AN ANALYTICAL INVESTIGATION INTO THE STIFFNESS PROPERTIES OF REINFORCED AND PRESTRESSED CONCRETE SECTIONS

A Thesis

By

M. A. RASHID

Submitted to the Department of Civil Engineering, Bangladesh University of Engineering and Technology, Dhaka, in partial fulfillment of the requirements for the degree

of

MASTER OF SCIENCE IN CIVIL ENGINEERING

January, 1988

AN ANALYTICAL INVESTIGATION INTO THE STIFFNESS PROPERTIES OF REINFORCED AND PRESTRESSED CONCRETE SECTIONS

A Thesis
By
M. A. RASHID

Approved as to style and content by

Dr. M. A. Rouf
Associate Professor,
Dept. of Civil Engineering, BUET, Dhaka. — Chairman

Dr. Md. Alee Murtuza
Professor and Head,
Dept. of Civil Engineering, BUET, Dhaka. — Member

Dr. J. R. Chowdhury
Professor,
Dept. of Civil Engineering, BUET, Dhaka. — Member

Dr. M.Shamim-uz-Zaman Bosunia
Professor,
Dept. of Civil Engineering, BUET, Dhaka. — Member

Mr. Matinur Rashid
Director,
B.I.T., Chittagong. — Member (External)

January,1988

ACKNOWLEDGEMENT

The author is indebted to Dr. M.A. Rouf for his careful guidance, supervision and encouragement throughout the work of the thesis. His personal interest, advice and constructive criticism have been invaluable and made this thesis work possible.

The author wishes to thank Professor Md. Alee Murtuza, Head, Department of Civil Engineering for his kind co-operation.

Profound gratitude is expressed to Professor Alamgir Habib and Professor M.Shamim-uz-Zaman for their encouragement throughout the course. The author is very much indebted to Professor A.F.M. Anwarul Haque, Director of Student's Welfare, BUET, for the encouragement of this study.

The author is grateful to Mr. Kazi Ataul Haque, Director, Housing and Building Research Institute, for giving permission to carryout this course and for his encouragement and co-operation. The author expresses his gratitude and thanks to his colleague, specially Mr. Shafiul Alam Khan, for their co-operation.

The author thanks Mr. M. A. Malek for his efficient typing and Mr. Sohrab for drafting figures of the thesis work.

ABSTRACT

The work was done to develop stiffness properties applicable to reinforced and prestressed concrete members loaded under bending and/or axial force with due consideration to material properties. This was realised as the first step in developing a numerical model for the limit state analysis of reinforced and prestressed concrete structures.

The limit state theory is becoming popular for design purposes. The method necessitates the use of appropriate material properties. It is well established experimentally that the stress-strain characteristics of concrete in compression is parabolic and that of steel can be idealised to different elasto-plastic form. These have been used throughout the work.

The usual section of concrete structures can be considered to consist of four basic elements, such as top flange, web, bottom flange and reinforcing and/or prestressing steel. It was found convenient to develop the stiffness properties separately for these sectional elements. Combination of stiffnesses for different sectional elements present in a section gives the complete stiffness properties of the whole section. The properties of a section was then utilised to develop stiffness properties of a beam-column element.

The stiffness properties developed correspond to a particular strain level at the section. The stiffness terms were derived by applying first principle. An alternative approach is also presented. Most of the stiffness terms are presented in the form of equations, suitable for developing a numerical model. The variation of some of the terms with strain are presented graphically.

Load deflection behaviour of concrete members is important in understanding the possible modes of failure. Such behaviour mainly depend upon moment-curvature characteristics of section. Moment-curvature characteristics of rectangular reinforced concrete section are, therefore, presented, which indicates brittle mode of failure with increasing amount of reinforcement and/or axial force, as expected.

LIST OF NOTATIONS

a, b, e = Co-efficients of a quadratic equation.

A_o = Gross area of a section.

A_b, A_t, A_w = Gross area of bottom flange, top flange and web of a section, respectively.

A_c = Area of un-cracked portion of a cracked web.

A_s = Area of steel in a section.

B, B_b, B_t = Width of web, bottom and top flange, respectively.

c = Curvature of a section.

d = Distance of steel from extreme compression fibre in a section.

D, D_b, D_t = Ratio of un-cracked depth to total depth of a web, bottom flange and top flange, respectively.

E_o = Initial tangent modulus of elasticity of concrete in compression($=2\sigma_m/\varepsilon_m$).

E_s = Modulus of elasticity of steel.

F = Axial stress ratio in a section(σ/σ_m).

I_o = Moment of inertia of a rectangular section.

I_c = Moment of inertia of uncracked portion of cracked web.

I_b, I_t, I_w = Moment of inertia of bottom flange, top flange and web element, respectively in a section.

L = Distance of centroidal axis of a section from its extreme compression fibre.

M = Total bending moment carried by the section.

M_c, M_s = Bending moment carried by concrete and steel in a section.

n = Modular ratio($=E_s/E_c$).

p = Tension steel ratio($=A_s/A_c$).

P = Total axial load carried by the section.

P_c, P_s = Axial load carried by concrete and steel, respectively in a section.

R = Ratio of strains at the bottom most fiber to the top most fiber of a section($=\varepsilon_1/\varepsilon_2$).

R_b, R_t = Ratio of strains at the bottom fiber to the top fiber of an un-cracked bottom and top flange, respectively.

$[S]$ = Stiffness matrix of a section.

$[S]^w, [S]^b, [S]^t$ = Contribution to the stiffness matrix of a section due to the un-cracked web, bottom flange and top flange, respectively.

$[S]^{wc}, [S]^{bc}, [S]^{tc}$ = Contribution to the stiffness matrix of a section due to the cracked web, bottom flange and top flange, respectively.

T = Depth of a section.

T_b, T_t = Depth of bottom and top flange, respectively.

y = Distance of a fiber from the zero strain in a strain diagram.

dP, dM = Total differential of axial force(P) and bending moment(M) in a section.

$d\sigma, d\varepsilon$ = Derivatives of stress and strain, respectively.

$dF, d\delta$ = Derivatives of force and displacement, respectively.

$dc, d\varepsilon_L$ = Derivatives of curvature and strain at the centroidal plane of a section, respectively.

$\delta P_{1/2}, \delta V_{1/2}, \delta M_{1/2}$ = Incremental nodal loads in a beam-column element.

$\delta u_{1/2}, \delta v_{1/2}, \delta\theta_{1/2}$ = Incremental nodal displacements in a beam-column element.

δP = Increment in axial force of a section.

δM = Increment in bending moment of a section.

$\delta\sigma$ = Increment in stress at a fiber of a section.

$\delta\varepsilon$ = Incremental strain applied in a section.

$\delta\theta$ = Incremental rotation applied in a section.

$\theta, \theta_1, \theta_2, \theta'$ = Slope of the diferrent portion of stress-strain curve for prestressing/reinforcing steel.

ε = Strain in general and equal to $\varepsilon_1/\varepsilon_m$ in particular.

ε_y = Strain at any fiber of concrete and also yield strain of steel.

ε_y' = Strain at a fiber of a concrete section after applying an incremental strain or rotation.

$\varepsilon_s, \varepsilon_s'$ = Tensile strain in steel before and after applying an incremental strain or rotation, respectively.

ε_{se} = Net tensile strain in steel required to produce effective prestress.

ε_{ce} = Compressive strain in concrete at the level of steel just after prestressing.

ε_{cs} = Strain in concrete at the level of steel.

ε_1 = Strain at the extreme compressive fiber of a section.

ε_2 = Strain at the bottom most fiber of a section.

ε_3 = Strain at the bottom most fiber of top flange.

ε_4 = Strain at the top most fiber of bottom flange.

ε_L = Strain at the centroidal axis of a section.

ε_m = Concrete strain corresponding to maximum stress(σ_m).

σ_m = Maximum stress in the stress-strain curve of concrete in compression.

σ_y, σ_y' = Stress in a fiber at a distance of y from zero strain before and after applying an incremental strain or rotation, respectively.

σ = Average axial stress in a section(=P/A_o).

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 General

Non-linear analysis of structures has become popular with the availability of different types of computers. The non-linear analysis involves much mathematical computations which may easily be solved by computer programmes.

Various analytical procedures are available for designing structures. Among them the limit state theory is increasingly used now-a-days, because it provides the history of structural behaviour which is essential for a safe and serviceable structure and presents the idea of the margin of safety actually employed.

The limit state theory necessitates the use of material properties which closely represent its behaviour. The stress-strain relationship of concrete and steel being non-linear, it is essential to develop stiffness properties of a section considering non-linear material properties.

1.2 The Problem

Various methods of analysis and design are available for reinforced and prestressed concrete structures. These are

1. Working-stress method,
2. Ultimate strength method,
3. Moment-curvature analysis,
4. Strain compatibility method,
5. Limit state method, etc.

The working or permissible stress method of design pioneered by the German Professor Morsch, is also sometimes referred to as the elastic theory of design. In this method the permissible stresses in concrete and steel are assumed to be a fraction of specified strength of the individual material and a constant modular ratio is assumed for all loading conditions with the elastic behaviour of concrete and steel.

In prestressed concrete structures, stresses in concrete due to prestress are computed by elastic theory. This method assumes that the concrete section is not cracked. It is believed that the elastic theory is sufficiently accurate upto the point of cracking and cannot be used to predict the ultimate strength. The exact analysis for the ultimate strength of a prestressed concrete section under flexure is a complicated theoretical problem, because both steel and concrete are generally stressed beyond their elastic range.

The moment curvature analysis(1) is capable of predicting the behaviour of bonded prestressed concrete flexural members throughout the total load range from initial loading to failure. The method utilises the nonlinear material properties and tests have shown the results of the analysis to be quite reliable. Another method of estimating the flexural strength of prestressed concrete members is based on the compatibility of strains and equilibrium of forces on the section(2). The basic theory is applicable to all structural concrete sections, whether reinforced or prestressed.

The inadequacy of the working load design, in predicting ultimate load of structure, was recognized after the First World War. The factor of safety applied to the constituent materials does not present a realistic picture of the safety against the collapse of the composite material like reinforced concrete used in the structural component. Then the ultimate load method was introduced. The main feature of the ultimate load method of design is the applications of varying load factors for different types of loads to arrive at the required ultimate load for which the member is to be designed. The ultimate load method of design ensures only the safety of the structure against the collapse limit state and does not present any information about the behaviour of the structure at service load and the range between service and collapse load.

A comprehensive knowledge of the behaviour of structural concrete elements under different types of loading is essential for producing safe, serviceable and economic design of concrete structures. The limit state design philosophy recognizes the need to provide safe and serviceable structures at an economic price and at the same time presents a clearer idea of the margin of safety actually employed to cover uncertainity and ignorance of the function and the performance of structure in actual practice. The approach is being adopted increasingly by different codes. The influence of limit state design philosophy is evident in the revised American Code(3) and the unified British Code(4). The Indian Standard Codes for structural concrete(5,6) are also being revised to incorporate the limit state concepts.

There are several limit states at which a structure ceases its intended function. The most important among them are the limit state of collapse, excessive deflection and cracking. Each of these limit states may be attained due to different types of loading configuration. Thus it is apparent that to design a structure following limit state concept, its behaviour throughout the load range is very important. The limit state theory thus necessitates the study of the behaviour of structures, which again requires to provide due consideration to the material properties before any such study is undertaken. Concrete being non-linear material under compression, it was therefore decided to develop the stiffness properties for flexural elements considering parabolic stress-strain relationship.

1.3 Objective and Scope

The main objective of this research work was to develop the stiffness properties of various prestressed and reinforced concrete sections and a beam-column element with due consideration to the stress-strain characteristics of the constituent materials.

The stiffness properties developed in this work may be utilised to develop a numerical model for limit state analysis of prestressed and reinforced concrete structures. The numerical model can then be used to study the behaviour of reinforced and prestressed concrete structures, i.e. to evaluate their load-displacement relationship, extent of cracking etc. throughout the loading stages upto the collapse.

Therefore, the stiffness properties developed provide a basic tool for developing a numerical model to monitor the true behaviour of reinforced and prestressed concrete structures.

1.4 Thesis Outline

The knowledge of the fundamental properties of materials used in a structure is very much essential before any study is made on the behaviour of the structure. Stress-strain relationships of concrete and steel are, therefore, presented in Chapter 2.

Early on during the work it was realised that stiffness properties of reinforced and prestressed concrete sections have not been well defined which are essential for developing numerical model for their analysis. The numerical formulations of these properties is presented in Chapter 3. The stiffness properties of a beam-column element was then developed and presented at the end of this chapter.

The basic approach utilised to develop the stiffness properties of sectional elements was then used to study the moment-curvature relationship of a singly reinforced rectangular concrete section. This is presented in Chapter 4. Discussions and conclusions on this work are also presented in this chapter.

It is hoped that the work carried out will provide a useful tool in developing a numerical model which will be able to study the behaviour of reinforced and prestressed concrete structures.

CHAPTER 2

PROPERTIES OF MATERIALS FOR REINFORCED AND PRESTRESSED CONCRETE

2.1 General

Concrete has been considered as universal building material. It can be deposited and made to fill forms or moulds of almost any practical shape. Its high fire and weather resistance are evident advantages. Its compressive strength is high which makes it suitable for members primarily subjected to compression. It has been found possible to use steel to reinforce concrete, mainly in those places where its small tensile strength would limit the carrying capacity of the member. This combination of two materials is known as Reinforced Concrete.

In more recent times a special way has been found to use steel and concrete of very high strength in combination known as

Prestressed Concrete. The steel, mostly in the shape of wires or strands but sometimes as bars, are embedded in the concrete under high tension and are held in equilibrium by compressive stresses in the surrounding concrete after hardening. Because of this precompression, the concrete in a flexural member will crack on the tension side at a much larger load than when not so precompressed. This reduces radically both the deflection and the tensile crack at service load. Moreover, high strength concrete has a higher modulus of elasticity and smaller ultimate creep strain which results smaller loss of prestress in steel. The use of high strength materials reduces the cross-sectional dimensions of structural elements and also the dead weight.

Concrete in a wide range of strength properties can be obtained by appropriate adjustment of the proportions of the constituent materials. Special cement, special aggregates and special curing methods permit an even wider variety of properties to be obtained. These properties depend to a very substantial degree on the proportions of the mix, on the thoroughness with which the various constituents are intermixed, and on the condition of humidity and temperature(curing). However, all these fundamental properties of concrete are out of the scope of this study.

To develop a numerical model for analysing prestressed and reinforced concrete flexural members, stiffness properties are essential. And to determine stiffness properties the material characteristics such as stress-strain properties of concrete and steel are important. The parabolic stress-strain relationship for concrete in compression is adapted in this work. The stress-strain relationship of steel is assumed to be elasto-plastic.

2.2 Stress-strain Characteristics of Concrete in Compression

To understand the behaviour of any material under stress, it is necessary to establish its stress-strain characteristic. Since concrete is mainly used for compression, its compressive stress-strain properties is of primary interest.

It is well known that the stress-strain properties of concrete in compression is not linear even at normal level of stress as indicated in Fig.2.1, but for 30 to 50% of its crushing strength, it is assumed to be linear. The figure shows a typical set of curves, obtained for concrete of various cylinder strengths at 28 days. All the curves have somewhat similar character. They consist of an initial relatively straight elastic portion in which stress and strain are closely proportional, then begin to curve to the horizontal, reaching the maximum stress(the compressive strength) at a strain of approximately 0.002 and finally show a descending branch. It is also seen that concretes of lower strength are less brittle, i.e fracture at a larger maximum strain, than high strength concretes.

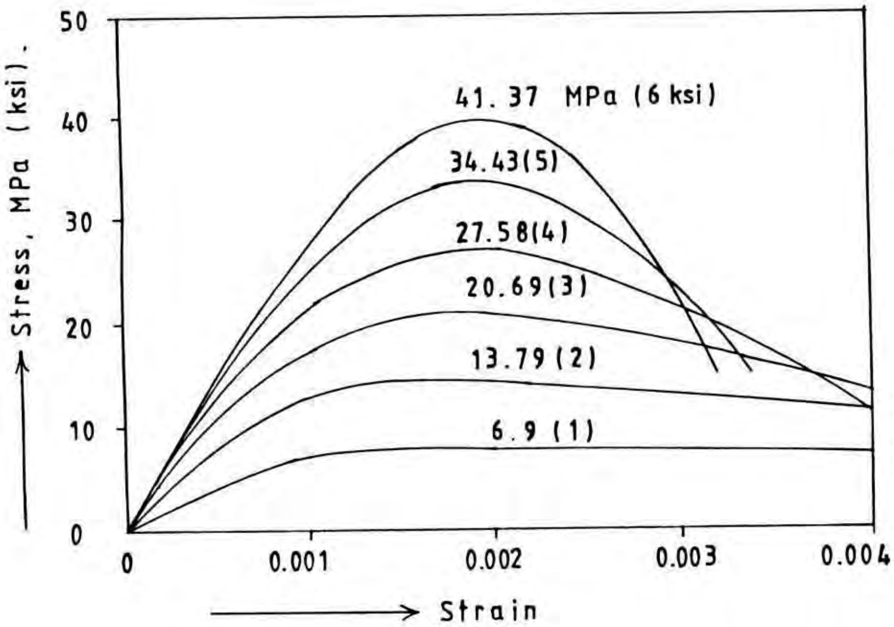


Figure 2.1 Typical concrete stress-strain curves(7).

Since 1899, many investigators have tried to represent the relationship by standard mathematical curves, e.g. a parabola, hyperbola, ellipse, cubic parabola, or combinations like a parabola with a straight line or a sine wave with a cubic parabola and so on(8). Some of which are mentioned below:

(a) Exponential form of equation as proposed by Desaye and Krishnan(9).

$$\sigma = E\varepsilon / [1 + (\varepsilon / \varepsilon_m)^2] \qquad 2.1$$

(b) Exponential form of equation as proposed by Smith and

Young(10).

$$\sigma = E\,\varepsilon^{m} \qquad 2.2$$

(c) Combination of parabola for the ascending part and a straight line of negative or zero slope for the falling branch due to Hognestad(11) and as adopted by British and Indian Codes(4,5,6).

$$\sigma = [2\varepsilon/\varepsilon_m - (\varepsilon/\varepsilon_m)^2]\sigma_m\ , \quad \text{when } \varepsilon \leqslant \varepsilon_m\ .$$
$$\sigma = [1-100(\varepsilon - \varepsilon_m)]\sigma_m\ , \quad \text{when } \varepsilon > \varepsilon_m\ . \qquad 2.3$$

(d) Parabola for the whole curve as adopted in moment-curvature analysis of prestressed concrete beams(1).

$$\sigma = [2\varepsilon/\varepsilon_m - (\varepsilon/\varepsilon_m)^2]\sigma_m \qquad 2.4$$

A comparison of Eqs.2.1, 2.2 and 2.4 with the experimental results is reproduced in Fig.2.2. As the Eq.2.3 is a combination of two curves, so it will be not simple for integration, which is necessary for evaluating the stiffness properties analytically. Moreover, on the falling branches the Eq.2.4 can be considered even more accurate than all other equations. From the figure, it appears that the parabolic equation represents the material properties fairly well and hence it was decided to use the parabolic stress-strain relationship(Eq.2.4) for this work as presented in Fig.2.3.

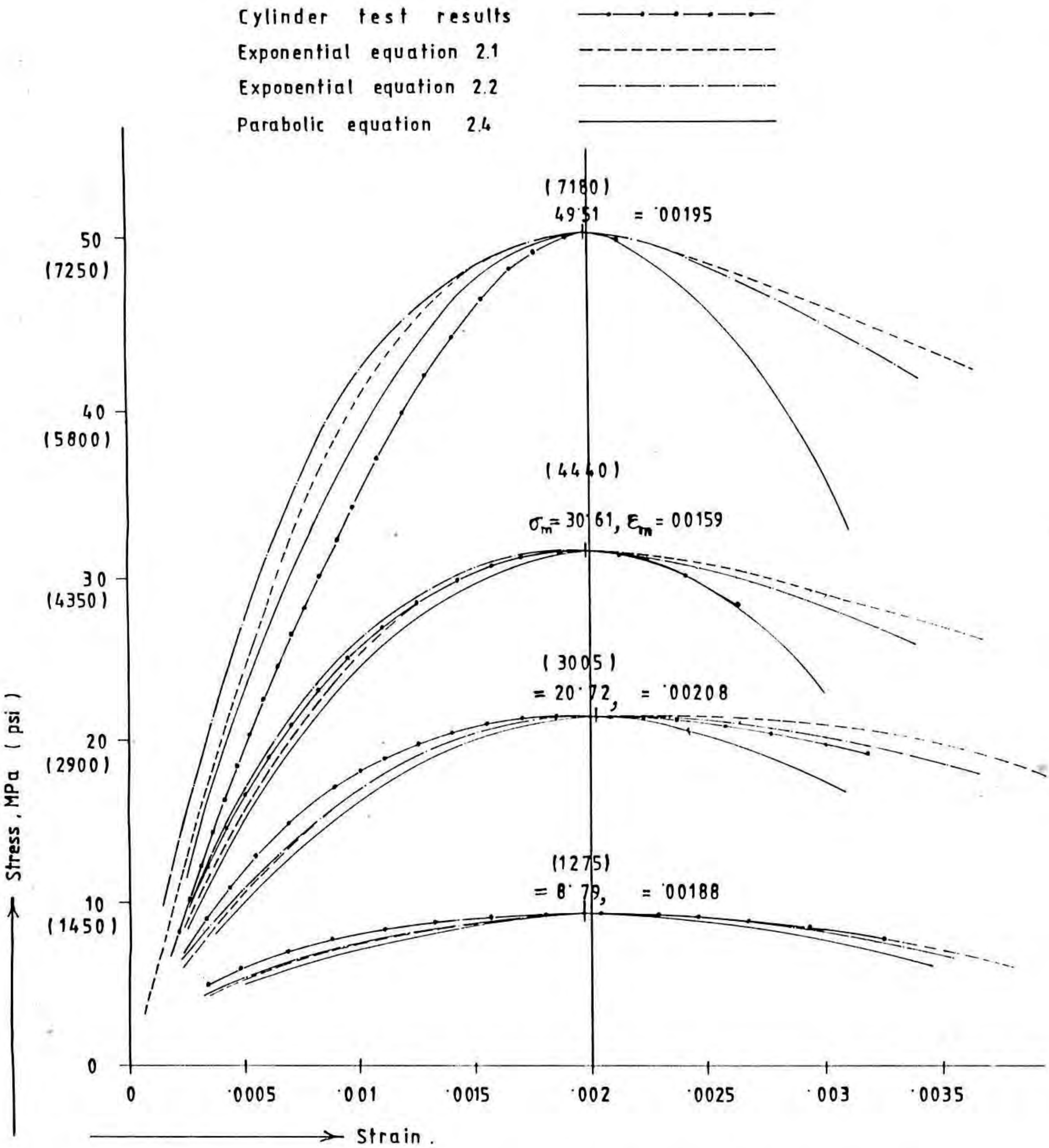


Figure : 2.2 Stress-strain curves of concrete given by different equation and test results.

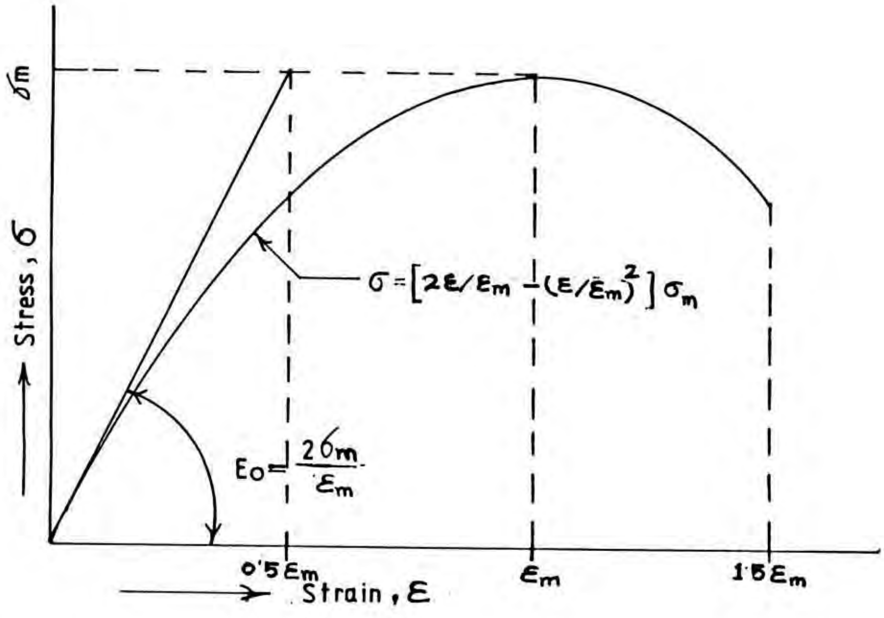


Figure 2.3 Idealised stress-strain curve of concrete.

The curve indicates that the young's modulus of elasticity varies with strain and is given by

$E = d\sigma/d\varepsilon = (d/d\varepsilon)\ [2\varepsilon/\varepsilon_m - (\varepsilon/\varepsilon_m)^2]\,\sigma_m$

or $E = (2/\varepsilon_m - 2\varepsilon/\varepsilon_m^2)\,\sigma_m$

so $E = 2(1-\varepsilon/\varepsilon_m)\,\sigma_m/\varepsilon_m$ 2.5

Thus the initial tangent modulus at zero strain is given as $E_o = 2\sigma_m/\varepsilon_m$ 2.6

and hence tangent modulus can be expressed as

$E = E_o(1-\varepsilon/\varepsilon_m)$ 2.7

2.3 Strength of Concrete in Tension

Concrete is very weak in tension and hence the tensile strength of concrete can be ignored in designing the reinforced and prestressed concrete structures. Shear and torsional resistance of concrete structure primarily depend on the tensile strength of concrete. Since this study is more concerned with the flexural behaviour of concrete structures, it is felt that the contribution from tensile strength to overall flexural strength would be small and has therefore been neglected.

