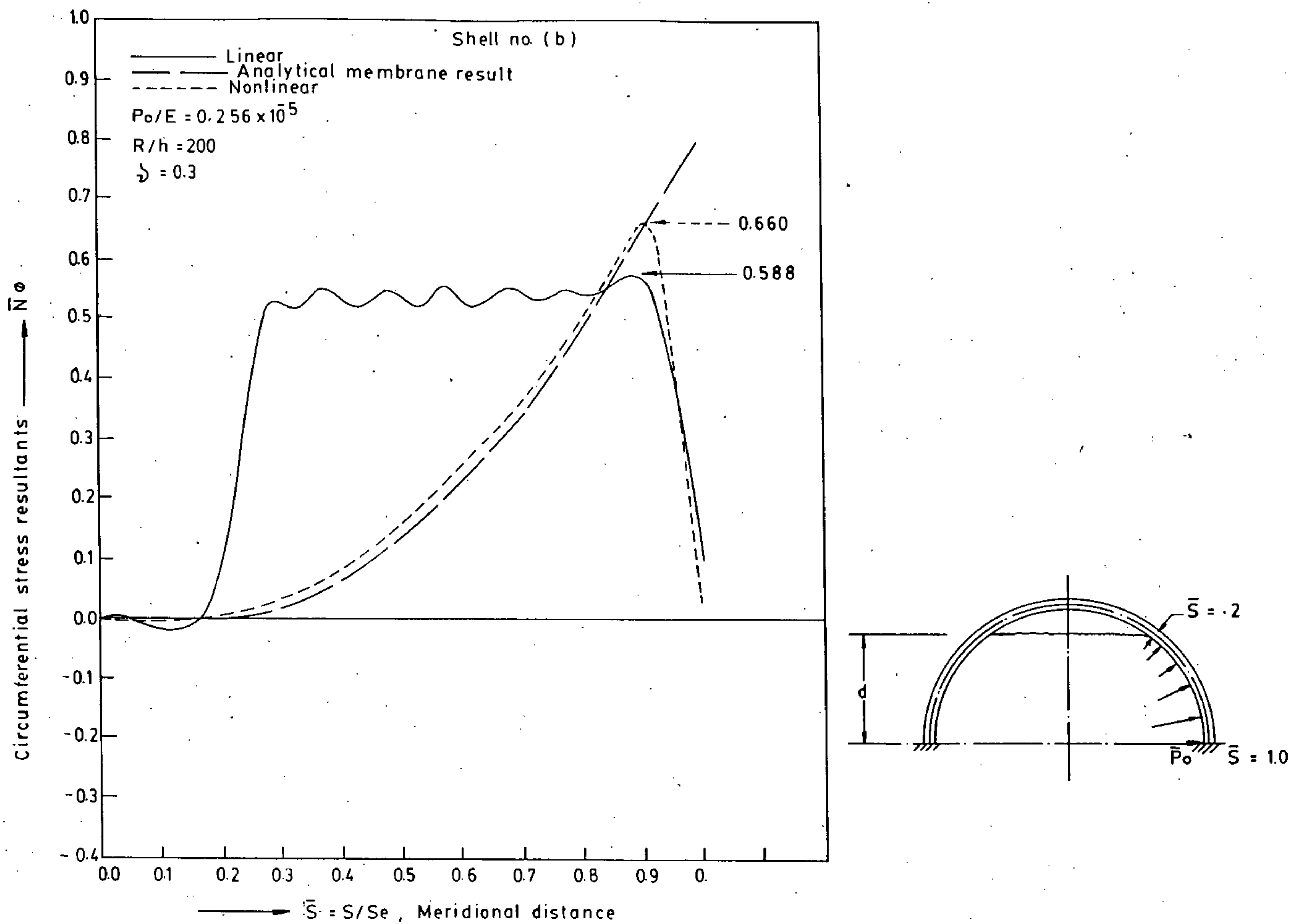


Fig. 37 Distribution of circumferencial stress resultants, in shell no. (b)

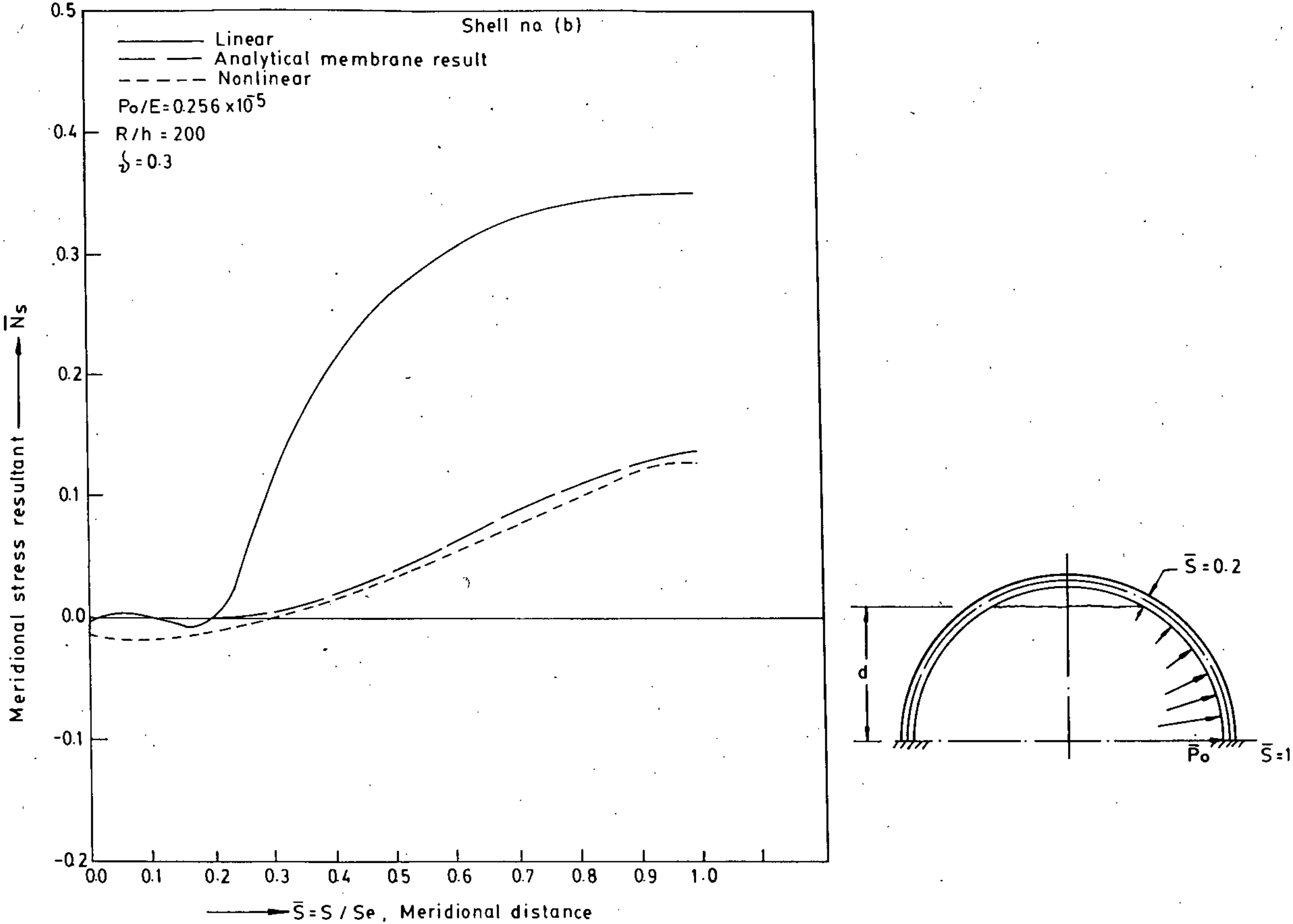


Fig. 38 Distribution of meridional stress resultant, $\bar{N}_s$, in shell no. (b)

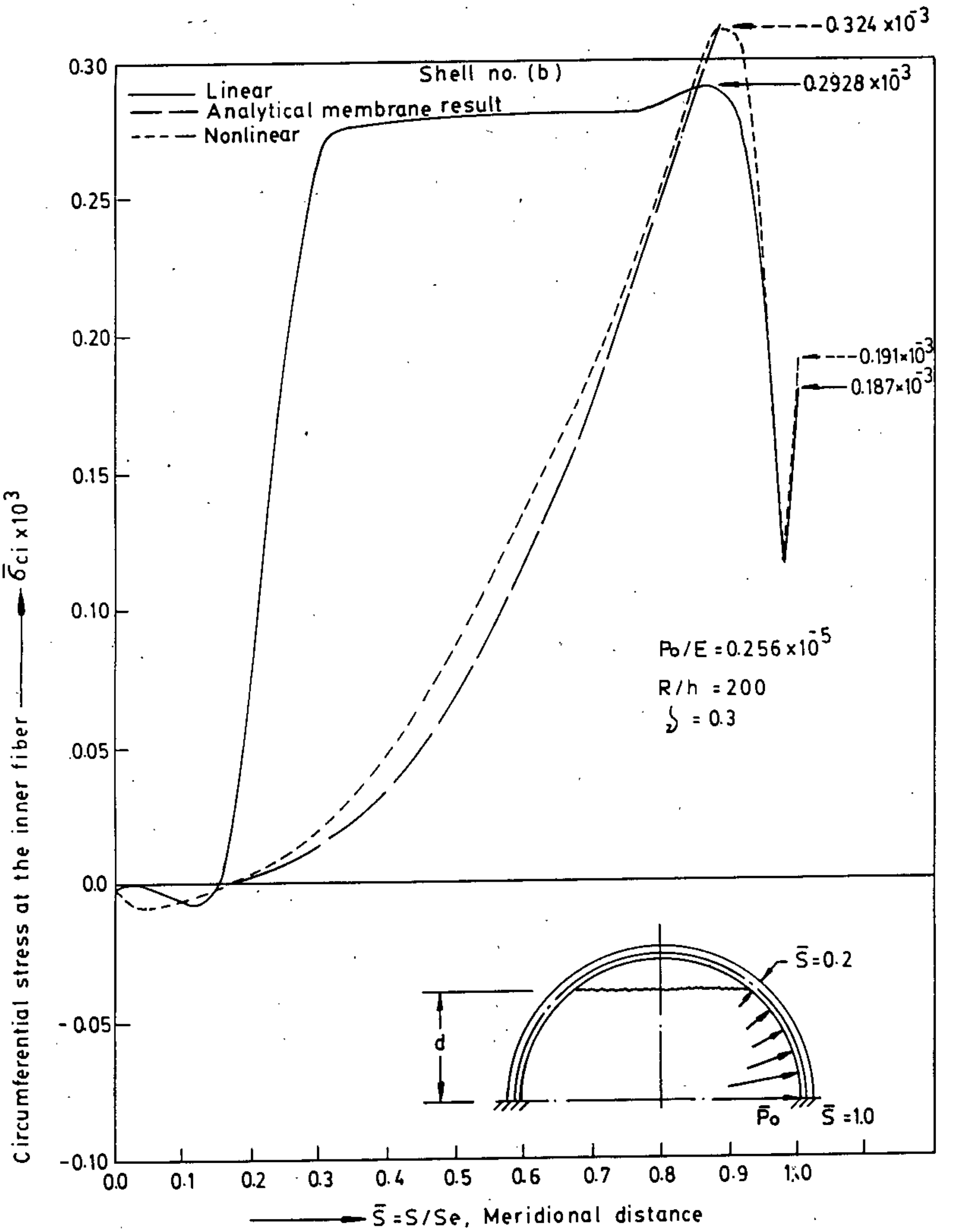


Fig. 39 Distribution of circumferential stress at the inner fiber in shell no.(b)