2.4 Stress-strain Characteristic of Steel

Steel is used in two different ways in concrete structures, such as reinforcing steel in reinforced concrete structure and as prestressing steel in prestressed concrete structure. Steel for these two uses are different.

The most common type of reinforcing steel is in the form of round bar available in a large range of diameters. Prestressing steel is used in three forms; strands, wire and bar. To determine the character of steel two main numerical characteristics are important. These are yield point and modulus of elasticity. The modulus of elasticity is practically same for all reinforcing steel as $E=29x10^6$ psi($20x10^4$MPa). Typical stress-strain curves of reinforcing and prestressing steel are shown in Fig.2.4.

The reinforcing steel shows an elastic portion followed by a yield plateau, with further strain the stress begins to increase again through strain hardening process at a slower rate. In contrast to reinforcing steel, prestressing steel has no definite yield plateau, that is, they do not yield at a constant or nearly constant stress. Yielding develops gradually. In the inelastic range the curve continues to rise smoothly until the maximum strength is reached. Since discontinuous yielding is not observed, so the yield strength is somewhat arbitrarily defined as the stress at a total elongation of 1% for strand and wire and at 0.2% permanent strain for high-strength bars(7).

The stress-strain curves for reinforcing steel in tension and compression are generally assumed to be identical. Tests have shown that this is a reasonable assumption(11). On the other hand, stress-strain characteristics of prestressing steel usually refer to the action of tensile load.

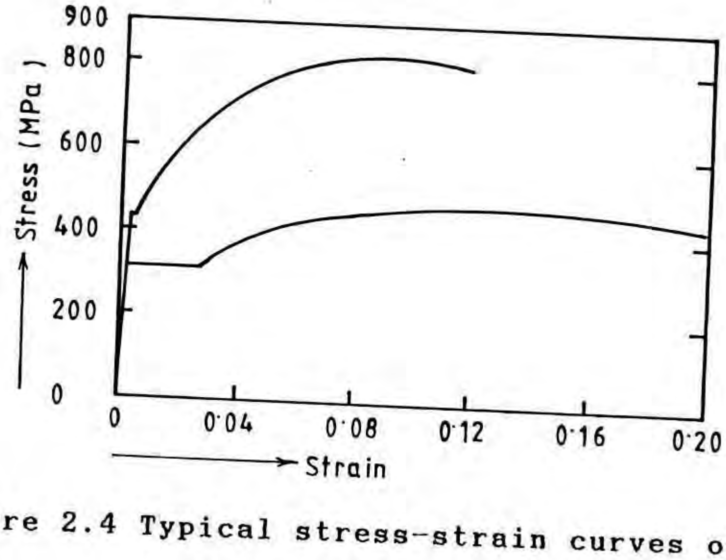
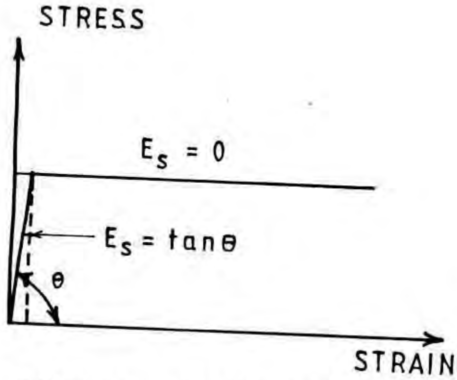
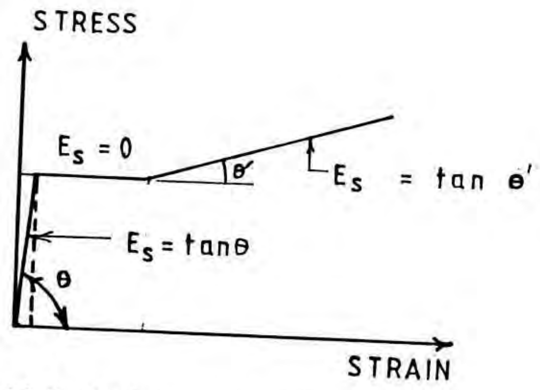


Figure 2.4 Typical stress-strain curves of steel(Ref.11).

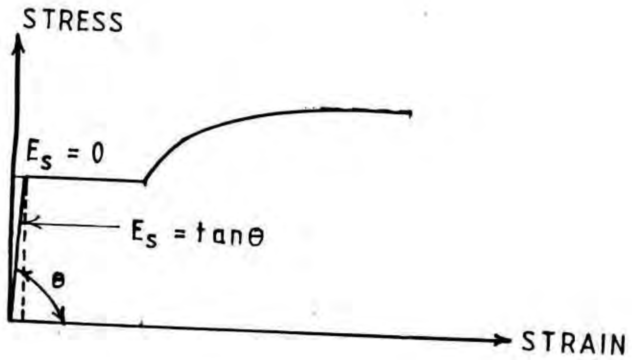
To develop a numerical model it is necessary to idealise the shape of the stress-strain curve. Generally the curve for reinforcing steel is simplified by idealising it as two straight lines(11), shown in Fig.2.5a. Two other form of idealizations are also shown in the figure. For prestressing steel, the British(4) and Indian Codes(5,6) recommend the idealised form as shown in Fig.2.6.



a) Elastic plastic approximation.



b) Triliniar approximation.



c) Complete curve.

Figure 2.5 Idealisation of stress-strain curve for reinforcing steel(Ref.11,pp.41).

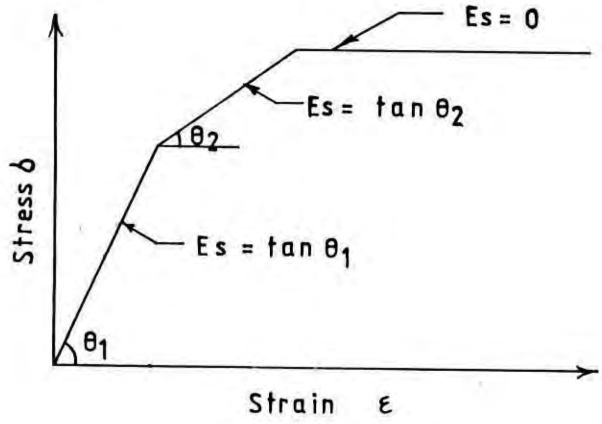


Figure. 2.6 Idealisation of stress-strain curve for prestressing steel(Ref.4).

For developing fundamental stiffness properties of reinforced and prestressed concrete structures, any form of idealisation which is a combination of straight lines can be used. Thus the stiffness matrices developed and presented in next chapter are suitable for any form of idealisation shown in Fig.2.5 and 2.6 except that in Fig.2.5(c).

CHAPTER 3

STIFFNESS PROPERTIES OF SECTIONAL ELEMENTS

3.1 Introduction

The increasing emphasis on the use of reinforced and prestressed concrete structures has stimulated the development of stiffness properties which are essential for developing numerical model suitable for analysis and design of concrete structures. The analysis of structures following linear and elastic relationship is straight forward. For a non-linear material, such as concrete, analysis is far from straight-forward and requires a fundamental look at its properties. This chapter develops, from first principle, expressions for axial and bending stiffness. The demand for such a fundamental look is to develop computer programme which describe more accurately the response of prestressed and reinforced concrete structures to loads upto the ultimate limit state. For concrete sections under flexure, linear strain distribution which has been used satisfactorily for many years is also assumed here. Parabolic stress-strain relationship (Eq.2.4) is used to determine the stress distribution corresponding to the linear strain distribution. Tensile strength of concrete, being small compared with compressive strength, adds little to the overall strength of concrete and hence has been neglected in the derivation. The stress-strain relationship for prestressing tendon and mild steel are assumed to be any combination of straight lines as discussed in Art.2.4.

The stiffness terms were developed separately for different cracked and uncracked sectional elements, prestressing and reinforcing steel. The stiffness matrix of a section can be obtained by algebraic summation of matrices corresponding to the different elements comprising the section and prestressing/reinforcing steel.

The stiffness terms developed correspond to a particular level of strain present at a section. The expressions can be derived directly from first principles and also by alternative approach using differential calculus. Most of the derivations presented here are done by direct approach.

3.2 Shapes of Concrete Sections

Concrete is advantageous to fabricate in any desired shape. The suitability of these shapes depend upon the particular requirements. The typical shapes frequently used for concrete structures are shown in Fig.3.1.

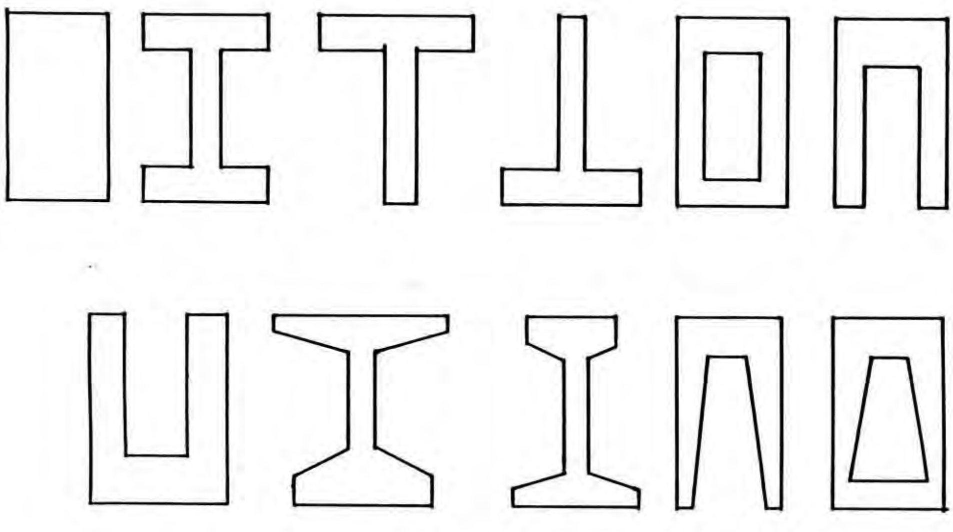


Figure 3.1 Shapes of concrete sections.

For the purpose of developing stiffness properties, a section can be considered to be consisting of one or more rectangular sectional elements. Thus the primary sectional elements are web, top flange, bottom flange and steel. The stiffness properties, therefore, are developed separately for each of these sectional elements. Stiffness properties for a complete section may then be obtained by superposition of the corresponding properties of its sectional elements.

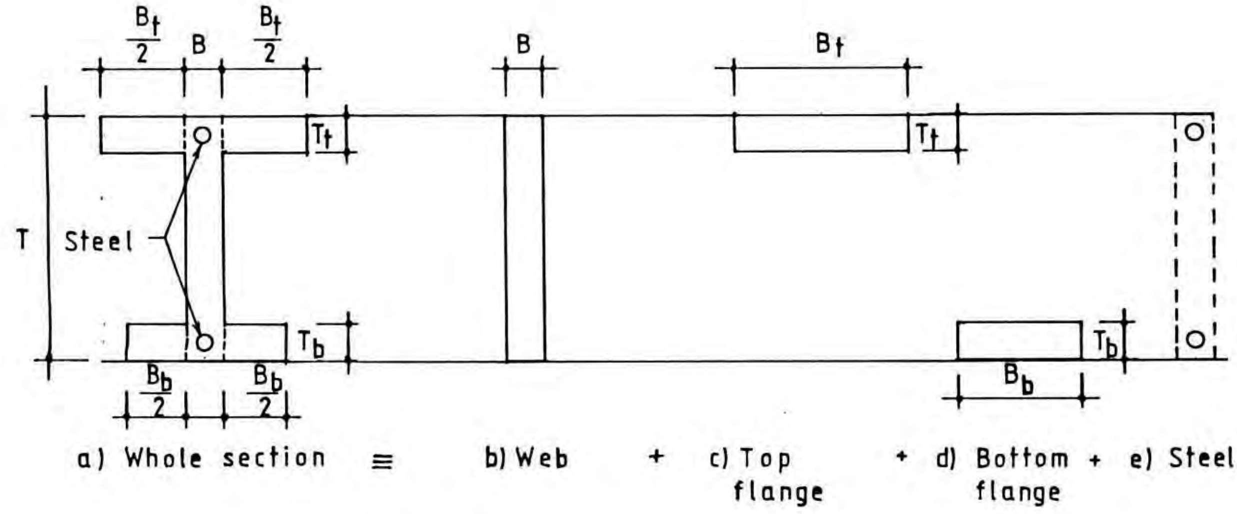


Figure 3.2 Sectional elements in a section.

Reinforced and prestressed concrete structures undergo different stages of loading which cause different types of strain distribution in a section. For the purpose of developing the stiffness properties of a section it is necessary to consider all the different strain distributions as shown in Fig.3.3.

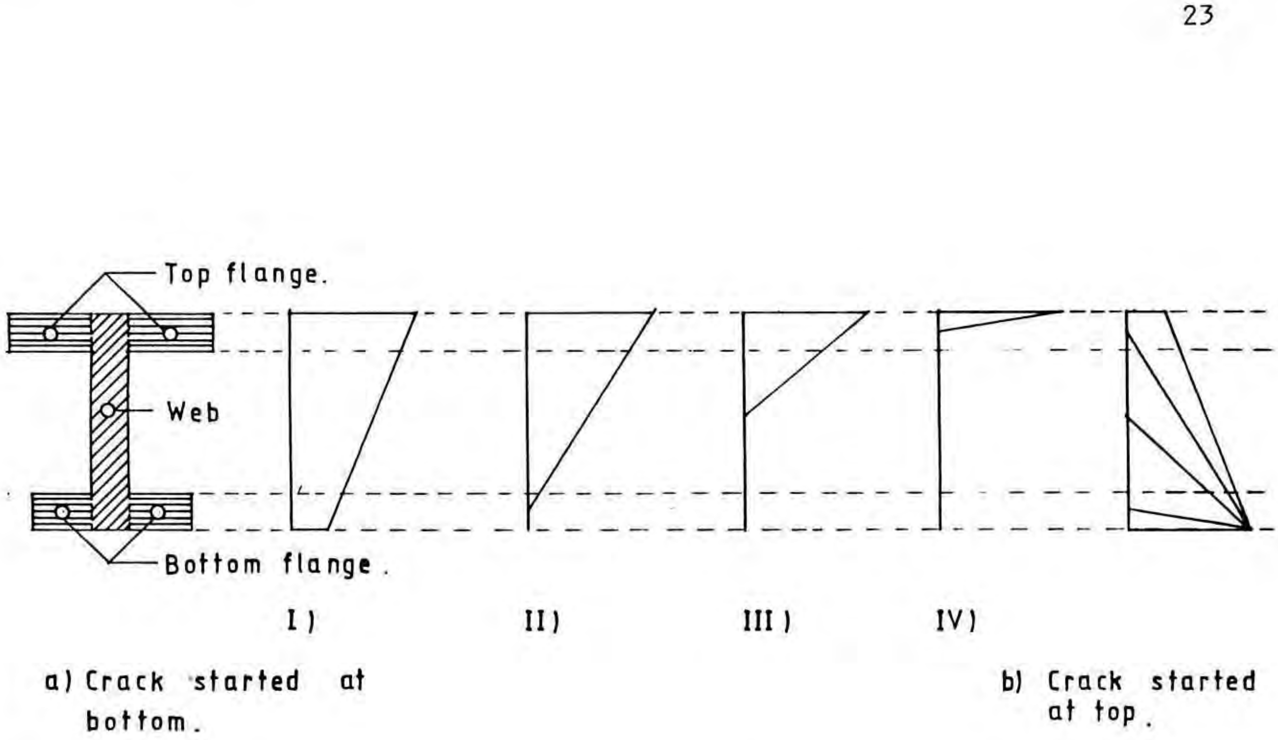


Figure 3.3 Different stages of strain distribution.

a) Crack started at bottom

i) Uncracked web, top flange and bottom flange.

ii) Uncracked top flange, partially cracked web and bottom flange.

iii) Uncracked top flange, partially cracked web and fully cracked bottom flange.

iv) Top flange and web are partially cracked and bottom flange is fully cracked.

b) Crack started from top[(reverse cases of(a)].

Three different stages of strain distributions may be observed in the elements. These are uncracked stage, partially cracked stage and fully cracked stage. When the element is completely cracked, for example the bottom flange in case of a(iii) and a(iv) in Fig.3.3, it will contribute nothing to the stiffness and hence can be ignored. The strain distributions for cases (b) are just reverse of those in case (a), where the crack starts at top and gradually propagates to the bottom. Stiffness for these cases can be obtained from the corresponding case of (a) by turning the section upside down, as explained in the article 3.14. The contribution to the stiffness for prestressing or reinforcing steel can easily be determined once the strain is known at the level of steel.

Further it is to be noted that the stiffness properties of different sectional elements are developed in this chapter assuming the elements have unvarying depth and width. Hence stiffness properties derived can not be applied for the section having tapered flange and/or web. Stiffness properties for these sections need separate consideration. It is however, possible to idealise these sections as explained qualitatively in Fig.3.4, so

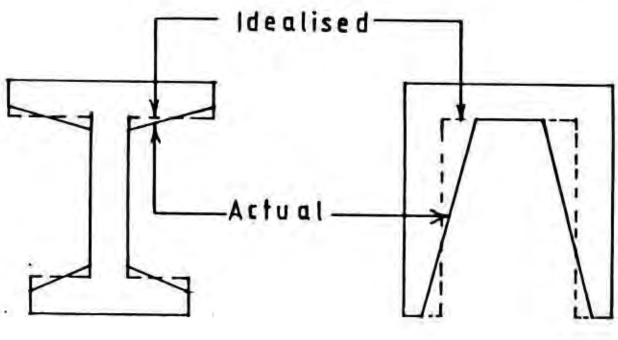


Figure 3.4 Idealisation of Tapering sections.

that the area of an element does not change. It is then possible to establish their stiffness properties utilising the properties developed in this chapter.

3.3 Definition of Stiffness

In linear elasticity, stiffness of an element is defined as its response to load. Instantaneous axial stiffness, for example, is defined as:

$$\text{Axial Stiffness} = \frac{\text{Load increment}}{\text{Axial deformation increment}}$$

For a member with uniform cross-sectional area:

Stiffness $EA = \delta P/\delta \varepsilon$

where $\delta\varepsilon$ = increment of axial deformation per unit length, and

δP = increment of axial load .

Stiffness is constant along the length of a member and is defined by the product of A(cross-sectional area of the section) and E(modulus of elasticity of the material).

If non-linear behaviour is assumed, the stiffness becomes a function of not only E and A but also of the state of strain and hence stress. Since this can vary along the length it is no longer possible to develop explicit expression for overall stiffness, but it is possible to derive expressions for stiffness corresponding to a particular section and strain level. For non-linear stress-strain relationship the axial and bending stiffnesses interact, i.e. the axial and bending stiffnesses are obtained in the form of 2x2 matrix as follows:

$$\begin{bmatrix} \delta P \\ \delta M \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \delta \varepsilon \\ \delta \theta \end{bmatrix}$$

The theoretical derivations of the axial stiffness and the corresponding coupling terms were eventually achieved by applying a uniform strain increment($\delta\varepsilon$) and calculating the consequential increase in load(δP) and moment(δM). The appropriate values were then obtained by making the strain increment equal to unity as in the following matrix equation,

$$\begin{bmatrix} \delta P \\ \delta M \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

from which $\delta P = S_{11}$ and $\delta M = S_{21}$

Similarly, the bending stiffness terms were obtained by applying an incremental rotation to the section and calculating the consequential increase in bending moment and axial load.

It must be stressed that these stiffness factors apply at the section being considered and corresponding to a particular level of strain in the section. The above procedure may be followed for different sectional element and prestressing/reinforcing steel of a section. The algebraic sum of stiffness matrices for corresponding elements and steel gives the complete stiffness matrix of a section.

3.4 Stiffness Matrix-Uncracked Web

The state of strain and stress in uncracked web of a concrete section under axial force(P) and bending moment(M) is shown in Fig.3.5. It will be observed that the state of strain in concrete can be defined by two independent non-dimensional terms; $\varepsilon_1/\varepsilon_m$ (the maximum strain ratio in the section) and $R=\varepsilon_2/\varepsilon_1$ (the ratio of minimum to maximum strain in the section).

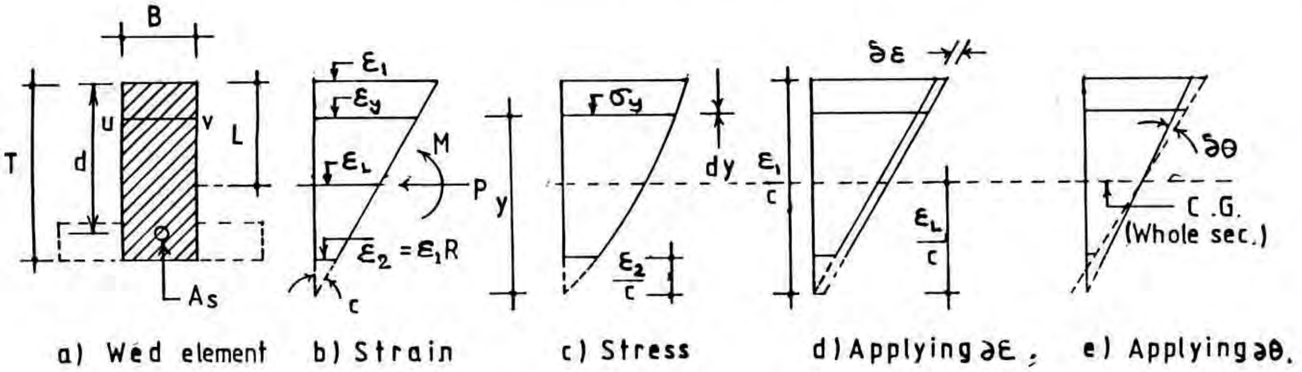


Figure 3.5 The state of strain and stress in uncracked web.

The basic approach of the derivation of stiffness terms are presented in Ref.(12) and (13) developed for cracked and uncracked rectangular section of structural masonry. The stiffness matrix of rectangular uncracked section is given by

$$[S] = \begin{bmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{bmatrix}$$

where,

$$S_{11}^* = \delta P/\delta \varepsilon = BE_o \int_{\varepsilon_2/c}^{\varepsilon_1/c} (1 - cy/\varepsilon_m)\, dy$$

$$= E_o A_w [1 - (1+R)\varepsilon_1/2\varepsilon_m] \qquad 3.1$$

$$S_{12}^* = S_{21}^* = \partial P/\partial \theta = \delta M/\partial \varepsilon = BE_o \int_{\varepsilon_2/c}^{\varepsilon_1/c} (1 - cy/\varepsilon_m)(y - \varepsilon_2/c - T/2)\, dy$$

$$= -E_o I_w [(1-R)\varepsilon_1/\varepsilon_m]/T \qquad 3.2$$

$$\text{and } S_{22}^* = \delta M/\partial \theta = BE_o \int_{\varepsilon_2/c}^{\varepsilon_1/c} (1 - cy/\varepsilon_m)(y - \varepsilon_2/c - T/2)^2\, dy$$

$$= E_o I_w [1 - (1+R)\varepsilon_1/2\varepsilon_m] \qquad 3.3$$

The stiffness terms are related to the centroidal axis of the rectangular section. The stiffness terms to be developed in this work are to be related to the centroidal axis of the whole section of which the rectangular web forms a part, and hence will be different from those presented in Ref(12) and(13). The expressions, just presented will be used to save the efforts of integration in obtaining the expressions developed and presented in this work.

From the strain distribution(Fig.3.5b) and by simple geometry, strain at a fiber uv is, $\varepsilon_y = cy$ 3.4

where curvature, $c = (\varepsilon_1 - \varepsilon_2)/T = \varepsilon_1(1-R)/T$ 3.5

ε_1=top fibre strain of the web section, ε_2=bottom fibre strain of the web section, T=depth of the web section, R=the ratio of minimum to maximum strain in the section, and y=distance of a fibre from the zero strain(Fig.3.5b).

By the parabolic stress-strain relationship, stress at the fiber uv is, $\sigma_y = [2\varepsilon_y/\varepsilon_m - (\varepsilon_y/\varepsilon_m)^2]\sigma_m$

which by Eq.3.4 becomes, $\sigma_y = [2cy/\varepsilon_m - c^2y^2/\varepsilon_m^2]\sigma_m$ 3.6

where ε_m and σ_m are the strain and stress at the peak of the parabolic stress-strain curve (Fig.2.3), respectively.

Applying a uniform strain increment($\delta\varepsilon$) in the section (Fig.3.5d), the strain at the fiber uv increases to;

$\varepsilon_y' = cy + \delta\varepsilon$ 3.7

and hence the corresponding stress at that fiber, by parabolic stress-strain relationship, increases to,

$\sigma_y' = [2\varepsilon_y'/\varepsilon_m - (\varepsilon_y'/\varepsilon_m)^2]\sigma_m$

Replacing ε_y' by Eq.3.7, the increased stress at the fiber uv is,

$$\sigma_y' = [2(cy+\delta\varepsilon)/\varepsilon_m - (cy+\delta\varepsilon)^2/\varepsilon_m^2]\sigma_m$$

and neglecting higher order terms

$$\sigma_y' = [2cy/\varepsilon_m + 2\delta\varepsilon/\varepsilon_m - c^2y^2/\varepsilon_m^2 - 2cy\delta\varepsilon/\varepsilon_m^2]\sigma_m \qquad 3.8$$

Hence, an increase in stress at the fiber uv, from Eqs.3.6 and 3.8 is given by,

$$\delta\sigma = \sigma_y' - \sigma_y = [2\delta\varepsilon/\varepsilon_m - 2cy\delta\varepsilon/\varepsilon_m^2]\sigma_m = (2\sigma_m/\varepsilon_m)[1-cy/\varepsilon_m]\delta\varepsilon$$

Replacing$(2\sigma_m/\varepsilon_m)$ by E_o(initial tangent modulus) the increase in stress becomes, $\delta\sigma = E_o(1-cy/\varepsilon_m)\delta\varepsilon$ 3.9

Therefore, an increase in axial force in the section is given by

$$\delta P = \int_{\varepsilon_2/c}^{\varepsilon_1/c} B\,\delta\sigma\,dy \qquad 3.10$$

and by taking moment about the centroidal plane of the whole section, the increase in the bending moment in the section is given by,

$$\delta M = \int_{\varepsilon_2/c}^{\varepsilon_1/c} B\,\delta\sigma(y-T+L-\varepsilon_2/c)\,dy \qquad 3.11$$

where B is the width of the section, $\delta\sigma$ is the increase in stress at the fiber uv, dy is the depth of the fiber uv and L is depth of the centroidal axis of the whole section(not only the rectangular web) from top fiber.

Substituting $\delta\sigma$ from Eq.3.9 into Eq.3.10 increase in axial force in the section is obtained as,

$$\delta P = BE_o \delta\varepsilon \int_{\varepsilon_2/c}^{\varepsilon_1/c} (1-cy/\varepsilon_m)\ dy \tag{3.12}$$

Using the relationship expressed in Eq.3.1, we get,

$$\delta P = E_o A_w [1-(1+R)\varepsilon_1/2\varepsilon_m]\delta\varepsilon \tag{3.13}$$

Substituting $\delta\sigma$ from Eq.3.9 into Eq.3.11, the increase in the bending moment in the web is given as,

$$\delta M = BE_o \delta\varepsilon \int_{\varepsilon_2/c}^{\varepsilon_1/c} (1-cy/\varepsilon_m)(y-T+L-\varepsilon_2/c)\ dy \tag{3.14}$$

or $\delta M = BE_o \delta\varepsilon \int_{\varepsilon_2/c}^{\varepsilon_1/c} (1-cy/\varepsilon_m)(y-T/2-\varepsilon_2/c)\ dy + (L-T/2)BE_o\delta\varepsilon \int_{\varepsilon_2/c}^{\varepsilon_1/c}(1-cy/\varepsilon_m)dy$

Using the expressions of Eq.3.1 and Eq.3.2,

$$\delta M = -(E_o I_w/T)[(1-R)\varepsilon_1/\varepsilon_m]\delta\varepsilon + (L-T/2)E_o A_w[1-(1+R)\varepsilon_1/2\varepsilon_m] \tag{3.15}$$

where A_w and I_w are the cross-sectional area and the moment of inertia of the web section, respectively.

Making $\delta\varepsilon$ to unity, as mentioned before, the expressions of δP and δM gives two terms of the stiffens matrix,

$$S_{11}{}^{w} = E_o A_w[1-(1+R)\varepsilon_1/2\varepsilon_m] \tag{3.16}$$

and $S_{21}{}^{w} = -(E_o I_w/T)[(1-R)\varepsilon_1/\varepsilon_m]$

$$+(L-T/2)E_o A_w[1-(1+R)\varepsilon_1/2\varepsilon_m] \tag{3.17}$$

After applying an incremental rotation($\delta\theta$) about the centroidal plane(Fig.3.5e), the strain at the fiber uv increases to, $\varepsilon_y' = cy + (y-\varepsilon_2/c+L-T)\delta\theta$ 3.18

And the corresponding stress at that fiber increases to,

$$\sigma_y' = [2\varepsilon_y'/\varepsilon_m - (\varepsilon_y'/\varepsilon_m)^2]\,\sigma_m$$

Replacing ε_y' by Eq.3.18 and neglecting the higher order terms,

$$\sigma_y' = [2cy/\varepsilon_m + 2\delta\theta(y - \varepsilon_2/c + L - T)/\varepsilon_m$$
$$-c^2y^2/\varepsilon_m^2 - 2cy\delta\theta(y - \varepsilon_2/c + L - T)/\varepsilon_m^2\,]\,\sigma_m \qquad 3.19$$

Hence the increase in stress at the fiber uv, from Eq.3.6 and Eq.3.19, is

$$\delta\sigma = \sigma_y' - \sigma_y = (2\sigma_m/\varepsilon_m)\delta\theta\,[(1 - cy/\varepsilon_m)(y - \varepsilon_2/c + L - T)]$$

which after putting $E_o = 2\sigma_m/\varepsilon_m$ becomes,

$$\delta\sigma = E_o\delta\theta\,[(1 - cy/\varepsilon_m)(y - \varepsilon_2/c + L - T)] \qquad 3.20$$

As before, the increase in axial force and bending moment in the section is given by,

$$\delta P = \int_{\varepsilon_2/c}^{\varepsilon_1/c} B\,\delta\sigma\,dy \quad \text{and} \quad \delta M = \int_{\varepsilon_2/c}^{\varepsilon_1/c} B\,\delta\sigma(y - \varepsilon_2/c + L - T)\,dy$$

Substituting $\delta\sigma$ from Eq.3.20, we get

$$\delta P = BE_o\delta\theta\int_{\varepsilon_2/c}^{\varepsilon_1/c} (1 - cy/\varepsilon_m)(y - \varepsilon_2/c + L - T)\,dy \qquad 3.21$$

and $$\delta M = BE_o\delta\theta\int_{\varepsilon_2/c}^{\varepsilon_1/c} (1 - cy/\varepsilon_m)(y - \varepsilon_2/c + L - T)^2\,dy \qquad 3.22$$

Comparing Eq.3.21 with Eq.3.14, it is observed that both are basically the same equation except for the terms $\delta\varepsilon$ and $\delta\theta$ and hence from Eq.3.15,

$$\delta P = -(E_o I_w/T)[(1-R)\varepsilon_1/\varepsilon_m]\delta\theta + (L - T/2)E_o A_w[1 - \varepsilon_1(1+R)/2\varepsilon_m]\delta\theta \qquad 3.23$$

Expanding Eq.3.22 we get,

$$\delta M = BE_o\delta\theta\int_{\varepsilon_2/c}^{\varepsilon_1/c}(1-cy/\varepsilon_m)(y-\varepsilon_2/c-T/2)^2\,dy\ +$$

$$+2BE_o\delta\theta(L-T/2)\int_{\varepsilon_2/c}^{\varepsilon_1/c}(1-cy/\varepsilon_m)(y-\varepsilon_2/c-T/2)\,dy$$

$$+\ BE_o\delta\theta(L-T/2)^2\int_{\varepsilon_2/c}^{\varepsilon_1/c}(1-cy/\varepsilon_m)\,dy$$

Using the expressions in Eqs.3.1, 3.2 and 3.3, we get,

$$\delta M = E_o I_w[1-(1+R)\varepsilon_1/2\varepsilon_m]\delta\theta - 2(L-T/2)E_o I_w\delta\theta[(1-R)\varepsilon_1/\varepsilon_m]/T$$
$$+(L-T/2)^2 E_o A_w[1-(1+R)\varepsilon_1/2\varepsilon_m]\delta\theta \qquad 3.24$$

As explained before, making $\delta\theta$ equal to unity, the expressions of δP and δM in Eqs.3.23 and 3.24 respectively gives two other terms of the stiffness matrix of the web;

$$S_{12}^{w} = -(E_o I_w/T)[(1-R)\varepsilon_1/\varepsilon_m] + (L-T/2)E_o A_w[1-(1+R)\varepsilon_1/2\varepsilon_m] \qquad 3.25$$

and $S_{22}^{w} = E_o I_w[1-(1+R)\varepsilon_1/2\varepsilon_m] - 2(L-T/2)(E_o I_w/T)[(1-R)\varepsilon_1/\varepsilon_m]$

$$+(L-T/2)^2 E_o A_w[1-(1+R)\varepsilon_1/2\varepsilon_m] \qquad 3.26$$

Thus the complete stiffness matrix due to the concrete in the web element can be expressed as,

$$[S]^{w} = \begin{bmatrix} S_{11}^{w} & S_{12}^{w} \\ S_{21}^{w} & S_{22}^{w} \end{bmatrix}$$

while S_{11}^{w}, S_{12}^{w}, S_{21}^{w} and S_{22}^{w} are given by Eqs.3.16, 3.17, 3.25 and 3.26 respectively.

3.5 Stiffness Matrix-Cracked Web

The state of strain and stress in cracked web of a concrete section under axial force(p) and bending moment(M) is shown in Fig.3.6.

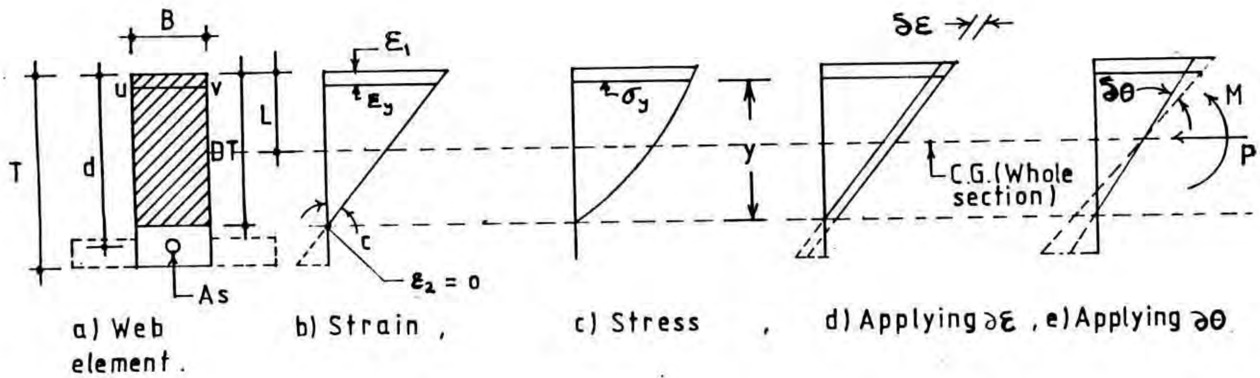


Figure 3.6 State of strain and stress in cracked web.

Strain distribution(Fig.3.6b) in a cracked web can be considered to be a particular case of that in an uncracked web, where the strain at the bottom fibre of the uncracked zone is zero. Thus the expressions developed for the uncracked web apply but in reference to the geometric properties of the uncracked zone.

Hence, putting R=0 and changing the terms representing geometrical properties, in Eqs.3.15, 3.16, 3.25 and 3.26, we get,

$$S_{11}{}^{wc} = E_o A_c [1-\varepsilon_1/2\varepsilon_m]$$

$$S_{21}{}^{wc} = S_{12}{}^{wc} = -E_o I_c (\varepsilon_1/\varepsilon_m)/DT + (L-DT/2)E_o A_c (1-\varepsilon_1/2\varepsilon_m)$$

and $S_{22}^{wc} = E_o I_c(1-\varepsilon_1/2\varepsilon_m) - 2(L-DT/2)E_o I_c(\varepsilon_1/\varepsilon_m)/DT$
$+ (L-DT/2)^2 E_o A_c[1-\varepsilon_1/2\varepsilon_m]$

where from Fig.3.6, D=Depth ratio i.e ratio of the depth of the uncracked zone(DT) to the original depth(T), DT=Depth of the uncracked zone of the web section, $A_c = BDT = A_w D$ = cross-sectional area of uncracked zone and $I_c = B(DT)^3/12 = BT^3D^3/12 = I_w D^3$ = second moment of area of the uncracked zone about the centroidal plane of the uncracked zone.

Utilizing the cross-sectional equivalences just mentioned, the above expressions become,

$$S_{11}^{wc} = E_o A_w D(2-\varepsilon_1/\varepsilon_m)/2 \qquad 3.28$$

$$S_{12}^{wc} = S_{21}^{wc} = -(E_o I_w D^3/DT)(\varepsilon_1/\varepsilon_m) + (L-DT/2)E_o A_w D(2-\varepsilon_1/\varepsilon_m)/2$$

$$= -(E_o I_w D^2/T)(\varepsilon_1/\varepsilon_m) + (L-DT/2)E_o A_w D(2-\varepsilon_1/\varepsilon_m)/2 \qquad 3.29$$

$$S_{22}^{wc} = (E_o I_w D^3/2)(2-\varepsilon_1/\varepsilon_m) - 2(L-DT/2)(E_o I_w D^2/T)(\varepsilon_1/\varepsilon_m)$$
$$+ (L-DT/2)^2(E_o A_w D/2)(2-\varepsilon_1/\varepsilon_m) \qquad 3.30$$

Thus the stiffness matrix of a cracked web, with respect to the centroidal axis of a whole section, can be represented as

$$[S]^{wc} = \begin{bmatrix} S_{11}^{wc} & S_{12}^{wc} \\ S_{21}^{wc} & S_{22}^{wc} \end{bmatrix}$$

where the stiffness terms are given by Eqs.3.28, 3.29 and 3.30.

3.6 Stiffness Matrix-Uncracked Top Flange

The general state of strain and stress in uncracked top flange of a concrete section under axial force(P) and bending moment(M) is shown in Fig.3.7.

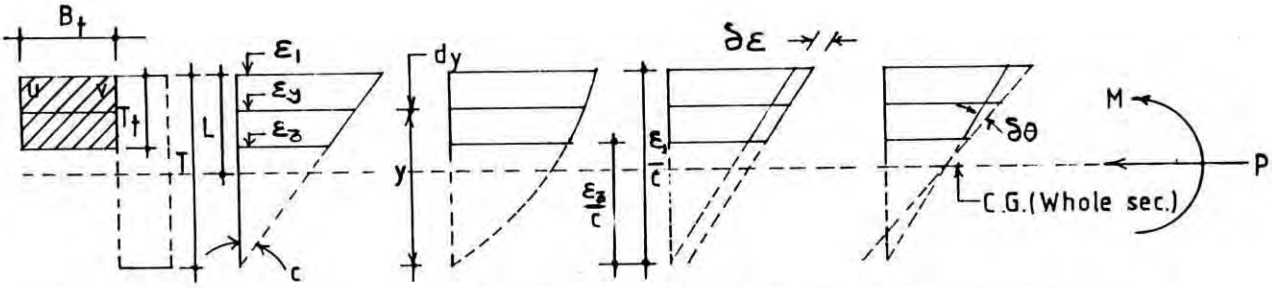


Figure 3.7 State of strain and stress in uncracked top flange.

The stiffness matrix for rectangular web is developed in previous sections. The stiffness matrices for the top flange will be developed in this section.

From the strain distribution(Fig.3.7b), the strain at a fiber uv is $\varepsilon_y = cy$. Where curvature, $c = (\varepsilon_1 - \varepsilon_3)/T_t = (1-R_3)\varepsilon_1/T_t$ and $R_3 = \varepsilon_3/\varepsilon_1$. The corresponding stress at the fiber uv is

$$\sigma_y = [2cy/\varepsilon_m - c^2 y^2/\varepsilon_m^2]\,\sigma_m$$

Applying a uniform strain increment($\partial\varepsilon$) in the section (Fig.3.7d) the strain at the fiber uv becomes, $\varepsilon_y' = cy+\partial\varepsilon$. And thereby as before, the stress increases to,

$$\sigma_y' = [2(cy+\partial\varepsilon)/\varepsilon_m - (cy+\partial\varepsilon)^2/\varepsilon_m]\,\sigma_m$$

Neglecting the higher order terms, σ_y' becomes

$$\sigma_y' = [2cy/\varepsilon_m + 2\partial\varepsilon/\varepsilon_m - c^2y^2/\varepsilon_m - 2cy\partial\varepsilon/\varepsilon_m]\,\sigma_m$$

Hence, an increase in stress at the fiber uv is,

$$\partial\sigma = \sigma_y' - \sigma_y = [2\partial\varepsilon/\varepsilon_m - 2cy\partial\varepsilon/\varepsilon_m^2]\,\sigma_m$$

Putting $2\sigma_m/\varepsilon_m = E_o$, the expression becomes $\partial\sigma = E_o(1-cy/\varepsilon_m)\partial\varepsilon$

Therefore considering only the top flange of the section, an increase in axial force is

$$\partial P = \int_{\varepsilon_3/c}^{\varepsilon_1/c} Bt\,\partial\sigma\,dy \qquad 3.32$$

and taking moment about the centroidal plane of the whole section, the increase in bending moment is

$$\partial M = \int_{\varepsilon_3/c}^{\varepsilon_1/c} Bt\,(y-\varepsilon_3/c+L-Tt)\,\partial\sigma\,dy \qquad 3.33$$

The expressions of ∂P and ∂M just obtained are the same as those of uncracked web given by Eqs.3.12 and 3.14, if the corresponding terms related to geometrical properties are interchanged.

Therefore, with the help of Eqs.3.13, 3.15, 3.16 and 3.17 we can write

$$S_{11}^{t} = E_o A_t [1-(1+R_t)\varepsilon_1/2\varepsilon_m] \tag{3.34}$$

$$S_{21}^{t} = -E_o I_t [(1-R_t)\varepsilon_1/\varepsilon_m]/T_t + (L-T/2)E_o A_t [1-(1+R_t)\varepsilon_1/2\varepsilon_m] \tag{3.35}$$

where T_t, A_t, I_t are thickness, area and moment of inertia of the top flange, respectively. R_t=Ratio of strains at the bottom fibre(ε_3) to the top fibre(ε_1) of the top flange.

For determining other two terms, applying an incremental rotation($\delta\theta$) about the centroidal plane of the whole section (Fig.3.7e), the strain in the fiber uv increases to,

$$\varepsilon_y' = cy + (y-\varepsilon_3/c + L-T_t)\delta\theta$$

and thereby the stress increases to,

$$\sigma_y' = [2\varepsilon_y'/\varepsilon_m - (\varepsilon_y'/\varepsilon_m)^2]\sigma_m$$

which after substituting ε_y' and neglecting the higher order terms, we get,

$$\sigma_y' = [2cy/\varepsilon_m + 2\delta\theta(y-\varepsilon_3/c + L-T_t)/\varepsilon_m - c^2y^2/\varepsilon_m^2$$
$$- 2cy\delta\theta(y-\varepsilon_3/c + L-T_t)/\varepsilon_m^2]\sigma_m$$

Hence the increase in stress at the fiber uv is

$$\delta\sigma = \sigma_y' - \sigma_y = [2\delta\theta(y-\varepsilon_3/c + L-T_t)/\varepsilon_m - 2cy\delta\theta(y-\varepsilon_3/c + L-T_t)/\varepsilon_m^2]\sigma_m$$

which after putting $2\sigma_m/\varepsilon_m = E_o$, becomes;

$$\delta\sigma = E_o(1-cy/\varepsilon_m)(y-\varepsilon_3/c + L-T_t)\delta\theta$$

As before, considering only the top flange the increase in axial force in the section is

$$\delta P = \int_{\varepsilon_3/c}^{\varepsilon_1/c} B_t\, \delta\sigma\, dy$$

and taking moment about the centroidal plane of the whole section, the increasing in bending moment is

$$\delta M = \int_{\varepsilon_3/c}^{\varepsilon_1/c} B_t (y-\varepsilon_3/c + L-T_t)\delta\sigma \, dy$$

putting $\delta\sigma$ from above equation, we get

$$\delta P = B_t E_o \delta\theta \int_{\varepsilon_3/c}^{\varepsilon_1/c} (1-cy/\varepsilon_m)(y-\varepsilon_3/c + L-T_t) \, dy$$

$$\text{and } \delta M = B_t E_o \delta\theta \int_{\varepsilon_3/c}^{\varepsilon_1/c} (1-cy/\varepsilon_m)(y-\varepsilon_3/c + L-T_t)^2 \, dy$$

As before, the expressions just obtained for δP and δM are similar to those for web(Eqs.3.21 and 3.22), and hence from Eqs.3.25 and 3.26, we can write as,

$S_{12}^t = S_{21}^t$ (same as Eq.3.35) and

$$S_{22}^t = E_o I_t [1-(1+R_t)\varepsilon_1/2\varepsilon_m] - 2(L-T_t/2)E_o I_t [(1-R_t)\varepsilon_1/\varepsilon_m]/T_t + (L-T_t/2)^2 E_o A_t [1-(1+R_t)\varepsilon_1/2\varepsilon_m] \quad 3.36$$

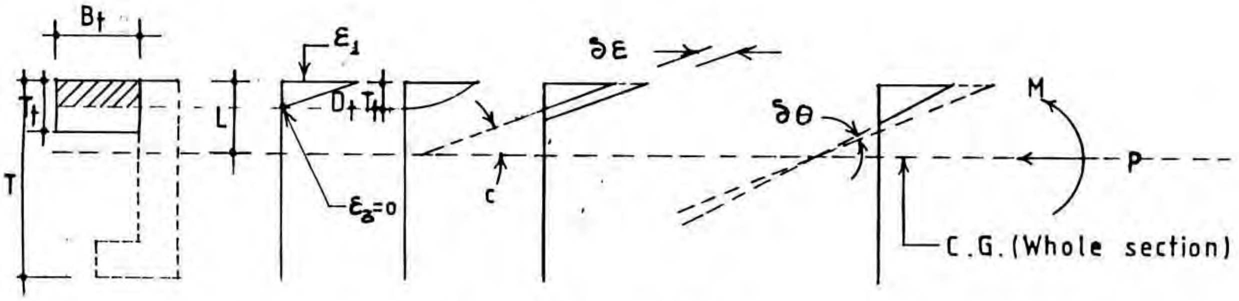
Thus the contribution to the stiffness matrix due to the top flange of a section is given by

$$[S]^t = \begin{bmatrix} S_{11}^t & S_{12}^t \\ S_{21}^t & S_{22}^t \end{bmatrix} \quad 3.37$$

While the terms are expressed by Eqs.3.34, 3.35 and 3.36.

3.7 Stiffness Matrix-Cracked Top Flange

The general state of strain and stress distribution for a cracked top flange is shown in Fig.3.8.



a)Top flange , b)Strain , c) Stress ,d) Applying $\partial\varepsilon$, e) Applying $\partial\theta$,

Figure 3.8 State of strain and stress in cracked top flange.

In determining the stiffness matrix for cracked web it is demonstrated that it can be obtained from the expressions for uncracked web. Applying similar principles, that is making Rt=0 and replacing the terms representing geometric properties, we get from Eqs.3.34, 3.35 and 3.36,

$$S_{11}^{tc} = E_o A_t D_t [1-\varepsilon_1/2\varepsilon_m] \qquad 3.38$$

$$S_{12}^{tc} = S_{21}^{tc} = -E_o I_t D_t^2 [\varepsilon_1/\varepsilon_m]/T_t + (L-T_t D_t/2) E_o A_t D_t [1-\varepsilon_1/2\varepsilon_m] \qquad 3.39$$

and $$S_{22}^{tc} = E_o I_t D_t^3 (1-\varepsilon_1/2\varepsilon_m) - 2(L-T_t D_t/2) E_o I_t D_t^2 (\varepsilon_1/\varepsilon_m)/T_t + (L-T_t D_t/2)^2 E_o A_t D_t (1-\varepsilon_1/2\varepsilon_m) \qquad 3.40$$

where $D_t T_t$ is the depth of uncracked zone of cracked top flange.

Hence the stiffness matrix for the cracked top flange can be given as,

$$[S]^{tc} = \begin{bmatrix} S_{11}{}^{tc} & S_{12}{}^{tc} \\ S_{21}{}^{tc} & S_{22}{}^{tc} \end{bmatrix} \qquad 3.41$$

3.8 Stiffness Matrix-Uncracked Bottom Flange

General state of strain, while the bottom flange of a section remains uncracked, is shown in Fig. 3.9.

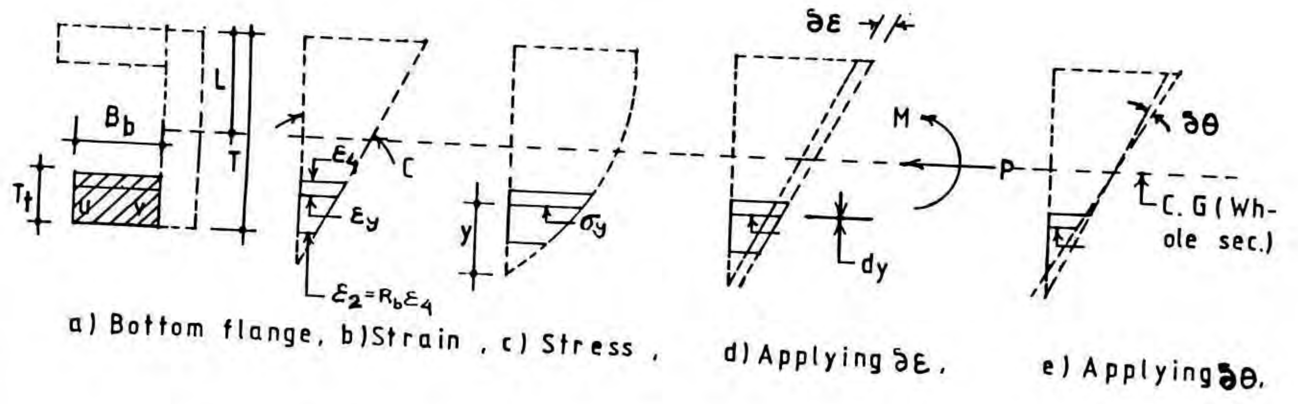


Figure 3.9 State of strain and stress in uncracked bottom flange.

From the strain distribution (Fig.3.9b) the strain at a fiber uv is $\varepsilon_y = cy$, where curvature $c = (\varepsilon_4 - \varepsilon_2)/T_b = \varepsilon_4(1-R_b)/T_b$. And the corresponding stress, by the parabolic stress-strain relationship is

$$\sigma_y = [2\varepsilon_y/\varepsilon_m - (\varepsilon_y/\varepsilon_m)^2]\sigma_m = [2cy/\varepsilon_m - (cy/\varepsilon_m)^2]\sigma_m$$

Applying a uniform strain increment($\delta\varepsilon$) in the section (Fig.3.9d), the strain in the fiber uv increases to($cy+\delta\varepsilon$) and thereby the stress increases to

$$\sigma_y' = [2(cy+\delta\varepsilon)/\varepsilon_m - (cy+\delta\varepsilon)^2/\varepsilon_m^2]\sigma_m$$

Neglecting the higher order terms, the expression becomes,

$$\sigma_y' = [2cy/\varepsilon_m + 2\delta\varepsilon/\varepsilon_m - c^2y^2/\varepsilon_m^2 - 2cy\delta\varepsilon/\varepsilon_m^2]\sigma_m$$

Hence the increase in stress after simplification, is

$\delta\sigma = \sigma_y' - \sigma_y = E_o(1-cy/\varepsilon_m)\delta\varepsilon$, where again $E_o = 2\sigma_m/\varepsilon_m$.