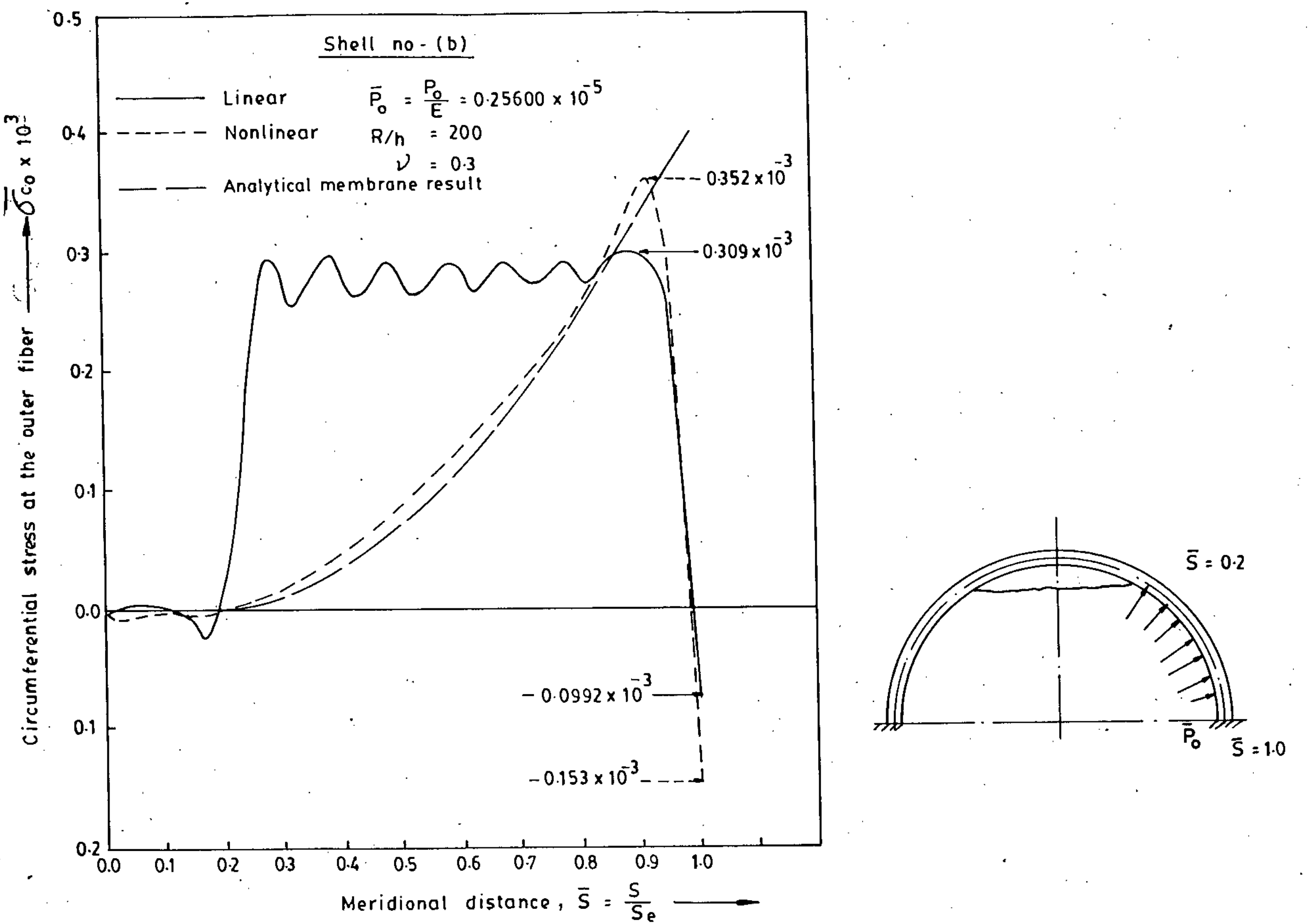


Fig. 40 Distribution of circumferential stress at the outer fiber in shell no (b)

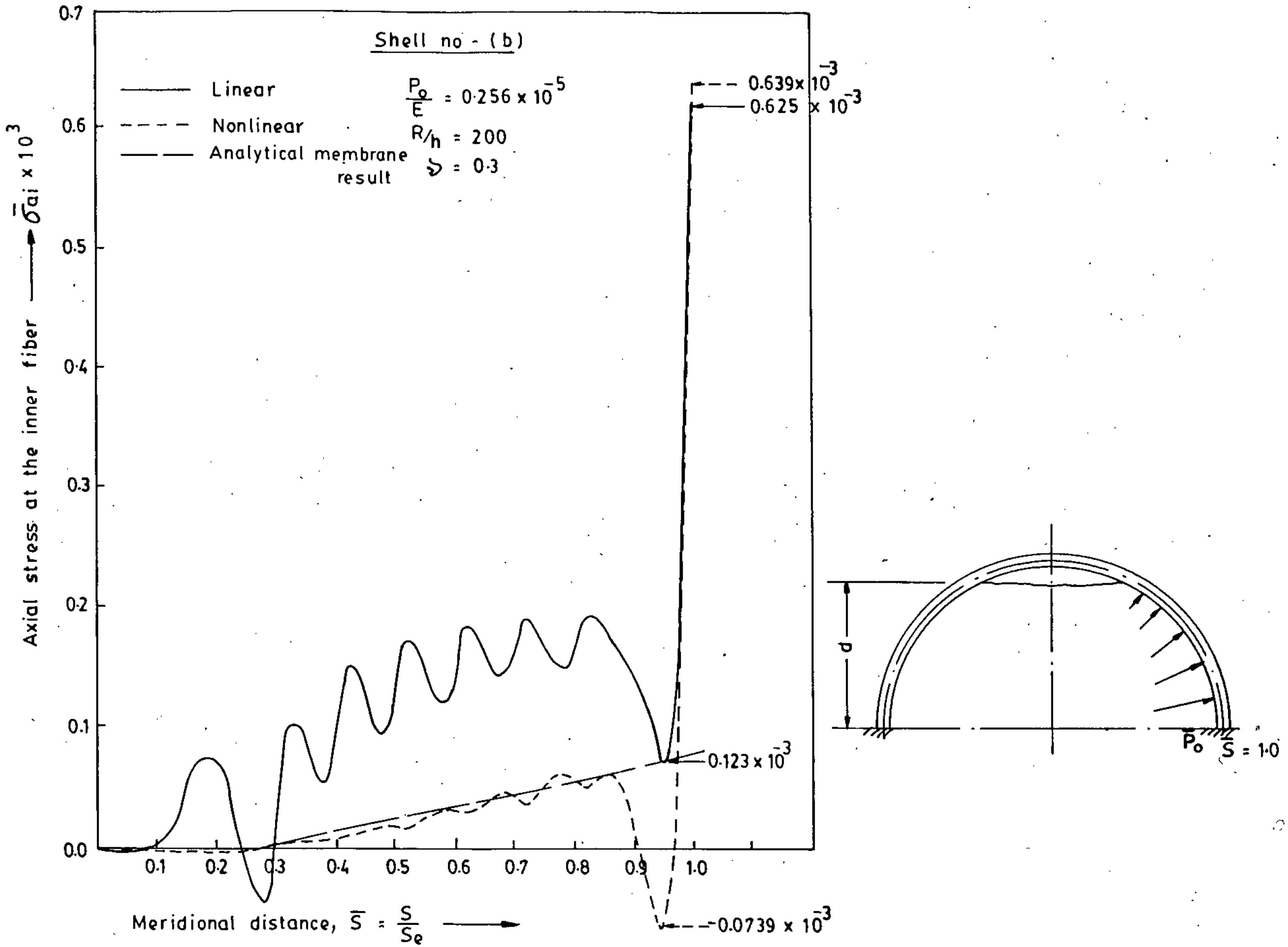


Fig. 41 Distribution of axial stress at the inner fiber in shell no - (b)

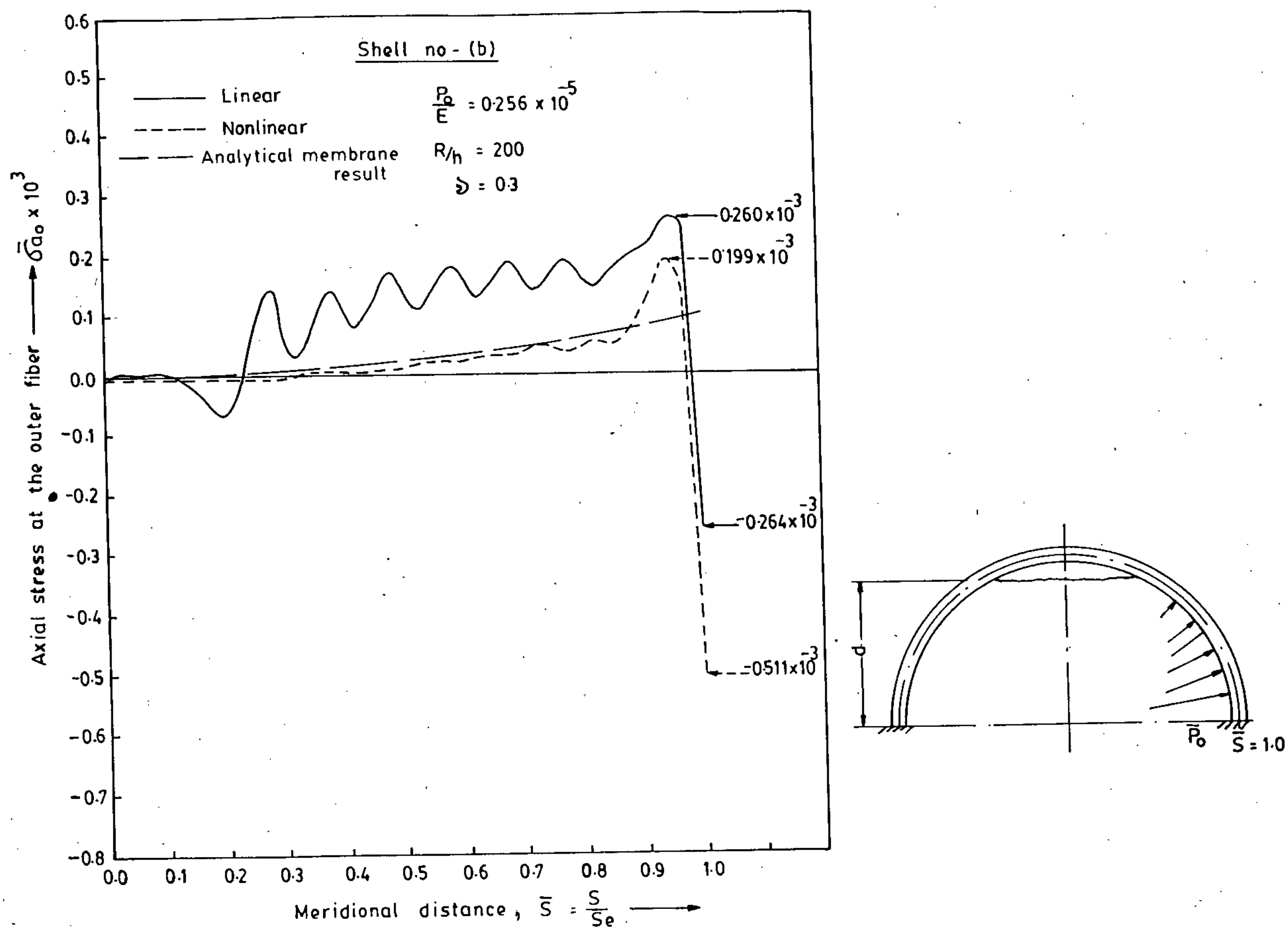


Fig. 42 Distribution of axial stress at the outer fiber in shell no (b)

TABLE - 1

Analytical and Computational Solutions of pure Cylindrical shell with one end fixed.