Considering the bottom flange of the section, the increase in axial force is

$$\delta P = \int_{\varepsilon_2/c}^{\varepsilon_4/c} B_b\,\delta\sigma\,dy = B_b E_o \delta\varepsilon \int_{\varepsilon_2/c}^{\varepsilon_4/c} (1-cy/\varepsilon_m)\,dy$$

and taking moment about the centroidal plane of the section the increase in bending moment is

$$\delta M = -\int_{\varepsilon_2/c}^{\varepsilon_4/c} B_b(T-L-y+\varepsilon_2/c)\delta\sigma\,dy$$

(-ve sign, because the moment is opposite to the +ve sign convention chosen, as shown in Fig.3.9). Expanding δM we get,

$$\delta M = B_b E_o \delta\varepsilon \int_{\varepsilon_2/c}^{\varepsilon_4/c} (1-cy/\varepsilon_m)(-T+L+y-\varepsilon_2/c)\,dy$$

$$= B_b E_o \delta\varepsilon \int_{\varepsilon_2/c}^{\varepsilon_4/c} (1-cy/\varepsilon_m)(y-\varepsilon_2/c- T_b/2)\ dy$$

$$-B_b E_o \delta\varepsilon\ (T-L-T_b/2) \int_{\varepsilon_2/c}^{\varepsilon_4/c} (1-cy/\varepsilon_m)\ dy \qquad 3.42$$

The expression of δP is similar to that of uncracked web given by Eq.3.12 and hence

$$S_{11}{}^b = E_o A_b [1-(1+R_b)\varepsilon_4/2\varepsilon_m] \qquad 3.43$$

where A_b, I_b, T_b and B_b are cross-sectional area, moment of inertia, depth and width of the bottom flange respectively. And $R_b = \varepsilon_2/\varepsilon_4$ as mentioned before.

The expression of δM, in Eq.3.42 has two parts. The first part is analogous to the first part of that of uncracked web given in Eq.3.14 while the second part is a multiplier of the expression of δP in Eq.3.12. Thus

$$S_{12}{}^b = \delta M/\delta\varepsilon = E_o I_b [(1-R_b)\varepsilon_4/\varepsilon_m]/T_b$$

$$-(T-L-T_b/2) E_o A_b [1-(1+R_b)\varepsilon_4/2\varepsilon_m] \qquad 3.44$$

Applying an incremental rotation($\delta\theta$) about the centroidal plane of the section(Fig.3.9c) the strain in the fiber uv reduces to $\varepsilon_y' = cy - \delta\theta(T-L-y+\varepsilon_2/c) = cy + \delta\theta(y-\varepsilon_2/c +L-T)$

and the corresponding stress, neglecting higher order terms, is

$$\sigma_y' = [2cy/\varepsilon_m + 2\delta\theta(y-\varepsilon_2/c +L-T)/\varepsilon_m - c^2 y^2/\varepsilon_m^2$$

$$-2cy\delta\theta(y-\varepsilon_2/c +L-T)/\varepsilon_m^2]\sigma_m$$

The change in stress is

$$\partial\sigma = \sigma_y' - \sigma_y = E_o\partial\theta(1-cy/\varepsilon_m)(y-\varepsilon_2/c + L - T)$$

$$= E_o\partial\theta(1-cy/\varepsilon_m)\{(y-\varepsilon_2/c - T_b/2)-(T-L-T_b/2)\}$$

The change in axial force in the section is, therefore,

$$\partial P = \int_{\varepsilon_2/c}^{\varepsilon_4/c} B_b\partial\sigma\ dy = B_b E_o\partial\theta\int_{\varepsilon_2/c}^{\varepsilon_4/c}(1-cy/\varepsilon_m)\{(y-\varepsilon_2/c - T_b/2)-(T-L-T_b/2\}\ dy$$

which is same as the expression of ∂M given in Eq.3.42 except the term $\partial\theta$, and hence from Eq.3.44, we can write

$$S_{21}{}^{b} = \partial P/\partial\theta = -E_o I_b[(1-R_b)\varepsilon_4/\varepsilon_m]/T_b$$
$$-(T-L-T_b/2)A_b E_o[1-(1+R_b)\varepsilon_4/2\varepsilon_m] \qquad 3.45$$

Taking moment about the centroidal plane of the whole section, the change in bending moment is

$$\partial M = -\int_{\varepsilon_2/c}^{\varepsilon_4/c} B_b(T-L-y+\varepsilon_2/c)\partial\sigma\ dy$$

$$= B_b E_o\partial\theta\int_{\varepsilon_2/c}^{\varepsilon_4/c}(1-cy/\varepsilon_m)\{(y-\varepsilon_2/c - T_b/2)-(T-L-T_b/2)\}^2\ dy$$

$$= B_b E_o\partial\theta\int_{\varepsilon_2/c}^{\varepsilon_4/c}(1-cy/\varepsilon_m)(y-\varepsilon_2/c - T_b/2)^2\ dy$$

$$-2B_b E_o\partial\theta\ (T-L-T_b/2)\int_{\varepsilon_2/c}^{\varepsilon_4/c}(1-cy/\varepsilon_m)(y-\varepsilon_2/c - T_b/2)\ dy$$

$$+B_b E_o\partial\theta\ (T-L-T_b/2)^2\int_{\varepsilon_2/c}^{\varepsilon_4/c}(1-cy/\varepsilon_m)\ dy$$

The expression of ∂M just presented has three parts. The first part is similar to the expression of S_{22} given by Eq.3.3

while the second and third part are a constant multiplier of the expressions of S_{12} and S_{11} given by Eqs.(3.2) and (3.1) respectively. Therefore, for an uncracked bottom flange

$$S_{22}^{b} = \partial M/\partial\theta = E_o I_b[1-\varepsilon_4(1+R_b)/2\varepsilon_m] +2(T-L-T_b/2)E_o I_b[\varepsilon_4(1-R_b/\varepsilon_m)]/T_b$$
$$+(T-L-T_b/2)^2 E_o A_b[1-\varepsilon_4(1+R_b)/2\varepsilon_m] \qquad 3.46$$

Thus the stiffness matrix for the uncracked bottom flange is given as

$$[S]^{b} = \begin{bmatrix} S_{11}^{b} & S_{12}^{b} \\ S_{21}^{b} & S_{22}^{b} \end{bmatrix} \qquad 3.47$$

3.9 Stiffness Matrix-Cracked Bottom Flange

In determining stiffness matrix for cracked web and top flange it is demonstrated before that the cracked element is a special case of the uncracked sectional element. Hence similarly as before, making $R_b=0$ and replacing terms related to geometric properties, we get,

$$S_{11}^{bc} = E_o D_b A_b[1-\varepsilon_4/2\varepsilon_m] \qquad 3.48$$

$$S_{12}^{bc} = S_{21}^{bc} = -E_o D_b^2 I_b(\varepsilon_4/\varepsilon_m)/T_b$$
$$-(T-L-T_b/2)E_o A_b D_b(1-\varepsilon_4/2\varepsilon_m) \qquad 3.49$$

$$S_{22}^{bc} = E_o D_b^3 I_b[1-\varepsilon_4/2\varepsilon_m] +2(T-L-T_b/2)E_o D_b^2 I_b(\varepsilon_4/\varepsilon_m)/T_b$$
$$+(T-L-T_b/2)^2 E_o D_b A_b[1-\varepsilon_4/2\varepsilon_m] \qquad 3.50$$

where $D_b T_b$ is the depth of uncracked zone of the cracked bottom flange.

So the stiffness matrix for the cracked bottom flange can be given as

$$[S]^{bc} = \begin{bmatrix} S_{11}^{bc} & S_{12}^{bc} \\ S_{21}^{bc} & S_{22}^{bc} \end{bmatrix} \qquad 3.51$$

3.10 Stiffness Matrix-Prestressing Steel

General state of stress and strain in an cracked and uncracked prestressed concrete section is shown in Fig.3.10 and 3.11. In the figures it is observed that, to define the state of strain in prestressing steel three independent terms of strain are essential. Thus at any instant, strain in prestressing steel can be expressed as $\varepsilon_s = \varepsilon_{se} + \varepsilon_{ce} + \varepsilon_{cs}$

where ε_s = Tensile strain in steel,

ε_{se} = Net tensile strain in steel required to produce the effective prestress,

ε_{ce} = Compressive strain in concrete at the level of steel just after prestressing, and

ε_{cs} = Strain in concrete at the level of steel (+ve if tensile).

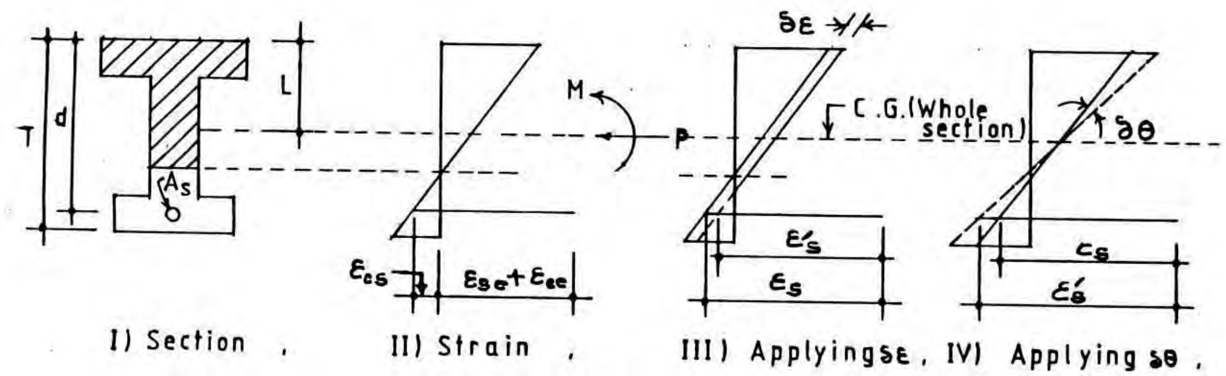


Figure 3.10 State of strain in prestressing steel of a cracked section.

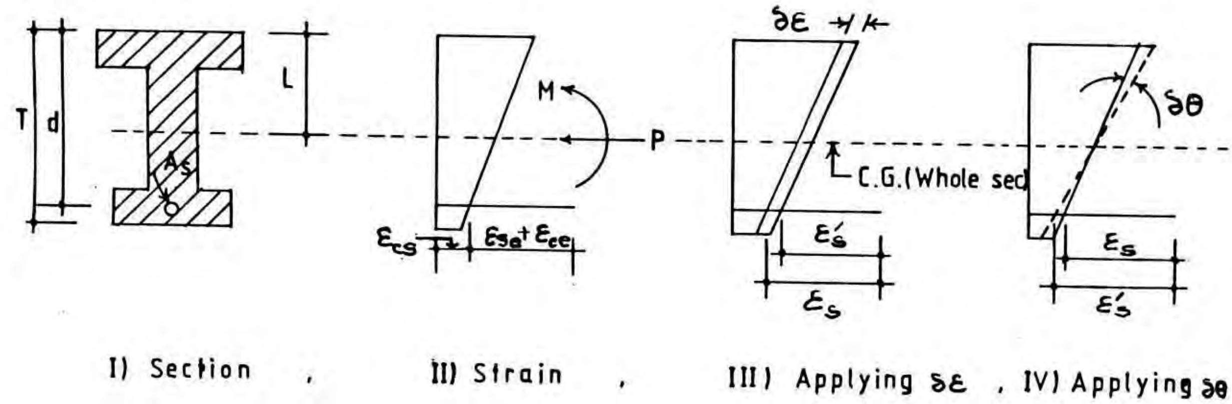


Figure 3.11 State of strain in prestressing steel of an uncracked section.

For the purpose of determining the contribution of prestressing steel to the overall stiffness of a prestress concrete section, a uniform strain increment($\delta\varepsilon$) and a rotational increment($\delta\theta$) will be applied, as has been done before. And from the consequential increase of axial force and bending moment in the section, the corresponding terms of stiffness will be determined.

Applying a uniform strain increment($\delta\varepsilon$) the tensile strain in steel, becomes $\varepsilon_s' = \varepsilon_s - \delta\varepsilon$

and hence the change in tensile stress in steel is

$\delta\sigma = E_s(\varepsilon_s' - \varepsilon_s) = -E_s\delta\varepsilon$

Where E_s=Young's modulus of elasticity of steel, assuming it is not yielding($\varepsilon_s < \varepsilon_y$). The change in axial force and bending moment in the section are

$\delta P = A_s\delta\sigma = -E_sA_s\delta\varepsilon$ (tensile) $= E_sA_s\delta\varepsilon$ (compressive)

and $\delta M = \delta\sigma A_s(d-L) = -E_sA_s(d-L)\delta\varepsilon$

where, d is the distance of the steel from the top fiber of the section, L is the distance of c.g of the whole section from the top fiber and A_s is the area of prestressing steel. Making $\delta\varepsilon$ equal to unity as before, we get

$S_{11}{}^{P} = +\delta P = E_sA_s$ 3.52

and $S_{21}{}^{P} = \delta M = -E_sA_s(d-L)$ 3.53

Applying an incremental rotation($\delta\theta$) as before, the change in stress in steel is $\delta\sigma = E_s(d-L)\delta\theta$

Hence the change in axial force and bending moment in the section are $\delta P = A_s\delta\sigma = E_sA_s(d-L)\delta\theta$ (tensile)

$= -E_sA_s(d-L)\delta\theta$ (compression)

and $\delta M = A_s(d-L)\delta\sigma = E_sA_s(d-L)^2\delta\theta$

Making δe equal to unity, we get,

$$S_{12}^{P} = \delta P = -E_s A_s (d-L) \tag{3.54}$$

and $S_{22}^{P} = \delta M = E_s A_s (d-L)^2$ 3.55

Thus the contribution to the stiffness matrix for prestressed concrete sections due to prestressing steel is given by,

$$[S]^{P} = \begin{bmatrix} S_{11}^{P} & S_{12}^{P} \\ S_{21}^{P} & S_{22}^{P} \end{bmatrix} = \begin{bmatrix} E_s A_s & -E_s A_s (d-L) \\ -E_s A_s (d-L) & E_s A_s (d-L)^2 \end{bmatrix} \tag{3.56}$$

It is to be noted that the stiffness terms for prestressing steel appears to be of linear form, due to the fact that the basic stress-strain characteristics of steel is a combination of two or three linear parts as presented in Figs.2.5a, 2.5b and 2.6.

The expressions presented here are related to the first part of the curve representing linear elastic stress-strain relationship for steel(before yielding is started). The stiffness terms corresponding to any other part of the stress-strain curve (plastic or strain hardening range) can easily be obtained from the expressions presented by just replacing the term E_s, with the slope of the corresponding portion of the stress-strain curve.

3.11 Stiffness Matrix-Reinforcing Steel.

Unlike the prestressing steel for reinforcing steel only one term is required to express the strain in steel. Thus at any stage of loading, strain in steel is $\varepsilon_s = \varepsilon_{cs}$. Where, as before, ε_{cs}=the strain of concrete at the level of steel(+ve if tensile).

Following the same approach as that for prestressing steel it can be observed that the difference in expressing strain for reinforcing steel dose not make any difference in the expression of stiffness terms and hence the contribution to the stiffness of a section is given as,

$$[S]^R = \begin{bmatrix} E_s A_s & -E_s A_s (d-L) \\ -E_s A_s (d-L) & E_s A_s (d-L)^2 \end{bmatrix} \qquad 3.57$$

where, E_s represents the slope of the appropriate part of the stress-strain curve(Fig.2.5a, 2.5b and 2.6) depending upon the strain present in steel.

3.12 Alternate Approach

Derivation of the stiffness terms can also be done alternatively using differential calculus and a Jacobian matrix.

Results obtained are identical to those derived in the previous section.

The approach is equally applicable for different uncracked and cracked sectional elements but it is presented only for the uncracked web element.

The state of strain and stress in an uncracked web of a concrete section under axial force(P) and bending moment(M) is shown in Fig.3.5. From the strain distribution(Fig.3.5b) by simple geometry, the strain at a fiber uv is $\varepsilon_y = cy$, where the curvature, $c = (\varepsilon_1 - \varepsilon_2)/T = (1-R)\varepsilon_1/T$

By the parabolic stress-strain relationship, the corresponding stress at the fiber uv is

$$\sigma_y = [2cy/\varepsilon_m - (cy/\varepsilon_m)^2]\sigma_m = (2\sigma_m/\varepsilon_m)(cy - c^2y^2/2\varepsilon_m)$$

$$\therefore \sigma_y = E_o(cy - c^2y^2/2\varepsilon_m) \qquad 3.58$$

Integrating over the whole depth the axial force in the section carried by the concrete is given as

$$P = \int_{\varepsilon_L/c - T + L}^{\varepsilon_L/c + L} B\,\sigma_y\,dy$$

where as before,

c=curvature of the section, L=depth of centroidal plane of the section from top fiber, T=depth of the web, B=width of the web element and ε_L =strain at the centroidal plane of the whole section = $(L-T)\varepsilon_1/T + (L)\varepsilon_2/T = \varepsilon_1 - (1-R)\varepsilon_1 L/T$.

Replacing σ_y by Eq.3.58 the axial force carried by concrete is,

$$P = BE_o\int_{L-T+\varepsilon_L/c}^{L+\varepsilon_L/c} (cy - c^2y^2/2\varepsilon_m)\,dy = BE_o\left[cy^2/2 - c^2y^3/6\varepsilon_m\right]_{L-T+\varepsilon_L/c}^{L+\varepsilon_L/c} \qquad 3.59$$

$$= BE_o[(c/2)\{(\varepsilon_L/c + L)^2 - (\varepsilon_L/c - T + L)^2\} - (c^2/6\varepsilon_m)\{(\varepsilon_L/c + L)^3 - (\varepsilon_L/c - T + L)^3\}]$$

$$= BE_o[\{(L+\varepsilon_L/c)^2 - (L+\varepsilon_L/c)^2 + 2T(L+\varepsilon_L/c) - T^2\}c/2 - \{(L+\varepsilon_L/c)^3 - (L+\varepsilon_L/c)^3 + 3(L+\varepsilon_L/c)^2T - 3(L+\varepsilon_L/c)T^2 + T^3\}c^2/6\varepsilon_m]$$

$$= BE_o[(2TL - T^2 + 2T\varepsilon_L/c)c/2 - (3T\varepsilon_L^2/c^2 + 6TL\varepsilon_L/c + 3TL^2 + T^3 - 3T^2\varepsilon_L/c - 3T^2L)c^2/6\varepsilon_m]$$

$$= BE_o[T\varepsilon_L + cTL - cT^2/2 - T\varepsilon_L^2/2\varepsilon_m - c^2TL^2/2\varepsilon_m - cTL\varepsilon_L/\varepsilon_m - c^2T^3/6\varepsilon_m + cT^2\varepsilon_L/2\varepsilon_m + c^2LT^2/2\varepsilon_m]$$

Taking moment of stress diagram about the centroidal plane, the bending moment carried by the concrete is,

$$M = \int_{L-T+\varepsilon_L/c}^{L+\varepsilon_L/c} B(y - \varepsilon_L/c)\sigma_y\,dy$$

Replacing σ_y by Eq.3.58

$$M = BE_o\int_{L-T+\varepsilon_L/c}^{L+\varepsilon_L/c} (y - \varepsilon_L/c)(cy - c^2y^2/2\varepsilon_m)\,dy$$

$$= BE_o\int_{L-T+\varepsilon_L/c}^{L+\varepsilon_L/c} y(cy - c^2y^2/2\varepsilon_m)\,dy - (BE_o\varepsilon_L/c)\int_{L-T+\varepsilon_L/c}^{L+\varepsilon_L/c} (cy - c^2y^2/2\varepsilon_m)\,dy$$

$$= M_1 + M_2 \tag{3.60}$$

The second part of the expression above is just$(-\varepsilon_L/c)$ times the expression of P(Eq.3.59). So

$$M_2 = -(BE_o\varepsilon_L/c)[T\varepsilon_L + cTL - cT^2/2 - T\varepsilon_L^2/2\varepsilon_m - Tc^2L^2/2\varepsilon_m - cTL\varepsilon_L/\varepsilon_m$$
$$-c^2T^3/6\varepsilon_m + cT^2\varepsilon_L/2\varepsilon_m + c^2LT^2/2\varepsilon_m]$$
$$= BE_o[-T\varepsilon_L^2/c - TL\varepsilon_L + T^2\varepsilon_L/2 + T\varepsilon_L^3/2c\varepsilon_m + TcL^2\varepsilon_L/2\varepsilon_m + TL\varepsilon_L^2/\varepsilon_m$$
$$+cT^3\varepsilon_L/6\varepsilon_m - T^2\varepsilon_L^2/2\varepsilon_m - cLT^2\varepsilon_L/2\varepsilon_m]$$

Now integrating the expression of M_1, we get

$$M_1 = BE_o[cy^3/3 - c^2y^4/8\varepsilon_m]_{L-T+\varepsilon_L/c}^{L+\varepsilon_L/c}$$

$$= BE_o[\{T^3 - 3T^2(L+\varepsilon_L/c) + 3T(L+\varepsilon_L/c)^2\}c/3 - \{4T(L+\varepsilon_L/c)^3$$
$$-6T^2(L+\varepsilon_L/c)^2 + 4T^3(L+\varepsilon_L/c) - T^4\}c^2/8\varepsilon_m]$$

Since from basic algebra

$$(L+\varepsilon_L/c)^3 - (L-T+\varepsilon_L/c)^3$$
$$= (L+\varepsilon_L/c)^3 - \{(L+\varepsilon_L/c) - T\}^3$$
$$= (L+\varepsilon_L/c)^3 - (L+\varepsilon_L/c)^3 + 3(L+\varepsilon_L/c)^2T - 3(L+\varepsilon_L/c)T^2 + T^3$$
$$= T^3 - 3T^2(L+\varepsilon_L/c) + 3T(L+\varepsilon_L/c)^2$$

and

$$(L+\varepsilon_L/c)^4 - \{(L+\varepsilon_L/c)-T\}^4$$
$$= (L+\varepsilon_L/c)^4 - (L+\varepsilon_L/c)^4 + 4T(L+\varepsilon_L/c)^3 - 6T^2(L+\varepsilon_L/c)^2$$
$$+4T^3(L+\varepsilon_L/c) - T^4$$
$$= 4T(\varepsilon_L^3/c^3 + 3L\varepsilon_L^2/c^2 + 3L^2\varepsilon_L/c + L^3) - 6T^2(\varepsilon_L^2/c^2 + 2L\varepsilon_L/c + L^2)$$
$$+4T^3(L+\varepsilon_L/c) - T^4$$

Thus

$$
\begin{aligned}
M_1 = & BE_o[(T^3 - 3T^2\varepsilon_L/c - 3T^2L + 3T\varepsilon_L^2/c^2 + 6TL\varepsilon_L/c + 3TL^2)c/3 \\
& -(4T\varepsilon_L^3/c^3 + 12TL\varepsilon_L^2/c^2 + 12TL^2\varepsilon_L/c + 4TL^3 - 6T^2\varepsilon_L^2/c^2 - 12LT^2\varepsilon_L/c \\
& -6T^2L^2 + 4T^3\varepsilon_L/c + 4T^3L - T^4)c^2/8\varepsilon_m] \\
= & BE_o[cT^3/3 - T^2\varepsilon_L - LT^2c + T\varepsilon_L^2/c + cTL^2 + 2TL\varepsilon_L - T\varepsilon_L^3/2c\varepsilon_m \\
& -3LT\varepsilon_L^2/2\varepsilon_m - 3TL^2c\varepsilon_L/2\varepsilon_m - TL^3c^2/2\varepsilon_m + 3T^2\varepsilon_L^2/4\varepsilon_m + 3T^2c^2L^2/4\varepsilon_m \\
& +3cLT^2\varepsilon_L/2\varepsilon_m - cT^3\varepsilon_L/2\varepsilon_m - c^2LT^3/2\varepsilon_m + c^2T^4/8\varepsilon_m]
\end{aligned}
$$

It can be observed that two encircled terms in both the expression of M_1 and M_2 have the same value with opposite sign and hence they will cancel one another. Therefore, these terms will be ignored in obtaining partial derivatives followed.