SHELL PARAMETERS : $P_o/E = 0.256 \times 10^{-5}$, $R/h = 200$, $\nu = 0.3$

Meridional distance from the appex, $\bar{S}$	Radial Displacements, u in inch			Circumferential stress Resultant, N_θ lb/ inch			Axial Bending Moments, M_s inch lb. per inch		
	Analytical	Computational		Analytical	Computational		Analytical	Computational	
		Linear	Non-linear		Linear	Non-linear		Linear	Non-linear
1.0	0.0	0.0	0.0	0.0	0.2279	-11.776	765175.1	773292.14	660476.27
0.95	1.109	1.12	0.9433	166350	168219.27	141511.87	-159792.7	-153315.06	-139186.62
0.90	0.9919	1.304	0.9880	148785	195749.47	148216.79	- 7562.65	- 9305.61	- 11116.31
0.85	0.8477	1.24	0.770	127155	18 6231.1	115596.64	4806.17	- 3144.02	4816.04
0.80	0.680	1.213	0.5924	102000	182090.54	88868.82	- 183.95	- 2564.66	255.75
0.75	0.5108	1.176	0.4216	76620	176469.76	67576.40	- 112.02	- 3221.10	1169.13
0.70	0.3404	1.11	0.2641	51060	167808.89	39623.15	13.65	-13026.88	4581.02
0.65	0.1702	0.909	0.1476	25530	136377.01	22142.17	1.86	-65105.90	0.01025
0.60	1.79×10^{-7}	0.224	0.0457	0.02685	33602.20	6858.05	-0.5097	76485.19	-3.49×10^{-5}
0.55		-0.030	-0.0561		-4517.04	-8425.25		18860.36	-3.47×10^{-5}
0.50		4.27×10^{-3}	-0.0158		640.68	-23708.15		0.0165	-4.126×10^{-5}

TABLE- 2

LINEAR AND NONLINEAR RESULTS OF THE COMPOSITE SHELL NO.1 (Figure - 1)

SHELL PARAMETERS : Thickness ratio, R/h = 200 ; Poisson's ratio, = 0.3

Base Pressure, P_o /E = 0.25600 E-05

LINEAR RESULTS

$\bar{s}$	$\bar{u}$	$\bar{\beta}$	$\bar{w}$	$\bar{V}$	$\bar{H}$	$\bar{M}_s$
0.10000000E+01	[illegible]	[illegible]	[illegible]	[illegible]E+00	[illegible]79E-01	0.25837714E+00
0.90000000E+00	0.25458162E+00	-.21825883E-04	-.250[illegible]	[illegible]E+00	-.14[illegible]79E-03	-.34884877E-02
0.80000000E+00	[illegible]4744[illegible]04E+00	-.52691171E-03	-.3[illegible]	[illegible]261299E+00	-.751[illegible]E-02	-.32171487E-02
0.70000000E+00	[illegible]	[illegible]	[illegible]	[illegible]261224E+00	[illegible]E-01	0.27524725E+00
0.60000000E+00	0.46554481E+00	-.21599618E-03	-.4915[illegible]	[illegible]E+00	0.151[illegible]E+00	-.75020643E-02
0.50000000E+00	0.30434[illegible]E+00	-.19797030E-02	-.320451[illegible]	[illegible]97[illegible]006E+00	0.194[illegible]E+00	0.86473238E-01
0.40000000E+00	[illegible]E+00	[illegible]E-03	[illegible]	[illegible]723377E+00	0.14173411E+00	-.16533845E-02
0.30000000E+00	0.26713595E+00	0.38071029E-04	-.2799[illegible]	[illegible]E-01	[illegible]E-01	-.36565762E-02
0.20000000E+00	0.10294861E-01	0.49732331E-03	[illegible]	[illegible]E-08	-.74[illegible]E-03	0.11585519E-01
0.10000000E+00	[illegible]E-04	-.110[illegible]E-0[illegible]	[illegible]	-.91098017E-11	0.11[illegible]E-03	-.17779222E-03
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.113[illegible]E-01	[illegible]E+00	-.[illegible]17E-04	-.37291461E-05

NONLINEAR RESULTS

$\bar{s}$	$\bar{u}$	$\bar{\beta}$	$\bar{w}$	$\bar{V}$	$\bar{H}$	$\bar{M}_s$
0.10000000E+01	0.00000000E+00	0.00000000E+00	0.0[illegible]	[illegible]E-01	-.47[illegible]E-01	0.23291764E+00
0.90000000E+00	[illegible]7677E+00	0.7430[illegible]	0.19[illegible]	[illegible]47943E-01	0.35[illegible]E-02	-.32958493E-02
0.80000000E+00	0.51[illegible]E+00	0.527[illegible]	[illegible]E-01	[illegible]E-01	0.15[illegible]	-.16758771E-02
0.70000000E+00	[illegible]	[illegible]	[illegible]	[illegible]E-01	0.1[illegible]E-01	0.44723281E-01
0.60000000E+00	0.27161508E+00	0.2735009[illegible]	0.37[illegible]	[illegible]2252[illegible]E-01	0.3777[illegible]11E-01	-.27610084E-02
0.50000000E+00	0.15[illegible]E+00	[illegible]	0.115[illegible]	0.37312069E-01	0.236629[illegible]E-01	0.44253918E-02
0.40000000E+00	0.9710[illegible]E-01	[illegible]	[illegible]	[illegible]914180E-01	0.1[illegible]E-01	0.52273825E-03
0.30000000E+00	0.34180546E-01	[illegible]E-03	0.2[illegible]	-.5185292[illegible]E-07	-.352507[illegible]E-03	0.23710544E-03
0.20000000E+00	[illegible]5794[illegible]20E-02	[illegible]	[illegible]	-.248[illegible]245E-07	-.1989931[illegible]E-01	0.24605841E-03
0.10000000E+00	-.272[illegible]E-0[illegible]	[illegible]	[illegible]	[illegible]E-07	-.2[illegible]E-01	0.25229427E-03
0.00000000E+00	0.00000000E+00	0.00000000E+00	[illegible]	[illegible]0000E+00	-.4[illegible]16853E-01	0.35103463E-03

TABLE-3

LINEAR AND NONLINEAR RESULTS OF THE COMPOSITE SHELL No.2 (Figure - 1)

SHELL PARAMETERS : Thickness ratio, R/h = 300 ; Poisson's ratio, = 0.3

Base Pressure, P_o/E = 0.35600E-05

LINEAR RESULTS

$\bar{s}$	$\bar{u}$	$\bar{\beta}$	$\bar{w}$	$\bar{V}$	$\bar{H}$	$\bar{M}_s$
0.10000000E+01	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.33241469E+00	-.39579432E-01	0.26192619E+00
0.90000000E+00	0.83841123E+00	-.37867938E-03	-.22291357E-01	0.33241384E+00	-.15419394E-02	0.15219768E-02
0.80000000E+00	0.82952063E+00	-.16135494E-02	-.33723846E-01	0.33241310E+00	-.20576759E-02	0.25832563E-02
0.70000000E+00	0.47244116E+00	0.24062386E-02	-.62336397E-01	0.33241238E+00	0.51232831E-01	0.33740678E+00
0.60000000E+00	0.47439321E+00	-.90974884E-03	-.45456486E+00	0.29057319E+00	0.15128188E+00	0.13068755E-02
0.50000000E+00	0.25214165E+00	-.57841784E-02	-.34139273E+00	0.22975983E+00	0.19697299E+00	0.10863361E+00
0.40000000E+00	0.48711881E+00	0.19345793E-03	-.53505635E+00	0.14702366E+00	0.14187554E+00	0.42099307E-03
0.30000000E+00	0.27081153E+00	-.56882530E-03	-.35421704E+00	0.64430798E-01	0.57597000E-01	-.40668706E-02
0.20000000E+00	0.81124842E-02	0.94168332E-03	-.72816558E-01	0.33386111E-08	-.32139607E-03	0.81905517E-02
0.10000000E+00	-.42952181E-05	-.10792446E-09	-.61560103E-01	0.37164134E-08	-.18190877E-04	-.18675455E-03
0.00000000E+00	0.00000000E+00	0.00000000E+00	-.64761051E-01	0.00000000E+00	-.72263382E-01	0.73327005E+00

NONLINEAR RESULTS

$\bar{s}$	$\bar{u}$	$\bar{\beta}$	$\bar{w}$	$\bar{V}$	$\bar{H}$	$\bar{M}_s$
0.10000000E+01	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.91090517E-01	-.30555702E-01	0.23223034E+00
0.90000000E+00	0.77048766E+00	0.19736775E-02	0.24337011E-01	0.91717627E-01	0.21155802E-02	0.18172440E-02
0.80000000E+00	0.61711523E+00	0.12839154E-02	0.42944164E-01	0.91615689E-01	0.17232238E-02	0.29217480E-03
0.70000000E+00	0.17899861E+00	0.23276534E-02	0.60859246E-01	0.91394097E-01	0.15513945E-01	0.54187772E-01
0.60000000E+00	0.27589997E+00	0.11617558E-02	0.27942029E-01	0.64826546E-01	0.36823850E-01	-.82675039E-03
0.50000000E+00	0.15390859E+00	0.53508036E-03	0.11056628E+00	0.37319531E-01	0.33371273E-01	0.56503274E-02
0.40000000E+00	0.98028624E-01	0.96900973E-03	0.17277295E+00	0.13017508E-01	0.15742986E-01	0.61362066E-03
0.30000000E+00	0.31253329E-01	0.29314505E-03	0.24997797E+00	-.25308933E-07	0.60082182E-03	0.20795023E-03
0.20000000E+00	0.30159577E-02	0.16308051E-03	0.27756809E+00	-.29313980E-07	-.15134582E-01	0.22582972E-03
0.10000000E+00	-.25472067E-02	0.17504721E-09	0.29293362E+00	-.19157453E-07	-.18545574E-01	0.30305004E-03
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.29347014E+00	0.00000000E+00	-.23354220E-01	-.20193880E-01

TABLE-4

LINEAR AND NONLINEAR RESULTS OF THE COMPOSITE SHELL NO.3 (Figure-2)

SHELL PARAMETERS : Thickness ratio, R/h = 300; Poisson's ratio, = 0.3

Base Pressure, P_o/E = 0.50000E-05

LINEAR RESULTS

$\bar{s}$	$\bar{u}$	$\bar{\beta}$	$\bar{w}$	$\bar{V}$	$\bar{H}$	$\bar{M}_s$
0.10000000E+01	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.31660741E+00	-.39883291E-01	0.25323244E+00
0.90000000E+00	0.84225923E+00	-.54331499E-03	-.19239535E-01	0.31650660E+00	-.15500650E-02	0.15416811E-02
0.80000000E+00	0.82764835E+00	-.20031432E-02	-.27596428E-01	0.31600584E+00	-.20352175E-02	0.16305868E-02
0.70000000E+00	-.97333188E-01	0.34060352E-02	-.43111834E-01	0.31660511E+00	0.34437080E-01	0.22425949E+00
0.60000000E+00	0.47564182E+00	-.18626219E-02	-.24131152E+00	0.28340917E+00	0.11834577E+00	0.79393874E-03
0.50000000E+00	0.43313397E-01	-.92222705E-02	-.23387726E-01	0.23103205E+00	0.18229643E+00	0.18831959E+00
0.40000000E+00	0.52474929E+00	0.35529910E-03	-.51812393E+00	0.14952311E+00	0.14390532E+00	0.68946316E-03
0.30000000E+00	0.18837841E+00	-.23468757E-02	-.17210569E+00	0.72226035E-01	0.76892120E-01	0.54394720E-01
0.20000000E+00	0.18520147E-01	0.34540569E-02	0.20706392E+00	0.35354194E-07	-.53135272E-03	0.16925363E-01
0.10000000E+00	0.11973823E-03	-.60103717E-05	0.24925014E+00	-.12473309E-10	0.18027175E-03	-.28043430E-03
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.25003268E+00	0.00000000E+00	0.82891014E-04	0.36710528E-06

NONLINEAR RESULTS

$\bar{s}$	$\bar{u}$	$\bar{\beta}$	$\bar{w}$	$\bar{V}$	$\bar{H}$	$\bar{M}_s$
0.10000000E+01	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.74821039E-01	-.39559526E-01	0.23288686E+00
0.90000000E+00	0.77292304E+00	0.27866544E-02	0.28994720E-01	0.75721629E-01	0.21670003E-02	0.17678764E-02
0.80000000E+00	0.61465341E+00	0.19017651E-02	0.55473955E-01	0.75571272E-01	0.18451897E-02	0.36575660E-03
0.70000000E+00	0.28545205E+00	0.30669185E-02	0.72877033E-01	0.75332893E-01	0.10350659E-01	0.31005753E-01
0.60000000E+00	0.26887409E+00	0.16362750E-02	0.85403113E-01	0.54254024E-01	0.25522455E-01	-.45680605E-03
0.50000000E+00	0.13012547E+00	0.76542021E-03	0.15639139E+00	0.31455373E-01	0.26299018E-01	0.76912481E-02
0.40000000E+00	0.90108584E-01	0.12889967E-02	0.21225368E+00	0.97060245E-02	0.12794180E-01	0.44696204E-03
0.30000000E+00	0.23435593E-01	0.39553423E-03	0.23362985E+00	0.95333421E-07	0.30555690E-03	0.17002071E-03
0.20000000E+00	0.40102288E-02	0.25315694E-03	0.32298326E+00	-.22063356E-07	-.11652850E-01	0.17287201E-03
0.10000000E+00	-.15446524E-02	0.13081131E-03	0.34871340E+00	-.18897317E-07	-.18609355E-01	0.17460486E-03
0.00000000E+00	0.00000000E+00	0.00000000E+00	0.35753905E+00	0.00000000E+00	-.15747362E-01	0.13215930E-03

TABLE–5

LINEAR AND NONLINEAR RESULTS OF BUILT IN EDGE HEMISPHERICAL SHELL No. b

SHELL PARAMETERS : Thickness ratio, R/h = 200 Poisson's ratio = 0.3

Base Pressure, P_o/E = 0.25600E-05

LINEAR RESULTS

$\bar{s}$	$\bar{u}$	$\bar{\beta}$	$\bar{w}$	$\bar{V}$	$\bar{H}$	$\bar{M}_s$
0.10000000E+01	0.00000000E+00	0.00000000E+00	0.00000000E+00	-.45274476E+00	-.27760123E-01	0.14481831E+00
0.90000000E+00	0.47100563E+00	-.27496056E-03	[illegible]	0.34611147E+00	0.64305[illegible]E-01	-.10949913E-01
0.80000000E+00	0.43157134E+00	-.12639872E-03	-.951[illegible]69E-01	0.32608132E+00	0.10347705E+00	-.36158849E-05
0.70000000E+00	0.40404562E+00	[illegible]	[illegible]	0.29343582E+00	0.14151295E+00	-.90628357E-03
0.50000000E+00	0.37065465E+00	-.17751017E-03	[illegible]	0.24911940E+00	0.17735758E+00	-.11709898E-02
0.50000000E+00	0.35111929E+00	-.13964866E-03	[illegible]	0.19426508E+00	0.1[illegible]77625E+00	-.17733262E-02
0.40000000E+00	[illegible]	[illegible]	[illegible]	0.13033406E+00	0.17107956E+00	-.29407516E-02
0.30000000E+00	0.22729505E+00	0.14281398E-03	0.2374[illegible]E-01	0.60251931E-01	0.10251747E+00	-.12118682E-01
0.20000000E+00	0.28917192E-01	0.16868106E-02	0.54812[illegible]E+00	0.21558299E-06	0.72145947E-03	0.25419700E-01
0.10000000E+00	-.1327[illegible]E-02	-.12402179E-03	[illegible]	-.56798346E-09	[illegible]E-03	-.11410020E-02
0.00000000E+00	0.00000000E+00	0.00000000E+00	[illegible]	0.00000000E+00	0.1[illegible]21094[illegible]E-02	-.11175974E-03

NONLINEAR RESULTS

$\bar{s}$	$\bar{u}$	$\bar{\beta}$	$\bar{w}$	$\bar{V}$	$\bar{H}$	$\bar{M}_s$
0.10000000E+01	0.00000000E+00	0.00000000E+00	0.00000000E+00	0.12535840E+00	-.39602154E-01	0.18741286E+00
0.90000000E+00	0.61586381E+00	0.32517717E-03	[illegible]	0.11679632E+00	0.23191935E-01	-.15282372E-01
0.80000000E+00	0.44516443E+00	0.6486[illegible]E-03	[illegible]	0.96037065E-01	0.33103922E-01	0.14455020E-02
0.70000000E+00	[illegible]	[illegible]	[illegible]	[illegible]	0.37709369E-01	0.26375037E-03
0.50000000E+00	0.19902775E+00	0.45491009E-03	[illegible]	0.44206254E-01	0.34130420E-01	0.35582628E-03
0.50000000E+00	0.11149540E+00	0.37552742E-03	[illegible]	0.22544805E-01	0.24543677E-01	0.44569887E-03
0.40000000E+00	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.69931398E-03
0.30000000E+00	0.17874220E-01	0.1571[illegible]E-03	[illegible]	0.55671567E-02	0.27999952E-03	0.40501540E-03
0.20000000E+00	0.30513417E-02	0.11156146E-03	[illegible]	[illegible]	[illegible]	0.40925953E-03
0.10000000E+00	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.41189418E-03
0.00000000E+00	0.00000000E+00	0.00000000E+00	[illegible]	0.00000000E+00	-.1929356[illegible]E-01	0.41249905E-03

APPENDIX – A

PROGRAMMING FEATURES

A-1 : GENERAL FEATURES

The Computer program used in the present investigation is adopted from that of Uddin (46) with necessary modifications to suit the requirements of solving stability problems of axisymmetric composite shells under axially varying internal pressure. The program is based on Reissner's nonlinear theory of axisymmetric deformation of shells (36) while the multisegment method developed by Kalnins and Lestingi (24) takes care of the solution of the governing equations and the integration process is carried out by a predictor - corrector method. The predictor and the corrector are respectively given by formulas (19.16) and (19.17) of Ref. (29). To secure the six starting values necessary for the application of this pair of predictor and corrector, the six-point forward difference formulas (19.10 - 19.14) of Ref. (29) are being used. It should be noted that all these formulas contain error of the order of H^7, where H is the distance between two consecutive computational points, thus they are highly sophisticated. The program will produce nonlinear results for increasing steps of louding up to the number of steps as directed. In part A of the program the necessary information required for the solution of problem is read in. Part B of the program deals with the problem of adjusting the given boundary conditons with regard to the solution of the matrix equations. In part C, R, called 'RC' is determined for composite shells. Part D of the program is concerned with the calculation of the normalised constants involving shell parameters, material constant, and loading; under the part E of the program the output of the results is handled. The remaining portion of the program deals with the integration of the different systems of differential equations and the solutio of matrix equaitons. Each segment of the shell is divided into twenty-one computational points.

A-2 : TREATMENT OF BOUNDARY CONDITONS

Equations (3.37) written in terms of the normalised fundamental variables and in accordance with the statement of equation (2.82) appear as

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\bar{u} \\ \bar{\beta} \\ \bar{w} \\ \bar{V} \\ \bar{H} \\ \bar{M}_s
\end{bmatrix}
=
\begin{bmatrix}
\bar{u} \\ \bar{\beta} \\ \bar{w} \\ \bar{V} \\ \bar{H} \\ \bar{M}_s
\end{bmatrix}
\quad \ldots\ldots \text{A-1}
$$

In the matrix equation (A-1) the elements of the column matrix on the left hand side remain in the same order, whereas, those on the right hand side should be arranged in such a manner that the three prescribed elements at the boundary become the first three elements of this column matrix. According to equation (2.82) if $\bar{u}$ is specified at the boundary, the first and fifth rows of the unit matrix of (A-1) remain the same, while specification of $\bar{H}$ at the boundary will require the inter change of these two rows which will interchange $\bar{u}$ and $\bar{H}$ in the column matrix on the right hand side. Similarly, if $\bar{\beta}$ is specified at the boundary, the second and the last rows remain as they are, and interchanged when $\bar{M}_s$ is specified. Lastly, the third and the fourth rows of the unit - matrix are kept the same or interchanged depending on whether $\bar{W}$ or $\bar{V}$ is specified at the boundary. The same operation

is carried out for both the boundary points. The transformed unit matrices of (A-1) are then designated by T_1 at the starting boundary and by T_{M+1} at the finishing boundary.

A-3 : ON THE USE OF THE PROGRAM

In order to use the program for obtaining solutions of different problems, knowledge of the definition of input and output variables is essential. Variables used in the program with their definition are given in the table at the end of Appendix A.

Necessary information to be read in are :

Card No. 33 : This card reads in the amount of loading step EM1 and the number of loading steps SOB1. If at any loading the solution fails to converge, the loading step EM1 is automatically halved by the program and the solution for the new loading is attempted.

Card No. 35 : M, the number of segments of the shell meridian, and IZ, indicator of the type of problem, are read in by this card.

The indicator IZ will have different values depending upon the type of problem to be solved. The appropriate values of JZ in accordance with the types of problems are given below in tabular form.

Type of Problem	Value of IZ
Spherical head puressure vessel	1
Flat end pressure vessel	2
Conical head pressure vessel	3
Ellipsoidal head pressure vessel	4
General case of composite shell	5

Card No. 38 :

This card is used only for the general case of composite shell and will be skipped over in case of pressure vessel problems. It reads in the value of IG(I) which indicate the type of the segment S_i. The quantity IG(I) may have any one of the values given below in tabular form depending upon the type of the Segment S_i

Type of Segment S_i	Value of IG (1)
Line element	1
Circular element	2
Elliptic element	3

Card No. 40 : This card also is used only for the general case of composite shells and skipped otherwise. It reads in the values of APH(I) which indicate the starting value of the merdional angle $(\phi_o)_i$ for the segment S_i.