After obtaining the expression of axial force(P) and bending moment(M) corresponding to the strain and stress distribution of a section it is possible to find the total differential(14) of P and M using the principles of differential calculus. The total differential is given by

$dP = (\partial P/\partial\varepsilon_L)\, d\varepsilon_L + (\partial P/\partial c)\, dc$ and $dM = (\partial M/\partial\varepsilon_L)\, d\varepsilon_L + (\partial M/\partial c)\, dc$
which can be written in matrix form as

$$
\begin{bmatrix} dP \\ dM \end{bmatrix} = \begin{bmatrix} \dfrac{\partial P}{\partial \varepsilon_L} & \dfrac{\partial P}{\partial c} \\ \dfrac{\partial M}{\partial \varepsilon_L} & \dfrac{\partial M}{\partial c} \end{bmatrix} \begin{bmatrix} d\varepsilon_L \\ dc \end{bmatrix}
$$

or $[dF] = [S][d\ ]$

in which [S], the Jacobian relating incremental internal forces with increment deformations of the section, is the required stiffness matrix of the sectional element. Thus from Eq.3.59,

$$\partial P/\partial \varepsilon_L = BE_o T[1-\varepsilon_L/\varepsilon_m -cL/\varepsilon_m +cT/2\varepsilon_m]$$

$$= BE_o T[1-\varepsilon_L/\varepsilon_m +cT/2\varepsilon_m -(L/T)cT/\varepsilon_m\ ]$$

Using the relationship, stated before;

$$\varepsilon_L = \varepsilon_1 -(L/T)\varepsilon_1(1-R) \quad \text{and} \quad c = \varepsilon_1\ (1-R)/T$$

we get

$$\partial P/\partial \varepsilon_L = A_w E_o [1-\varepsilon_1/\varepsilon_m +(L/T)(1-R)\varepsilon_1/\varepsilon_m +(1-R)\varepsilon_1/2\varepsilon_m -(L/T)(1-R)\varepsilon_1/\varepsilon_m]$$

$$= A_w E_o [1-(2-1+R)\varepsilon_1/2\varepsilon_m\ ]$$

$$\therefore \partial P/\partial \varepsilon_L = A_w E_o [1\ -(1-R)\varepsilon_1/2\varepsilon_m] \qquad 3.61$$

Differentiating the expression of P with respect to curvature, c.

$$\partial P/\partial c = BTE_o [L-\ T/2\ -cL^2/\varepsilon_m -L\varepsilon_L/\varepsilon_m -cT^2/3\varepsilon_m +T\varepsilon_L/2\varepsilon_m$$
$$+cLT/\varepsilon_m\ ] \qquad 3.62$$

$$= A_w E_o [(L-T/2)-(L^2/T)(1-R)\varepsilon_1/\varepsilon_m -L\varepsilon_1/\varepsilon_m +(L^2/T)(1-R)\varepsilon_1/\varepsilon_m$$
$$-T(1-R)\varepsilon_1/3\varepsilon_m +T\varepsilon_1/2\varepsilon_m -L(1-R)\varepsilon_1/2\varepsilon_m +L(1-R)\varepsilon_1/\varepsilon_m\ ]$$

$$= A_w E_o [(L-T/2)\ -(L-T/2)\varepsilon_1/\varepsilon_m +(1-R)(L-T/2)\varepsilon_1/2\varepsilon_m$$
$$-T(1-R)\varepsilon_1/12\varepsilon_m\ ]$$

$$= A_w E_o [(L-T/2)\{1\ -\varepsilon_1/\varepsilon_m +(1-R)\varepsilon_1/2\varepsilon_m\}\ -T(1-R)\varepsilon_1/12\varepsilon_m\ ]$$

$$=-(E_o A_w T/12)(1-R)\varepsilon_1/\varepsilon_m +(L-T/2)A_w E_o [1-\{1-(1-R)/2\}\varepsilon_1/\varepsilon_m]$$

$$=-(E_o I_w/T)[(1-R)\varepsilon_1/\varepsilon_m]+(L-T/2)A_w E_o [1-\{(1+R)/2\}\varepsilon_1/\varepsilon_m] \qquad 3.63$$

The expression of $\partial P/\partial \varepsilon_L$ and $\partial P/\partial c$ in Eqs.3.61 and 3.63 respectively can be found to be the same as obtained previously in Eqs.3.16 and 3.17. Similarly, differentiating the expression of M in Eq.3.60, we get

$$\partial M/\partial \varepsilon_L = \partial M_1/\partial \varepsilon_L + \partial M_2/\partial \varepsilon_L$$

$$= BE_o[-T^2 +2TL-3LT\varepsilon_L/\varepsilon_m -3TL^2c/2\varepsilon_m +6T^2\varepsilon_L/4\varepsilon_m +3cLT^2/2\varepsilon_m$$

$$-cT^3/2\varepsilon_m] +BE_o[-TL -cTL^2/2\varepsilon_m +2TL\varepsilon_L/\varepsilon_m +cT^3/6\varepsilon_m$$

$$-T^2\varepsilon_L/\varepsilon_m -cLT^2/2\varepsilon_m +T^2/2]$$

$$= BE_oT[L- T/2 -cL^2/\varepsilon_m -L\varepsilon_L/\varepsilon_m -cT^2/3\varepsilon_m +T\varepsilon_L/2\varepsilon_m$$

$$+cTL/\varepsilon_m] \qquad 3.64$$

The expression is the same as that obtained for $\partial P/\partial c$ in Eq.3.62 and hence the final expression would be the same as in Eq.3.63 which is the same as that for corresponding term of stiffness S_{21} found earlier in Eq.3.17. Finally differentiating M with respect to curvature,c.

$$\partial M/\partial c = \partial M_1/\partial c + \partial M_2/\partial c$$

$$= BE_o[T^3/3 -T^2L +TL^2 -3TL^2\varepsilon_L/2\varepsilon_m -cTL^3/\varepsilon_m +6T^2cL^2/4\varepsilon_m$$

$$+3LT^2\varepsilon_L/2\varepsilon_m -T^3\varepsilon_L/2\varepsilon_m -cLT^3/\varepsilon_m +cT^4/4\varepsilon_m]$$

$$+BE_o[TL^2\varepsilon_L/2\varepsilon_m +T^3\varepsilon_L/6\varepsilon_m -LT^2\varepsilon_L/2\varepsilon_m]$$

$$= BE_o[T^3/3 -LT^2 +TL^2 -TL^2\varepsilon_L/\varepsilon_m -TL^3c/\varepsilon_m +3cT^2L^2/2\varepsilon_m$$

$$+LT^2\varepsilon_L/\varepsilon_m -T^3\varepsilon_L/3\varepsilon_m -cLT^3/\varepsilon_m +cT^4/4\varepsilon_m] \qquad 3.65$$

While the corresponding term of stiffness matrix obtained directly is given by Eq.3.26 as

$$S_{22} = E_oI_w[1 -(1+R)\varepsilon_1/2\varepsilon_m] -2(L-T/2)(E_oI_w/T)[(1-R)\varepsilon_1/\varepsilon_m]$$

$$+(L-T/2)^2E_oA_w[1-(1+R)\varepsilon_1/2\varepsilon_m]$$

$$= [S_{22}]^{i} + [S_{22}]^{ii} +[S_{22}]^{iii}$$

Using the reverse relationship(Fig.3.5)

$$\varepsilon_1 = \varepsilon_L + Lc$$

$$R = \varepsilon_2/\varepsilon_1 = \{\varepsilon_L-c(T-L)\}/(\varepsilon_L+cL) = (\varepsilon_L+Lc-Tc)/(\varepsilon_L+cL)$$

We get,

$$[S_{22}]^{i} = E_oI_w[1 -(1+R)\varepsilon_1/2\varepsilon_m]$$

$$= E_oI_w[1-(\varepsilon_L+Lc)\{1+(\varepsilon_L+Lc-Tc)/(\varepsilon_L+Lc)\}/2\varepsilon_m]$$

$$= E_oI_w[1-(\varepsilon_L+Lc)(\varepsilon_L+Lc+\varepsilon_L+Lc-Tc)/2\varepsilon_m(\varepsilon_L+Lc)]$$

$$= (E_oBT^3/12)[1 -\varepsilon_L/\varepsilon_m -Lc/\varepsilon_m +Tc/2\varepsilon_m]$$

$$= BE_o[T^3/12 -T^3\varepsilon_L/12\varepsilon_m -cLT^3/12\varepsilon_m +cT^4/24\varepsilon_m]$$

$$[S_{22}]^{ii} = -2(L-T/2)(E_oI_w/T)[(1-R)\varepsilon_1/\varepsilon_m]$$

$$= -2(L-T/2)(E_oBT^3/12T)[\{(1-(\varepsilon_L+Lc-Tc)/(\varepsilon_L+Lc)\}(\varepsilon_L+Lc)/\varepsilon_m]$$

$$= E_oB(-T^2L/6 +T^3/12)[(\varepsilon_L+Lc-\varepsilon_L-Lc+Tc)(\varepsilon_L+Lc)/(\varepsilon_L+Lc)\varepsilon_m]$$

$$= E_oB(T^3/12 -T^2L/6)Tc/\varepsilon_m$$

$$= E_oB[T^4c/12\varepsilon_m -cLT^3/6\varepsilon_m]$$

$$[S_{22}]^{iii} = (L-T/2)2E_oA_w[1-(1+R)\varepsilon_1/2\varepsilon_m]$$

$$= BE_oT(L^2-LT+T^2/2)[1-\varepsilon_L/\varepsilon_m -Lc/\varepsilon_m +Tc/2\varepsilon_m]$$

$$= BE_o(TL^2-T^2L+T^3/2)(1-\varepsilon_L/\varepsilon_m -Lc/\varepsilon_m +Tc/2\varepsilon_m]$$

$$= BE_o[TL^2-TL^2\varepsilon_L/\varepsilon_m -TL^3c/\varepsilon_m +T^2cL^2/2\varepsilon_m -LT^2 +LT^2\varepsilon_L/\varepsilon_m$$
$$+L^2cT^2/\varepsilon_m -cLT^3/2\varepsilon_m +T^3/4 -T^3\varepsilon_L/4\varepsilon_m -cLT^3/4\varepsilon_m +cT^4/8\varepsilon_m]$$

Therefore, $S_{22} = [S_{22}]^{i} + [S_{22}]^{ii} + [S_{22}]^{iii}$

$$= BE_o[T^3/12 -T^3\varepsilon_L/12\varepsilon_m -cLT^3/12\varepsilon_m +cT^4/24\varepsilon_m +cT^4/12\varepsilon_m$$
$$-cLT^3/6\varepsilon_m +TL^2 -TL^2\varepsilon_L/\varepsilon_m -cTL^3/\varepsilon_m +cL^2T^2/2\varepsilon_m -LT^2$$
$$+LT^2\varepsilon_L/\varepsilon_m +cL^2T^2/\varepsilon_m -cLT^3/2\varepsilon_m +T^3/4 -T^3\varepsilon_L/4\varepsilon_m -cLT^3/4\varepsilon_m$$
$$+cT^4/8\varepsilon_m]$$

$$= BE_o[T^3/3 -T^3\varepsilon_L/3\varepsilon_m -cLT^3/\varepsilon_m +TL^2 -TL^2\varepsilon_L/\varepsilon_m -cTL^3/\varepsilon_m$$
$$+3cL^2T^2/2\varepsilon_m -LT^2 +LT^2\varepsilon_L/\varepsilon_m +cT^4/4\varepsilon_m]$$

Which can be seen to be the same as that of $\partial M/\partial c$ in Eq.3.65.

It is therefore proved that the alternative approach of using differential calculus can also be used in obtaining the stiffness terms for a sectional element.

3.13 Variation of Stiffness Terms

The stiffness terms derived in previous articles can be observed to be a function of the geometric properties, the state of strain in the sectional element and the location of the geometric centroid of the whole section.

The variation of these terms for web element are presented in Figs.3.12 to 3.18 in their non-dimensional form as a function of extreme fibre strain.

The stiffness terms obtained for the top flange element can be seen to be the same as those for web element except the terms presenting sectional properties. Thus the curves presented are equally applicable for top flange element. The terms for bottom flange element are quite different from those for other two elements. The variation of these terms are, however, not presented.

Two sets of curves are presented showing the variation of stiffness for uncracked and cracked web element. The curves for variation of stiffness terms due to reinforcing steel are also presented. Each figure contains three sets of curves for three different ratios of the depth of the centriodal axis to the total depth of the whole section(L/T). The curves show linear variation of stiffness terms with strain. The curves at L/T=0.5 can be considered for the rectangular section. It is worth mentioning that at zero strain the diagonal terms $S_{11}=E_o A_o$ and $S_{22}=E_o I_o$ while the off-diagonal terms are zero. These values of stiffnesses are well-known in linear elastic analysis of structures.

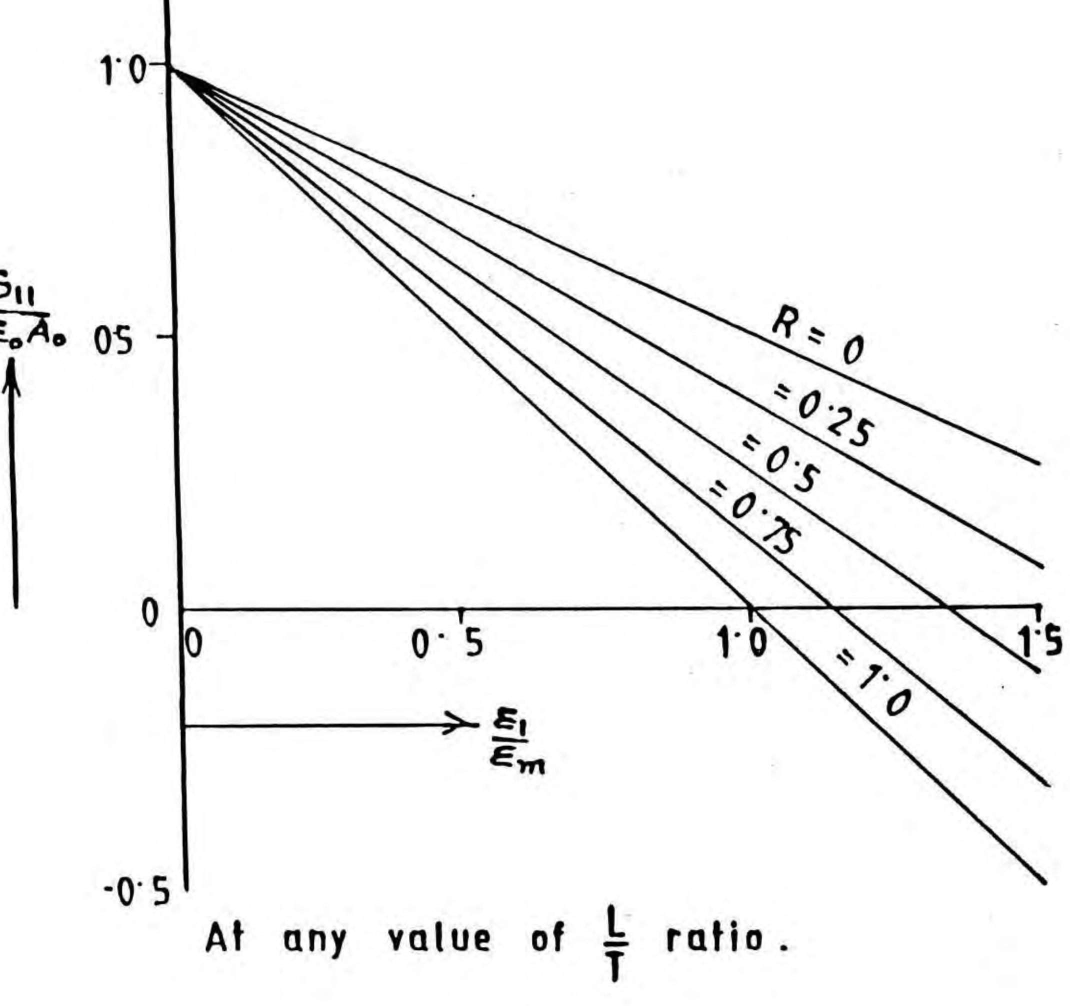


Figure 3.12 Variation of S_{11} term with strain for uncracked web element.

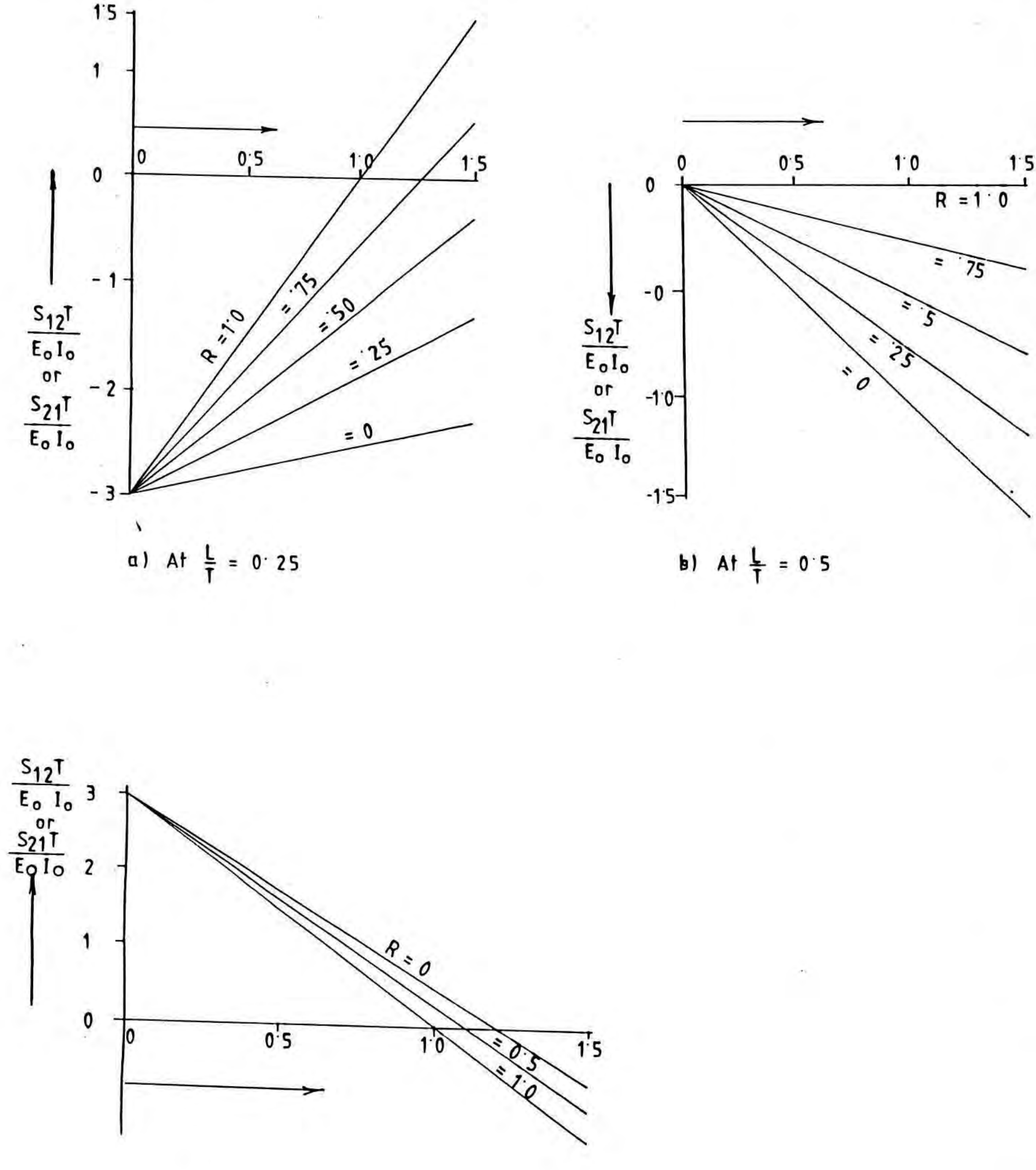


Figure 3.13 Variation of S_{12} and S_{21} terms with strain for uncracked web element.

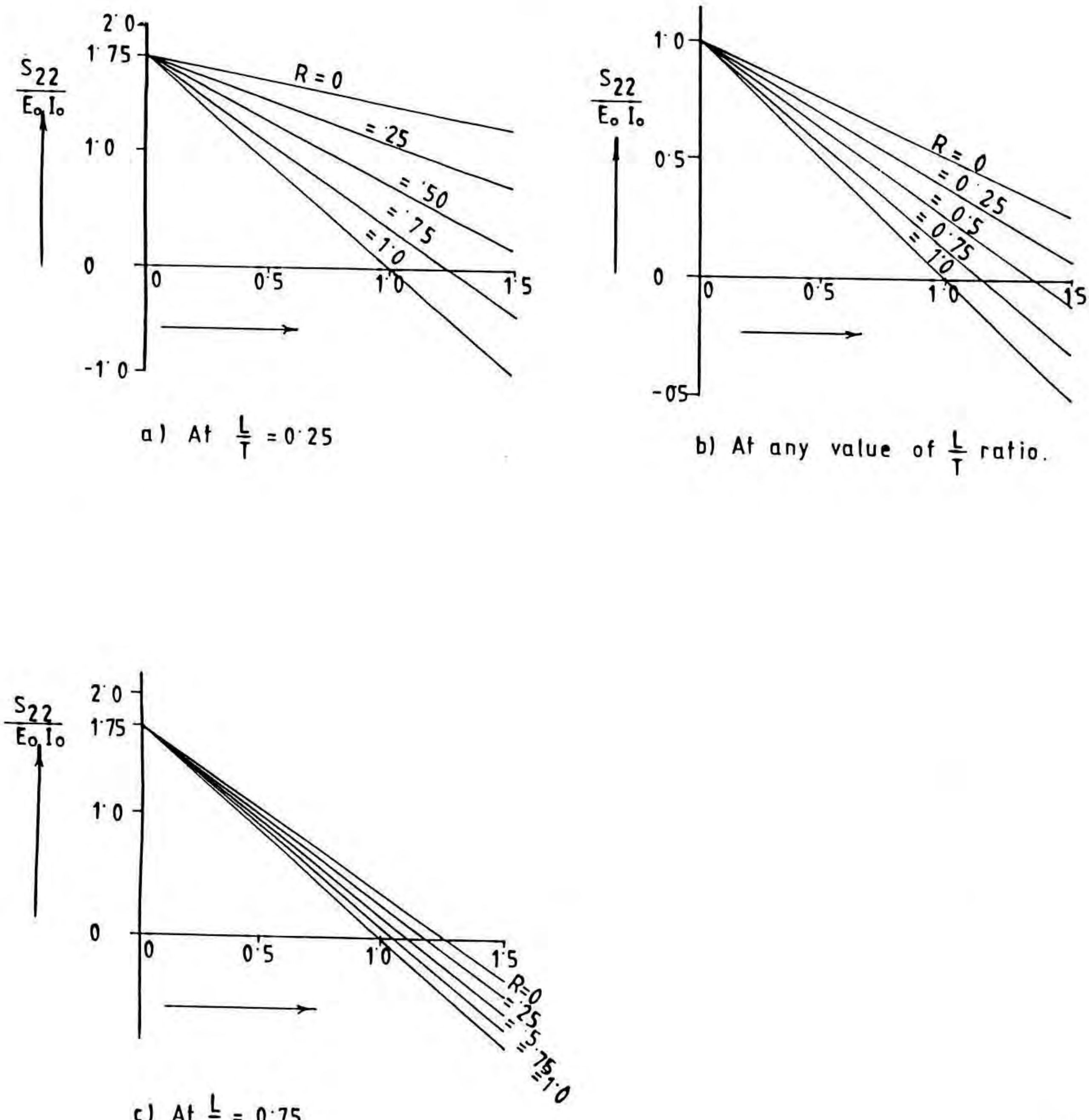
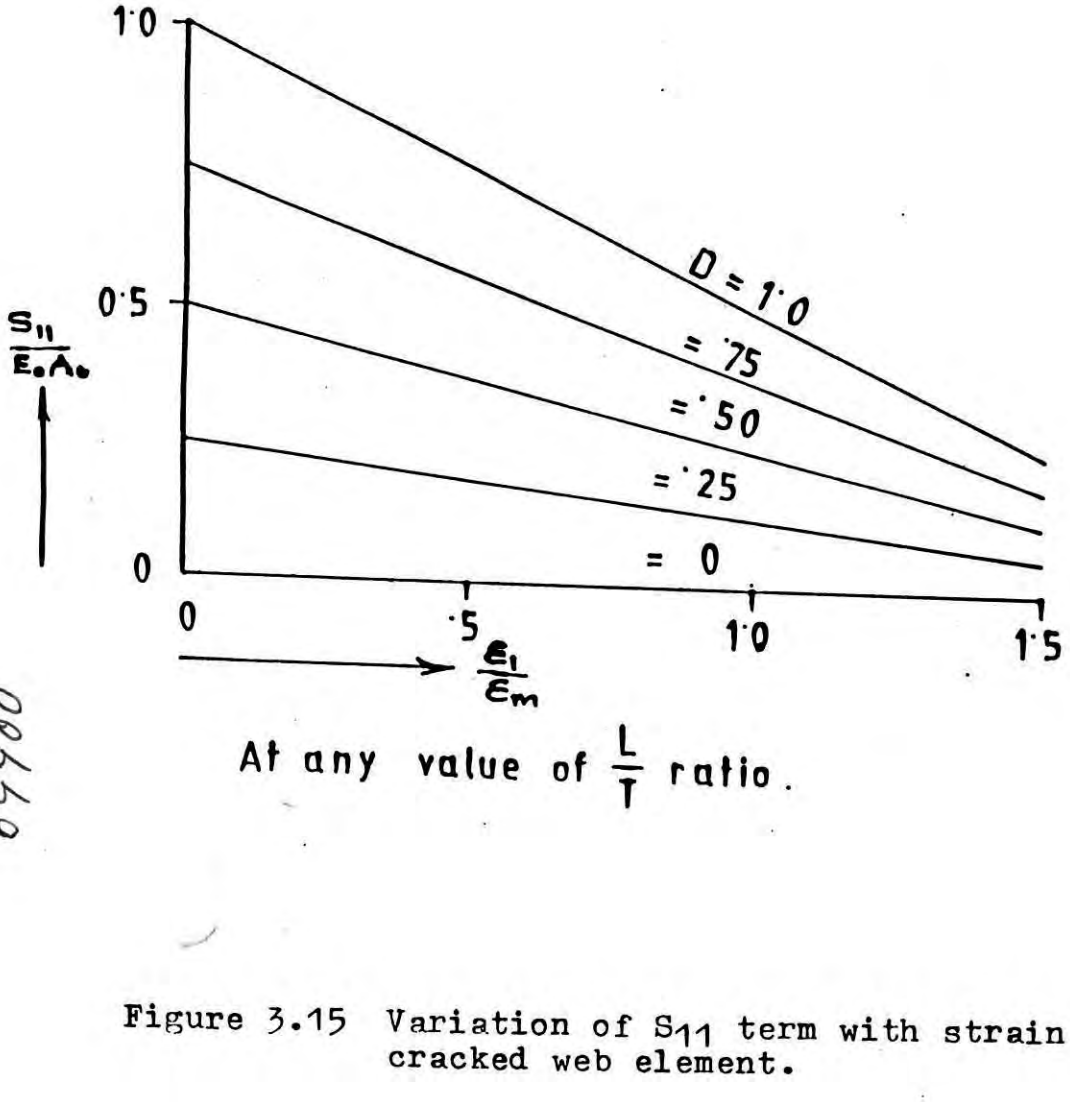


Figure 3.14 Variation of S_{22} term with strain for uncracked web element.