Card No. 42 : Like cards No.38 and 40 this card is ignored for pressure vessel problems and is used only for composite shells. The value of 'RC', the ratio of the total length of the shell meridian to the radius at the base of the shell, is read in by this card. In case of a shell which is open at the top the length of the meridian should be measured from the center of the open top ; so that the value of $\bar{s}$ at the edge of the open top is different from zero. This is necessary because $\bar{s} = 0$ is associated with the specialised equations valid only at the apex.

Card No. 44 : This card reads in the values of Poisson's ratio 'AN', normalized load 'EMO' at the base ($\bar{s} = 1.0$), meridional angle of the spherical cap 'PHI' at the juncture the semi-angle 'ALP' of the conical head, the ratio 'ER' of the minor to major axes of the ellipsoidal head and the ratio 'XL' of the radius at the juncture of the sperical tipping of conical head to the radius of the cylindrical part. 'EM2' is the same as 'EMO' for operation facilities only. The four quantities of this card, namely 'PHI','ALP', 'ER', and 'XL' are not needed for general case of composite shells, and thus can be assigned arbitrary values.

Card No.46 : This card reads in the thickness ratios Tk (I) for the segments s_i , i = 1,2 ....M

Card No. 50 : This card reads in the values of the independent variables X(J,1) and the initial values of the six fundamental variables X (J, I), I = 2,7) for the nodal points J, (J = 1, M +1), For the general case of composite shells the nodal point (J=1) coincides with the base of the shell where X (1,1) = 1.0

CArd No. 52 : The boundary values of any three of the six fundamental variables at the starting boundary are accepted through this card. These are, for clamped edges

$$X(1,1) = \bar{H} = 0.0$$
$$X(2,1) = \bar{\beta} = 0.0 \quad \ldots \text{A-2}$$
$$X(3,1) = \bar{w} = 0.0$$

Card No. 54 : This card reads in the three prescribed boundary conditions at the final boundary. For the general case of composite shell with no hole at the apex, they are -

$$XY(1,1) = \bar{u} = 0.0$$
$$XY(2,1) = \bar{\beta} = 0.0 \quad \ldots\ldots\ldots \text{A-3}$$
$$XY(3,1) = \bar{V} = 0.0$$

Card No. 56 : The values of the boundary condition indicators at the starting boundary are read in by this card. The appropriate values of the indicators 'IS1', 'IS2', and 'IS3' are given in the following table.

Specified quantity	Indicator and its value
$\bar{u}$	ISI = 0
$\bar{\beta}$	IS2 = 0
$\bar{w}$	IS3 = 0
$\bar{V}$	IS3 = 1
$\bar{H}$	IS1 = 1
$\bar{M}_s$	IS2 = 1

Card No.58 : Here the values of the boundary condition indicators at the final boundary are read in. Their appropriate values are given in the above table where the quantities 'IS1', 'IS2', and 'IS3' should be replaced by 'IF1', 'IF2', and 'IF3', respectively.

A-4 : OUTPUT OF THE PROGRAM

The first output will be the given initial nodal values of the independent variable s and the six fundamental variables $\bar{u}$, $\bar{\beta}$,

$\bar{\omega}$, $\bar{V}$, $\bar{H}$, and $\bar{M}_s$ in their written order columnwise and in tabular form. The second output gives the value of number of of pass and residue - the sum of the differences of the absolute values of the fundamental variables at the nodal points of the two recent consequtive passes.

The first out-put is then repeated for solution based on linear theory. The next output presents the details of the solution based on the linear thery. Here the following quantities are printed out in tabular form and in the order of $\bar{s}$, $\bar{u}$, $\bar{w}$, $\bar{M}_\theta$, $\bar{M}_s$, $\bar{N}_\theta$, $\bar{N}_s$, $\bar{\sigma}_{ci}$, $\bar{\sigma}_{co}$, $\bar{\sigma}_{ai}$, $\bar{\sigma}_{ao}$, $\bar{P}$ columnwise. For each segment these quantities are printed out at twelve equispaced points.

A-5, DEFINITION OF COMPUTER VARIABLES

Variable	Definition
EMO	P_o/E, normalized load at the base
EM	P/E, normalized load at any point on the meridian
EM1	Increasing step of EMO
SOB1	Number of desired loading step
M	Number of segments on the shell meridian
IZ	Indicator of the type of problem(IZ=5, for composite shell)

RC	Constant $\overline{R}$ = s_e/R
AN	Poisson's ratio,
ER	E/lipticity ratio, B/A
Tk(I)	R/h, Thickness ratio for segment s_i
X(I,1)	$\overline{s}$ at the nodal point I
X(I,2)	$\overline{u}$ at the nodalpoint I
X(I,3)	$\overline{\beta}$ at the nodal point I
X(I,4)	$\overline{w}$ at the nodal point I
X(I,5)	$\overline{V}$ at the nodal point I
X(I,6)	$\overline{H}$ at the nodal point I
X(I,7)	$\overline{M_s}$ at the nodal point I
XX(1,1)	$\overline{u}$ or $\overline{H}$ at the starting boundary
XX(2,1)	$\overline{\beta}$ or $\overline{M_s}$ at the starting boundary
XX(3,1)	$\overline{w}$ or $\overline{V}$ at the starting boundary
XY(1,1)	$\overline{u}$ or $\overline{H}$ at the finishing boundary
XY(2,1)	$\overline{\beta}$ or $\overline{M_s}$ at the finishing boundary
XY(3,1)	$\overline{w}$ or $\overline{V}$ at the finishing boundary
IS1,IS2,IS3	Indicators of boundary Conditions at thestarting boundary
IF1,IF2,IF3	Indicators of boundary Conditions at the finishing boundary
NP	Number of pass; NP=1 indicates linear solutions.
T7(N)	$\overline{N_\theta} = N_\theta /(P_oR)$
T2Z(N)	$\overline{N_s} = N_s/(P_oR)$
T9(N)	$\overline{M_\theta} = M_\theta /(P_o.R.h)$

y(1,N)	$\bar{s}$	=	s/s_e
y(2,N)	$\bar{u}$	=	$uEh/(P_o.R^2)$
y(3,N)	$\bar{\beta}$	=	β
y(4,N)	$\bar{w}$	=	$wEh/(P_o.R^2)$
y(5,N)	$\bar{V}$	=	$V/(P_o.R)$
y(6,N)	$\bar{H}$	=	$H/(P_o.R)$
y(7,N)	$\bar{M}_s$	=	$M_s/(P_o.R.h)$
N	Point in a segment at which the variables are evaluated.		