69900

Figure 3.15 Variation of S_{11} term with strain for cracked web element.

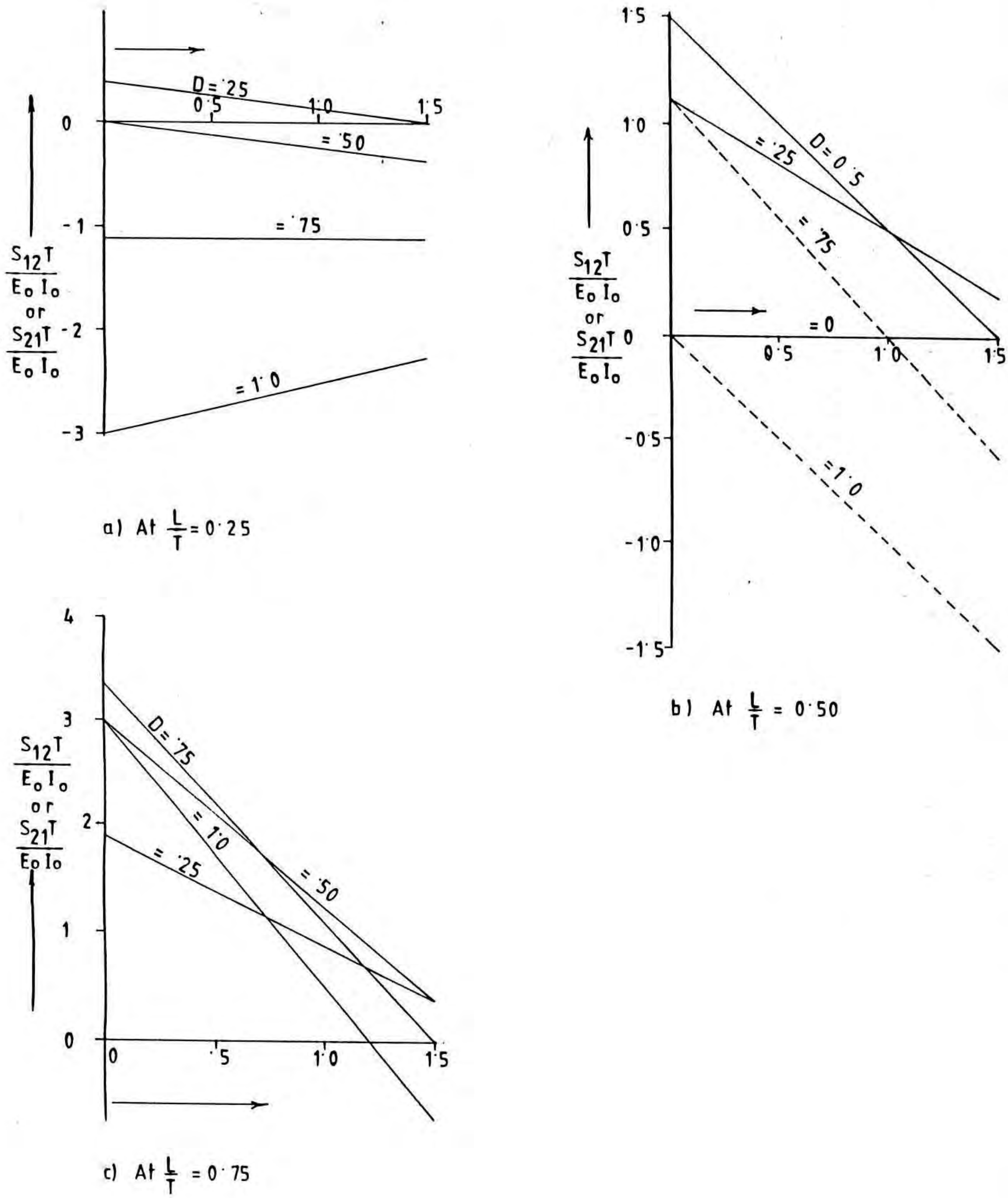


Figure 3.16 Variation of S_{12} and S_{21} terms with strain for cracked web element.

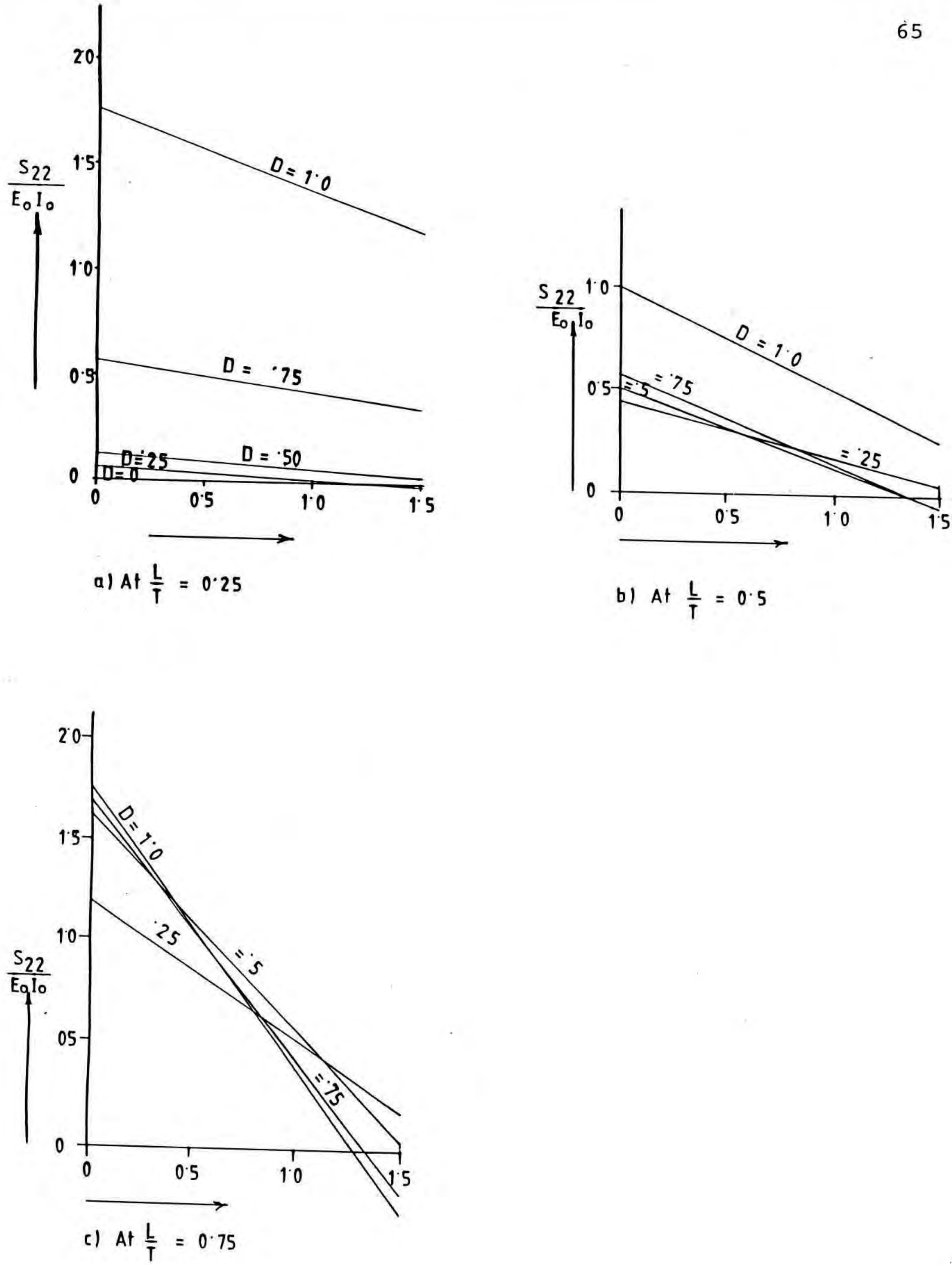


Figure 3.17 Variation of S_{22} term with strain for cracked web element.

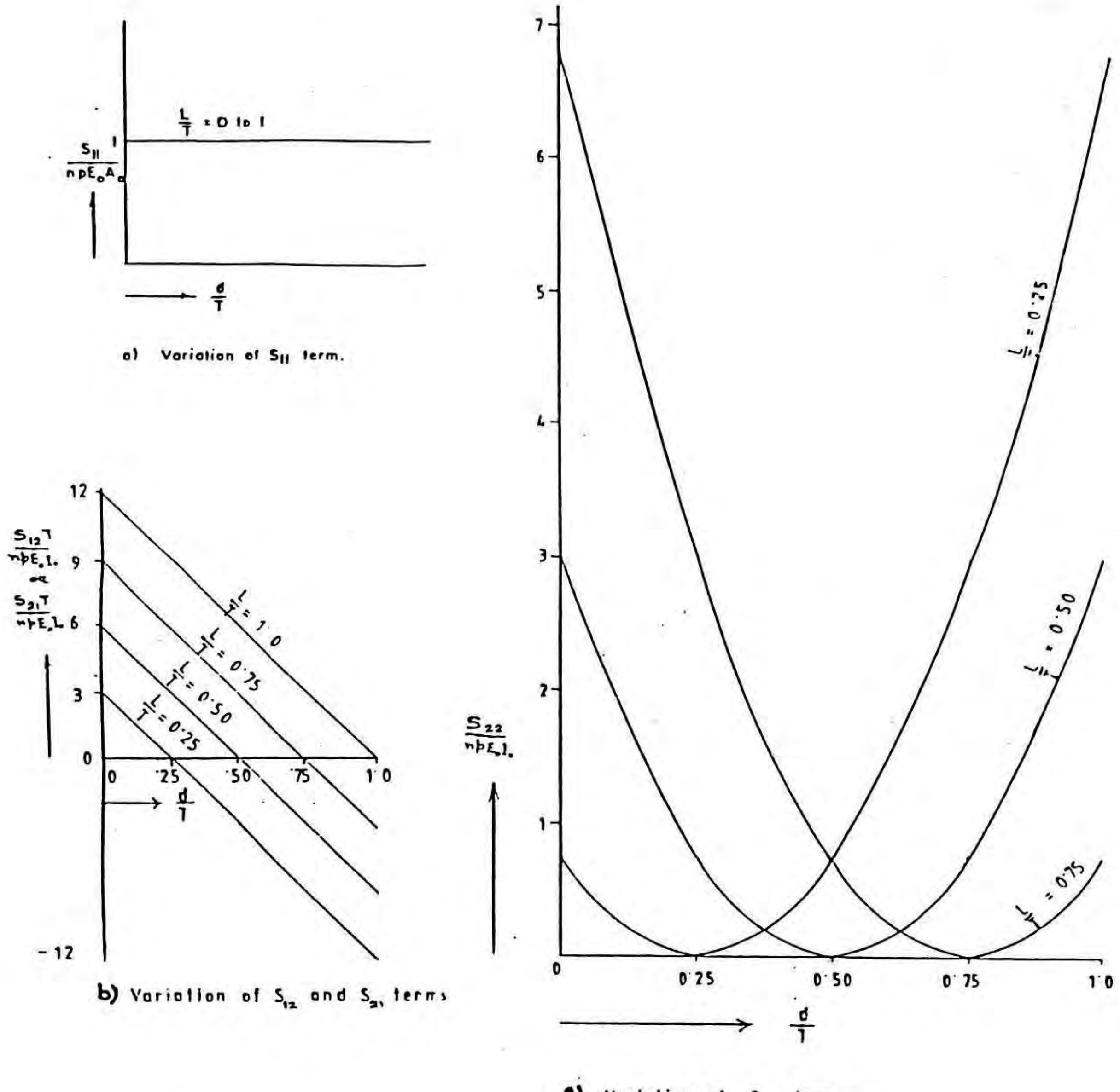


Figure 3.18 Variation of stiffness terms for reinforcing steel with d/T ratio.

3.14 Stiffness of a Section

Stiffness matrix for different sectional elements are developed and presented in the previous articles. Stiffness matrix for a section of reinforced or prestressed concrete can be obtained by matrix addition of the stiffness matrices of corresponding sectional elements and reinforcement present in the section. Thus for I, T and rectangular section, the stiffness can be written as follows,

For I-section, $[S] = [S]^{t} + [S]^{w} + [S]^{b} + [S]^{P/R}$

For T-section, $[S] = [S]^{t} + [S]^{w} + [S]^{P/R}$

For rectangular section, $[S] = [S]^{w} + [S]^{P/R}$

where, t, w, b, P and R represents the top flange, web, bottom flange, prestressing and reinforcing steel, respectively. since stiffness matrices for all the sectional elements are (2x2), the complete stiffness of a section will be a (2x2) matrix and can be written as,

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

It is to be noted that for deriving the stiffness matrices for different sectional elements it is tacitly assumed that the curvature of the section is positive(Fig.3.19b), or in other words the strain at the top most fibre is greater than at the bottom most fibre. For sections, where opposite situation arises as shown in Fig.3.19a, it is not possible to obtain the stiffness matrix directly from the derivations as presented before. The section is to be inverted upside down so that the curvature appears to be positive for the inverted section. The stiffness of this inverted section can be determined using the derivations presented. The stiffness thus obtained is the desired matrix but the off-diagonal terms would be multiplied by(-1) for converting the stiffness terms with respect to the positive sign convention as shown in Fig.3.19a.

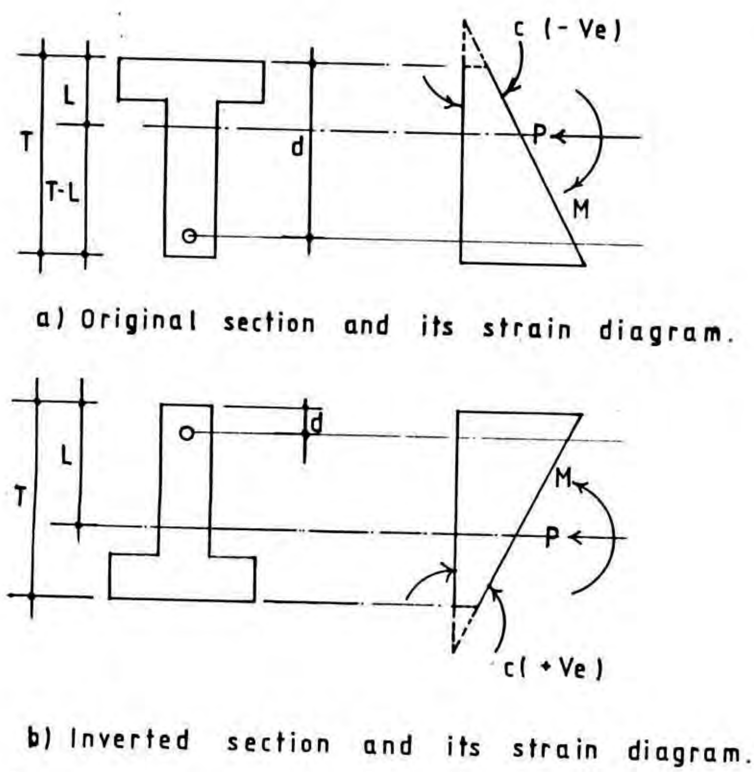


Figure 3.19 Section having +ve/-ve curvature.

Thus the stiffness matrix for the T-section shown in Fig.3.19a, is

$$[S] = \begin{bmatrix} S_{11} & -S_{12} \\ -S_{21} & S_{22} \end{bmatrix}$$

where S_{11}, S_{12}, S_{21} and S_{22} are the stiffness terms for the inverted T section in Fig.3.19b, given as

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

3.15 Stiffness of a Beam-column Element

To utilize the powerful techniques of finite element in structural analysis the first step is descretization, that is, the division of a structural continua into finite elements. Elements can be of different shape, size and even of different class. The next step is to determine the stiffness matrix of different types of element.

Reinforced and prestressed concrete flexural member can be conveniently represented by the conventional two noded beam-column element, having three degrees of freedom at each node(Fig.3.20).

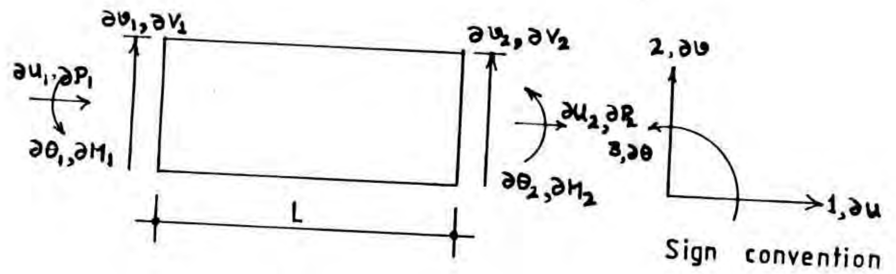


Figure 3.20 Increment in nodal forces and displacements for an incremental load in a beam-column element.

It is mentioned in the previous chapter that the strain distribution varies over the length of a member and so the stiffness properties. However, using a large number of small elements the variation of stiffness properties over the length can be minimized and stiffness properties corresponding to the middle section of the element can be considered to be that of the element. This simplification makes it possible to derive the stiffness properties analytically.

The approach has been utilised in developing a numerical model for limit state analysis of brickwork structures. The complete derivation of the stiffness of the beam-column element is presented in Ref.12. The 6x6 stiffness matrix for the element in Fig.3.20 can be written as

$$\begin{bmatrix} \delta P_1 \\ \delta V_1 \\ \delta M_1 \\ \delta P_2 \\ \delta V_2 \\ \delta M_2 \end{bmatrix} = \begin{bmatrix} S_{11}/L & 0 & -S_{21}/L & -S_{21}/L & 0 & S_{12}/L \\ 0 & 12S_{22}/L^3 & 6S_{22}/L^2 & 0 & -12S_{22}/L^3 & 6S_{22}/L^2 \\ -S_{21}/L & 6S_{22}/L^2 & 4S_{22}/L & S21/L & -6S_{22}/L^2 & 2S_{22}/L \\ -S_{11}/L & 0 & S_{12}/L & S_{11}/L & 0 & -S_{12}/L \\ 0 & -12S_{22}/L^3 & -6S_{22}/L^2 & 0 & 12S_{22}/L^3 & -6S_{22}/L^2 \\ S_{21}/L & 6S_{22}/L^2 & 2S_{22}/L & -S_{21}/L & -6S_{22}/L^2 & 4S_{22}/L \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta v_1 \\ \delta \theta_1 \\ \delta u_2 \\ \delta v_2 \\ \delta \theta_2 \end{bmatrix}$$

Where S_{11}, S_{12}, S_{21}, and S_{22} are the stiffness terms corresponding to the sectional properties and state of strain at the middle of the element.

Thus stiffness matrices can be developed corresponding to a particular strain level at the mid-section of the element. As the derivation is based on the properties at the mid-section of the element, the whole element is to be considered cracked if the mid-section is cracked, otherwise it is to be considered as uncracked.

CHAPTER 4

MOMENT CURVATURE RELATIONSHIP AND DISCUSSION

4.1 Introduction

Load-deflection behaviour of concrete members, as indicated in Fig.4.1, upto and beyond yielding is important in understanding the possible modes of failure. In general, brittle failure of member is not desirable, rather, it should undergo a large deflection at the maximum load. In the other words, the concrete members should be ductile, so that, there is an ample warning of failure and sudden collapse. The ductile property of the members at the critical sections is important for considering the possible distributions of bending moment, shear force and axial load in the design of statically indeterminate structures.

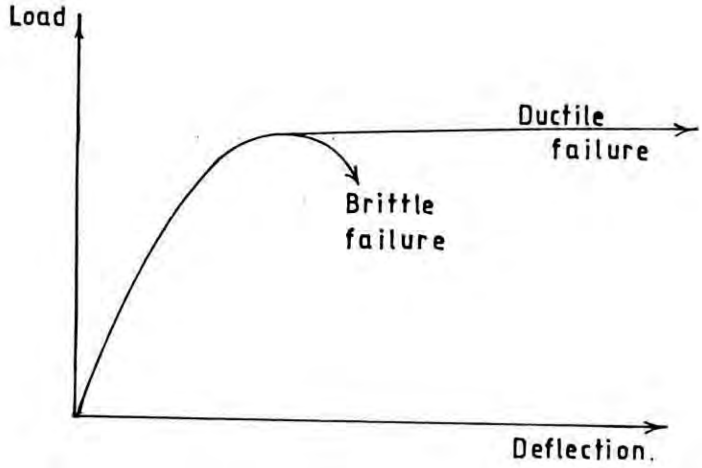


Figure 4.1 Load-deflection behaviour of a flexural member.

Such load-deformation characteristics are mainly dependent on the moment-curvature characteristics of sections since most of the deformations of members of normal proportions arise from strains associated with flexure. In this chapter the moment-curvature characteristics of singly reinforced rectangular section under varying axial load are presented and compared to those present in Ref.11.

Theoretical moment-curvature curves for reinforced concrete sections with flexure and axial load can be derived on the basis of following assumptions:

i) strain distribution in a section is linear i.e. plane sections before bending remain plane after bending.

ii) stress-strain curves for concrete is parabolic(Eq.2.4) and for steel is elasto-plastic. The assumptions as stated earlier, are similar to those used for the determination of flexural strength of sections.

The moment-curvature relationship for a section under a given axial load is determined by taking different increment of concrete strain at the extreme compression fibre of a section. For each value of such strain, strain distribution in the section is obtained satisfying the force equilibrium in the section. The strain distribution so found is then used to determine the moment and curvature corresponding to that value of extreme compression fibre strain.

4.2 Equilibrium of Axial Forces and Selection of Strain Distribution

The general state of strain in uncracked and cracked rectangular section(web element) is presented earlier in Fig.3.5 and 3.6 respectively. For such a section the centroidal axis will be at the middle of the section ignoring the shift due to steel, i.e. L=T/2.

For rectangular sections with general state of strain distribution the axial force(P_c) and bending moment(M_c) carried by concrete only is given as follows(12, 13).

For uncracked section:

$$P_c\,(\text{compressive}) = E_o A_o \varepsilon_m [3(1+R)\varepsilon - (1+R+R^2)\varepsilon^2]/6 \qquad 4.1$$

and $$M_c(\circlearrowright) = E_o I_o \varepsilon \varepsilon_m \;(1-R)[1-\varepsilon\,(1+R)/2]/T \qquad 4.2$$

and when the section is cracked,

$$P_c\,(\text{compressive}) = E_o A_o \varepsilon_m D(3\varepsilon - \varepsilon^2)/6 \qquad 4.3$$

and $$M_c(\circlearrowright) = E_o I_o \varepsilon_m [2D(3\varepsilon - \varepsilon^2) + D^2(\varepsilon^2 - 4\varepsilon)]/2T \qquad 4.4$$

Where, $\varepsilon = \varepsilon_1/\varepsilon_m$ and ε_1 is the extreme compressive fibre strain as indicated in the figures.

From the strain distribution in an uncracked section (Fig.3.5) the strain in steel, by geometry, is

$$\varepsilon_s = \varepsilon_1 - cd \;(\text{compressive})$$

and hence the axial force (P_s) and bending moment (M_s) carried by the steel, if there is no yielding, is

$$P_s\,(\text{compressive}) = A_s E_s \varepsilon_s = A_s E_s (\varepsilon_1 - cd)$$

and $M_s(\circlearrowright) = -P_s(d-T/2)$

Dividing by $E_o A_o$ and replacing c by Eq.3.5, we get

$$P_s = (E_s A_s / E_o A_o)[\varepsilon_1 - d\varepsilon_1(1-R)/T] E_o A_o$$

or $$P_s\,(\text{compressive}) = npE_o A_o \varepsilon_1 [1-d(1-R)/T] \qquad 4.5$$

and $$M_s = -P_s(d-T/2) \qquad 4.6$$

where, n=modular ratio(E_s/E_o) and p=the steel ratio, (A_s/A_o).

Similarly in a cracked section(Fig.3.6) the strain in steel is given by $\varepsilon_s(\text{tensile}) = (d-DT)c$

and hence axial force and moment carried by steel are

$$P_s\,(\text{tensile}) = A_s E_s \varepsilon_s = A_s E_s c(d-DT)$$

and $M_s(\circlearrowright) = P_s(d-T/2)$

As before, dividing and multiplying by E_oA_o and replacing c by ε_1/DT, we get

$$P_s(\text{tensile}) = npE_oA_o(d-DT)\varepsilon_1/DT = npE_oA_o(d/DT-1)\varepsilon_1 \quad 4.7$$

and $M_s(\circlearrowright) = P_s(d-T/2)$ 4.8

when steel starts yielding($\varepsilon_s \geqslant \varepsilon_y$) the axial force and moment due to steel in uncracked section is given as

$$P_s(\text{compressive}) = E_sA_s\varepsilon_y = npE_oA_o\varepsilon_y \quad 4.9$$

and $M_s(\circlearrowright) = -P_s(d-T/2)$ 4.10

In cracked section

$$P_s(\text{tensile}) = E_sA_s\varepsilon_y = npE_oA_o\varepsilon_y \quad 4.11$$

and $M_s(\circlearrowright) = P_s(d-T/2)$ 4.12

In an uncracked section the equilibrium of axial forces can be written as $P = P_c + P_s$ 4.13

where, P=axial force applied in the section by external load (compressive). P_s and P_c are the internal resisting axial force (compressive) due to steel and concrete respectively. Replacing P_c and P_s by Eqs.4.1 and 4.5 assuming steel is not yielding, we get, $E_oA_o\varepsilon_m[3\varepsilon(1+R)-\varepsilon^2(1+R+R^2)]/6 + npE_oA_o\varepsilon_1[1-d(1-R)/T] = P$

where $\varepsilon = \varepsilon_1/\varepsilon_m$, $P=A_o\sigma$ and σ=average stress in the section due to externally applied axial force(=P/A_o).

Dividing both sides by $E_oA_o\varepsilon_m/6$ and replacing E_o by $2\sigma_m/\varepsilon_m$

$$3\varepsilon(1-R) - \varepsilon^2(1+R+R^2) + 6np\varepsilon(1-d/T+dR/T)$$

$$=6\sigma/E_o\varepsilon_m = 6\sigma/2\sigma_m = 3\sigma/\sigma_m$$

Rearranging the terms, we get

$$\varepsilon^2R^2 + R(\varepsilon^2-3\varepsilon-6np\varepsilon d/T) + 3\sigma/\sigma_m - 3\varepsilon + \varepsilon^2 - 6np\varepsilon(1-d/T) = 0$$

which is a quadratic equation in R and hence

$$R_{1/2} = [-b \pm \sqrt{(b^2-4ae)}]/2a \qquad 4.14$$

where $a = \varepsilon^2$, $b = \varepsilon^2 - 3\varepsilon - 6np\,\varepsilon\, d/T$

and $e = 3\sigma/\sigma_m - 3\varepsilon + \varepsilon^2 - 6np\,\varepsilon\,(1-d/T)$

When the steel is yielding, substituting P_c and P_s from Eqs.4.1 and 4.9 in Eq.4.13, we get

$$E_o A_o \varepsilon_m [3\varepsilon(1+R) - (1+R+R^2)\varepsilon^2]/6 + np\varepsilon_y E_o A_o = P = \sigma A_o$$

Dividing both sides by $E_o A_o \varepsilon_m/6$ and substituting $E_o = 2\sigma_m/\varepsilon_m$, we get,

$$3\varepsilon(1+R) - (1+R+R^2)\varepsilon^2 + 6np\varepsilon_y/\varepsilon_m = 6\sigma/E_o\varepsilon_m = 3\sigma/\sigma_m$$

Rearranging the terms, we get,

$$\varepsilon^2 R^2 + R(-3\varepsilon + \varepsilon^2) - 3\varepsilon + \varepsilon^2 - 6np\varepsilon_y/\varepsilon_m + 3\sigma/\sigma_m = 0$$

which is again a quadratic equation in R and hence

$$R_{3/4} = [-b \pm \sqrt{(b^2-4ae)}]/2a \qquad 4.15$$

where $a = \varepsilon^2$, $b = \varepsilon^2 - 3\varepsilon$ and $e = -3\varepsilon + \varepsilon^2 - 6np\varepsilon_y/\varepsilon_m + 3\sigma/\sigma_m$

From the solution of the equilibrium equation of axial forces it appears that if the section remains uncracked it is possible to get four sets of strain distributions(R_1, R_2, R_3 and R_4) corresponding to any particular value of extreme compressive fibre strain(ε_1).

Before starting the selection of correct strain distribution, possible distributions assuming the section is cracked are essential.

In a cracked section steel is always in tension and hence the equilibrium equation can be written as $P = P_c - P_s$. Assuming the steel is not yielding from Eqs.4.3 and 4.7, the equilibrium equation becomes

$$E_o A_o \varepsilon_m D(3\varepsilon - \varepsilon^2)/6 - npE_o A_o \varepsilon_1 [d/DT-1] = P = A_o \sigma$$

Dividing both sides by $E_o A_o \varepsilon_m/6$ and substituting $E_o = 2\sigma_m/\varepsilon_m$, as before, we get

$$D(3\varepsilon - \varepsilon^2) - 6np\varepsilon(d/DT-1) = 3\sigma/\sigma_m$$

or $D^2\varepsilon(3-\varepsilon) - 6np\varepsilon d/T + 6np\varepsilon D - 3D\sigma/\sigma_m = 0$

or $D^2(3-\varepsilon) - 6npd/T + D(6np-3\sigma/\sigma_m\varepsilon) = 0$

or $D^2(3-\varepsilon)/6np + D(1-\sigma/\sigma_m 2np\varepsilon) - d/T = 0$

which is a quadratic equation in D and hence

$$D_{1/2} = [-b \pm \sqrt{(b^2-4ae)}/2a \tag{4.16}$$

Where $a = (3-\varepsilon)/6np$, $b = 1-(\sigma/\sigma_m)/2np\varepsilon$ and $e = -d/T$

When yielding takes place in steel, using Eqs.4.3 and 4.11 the equilibrium equation of axial forces becomes as

$$E_o A_o \varepsilon_m D(3\varepsilon - \varepsilon^2)/6 - np\varepsilon_y E_o A_o = A_o \sigma$$

Similarly as before, we get

$$D(3\varepsilon - \varepsilon^2) - 6np\varepsilon_y/\varepsilon_m = 3\sigma/\sigma_m$$

$$D_3 = (6np\varepsilon_y/\varepsilon_m + 3\sigma/\sigma_m)/(3\varepsilon - \varepsilon^2) \tag{4.17}$$

Therefore, assuming the section is cracked, three more strain distributions(D_1, D_2 and D_3) can be obtained. Thus seven possible strain distributions are obtained corresponding to each value of extreme fibre strain of the section.

Preliminary selection of consistent strain distribution(s) is done from the basic characteristics of strain distributions that is the value of R and D must lie within 0 and 1.

The strain distribution(s) thus selected are then used to calculate the value of axial forces carried by concrete(P_c) and steel(P_s) and also the equilibrium of axial forces is checked. The particular strain distribution which satisfies the corresponding equilibrium equation is the appropriate one.

The solution of appropriate strain distribution can also be obtained graphically as presented in Fig.4.2. In the figure, the calculated value of total axial load($P_c \pm P_s$) is plotted corresponding to different assumed strain distributions. The figure can be used for any singly reinforced rectangular section having tension steel ratio of 2.5%. All the points correspond to a particular value of extreme fibre strain($\varepsilon_1/\varepsilon_m$) are joined together and thus several curves for different values of that strain are plotted. Similar curves for other steel ratios can also be produced.

The horizontal dotted line, in Fig.4.2 represents the case of the section having an axial stress ratio of 0.4 ($P/A_o\sigma_m = \sigma/\sigma_m = 0.4$). For such a section, it appears that:

i) there is no consistent strain distribution for extreme fibre strain ratio($\varepsilon_1/\varepsilon_m$) upto around 0.2,

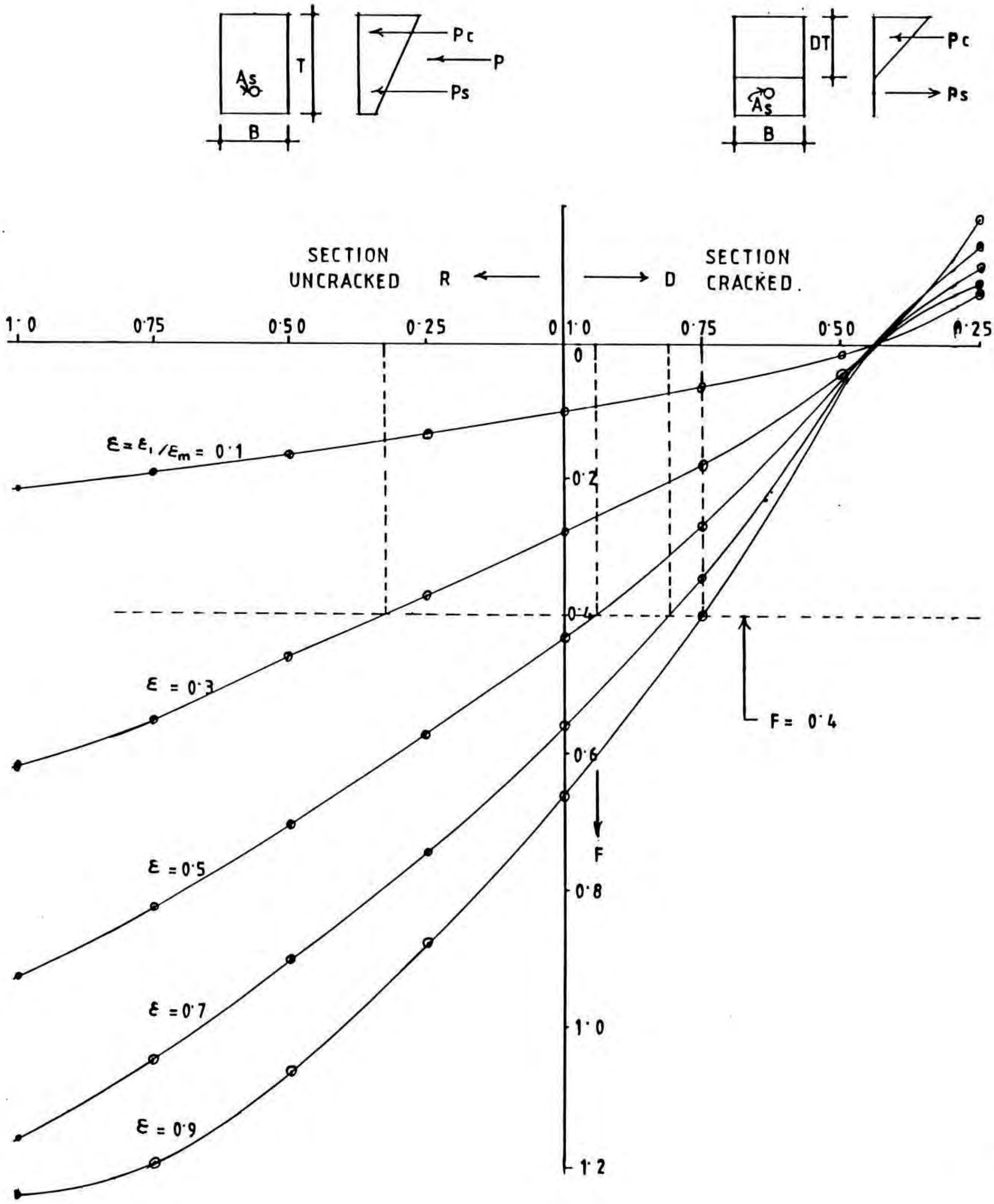


Figure: 4.2 : F Vs R and D curve of a section for various values of ε (p = 2·5%).

ii) for the extreme fibre strain ratio of 0.3 the section remains uncracked and the value of R is around 0.325,

iii) the section becomes cracked when the extreme fibre strain ratio is 0.46, and

iv) with further increment of strain the section gets further cracking indicated by the reducing value of D.

Thus the figure is also helpful in understanding the cracking behaviour of a section while maintains a constant axial load. The figure also indicates that for higher value of axial load the section might not be cracked. It can also be observed that when there is no axial load($F=\sigma/\sigma_m=0$) the section is always cracked and value of D slightly varies around 0.44 which is well known in elastic design of singly reinforced rectangular concrete sections.

4.3 Moment-Curvature Relationship

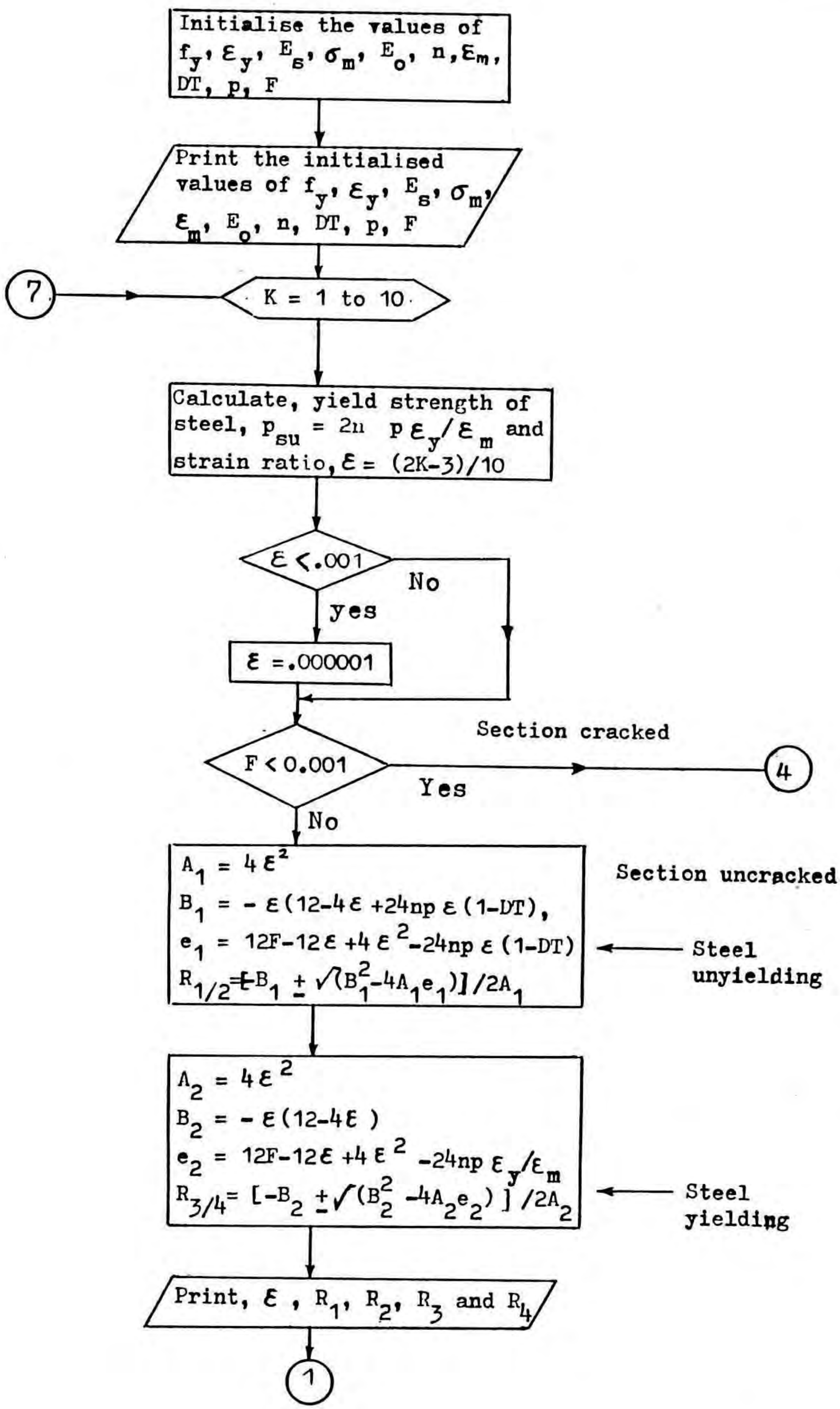
The appropriate strain distribution, as obtained in the last article, can be used to calculate the curvature and bending moment carried by concrete(M_c) and by steel(M_s) in the section using the relevant equations presented in the last article. Total bending moment(M) present in the section is calculated by algebraic summation of M_c and M_s.

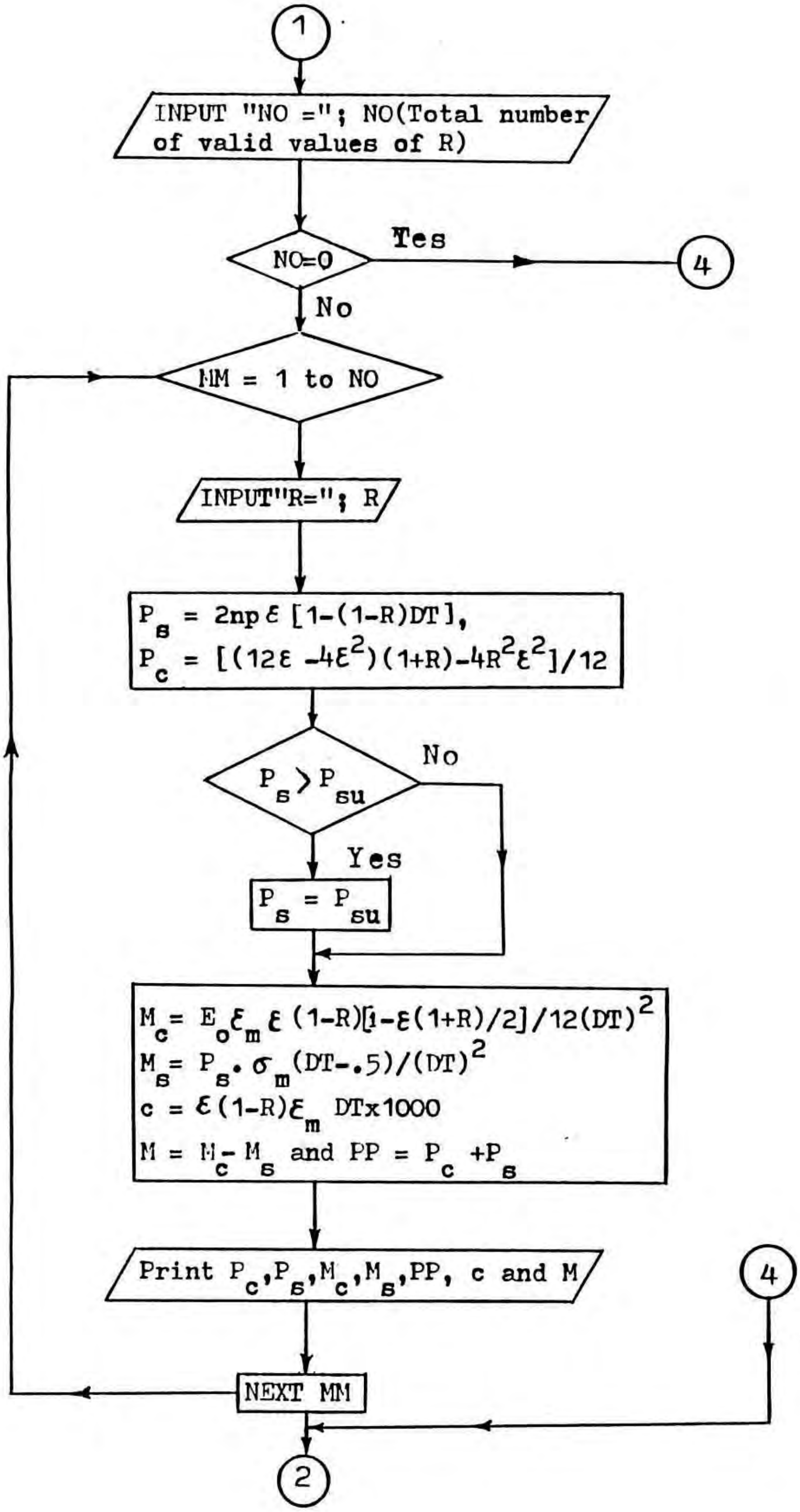
The above procedure is repeated for each value of the extreme compressive fibre strain. The solution of the corresponding strain distribution and the calculated values of moment and curvature give just one point of the moment-curvature curve. Repetitions of the procedure for other values of the extreme fibre strain give several other points of the curve. Plotting of all these points gives a moment curvature curve for the section under a given axial load.

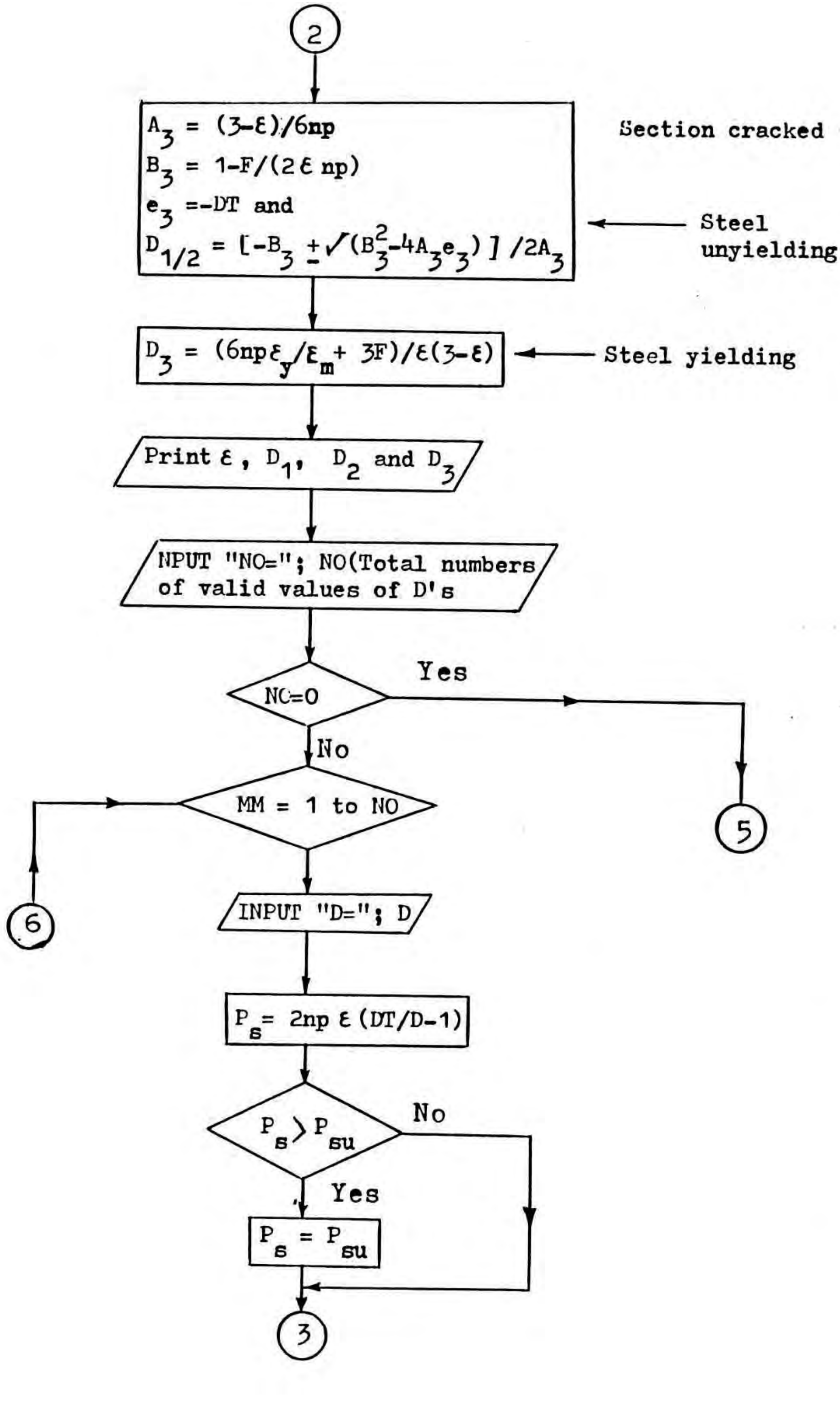
The procedure presented above is lengthy for hand calculation but can be conveniently used by using a computer. An interactive computer programme in BASIC, therefore, was developed. The flowchart of the programme is presented in Fig.4.3. The programme was run on IBM PC microcomputer.

Moment-curvature curves thus obtained for rectangular singly reinforced concrete section are shown in Figs.4.4, 4.5 and 4.6. The figures presented are suitable for rectangular section having material properties as depicted in the Fig.4.5 and with a tension steel ratio of either 1.25% or 2.50% or 3.75%. Three separate figures are presented for three different steel ratios. Each figure contains several curves correspond to different magnitude of axial load ratios($P/P_m = \sigma A_o / \sigma_m A_o = F$).