APPENDIX -B

PROGRAM LISTING

```
*      ***************************************************************************
*      ***************************************************************************
*=====>STRESSES AT THE JUNCTIONS OF AXISYMMETRIC  SHELLS  UNDER
*       AXIALLY  VARYING  LOAD.
*
*     ###########################################################################
*
      DIMENSION IG(10)
      REAL*8  X(11,7),Y(7,21),Z(7,6),Y1(7,21),Y2(11,3),Y3(11,3),F(7,21)
      REAL*8  H(32),APH(10),TK(10),X7(11,7),AK(4),T22(21),Z2(3,1),TSL
      REAL*8  AY(3,1),BY(3,1),EM(21),FX(21),HH(21),ZA(21),ZB(21)
      REAL*8  TS1(3,3),TS2(3,3),TS3(3,3),TS4(3,3),TF1(3,3),TF2(3,3),TCL
      REAL*8  TF3(3,3),TF4(3,3),A14(3,1),A15(3,1),A16(3,1),A17(3,1),TSC
      REAL*8  A18(3,3),C(11,3,3),A(11,3),E(11,3,3),B(11,3),X1(3,1),RA
      REAL*8  X2(3,1),C1(21),C2(21),T7(21),T9(21),T10(21),R(21),PH(21)
      REAL*8  R0(21),Z1(3,1),A1(3,3),A2(3,3),A3(3,3),A4(3,3),A6(3,3)
      REAL*8  A7(3,3),A8(3,3),A9(3,1),A10(3,1),A11(3,1),A12(3,1),BH6,EM2
      REAL*8  XX(3,1),XY(3,1),AB(3,3),U(6,6),ZF(21),HL(10),EMO,TH
      REAL*8  PB2,RC,AKL,EL,DR,FL,TO,TL,ZZ,FF,P3,DP,PHI,ALP,T3,T21,TM,PR
      OPEN(UNIT=8,FILE='IN',STATUS='OLD')
      OPEN(UNIT=9,FILE='OUT',STATUS='NEW')
      NP=0
      IN=1
      SOB2=0.0
      SS=1.0
      N2=6
      N3=3
      PB2=1.5707963268
*      ***************************************************************************
*                            PART-A
*                 READING IN INFORMATION
*      ***************************************************************************
      READ(8,110)EM1,SOB1
      WRITE(9,110)EM1,SOB1
25    READ(8,59)M,IZ
      WRITE(9,59)M,IZ
      IF(IZ-5)515,516,516
516   READ(8,59)(IG(I),I=1,M)
      WRITE(9,59)(IG(I),I=1,M)
      READ(8,110)(APH(I),I=1,M)
      WRITE(9,110)(APH(I),I=1,M)
      READ(8,110)RC,RA,BH6
      WRITE(9,110)RC,RA,BH6
515   READ(8,110)AN,EMO,PHI,ALP,ER,XL,EM2
      WRITE(9,110)AN,EMO,PHI,ALP,ER,XL,EM2
      READ(8,1100)(TK(I),I=1,M)
      WRITE(9,1100)(TK(I),I=1,M)
1100  FORMAT(10F5.1)
      MO=M+1
      READ(8,41)((X(J,I),I=1,7),J=1,MO)
      WRITE(9,41)((X(J,I),I=1,7),J=1,MO)
      READ(8,41)(XX(I,1),I=1,3)
      WRITE(9,41)(XX(I,1),I=1,3)
      READ(8,41)(XY(I,1),I=1,3)
      WRITE(9,41)(XY(I,1),I=1,3)
```

```
      READ(8,59)IS1,IS2,IS3
      WRITE(9,59)IS1,IS2,IS3
      READ(8,59)IF1,IF2,IF3
      WRITE(9,59)IF1,IF2,IF3
*     ****************************************************************
*                                 PART-B
*                      BOUNDARY CONDITION TREATMENT
*     ****************************************************************
      DO 21 I=1,N3
      DO 21 J=1,N3
      TS1(I,J)=0.0
      TS2(I,J)=0.0
      TS3(I,J)=0.0
      TS4(I,J)=0.0
      TF4(I,J)=0.0
      TF3(I,J)=0.0
      TF2(I,J)=0.0
21    TF1(I,J)=0.0
      IF(IS1)23,23,24
23    TS1(1,1)=1.0
      TS4(2,2)=1.0
      GO TO 27
24    TS2(1,2)=1.0
      TS3(2,1)=1.0
27    IF(IS2)28,28,29
28    TS1(2,2)=1.0
      TS4(3,3)=1.0
      GO TO 30
29    TS2(2,3)=1.0
      TS3(3,2)=1.0
30    IF(IS3)33,33,34
33    TS1(3,3)=1.0
      TS4(1,1)=1.0
      GO TO 35
34    TS2(3,1)=1.0
      TS3(1,3)=1.0
35    IF(IF1)36,36,37
36    TF2(1,2)=1.0
      TF3(2,1)=1.0
      GO TO 38
37    TF1(1,1)=1.0
      TF4(2,2)=1.0
38    IF(IF2)39,39,40
39    TF2(2,3)=1.0
      TF3(3,2)=1.0
      GO TO 819
40    TF1(2,2)=1.0
      TF4(3,3)=1.0
819   IF(IF3)84,84,87
84    TF2(3,1)=1.0
      TF3(1,3)=1.0
      GO TO 88
87    TF1(3,3)=1.0
      TF4(1,1)=1.0
88    CONTINUE
```

```
      DO 31 J=1,M
      HH(J)=X(J+1,1)-X(J,1)
31    H(J)=(X(J+1,1)-X(J,1))*.05
*     *****************************************************************
*                          PART-C
*                   CALCULATION OF RC
*     *****************************************************************
      GO TO (401,402,403,404,405),IZ
401   RC=PHI/DSIN(PHI)
      GO TO 405
402   RC=1.
      GO TO 405
403   RC=(1.-XL)/DSIN(ALP)+(PB2-ALP)*XL/DCOS(ALP)
      X(M,1)=(PB2-ALP)*XL/DCOS(ALP)/RC
      GO TO 405
404   I=1
      AL=1.
      BL=2.
      AKL=1.-ER**2.
      EL=1.
      CL=1.
406   EL=EL*(AL/BL)**2.
      FL=EL*AKL**I/AL
      CL=CL-FL
      AL=AL+2.
      BL=BL+2.
      I=I+1
      IF(DABS(FL)-.1E-08)407,407,406
407   RC=PB2*CL
405   CONTINUE
      IF(IZ-5)521,522,522
521   DP=PB2
      GO TO 523
522   DP=APH(I)
523   DR=1./RC
*     DP=PB2
26    DO 1 J1=1,M
*     *****************************************************************
*                            PART-D
*                   CALCULATION OF CONSTANTS
*     *****************************************************************
      TZ=1.+AN
      T1=RC*(1.-AN*AN)
      T=TK(J1)
      TO=1.0/(12.0*T1*EM2*T*T)
      TL=RC/T/EM2
      T21=EM2*T
*     TH=1.0/PB2*DCOS(APH(9))
      TSL=HH(J1)*DSIN(APH(1))*3.0
      TCL=RA*(DCOS(BH6)-DCOS(APH(4)))
      TSC=HH(J1)*DSIN(APH(7))*2.0
      TP=RA*(DCOS(APH(9))-DCOS(APH(8)))
*     TQ=RB*(1.0-DCOS(APH(9)))
      TH=-TSL+TCL-TSC+TP
*     WRITE(6,*)TSL,TCL,TP,TH
```

```
      N=1
      DO 32 I=1,7
32    Y(I,N)=X(J1,I)
      DO 300 I=1,21
      IF(I-21)312,313,313
312   Y(1,I+1)=Y(1,I)+H(J1)
313   IF(Y(1,I)-1.)306,308,305
308   IF(IZ-5)306,305,305
305   PH(I)=PB2
      RO(I)=1./RC
      HN=FLOAT(I-1)
      ZA(I)=HN*H(J1)*DSIN(PH(I))
*     ZA(I)=HN*RO(I)*DCOS(PH(I))
      ZB(I)=ZA(I)/TH
      FX(I)=1.0-ZB(I)
      EM(I)=EMO*FX(I)
      TM=EM(I)*T*T
      PR=EM(I)*T
      WRITE(6,*)EM(I),TH,SOB2
      GO TO 300
306   GO TO (301,302,303,304,509),IZ
301   PH(I)=Y(1,I)*PHI
      RO(I)=DSIN(PH(I))/PHI
      GO TO 300
302   PH(I)=0.
      RO(I)=Y(1,I)
      GO TO 300
303   IF(Y(1,I)-X(M,1))307,309,309
309   PH(I)=PB2-ALP
      RO(I)=XL/RC+(Y(1,I)-X(M,1))*DSIN(ALP)
      GO TO 300
307   PH(I)=Y(1,I)*RC/XL*DCOS(ALP)
      RO(I)=XL*DSIN(PH(I))/RC/DCOS(ALP)
      GO TO 300
304   PH(I)=DP
      RO(I)=DR
      ZZ=PH(I)
      DO 310 J=1,4
      FF=RC/ER**2.*(ER**2.+AKL*DSIN(ZZ)*DSIN(ZZ))**1.5
      AK(J)=H(J1)*FF
      GO TO (311,311,314,310),J
311   V=.5
      GO TO 316
314   V=1.0
316   ZZ=PH(I)+V*AK(J)
310   CONTINUE
      DP=PH(I)+(AK(1)+AK(4)+2.*(AK(3)+AK(2)))/6.
      DR=DSIN(DP)/RC/(ER**2.+AKL*DSIN(DP)*DSIN(DP))**.5
      GO TO 300
509   IJK=IG(J1)
      GO TO (510,511,304),IJK
510   PH(I)=APH(J1)
      RO(I)=DR
      DR=RO(I)+H(J1)*DCOS(APH(J1))
      HN=FLOAT(I-1)
```

```
      ZA(I)=HN*H(J1)*DSIN(PH(I))
      IF(J1-6)959,858,858
959   RML=FLOAT(J1-1)
      HL(J1)=TH+(RML*HH(J1)*DSIN(APH(J1)))
      GO TO 757
858   HMM=FLOAT(J1-6)
      HL(J1)=TH+(TSL-TCL+(HMM*HH(J1)*DSIN(APH(7))))
757   CONTINUE
      ZB(I)=ZA(I)/HL(J1)
      FX(I)=1.0+ZB(I)
      EM(I)=EMO*FX(I)
      TM=EM(I)*T*T
      PR=EM(I)*T
      WRITE(5,*)EM(I),HL(J1),SOB2
      GO TO 300
511   RM=FLOAT(I-1)
      PH(I)=APH(J1)+RM*H(J1)*DSIN(APH(J1))/DR
      RO(I)=DR*DSIN(PH(I))/DSIN(APH(J1))
C     ZE(I)=RO(I)/DTAN(PH(I))
C     ZO(I)=DR/DTAN(DP)
C     ZF(I)=ZE(I)-ZO(I)
      ZF(I)=(DCOS(PH(I))-DCOS(DP))*RA
*     GO TO (252,252,353,353,877,877,9876,9876,444,444),J1
      IF(J1-8)252,353,444
252   HNN=FLOAT(J1-4)
      HL(J1)=TH+TSL-HNN*RA*(DCOS(APH(5))-DCOS(APH(4)))
      GO TO 333
353   HMN=FLOAT(J1-8)
*     HL(J1)=TH+(TSL-TCL+TSC)-HMN*TP
      HL(J1)=TP
      GO TO 333
*77   HNM=FLOAT(J1-5)
*     HL(J1)=TH-(DCOS(APH(5))/PB2)-HNM*((DCOS(APH(6))-DCOS(APH(5)))/PB2)
*     GO TO 333
*876  HMM=FLOAT(J1-7)
*     HL(J1)=TH-(DCOS(APH(7))/PB2)-HMM*((DCOS(APH(8))-DCOS(APH(7)))/PB2)
333   CONTINUE
      FX(I)=1.0-ZF(I)/HL(J1)
      EM(I)=EMO*FX(I)
      TM=EM(I)*T*T
      PR=EM(I)*T