The figures clearly indicate that the section remains more ductile when there is zero or little axial load and it becomes


Initialise the values of f_y, ε_y, E_s, σ_m, E_o, n, ε_m, DT, p, F
Print the initialised values of f_y, ε_y, E_s, σ_m, ε_m, E_o, n, DT, p, F
7
K = 1 to 10
Calculate, yield strength of steel, $p_{su} = 2n\ p\,\varepsilon_y/\varepsilon_m$ and strain ratio, $\varepsilon = (2K-3)/10$
$\varepsilon < .001$
No
yes
$\varepsilon = .000001$
Section cracked
$F < 0.001$
Yes
4
No
$A_1 = 4\varepsilon^2$
$B_1 = -\varepsilon(12-4\varepsilon+24np\,\varepsilon(1-DT)$,
$e_1 = 12F-12\varepsilon+4\varepsilon^2-24np\,\varepsilon(1-DT)$
$R_{1/2} = [-B_1 \pm \sqrt{(B_1^2-4A_1e_1)}]/2A_1$
Section uncracked
Steel unyielding
$A_2 = 4\varepsilon^2$
$B_2 = -\varepsilon(12-4\varepsilon)$
$e_2 = 12F-12\varepsilon+4\varepsilon^2-24np\,\varepsilon_y/\varepsilon_m$
$R_{3/4} = [-B_2 \pm \sqrt{(B_2^2-4A_2e_2)}]/2A_2$
Steel yielding
Print, ε, R_1, R_2, R_3 and R_4
1


1
INPUT "NO ="; NO(Total number of valid values of R)
NO=0
Tes
4
No
MM = 1 to NO
INPUT"R="; R
$P_s = 2np\varepsilon\,[1-(1-R)DT]$,
$P_c = [(12\varepsilon - 4\varepsilon^2)(1+R) - 4R^2\varepsilon^2]/12$
$P_s > P_{su}$
No
Yes
$P_s = P_{su}$
$M_c = E_o\varepsilon_m\varepsilon(1-R)[1-\varepsilon(1+R)/2]/12(DT)^2$
$M_s = P_s \cdot \sigma_m(DT-.5)/(DT)^2$
$c = \varepsilon(1-R)\varepsilon_m\, DT\times 1000$
$M = M_c - M_s$ and $PP = P_c + P_s$
Print P_c, P_s, M_c, M_s, PP, c and M
4
NEXT MM
2


2
$A_3 = (3-\varepsilon)/6np$
$B_3 = 1-F/(2\varepsilon np)$
$e_3 = -DT$ and
$D_{1/2} = [-B_3 \pm \sqrt{(B_3^2-4A_3e_3)}\,]/2A_3$
Section cracked
Steel unyielding
$D_3 = (6np\varepsilon_y/\varepsilon_m + 3F)/\varepsilon(3-\varepsilon)$
Steel yielding
Print ε, D_1, D_2 and D_3
NPUT "NO="; NO(Total numbers of valid values of D's
NO=0
Yes
No
MM = 1 to NO
5
6
INPUT "D="; D
$P_s = 2np\,\varepsilon\,(DT/D-1)$
$P_s > P_{su}$
No
Yes
$P_s = P_{su}$
3

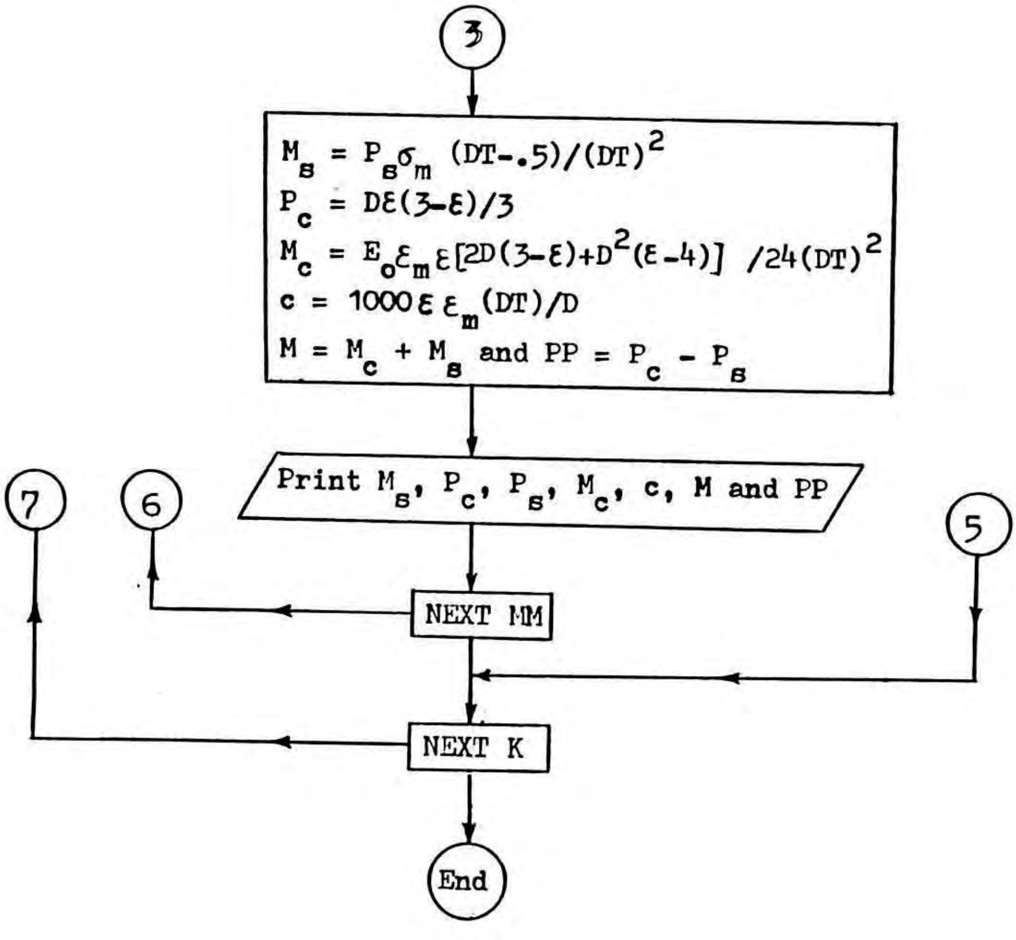


Figure 4.3 Flowchart for moment-curvature analysis.

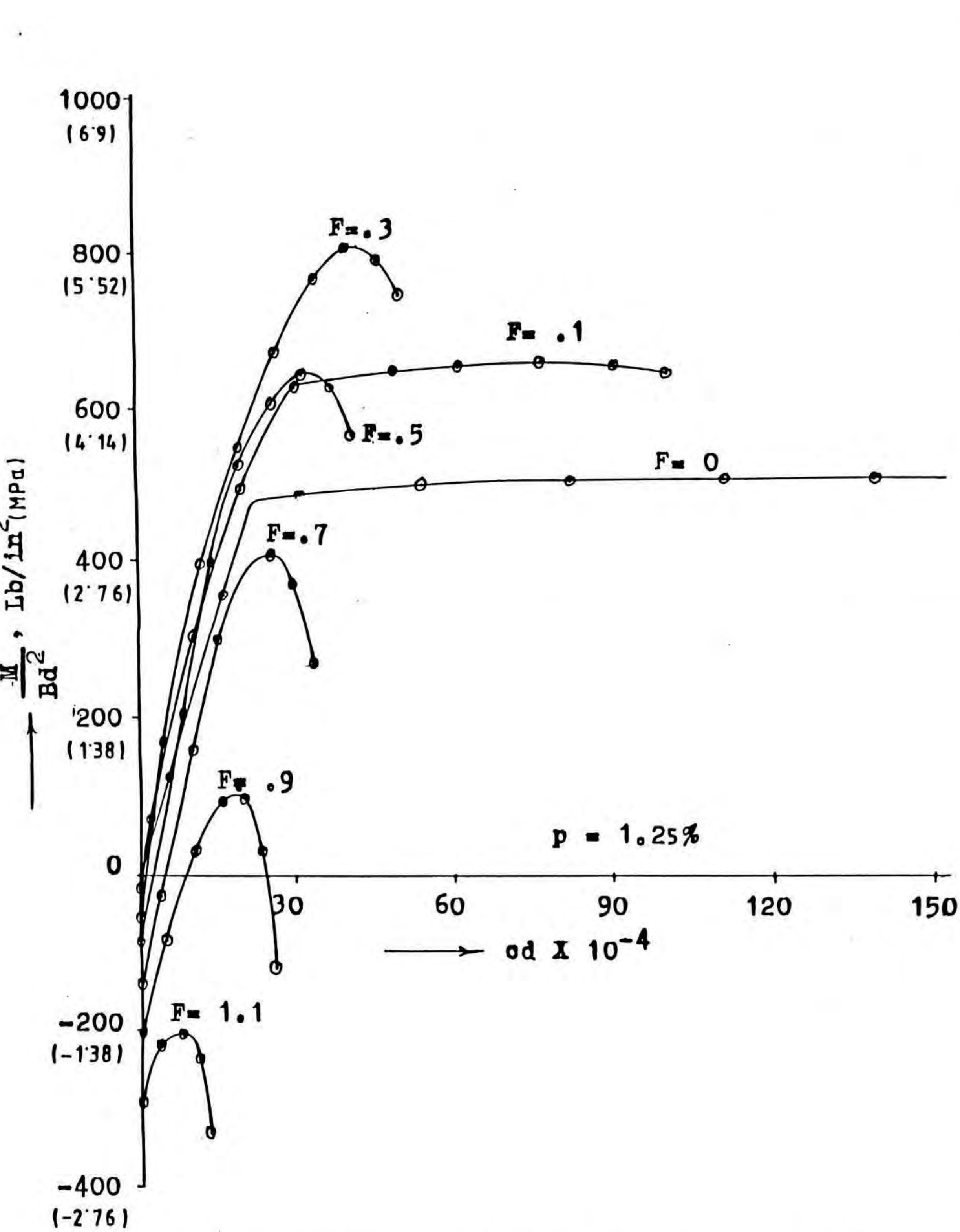


Figure 4.4: Theoritical moment-curvature relationships of a R.C.C. section for p=1.25%.

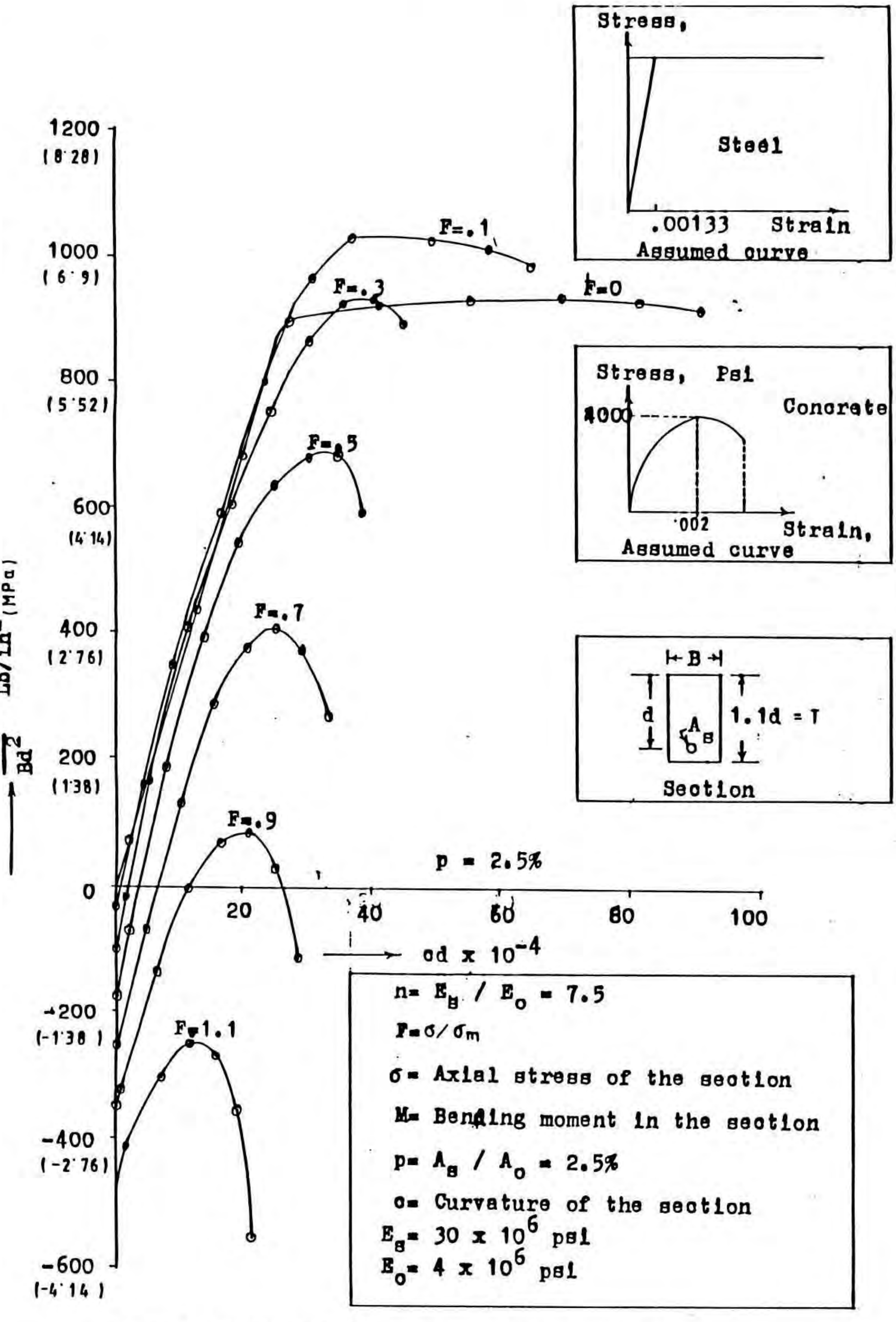


Figure 4.5 : Theoritical moment-curvature relationships of a R.C.C. section for p-2·5 %.

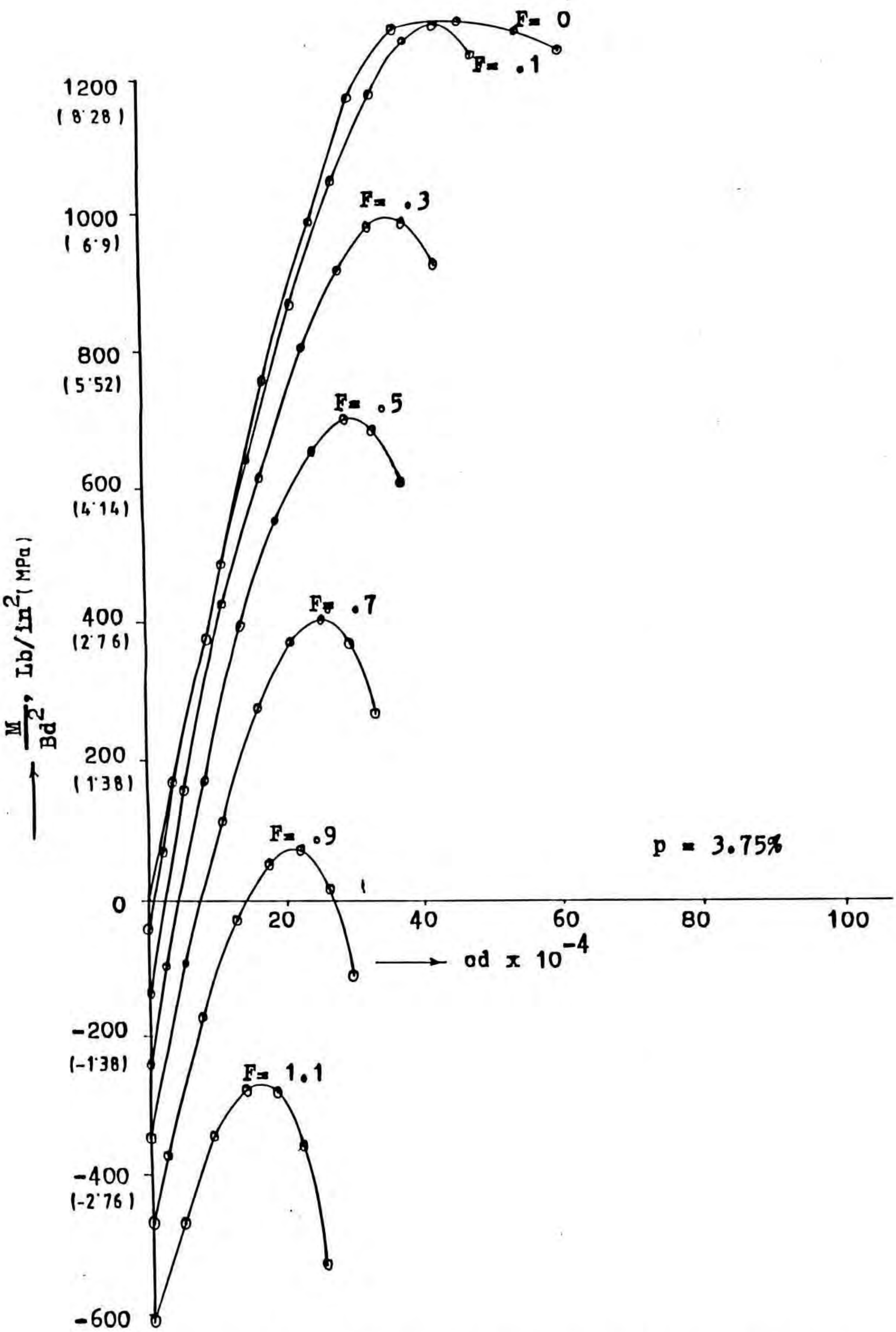


Figure 4.6: Theoritical moment-curvature relationships of a R.C.C. section for p = 3·75%.

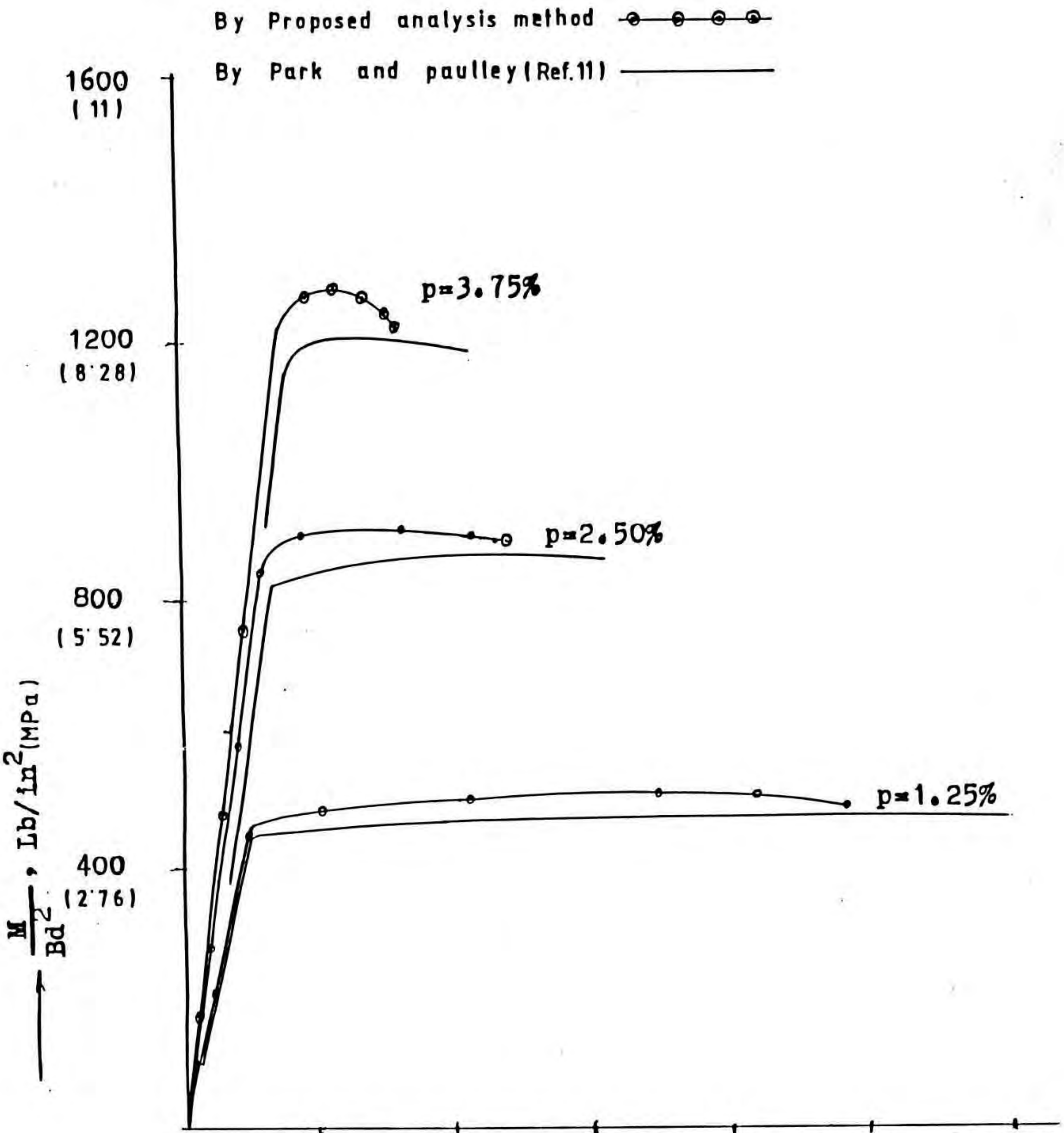


Figure 4.7 : Theoritical moment-curvature curves of a flexural R.C.C. section.

more brittle with increasing axial load. It is further observed that the section remains more ductile for smaller value of steel ratio, as expected.

In Fig.4.7 moment-curvature curves under zero axial load presented in comparison to similar curves presented in Ref.11. The curves can be observed to match each other favorably unless that for higher steel ratio. For higher steel ratio the difference is more pronounced, probably due to the difference in assumed stress-strain curves of concrete. It is parabolic, as assumed throughout the work presented in this thesis while in Ref.11 it is parabolic upto the peak stress and the rest of the curve is straight line, as shown in Fig.4.8.

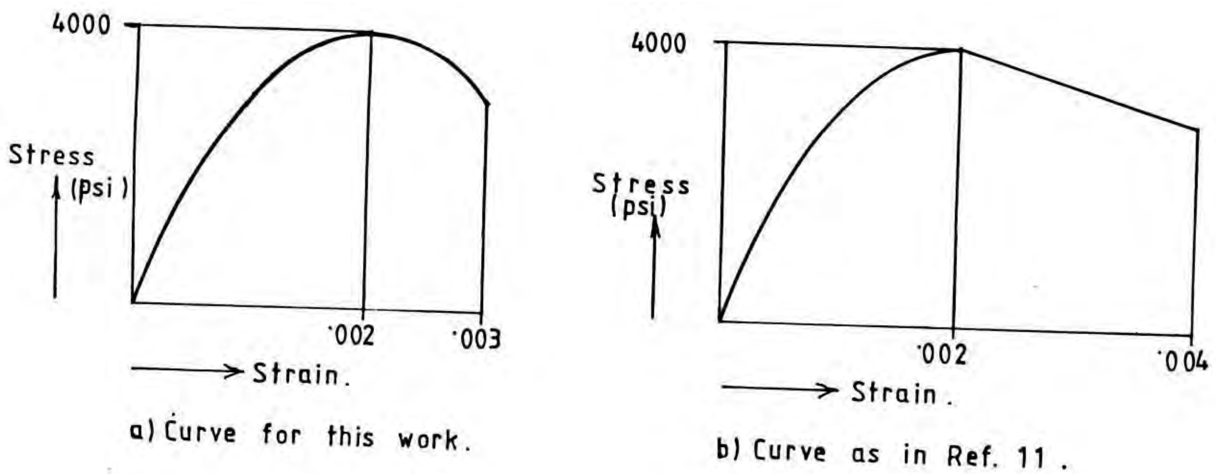


Figure 4.8 Assumed stress-strain curve of concrete for moment-curveture curves.

In the Fig.4.8, the curve in (b) is more flatter than the curve in (a) beyond the peak stress and this causes the moment-curvature curve for p=3.75% to be less flatter than that of Ref.11. The curves of Ref.11 can also be seen to be relatively longer which may indicate that they are relatively more ductile. This difference arises because the stress-strain curve of concrete in Fig.4.8b terminates at a higher strain of 0.004 while that in Fig.4.8a terminates at 0.003. In general, it can be concluded that the curves, represent the moment-curvature relationship, are quite well, as expected for a reinforced concrete section.

4.4 Conclusions and Limitations

The stiffness properties developed applying first principles is a first step in developing a numerical model which can be utilised for limit state analysis of reinforced and prestressed concrete members under flexure and axial load. The numerical model can thus be useful to understand the behaviour of reinforced and prestressed concrete members throughout their loading history upto collapse.

It is to be noted that the stiffness properties developed for different sectional elements correspond to a particular level of strain in the element. Combining the stiffness matrices for different type of sectional elements present in a section, complete stiffness matrix of the section is obtained.

The equations and basic assumptions used to obtain the stiffness matrices are also utilised to study the moment-curvature relationship of a rectangular section. The moment-curvature curves are found quite well to represent the relationship as expected for a reinforced concrete section and in comparison to similar curves presented by others.

The explicit stiffness matrix developed for a beam-column element would save much computational efforts by avoiding numerical integration usually carried out for evaluating stiffness matrices of finite elements. The matrix developed correspond to a particular level of strain present in the mid-section of the element.

The stiffness matrices developed in this work followed a similar work carried out for structural brickwork(15). A numerical model was then developed for limit state analysis of structural brickwork member and particularly applied to brickwork arches. The numerical model was able to study the behaviour of arches throughout the loading history. It was further demonstrated that the model predicting both the crushing and instability mode of failure automatically. Thus it is hoped that the stiffness matrix developed in this work will enable us to develop a numerical model having similar capabilities.

4.5 Recommendations for Further Study

Utilising the stiffness properties presented in this work, a numerical model should be developed to study the load-deflection behaviour, extent of cracking with load, crushing and instability failure patterns of reinforced and prestressed concrete structures.

The stiffness matrices developed in this work are based on the parabolic stress-strain relationship of concrete in compression and elasto-plastic stress-strain characteristics of reinforcing and prestressing steel. Applying the same principle, similar stiffness matrices can be developed correspond to other type of stress-strain characteristics if found suitable for a particular concrete and steel.

REFERENCES

1. T.Y. LIN and NED H. BURNS
   'Design of Prestressed Concrete Structures', 3rd Edition, John Wiley and Sons, 1981.

2. N. KRISHNA RAJU
   'Prestressed Concrete', Tata McGraw-Hill Publishing Company Limited, New Delhi, India.

3. ACI: 318-77; ACI Standard Building Code Requirement for Reinforced Concrete, Detroit, Michigan, 1977.

4. CP 110 Part1: 1972; Code of Practice for the Structural Use of Concrete, British Standard Institution, London, Nov. 1972.

5. IS 456-1979, Indian Standard Code of Practice for Plain and Reinforced Concrete for General Building Construction, Indian Standards Institution, New Delhi, 1977(3rd Revision).

6. IS:1343, Indian Code of Practice for Prestressed Concrete (draft revision), Indian Standards Institution, New Delhi, Oct.1975.

7. GEORGE WINTER and ARTHUR H. NILSON
'Design of Concrete Structures', 9th Edition, McGraw-Hill International Book Company.

8. HOGNESTAD, E. 'Confirmation of inelastic Stress Distribution in Concrete', Proceedings, ASCE, V.83, ST2, March,1957.

9. PRAKASH DESAYI and S. KRISHNAN
'Equations for the Stress-strain Curve of Concrete', J. Am. Conc. Inst., March,1964.

10. SMITH, G.M. and YOUNG, L.E.
'Ultimate Theory in Flexure by Exponential Function', ACI Journal, Proceedings V.52, No.3, Nov.1955.

11. R. PARK and T. PAULAY
'Reinforced Concrete Structures', A Willey Interscience Publication.

12. M.A. ROUF
'Fundamental Properties of Brickwork with Particular Emphasis to Brickwork Arches', Ph.D. Thesis, University of Liverpool, September 1984.

13. F. SAWKO and M.A. ROUF

'On the Stiffness Properties of Masonry', Proceedings of Inst. of Civil Engineers, TN 381, March 1984.

14. I.S. SOKOLNIKOFF and R.M. REDHEFFER

'Mathematics of Physics and Engineering', McGraw-Hill, N.Y. 1958.

15. F. SAWKO and M.A. ROUF

'A Proposed Numerical Model for Structural Masonry', Masonry International, No.5, July 1985.