      WRITE(5,*)EM(I),HL(J1),SOB2
      GO TO 300
444   HMNI=FLOAT(J1-9)
      HL(J1)=ZF(I)
*     FX(I)=1.0-ZF(I)/HL(J1)
      FX(I)=0.0
      EM(I)=EMO*FX(I)
      TM=EM(I)*T*T
      PR=EM(I)*T
      WRITE(6,*)EM(I),HL(J1),SOB2
*     GO TO 300
*77   HL(I)=TH+TSL-TCL+TSC-TP
*     FX(I)=1.0-ZF(I)/ZF(I)
*     EM(I)=EMO*FX(I)
```

```
*       TM=EM(I)*T*T
*       PR=EM(I)*T
300     CONTINUE
        DR=RD(21)
        IF(IZ-5)512,513,513
512     DP=PH(21)
        GO TO 514
513     DP=APH(J1+1)
        EMD=EM(21)
514     NI=1
*       ***********************************************************************
*                             PART-E
*               INTEGRATION OF FUNDAMENTAL SET
*       ***********************************************************************
60      NO=0
46      CONTINUE
        IF(NP-1)111,111,112
112     IF(Y(1,N)-.1E-06)198,198,199
198     F(2,N)=T1*Y(5,N)/TZ
        F(3,N)=Y(7,N)/TD/TZ
        TQ=TL+F(2,N)
        F(5,N)=TQ*PR/2.
        F(4,N)=0.0
        F(6,N)=0.0
        F(7,N)=0.0
        GO TO 200
199     T2=Y(2,N)/RD(N)
        T3=PH(N)-Y(3,N)
        C1(N)=DCOS(T3)
        C2(N)=DSIN(T3)
        T4=(DSIN(PH(N))-DSIN(T3))/RD(N)
        T5=Y(6,N)*C1(N)+Y(5,N)*C2(N)
        T22(N)=T5
        T8=T1*T5-AN*T2
        T6=(Y(7,N)-AN*TD*T4)/TD
        T7(N)=(T2+AN*T8)/T1
        T9(N)=TD*(T4+AN*T6)
        T10(N)=TL+T8
        R(N)=TL*RD(N)+Y(2,N)
        F(2,N)=T10(N)*C1(N)-DCOS(PH(N))*TL
        F(3,N)=T5
        F(4,N)=T10(N)*C2(N)-DSIN(PH(N))*TL
        F(5,N)=-T10(N)*(Y(5,N)*C1(N)/R(N)-PR*C1(N))
        F(6,N)=-T10(N)*((Y(6,N)*C1(N)-T7(N))/R(N)+PR*C2(N))
        F(7,N)=(T10(N)*C1(N)/R(N))*(T9(N)-Y(7,N))-T10(N)*(Y(6,N)*C2(N)-Y(5
     C,N)*C1(N))*TM
        GO TO 200
111     C1(N)=DCOS(PH(N))
        C2(N)=DSIN(PH(N))
        IF(Y(1,N)-.1E-06)598,598,599
598     F(2,N)=T1*Y(5,N)/TZ
        F(3,N)=Y(7,N)/TD/TZ
        F(4,N)=0.0
        F(5,N)=FX(N)*RC/2.
        F(6,N)=0.0
```

```
      F(7,N)=0.0
      GO TO 200
599   T2=Y(2,N)/RO(N)
      T4=Y(3,N)*C1(N)/RO(N)
      T5=Y(6,N)*C1(N)+Y(5,N)*C2(N)
      T22(N)=T5
      T8=T1*T5-AN*T2
      T6=Y(7,N)/T0-AN*T4
      T7(N)=(T2+AN*T8)/T1
      T9(N)=(T4+AN*T6)*T0
      F(2,N)=T8*C1(N)+Y(3,N)*C2(N)*TL
      F(3,N)=T6
      F(4,N)=T8*C2(N)-Y(3,N)*C1(N)*TL
      F(5,N)=-(Y(5,N)/RO(N)-FX(N)*RC)*C1(N)
      F(6,N)=-(Y(6,N)*C1(N)-T7(N))/RO(N)-FX(N)*RC*C2(N)
      TX=-(Y(7,N)-T9(N))/RO(N)
      F(7,N)=TX*C1(N)-RC*T*(Y(6,N)*C2(N)-Y(5,N)*C1(N))
200   IF(N-2)42,43,43
43    IF(N-6)44,47,45
44    N=N+1
      GO TO 46
42    DO 81 J=2,6
      P2=FLOAT(J-1)
      P3=P2*H(J1)
      Y(1,J)=Y(1,1)+P3
      DO 81 I=2,7
81    Y(I,J)=Y(I,1)+P3*F(I,1)
      N=2
      IP=1
      GO TO 46
47    DO 48 I=2,7
      Z(I,2)=Y(I,1)+(H(J1)/1440.)*(493.*F(I,1)+1337.*F(I,2)-618.*F(I,3)+
     +302.*F(I,4)-83.*F(I,5)+9.*F(I,6))
      Z(I,3)=Y(I,1)+(H(J1)/90.)*(28.*F(I,1)+129.*F(I,2)+14.*F(I,3)+14.*
     +F(I,4)-6.*F(I,5)+F(I,6))
      Z(I,4)=Y(I,1)+(3.*H(J1)/160.)*(17.*F(I,1)+73.*F(I,2)+38.*(F(I,3)+
     +F(I,4))-7.*F(I,5)+F(I,6))
      Z(I,5)=Y(I,1)+(4.*H(J1)/90.)*(7.*(F(I,1)+F(I,5))+32.*(F(I,2)+F(I,4
     +))+12.*F(I,3))
48    Z(I,6)=Y(I,1)+(5.*H(J1)/288.)*(19.*(F(I,1)+F(I,6))+75.*(F(I,2)+
     +F(I,5))+50.*(F(I,4)+F(I,3)))
      R1=0.0
      IP=IP+1
      DO 49 I=2,7
      DO 49 J=2,6
      R1=DABS(Y(I,J)-Z(I,J))+R1
49    Y(I,J)=Z(I,J)
      IF(IP-15) 141,45,45
141   IF(R1-.1E-07)45,45,50
50    N=2
      GO TO 46
45    IF(N0-1)53,53,55
53    N=N+1
      IF(N-21) 61,61,62
61    Y(1,N)=Y(1,N-1)+H(J1)
```

```
      DO 51 I=2,7
51    Y(I,N)=Y(I,N-5)+(.3*H(J1))*(11.*(F(I,N-5)+F(I,N-1))-14.*(F(I,N-4)+
     +F(I,N-2))+26.*F(I,N-3))
99    ND=2
      IP=1
      GO TO 46
55    R1=0.0
      IP=IP+1
      DO 56 I=2,7
      Z(I,1)=Y(I,N-5)+(.3*H(J1))*(F(I,N-6)+5.*F(I,N-5)+F(I,N-4)+6.*F(I,N
     *-3)+F(I,N-2)+5.*F(I,N-1)+F(I,N))
      R1=R1+DABS(Y(I,N)-Z(I,1))
56    Y(I,N)=Z(I,1)
      IF(IP-10) 142,60,60
142   IF(R1-.1E-07) 60,46,46
60    IF(NP-1) 662,762,912
912   IF(AA-0.1) 911,911,914
914   IF(NP-10) 662,911,911
911   IN=2
      GO TO 764
762   RRR=0.0
      DO 763 I=2,7
763   RRR=RRR+DABS(Y(I,21)-X(J1+1,I))
      IF(RRR-.1) 764,764,766
766   WRITE(9,767)
767   FORMAT(2X,'SEGMENT IS TOO LONG')
764   CONTINUE
*     ******************************************************************
*                                PART-F
*                         OUTPUT OF RESULTS
*     ******************************************************************
      WRITE(9,508)
      WRITE(9,507)
      DO 793 N=1,21,4
      ST1=(T7(N)+T9(N)*6.)*T21
      ST2=(T7(N)-T9(N)*6.)*T21
      ST3=(T22(N)+Y(7,N)*6.)*T21
      ST4=(T22(N)-Y(7,N)*6.)*T21
793   WRITE(9,105)Y(1,N),Y(2,N),Y(4,N),T9(N),Y(7,N),T22(N),T7(N),ST1,ST2
     &,ST3,ST4,EM(N)
      GO TO 1
*     ******************************************************************
*                  INTEGRATION OF DERIVED SET STARTS
*     ******************************************************************
662   N1=N1+1
      N=1
      Y1(1,N)=X(J1,1)
      DO 63 I=2,7
63    Y1(I,N)=0.0
      Y1(N1,N)=1.0
90    ND=0
76    CONTINUE
      IF(NP-1) 113,113,114
114   IF(Y1(1,N)-.1E-06) 201,201,202
201   F(2,N)=T1*Y1(5,N)/TZ
```

```
      F(3,N)=Y1(7,N)/TO/TZ
      F(5,N)=F(2,N)*PR/2.
      F(4,N)=0.0
      F(6,N)=0.0
      F(7,N)=0.0
      GO TO 203
202   T2=Y1(2,N)/RO(N)
      T3=Y1(3,N)*C1(N)/RO(N)
      T4=Y1(5,N)*C1(N)+Y1(6,N)*C2(N)-Y1(3,N)*(Y(5,N)*C1(N)-Y(6,N)*C2(N))
      T5=T1*T4-AN*T2
      T6=Y1(7,N)/TO-AN*T3
      Q1=(T2+AN*T5)/T1
      T8=TO*(T3+AN*T6)
      F(2,N)=T5*C1(N)+T10(N)*Y1(3,N)*C2(N)
      F(4,N)=T5*C2(N)-T10(N)*Y1(3,N)*C1(N)
      F(3,N)=T6
      TA=(Y(6,N)*C1(N)-T7(N))/R(N)
      F(6,N)=-T5*(TA+PR*C2(N))-T10(N)*((Y1(6,N)*C1(N)+Y1(3,N)*Y(6,N)*
     #C2(N)-Q1-TA*Y1(2,N))/R(N)-PR*Y1(3,N)*C1(N))
      F(5,N)=-F(2,N)*(Y(5,N)/R(N)-PR)-T10(N)*C1(N)*(Y1(5,N)-Y(5,N)*
     #Y1(2,N)/R(N))/R(N)
      TX=(T9(N)-Y(7,N))/R(N)
      F(7,N)=F(2,N)*(TX+TM*Y(5,N))+T10(N)*(C1(N)*(TM*Y1(5,N)+(-Y1(7,N)+
     #T8-TX*Y1(2,N))/R(N))-TM*C2(N)*Y1(6,N))-TM*F(4,N)*Y(6,N)
      GO TO 203
113   IF(Y1(1,N)-.1E-06) 501,501,502
501   F(2,N)=T1*Y1(6,N)/TZ
      F(3,N)=Y1(7,N)/TO/TZ
      F(4,N)=0.0
      F(5,N)=0.0
      F(6,N)=0.0
      F(7,N)=0.0
      GO TO 203
502   T2=Y1(2,N)/RO(N)
      T4=Y1(3,N)*C1(N)/RO(N)
      T5=Y1(5,N)*C1(N)+Y1(6,N)*C2(N)
      T8=T1*T5-AN*T2
      T6=Y1(7,N)/TO-AN*T4
      T7(N)=(T2+AN*T8)/T1
      T9(N)=(T4+AN*T6)*TO
      F(2,N)=T8*C1(N)+Y1(3,N)*C2(N)*TL
      F(3,N)=T6
      F(4,N)=T8*C2(N)-Y1(3,N)*C1(N)*TL
      F(5,N)=-Y1(5,N)/RO(N)*C1(N)
      F(6,N)=-(Y1(6,N)*C1(N)-T7(N))/RO(N)
      TX=-(Y1(7,N)-T9(N))/RO(N)
      F(7,N)=TX*C1(N)-RC*T*(Y1(6,N)*C2(N)-Y1(5,N)*C1(N))
203   IF(N-2) 72,73,73
73    IF(N-6) 74,77,75
74    N=N+1
      GO TO 75
72    DO 82 J=2,6
      P2=FLOAT(J-1)
      P3=P2*H(J1)
      Y1(1,J)=Y1(1,1)+P3
```

```
      DO 82 I=2,7
82    Y1(I,J)=Y1(I,1)+P3*F(I,1)
      N=2
      IP=1
      GO TO 76
77    DO 78 I=2,7
      Z(I,2)=Y1(I,1)+(H(J1)/1440.)*(493.*F(I,1)+1337.*F(I,2)-618.*F(I,3)
     ++302.*F(I,4)-83.*F(I,5)+9.*F(I,6))
      Z(I,3)=Y1(I,1)+(H(J1)/90.)*(28.*F(I,1)+129.*F(I,2)+14.*F(I,3)
     ++14.*F(I,4)-6.*F(I,5)+F(I,6))
      Z(I,4)=Y1(I,1)+(3.*H(J1)/160.)*(17.*F(I,1)+73.*F(I,2)+38.*(F(I,3)
     ++F(I,4))-7.*F(I,5)+F(I,6))
      Z(I,5)=Y1(I,1)+(4.*H(J1)/90.)*(7.*(F(I,1)+F(I,5))+32.*(F(I,2)
     ++F(I,4))+12.*F(I,3))
78    Z(I,6)=Y1(I,1)+(5.*H(J1)/288.)*(19.*(F(I,1)+F(I,6))+75.*(F(I,2)
     ++F(I,5))+50.*(F(I,4)+F(I,3)))
      R1=0.0
      IP=IP+1
      DO 79 I=2,7
      DO 79 J=2,6
      R1=DABS(Y1(I,J)-Z(I,J))+R1
79    Y1(I,J)=Z(I,J)
      IF(IP-15) 143,75,75
143   IF(R1-.1E-06) 75,75,80
80    N=2
      GO TO 76
75    IF(NO-1)83,83,85
83    N=N+1
      IF(N-21) 91,91,92
91    Y1(1,N)=Y1(1,N-1)+H(J1)
      DO 95 I=2,7
95    Y1(I,N)=Y1(I,N-6)+(.3*H(J1))*(11.*(F(I,N-5)+F(I,N-1))-14.*(F(I,N-4
     +)+F(I,N-2))+26.*F(I,N-3))
101   NO=2
      IP=1
      GO TO 76
85    R1=0.0
      IP=IP+1
      DO 86 I=2,7
      Z(I,1)=Y1(I,N-6)+(.3*H(J1))*(5.*F(I,N-5)+F(I,N-6)+F(I,N-4)+6.*
     #F(I,N-3)+F(I,N-2)+5.*F(I,N-1)+F(I,N))
      R1=R1+DABS(Y1(I,N)-Z(I,1))
86    Y1(I,N)=Z(I,1)
      IF(IP-10) 144,90,90
144   IF(R1-.1E-07) 90,76,76
92    DO 22 J=1,N2
22    U(N1-1,J)=Y1(J+1,21)
      IF(N1-7) 662,96,96
104   FORMAT(7E14.8)
59    FORMAT(10I2)
508   FORMAT(//,8X,'DISTANCE',5X,'DISPLACEMENTS',9X,'MOMENTS',8X,'STRESS
     # RESULTANTS',5X,'CIRCUM. STRESS',7X,'AXIAL STRESS',5X,'INTERNAL')
507   FORMAT(3X,'FROM APEX',2X,'RADIAL',5X,'AXIAL',3X,'CIRCUM.',5X,'AXIA
     #L',3X,'CIRCUM',5X,'AXIAL',6X,'INNER',5X,'OUTER',5X,'INNER',5X,'OUT
     #ER',5X,'PRESSURE')
```

```
41     FORMAT(7E11.5)
411    FORMAT(7E14.8)
211    FORMAT(7F7.5)
110    FORMAT(7E11.5)
105    FORMAT(12E11.5)
505    FORMAT(//,2X,'NO. OF PASS=',I3,2X,'RESIDUE=',E14.8)
*      ******************************************************************
*                             PART-G
*          SOLUTION OF MATRIX EQUATION STARTS
*      ******************************************************************
96     NI=J1
       DO 4 I=1,N3
       DO 4 J=1,N3
       A1(J,I)=U(I,J)
       A2(J,I)=U(I+3,J)
       A3(J,I)=U(I,J+3)
       A4(J,I)=U(I+3,J+3)
       X1(I,1)=X(NI,I+1)
       X2(I,1)=X(NI,I+4)
       Y3(NI+1,I)=Y(I+1,21)
4      Y2(NI+1,I)=Y(I+4,21)
       DO 20 I=1,N3
       AY(I,1)=Y3(NI+1,I)
20     BY(I,1)=Y2(NI+1,I)
       CALL MATM(A1,X1,A9,N3,N3,1)
       CALL MATM(A2,X2,Z1,N3,N3,1)
       CALL MATS(A9,Z1,N3,1)
       CALL MATS3(Z1,N3,1)
       CALL MATS(AY,Z1,N3,1)
       CALL MATM(A3,X1,A9,N3,N3,1)
       CALL MATM(A4,X2,Z2,N3,N3,1)
       CALL MATS(A9,Z2,N3,1)
       CALL MATS3(Z2,N3,1)
       CALL MATS(BY,Z2,N3,1)
       IF(NI-1) 6,6,7
6      CALL MATM(A1,TS1,A6,N3,N3,N3)
       CALL MATM(A1,TS2,A7,N3,N3,N3)
       CALL MATM(A2,TS3,A1,N3,N3,N3)
       CALL MATS(A6,A1,N3,N3)
       CALL MATM(A2,TS4,A6,N3,N3,N3)
       CALL MATS(A6,A7,N3,N3)
       CALL MATM(A3,TS1,A6,N3,N3,N3)
       CALL MATM(A3,TS2,A8,N3,N3,N3)
       CALL MATM(A4,TS3,A3,N3,N3,N3)
       CALL MATS(A6,A3,N3,N3)
       CALL MATM(A4,TS4,A6,N3,N3,N3)
       CALL MATS(A6,A8,N3,N3)
       DO 2 I=1,N3
       DO 2 J=1,N3
       A4(I,J)=A8(I,J)
2      A2(I,J)=A7(I,J)
       CALL MATI(A2,A6,N3)
       CALL MATM(A4,A6,A7,N3,N3,N3)
       CALL MATI(A7,A8,N3)
       CALL MATM(A1,XX,A9,N3,N3,1)
```

```
      CALL MATS(Z1,A9,N3,1)
      CALL MATS3(A9,N3,1)
      CALL MATM(A3,XX,A10,N3,N3,1)
      CALL MATS(Z2,A10,N3,1)
      CALL MATM(A4,A6,A7,N3,N3,N3)
      CALL MATM(A7,A9,A11,N3,N3,1)
      CALL MATS(A11,A10,N3,1)
      CALL MATS3(A10,N3,1)
      GO TO 8
7     IF(N1-M) 3,5,5
5     CALL MATM(TF1,A1,A6,N3,N3,N3)
      CALL MATM(TF3,A1,A7,N3,N3,N3)
      CALL MATM(TF2,A3,A1,N3,N3,N3)
      CALL MATS(A6,A1,N3,N3)
      CALL MATM(TF4,A3,A6,N3,N3,N3)
      CALL MATS(A6,A7,N3,N3)
      CALL MATM(TF1,A2,A6,N3,N3,N3)
      CALL MATM(TF3,A2,A18,N3,N3,N3)
      CALL MATM(TF2,A4,A2,N3,N3,N3)
      CALL MATS(A6,A2,N3,N3)
      CALL MATM(TF4,A4,A6,N3,N3,N3)
      CALL MATS(A6,A18,N3,N3)
      CALL MATM(TF1,Z1,A14,N3,N3,1)
      CALL MATM(TF3,Z1,A15,N3,N3,1)
      CALL MATM(TF2,Z2,Z1,N3,N3,1)
      CALL MATS(A14,Z1,N3,1)
      CALL MATM(TF4,Z2,A14,N3,N3,1)
      CALL MATS(A14,A15,N3,1)
      DO 19 I=1,N3
      Z2(I,1)=A15(I,1)
      DO 19 J=1,N3
      A3(I,J)=A7(I,J)
19    A4(I,J)=A18(I,J)
3     CALL MATM(A1,A8,A7,N3,N3,N3)
      CALL MATS(A2,A7,N3,N3)
      CALL MATI(A7,A6,N3)
      CALL MATM(A1,A8,A7,N3,N3,N3)
      CALL MATM(A7,A10,A9,N3,N3,1)
      CALL MATS(Z1,A9,N3,1)
      CALL MATS3(A9,N3,1)
      CALL MATM(A3,A8,A7,N3,N3,N3)
      CALL MATM(A7,A10,A11,N3,N3,1)
      CALL MATS(A4,A7,N3,N3)
      CALL MATM(A6,A9,A12,N3,N3,1)
      CALL MATM(A7,A12,A10,N3,N3,1)
      CALL MATS(A11,A10,N3,1)
      CALL MATS(Z2,A10,N3,1)
      CALL MATS3(A10,N3,1)
      CALL MATM(A3,A8,A7,N3,N3,N3)
      CALL MATS(A4,A7,N3,N3)
      CALL MATM(A7,A6,A1,N3,N3,N3)
      CALL MATI(A1,A8,N3)
      IF(N1-M)8,9,9
9     CALL MATS(XY,A10,N3,1)
8     DO 1 I=1,N3
```

```
      DO 1 J=1,N3
      E(N1,I,J)=A6(I,J)
      C(N1,I,J)=A8(I,J)
      A(N1,I)=A9(I,1)
      B(N1,I)=A10(I,1)
1     CONTINUE
      EM0=EM2
      IF(NP-1) 117,115,117
117   GO TO (718,108),IN
718   AA=0.0
      DO 15 I1=1,M
      N1=M-I1+1
      DO 10 I=1,N3
      DO 10 J=1,N3
      A6(I,J)=E(N1,I,J)
      A8(I,J)=C(N1,I,J)
      A9(I,1)=A(N1,I)
10    A10(I,1)=B(N1,I)
      IF(N1-M) 11,12,12
12    CALL MATM(A8,A10,A11,N3,N3,1)
      CALL MATS(A11,A9,N3,1)
      CALL MATM(A6,A9,A12,N3,N3,1)
      CALL MATM(TF1,A11,A14,N3,N3,1)
      CALL MATM(TF2,XY,A15,N3,N3,1)
      CALL MATM(TF3,A11,A16,N3,N3,1)
      CALL MATM(TF4,XY,A17,N3,N3,1)
      DO 89 I=1,N3
      X(M0,I+1)=A15(I,1)+A14(I,1)
89    X(M0,I+4)=A17(I,1)+A16(I,1)
      GO TO 15
11    CALL MATS(A12,A10,N3,1)
      CALL MATM(A8,A10,A11,N3,N3,1)
      CALL MATS(A11,A9,N3,1)
      CALL MATM(A6,A9,A12,N3,N3,1)
      DO 17 I=1,N3
17    X(N1+1,I+1)=A11(I,1)
      IF(N1-1) 93,93,16
93    CALL MATM(TS1,XX,A14,N3,N3,1)
      CALL MATM(TS2,A12,A15,N3,N3,1)
      CALL MATM(TS3,XX,A16,N3,N3,1)
      CALL MATM(TS4,A12,A17,N3,N3,1)
      DO 98 I=1,N3
      X(1,I+1)=A15(I,1)+A14(I,1)
98    X(1,I+4)=A17(I,1)+A16(I,1)
      GO TO 18
16    DO 13 I=1,N3
13    X(N1,I+4)=A12(I,1)
18    DO 15 I=1,N3
      AA=DABS(Y3(N1+1,I)-X(N1+1,I+1))+AA
15    AA=DABS(Y2(N1+1,I)-X(N1+1,I+4))+AA
115   NP=NP+1
      RES=AA/SS
      SS=AA
      WRITE(9,505)NP,AA
      IF(NP-5) 151,152,152
```

```
152   IF(RES-1.0) 151,151,153
153   DO 154 I=2,7
      DO 154 J=1,MO
154   X(J,I)=X7(J,I)
      EMO=EMO-EM1
      EM1=EM1/2.
      EMO=EMO+EM1
      NP=3
151   WRITE(9,104)((X(J,I),I=1,7),J=1,MO)
      GO TO 405
108   DO 155 I=2,7
      DO 155 J=1,MO
155   X7(J,I)=X(J,I)
      IN=1
      NP=3
      AA=1.0
      SOB2=SOB2+1.0
      EMO=EMO+EM1
      IF(ABS(EM1)-.1E-08) 109,109,1011
1011  IF(SOB2-SOB1) 405,405,109
109   STOP
      END
*     **********************************************************************
*                                   SUBROUTINES
*     **********************************************************************
      SUBROUTINE MATI(A5,B5,K1)
      REAL*8 A5(3,3),B5(3,3)
      P=0.0
      DO 9 L=1,3
      DO 9 K=1,3
      GO TO (2,3,4),L
2     I1=L+1
      I2=L+2
      GO TO 5
3     I1=L+1
      I2=1
      GO TO 5
4     I1=1
      I2=2
5     GO TO (6,7,8),K
6     J1=K+1
      J2=K+2
      GO TO 9
7     J1=K+1
      J2=1
      GO TO 9
8     J1=1
      J2=2
9     B5(K,L)=A5(I1,J1)*A5(I2,J2)-A5(I2,J1)*A5(I1,J2)
      DO 11 L=1,3
11    P=P+A5(1,L)*B5(L,1)
      DO 12 L=1,3
      DO 12 K=1,3
12    B5(L,K)=B5(L,K)/P
      RETURN
```

```
      END
*
*
      SUBROUTINE MATS(A5,B5,L,K)
      REAL*8 A5(3,3),B5(3,3)
      DO 99 L1=1,L
      DO 99 K1=1,K
99    B5(L1,K1)=A5(L1,K1)+B5(L1,K1)
      RETURN
      END
*
*
      SUBROUTINE MATSB(A5,L,K)
      REAL*8 A5(3,3)
      DO 98 L1=1,L
      DO 98 K1=1,K
98    A5(L1,K1)=-A5(L1,K1)
      RETURN
      END
*
*
      SUBROUTINE MATM(A5,B5,C5,L,K,K2)
      REAL*8 A5(3,3),B5(3,3),C5(3,3)
      DO 97 L1=1,L
      DO 97 K1=1,K2
      C5(L1,K1)=0.0
      DO 97 J1=1,K
97    C5(L1,K1)=C5(L1,K1)+A5(L1,J1)*B5(J1,K1)
      RETURN
      END
```